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► **To cite this version:**

S Konate, Marcu-Antone Orsoni. SAMPLING CONSTANTS IN GENERALIZED FOCK SPACES. 2021. hal-03174991v2

HAL Id: hal-03174991

<https://hal.science/hal-03174991v2>

Preprint submitted on 3 Oct 2021 (v2), last revised 6 Apr 2024 (v4)

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SAMPLING CONSTANTS IN GENERALIZED FOCK SPACES

S. KONATE & M.-A. ORSONI

ABSTRACT. We prove several results related to a Logvinenko-Sereda type theorem on dominating sets for generalized doubling Fock spaces. In particular, we give a precise polynomial dependence of the sampling constant on the relative density parameter γ of the dominating set. Our method is an adaptation of that used in [HKKO21] for the Bergman spaces and is based on a Remez-type inequality and a covering lemma related to doubling measures.

1. INTRODUCTION

Sampling problems are central problems in signal theory and cover, for instance, sampling sequences and so-called dominating sets which allow to recover the norm of a signal — defined by an integration over a given domain — from the integration on a subdomain (precise definitions will be given later). They have been considered in a large setting of situations (see e.g. the survey [FHR17]), including the Fock space and its generalized version (see [Sei04, MMOC03, OC98, Lin00, JPR87, LZ19]). In this paper we will focus on the second class of problems, i.e. dominating sets. Conditions guaranteeing that a set is dominating have been established rather long ago (in the 70's for the Paley-Wiener space and in the 80's for the Bergman and Fock spaces, see e.g. the survey [FHR17] and references therein). More recently, people got interested in estimates of the sampling constants which give an information on the tradeoff between the cost of the sampling and the precision of the estimates. The major paper in this connection is by Kovrijkine [Kov01] who gave a method to consider this problem in the Paley-Wiener space. Subsequently, his method was adapted to several other spaces (see [HJK20, JS21, HKKO21, GW21, BJPS21]). In this paper, inspired by the methods in [HKKO21], we will discuss the case of generalized doubling Fock spaces. For these, the paper [MMOC03] provides a wealth of results that allow to translate the main steps of [HKKO21] to this new framework.

1.1. Doubling subharmonic functions and generalized Fock spaces. A function $\phi : \mathbb{C} \rightarrow \mathbb{R}$ of class C^2 is said to be *subharmonic* if $\Delta\phi \geq 0$, and *doubling subharmonic* if it is subharmonic and the (non-negative) measure $\mu := \Delta\phi$ is doubling, i.e. there exists a

2010 *Mathematics Subject Classification.* 30H20.

Key words and phrases. Dominating sets, reverse Carleson measure, Fock space, sampling.

The research of the first author is partially supported by Banque Mondiale via the Projet d'Appui au Développement de l'Enseignement Supérieur du Mali. The research of the second author is partially supported by the project ANR-18-CE40-0035 and by the Joint French-Russian Research Project PRC CNRS/RFBR 2017–2019.

constant C such that for every $z \in \mathbb{C}$ and $r > 0$,

$$\mu(D(z, 2r)) \leq C\mu(D(z, r)). \quad (1.1)$$

The minimal constant satisfying (1.1) is denoted by C_μ and called *doubling constant* for μ . Here $D(z, r)$ denotes the standard open euclidean disk of center $z \in \mathbb{C}$ and radius $r > 0$. Such a doubling subharmonic function induces a new metric on the complex plane \mathbb{C} which is more adapted to the complex analysis we shall work with. We postpone the advanced results on doubling measures to Section 2 but let us introduce some basic objects related to it. First, we can associate with a ϕ as above a function $\rho : \mathbb{C} \rightarrow \mathbb{R}^+$, such that

$$\mu(D(z, \rho(z))) = 1.$$

To get an idea, assuming ϕ suitably regularized, we have $\Delta\phi \asymp \rho^{-2}$ (see [MMOC03, Theorem 14 and Remark 5]). Next, denote $D^r(z) := D(z, r\rho(z))$ the disk adapted to the new metric and write $D(z) := D^1(z)$ the disk with unit mass for the measure μ . Finally, with these definitions in mind we can introduce the following natural density. A measurable set E is (γ, r) -dense (with respect to the above metric), if for every $z \in \mathbb{C}$

$$\frac{|E \cap D^r(z)|}{|D^r(z)|} \geq \gamma.$$

Here $|F|$ denotes planar Lebesgue measure of a measurable set F . We will just say that the set is *relatively dense* if there is some $\gamma > 0$ and some $r > 0$ such that the set is (γ, r) -dense. We shall see in the next subsection that relative density is the right notion to characterize the dominating sets.

For a doubling subharmonic function $\phi : \mathbb{C} \rightarrow \mathbb{R}$, and $1 \leq p < +\infty$, we define the *doubling Fock space*, by

$$\mathcal{F}_\phi^p = \{f \in \text{Hol}(\mathbb{C}) : \|f\|_{p,\phi}^p := \int_{\mathbb{C}} |f|^p e^{-p\phi} dA < +\infty\},$$

where dA is planar Lebesgue measure on \mathbb{C} . These spaces appear naturally in the study of the Cauchy-Riemann equation (see [Chr91, COC11]) and are well studied objects. When $\phi(z) = |z|^2$, \mathcal{F}_ϕ^p is the classical Fock space (see the textbook [Zhu12] for more details).

1.2. Dominating sets and sampling constants. A measurable set $E \subset \mathbb{C}$ will be called *dominating* for \mathcal{F}_ϕ^p if there exists $C > 0$ such that

$$\int_E |f|^p e^{-p\phi} dA \geq C^p \int_{\mathbb{C}} |f|^p e^{-p\phi} dA, \quad \forall f \in \mathcal{F}_\phi^p. \quad (1.2)$$

The biggest constant C appearing in (1.2) is called the *sampling constant*. We will use the notation $L_\phi^p(F)$ for the Lebesgue space on a measurable set $F \subset \mathbb{C}$ with respect to the measure $e^{-p\phi} dA$.

The notion of dominating sets can be defined for several spaces of analytic functions and their characterization has been at the center of an intensive research starting with a famous result of Panejah [Pan62, Pan66], and Logvinenko and Sereda [LS74] for the Paley-Wiener space which consists of all functions in L^2 whose Fourier transform is supported

on a compact interval. For corresponding results in the Bergman space we refer to [Lue81, Lue85, Lue88, GW21]. As to Fock spaces, this has been done by Jansen, Peetre and Rochberg in [JPR87] for the classical case $\varphi(z) = |z|^2$ and by Lou and Zhuo in [LZ19, Theorem A] for generalized doubling Fock spaces. It turns out that a set E is dominating if and only if it is relatively dense. This characterization also holds for the other spaces of analytic functions mentioned above with an adapted notion of relative density.

Once these conditions established for being dominating, a second question of interest is to know whether the sampling constants can be estimated in terms of the density parameters (γ, r) . Kovrijkine answered this question in [Kov01] giving a sharp estimate on the sampling constants for the Paley-Wiener spaces, with a polynomial dependence on γ (see also [Rez10] for sharp constants in some particular geometric settings). Such a dependence is used for example in control theory (see [EV18, ES21, BJPS21] and the references therein). Kovrijkine's method involves Remez-type and Bernstein inequalities. It has been used in several spaces in which the Bernstein inequality holds (see for instance [HJK20] for the model space, [BJPS21] for spaces spanned by Hermite functions, or [ES21] for general spectral subspaces), and also to settings where a Bernstein inequality is not at hand (e.g. Fock and polyanalytic Fock spaces [JS21] or Bergman spaces [HKKO21, GW21]). In [HKKO21], the authors developed a machinery based on a covering argument to circumvent Bernstein's inequality and the aim of this paper is to adapt this method to doubling Fock spaces.

1.3. Main results. It can be deduced from Lou and Zhuo's result [LZ19, Theorem A] that if E is (γ, r) -dense then there exist some constants ε_0 and $c > 0$ depending only on r such that inequality (1.2) holds for every

$$C^p \geq c\gamma\varepsilon_0^{\frac{2(p+2)}{\gamma}}.$$

This last estimate gives an exponential dependence on γ . In the spirit of the work [Kov01], we improve it providing a polynomial estimate in γ with a power depending suitably on r .

Theorem 1. *Let ϕ be a doubling subharmonic function and $1 \leq p < +\infty$. Given $r > 1$, there exists L such that for every measurable set $E \subset \mathbb{C}$ which is (γ, r) -dense, we have*

$$\|f\|_{L_\phi^p(E)} \geq \left(\frac{\gamma}{c}\right)^L \|f\|_{p,\phi} \quad (1.3)$$

for every $f \in \mathcal{F}_\phi^p$. Here, the constants c and L depend on r , and for L we can choose

$$L \leq \lambda r^{\frac{1}{\kappa}} + \frac{1}{p}(\lambda' + \lambda'' \ln(1+r))$$

where κ is a constant depending on ϕ (see (2.1) below); and $\lambda, \lambda', \lambda''$ are some universal constants depending also only on the space.

Observe that we are mainly interested in the case when r is big, for instance $r \geq 1$ (in case E is (γ, r) -dense for some $r < 1$ we can also show that E is $(\tilde{\gamma}, 1)$ -dense for some $\tilde{\gamma} \asymp \gamma$ where underlying constants are universal).

The proof of Theorem 1 follows the scheme presented in [HKKO21]. We will recall the necessary results from that paper. The main new ingredients come from [MMOC03] and concern a finite overlap property and a lemma allowing to express the (subharmonic) weight locally as (the real part of) a holomorphic function.

In [LZ19, Theorem 7], proving that the relative density is a necessary condition for domination, it is shown that $\gamma \gtrsim C^p$. Hence, we cannot expect better than a polynomial dependance in γ of the sampling constant C . In this sense, our result is optimal.

As noticed in a remark in [Lue81, p.11] and after Theorem 2 in [HKKO21] for the Bergman space, there is no reason a priori why a holomorphic function for which the integral $\int_E |f|^p e^{-p\phi} dA$ is bounded for a relative dense set E should be in \mathcal{F}_ϕ^p . Outside the class \mathcal{F}_ϕ^p relative density is in general not necessary for domination. For this reason, we always assume that we test on functions $f \in \mathcal{F}_\phi^p$.

As a direct consequence of Theorem 1, we obtain a bound for the norm of the inverse of a Toeplitz operator T_φ . We remind that for any bounded measurable function φ , the Toeplitz operator T_φ is defined on \mathcal{F}_ϕ^2 by $T_\varphi f = \mathbf{P}(\varphi f)$ where \mathbf{P} denotes the orthogonal projection from $L_\phi^2(\mathbb{C})$ onto \mathcal{F}_ϕ^2 . As remarked in [LZ19, Theorem B], for a non-negative function φ , T_φ is invertible if and only if $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is a dominating set for some $s > 0$. Tracking the constants we obtain

Corollary 1. *Let φ be a non-negative bounded measurable function. The operator T_φ is invertible if and only if there exists $s > 0$ such that $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is (γ, r) -dense for some $\gamma > 0$ and $r > 0$. In this case, we have*

$$\|T_\varphi^{-1}\| \leq \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - \left(\frac{s}{\|\varphi\|_\infty}\right)^2} \left(\frac{\gamma}{c}\right)^{2L}}.$$

Notice that the right-hand side behaves as γ^{-L} as $\gamma \rightarrow 0$. For the sake of completeness, we will give a proof of the reverse implication at the end of Section 4.

The paper is organized as follows. In Section 2 we recall several results from [MMOC03] concerning doubling measures and subharmonic functions. In Section 3, we introduce planar Remez-type inequalities which will be a key ingredient in the proof of Theorem 1. Finally, we prove the covering lemma and Theorem 1 in Section 4 and deduce Corollary 1.

2. REMINDERS ON DOUBLING MEASURES

A non-negative doubling measure μ (see (1.1) for the definition) determines a metric on \mathbb{C} to which the usual notions have to be adapted to. In this section, we recall several geometric measure theoretical tools in connection with doubling measures that we will need later and which come essentially from the paper [MMOC03, Chapter 2]. Actually, the results of the present section can be translated in terms of the distance induced by the metric $\rho(z)^{-2} dz \otimes d\bar{z}$ but we will not exploit this point of view here.

We start with two geometric estimates. The first is given by [MMOC03, Equation (4)] and will be used several times in this article: there exists $\kappa > 0$ such that for all $z \in \mathbb{C}$ and $r > 1$,

$$r^\kappa \lesssim \mu(D^r(z)) \lesssim r^{\frac{1}{\kappa}}. \quad (2.1)$$

The second is a part of Corollary 3 of [MMOC03] that we recall here.

Corollary 2 (Corollary 3 of [MMOC03]). *For every $r > 0$, there exists $\kappa \geq 0$ such that if $\zeta \in D^r(z)$, then*

$$\rho(z) \lesssim \rho(\zeta)(1+r)^\kappa.$$

We say that a sequence $(a_n)_{n \in \mathbb{N}}$ is ρ -separated if there exists $\delta > 0$ such that

$$|a_i - a_j| \geq \delta \max(\rho(a_i), \rho(a_j)), \quad \forall i \neq j.$$

This means that the disks $D^\delta(a_n)$ are pairwise disjoint. We will cover \mathbb{C} by disks satisfying a finite overlapping property and such that the sequence formed by their centers is ρ -separated. In [MMOC03], the authors construct a decomposition of \mathbb{C} into rectangles R_k (so-called quasi-squares): $\mathbb{C} = \bigcup_k R_k$, and two such rectangles can intersect at most along sides. Denoting by a_k the center of R_k , Theorem 8(c) of [MMOC03] claims in particular that there is $r_0 \geq 1$, such that for every k ,

$$\frac{\rho(a_k)}{r_0} \leq \text{diam } R_k \leq r_0 \rho(a_k).$$

In other words

$$D^{1/(2Cr_0)}(a_k) \subset R(a_k) \subset D^{r_0/2}(a_k) \quad (2.2)$$

with $C = \sqrt{1+e^2}$. As a consequence we get the required covering $\mathbb{C} = \bigcup D^{r_0/2}(a_k)$ and the sequence (a_k) is ρ -separated (take $\delta = 1/(2Cr_0)$).

For such ρ -separated sequences, the doubling measure μ satisfies a summability condition. The following lemma is essentially contained in the proof of [MMOC03, Lemma 6] and [MMOC03, Lemma 5.b)].

Lemma 1. *Let $m > 1 + \frac{1}{\kappa}$. There exists $C(\phi, m) > 0$ such that for every $r \geq 1$*

$$\sup_{z \in \mathbb{C}} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \leq C(\phi, m) r^{\frac{1}{\kappa} - \frac{m}{1+\kappa}}.$$

It will be shown in the proof that $C(\phi, m) \sim \frac{1}{m-1-\frac{1}{\kappa}}$ when m tends to $1 + \frac{1}{\kappa}$. The proof also shows that $r \geq 1$ is a rather technical condition and can be replaced by $r \geq \hat{r} > 0$.

Proof. Let us start with $r > 1$ large, to be fixed later. Since the sequence $(a_k)_{k \in \mathbb{N}}$ is separated, we can take δ such that $0 < \delta < 1/4$ and the disks $(D^\delta(a_k))_{k \in \mathbb{N}}$ are pairwise disjoint. There exists $i \geq 3$ such that $\frac{1}{2^i} < \delta$. Therefore $1 = \mu(D^1(a_k)) \leq C_\mu^i \mu(D^\delta(a_k))$

where C_μ is the doubling constant. Hence for every $z \in \mathbb{C}$

$$\begin{aligned} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &= \sum_{a_k \notin D^r(z)} \frac{1}{\mu(D^\delta(a_k))} \int_{D^\delta(a_k)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m d\mu(\zeta) \\ &\leq C_\mu^i \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m d\mu(\zeta). \end{aligned} \quad (2.3)$$

Now, we claim that $z \notin D^\delta(a_k)$ whenever $a_k \notin D^r(z)$. Indeed, suppose that $z \in D^\delta(a_k)$. There are two cases. First, if $\delta\rho(a_k) < r\rho(z)$ then $|z - a_k| < \delta\rho(a_k) < r\rho(z)$ and so $a_k \in D^r(z)$ which is a contradiction. Secondly if $\delta\rho(a_k) \geq r\rho(z)$ then $D^r(z) \subset D^{2\delta}(a_k)$ (see Figure 1). Therefore $1 < \mu(D^r(z)) \leq \mu(D^{2\delta}(a_k)) < 1$ which is a contradiction.

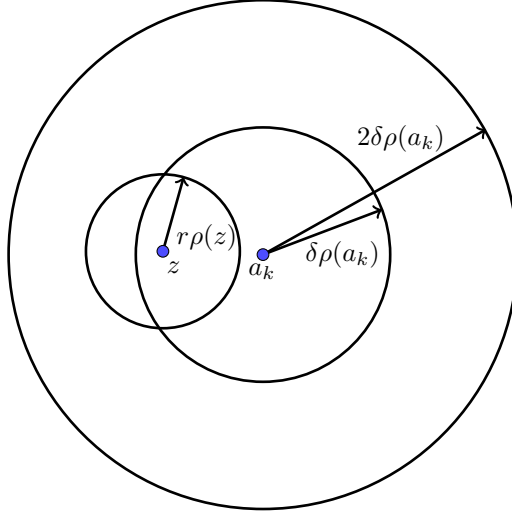


FIGURE 1. The disks $D^\delta(a_k)$ and $D^r(z)$.

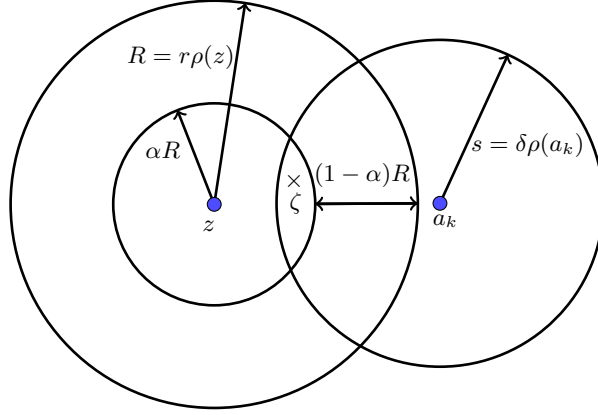
Thus, if $a_k \notin D^r(z)$ and $\zeta \in D^\delta(a_k)$, we have

$$|z - \zeta| \leq |z - a_k| + |a_k - \zeta| \leq 2|z - a_k|. \quad (2.4)$$

Also, by Corollary 2, if $\zeta \in D^\delta(a_k)$ then there exists $C > 0$ such that $\rho(a_k) \leq C(1 + \delta)^\kappa \rho(\zeta)$. With inequality (2.3), we obtain

$$\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \leq C_\mu^i (2C(1 + \delta)^\kappa)^m \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta)$$

Now, setting $\alpha = 1 - 2\delta < 1$ we have $D^{\alpha r}(z) \cap D^\delta(a_k) = \emptyset$. Indeed, if $\zeta \in D^{\alpha r}(z) \cap D^\delta(a_k)$, then $|\zeta - z| \leq \alpha r\rho(z)$ and $|a_k - \zeta| \leq \delta\rho(a_k)$. Let $R = r\rho(z)$ and $s = \delta\rho(a_k)$, then $s > (1 - \alpha)R$, since $a_k \notin D^r(z) = D(z, R)$ (see Figure 2).


 FIGURE 2. The disks $D(a_k, s)$, $D(z, R)$ and $D(z, \alpha R)$.

Therefore, $|z - \zeta| \leq \alpha R < \frac{s\alpha}{1-\alpha}$ and it follows by the triangular inequality

$$\begin{aligned} |z - a_k| &\leq |\zeta - a_k| + |\zeta - z| \leq \delta\rho(a_k) + \frac{s\alpha}{1-\alpha} \\ &= \delta\rho(a_k) + \frac{\delta\rho(a_k)\alpha}{1-\alpha} \\ &= \frac{\delta\rho(a_k)}{1-\alpha}. \end{aligned}$$

So $z \in D^{\frac{\delta}{1-\alpha}}(a_k)$. Since, $a_k \notin D^r(z)$ this implies $D^r(z) \subset D^{\frac{2\delta}{1-\alpha}}(a_k)$. But

$$1 = \mu(D^1(z)) < \mu(D^r(z)) \leq \mu(D^{\frac{2\delta}{1-\alpha}}(a_k)) = \mu(D^1(a_k)) = 1.$$

Contradiction. Hence,

$$\begin{aligned} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta) \\ &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta). \end{aligned}$$

This leads by Fubini's theorem

$$\begin{aligned} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} m \int_0^{\rho(\zeta)/|z-\zeta|} t^{m-1} dt d\mu(\zeta) \\ &= C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} m \int_0^\infty \mathbf{1}_{\{0 < t < \rho(\zeta)/|z-\zeta|\}} t^{m-1} dt d\mu(\zeta) \\ &= C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^\infty m t^{m-1} \int_{\zeta \notin D^{\alpha r}(z)} \mathbf{1}_{\{\zeta \in \mathbb{C} \mid 0 < t < \rho(\zeta)/|z-\zeta|\}} d\mu(\zeta) dt \end{aligned}$$

Note that

$$|z - \zeta| < \frac{\rho(\zeta)}{t} \quad (2.5)$$

is equivalent to $z \in D^{1/t}(\zeta)$.

Now, we claim that $t < 1$ for r large enough when $0 < t < \rho(\zeta)/|z - \zeta|$ and $\zeta \notin D^{\alpha r}(z)$. Indeed, if $t \geq 1$ and $z \in D^{1/t}(\zeta)$ then by Corollary 2 there exists $C > 0$ such that $\rho(\zeta) \leq C\rho(z)(1 + \frac{1}{t})^\kappa$. Replacing in (2.5), we obtain $|z - \zeta| < C\rho(z)(1 + \frac{1}{t})^\kappa/t \leq C\rho(z)2^\kappa/t$. Moreover, since $\zeta \notin D^{\alpha r}(z)$, we have

$$\rho(z)\alpha r \leq |z - \zeta| < C2^\kappa/t\rho(z).$$

Hence, $\alpha r < C2^\kappa/t$ or equivalently

$$t < \frac{C2^\kappa}{\alpha r}.$$

Finally, fixing $r > r' := \frac{C2^\kappa}{\alpha}$, we obtain a contradiction with $t \geq 1$, so that indeed $t < 1$.

Now, since $0 < t < 1$ and repeating the same arguments, by Corollary 2 we have $\rho(\zeta) \leq C\rho(z)(1 + \frac{1}{t})^\kappa \leq C\rho(z) \left(\frac{2}{t}\right)^\kappa$, and with inequality (2.5), this leads to $\zeta \in D^\beta(z)$ where $\beta = \frac{C}{t} \left(\frac{2}{t}\right)^\kappa = \frac{C2^\kappa}{t^{\kappa+1}}$. Also,

$$\zeta \in D^\beta(z) \setminus D^{\alpha r}(z) \iff \rho(z)\alpha r \leq |z - \zeta| < \beta\rho(z) \implies \left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}} > t.$$

Thus, $0 < t < \left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}} < 1$ since r is fixed to be larger than $r' = \frac{C2^\kappa}{\alpha}$. It follows

$$\begin{aligned} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|}\right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-1} \int_{\zeta \in D^\beta(z) \setminus D^{\alpha r}(z)} d\mu(\zeta) dt \\ &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-1} \mu(D^\beta(z)) dt \\ &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-1} \beta^{\frac{1}{\kappa}} dt \end{aligned}$$

where the last inequality comes from (2.1). Hence we obtain

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-1} \left(\frac{C2^\kappa}{t^{\kappa+1}} \right)^{\frac{1}{\kappa}} dt \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-2-\frac{1}{\kappa}} dt \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \frac{m}{m-1-\frac{1}{\kappa}} \left[t^{m-1-\frac{1}{\kappa}} \right]_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \frac{m}{m-1-\frac{1}{\kappa}} \left(\frac{C2^\kappa}{\alpha} \right)^{\frac{m-1-\frac{1}{\kappa}}{1+\kappa}} \left(\frac{1}{r} \right)^{\frac{m-1-\frac{1}{\kappa}}{1+\kappa}} \\
&= 2^{m(1+\frac{\kappa}{1+\kappa})} C^{m+\frac{m}{1+\kappa}} C_\mu^i (1 + \delta)^{\kappa m} \frac{m}{m-1-\frac{1}{\kappa}} \alpha^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} r^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} \\
&= C_1 r^{-\frac{m+1+\frac{1}{\kappa}}{1+\kappa}}.
\end{aligned}$$

with $C_1 := 2^{m(1+\frac{\kappa}{1+\kappa})} C^{m+\frac{m}{1+\kappa}} C_\mu^i (1 + \delta)^{\kappa m} \frac{m}{m-1-\frac{1}{\kappa}} \alpha^{\frac{1}{\kappa}-\frac{m}{1+\kappa}}$ and $m > 1 + \frac{1}{\kappa}$. In particular, $C_1 \sim 1/(m-1-1/\kappa)$, where the underlying constants (κ, C_μ, δ , etc.) only depend on the space and the reference sequence (a_k) . Hence the result is proved for $r > r'$.

Let us prove now that the result holds for $1 \leq r \leq r'$. Let $z \in \mathbb{C}$. Pick z_1 satisfying $D^{2r'}(z) \cap D^{2r'}(z_1) = \emptyset$.

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq \sum_{k \in \mathbb{N}} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \\
&\leq \sum_{a_k \notin D^{2r'}(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m + \sum_{a_k \notin D^{2r'}(z_1)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \\
&\leq 2C_1 (2r')^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} \leq C_2 r^{\frac{1}{\kappa}-\frac{m}{1+\kappa}}.
\end{aligned}$$

where, noticing that $\frac{1}{\kappa} - \frac{m}{1+\kappa} < 0$, we write $C_2 := C_1 2^{1+\frac{1}{\kappa}-\frac{m}{1+\kappa}}$. Thus, the proof is complete for $C(\phi, m) = \max(C_1, C_2) \sim \frac{1}{m-1-1/\kappa}$. \square

We end this section by focusing on the particular case of a doubling measure μ given by the Laplacian $\Delta\phi$ of a subharmonic function ϕ which is the case of interest in this paper. We will need to control the value of ϕ in a disk by the value at its center. For this, we can use [MMOC03, Lemma 13] that we state as follows: for every $\sigma > 0$, there exists $A = A(\sigma) > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{z \in D^\sigma(a_k)} |\phi(z) - \phi(a_k) - \mathbf{h}_{a_k}(z)| \leq A(\sigma) \tag{2.6}$$

where h_{a_k} is a harmonic function in $D^\sigma(a_k)$ with $h_{a_k}(a_k) = 0$. Moreover, in view of the proof of [MMOC03, Lemma 13] we have

$$A(\sigma) \leq C \sup_{k \in \mathbb{N}} \mu(D^\sigma(a_k)) \leq \tilde{C} \sigma^{\frac{1}{\kappa}} \quad (2.7)$$

with \tilde{C} depending only on the space and where the inequalities come from [MMOC03, Lemma 5(a)] and the estimate (2.1). This last result will allow us to translate the subharmonic weight locally into a holomorphic function.

3. REMEZ-TYPE INEQUALITIES

Let us introduce central results of the paper [AR07]. Let G be a (bounded) domain in \mathbb{C} . Let $0 < s < |\overline{G}|$ (Lebesgue measure of \overline{G}). Denoting Pol_n the space of complex polynomials of degree at most $n \in \mathbb{N}$, we introduce the set

$$P_n(\overline{G}, s) = \{p \in \text{Pol}_n : |\{z \in \overline{G} : |p(z)| \leq 1\}| \geq s\}.$$

Next, let

$$R_n(z, s) = \sup_{p \in P_n(\overline{G}, s)} |p(z)|.$$

This expression gives the biggest possible value at z of a polynomial p of degree at most n and being at most 1 on a set of measure at least s . In particular Theorem 1 from [AR07] claims that for $z \in \partial G$, we have

$$R_n(z, s) \leq \left(\frac{c}{s}\right)^n. \quad (3.1)$$

This result corresponds to a generalization to the two-dimensional case of the Remez inequality which is usually given in dimension 1. In what follows we will essentially consider G to be a disk or a rectangle. By the maximum modulus principle, the above constant gives an upper estimate on G for an arbitrary polynomial of degree at most n which is bounded by one on a set of measure at least s . Obviously, if this set is small (s close to 0), i.e. p is controlled by 1 on a small set, then the estimate has to get worse.

Remark 1. *Let us make another observation. If c is the constant in (3.1) associated with the unit disk $G = \mathbb{D} = D(0, 1)$, then a simple argument based on homothecy shows that the corresponding constant for an arbitrary disk $D(0, r)$ is cr^2 (considering $D(0, r)$ as underlying domain, the constant c appearing in [AR07, Theorem 1] satisfies $c > 2m_2(D(0, r))$). So, in the sequel we will use the estimate*

$$R_n(z, s) \leq \left(\frac{cr^2}{s}\right)^n, \quad (3.2)$$

where c does not depend on r .

Up to a translation, the following counterpart of Kovrijkine's result for the planar case has been given in [HKKO21]:

Lemma 2. *Let $0 < r < R$ be fixed. Let $w \in \mathbb{C}$. There exists a constant $\eta > 0$ such that the following holds. Let ϕ be analytic in $D^R(w)$, and let $E \subset D^r(w)$ be a planar measurable set of positive measure, and let $z_0 \in D^r(w)$. If $|\phi(z_0)| \geq 1$ and $M = \sup_{z \in D^R(w)} |\phi(z)|$ then*

$$\sup_{z \in D^r(w)} |\phi(z)| \leq \left(\frac{cr^2 \rho(w)^2}{|E|} \right)^{\eta \ln M} \sup_{z \in E} |\phi(z)|,$$

where c does not depend on r , and

$$\eta \leq c'' \frac{R^4}{(R-r)^4} \ln \frac{R}{R-r}$$

for an absolute constant c'' .

The corresponding case for p -norms is deduced exactly as in Kovrijkine's work.

Corollary 3. *Let $0 < r < R$ be fixed. Let $w \in \mathbb{C}$. There exists a constant $\eta > 0$ such that following holds. Let ϕ be analytic in $D^R(w)$ and let $E \subset D^r(w)$ be a planar measurable set of positive measure and let $z_0 \in D^r(w)$. If $|\phi(z_0)| \geq 1$ and $M = \max_{z \in D^R(w)} |\phi(z)|$ then for $p \in [1, +\infty)$ we have*

$$\|\phi\|_{L^p(D^r(w))} \leq \left(\frac{cr^2 \rho(w)^2}{|E|} \right)^{\eta \ln M + \frac{1}{p}} \|\phi\|_{L^p(E)}.$$

The estimates on η are the same as in the lemma. The constant c does not depend on r .

4. PROOF OF THEOREM 1

The key ingredient in the proof of Theorem 1 and the main contribution of this paper is the existence and the estimate of the covering constant. Indeed, we have shown in Section 2 that the disks $(D^s(a_k))_{k \in \mathbb{N}}$ cover the complex plane \mathbb{C} for $s > r_0/2$. Now, we shall prove that there exists a covering constant N depending on the covering radius $s \geq r_0/2$. This means that the number of overlapping disks cannot exceed N . We denote by χ_F the characteristic function of a measurable set F in \mathbb{D} . We are now in a position to prove the covering lemma.

Lemma 3. *For $s > 0$, there exists a constant N such that*

$$\sum_k \chi_{D^s(a_k)} \leq N.$$

Moreover for every $\varepsilon > 0$, there exists some universal constant $c_{ov} := c_{ov}(\phi, \varepsilon) > 0$ such that

$$N \leq c_{ov} (1+s)^{1+\frac{1}{\kappa}+\frac{\kappa\varepsilon}{1+\kappa}}$$

where κ is given in (2.1) and depends on ϕ . The constant c_{ov} satisfies $c_{ov}(\phi, \varepsilon) \sim \frac{1}{\varepsilon}$ when $\varepsilon \rightarrow 0$.

Obviously, the constant N is at least equal to 1. Remind that in order to cover the complex plane \mathbb{C} , we need to require that $s \geq r_0/2$.

Proof of Lemma 3. We will show that this lemma is a consequence of Lemma 1. We start proving that for $s > 1$,

$$\#\{k : a_k \in D^s(z)\} \lesssim C_\mu^{i+1} s^{\frac{1}{\kappa}} \quad (4.1)$$

for all $z \in \mathbb{C}$. Take $0 < \delta < \frac{1}{2}$ such that the disks $(D^\delta(a_k))_{k \in \mathbb{N}}$ are pairwise disjoint. Since the measure is doubling, for $i \geq 3$ satisfying $\frac{1}{2^i} \leq \delta$ we have

$$\begin{aligned} \#\{k : a_k \in D^s(z)\} &= \sum_{a_k \in D^s(z)} \mu(D^1(a_k)) \leq C_\mu^i \sum_{a_k \in D^s(z)} \mu(D^\delta(a_k)) \\ &= C_\mu^i \mu \left(\bigsqcup_{a_k \in D^s(z)} D^\delta(a_k) \right) \end{aligned} \quad (4.2)$$

where \bigsqcup means that the union is disjoint.

Since the disks $D^\delta(a_k)$ are pairwise disjoint, the euclidean radii of $D^\delta(a_k)$ are smaller than that of $D^s(z)$, i.e. $\delta \rho(a_k) < s \rho(z)$. Indeed, otherwise we have $D^s(z) \subset D^{2\delta}(a_k)$ which leads to $1 < \mu(D^s(z)) \leq \mu(D^{2\delta}(a_k)) < 1$, which is a contradiction. Hence, $D^\delta(a_k) \subset D^{2s}(z)$ and so by (4.2) we have

$$\#\{k : a_k \in D^s(z)\} \leq C_\mu^i \mu(D^{2s}(z)) \leq C_\mu^{i+1} \mu(D^s(z)).$$

Then the inequality follows from equation (2.1)

$$\#\{k : a_k \in D^s(z)\} \leq C_\mu^{i+1} C s^{\frac{1}{\kappa}}$$

where C_μ is the doubling constant and C is a constant hidden in (2.1).

Let $m > 1 + \frac{1}{\kappa}$ and write

$$\Gamma := \sup_{z \in \mathbb{C}} \sum_{a_k \notin D^s(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m < +\infty.$$

We claim that

$$N \leq 2\Gamma s^m + C_\mu^{i+1} C s^{\frac{1}{\kappa}}. \quad (4.3)$$

This will imply that $N \leq (2C(\phi, m) + C_\mu^{i+1} C) s^{\frac{1}{\kappa} + \frac{\kappa m}{1+\kappa}}$ since $\Gamma \leq C(\phi, m) s^{\frac{1}{\kappa} - \frac{m}{1+\kappa}}$ by Lemma 1. Thus, taking $m = 1 + \frac{1}{\kappa} + \varepsilon$ for $\varepsilon > 0$, we obtain $N \lesssim s^{1 + \frac{1}{\kappa} + \frac{\kappa \varepsilon}{1+\kappa}}$ for $s > 1$. Finally, we obtain for $s > 0$,

$$N \leq c_{ov}(\varepsilon) (1 + s)^{1 + \frac{1}{\kappa} + \frac{\kappa \varepsilon}{1+\kappa}}$$

with an absolute constant $c_{ov}(\varepsilon) := 2C(\phi, m) + C_\mu^{i+1} C > 0$ which depends only on ϕ . Moreover, since $C(\phi, m) \sim \frac{1}{m-1-\frac{1}{\kappa}}$ when $m \rightarrow 1 + \frac{1}{\kappa}$, we have $c_{ov}(\varepsilon) \sim \frac{1}{\varepsilon}$ when $\varepsilon \rightarrow 0$.

In order to show the claim (4.3), by contradiction we assume that it does not hold. Then setting

$$A_z = \{k \in \mathbb{N} : z \in D^s(a_k)\}, \quad \text{for } z \in \mathbb{C}$$

there exists z_0 such that

$$\#A_{z_0} > 2\Gamma s^m + C_\mu^{i+1} C s^{\frac{1}{\kappa}}. \quad (4.4)$$

But then

$$\Gamma \geq \sum_{a_k \notin D^s(z_0)} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m \geq \sum_{\substack{k \in A_{z_0} \\ a_k \notin D^s(z_0)}} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m.$$

Since $z_0 \in D^s(a_k)$ for $k \in A_{z_0}$ and combining (4.1) and (4.4) we get

$$\Gamma \geq \sum_{\substack{k \in A_{z_0} \\ a_k \notin D^s(z_0)}} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m \geq \frac{\#A_{z_0} - \#\{k : a_k \in D^s(z_0)\}}{s^m} > \frac{2\Gamma s^m}{s^m} = 2\Gamma,$$

which is a contradiction. \square

As in [HKKO21] for the Bergman space, we introduce good disks. Fix $r_0/2 \leq s < t$ where $r_0/2$ is the radius such that the disks $D^{r_0/2}(a_k)$ cover the complex plane. For $K > 1$ the set

$$I_f^{K\text{-good}} = \{k : \|f\|_{L_\phi^p(D^t(a_k))} \leq K \|f\|_{L_\phi^p(D^s(a_k))}\}$$

will be called the set of K -good disks for (t, s) (in order to keep notation light we will not include s and t as indices). This set depends on f .

The following proposition has been shown in [HKKO21] for the Bergman space, but its proof, implying essentially the finite overlap property, is exactly the same.

Proposition 1. *Let $r_0/2 \leq s < t$. For every constant $c \in (0, 1)$, there exists K such that for every $f \in \mathcal{F}_\phi^p$ we have*

$$\sum_{k \in I_f^{K\text{-good}}} \|f\|_{L_\phi^p(D^s(a_k))}^p \geq c \|f\|_{p,\phi}^p.$$

One can pick $K^p = N(t)/(1 - c)$ where $N(t)$ corresponds to the overlapping constant from Lemma 3 for the "radius" t .

We are now in a position to prove the theorem.

Proof of Theorem 1. Take $s = \max(r, r_0/2)$ and $t = 4s$. As noted in [HKKO21], E is $(\tilde{\gamma}, s)$ -dense, where $\tilde{\gamma} = c\gamma$ and c is a multiplicative constant. Hence we can assume that E is (γ, s) -dense.

Let h_{a_k} be the harmonic function introduced in (2.6) for $\sigma = t$. Since h_{a_k} is harmonic on $D^t(a_k)$, there exists a function H_{a_k} holomorphic on $D^t(a_k)$ with real part h_{a_k} .

Now, given f with $\|f\|_{p,\phi} = 1$, let $g = f e^{-(H_{a_k} + \phi(a_k))}$ for $k \in I_f^{K\text{-good}}$, and set

$$h = c_0 g, \quad c_0 = \left(\frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p dA(z)} \right)^{1/p}.$$

Again there is $z_0 \in D^s(a_k)$ with $|h(z_0)| \geq 1$.

Set $R = 2s$. We have to estimate the maximum modulus of h on $D^R(a_k)$ in terms of a local integral of h . To that purpose, we can assume $h \in A^p(D^t(a_k))$. Indeed, applying (2.6) we have

$$\begin{aligned} \int_{D^t(a_k)} |h|^p dA &= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p dA} \int_{D^t(a_k)} |g|^p dA \\ &= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |f|^p e^{-p(h_{a_k} + \phi(a_k))} dA} \int_{D^t(a_k)} |f|^p e^{-p(h_{a_k} + \phi(a_k))} dA \\ &\leq \frac{\pi s^2 \rho(a_k)^2 e^{2pA(t)}}{\int_{D^s(a_k)} |f|^p e^{-p\phi(z)} dA(z)} \int_{D^t(a_k)} |f|^p e^{-p\phi(z)} dA(z). \\ &\leq \pi s^2 \rho(a_k)^2 e^{2pA(t)} K^p \end{aligned}$$

where the last inequality comes from the fact that k is K -good for (s, t) . Hence $h \in A^p(D^t(a_k))$ and it follows

$$M^p := \max_{z \in D^R(a_k)} |h(z)|^p \leq \frac{C}{s^2 \rho(a_k)^2} \int_{D^t(a_k)} |h|^p dA \leq c_s K^p$$

where $c_s = C e^{2pA(4s)}$, $A(4s)$ comes from (2.6) and C is an absolute constant.

Now, setting $\tilde{E} = (E \cap D^s(a_k))$ we get using Corollary 3 applied to h :

$$\int_{D^s(a_k)} |h(z)|^p dA(z) \leq \left(\frac{cs^2 \rho(a_k)^2}{|\tilde{E}|} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |h(z)|^p dA(z)$$

The factor $s^2 \rho(a_k)^2$ appearing inside the brackets is a rescaling factor (see Remark 1). Again, by homogeneity we can replace in the above inequality h by g . Note also that $\pi s^2 \rho(a_k)^2 / |\tilde{E}|$ is controlled by $1/\gamma$. This yields

$$\begin{aligned} \int_{D^s(a_k)} |f|^p e^{-p\phi(z)} dA(z) &\leq e^{pA(t)} \int_{D^s(a_k)} |g(z)|^p dA(z) \\ &\leq e^{pA(t)} \left(\frac{cs^2 \rho(a_k)^2}{|\tilde{E}|} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |g(z)|^p dA(z) \\ &\leq e^{pA(t)} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |g(z)|^p dA(z) \\ &= e^{2pA(t)} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |f(z)|^p e^{-p\phi(z)} dA(z), \end{aligned}$$

where c_1 is an absolute constant.

Summing over all K -good k , and using Lemma 3 and Proposition 1 we obtain the required result

$$c \|f\|_{p, \phi} \lesssim \left(\frac{c_1}{\gamma} \right)^{\eta \ln M+1/p} \|f\|_{L_\phi^p(E)}$$

where

$$\begin{aligned} \ln(M) &\leq \ln(c_s^{1/p} K) = \ln\left(\left(c_s \frac{N(4s)}{1-c}\right)^{\frac{1}{p}}\right) \leq \frac{1}{p} \ln\left(C e^{2pA(4s)} c_{ov}(\varepsilon) \frac{(1+4s)^{1+\frac{1}{\kappa}+\frac{\kappa\varepsilon}{1+\kappa}}}{1-c}\right) \\ &\leq 2A(4s) + \frac{1}{p} \left[\ln(C c_{ov}(\varepsilon)) + \left(1 + \frac{1}{\kappa} + \frac{\kappa\varepsilon}{1+\kappa}\right) \ln(1+4s) - \ln(1-c) \right] \end{aligned}$$

and in view of Lemma 2

$$\eta \leq c'' \times 2^4 \ln 2$$

c comes from Proposition 1, $c_{ov}(\varepsilon) \sim \frac{1}{\varepsilon}$ from Lemma 3, κ from (2.1) and C is absolute. In particular, fixing $\varepsilon > 0$ and $c \in (0, 1)$, and noticing that $r \asymp s = \max(r, r_0/2)$ for $r > 1$, we obtain

$$\ln M \leq 2A(4r) + \frac{1}{p}(C' + C'' \ln(1+r))$$

where C' and C'' depends only on the space. In view of (2.7), we finally obtain

$$\ln M \leq \widehat{C} r^{\frac{1}{\kappa}} + \frac{1}{p}(C' + C'' \ln(1+r))$$

with \widehat{C} depending only on the space. Notice that we can optimize the constants C' and C'' in $\varepsilon > 0$ depending on r (C' involves $c_{ov}(\varepsilon) \sim \frac{1}{\varepsilon}$ and C'' involves $1 + \frac{1}{\kappa} + \frac{\kappa\varepsilon}{1+\kappa}$). However, this optimization will not really improve the result. \square

Finally, we give a short proof of the reverse implication in Corollary 1 which is a straightforward adaptation to the doubling Fock space of Luecking's proof of [Lue81, Corollary 3].

Proof. First, assume that $\varphi \leq 1$. Therefore $s \leq 1$. Since E is (γ, r) -dense, it is a dominating set. So

$$\int_{\mathbb{C}} \varphi^2 |f|^2 e^{-2\phi} dA \geq s^2 \int_E |f|^2 e^{-2\phi} dA \geq s^2 C^2 \|f\|_{2,\phi}^2,$$

where C is the sampling constant. Then

$$\begin{aligned} \|(I - T_\varphi)f\|_{2,\phi}^2 &= \|T_{1-\varphi}f\|_{2,\phi}^2 \\ &= \|\mathbf{P}[(1-\varphi)f]\|_{2,\phi}^2 \\ &\leq \|(1-\varphi)f\|_{2,\phi}^2 \\ &\leq \int_{\mathbb{C}} (1-\varphi^2) |f|^2 e^{-2\phi} dA \\ &\leq (1 - s^2 C^2) \|f\|_{2,\phi}^2 \end{aligned}$$

Hence $\|I - T_\varphi\| < 1$. So T_φ is invertible and

$$\|T_\varphi^{-1}\| \leq \frac{1}{1 - \|I - T_\varphi\|} \leq \frac{1}{1 - \sqrt{1 - s^2 C^2}}.$$

Using Theorem 1, we obtain

$$\|T_\varphi^{-1}\| \leq \frac{1}{1 - \sqrt{1 - s^2 \left(\frac{\gamma}{c}\right)^{2L}}}.$$

Finally, for a general φ , let $\psi = \frac{\varphi}{\|\varphi\|_\infty}$. Then $\psi \leq 1$, $E = \{\psi \geq \frac{s}{\|\varphi\|_\infty} = s'\}$ and $T_\varphi = \|\varphi\|_\infty T_\psi$. So, applying the previous discussion to ψ , it follows

$$\|T_\varphi^{-1}\| = \|T_\psi^{-1}\| \|\varphi\|_\infty \leq \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - (s')^2 \left(\frac{\gamma}{c}\right)^{2L}}} = \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - \left(\frac{s}{\|\varphi\|_\infty}\right)^2 \left(\frac{\gamma}{c}\right)^{2L}}}.$$

□

Acknowledgment. This work is part of the first named author's thesis supervised by Andreas Hartmann. The authors would like to thank him for many helpful discussions.

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