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SAMPLING CONSTANTS IN GENERALIZED FOCK SPACES

S. KONATE & M.-A. ORSONI

ABSTRACT. We discuss sampling constants for dominating sets in generalized Fock spaces.

1. INTRODUCTION

Sampling problems are central problems in signal theory, and cover, for instance, sampling sequences and so-called dominating sets which allow to recover the norm of a signal — defined by an integration over a given domain — from the integration on a subdomain (precise definitions will be given later). In this paper we will focus on the second class of problems, i.e. dominating sets. Once conditions established for being dominating, a second central question is to know whether the sampling constants can be estimated. Conditions guaranteeing that a set is dominating have been established rather long ago (in the 70's for the Paley-Wiener space and in the 80's for the Bergman and Fock spaces, see e.g. the survey [FHR17] and references therein). More recently, people got interested in estimates of the sampling constants which give an information on the tradeoff between the cost of the sampling and the precision of the estimates. The central paper in this connection is by Kovrijkine [Kov01] who gave a method to consider this problem in the Paley-Wiener space establishing a polynomial dependence of the sampling constant on an underlying density (see also [Rez10]). His method involves Remez-type and Bernstein inequalities. Subsequently, his method was adapted to other spaces (see for instance [HJK17] for the model space where weighted Bernstein inequalities hold), and also to settings where a Bernstein inequality is not at hand (e.g., Fock and polyanalytic Fock spaces [JS] and Bergman spaces [HKKO20]). In this paper, inspired by methods in [HKKO20], which are based on an Andrievskii-Ruscheweyh estimate replacing the Remez inequality and an alternate covering argument to circumvent Bernstein's inequality, we will discuss the case of generalized Fock spaces. As it turns out, the machinery from [HKKO20] applies to this more general setting. Indeed, the paper [MMOC03] contains a wealth of results for generalized Fock spaces that allow to translate the main steps of [HKKO20] to this new framework.

We recall that sampling problems have been considered in a large setting of situations, including the Fock space and its generalized version (see e.g. the survey [FHR17]). In

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classical Fock spaces, sampling sequences have been characterized by Seip (see e.g. the book [Sei04]), and for their generalized counterparts this had been done in [MMOC03]. Dominating sets have been characterized by Jansen, Peetre and Rochberg in [JPR87] for the classical Fock space and by Lou and Zhuo in [LZ19] for doubling Fock spaces (see definitions below). More general sampling measures have been discussed by Ortega-Cerdà [OC98]. We would also like to mention some work by Lindholm [Lin00].

Let us introduce some notation. For a subharmonic function $\phi : \mathbb{C} \rightarrow \mathbb{R}$, and $1 < p < +\infty$, we define the generalized Fock space by

$$\mathcal{F}_\phi^p = \{f \in \text{Hol}(\mathbb{C}) : \|f\|_{p,\phi}^p = \int_{\mathbb{C}} |f|^p e^{-p\phi} dA < +\infty\}$$

where dA is planar Lebesgue measure on \mathbb{C} . A measurable set $E \subset \mathbb{C}$ will be called dominating if there exists $C > 0$ such that

$$\int_E |f|^p e^{-p\phi} dA \geq C^p \int_{\mathbb{C}} |f|^p e^{-p\phi} dA, \quad \forall f \in \mathcal{F}_\phi^p. \quad (1.1)$$

We will use the notation $L_\phi^p(F)$ for the Lebesgue space on a measurable set $F \subset \mathbb{C}$ with respect to $e^{-p\phi} dA$.

We need some more notation. A subharmonic function $\phi : \mathbb{C} \rightarrow \mathbb{R}$ is called doubling if the measure $\Delta\phi$ is doubling, i.e., there exists a constant C_μ such that for every $z \in \mathbb{C}$ and $r > 0$,

$$\mu(D(z, 2r)) \leq C_\mu \mu(D(z, r)).$$

We will sometimes call \mathcal{F}_ϕ^p doubling if ϕ is doubling. We can associated with ϕ a function $\rho : \mathbb{C} \rightarrow \mathbb{R}^+$, such that

$$\mu(D(z, \rho(z))) = 1.$$

Here $D(z, r)$ is a standard euclidean disk. Assuming ϕ suitably regularized, we can assume $\Delta\phi = \rho^{-2}$ (see [MMOC03]?). We denote $D^r(z) = D(z, r\rho(z))$, and $D(z) = D^1(z)$. With these definitions in mind we can introduce the following natural density. A measurable set E is (γ, r) -dense, if

$$\frac{|E \cap D^r(z)|}{|D^r(z)|} \geq \gamma.$$

Here $|F|$ denotes planar Lebesgue measure of a measurable set F . We will just say that the set is relatively dense if there is some $\gamma > 0$ and some $r > 0$ such that the set is (γ, r) -dense.

It follows from Lou, Zhuo's result [LZ19, Theorem A] that a set E is dominating if and only if it is relatively dense. More precisely, they proved that if E is (γ, r) -dense then there exist some constants ε_0 and $c > 0$ depending only on r such that inequality (1.1) holds for every

$$C^p \geq c\gamma\varepsilon_0^{\frac{2(p+2)}{\gamma}}.$$

Our main result is the following

Theorem 1. *Let ϕ be a subharmonic function and $1 \leq p < +\infty$. Given $r > 1$, there exists L such that for every measurable set $E \subset \mathbb{C}$ which is (γ, r) -dense, we have*

$$\|f\|_{L^p_\phi(E)}^p \geq \left(\frac{\gamma}{c}\right)^L \|f\|_{p,\phi}^p \tag{1.2}$$

for every $f \in \mathcal{F}_\phi^p$. Here, the constants c and L depend on r , and for L we can choose

$$L \leq \lambda r^{\frac{1}{\kappa}} + \frac{1}{p}(\lambda' + \lambda'' \ln(1+r))$$

where κ is a constant given in (3.2) depending on the space; and $\lambda, \lambda', \lambda''$ are some universal constants depending also only on the space.

Observe that we are mainly interested in the case when r is big, for instance $r \geq 1$ (in case E is (γ, r) -dense for some $r < 1$ we can also show that E is $(\tilde{\gamma}, 1)$ -dense for some $\tilde{\gamma} \simeq \gamma$ where underlying constants are universal).

It should be noted that there is a competing relation between γ and r . The density γ can be as small as we want for a given r (take $E = \mathbb{C} \setminus D(0, r\sqrt{1-\varepsilon})$ for $0 < \varepsilon < 1$, then $\gamma = \varepsilon$), but fixing a relative dense set E (and so the sampling constant C) it can become rather big when we choose a bigger radius. In a sense one needs to optimize $L \ln \gamma$, and L depends on r .

Here is another observation. Though this might be obvious, it should be observed that there is no reason a priori why a holomorphic function for which the integral $\int_E |f|^p e^{-p\phi} dA$ is bounded for a relative dense set E should be in \mathcal{F}_ϕ^p . Outside the class \mathcal{F}_ϕ^p relative density is in general not necessary for domination (see also a remark in [Lue81, p.11]).

In [LZ19, Theorem 7], proving that the relative density is a necessary condition for domination, it is shown that $\gamma \gtrsim C^p$. Hence, we cannot expect more than a polynomial dependance in γ of the sampling constant C . In this sense, our result is optimal.

As a direct consequence, we obtain a bound for the norm of the inverse of a Toeplitz operator T_φ . We remind that for any bounded measurable function φ , the Toeplitz operator T_φ is defined on \mathcal{F}_ϕ^2 by $T_\varphi f = \mathbf{P}(\varphi f)$ where \mathbf{P} denotes the orthogonal projection from $L^2_\phi(\mathbb{C})$ onto \mathcal{F}_ϕ^2 . As remarked in [LZ19, Theorem B], for a non-negative function φ , T_φ is invertible if and only if $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is a dominating set for some $s > 0$. Tracking the constant we obtain

Corollary 1. *Let φ be a non-negative bounded measurable function. The operator T_φ is invertible if and only if there exists $s > 0$ such that $E_s = \{z \in \mathbb{C} : \varphi(z) > s\}$ is (γ, r) -dense for some $\gamma > 0$ and $r > 0$. In this case, we have*

$$\|T_\varphi^{-1}\| \leq \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - \left(\frac{s}{\|\varphi\|_\infty}\right)^2 \left(\frac{\gamma}{c}\right)^L}}.$$

Notice that the right-hand side behaves as γ^{-L} as $\gamma \rightarrow 0$.

For the sake of completeness, we give a proof of the reverse implication which is a straightforward adaptation to the doubling Fock space of Luecking's proof of [Lue81, Corollary 3].

Proof. First, assume that $\varphi \leq 1$. Therefore $s \leq 1$. Since E is (γ, r) -dense, it is a dominating set. So

$$\int_{\mathbb{C}} \varphi^2 |f|^2 e^{-2\phi} dA \geq s^2 \int_E |f|^2 e^{-2\phi} dA \geq s^2 C^2 \|f\|_{2,\phi}^2.$$

Then

$$\begin{aligned} \|(I - T_\varphi)f\|_{2,\phi}^2 &= \|T_{1-\varphi}f\|_{2,\phi}^2 \\ &= \|\mathbf{P}[(1-\varphi)f]\|_{2,\phi}^2 \\ &\leq \|(1-\varphi)f\|_{2,\phi}^2 \\ &\leq \int_{\mathbb{C}} (1-\varphi^2) |f|^2 e^{-2\phi} dA \\ &\leq (1 - s^2 C^2) \|f\|_{2,\phi}^2 \end{aligned}$$

Hence $\|I - T_\varphi\| < 1$. So T_φ is invertible and

$$\|T_\varphi^{-1}\| \leq \frac{1}{1 - \|I - T_\varphi\|} \leq \frac{1}{1 - \sqrt{1 - s^2 C^2}}.$$

Using Theorem 1, we obtain

$$\|T_\varphi^{-1}\| \leq \frac{1}{1 - \sqrt{1 - s^2 \left(\frac{\gamma}{c}\right)^L}}.$$

Finally, for a general φ , let $\psi = \frac{\varphi}{\|\varphi\|_\infty}$. Then $\psi \leq 1$, $E = \{\psi \geq \frac{s}{\|\varphi\|_\infty} = s'\}$ and $T_\varphi = \|\varphi\|_\infty T_\psi$. So, applying the previous discussion to ψ , the results follows.

$$\|T_\varphi^{-1}\| = \|T_\psi^{-1}\| \|\varphi\|_\infty \leq \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - (s')^2 \left(\frac{\gamma}{c}\right)^L}} = \frac{\|\varphi\|_\infty}{1 - \sqrt{1 - \left(\frac{s}{\|\varphi\|_\infty}\right)^2 \left(\frac{\gamma}{c}\right)^L}}$$

□

The proof of Theorem 1 follows the scheme presented in [HKKO20]. We will recall the necessary results from that paper. The main new ingredients come from [MMOC03] and concern a finite overlap property and a Lemma allowing to translate the (subharmonic) weight locally into a holomorphic function.

2. REMEZ-TYPE INEQUALITIES

Let us introduce a central results of the paper [AR07]. Let G be a (bounded) domain in \mathbb{C} . Let $0 < s < |\overline{G}|$ (Lebesgue measure of \overline{G}). Denoting Pol_n the space of complex polynomials of degree at most $n \in \mathbb{N}$, we introduce the set

$$P_n(\overline{G}, s) = \{p \in \text{Pol}_n : |\{z \in \overline{G} : |p(z)| \leq 1\}| \geq s\}.$$

Next, let

$$R_n(z, s) = \sup_{p \in P_n(\overline{G}, s)} |p(z)|.$$

This expression gives the biggest possible value at z of a polynomial p of degree at most n and being at most 1 on a set of measure at least s . In particular Theorem 1 from [AR07] claims that for $z \in \partial G$, we have

$$R_n(z, s) \leq \left(\frac{c}{s}\right)^n. \tag{2.1}$$

This result corresponds to a generalization to the two-dimensional case of the Remez inequality which is usually given in dimension 1. In what follows we will essentially consider G to be a disk or a rectangle. By the maximum modulus principle, the above constant gives an upper estimate on G for an arbitrary polynomial of degree at most n which is bounded by one on a set of measure at least s . Obviously, if this set is small (s close to 0), i.e. p is controlled by 1 on a small set, then the estimate has to get worse.

Remark 1. *Let us make another observation. If c is the constant in (2.1) associated with the unit disk $G = \mathbb{D} = D(0, 1)$, then a simple argument based on homothecy shows that the corresponding constant for an arbitrary disk $D(0, r)$ is cr^2 (considering $D(0, r)$ as underlying domain, the constant c appearing in [AR07, Theorem 1] satisfies $c > 2 \times m_2(D(0, r))$). So, in the sequel we will use the estimate*

$$R_n(z, s) \leq \left(\frac{cr^2}{s}\right)^n, \tag{2.2}$$

where c does not depend on r .

Up to a translation, the following counterpart of Kovrijkine's result for the planar case has been given in [HKKO20]:

Lemma 1. *Let $0 < r < R$ be fixed. Let $w \in \mathbb{C}$. There exists a constant $\eta > 0$ such that the following holds. Let ϕ be analytic in $D^R(w)$, and let $E \subset D^r(w)$ be a planar measurable set of positive measure, and let $z_0 \in D^r(w)$. If $|\phi(z_0)| \geq 1$ and $M = \max_{z \in D^R(w)} |\phi(z)|$ then*

$$\sup_{z \in D^r(w)} |\phi(z)| \leq \left(\frac{cr^2 \rho(w)^2}{|E|}\right)^{\eta \ln M} \sup_{z \in E} |\phi(z)|,$$

where c does not depend on r , and

$$\eta \leq c'' \frac{R^4}{(R-r)^4} \ln \frac{R}{R-r}$$

for an absolute constant c'' .

The corresponding case for p -norms is deduced exactly as in Kovrijkine's work.

Corollary 2. *Let $0 < r < R$ be fixed. Let $w \in \mathbb{C}$. There exists a constant $\eta > 0$ such that following holds. Let ϕ be analytic in $D^R(w)$ and let $E \subset D^r(w)$ be a planar measurable set of positive measure and let $z_0 \in D^r(w)$. If $|\phi(z_0)| \geq 1$ and $M = \max_{z \in D^R(w)} |\phi(z)|$ then for $p \in [1, +\infty)$ we have*

$$\|\phi\|_{L^p(D^r(w))} \leq \left(\frac{cr^2 \rho(w)^2}{|E|} \right)^{\eta \ln M + \frac{1}{p}} \|\phi\|_{L^p(E)}.$$

The estimates on η are the same as in the lemma. The constant c does not depend on r .

3. PROOF OF THEOREM 1

We will cover \mathbb{C} by disks satisfying certain properties and satisfying a finite covering property. Denote by χ_F the characteristic function of a measurable set F in \mathbb{D} .

In [MMOC03], the authors construct a decomposition of \mathbb{C} into so-called quasi-squares R_k : $\mathbb{C} = \bigcup_k R_k$, and two such quasi-squares can intersect at most along sides. Quasi-squares are rectangles for which the ratio between length and height is uniformly controlled, but this will not be of importance here. Denoting by a_k the center of R_k , Theorem 8(c) of [MMOC03] claims in particular that there is $r_0 \geq 1$, such that for every k ,

$$\frac{\rho(a_k)}{r_0} \leq \text{diam } R_k \leq r_0 \rho(a_k).$$

In other words

$$D^{1/(2Cr_0)}(a_k) \subset R_k \subset D^{r_0/2}(a_k) \quad (3.1)$$

with $C = \sqrt{1 + e^2}$ and $1/e \leq L/l \leq e$. Hence $\mathbb{C} = \bigcup D^{r_0/2}(a_k)$.

We will say that a sequence $(a_n)_{n \in \mathbb{N}}$ is ρ -separated if there exists $\delta > 0$ such that

$$|a_i - a_j| \geq \delta \max(\rho(a_i), \rho(a_j)), \quad \forall i \neq j.$$

This means that the disks $D^\delta(a_n)$ are pairwise disjoint.

Now, we shall prove that there exists a covering constant N depending on the covering radius $s \geq r_0$. For that, the key point is the following geometric estimate given by [MMOC03, Equation (4)]. There exists $\kappa > 0$ such that for all $z \in \mathbb{C}$ and $r > 1$,

$$r^\kappa \lesssim \mu(D^r(z)) \lesssim r^{\frac{1}{\kappa}}. \quad (3.2)$$

This permits to prove the following lemma which is essentially contained in the proof of [MMOC03, Lemma 6] and [MMOC03, Lemma 5.b)].

Lemma 2. *Let $m > 1 + \frac{1}{\kappa}$. There exists $C(\phi, m) > 0$ such that for every $r \geq 1$*

$$\sup_{z \in \mathbb{C}} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \leq C(\phi, m) r^{\frac{1}{\kappa} - \frac{m}{1+\kappa}}.$$

It will be shown in the proof that $C(\phi, m) \sim \frac{1}{m-1-\frac{1}{\kappa}}$ when m tends to $1 + \frac{1}{\kappa}$. The proof shows that $r \geq 1$ is a rather technical condition and can be replaced by $r \geq \widehat{r} > 0$.

Proof. We will use Corollary 3 of [MMOC03] that we recall here.

Corollary 3 (of [MMOC03]). *For every $r > 0$, there exists $\kappa \geq 0$ such that if $\zeta \in D^r(z)$, then*

$$\rho(z) \lesssim \rho(\zeta)(1+r)^\kappa.$$

Let us start with $r > 1$ large, to be fixed later. Since the sequence $(a_k)_{k \in \mathbb{N}}$ is separated, we can assume $0 < \delta < 1/4$ such that the disks $(D^\delta(a_k))_{k \in \mathbb{N}}$ are pairwise disjoint.

$$\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m = \sum_{a_k \notin D^r(z)} \frac{1}{\mu(D^\delta(a_k))} \int_{D^\delta(a_k)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m d\mu(\zeta)$$

There exists $i \geq 3$ such that $\frac{1}{2^i} < \delta$. Therefore $1 = \mu(D^1(a_k)) \leq C_\mu^i \mu(D^\delta(a_k))$ where C_μ is the doubling constant. Hence

$$\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \leq C_\mu^i \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m d\mu(\zeta). \quad (3.3)$$

Now, we claim that if $a_k \notin D^r(z)$ then $z \notin D^\delta(a_k)$. Indeed, if $z \in D^\delta(a_k)$ then $D^r(z) \subset D^{2\delta}(a_k)$ (see Figure 1). Therefore $1 < \mu(D^r(z)) \leq \mu(D^{2\delta}(a_k)) < 1$ which is a contradiction.

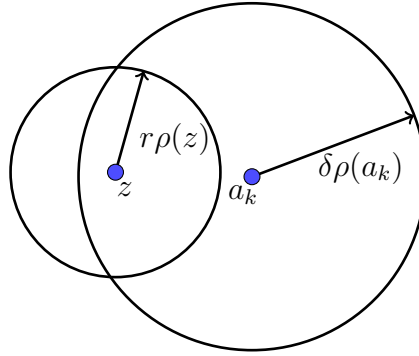


FIGURE 1. The disks $D^\delta(a_k)$ and $D^r(z)$.

Thus, if $a_k \notin D^r(z)$ and $\zeta \in D^\delta(a_k)$, we have

$$|z - \zeta| \leq |z - a_k| + |a_k - \zeta| \leq 2|z - a_k|. \quad (3.4)$$

Also, by Corollary 3, if $\zeta \in D^\delta(a_k)$ then there exists $C > 0$ such that $\rho(a_k) \leq C(1+\delta)^\kappa \rho(\zeta)$. With inequality (3.3), we obtain

$$\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \leq C_\mu^i (2C(1+\delta)^\kappa)^m \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta)$$

Now, setting $\alpha = 1 - 2\delta < 1$ we have $D^{\alpha r}(z) \cap D^\delta(a_k) = \emptyset$. Indeed, if $\zeta \in D^{\alpha r}(z) \cap D^\delta(a_k)$, then $|\zeta - z| \leq \alpha r \rho(z)$ and $|a_k - \zeta| \leq \delta \rho(a_k)$. Let $R = r \rho(z)$ and $s = \delta \rho(a_k)$, then $s > (1 - \alpha)R$, since $a_k \notin D^r(z) = D(z, R)$ (see Figure 2).

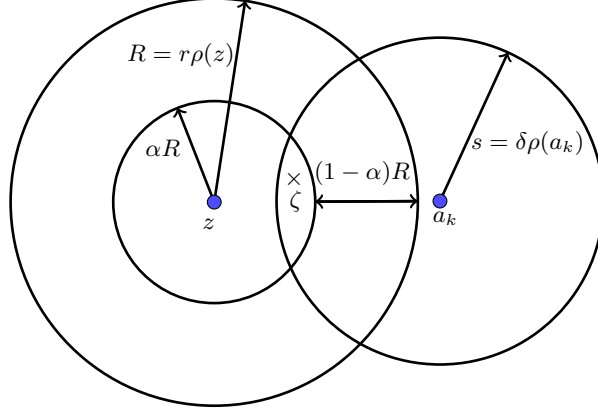


FIGURE 2. The disks $D(a_k, s)$, $D(z, R)$ and $D(z, \alpha R)$.

Therefore, $|z - \zeta| \leq \alpha R < \frac{s\alpha}{1-\alpha}$ and it follows by the triangular inequality

$$\begin{aligned} |z - a_k| &\leq |\zeta - a_k| + |\zeta - z| \leq \delta \rho(a_k) + \frac{s\alpha}{1-\alpha} \\ &= \delta \rho(a_k) + \frac{\delta \rho(a_k) \alpha}{1-\alpha} \\ &= \frac{\delta \rho(a_k)}{1-\alpha}. \end{aligned}$$

So $z \in D^{\frac{\delta}{1-\alpha}}(a_k)$. Since, $a_k \notin D^r(z)$ this implies $D^r(z) \subset D^{\frac{2\delta}{1-\alpha}}(a_k)$. But

$$1 = \mu(D^1(z)) < \mu(D^r(z)) \leq \mu(D^{\frac{2\delta}{1-\alpha}}(a_k)) = \mu(D^1(a_k)) = 1.$$

Contradiction. Hence,

$$\begin{aligned} \sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \sum_{a_k \notin D^r(z)} \int_{D^\delta(a_k)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta) \\ &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} \left(\frac{\rho(\zeta)}{|z - \zeta|} \right)^m d\mu(\zeta). \end{aligned}$$

This leads by Fubini's theorem

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} m \int_0^{\rho(\zeta)/|z - \zeta|} t^{m-1} dt d\mu(\zeta) \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m \int_{\zeta \notin D^{\alpha r}(z)} m \int_0^\infty \mathbb{1}_{\{0 < t < \rho(\zeta)/|z - \zeta|\}} t^{m-1} dt d\mu(\zeta) \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^\infty m t^{m-1} \int_{\zeta \notin D^{\alpha r}(z)} \mathbb{1}_{\{\zeta \in \mathbb{C} \mid 0 < t < \rho(\zeta)/|z - \zeta|\}} d\mu(\zeta) dt
\end{aligned}$$

Note that

$$|z - \zeta| < \frac{\rho(\zeta)}{t} \quad (3.5)$$

is equivalent to $z \in D^{1/t}(\zeta)$.

Now, we claim that $t < 1$ for r large enough when $0 < t < \rho(\zeta)/|z - \zeta|$ and $\zeta \notin D^{\alpha r}(z)$. Indeed, if $t \geq 1$ and $z \in D^{1/t}(\zeta)$ then by Corollary 3 there exists $C > 0$ such that $\rho(\zeta) \leq C\rho(z)(1 + \frac{1}{t})^\kappa$. Replacing in (3.5), we obtain $|z - \zeta| < C\rho(z)(1 + \frac{1}{t})^\kappa/t \leq C\rho(z)2^\kappa/t$. Moreover, since $\zeta \notin D^{\alpha r}(z)$, we have

$$\rho(z)\alpha r \leq |z - \zeta| < C2^\kappa/t\rho(z).$$

Hence, $\alpha r < C2^\kappa/t$ or equivalently

$$t < \frac{C2^\kappa}{\alpha r}.$$

Finally, fixing $r > r' := \frac{C2^\kappa}{\alpha}$, we obtain a contradiction with $t \geq 1$, so that indeed $t < 1$.

Now, since $0 < t < 1$ and repeating the same arguments, by Corollary 3 we have $\rho(\zeta) \leq C\rho(z)(1 + \frac{1}{t})^\kappa \leq C\rho(z) \left(\frac{2}{t}\right)^\kappa$, and with inequality (3.5), this leads to $\zeta \in D^\beta(z)$ where $\beta = \frac{C}{t} \left(\frac{2}{t}\right)^\kappa = \frac{C2^\kappa}{t^{\kappa+1}}$. Also,

$$\zeta \in D^\beta(z) \setminus D^{\alpha r}(z) \iff \rho(z)\alpha r \leq |z - \zeta| < \beta\rho(z) \implies \left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}} > t.$$

Thus, $0 < t < \left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}} < 1$ since r is fixed to be larger than $r' = \frac{C2^\kappa}{\alpha}$. It follows

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} m t^{m-1} \int_{\zeta \in D^\beta(z) \setminus D^{\alpha r}(z)} d\mu(\zeta) dt \\
&\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} m t^{m-1} \mu(D^\beta(z)) dt \\
&\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} m t^{m-1} \beta^{\frac{1}{\kappa}} dt
\end{aligned}$$

where the last inequality comes from (3.2). Hence we obtain

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq C_\mu^i (2C(1 + \delta)^\kappa)^m \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-1} \left(\frac{C2^\kappa}{t^{\kappa+1}} \right)^{\frac{1}{\kappa}} dt \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \int_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} mt^{m-2-\frac{1}{\kappa}} dt \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \frac{m}{m-1-\frac{1}{\kappa}} \left[t^{m-1-\frac{1}{\kappa}} \right]_0^{\left(\frac{C2^\kappa}{\alpha r}\right)^{\frac{1}{1+\kappa}}} \\
&= C_\mu^i (2C(1 + \delta)^\kappa)^m 2C^{\frac{1}{\kappa}} \frac{m}{m-1-\frac{1}{\kappa}} \left(\frac{C2^\kappa}{\alpha} \right)^{\frac{m-1-\frac{1}{\kappa}}{1+\kappa}} \left(\frac{1}{r} \right)^{\frac{m-1-\frac{1}{\kappa}}{1+\kappa}} \\
&= 2^{m(1+\frac{\kappa}{1+\kappa})} C^{m+\frac{m}{1+\kappa}} C_\mu^i (1 + \delta)^{\kappa m} \frac{m}{m-1-\frac{1}{\kappa}} \alpha^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} r^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} \\
&= C_1 r^{-\frac{m+1+\frac{1}{\kappa}}{1+\kappa}}.
\end{aligned}$$

with $C_1 := 2^{m(1+\frac{\kappa}{1+\kappa})} C^{m+\frac{m}{1+\kappa}} C_\mu^i (1 + \delta)^{\kappa m} \frac{m}{m-1-\frac{1}{\kappa}} \alpha^{\frac{1}{\kappa}-\frac{m}{1+\kappa}}$ and $m > 1 + \frac{1}{\kappa}$. In particular, $C_1 \simeq 1/(m-1-1/\kappa)$, where the underlying constants (κ, C_μ, δ , etc.) only depend on the space and the reference sequence (a_k) . Hence the result is proved for $r > r'$.

Let us prove now that the result holds for $1 \leq r \leq r'$. Let $z \in \mathbb{C}$. Pick z_1 satisfying $D^{2r'}(z) \cap D^{2r'}(z_1) = \emptyset$.

$$\begin{aligned}
\sum_{a_k \notin D^r(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m &\leq \sum_{k \in \mathbb{N}} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \\
&\leq \sum_{a_k \notin D^{2r'}(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m + \sum_{a_k \notin D^{2r'}(z_1)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m \\
&\leq 2C_1 (2r')^{\frac{1}{\kappa}-\frac{m}{1+\kappa}} \leq C_2 r^{\frac{1}{\kappa}-\frac{m}{1+\kappa}}.
\end{aligned}$$

where, noticing that $\frac{1}{\kappa} - \frac{m}{1+\kappa} < 0$, we write $C_2 := C_1 2^{1+\frac{1}{\kappa}-\frac{m}{1+\kappa}}$. Thus, the proof is complete for $C(\phi, m) = \max(C_1, C_2) \sim \frac{1}{m-1-1/\kappa}$. \square

We are now in a position to prove the covering lemma.

Lemma 3. *For $s > 0$, there exists a constant N such that*

$$\sum_k \chi_{D^s(a_k)} \leq N.$$

Moreover for every $\varepsilon > 0$, there exists some universal constant $c_{ov} := c_{ov}(\phi, \varepsilon) > 0$ such that

$$N \leq c_{ov} (1 + s)^{1+\frac{1}{\kappa}+\frac{\kappa\varepsilon}{1+\kappa}}$$

where κ is given in (3.2) and depends on ϕ . The constant c_{ov} satisfies $c_{ov}(\phi, \varepsilon) \simeq \frac{1}{\varepsilon}$ when $\varepsilon \rightarrow 0$.

Obviously, the constant N is at least equal to 1. Remind that in order to cover the complex plane \mathbb{C} , we need to require that $s \geq r_0$.

Proof of Lemma 3. This is a consequence of Lemma 2. Indeed, let $m > 1 + \frac{1}{\kappa}$ and write

$$\Gamma := \sup_{z \in \mathbb{C}} \sum_{a_k \notin D^s(z)} \left(\frac{\rho(a_k)}{|z - a_k|} \right)^m < +\infty.$$

We start proving that for $s > 1$, $\#\{k : a_k \in D^s(z)\} \leq C_\mu^{i+1} s^\kappa$ for all $z \in \mathbb{C}$. Take $0 < \delta < \frac{1}{2}$ such that the disks $(D^\delta(a_k))_{k \in \mathbb{N}}$ are pairwise disjoint. Since the measure is doubling, for $i \geq 3$ satisfying $\frac{1}{2^i} \leq \delta$ we have

$$\begin{aligned} \#\{k : a_k \in D^s(z)\} &= \sum_{a_k \in D^s(z)} \mu(D^1(a_k)) \leq C_\mu^i \sum_{a_k \in D^s(z)} \mu(D^\delta(a_k)) \\ &= C_\mu^i \mu \left(\bigsqcup_{a_k \in D^s(z)} D^\delta(a_k) \right) \end{aligned} \quad (\star)$$

where \bigsqcup means that the union is disjoint.

Since the disks $D^\delta(a_k)$ are pairwise disjoint, the euclidean radii of $D^\delta(a_k)$ are smaller than that of $D^s(z)$, i.e. $\delta \rho(a_k) < s \rho(z)$. Indeed, otherwise we have $D^s(z) \subset D^{2\delta}(a_k)$ which leads to $1 < \mu(D^s(z)) \leq \mu(D^{2\delta}(a_k)) < 1$, which is a contradiction. Hence, $D^\delta(a_k) \subset D^{2s}(z)$ and so by (\star) we have

$$\#\{k : a_k \in D^s(z)\} \leq C_\mu^i \mu(D^{2s}(z)) \leq C_\mu^{i+1} \mu(D^s(z)).$$

Then the inequality follows from equation (3.2)

$$\#\{k : a_k \in D^s(z)\} \leq C_\mu^{i+1} C s^{\frac{1}{\kappa}}$$

where C_μ is the doubling constant and C is a constant hidden in (3.2).

Now, given $z \in \mathbb{C}$, set

$$A_z = \{k \in \mathbb{N} : z \in D^s(a_k)\}.$$

We claim that

$$N \leq 2\Gamma s^m + C_\mu^{i+1} C s^{\frac{1}{\kappa}}. \quad (3.6)$$

This will imply that $N \leq (2C(\phi, m) + C_\mu^{i+1} C) s^{\frac{1}{\kappa} + \frac{\kappa m}{1+\kappa}}$ since $\Gamma \leq C(\phi, m) s^{\frac{1}{\kappa} - \frac{m}{1+\kappa}}$ by Lemma 2. Thus, taking $m = 1 + \frac{1}{\kappa} + \varepsilon$ for $\varepsilon > 0$, we obtain $N \lesssim s^{1 + \frac{1}{\kappa} + \frac{\kappa \varepsilon}{1+\kappa}}$ for $s > 1$. Finally, we obtain for $s > 0$,

$$N \leq c_{ov}(\varepsilon) (1 + s)^{1 + \frac{1}{\kappa} + \frac{\kappa \varepsilon}{1+\kappa}}$$

with an absolute constant $c_{ov}(\varepsilon) := 2C(\phi, m) + C_\mu^{i+1} C > 0$ which depends only on ϕ . Moreover, since $C(\phi, m) \simeq \frac{1}{m-1-\frac{1}{\kappa}}$ when $m \rightarrow 1 + \frac{1}{\kappa}$, we have $c_{ov}(\varepsilon) \simeq \frac{1}{\varepsilon}$ when $\varepsilon \rightarrow 0$.

In order to show the claim (3.6), by contradiction we assume that it does not hold. Then there exists z_0 such that

$$\#A_{z_0} > 2\Gamma s^m + C_\mu^{i+1} C s^{\frac{1}{\kappa}}.$$

But then

$$\Gamma \geq \sum_{a_k \notin D^s(z_0)} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m \geq \sum_{\substack{k \in A_{z_0} \\ a_k \notin D^s(z_0)}} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m.$$

Since $z_0 \in D^s(a_k)$ for $k \in A_{z_0}$, we have

$$\Gamma \geq \sum_{\substack{k \in A_{z_0} \\ a_k \notin D^s(z_0)}} \left(\frac{\rho(a_k)}{|z_0 - a_k|} \right)^m \geq \frac{\#A_{z_0} - \#\{k : a_k \in D^s(z_0)\}}{s^m} > \frac{2\Gamma s^m}{s^m} = 2\Gamma,$$

which is a contradiction. □

As in the Bergman space, we introduce good disks. Fix $r_0 \leq s < t$ where r_0 is the radius such that the disks $D^{r_0}(a_k)$ cover the complex plane. For $K > 1$ the set

$$I_f^{K\text{-good}} = \{k : \|f\|_{L_\phi^p(D^t(a_k))} \leq K \|f\|_{L_\phi^p(D^s(a_k))}\}$$

will be called the set of K -good disks for (t, s) (in order to keep notation light we will not include s and t as indices). This set depends on f .

The following proposition has been shown in [HKKO20] for the Bergman space, but its proof, implying essentially the finite overlap property, is exactly the same.

Proposition 1. *Let $r_0 \leq s < t$. For every constant $c \in (0, 1)$, there exists K such that for every $f \in \mathcal{F}_\phi^p$ we have*

$$\sum_{k \in I_f^{K\text{-good}}} \|f\|_{L_\phi^p(D^s(a_k))}^p \geq c \|f\|_{p, \phi}^p.$$

One can pick $K^p = N(t)/(1 - c)$ where $N(t)$ corresponds to the overlapping constant from Lemma 3 for the ‘‘radius’’ t .

Finally, we need to control the value of ϕ in a disk by the value at the center. For this, we can use [MMOC03, Lemma 13] which we state as follows: for every $\sigma > 0$, there exists $A = A(\sigma) > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{z \in D^\sigma(a_k)} |\phi(z) - \phi(a_k) - \mathbf{h}_{a_k}(z)| \leq A(\sigma) \tag{3.7}$$

where \mathbf{h}_{a_k} is a harmonic function in $D^\sigma(a_k)$ with $\mathbf{h}_{a_k}(a_k) = 0$. Moreover, in view of the proof of [MMOC03, Lemma 13] we have

$$A(\sigma) \leq C \sup_{k \in \mathbb{N}} \mu(D^\sigma(a_k)) \leq \tilde{C} \sigma^{\frac{1}{\kappa}} \tag{3.8}$$

with \tilde{C} depending only on the space and where the inequalities come from [MMOC03, Lemma 5(a)] and the estimate (3.2).

Notice that, since \mathbf{h}_{a_k} is harmonic on $D^\sigma(a_k)$, there exists a function H_{a_k} holomorphic on $D^\sigma(a_k)$ with real part \mathbf{h}_{a_k} .

We are now in a position to prove the theorem. Take $s = \max(r, r_0)$ and $t = 4s$. As noted in [HKKO20], E is $(\tilde{\gamma}, s)$ -dense, where $\tilde{\gamma} = c\gamma$ and c is a multiplicative constant. Hence we can assume that E is (γ, s) -dense.

Now, given f with $\|f\|_{p,\phi} = 1$, let $g = fe^{-(H_{a_k} + \phi(a_k))}$ for $k \in I_f^{K\text{-good}}$, and set

$$h = c_0 g, \quad c_0 = \left(\frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p dA(z)} \right)^{1/p}.$$

Again there is $z_0 \in D^s(a_k)$ with $|h(z_0)| \geq 1$.

Set $R = 2s$ and apply (3.7) with $\sigma = t = 4s$. We have to estimate the maximum modulus of h on $D^R(a_k)$ in terms of a local integral of h . To that purpose, we can assume $h \in A^p(D^t(a_k))$. Indeed, we have

$$\begin{aligned} \int_{D^t(a_k)} |h|^p dA &= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |g|^p dA} \int_{D^t(a_k)} |g|^p dA \\ &= \frac{\pi s^2 \rho(a_k)^2}{\int_{D^s(a_k)} |f|^p e^{-p(\mathbf{h}_{a_k} + \phi(a_k))} dA} \int_{D^t(a_k)} |f|^p e^{-p(\mathbf{h}_{a_k} + \phi(a_k))} dA \\ &\leq \frac{\pi s^2 \rho(a_k)^2 e^{2pA(t)}}{\int_{D^s(a_k)} |f|^p e^{-p\phi(z)} dA(z)} \int_{D^t(a_k)} |f|^p e^{-p\phi(z)} dA(z). \\ &\leq \pi s^2 \rho(a_k)^2 e^{2pA(t)} K^p \end{aligned}$$

where the last inequality comes from the fact that k is K -good for (s, t) . Hence $h \in A^p(D^t(a_k))$ and it follows

$$M^p := \max_{z \in D^R(a_k)} |h(z)|^p \leq \frac{C}{s^2 \rho(a_k)^2} \int_{D^t(a_k)} |h|^p dA \leq c_s K^p$$

where $c_s = Ce^{2pA(4s)}$, $A(4s)$ comes from (3.7) and C is an absolute constant.

Now, setting $\tilde{E} = (E \cap D^s(a_k))$ we get using Corollary 2 applied to h :

$$\int_{D^s(a_k)} |h(z)|^p dA(z) \leq \left(\frac{cs^2 \rho(a_k)^2}{|\tilde{E}|} \right)^{p\eta \ln M + 1} \int_{\tilde{E}} |h(z)|^p dA(z)$$

The factor $s^2 \rho(a_k)^2$ appearing inside the brackets is a rescaling factor (see Remark 1). Again, by homogeneity we can replace in the above inequality h by g . Note also that $\pi s^2 \rho(a_k)^2 / |\tilde{E}|$ is controlled by $1/\gamma$. This yields

$$\begin{aligned}
\int_{D^s(a_k)} |f|^p e^{-p\phi(z)} dA(z) &\leq e^{pA(t)} \int_{D^s(a_k)} |g(z)|^p dA(z) \\
&\leq e^{pA(t)} \left(\frac{cs^2 \rho(a_k)^2}{|\tilde{E}|} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |g(z)|^p dA(z) \\
&\leq e^{pA(t)} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |g(z)|^p dA(z) \\
&= e^{2pA(t)} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M+1} \int_{\tilde{E}} |f(z)|^p e^{-p\phi(z)} dA(z),
\end{aligned}$$

where c_1 is an absolute constant.

Summing over all K -good k , and using Lemma 3 and Proposition 1 we obtain the required result

$$c \|f\|_{p,\phi} \lesssim \left(\frac{c_1}{\gamma} \right)^{\eta \ln M+1/p} \|f\|_{L_\phi^p(E)}$$

where

$$\begin{aligned}
\ln(M) &\leq \ln(c_s^{1/p} K) = \ln \left(\left(c_s \frac{N(4s)}{1-c} \right)^{\frac{1}{p}} \right) \leq \frac{1}{p} \ln \left(C e^{2pA(4s)} c_{ov}(\varepsilon) \frac{(1+4s)^{1+\frac{1}{\kappa}+\frac{\kappa\varepsilon}{1+\kappa}}}{1-c} \right) \\
&\leq 2A(4s) + \frac{1}{p} \left[\ln(C c_{ov}(\varepsilon)) + \left(1 + \frac{1}{\kappa} + \frac{\kappa\varepsilon}{1+\kappa} \right) \ln(1+4s) - \ln(1-c) \right]
\end{aligned}$$

and in view of Lemma 1

$$\eta \leq c'' \times 2^4 \ln 2$$

c comes from Proposition 1, $c_{ov}(\varepsilon) \simeq \frac{1}{\varepsilon}$ from Lemma 3, κ from (3.2) and C is absolute. In particular, fixing $\varepsilon > 0$ and $c \in (0, 1)$, and noticing that $r \simeq s = \max(r, r_0)$ for $r > 1$, we obtain

$$\ln M \leq 2A(4r) + \frac{1}{p} (C' + C'' \ln(1+r))$$

where C' and C'' depends only on the space. In view of (3.8), we finally obtain

$$\ln M \leq \widehat{C} r^{\frac{1}{\kappa}} + \frac{1}{p} (C' + C'' \ln(1+r))$$

with \widehat{C} depending only on the space. Notice that we can optimize the constants C' and C'' in $\varepsilon > 0$ depending on r (C' involves $c_{ov}(\varepsilon) \simeq \frac{1}{\varepsilon}$ and C'' involves $1 + \frac{1}{\kappa} + \frac{\kappa\varepsilon}{1+\kappa}$). However, this optimization will not really improve the result.

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REFERENCES

- [AR07] Vladimir V. Andrievskii and Stephan Ruscheweyh. Remez-type inequalities in the complex plane. *Constr. Approx.*, 25(2):221–237, 2007.
- [FHR17] Emmanuel Fricain, Andreas Hartmann, and William T. Ross. A survey on reverse Carleson measures. In *Harmonic analysis, function theory, operator theory, and their applications*, volume 19 of *Theta Ser. Adv. Math.*, pages 91–123. Theta, Bucharest, 2017.
- [HJK17] Andreas Hartmann, Philippe Jaming, and Karim Kellay. Quantitative estimates of sampling constants. accepted for publication in *Amer. J. Math.*, arXiv: 1707.07880., July 2017.
- [HKKO20] A. Hartmann, D. Kamissoko, S. Konate, and M.-A. Orsoni. Dominating sets in Bergman spaces and sampling constants. preprint on HAL :, April 2020.
- [JPR87] Svante Janson, Jaak Peetre, and Richard Rochberg. Hankel forms and the Fock space. *Rev. Mat. Iberoamericana*, 3(1):61–138, 1987.
- [JS] P. Jaming and M. Speckbacher. Planar sampling sets of the short-time Fourier transform. preprint on ArXiv: 1906.02964.
- [Kov01] Oleg Kovrijkine. Some results related to the Logvinenko-Sereda theorem. *Proc. Amer. Math. Soc.*, 129(10):3037–3047, 2001.
- [Lin00] Niklas Lindholm. Sampling measures in Fock spaces in \mathbb{C}^n . Unpublished, 2000.
- [Lue81] Daniel H. Luecking. Inequalities on Bergman spaces. *Illinois J. Math.*, 25(1):1–11, 1981.
- [LZ19] Zengjian Lou and Zhengyuan Zhuo. A class of reverse Carleson measures on doubling Fock spaces. *Complex Anal. Oper. Theory*, 13(4):1795–1809, 2019.
- [MMOC03] N. Marco, X. Massaneda, and J. Ortega-Cerdà. Interpolating and sampling sequences for entire functions. *Geom. Funct. Anal.*, 13(4):862–914, 2003.
- [OC98] Joaquim Ortega-Cerdà. Sampling measures. *Publ. Mat.*, 42(2):559–566, 1998.
- [Rez10] Alexander Reznikov. Sharp constants in the Paneyah-Logvinenko-Sereda theorem. *C. R. Math. Acad. Sci. Paris*, 348(3-4):141–144, 2010.
- [Sei04] K. Seip. *Interpolation and sampling in spaces of analytic functions*, volume 33. American Mathematical Society, 2004.

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