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SPECIAL CLEAN AND PERSPECTIVE ELEMENTS AND PERSPECTIVE RINGS

XAVIER MARY

ABSTRACT. Motivated by the idea of perspectivity of rings and modules, we introduce left/right perspective elements of a ring. We notably prove that perspective elements form a proper subset of special clean elements. An equational characterization of these elements is given and examples are provided.

1. INTRODUCTION AND NOTATIONS

The object of this article is to study the link between perspective rings, as defined in [?], and special clean elements, as defined in [?], in an element-wise and ring-theoretical manner. To this end, we introduce and study a new class of elements, the perspective ones, and prove that they form a proper subset of special clean elements.

In the following, all rings will be assumed unital, and R will always denote a (unital) ring. Unless otherly stated, by modules we will always mean right modules over R .

Let R be a ring. As usual, we denote by $E(R)$ the set of idempotents of R , and by $U(R)$ the set of units of R . An element $a \in R$ is *clean* if it admits a decomposition $a = \bar{e} + u$ with $e \in E(R)$, $\bar{e} = 1 - e$, and $u \in U(R)$. If moreover $aR \cap \bar{e}R = 0$ then the decomposition is a (right) *special clean* decomposition. It is now well known that right special clean elements are also left special clean, so that $Ra \cap R\bar{e} = 0$, and it is also known that u^{-1} is an *inner inverse* of a , that is $au^{-1}a = a$ (see [?], [?]).

Generally, an element $b \in R$ that satisfies $aba = a$ (resp. $aba = a, bab = b$) is called an inner (resp. *reflexive*) inverse of a . We denote by $V(a)$ the set of reflexive inverses of a . If a admits an inner inverse in R (resp. in $U(R)$) then we say that a is *regular* (resp. *unit-regular*). As first noted by Hartwig and Luh [?], a is unit-regular iff it can be written as $a = fv$ for some idempotent $f \in E(R)$ and unit $v \in U(R)$. A commuting reflexive inverse of a exists iff a is *strongly regular* ($a \in a^2R \cap Ra^2$). It is unique if it exists, and called the *group inverse* of a . It is usually denoted by $a^\#$. The set of regular (resp. unit-regular, resp. strongly regular, resp. special clean) elements of R will be denoted by $reg(R)$ (resp. $ureg(R)$, resp. $R^\#$, resp. $sp.cl(R)$).

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As usual, we will denote by $r_R(a) = \{x \in R \mid ax = 0\}$ (resp. $l_R(a) = \{x \in R \mid xa = 0\}$) the right (resp. left) annihilator of a . If $a \in R$ is regular, then it is common knowledge that $aR = abR$ and $r_R(a) = (1 - ba)R$ for all $b \in V(a)$ (in particular, aR and $r_R(a)$ are direct summands in R_R). Conversely, for any regular element a , any complementary summand of aR is of the form $(1 - ab)R = r_R(b)$ for some $b \in V(a)$ (see Lemma 2.3 and its comment), and any complementary summand of $r_R(a)$ is of the form $baR = bR$ for some $b \in V(a)$ (see the proof of Theorem 3.1).

Let M be a module. Two direct summands A and B (in M) are *perspective* if they have a common complementary summand. The (right) module M itself is perspective if any two isomorphic direct summands of M are perspective in M . A ring R is a perspective ring if R_R is perspective as a (right) module over itself. As proved in [?, Theorem 3.3], the definition of a perspective ring is left/right symmetric.

Our first theorem (Theorem 2.4) gives a new characterization of special clean elements in terms of direct sums of right and left ideals. Our second theorem (Theorem 3.1) characterizes perspective rings in terms of a new class of elements, the (left/right) perspective ones. Then, Theorem 3.3 provides the reader with several equivalent characterizations of right perspective elements. In particular, perspective elements are characterized as a proper subset of special clean elements. In Section 4, we characterize perspective elements by solving equations in certain corner rings. With this methodology, we notably prove that regular elements that square to 0 are perspective 4.7. Section 5 considers perspectivity with regards to other conditions, notably stable range 1. In particular, we prove that left perspective elements are exactly the regular elements that satisfy “outer inverse right stable range 1” (Proposition 5.5). Finally, Section 6 is devoted to examples.

2. SPECIAL CLEAN ELEMENTS

Throughout the paper, the following facts about direct sums of principal ideals generated by idempotents will prove useful. The first part of the following lemma appears as [?, Exercise 21.4] and the the second part as [?, Lemma 3.1].

Lemma 2.1. *Let R be a ring and $e, f \in E(R)$. Then:*

- (1) $eR = fR \Leftrightarrow R\bar{e} = R\bar{f} \Leftrightarrow \{ef = f, fe = e\} \Leftrightarrow \{\bar{e}\bar{f} = \bar{e}, \bar{f}\bar{e} = \bar{f}\}$;
- (2) $eR \oplus \bar{f}R = R \Leftrightarrow R\bar{e} \oplus Rf = R$.

Also, we recall that an element a of a ring R is regular iff $aR = fR$ (alternatively $Rf = Ra$) for some idempotent $f \in E(R)$, and that a submodule A of a module $M = M_R$ is a direct summand iff it is of the form $A = eM$ for some idempotent $e \in \text{End}_R(M)$.

The following lemma, that can be directly deduced from [?, Theorem 4.1] (see also [?, Theorem 2.3 and Theorem 2.13], [?, Theorem 6.1]) will be crucial in the sequel.

Lemma 2.2. *Let R be a ring and $a \in R, e \in E(R)$. The following statements are equivalent:*

- (1) $u = a - \bar{e} \in U(R)$ and $aR \cap \bar{e}R = 0$;
- (1') $u = a - \bar{e} \in U(R)$ and $Ra \cap R\bar{e} = 0$;
- (2) $u = a - \bar{e} \in U(R)$ and $aR \oplus \bar{e}R = R$;

- (2') $u = a - \bar{e} \in U(R)$ and $Ra \oplus R\bar{e} = R$;
(3) $u = a - \bar{e} \in U(R)$ and $a = au^{-1}a$;
(4) $u = a - \bar{e} \in U(R)$, $z = u^{-1}au^{-1} \in V(a) \cap R^\#$ (z is a reflexive inverse of a which is strongly regular) and $zz^\# = e$;
(5) $aza = a, zaz = z$ and $zz^\# = e$ for some $z \in R^\#$.

Thus, the condition of being special clean is indeed left/right symmetric, as first observed independently in [?] and [?]. Moreover, it is a purely multiplicative notion as shown by (5) (since $z \in R^\#$ can be replaced by the multiplicative statement: z admits a commuting reflexive inverse). Some readers may be surprised by the use of the complementary idempotent \bar{e} , instead of e , in the clean decomposition $a = \bar{e} + u$. An argument in favor of this choice is that the multiplicative statement (5) makes sense even in a non-unital ring (see [?]), and that, under this convention, it then involves the idempotent e . A second argument is that, with this choice, the right submodules aR and eR are perspective.

It happens that the conditions (2) and (2'), which involve direct summands, have not been used directly to characterize special cleanness. More generally, while direct sums of right modules have been extensively studied, mixed-type decompositions (involving both right and left modules) have attracted less attention. Our next theorem claims that the direct sums conditions in (2) and (2') together actually imply invertibility of u (equivalently that a is special clean).

First, we prove the following lemma.

Lemma 2.3. *Let $a \in R$ and $e \in E(R)$ such that $aR \oplus \bar{e}R = R$. Then $Re = Rab$ for some $b \in V(a)$.*

In particular, by Lemma 2.1 any complementary summand of aR is of the form $(1 - ab)R = r_R(b)$ for some $b \in V(a)$.

Proof. Let $f \in E(R)$ be the idempotent which under left multiplication has image aR and kernel $\bar{e}R$. Thus $fR = aR$ and $\bar{f}R = \bar{e}R$. It follows that $f = ab$ for some $b \in R$, and $fa = a$. Therefore $aba = fa = a$. After replacing b with bab , we may additionally assume that $b \in V(a)$. Now, by Lemma 2.1(1), we have $Re = Rf = Rab$. \square

Theorem 2.4. *Let $a \in R$ and $e \in E(R)$ be such that $aR \oplus \bar{e}R = R$ and $Ra \oplus R\bar{e} = R$. Then $u = a - \bar{e}$ is a unit whose inverse u^{-1} is an inner inverse of a , and a is special clean.*

Proof. By Lemma 2.3 and its dual, there exist $b, c \in V(a)$ such that $Re = Rab$ and $eR = caR$. Let $z = cab$. Then $zaz = cabacab = cab = z$ and $aza = acaba = a$, so that $z \in V(a)$. We claim that z is group invertible with $zz^\# = e$. Indeed, $Rz = Raz = Racab = Rab = Re$ and dually $zR = zaR = cabaR = caR = eR$, so that $z = ze = ez$ and $e = zx = yz$ for some $x, y \in R$. Let $z' = yzx$. Then $z'z = yzxx = yez = yz = e$ and $zz' = zyxz = zex = zx = e$, with $z' \in V(z)$. It follows that $z' = z^\#$, the unique commuting reflexive inverse of z . We conclude the proof by applying the implication (5) \Rightarrow (3) in Lemma 2.2. \square

3. PERSPECTIVE ELEMENTS

In this section, we introduce a new class of elements, motivated by the notion of perspectivity. Then, we prove that this new class forms a proper subset of the class of special clean elements.

Second, we give a new element-wise characterization of perspective rings.

Theorem 3.1. *Let R be a ring. The following statements are equivalent:*

- (1) R is perspective;
- (2) For all regular $a \in R$, every complementary summand of $r_R(a)$ is perspective with aR .

Proof. We prove the two implications (1) \Rightarrow (2) and (2) \Rightarrow (1).

- (1) \Rightarrow (2) Write $R = B \oplus r_R(a)$. Left multiplication by a has kernel $r_R(a)$, but image aR . Thus left multiplication by a is an isomorphism from B to aR . By (1), we see that B and aR are perspective.
- (2) \Rightarrow (1) We prove that any two isomorphic direct summands of the right R -module R_R are perspective. Let $A \simeq A'$ be isomorphic direct summands of R_R . Then $A = fR$ and $A' = f'R$ for some isomorphic idempotents $f = ab, f' = ba$ with $a, b \in R$ reflexive inverses. In particular, $aR = abR = fR = A$. As $r_R(a) = (1 - ba)R = \bar{f}'R$ then $A' = f'R$ is a complementary summand of $r_R(a)$. By (2), A' is then perspective with $aR = A$.

□

In the same way that rings with stable range 1 or exchange rings for instance led to element-wise definitions, we propose the following definition.

Definition 3.2. Let R be a ring, and $a \in R$. We say that a is *right perspective* if it is regular and every complementary summand of $r_R(a)$ is perspective with aR . The *left perspective* elements are defined in a left-right symmetric way. An element is *perspective* when it is both left and right perspective.

The set of right (resp. left) perspective elements of R will be denoted by $per_r(R)$ (resp. $per_l(R)$).

The next theorem constitutes the central result of the article. It characterizes (left, right) perspective elements in terms of clean and special clean decompositions, reflexive inverses, idempotents and direct summands.

Theorem 3.3. *Let R be a ring, and $a \in R$. The following statements are equivalent:*

- (1) a is right perspective;
- (2) (aR, bR characterization) a is regular and for all $b \in V(a)$, aR and bR are perspective (as right R -submodules of R_R);
- (2') a is regular and for all $f \in E(R)$ such that $Ra = Rf$, aR and fR are perspective;
- (3) (Clean characterization) a is regular and for all $f \in E(R)$ such that $Ra = Rf$ there exists a clean decomposition $a = \bar{e} + u$ with $u \in U(R), e \in E(R)$ and $eR = fR$;
- (4) (Special clean characterization) a is regular and for all $f \in E(R)$ such that $Ra = Rf$ there exists a special clean decomposition $a = \bar{e} + u = au^{-1}a$ with $u \in U(R), e \in E(R)$ and $eR = fR$;

- (5) (*Group inverse characterization*) a is regular and for all $b \in V(a)$, there exists $z \in V(a) \cap R^\#$ such that $zR = bR$;
- (6) (*Idempotent characterization*) a is regular and for all $b \in V(a)$, there exists $e, g \in E(R)$ such that

$$abR = gR, Rg = Re \text{ and } eR = baR.$$

- (7) ($l_R(a), l_R(b)$ *characterization*) a is regular and for all $b \in V(a)$, $l_R(a)$ and $l_R(b)$ are perspective (as left R -submodules of ${}_R R$);
- (8) (*Dual characterization*) a is regular and any complementary summand of Ra is perspective with $l_R(a)$.

Proof. First, we observe that (2) and (2') are equivalent since for any idempotent $f \in E(R)$, $Ra = Rf$ iff a is regular and $f = ba$ for some $b \in V(a)$, and in this case $bR = fR$.

- (1) \Rightarrow (2) Assume that a is right perspective, and let $b \in V(a)$. Then $bR = baR$ is a complementary summand of $(1 - ba)R = r_R(a)$ and by assumption aR, bR are perspective.
- (2') \Rightarrow (3) Assume (2') and let $f \in E(R)$ be such that $Ra = Rf$. Let $e \in E(R)$ be the idempotent which under left multiplication has image aR and kernel B . Thus $fR = aR$ and $\bar{f}R = \bar{e}R$. It follows that $f = ab$ for some $b \in R$, and $fa = a$.
- (3) \Rightarrow (4) Assume (3) and let $f \in E(R)$ be such that $Ra = Rf$. Then $a = \bar{e} + u$ for some $e \in E(R), u \in U(R)$ such that $eR = fR$. By Lemma 2.1, $R\bar{e} = R\bar{f}$. Since $Ra \cap R\bar{e} = Rf \cap R\bar{f} = 0$, it then follows from Lemma 2.2 that $a = \bar{e} + u = au^{-1}a$.
- (4) \Rightarrow (5) Assume (4) and let $b \in V(a)$. Let also $f = ba \in E(R)$. Then $Ra = Rf$ and by assumption $a = \bar{e} + u = au^{-1}a$ for some $u \in U(R), e \in E(R)$ such that $eR = fR$. By Lemma 2.2 $e = zz^\#$ for some $z \in V(a) \cap R^\#$. Finally $zR = eR = fR = baR = bR$.
- (5) \Rightarrow (6) Assume (5) and let $b \in V(a)$. Let also $g = az$. Then $abR = aR = azR = gR, Rg = Raz = Rz = Rz^\#z = Re$ and $eR = fR = baR$.
- (6) \Rightarrow (7) Assume (6) and let $b \in V(a)$. Then $l_R(b) = R(1 - ba)$ and $l_R(a) = R(1 - ab)$. Also, by (6) and Lemma 2.1, $abR = gR, \bar{g}R = \bar{e}R$ and $eR = baR$ for some $e, g \in E(R)$. In particular, $abR \oplus \bar{e}R = R = baR \oplus \bar{e}R$ and by Lemma 2.1, $R(1 - ab) \oplus Re = R(1 - ba) \oplus Re$. Thus, $l_R(a)$ and $l_R(b)$ are perspective.
- (7) \Rightarrow (8) Assume (7) and let B be a complementary summand of Ra . Then $B = R(1 - ba)$ for some reflexive inverse b of a . Indeed, as B is a direct summand, $B = Rh$ for some $h \in E(R)$. By the dual of Lemma 2.3, $\bar{h}R = baR$ for some $b \in V(a)$, hence by Lemma 2.1 $B = Rh = R(1 - ba) = l_R(b)$. By (7), $B = l_R(b)$ and $l_R(a)$ are perspective.
- (8) \Rightarrow (1) Assume (8) and let B be a complementary summand of $r_R(a)$. Then $B = bR = baR$ for some reflexive inverse b of a . As $R(1 - ba) \oplus Rba = R$ and $Rba = Ra$, then $R(1 - ba)$ is perspective with $l_R(a) = R(1 - ab)$ by (8), and there exists $e \in E(R)$ such that $R(1 - ba) \oplus Re = R = R(1 - ab) \oplus Re$. By Lemma 2.1, we deduce that $baR \oplus \bar{e}R = R = abR \oplus \bar{e}R$, and B is perspective with $abR = aR$.

□

We make the following observations.

- It follows from (4) that a right perspective element is left special clean ($a = \bar{e} + u$ for some $e \in E(R), u \in U(R)$ such that $Ra \cap R\bar{e} = 0$) hence also right special clean. It is however an open question whether right perspectivity is equivalent to left perspectivity at the level of elements.

- From Lemma 2.2, it is straightforward to see that a special clean element $a = \bar{e} + u = au^{-1}a$, $e \in E(R)$, $u \in U(R)$ satisfies that $Ra = Rf$ and $eR = fR$ for the special idempotent $f = u^{-1}a$ (check that $fe = u^{-1}ae = u^{-1}ue = e$ and $ef = eu^{-1}a = (1 + u - a)u^{-1}a = u^{-1}a = f$). The special clean characterization (4) of right perspective elements expresses a strong strengthening of this property, which now holds for any idempotent $f \in E(R)$ such that $Ra = Rf$.
- From (6), and the well-known fact that, for any two idempotents $e, f \in E(R)$, $eR = fR$ iff $ef = f$ and $fe = e$, we deduce that being right perspective is a purely multiplicative notion (stable by semigroup isomorphism, see Proposition 3.5).
- A direct consequence of Theorem 3.1 and Theorem 3.3 is that a perspective ring R satisfies $\text{reg}(R) = \text{sp.cl}(R)$.
- Also, from (the duals of) Theorem 3.1 and (1) \Leftrightarrow (8) in Theorem 3.3, we recover that a ring R is perspective iff for all $a \in R$, any complementary summand of aR is perspective with $r_R(a)$, which is [?, Theorem 4.2, point (4)].

Let $a \in R$ be strongly regular. Then $e = aa^\# \in E(R)$ and $u = a - 1 + aa^\# \in U(R)$ (with $u^{-1} = a^\# - 1 + aa^\#$). Thus, $a = \bar{e} + u = au^{-1}a$ is special clean. We strengthen this fact by proving perspectivity.

Lemma 3.4. *Strongly regular elements are perspective. In particular, idempotents and units are perspective.*

Proof. Let $a \in R$ be strongly regular. Then aR and $r_R(a)$ are complementary direct summands, and right perspectivity follows from the definition. The proof of left perspectivity is dual. \square

We now prove that left/right perspectivity (as well as special cleanness) is stable under semigroup isomorphisms (of the multiplicative structures of the rings), a property that will be useful later.

Proposition 3.5. *Let $a \in R$ be right perspective (resp. left perspective, resp. special clean) and $\phi : (R, \cdot) \rightarrow (T, \cdot)$ be a semigroup isomorphism (of their multiplicative structure). Then $\phi(a)$ is right perspective (resp. left perspective, resp. special clean).*

Proof. We only prove the right perspective case, since the other two cases are similar.

Let a be right perspective element and $b \in V(a)$. Let also ϕ^{-1} be the inverse of ϕ . First, $\phi(b) \in V(\phi(a))$, and $\phi(a)$ is regular. Let $y \in V(\phi(a))$ and let $x = \phi^{-1}(y)$. Then $x \in V(a)$ and by the idempotent characterization of perspective elements (6) in Theorem 3.3, there exists $e, g \in E(R)$ such that

$$axR = gR, Rg = Re \text{ and } eR = xaR.$$

However, by Lemma 2.1, such equalities between principal ideals generated by idempotents reduce to the multiplicative identities

$$axg = g, gax = ax, eg = e, ge = g \text{ and } exa = xa, xae = e.$$

Finally, we compose by ϕ , and we conclude by using the implication (6) \Rightarrow (1) in Theorem 3.3. \square

One-sided perspectivity is therefore stable under similarity (a ring automorphism). It is also stable under multiplication by -1 (this is not the case for clean elements in general), or more generally under multiplication by any central unit $w \in R$.

Proposition 3.6. *Let $a \in R$ and $w \in U(R)$ be such that $aw = wa$. Then a is left (resp. right) perspective iff wa is left (resp. right) perspective.*

Proof. For any element $a \in R$ and any unit $w \in U(R)$ commuting with a , it holds that $aR = waR$ and $r_R(a) = r_R(wa)$. \square

The previous result was proved for special clean elements in [?, Theorem 2.8].

Let $a \in R$ be unit-regular with unit inverse v^{-1} and let $a' = v^{-1} - v^{-1}av^{-1}$. Then aR coincide with the right annihilator $r_R(a')$ of a' , and $a'R$ coincide with the right annihilator $r_R(a)$ of a . Thus a is left (resp. right)perspective iff a' is right (resp. left) perspective.

Proposition 3.7. *Let $a \in R$ be unit-regular with unit inner inverse v^{-1} . Then*

- (1) a is left (resp. right) perspective iff $a' = v^{-1} - v^{-1}av^{-1}$ is right (resp. left) perspective;
- (2) a is special clean iff $a' = v^{-1} - v^{-1}av^{-1}$ is special clean.

Proof. Let $a \in R$ be unit-regular with unit inner inverse v^{-1} and let $a' = v^{-1} - v^{-1}av^{-1}$. Then, a' is unit-regular with unit inner inverse v and $aR = av^{-1}R = (1 - va')R$, $(1 - v^{-1}a)R = a'vR = a'R$ (aR coincide with the right annihilator $r_R(a')$ of a' , and $a'R$ coincide with the right annihilator $r_R(a)$ of a).

- (1) It follows from the dual characterization (8) of Theorem 3.3 that a is left (resp. right) perspective iff a' is right (resp. left) perspective.
- (2) As $(a')' = v - va'v = a$, we have only to prove the implication. Assume that a is special clean. Then $aR \oplus \bar{e}R = R = Ra \oplus R\bar{e}$ for some $e \in E(R)$ by Lemma 2.2. It follows that $(1 - va')R \oplus \bar{e}R = R = Rv^{-1}a \oplus R\bar{e}$ and by Lemma 2.1 $Rva' \oplus Re = R = (1 - v^{-1}a)R \oplus eR$, or equivalently $Ra' \oplus Re = R = a'R \oplus eR$. By Theorem 2.4, a' is special clean. \square

We conclude this section by considering perspective elements with a unique special clean decomposition:

Corollary 3.8. *Let R be a ring, and $a \in R$. Then the following statements are equivalent:*

- (1) a is perspective and admits unique special clean decomposition ($a = \bar{e} + u = au^{-1}a$ for some unique $e \in E(R), u \in U(R)$);
- (2) $V(a)$ is a singleton;
- (3) a is strongly regular and $aa^\#$ is central.

Proof. Let $a \in R$.

- (1) \Rightarrow (2) Assume that a is perspective and uniquely special clean. Then $a = \bar{e} + u = au^{-1}a$ for some unique $e \in E(R), u \in U(R)$. Let b, b' be reflexive inverses of a . Then $f = ab$ and $f' = ab'$ satisfy that $aR = fR = f'R$ so that $\bar{e}R = \bar{f}R = \bar{f}'R$ (or equivalently $Rf = Re = Rf'$) by left perspectivity. Thus $f = f'$ since the two idempotents give rise to the same decomposition of R . Thus $ab = ab'$, and by symmetry $ba = b'a$. Then $b = bab = bab' = b'ab' = b'$.
- (2) \Rightarrow (3) Assume that a admits a unique reflexive inverse $b \in R$. It is well known that the set of reflexive inverses of a is the set of elements of the form $b' = b + (1 - ba)x(ab) + (ba)y(1 - ab) + (1 - ba)xay(1 - ab)$ for some $x, y \in R$. Since b is unique it follows that $(1 - ba)R(ab) = 0$ and

$(ba)R(1 - ab) = 0$, in particular $ab = ba^2b$ and $ba = ba^2b$ so that $ba = ab$ and a is strongly regular with group inverse $a^\# = b$. But then $e = aa^\#$ satisfies $eR\bar{e} = \bar{e}Re = 0$ and e is central.

(3) \Rightarrow (1) Assume that a is strongly regular and $e = aa^\#$ is central. First, by Lemma 3.4, a is perspective. Let $a = \bar{g} + v = av^{-1}a$ be a special clean decomposition of a , and let $z = v^{-1}av^{-1}$. As $f = av^{-1}$ satisfies $eR = aR = fR$ then $fe = e$ and $ef = f$. As e is central this implies that $f = e$, and $aa^\# = av^{-1}$. Dually $aa^\# = v^{-1}a$. It follows that $z = v^{-1}av^{-1} = v^{-1}aa^\# = a^\#aa^\# = a^\#$, which implies by Lemma 2.2 that $g = e$, and $v = (a - 1 + aa^\#)$. Finally, the special clean decomposition is unique.

□

In general, for an element a which is not perspective, we can only say that a is uniquely special clean iff $V(a) \cap R^\#$ is a singleton (a admits a unique reflexive inverse which is also strongly regular), since special clean decompositions are in bijective correspondance with strongly regular reflexive inverses from Lemma 2.2.

Example 3.9 shows that the hypothesis “ a is perspective” in Corollary 3.8 is not extraneous.

Example 3.9. Let $A = \begin{pmatrix} 8 & 3 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$. It is shown in [?] that A is uniquely clean in $\mathcal{M}_2(\mathbb{Z})$ with unique clean decomposition

$$\begin{pmatrix} 8 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix}.$$

Let $\bar{E} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix}$. We observe that the decomposition $A = \bar{E} + U$ is a special clean decomposition since $AU^{-1}A = A$. Thus A is uniquely special clean. However, A is not strongly regular since $A^2\mathcal{M}_2(\mathbb{Z}) \subseteq 8\mathcal{M}_2(\mathbb{Z})$, and A has no entry multiple of 8, so that A is not perspective (see also Example 6.1 for a direct proof of this statement). And $V(A)$ does not reduce to a singleton, since $B = \begin{pmatrix} 2 & 0 \\ -5 & 0 \end{pmatrix} \in V(A)$. On the other hand, $V(A) \cap R^\# = \{Z\}$, where $Z = U^{-1}AU^{-1} = \begin{pmatrix} -1 & 0 \\ 3 & 0 \end{pmatrix}$.

Corollary 3.10. *Let a be a regular element in a perspective ring R . Then the following statements are equivalent:*

- (1) a is uniquely special clean;
- (2) $V(a)$ is a singleton;
- (3) a is strongly regular and $aa^\#$ is central.

These results may be compared to [?, Corollary 2.6 and Theorem 4.4], where it is stated that the idempotent in the decomposition of a both uniquely clean and strongly clean element is central, and that in a von Neumann regular ring with clean corner rings (for instance a unit-regular ring), uniquely clean elements are the central idempotents. Another theorem to compare these results is [?, Theorem 5.4(3)], where it is stated that rings whose regular elements (equivalently idempotents) are uniquely special clean are precisely the abelian rings. Since idempotents are perspective, this also follows from Corollary 3.8.

4. EQUATIONAL RESOLUTION

In [?], the problem of deciding whether a unit-regular element is special clean or not is reduced to the resolution of an equation in certain corner rings. We recall the result here, and then adapt it to the case of left/right perspective elements. We use the Peirce decomposition relative to an idempotent (see [?] for the application to the present context) and write the associated matrices in upper-case letters. In this section, we use the special clean characterization of perspective elements.

Theorem 4.1 ([?, Theorem 2.1]). *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} . Let $f = av^{-1}$ and define a function $\varphi : \bar{f}Rf \times fR\bar{f} \rightarrow \bar{f}R\bar{f}$ by*

$$\varphi : (y, x) \mapsto yv_1x + yv_2 + v_3x + v_4 = (y + \bar{f})v(x + \bar{f})$$

where v has Peirce decomposition $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$ (in $fRf \oplus fR\bar{f} \oplus \bar{f}Rf \oplus \bar{f}R\bar{f}$).

Then the set of special clean decompositions of a is in one-to-one correspondence with the solution set for

$$\varphi(y, x) \in U(\bar{f}R\bar{f}).$$

Precisely, any invertible u such that $a - u$ is idempotent and $au^{-1}a = a$ has Peirce decomposition

$$U^{-1} = V^{-1} \begin{pmatrix} 1 & -x \\ -y & -\varphi(y, x) + yx \end{pmatrix} = V^{-1} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varphi(y, x) \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $\varphi(y, x) \in U(\bar{f}R\bar{f})$. And conversely, any $u \in R$ of this form is invertible and satisfies that $a - u$ is idempotent and $au^{-1}a = a$.

In this case, the idempotent $\bar{e} = a - u$ has the form

$$\bar{E} = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V.$$

Corollary 4.2. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. The element a is left perspective iff for all $x \in fR\bar{f}$, there exists $y \in \bar{f}Rf$ such that $\varphi(y, x) = (y + \bar{f})v(x + \bar{f}) \in U(\bar{f}R\bar{f})$.*

Proof. Let a be unit-regular with unit inner inverse v^{-1} and let $f = av^{-1}$. We use the left-right dual of characterization (4) of Theorem 3.3. Therefore, we consider an arbitrary idempotent $g \in E(R)$ such that $gR = aR$, and search for a special clean decomposition $a = \bar{e} + u = au^{-1}a$ that satisfies $Re = Rg$, or equivalently $\bar{e}R = \bar{g}R$.

As $gR = aR$ then $gR = av^{-1}R = fR$ and $fg = g, gf = f$. As $g = fg$ then $G = \begin{pmatrix} g_1 & g_2 \\ 0 & 0 \end{pmatrix}$ with $g_1 \in E(fRf)$, and as $gf = f$ then $g_1 = 1$ (in fRf , $g_1 = f$ in R). It then follows that $\bar{g}R = \begin{pmatrix} 0 & -g_2 \\ 0 & 1 \end{pmatrix} R = \begin{pmatrix} 0 & -g_2 \\ 0 & 1 \end{pmatrix} \bar{f}R$. And conversely any g of this form is an idempotent such that $aR = gR$.

Assume the existence of a special clean decomposition $a = \bar{e} + u = au^{-1}a$ of a . Then the idempotent \bar{e} is of the form $\bar{E} = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V$ for some x, y such that $\varphi(y, x)$ is a unit in $\bar{f}R\bar{f}$. But

$$\begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} VR = \begin{pmatrix} 0 & 0 \\ \varphi(y, x)^{-1}y & \varphi(y, x)^{-1} \end{pmatrix} R = \bar{f}R$$

so that $\bar{E}R = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \bar{f}R$. Finally $\bar{g}R = \bar{e}R$ iff $x = -g_2$ (one can also solve the two equations $\bar{G}\bar{E} = \bar{E}, \bar{E}\bar{G} = \bar{G}$). From the first one we deduce that $x = -g_2$ is necessary, and as in this case

$\bar{G} = \bar{E} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, x) \end{pmatrix}$ (this is also sufficient). Theorem 4.1 thus gives that a is special clean with decomposition $a = \bar{e} + u = au^{-1}a$ and $\bar{e}R = \bar{g}R$ iff there exists $y \in \bar{f}Rf$ such that $\varphi(y, -g_2) \in U(\bar{f}R\bar{f})$. As g_2 is arbitrary, we obtain the equivalence. \square

In Corollary 4.2, we characterize left perspectivity of a unit-regular element in terms of a specific unit inner inverse. In the following corollary, we instead allow the unit inner inverse to vary, which gives more freedom choosing an x such that $\varphi(y, x)$ is a unit (in a corner ring).

Corollary 4.3. *Let $a \in R$ be unit-regular. The element a is left perspective iff for each decomposition of the form $a = fv$ with $f \in E(R)$ and $v \in U(R)$, there exists $y \in \bar{f}Rf$ such that $\varphi(y, 0) = (y + \bar{f})v\bar{f} \in U(\bar{f}R\bar{f})$.*

Proof. One direction follows directly from Corollary 4.2 (let $x = 0$). To show the converse, let $f \in E(R)$ be such that $aR = fR$. From [?, Lemma 3.3] and the fact that idempotents are unit-regular, the condition $aR = fR$ implies that $a = fv$ for some unit $v \in U(R)$. By hypothesis, there exists $y \in \bar{f}Rf$ such that $\varphi(y, 0) = (y + \bar{f})v\bar{f} \in U(\bar{f}R\bar{f})$ and by Theorem 4.1, a admits a special clean decomposition of the form $a = \bar{e} + u$ with $u = a - \bar{e}$ and

$$\bar{E} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, 0)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V.$$

As V is invertible Then $\bar{E}R = \left\{ \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \right\} = \bar{F}R$ and a is left perspective. \square

By symmetric arguments, one can give similar characterizations of right perspective elements. However, these are in terms of idempotents that are left multiples of a by unit inner inverses. We can also describe right perspectivity in terms of the right multiples of a as follows.

Corollary 4.4. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Define, as before, $\varphi : \bar{f}Rf \times fR\bar{f} \rightarrow \bar{f}R\bar{f}$ by*

$$\varphi : (y, x) \mapsto yv_1x + yv_2 + v_3x + v_4 = (y + \bar{f})v(x + \bar{f}).$$

Then the element a is right perspective iff for all $y \in \bar{f}Rf$, there exists $x \in fR\bar{f}$ such that $\varphi(y, x) \in U(\bar{f}R\bar{f})$.

Proof. Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} , and let $f = av^{-1}$. Then a is similar to $b = vav^{-1}$ in R . Thus, by Proposition 3.5 a is right perspective iff b is right perspective. As b is unit-regular with unit inner inverse v^{-1} and $v^{-1}b = av^{-1} = f$, then by the left-right dual of Corollary 4.2, this happens iff for all $y \in \bar{f}Rf$, there exists $x \in fR\bar{f}$ such that $\varphi(y, x) \in U(\bar{f}R\bar{f})$. \square

Also, we deduce from Proposition 3.7, Corollary 4.2 and Corollary 4.4 that another equation can be used, based on the unit v^{-1} rather than v , and the corner ring fRf . We let the Peirce decomposition of v^{-1} be $V^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ (in $fRf \oplus fR\bar{f} \oplus \bar{f}Rf \oplus \bar{f}R\bar{f}$).

Corollary 4.5. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Define $\psi : fR\bar{f} \times \bar{f}Rf \rightarrow fRf$ by*

$$\psi : (x, y) \mapsto \mu_1 + \mu_2y + x\mu_3 + x\mu_4y = (x + f)v^{-1}(y + f).$$

- (1) The element a is left perspective iff for all $x \in fR\bar{f}$, there exists $y \in \bar{f}Rf$ such that $\psi(x, y) \in U(fRf)$;
- (2) The element a is right perspective iff for all $y \in \bar{f}Rf$, there exists $x \in fR\bar{f}$ such that $\psi(x, y) \in U(fRf)$.

Proof. Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. By Proposition 3.7, a is left perspective iff $a' = v^{-1} - v^{-1}av^{-1}$ is right perspective. As v is a unit inner inverse of a' and $\bar{g}' = va' = 1 - av^{-1} = \bar{f}$, this happens by the left-right dual of Corollary 4.2 iff for all $x \in g'Rg' = fR\bar{f}$, there exists $y \in g'R\bar{g}' = \bar{f}Rf$ such that $(x + \bar{g})v^{-1}(y + \bar{g}) = (x + f)v^{-1}(y + f) \in U(\bar{g}R\bar{g}) = U(fRf)$.

Dually, a is right perspective iff a' is left perspective and we conclude by Corollary 4.4. \square

By similar arguments, which we don't include here, one can now give an alternate characterization of special cleanness, using the function ψ .

Corollary 4.6. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Then the element a is special clean iff $\psi(x, y) = \mu_1 + \mu_2y + x\mu_3 + x\mu_4y = (x + f)v^{-1}(y + f) \in U(fRf)$ for some $x \in fR\bar{f}$, and $y \in \bar{f}Rf$.*

Finally, we consider regular elements $a \in \text{reg}(R)$ such that $a^2 = 0$. As they are special instances of doubly unit-regular elements, they are special clean ([?, Theorem 3.14] and [?]). We prove below that they are actually perspective.

Proposition 4.7. *Let R be a ring, and $a \in \text{reg}(R)$. If $a^2 = 0$, then a is perspective.*

Proof. Let $a \in \text{reg}(R)$ be such that $a^2 = 0$. By symmetry, it suffices to show that a is left perspective.

From [?], a admits an idempotent reflexive inverse e . It follows from Lemma 2.2 that $u = a - \bar{e}$ is a unit that satisfies $au^{-1}a = a$ (and $u^{-1}au^{-1} = e$). We consider the Peirce matrix decompositions, relative to the idempotent e , of the previous elements (in upper-case letters), and let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$. As $eae = e$ then $a_1 = 1$ (precisely, $a_1 = 1_{eRe} = e$), and as $aea = a$, then $a_4 = a_3a_2$. Finally, as $a^2 = 0$ then $1 + a_2a_3 = 0$ (in eRe). In particular, $a_2 + a_2a_3a_2 = 0$ (in $eR\bar{e}$) and $a_3 + a_3a_2a_3 = 0$ (in $\bar{e}Re$).

By Lemma 2.2, $U = A - \bar{E}_{11} = \begin{pmatrix} 1 & a_2 \\ a_3 & a_3a_2 - 1 \end{pmatrix}$ is a unit, and (for instance by using Schur complement) $U^{-1} = \begin{pmatrix} 2 & a_2 \\ a_3 & -1 \end{pmatrix}$. It follows that $F = AU^{-1} = \begin{pmatrix} 1 & 0 \\ a_3 & 0 \end{pmatrix}$. Observe that, for the moment, we cannot directly apply the previous results directly since the Peirce decompositions correspond to the idempotent e , not $f = au^{-1}$.

Therefore, we let $T = \begin{pmatrix} 1 & 0 \\ -a_3 & 1 \end{pmatrix} \in U(R)$, $V = TUT^{-1} = \begin{pmatrix} 0 & a_2 \\ -a_3 & -1 \end{pmatrix}$ and $V^{-1} = TV^{-1}T^{-1} = \begin{pmatrix} 1 & a_2 \\ -a_3 & -a_3a_2 - 1 \end{pmatrix}$. Let also $A' = TAT^{-1} = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}$. Then $A'V^{-1} = E_{11}$. By Proposition 3.5, a is left perspective iff $a' = tat^{-1}$ is left perspective. As v^{-1} is a unit inner inverse of a' and $a'v^{-1} = e$, this happens by Corollary 4.5 iff for all $x \in eR\bar{e}$, there exists $y \in \bar{e}Re$ such that $\psi(x, y) = 1 + a_2y - xa_3 + x(-a_3a_2 - 1)y \in U(eRe)$. Let any $x \in eR\bar{e}$. Then $\psi(x, -a_3xa_3) = 1_{eRe}$, hence a is left perspective. \square

5. PERSPECTIVITY, STABLE RANGE 1 AND OTHER CONDITIONS

Recall that a ring R (resp. element $a \in R$) has *stable range 1* (resp. left stable range 1), and we write $sr(R) = 1$ (resp. $sr_l(a) = 1$) if for all $a, b \in R$ (resp. for all $b \in R$) $xa + yb \in U(R)$ for some $x, y \in R$ implies that $a + cb \in U(R)$ for some $c \in R$. It is known that this property is left/right symmetric at the level of rings ($sr(R) = 1$ iff for all $a, b \in R$, $ax + by \in U(R)$ for some $x, y \in R$ implies that $a + bc \in U(R)$ for some $c \in R$) and that stable range 1 is inherited by corner rings.

As an application of Corollary 4.3, we deduce perspectivity from stable range 1 of the corner ring $\bar{f}R\bar{f}$. As before, we write Peirce decomposition of v^{-1} as $V^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ (in $fRf \oplus fR\bar{f} \oplus \bar{f}Rf \oplus \bar{f}R\bar{f}$).

Corollary 5.1. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. If $sr(\bar{f}R\bar{f}) = 1$ then a is perspective.*

Proof. Assume that $sr(\bar{f}R\bar{f}) = 1$. We first consider left perspectivity. As $V^{-1}V = I$ then $\mu_3v_2 + \mu_4v_4 = \bar{f}$ and $\mu_3v_1 + \mu_4v_3 = 0$. Let any $x \in fR\bar{f}$. Then $\mu_3(v_1x + v_2) + \mu_4(v_3x + v_4) = \bar{f}$ and by stable range 1, there exists $t \in \bar{f}R\bar{f}$ such that $t\mu_3(v_1x + v_2) + (v_3x + v_4) \in U(\bar{f}R\bar{f})$. Let $y = t\mu_3$. Then $\varphi(y, x) = y(v_1x + v_2) + (v_3x + v_4)$ is a unit in $\bar{f}R\bar{f}$ and as x is arbitrary, then a is left perspective by Corollary 4.3.

We now consider right perspectivity. As $VV^{-1} = I$ then $v_3\mu_2 + v_4\mu_4 = \bar{f}$ and $v_1\mu_2 + v_2\mu_4 = 0$. Let any $y \in \bar{f}Rf$. Then $(yv_1 + v_3)\mu_2 + (yv_2 + v_4)\mu_4 = \bar{f}$ and by stable range 1, there exists $t \in \bar{f}R\bar{f}$ such that $(yv_1 + v_3)\mu_2t + (yv_2 + v_4)\mu_4 \in U(\bar{f}R\bar{f})$. We conclude by setting $x = \mu_2t$ and applying Corollary 4.3. \square

Second, we consider stable range 1 of fRf .

Corollary 5.2. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. If $sr(fRf) = 1$ then a is perspective.*

Proof. Assume that $sr(fRf) = 1$. We let $\psi(x, y) = (x + f)v^{-1}(y + f) = \mu_1 + \mu_2y + x\mu_3 + x\mu_4y$ and consider left perspectivity. By Corollary 4.5, a is left perspective iff for all $x \in fR\bar{f}$, there exists $y \in \bar{f}Rf$ such that $\psi(x, y) \in U(fRf)$. As $V^{-1}V = I$ then $\mu_1v_1 + \mu_2v_3 = f$ and $\mu_3v_1 + \mu_4v_3 = 0$. Let any $x \in fR\bar{f}$. Then $(\mu_1 + x\mu_3)v_1 + (\mu_2 + x\mu_4)v_3 = f$ and by stable range 1, there exists $t \in fRf$ such that $(\mu_1 + x\mu_3) + (\mu_2 + x\mu_4)v_3t \in U(fRf)$. Let $y = v_3t$. Then $\psi(x, y) = \mu_1 + \mu_2y + x\mu_3 + x\mu_4y = (\mu_1 + x\mu_3) + (\mu_2 + x\mu_4)v_3t$ is a unit in fRf and a is left perspective. As above, we prove dually that a is right perspective. \square

Third, we deduce perspectivity of a under the assumption that a certain subring lies in the Jacobson radical $J(R)$ of R .

Corollary 5.3. *Let $a \in \text{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. If $\bar{f}RfR\bar{f} \in J(R)$, then a is perspective.*

Proof. Assume that $\bar{f}RfR\bar{f} \in J(R)$. As $V^{-1}V = I = V^{-1}V$ then $\mu_3v_2 + \mu_4v_4 = \bar{f} = v_3\mu_2 + v_4\mu_4$, and since $\mu_3v_2, v_3\mu_2 \in J(R) \cap \bar{f}R\bar{f} = J(\bar{f}R\bar{f})$ then v_4 is both left and right invertible hence a unit. Let any $x \in fR\bar{f}$ and $y \in \bar{f}Rf$. Then $\varphi(y, x) = yv_1x + yv_2 + v_3x + v_4$ is a unit in $\bar{f}R\bar{f}$ as a sum

of an element $(yv_1x + yv_2 + v_3x)$ in the Jacobson radical and a unit (v_4) . As x and y are arbitrary, then a is left and right perspective by Corollary 4.3. \square

Finally, we show that a ring is perspective under any of the two assumptions listed below. In [?], it is shown that any of these assumptions imply that R is special clean. In [?], it is shown that these assumptions imply perspectivity by module theoretical arguments.

Corollary 5.4. *Let R be a ring. In the following two cases, R is perspective:*

- (1) R has stable range 1;
- (2) for all $e \in E(R)$, $\bar{e}ReR\bar{e} \subset J(R)$.

In particular, if all skew-corner rings $eR\bar{e}$, $e \in E(R)$ are contained in $J(R)$ then R is perspective.

To conclude this section, we make the link between perspectivity, stable range 1 and (special) clean decompositions more precise at the level of elements. It is indeed known ([?, Theorem 3.5]) that a regular element of a ring has left (equivalently right) stable range 1 iff it is unit-regular. In rings with stable range 1 (in particular unit-regular ones) regular elements have right and left *idempotent stable range 1*, where $a \in R$ has right idempotent stable range 1 if for all $b \in R$, $ax + by \in U(R)$ for some $x, y \in R$ implies that $a + be \in U(R)$ for some $e \in E(R)$. We prove that left perspective elements are precisely regular elements with *outer inverse right stable range 1*, where $a \in R$ has outer inverse right stable range 1 if $aR + bR = R$ for some $b \in R$ implies that $a + bx \in U(R)$ for some outer inverse $x \in R$ of b .

Proposition 5.5. *Let R be a ring and $a \in \text{reg}(R)$. Then the following statements are equivalent:*

- (1) a is left perspective;
- (2) If $aR + bR = R$ for some $b \in R$ then a admits a special clean decomposition $a = \bar{e} + u = au^{-1}a$ for some $e \in E(R)$, $u \in U(R)$ such that $\bar{e}R \subseteq bR$;
- (3) If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b ($xbx = x$) such that $aR \cap bxR = 0$;
- (4) If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b (a has outer inverse right stable range 1);
- (5) If $aR + bR = R$ for some $b \in \text{reg}(R)$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b ;
- (6) If $aR + \bar{f}R = R$ for some $f \in E(R)$ then a admits a clean decomposition $a = \bar{e} + u$ for some $e \in E(R)$, $u \in U(R)$ such that $\bar{e}R \subseteq \bar{f}R$.

Proof.

- (1) \Rightarrow (2) Assume that a is left perspective and that $aR + bR = R$ for some $b \in R$. As a is regular then aR is a direct summand and it then follows from [?, Lemma 2.8] that there exists a submodule B of R_R such that $aR \oplus B = R$ and $B \subseteq bR$. Let f be the projection on aR parallel to B . Then $fR = aR$ and $\bar{f}R = B$. Since a is left perspective, there exists a special clean decomposition $a = \bar{e} + u = au^{-1}a$ with $\bar{e}R = \bar{f}R \subseteq bR$.
- (2) \Rightarrow (3) Assume (2) and let $b \in R$ be such that $aR + bR = R$. Then a admits a special clean decomposition $a = \bar{e} + u = au^{-1}a$ for some $e \in E(R)$, $u \in U(R)$ such that $\bar{e}R \subseteq bR$. By [?, Theorem 2.8], $-a$ admits a special clean decomposition $-a = \bar{e} + u' = (-a)(u')^{-1}(-a)$ with

$u' = -a - \bar{e} \in U(R)$. As $\bar{e}R \subseteq bR$ then there exists $y \in R$ such that $\bar{e} = by$. Let $x = yby = y\bar{e}$. Then $a + bx = a + byby = a + \bar{e} = -u'$ and $xbx = (yby)b(yby) = y\bar{e}^3 = x$. Finally, as the decomposition of a is special clean then $aR \cap \bar{e}R = aR \cap bxR = 0$.

(3) \Rightarrow (4) Straightforward.

(4) \Rightarrow (5) Straightforward.

(5) \Rightarrow (6) Assume (5) and assume also that $aR + \bar{f}R = R$ for some $f \in E(R)$. As in (1) \Rightarrow (2) there exists $h \in E(R)$ such that $aR \oplus \bar{h}R = R$ and $\bar{h}R \subseteq \bar{f}R$. By (5) there exists an outer inverse $x \in R$ of \bar{h} such that $-u' = a + \bar{h}x$ is a unit. Let $\bar{e} = \bar{h}x \in E(R)$. Then $-a = \bar{e} + u'$ is a clean decomposition of $-a$. As also $\bar{e}R \subseteq \bar{h}R$ and $aR \cap \bar{h}R = 0$ then $(-a)R \cap \bar{e}R = 0$ and the decomposition is actually a special clean decomposition. It follows from [?, Theorem 2.8] that a admits a special clean decomposition $a = \bar{e} + u$ with $u \in U(R)$ (and $\bar{e}R \subseteq \bar{h}R \subseteq \bar{f}R$ by construction).

(6) \Rightarrow (1) Assume (6) and let $f \in E(R)$ such that $aR = fR$. Then $aR \oplus \bar{f}R = R$ and a admits a clean decomposition $a = \bar{e} + u$ for some $e \in E(R), u \in U(R)$ such that $\bar{e}R \subseteq \bar{f}R$. As $aR \cap \bar{f}R = 0$ and $\bar{e}R \subseteq \bar{f}R$ then $aR \cap \bar{e}R = 0$ and the decomposition is special clean. Also, as $a - \bar{e} = u \in U(R)$ then $aR \oplus \bar{e}R = R$. It then follows that $fR \oplus \bar{e}R = R$, or equivalently that $R\bar{f} \oplus Re = R$ by Lemma 2.1. As $\bar{e}R \subseteq \bar{f}R$ then $\bar{e} = \bar{f}\bar{e}$ and $e\bar{f} = e\bar{f}(e + \bar{e}) = e\bar{f}e + e\bar{f}\bar{e} = e\bar{f}e \in R\bar{f} \cap Re = 0$, so that $\bar{e}\bar{f} = \bar{f}$ (hence $\bar{f}R \subseteq \bar{e}R$). We conclude by Lemma 2.2 that $a = \bar{e} + u = au^{-1}a$ with $\bar{e}R = \bar{f}R$ (equivalently $Re = Rf$), whence a is left perspective.

□

Due to the various characterizations of special clean elements (Lemma 2.2 and Theorem 2.4), we can replace the third statement in the proposition by any one of the following statements.

- If $aR + bR = R$ for some $b \in R$ then $u = a + bx \in U(R)$ for some x such that $xbx = x$ and $Ra \cap Rx = 0$ (since in this case $Rbx = Rx$);
- If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some x such that $xbx = x$ and $a(- (a + bx))^{-1}a = a$;
- If $aR + bR = R$ for some $b \in R$ then there exists x such that $xbx = x$, $Ra \oplus Rx = R$ and $aR \oplus bxR = R$.

6. EXAMPLES

In [?, Section 4], D. Khurana and T.Y. Lam study the cleanness of unit-regular matrices of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$. Next example studies perspectivity of such matrices.

Example 6.1. Let $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$. Then A is perspective iff $a = \pm 1$ or $(a = 0$ and $b \in \{-1, 0, 1\})$.

Indeed, A is unit-regular iff $a = b = 0$ or (a, b) is an unimodular row ($a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$). As the null matrix is perspective, we consider below (a, b) an unimodular row, and write down the proof for the left perspectivity of A only. We let the reader check that right perspectivity can be treated equivalently.

(1) First case $a = 0$.

As $(0, b)$ is unimodular then $b = 1$ or -1 . And as A is regular and $A^2 = 0$, then it is perspective by Proposition 4.7.

(2) Second case $a \neq 0$.

As (a, b) is unimodular, then $au + bv = 1$ for some $u, v \in \mathbb{Z}$. Let $V = \begin{pmatrix} a & b \\ -v & u \end{pmatrix}$. Then $V \in U(\mathcal{M}_2(\mathbb{Z}))$ with inverse $V^{-1} = \begin{pmatrix} u & -b \\ v & a \end{pmatrix}$, and $AV^{-1}A = A$, $AV^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Define as usual $\varphi(y, x) = yax + yb - vx + u$. As the only units in \mathbb{Z} are 1 and -1 , then $\varphi(y, x) = y(ax+b) + (-vx+u)$ is a unit iff $y(ax+b) + (-vx+u-1) = 0$ or $y(ax+b) + (-vx+u+1) = 0$. We introduce the two homographic functions (on the reals) $f_1(x) = \frac{-vx+u-1}{ax+b}$ and $f_2(x) = \frac{-vx+u+1}{ax+b}$. Then $f_1(x) = \frac{((u-1)a+vb)/a^2}{x+b/a} - \frac{v}{a}$ which takes an infinite number of non-integer values for integers x , unless $(u-1)a + vb = 0$. Symmetrically, f_2 takes an infinite number of non-integer values for integers x , unless $(u+1)a + vb = 0$. Thus, A is left perspective only if $(u-1)a + vb = 0$ and $\frac{v}{a} \in \mathbb{Z}$ or $(u+1)a + vb = 0$ and still $\frac{v}{a} \in \mathbb{Z}$. But $\frac{v}{a} \in \mathbb{Z}$ implies that a divides the determinant of V that is either 1 or -1 , hence $a = 1$ or $a = -1$. Finally, A with $a \neq 0$ is left perspective only if $a = \pm 1$, but this condition is also sufficient. Indeed, in this case either A or $-A$ is in $E(\mathcal{M}_2(\mathbb{Z}))$, and the claim follows by Lemma 3.4. Alternatively, we can use the equational characterization: by looking at the determinant, either f_1 or f_2 is the constant integer function $\frac{v}{a} \in \mathbb{Z}$, and $\varphi(y, x)$ is a unit for all $x \in \mathbb{Z}$ and $y = \frac{-v}{a}$.

Let F be a field. In [?], the authors prove that $\mathcal{M}_2(F[X])$ is not a perspective ring (since rings of the form $\mathcal{M}_2(R)$ are perspective iff R has stable range one [?, Theorem 5.12], and $F[X]$ does not have stable range one: $sr(F[X]) \geq 2$ by [?, Theorem A]). In the next example, we exhibit directly a regular element of $\mathcal{M}_2(F[X])$ that is neither left nor right perspective.

Example 6.2. Consider the matrix $A = E_{11}V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+X & X^2 \\ -1 & 1-X \end{pmatrix} = \begin{pmatrix} 1+X & X^2 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(F[X])$, where F is a field. Then A is regular with unit inner inverse $V^{-1} = \begin{pmatrix} 1-X & -X^2 \\ 1 & 1+X \end{pmatrix}$. The perspectivity function of A satisfies the formula below for all $p(X), q(X) \in F[X]$.

$$\varphi(q(X), p(X)) = q(X)(1+X)p(X) + q(X)X^2 - p(X) + 1 - X.$$

- As $\varphi(0, -X) = 1$, then A is special clean by Theorem 4.1.
- As $\varphi(q(X), 0) = 1 - X + q(X)X^2$ is not a unit for any $q(X) \in F[X]$, then A is not left perspective by Corollary 4.2.
- As $\varphi(1-X, p(X)) = 1 - X + X^2(1-X-p(X))$ is not a unit for any $p(X) \in F[X]$, then A is not right perspective by Corollary 4.4.

The question whether a left perspective element is always right perspective is open. The following example was investigated in an attempt to prove asymmetry of the notion by using one-sided invertible elements, yet symmetry was obtained.

Example 6.3. Let R be any non-commutative ring and consider $T = \mathcal{M}_3(R)$. We consider perspectivity of

$$A = E_{11}V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $u \in R$ is either invertible or only one-sided invertible.

The matrix $V = \begin{pmatrix} u & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is invertible with inverse $V^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & u & -u \\ 0 & 1 & 0 \end{pmatrix}$.

The perspectivity function of A satisfies the formula below for all $x = (x_1 \ x_2) \in E_{11}T\bar{E}_{11}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \bar{E}_{11}TE_{11}$.

$$\begin{aligned} \varphi(y, x) &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} u (x_1 \ x_2) + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} (1 \ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (x_1 \ x_2) + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} y_1 u x_1 + y_1 & y_1 u x_2 + 1 \\ y_2 u x_1 + y_2 + x_1 & x_2 + y_2 u x_2 + 1 \end{pmatrix}. \end{aligned}$$

(1) Assume first that u is a unit.

- Let $x_1^0 = -u^{-1}$, $x_2^0 = 0$. Then for any $y \in \bar{E}_{11}TE_{11}$ $\varphi(y, x^0) = \begin{pmatrix} 0 & 1 \\ -u^{-1} & 1 \end{pmatrix}$ is a unit in T , and A is right perspective.
- Let $y_1^0 = 0$, $y_2^0 = -u^{-1}$. Then for any $x \in E_{11}T\bar{E}_{11}$ $\varphi(y^0, x) = \begin{pmatrix} 0 & 1 \\ -u^{-1} & 1 \end{pmatrix}$ is a unit in T , and A is left perspective.

Alternatively, we could have used the fact that A is strongly regular (with group inverse $A^\# = \begin{pmatrix} u^{-1} & u^{-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$) and applied Lemma 3.4.

(2) Assume now that u is right but not left invertible, with right inverse u' (not right invertible). In this case it is easier to use Corollary 4.5 and consider the second function

$$\begin{aligned} \psi(x, y) &= 0 + (-1 \ 1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (x_1 \ x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_1 \ x_2) \begin{pmatrix} u & -u \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= (x_1 u - 1)(y_1 - y_2) + x_1 + x_2 y_1. \end{aligned}$$

- Let $x_1^0 = u'$, $x_2^0 = 0$ and assume that $\psi(x^0, y)$ is a unit for some $y \in \bar{E}_{11}TE_{11}$. Then $u\psi(x^0, y) = u(u'u - 1)(y_1 - y_2) + uu' = 1$ hence $u = \psi(x^0, y)^{-1}$, which is absurd since u is not left invertible. Thus, a is not left perspective.
- We now consider right perspectivity and let $y_1^0 = 0$, $y_2^0 = u'$. Then for any $x \in E_{11}T\bar{E}_{11}$ it holds that $\psi(x, y^0) = -(x_1 u - 1)u' + x_1 = u'$ is not a unit, and a is neither right perspective.

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