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Quasi type IV codes over a non-unital ring

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Abstract

There is a local ring I of order 4, without identity for the multiplication, defined by generators and relations as

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

We give a natural map between linear codes over I and additive codes over \mathbb{F}_4 , that allows for efficient computations. We study the algebraic structure of linear codes over this non-unital local ring, their generator and parity-check matrices. A canonical form for these matrices is given in the case of so-called nice codes. By analogy with \mathbb{Z}_4 -codes, we define residue and torsion codes attached to a linear I -code. We introduce the notion of quasi self-dual codes (QSD) over I , and Type IV I -codes, that is, QSD codes all codewords of which have even Hamming weight. This is the natural analogue of Type IV codes over the field \mathbb{F}_4 . Further, we define quasi Type IV codes over I as those QSD codes with an even torsion code. We give a mass formula for QSD codes, and another for quasi Type IV codes, and classify both types of codes, up to coordinate permutation equivalence, in short lengths.

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1 Introduction

While rings have been used as alphabets in Coding Theory for more than thirty years [14], it is only recently that non-unital rings have been used to that effect [1]. In that paper, the authors introduce the notion of Type IV codes over the ring E in the terminology of [8, 13], in analogy with Type IV codes over unital rings of order four [7].

In this paper we study codes over another non-unital commutative ring of order four, denoted by I in the classification of [8, 13]. We call I -code of length n any I -submodule of I^n . Such a code can be thought of as an additive \mathbb{F}_4 -code in the sense of [3] with an extra module structure. This allows us to use the additive code package of Magma [12] in computations, and might lead in the future to the discovery of new additive \mathbb{F}_4 -codes. In the ring I the notion of self-dual code must be adapted since the usual relation between the size of the code and that of its dual [15], does not hold in general. We thus introduce, in Section 2.4, following [1], the notion of quasi self-dual code (QSD), that is of an I -code of length n which is both self-orthogonal and of size 2^n . As with other local rings of order 4, residue and torsion codes play a fundamental role in the structure of QSD codes. Thus, in Sect. 3, we derive a canonical form for the generator matrix of any I -code, based on the residue and torsion codes. However, an I -code cannot be determined uniquely by its residue and torsion codes. But it is completely characterized by the triplet formed by these two codes and a map between the residue code and the cosets of the torsion code. This is similar to the technique used to derive mass formulas for self-dual codes over rings [2, 5, 9]. In Sect. 4 we give a construction of QSD codes from a pair of binary codes; we call this the multilevel construction in analogy with a similar construction over \mathbb{Z}_4 [4]. In Sect. 5, we derive a mass formula for QSD codes with the objective of classifying these codes. The method of calculation of this formula leads us to introduce the notion of quasi Type IV codes: QSD codes whose torsion code is even, and to derive a mass formula for them. Finally, we give a classification of QSD codes up to length 3 in Sect. 6, and some concluding remarks in Sect. 7.

2 Background material

2.1 Binary codes

Denote by $wt(x)$ the **(Hamming) weight** of $x \in \mathbb{F}_2^n$, or, in other words, the number of indices i where $x_i \neq 0$. A linear code is a subspace of \mathbb{F}_2^n , viewed as an \mathbb{F}_2 -vector space. The dual of a binary linear code C is denoted by C^\perp and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \mid \forall x \in C, (x, y) = 0\},$$

where $(x, y) = \sum_{i=1}^n x_i y_i$, denotes the standard inner product. A code C is **self-orthogonal** if it is included in its dual: $C \subseteq C^\perp$. A code is **even** if all its codewords have even weight. All self-orthogonal codes are even but not conversely. Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

2.2 Rings

Following [8], we define a ring on two generators a, b by its relations

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

Thus, I has characteristic two, and consists of four elements $I = \{0, a, b, c\}$, with $c = a + b$. The addition table is immediate from these definitions

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The multiplication table is as follows.

×	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

From this table, we infer that this ring is commutative, and without an identity element for the multiplication. It is local with maximal ideal $J = \{0, b\}$, and residue field $\mathbb{F}_2 = \{0, 1\}$, the finite field of order 2. Thus we have a **b -adic decomposition** as follows. Every element $i \in I$ can be written

$$i = as + bt,$$

where $s, t \in \mathbb{F}_2$ and where we have defined a natural **action** of \mathbb{F}_2 on I by the rule $r0 = 0r = 0$ and $r1 = 1r = r$ for all $r \in I$. Thus $a = 1a, b = 1b$ and $c = a1 + b1 = c1$. Note that for all $r \in I$, this action is “distributive” in the sense that $r(s \oplus t) = rs + rt$, where \oplus denotes the addition in \mathbb{F}_2 . On occasion we will use the **inner product notation** (x, r) for $x \in \mathbb{F}_2^n, r \in I^n$ to denote

$$(x, r) = \sum_{i=1}^n x_i r_i = \sum_{x_i=1} r_i.$$

Denote by $\alpha : I \rightarrow I/J \simeq \mathbb{F}_2$ the **map of reduction modulo J** . Thus $\alpha(0) = \alpha(b) = 0$, and $\alpha(a) = \alpha(c) = 1$. This map is extended in the natural way in a map from I^n to \mathbb{F}_2^n .

2.3 Modules

A **linear I -code C** of length n is an I -submodule of I^n . It can be described as the I -span of the rows of a **generator matrix**. An **\mathbb{F}_4 -code of length n** is just any non-empty subset of \mathbb{F}_4^n . The notion and notation of **Hamming weight** over \mathbb{F}_4 is the same as over \mathbb{F}_2 . An **additive code** of length n over \mathbb{F}_4 is an additive subgroup of \mathbb{F}_4^n . It is an \mathbb{F}_2 vector space with 4^k elements for some $k \leq n$ (here $2k$ is an integer, but k may be half-integral). Using a **generator matrix G** , such a code can be cast as the \mathbb{F}_2 -span of its rows. To every linear I -code C is attached an additive \mathbb{F}_4 -code $\phi(C)$ by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow 1, c \rightarrow \omega^2,$$

where $\mathbb{F}_4 = \mathbb{F}_2[\omega]$, extended naturally to \mathbb{F}_4^n . Trivially, the Hamming weight of $\phi(x)$ is $wt(x)$.

We use the Magma notation

$$[\langle 0, 1 \rangle, \dots, \langle i, A_i \rangle, \dots, \langle n, A_n \rangle]$$

for the **weight distribution** of an additive \mathbb{F}_4 -code code, where A_i is the number of codewords of weight i . Two I -codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other (and, consequently, the ϕ -image of one on the ϕ -image of the other). The **automorphism group $Aut(C)$** of an I -code C is the group of column permutations that leave C invariant.

2.4 Duality

Define an **inner product** on I^n , by writing $(x, y) = \sum_{i=1}^n x_i y_i$ for all $x, y \in I^n$.

The **dual C^\perp** of a linear I -code C is the module defined by

$$C^\perp = \{y \in I^n \mid \forall x \in C, (x, y) = 0\}.$$

Thus the dual of a module is a module. A code is **self-dual** if it is equal to its dual.

Remark 1 The **repetition code** of length 2 is the linear I -code defined by $R_2 := \{00, aa, bb, cc\}$. Its dual is $R_2^\perp = \{00, aa, bb, cc, 0b, b0, ac, ca\}$, a supercode of R_2 of size 8. In length one, we have $J^\perp = I$.

Remark 1 shows that the product of the sizes of a code and its dual is not always 4^n . A code C of length n is **nice** if $|C||C^\perp| = 4^n$. A linear I -code C is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

Clearly, C is **self-orthogonal** if and only if $C \subseteq C^\perp$.

A code of length n is **quasi self-dual** (or QSD for short) if it is self-orthogonal and of size 2^n . Note that the direct sum of two QSD codes is QSD.

Following a terminology from [7], a quasi self-dual code over I with all weights even is called a **Type IV** code. This notion is also stable by taking direct sums.

Remark 2 The repetition code of length 2 is quasi self-dual over I and is of Type IV. This shows, by taking direct sums of codes, that Type IV codes over I exist for all even lengths. We see that J is a quasi self-dual code over I . This shows, again by taking direct sums, that QSD codes exist for all integer lengths.

We introduce the notion of **Quasi Type IV** code (QT4) as a QSD code with an even torsion code (see next section for a definition of torsion code). Every Type IV code is quasi Type IV but not conversely as the next example shows. The motivation will appear in the mass formula section.

Example 1 The code with three generators $\begin{pmatrix} a & b & a & b \\ 0 & b & b & 0 \\ b & 0 & 0 & b \end{pmatrix}$ is QSD but not Type IV as the sum of first and second row has odd weight. But its torsion code with generator matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ is an even code.

3 Structure of linear codes

Let C be a code of length n over I . With this code we associate two binary codes of length n :

1. the **residue code** defined by $res(C) = \{ \alpha(y) \mid y \in C \}$,
2. the **torsion code** defined by $tor(C) = \{ x \in \mathbb{F}_2^n \mid bx \in C \}$.

Recall that the **Trace** function of \mathbb{F}_4 down to \mathbb{F}_2 is defined by $\forall z \in \mathbb{F}_4, Tr(z) = z + z^2$. In other words

$$Tr(0) = Tr(1) = 0; Tr(\omega) = Tr(\omega^2) = 1.$$

By distinguishing four cases and induction on n it can be checked that for all $x \in I^n$, we have $Tr(\phi(x)) = \alpha(x)$, and thus $res(C) = Tr(\phi(C))$. Similarly, we see that $tor(C)$ is the so-called **subfield subcode** of $\phi(C)$, that is $\phi(C) \cap \mathbb{F}_2^n$. Denote by α_C the restriction of α to C . We see that $tor(C)b = Ker(\alpha_C)$, and that $res(C) = Im(\alpha_C)$. By the first isomorphism theorem applied to the map α_C , we see that $|C| = |res(C)||tor(C)|$. There is a relationship between these two codes.

Lemma 1 If C is a linear I -code then $res(C) \subseteq tor(C)$.

Proof Write an arbitrary codeword z in b -adic decomposition form as $z = ax + by$, with x, y binary vectors. Since $\alpha(ax + by) = x$, we have $x \in res(C)$. Note that, by

definition of the residue code, any $x \in \text{res}(C)$ arises in that way. Multiplying the codeword $ax + by$ by a , we see that $bx \in C$, implying $x \in \text{tor}(C)$. \square

We let $k_1 = \dim(\text{res}(C))$, and $k_2 = \dim(\text{tor}(C)) - k_1$, a nonnegative quantity by Lemma 1, and say that C is of **type** (k_1, k_2) . It can be seen that C is free as an I -module if and only if $k_2 = 0$. Further, by a previous observation, $|C| = |\text{res}(C)||\text{tor}(C)| = 2^{2k_1+k_2}$. We give a characterization of the generator matrix of a linear code as a function of these invariants.

Theorem 1 Assume C is an I -linear code of length n and type (k_1, k_2) . Then, up to a permutation of columns, there exists a generator matrix G of C that is of the form

$$G = \begin{pmatrix} aI_{k_1} & aX & Y \\ 0 & bI_{k_2} & bZ \end{pmatrix},$$

where I_j denotes the identity matrix of order j , the matrix Y has entries in I , and X, Z are binary matrices.

Proof Write the generator matrices of $\text{res}(C)$ and $\text{tor}(C)$ as $G_1 = (I_{k_1} \ X \ \alpha(Y))$ and $G_2 = \begin{pmatrix} I_{k_1} & X & \alpha(Y) \\ 0 & I_{k_2} & Z \end{pmatrix}$, respectively, with Y a matrix of suitable dimensions with entries in I , and X, Z are binary matrices. By the first isomorphism theorem applied to the map α_C , the matrix G can be written on the form $\begin{pmatrix} R \\ bT \end{pmatrix}$, where

$$\text{tor}(C) = \text{res}(C) \oplus \langle T \rangle,$$

and $\alpha(R) = G_1$. By taking linear combinations with rows of T , the matrix R can be written in the above form. \square

Theorem 2 Assume C is a nice I -linear code of length n and type (k_1, k_2) . Then a parity check matrix H of C consistent with Theorem 1 is of the form

$$H = \begin{pmatrix} Y^t + aZ^tX^t & aZ^t & aI_{n-k_1-k_2} \\ bX^t & bI_{k_2} & 0 \end{pmatrix},$$

where I_j denotes the identity matrix of order j , and X, Y, Z are as in Theorem 1. In particular C^\perp is of type $(n - k_1 - k_2, k_2)$. If, furthermore C is self-dual, then $n = 2k_1 + k_2$.

Proof A direct calculation shows that $HG^t = 0$. This shows that, as I -modules, $\langle H \rangle = I^{n-k_1}H \subseteq C^\perp$. Equality follows by size comparison upon noticing that $\langle H \rangle$ has type $(n - k_1 - k_2, k_2)$, and upon observing that by the niceness hypothesis we have $|C||C^\perp| = 4^n$. The last assertion follows by unicity of the type $k_1 = n - k_1 - k_2$. \square

To prepare for the derivation of a mass formula, we develop a method similar to that of [9]. We define a map F from $\text{res}(C)$ to $\mathbb{F}_2^n/\text{tor}(C)$ by the formula

$$F(x) = \{y \in \mathbb{F}_2^n \mid ax + by \in C\}.$$

Since C is I -linear, it follows that F is \mathbb{F}_2 -linear. We write $\psi(C) = (res(C), tor(C), F)$. The following Theorem is immediate, but essential.

Theorem 3 *Keep the above notation. The map ψ is a one to one correspondence between linear I -codes C of length n , and triplets (C_1, C_2, F) such that C_1, C_2 are binary codes of length n , satisfying $C_1 \subseteq C_2$ and such that the map F is \mathbb{F}_2 -linear from C_1 to \mathbb{F}_2^n/C_2 .*

Proof Let C be an arbitrary linear I -code. Let $\psi(C) = (C_1, C_2, F)$, be given. Writing an arbitrary codeword of C as $ax + by$, with x, y binary vectors of length n , we see that

- $\alpha(ax + by) = x$, implying $x \in C_1$;
- $y \in F(x)$, by definition of F .

From there it follows that the code C is uniquely determined as

$$C = \bigcup_{x \in C_1} \bigcup_{y \in F(x)} ax + by.$$

□

4 Construction of QSD codes

We call the next construction of I -codes from binary codes the **multilevel construction**.

Theorem 4 *Let C_1 be a self-orthogonal binary code of length n , and let C_2 be a binary code of length n , where $C_1 \subseteq C_2$. The code C defined by the relation*

$$C = aC_1 + bC_2,$$

is a self-orthogonal code. Its residue code is C_1 and its torsion code is C_2 . If, furthermore, $|C_1||C_2| = 2^n$, then C is QSD.

Proof The code C is closed under addition, by linearity of C_1 and C_2 . Let us check it is closed under multiplication by a scalar of I . Note that $aC = a^2C_1 + abC_2 = bC_1 \subseteq bC_2 \subseteq C$. Likewise, we see that $cC = caC_1 + cbC_2 = bC_1 \in C$. Because $bI = \{0\}$, we get $bC = baC_1 + b^2C_2 = \{0\} \subseteq C$.

Thus C is I -linear.

For all x, x' in C_1 and y, y' in C_2 we have the inner products

$$(ax + by, ax' + by') = a^2(x, x') + ab(x, y') + ba(y, x') + b^2(y, y') = b(x, x') = 0$$

since $x \in C_1$ which is self orthogonal. Thus C is self-orthogonal. The residue and torsion codes are immediately derived from the definitions. Since $|C| = |C_1||C_2|$, the last assertion follows. \square

Note that for given residue and torsion codes many QSD codes may exist. As an example, the code C_1 generated by the matrix $\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}$ and the code C_2 generated by the matrix $\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$ have same residue and torsion codes.

5 Mass formulas

5.1 QSD codes

Define $N_I(n, k)$ as the number of QSD I -codes of length n and type $(k, n - 2k)$. Let $\Phi(N, K)$ denote the number of self-orthogonal binary codes having parameters $[N, K]$. The following explicit formula for this quantity can be found in [10, p.1]. It is valid for $K \geq 1$.

$$\Phi(N, K) = \begin{cases} \frac{\prod_{j=1}^K (2^{N+1-2j}-1)}{\prod_{j=1}^K (2^j-1)} & \text{if } n \geq 3 \text{ is odd,} \\ \frac{(2^{N-K}-1) \prod_{j=1}^{K-1} (2^{N-2j}-1)}{\prod_{j=1}^K (2^j-1)} & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

We need the following counting lemma. For information on q -binomial coefficients we refer the reader to [11, p. 443].

Lemma 2 *The number $E(q, n, r, s)$ of subspaces of dimension r of \mathbb{F}_q^n containing a given subspace of dimension $s < r$ is $\binom{n-s}{r-s}_q$, where $\binom{i}{j}_q$ denotes the q -binomial coefficient defined for integers $i \geq j$ by*

$$\binom{i}{j}_q = \frac{(q^i - 1)(q^{i-1} - 1) \dots (q^{i-j+1} - 1)}{(q^j - 1)(q^{j-1} - 1) \dots (q - 1)}.$$

Proof Follows by Lemma 3 of [6] with $i = j = s$ and $k = r$. \square

We also need the following simple Lemma.

Lemma 3 *If C is a self-orthogonal I -code, then its residue $res(C)$ is a binary self-orthogonal code.*

Proof By linearity of the map α it can be seen that $\forall x, y \in C$, we have

$$(\alpha(x), \alpha(y)) = \alpha((x, y)) = \alpha(0) = 0.$$

The result follows by $\text{res}(C) = \text{Im}(\alpha_C)$. □

We can now count the QSD codes of given length and residue dimension.

Theorem 5 *For all lengths n and type $(k, n - 2k)$, with $k > 0$ we have*

$$N_I(n, k) = \Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}.$$

Proof Let C be a QSD code of length n and type $(k, n - 2k)$. We reason in terms of the map ψ . Note the following three independent counts.

1. By Lemma 3, we know that $\text{res}(C)$ is self-orthogonal. By definition of Φ , there are $\Phi(n, k)$ self-orthogonal binary $[n, k]$ codes that can be used as $\text{res}(C)$.
2. The residue code is contained in $E(2, n, n - k, k) = \binom{n - k}{n - 2k}_2$ possible torsion codes by Lemma 2 for $r = k, s = n - k$, and $q = 2$.
3. The additive map F is arbitrary from $\text{res}(C)$ to $\mathbb{F}_2^n / \text{tor}(C)$. By the QSD hypothesis, these two vector spaces have dimension k . Hence, by considering the matrix of F , we see that there are 2^{k^2} choices for F .

The result follows by multiplying the three factors together. □

The following mass formula follows by the usual counting technique.

Corollary 1 *For given length n and type $(k, n - 2k)$, with $k > 0$, we have*

$$\sum_C \frac{1}{|\text{Aut}(C)|} = \frac{\Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}}{n!},$$

where C runs over distinct representatives of equivalence classes under column permutations of QSD codes of length n and type $(k, n - 2k)$.

5.2 Quasi type IV codes

Define $N_4(n, k)$ as the number of QT4 I -codes of length n and type $(k, n - 2k)$. The analogue of Lemma 2 is as follows.

Lemma 4 *The number $EE(n, r, s)$ of even codes of dimension r of \mathbb{F}_2^n containing a given subspace of dimension $s < r$ is $\binom{n - 1 - s}{r - s}_2$.*

Proof Follows by Lemma 3 of [6] with $i = j = s$ and $k = r$ for $q = 2$. Here C in the notation of [6] is the binary code of length n , and dimension $n - 1$, consisting of all even Hamming weight vectors. □

Using this Lemma the following analogue of Theorem 5 for QT4 codes can be shown. The easy proof is omitted.

Theorem 6 For all lengths n and type $(k, n - 2k)$, with $k > 0$ we have

$$N_4(n, k) = \Phi(n, k) \binom{n - 1 - k}{n - 2k}_2 2^{k^2}.$$

The following mass formula follows by the usual counting technique.

Corollary 2 For given length n , and type $(k, n - 2k)$, with $k > 0$, we have

$$\sum_C \frac{1}{|Aut(C)|} = \frac{\Phi(n, k) \binom{n-1-k}{n-2k}_2 2^{k^2}}{n!},$$

where C runs over distinct representatives of equivalence classes under column permutations of QT4 codes of length n and type $(k, n - 2k)$.

6 Short length classification ($n < 4$)

In the following, we classify, up to coordinate permutation, QSD codes. We construct QSD codes by means of the multilevel construction, and of the characterization of Theorem 1, in short lengths ($n < 4$) and rely on the mass formula to know the maximum number of equivalence classes.

For $n = 1$ there is just one QSD code with generator matrix (b) .

6.1 $n = 2$

There are two codes of length 2. Both are Type IV codes and their weight distribution is $[\langle 0, 1 \rangle, \langle 2, 3 \rangle]$. One is generated by $(a \ a)$ and the other one by $(a \ c)$.

6.2 $n = 3$

There are exactly six codes that are not permutation equivalent. All these codes have an automorphism group of order 2. The following four codes have a minimum distance equal to 1.

- One code has generator matrix

$$\begin{pmatrix} a & a & b \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution $[\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 3, 4 \rangle]$.

- One code has generator matrix

$$\begin{pmatrix} a & a & 0 \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution [$\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 5 \rangle$].

- The last two codes amongst these four share the same weight distribution [$\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle$]. Their generator matrices are

$$\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 & c \\ 0 & b & 0 \end{pmatrix}.$$

Two codes have minimum distance 2 and weight distribution [$\langle 0, 1 \rangle, \langle 2, 5 \rangle, \langle 3, 2 \rangle$]. Their generator matrices are

$$\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$$

and

$$\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}.$$

These two codes are quasi Type IV codes with, as torsion code, a binary code of parameters [3, 2, 2] and generator matrix $\begin{pmatrix} b & 0 & b \\ 0 & b & b \end{pmatrix}$. Note that we obtain two quasi Type IV codes, which is consistent with Corollary 1.

7 Conclusion

In this article, we have studied quasi self-dual codes over the non-unital commutative ring I of order four. The existence of codes that are not nice precludes any attempt to derive a general MacWilliams formula, since this would imply a relation between the size of a code and that of its dual. We have thus introduced QSD codes as an alternative to the concept of self-dual codes. For similar reasons, we have introduced the notion of quasi Type IV codes, in view of the difficulty of studying Type IV codes. We have derived a mass formula for QSD codes and for QT4 codes to classify them under coordinate permutation equivalence.

The main open problem arising from this work would be to derive a mass formula for Type IV codes. This would require a characterization of these codes akin to what exists over E [1], or over unital rings of Type IV [7]. It would be interesting to derive a classification under a notion of equivalence based on the symmetry of the

alphabet between a and c . It might be a simpler classification than the classification under column permutation of the present paper.

On the computational side, pushing the classification of QSD codes beyond length 3, is certainly a worthy project, and might lead to the discovery of new additive \mathbb{F}_4 -codes in lengths > 3 .

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References

1. Alahmadi, A., Bonnetcaze, A., Shoaib, H., Altassan, A., Bassafar, W., Solé, P.: Type IV codes over a non-unital ring, submitted
2. Balmaceda, J.M.L., Betty, R.A.L., Nemenzo, F.: Mass formula for self-dual codes over \mathbb{Z}_{p^2} . *Discrete Math.* **308**, 2984–3002 (2008)
3. Calderbank, A.R., Rains, E.M., Sloane, N.J.A.: Quantum error correction via codes over $GF(4)$. *IEEE Trans. Inf. Theory* **44**, 1369–1387 (1998)
4. Conway, J.H., Sloane, N.J.A.: Self-dual codes over the integers modulo four. *J. Comb. Theory Ser. A* **62**, 30–45 (1993)
5. Choi, W.: Mass formula of self-dual codes over Galois rings $GR(p^2, 2)$. *Korean J. Math.* **24**(4), 751–764 (2016)
6. Delsarte, P.: Association schemes and t-designs in regular semi-lattices. *J. Comb. Theory* **20**, 230–243 (1976)
7. Dougherty, S.T., Gaborit, P., Harada, M., Munemasa, A., Solé, P.: Type IV self-dual codes over rings. *IEEE Trans. Inf. Theory* **45**(7), 2345–2360 (1999)
8. Fine, B.: Classification of finite rings of order p^2 . *Math. Mag.* **66**(4), 248–252 (1993)
9. Gaborit, P.: Mass formulas for self-dual codes over \mathbb{Z}_4 and $\mathbb{F}_q + u_{\mathbb{F}_q}$ rings. *IEEE Trans. Inf. Theory* **42**, 1222–1228 (1996)
10. Hou, X.D.: On the number of inequivalent binary self-orthogonal codes. *Trans. Inf. Theory* **53**, 2459–2479 (2007)
11. MacWilliams, F.J., Sloane, N.J.A.: *The Theory of Error-Correcting Codes*. North-Holland, Amsterdam (1977)
12. <http://magma.maths.usyd.edu.au/magma/>
13. Raghavendran, R.: A class of finite rings. *Compos. Math.* **21**, 195–229 (1969)
14. Shi, M., Alahmadi, A., Solé, P.: *Codes and Rings: Theory and Practice*. Academic Press, Cambridge (2017)
15. Wood, Jay A.: Duality for modules over finite rings and applications to coding theory. *Am. J. Math.* **121**(3), 555–575 (1999)

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