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# Quasi Type IV codes over a non-unital ring

Adel Alahmadi\*, Alaa Altassan †, Widyan Basaffar ‡  
Alexis Bonnetcaze §, Hatoon Shoaib ¶, Patrick Solé ||\*\*

## Abstract

There is a local ring  $I$  of order 4, without identity for the multiplication, defined by generators and relations as

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

We study the algebraic structure of linear codes over this non-unital local ring, in particular their residue and torsion codes. We introduce the notion of quasi self-dual codes (QSD) over  $I$ , and Type IV  $I$ -codes, that is, QSD codes all codewords of which have even Hamming weight. Further, we define quasi Type IV codes over  $I$  as those QSD codes with an even torsion code. We give a mass formula for QSD codes, and another quasi Type IV codes, and classify both of them in short lengths.

**Keywords:** rings, codes, additive  $\mathbb{F}_4$ -codes, mass formulas, Type IV codes.

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# 1 Introduction

While rings have been used as alphabets in Coding Theory for more than thirty years [7], it is only recently that non-unital rings have been used to that effect [1]. In that paper, the authors introduce the notion of Type IV codes over the ring  $E$  in the terminology of [10, 16], in analogy with Type IV codes over unital rings of order four [8].

In this paper we study codes over another non-unital commutative ring of order four, denoted by  $I$  in the classification of [10, 16]. We call  $I$ -code of length  $n$  any  $I$ -submodule of  $I^n$ . Such a code can be thought of as an additive  $\mathbb{F}_4$ -code in the sense of [3] with an extra module structure. This allows us to use the additive code package of Magma [15] in computations, and might lead in the future to the discovery of new additive  $\mathbb{F}_4$ -codes. In the ring  $I$  the notion of self-dual code must be adapted since the usual relation between the size of the code and that of its dual [20], does not hold in general. We thus introduce, in Section 2.4, following [1], the notion of quasi self-dual code (QSD), that is of an  $I$ -code of length  $n$  which is both self-orthogonal and of size  $2^n$ . As with other local rings of order 4, residue and torsion codes play a fundamental role in the structure of QSD codes. Thus, in Section 3, we derive a canonical form for the generator matrix of any  $I$ -code, based on the residue and torsion codes. However, an  $I$ -code cannot be determined uniquely by its residue and torsion codes. But it is completely characterized by the triplet formed by these two codes and a map between the residue code and the cosets of the torsion code. This is similar to the technique used to derive mass formulas for self-dual codes over rings [2, 5, 11]. In Section 4 we give a construction of QSD codes from a pair of binary codes; we call this the multilevel construction in analogy with a similar construction over  $\mathbb{Z}_4$  [4]. In Section 5, we derive a mass formula for QSD codes with the objective of classifying these codes. The method of calculation of this formula leads us to introduce the notion of quasi Type IV codes: QSD codes whose torsion code is even, and to derive a mass formula for them. Finally, we give a classification of QSD codes up to length 3 in Section 6, and some concluding remarks in Section 7.

## 2 Background material

### 2.1 Binary codes

Denote by  $wt(x)$  the (Hamming) weight of  $x \in \mathbb{F}_2^n$ . The dual of a binary linear code  $C$  is denoted by  $C^\perp$  and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \mid \forall x \in C, (x, y) = 0\},$$

where  $(x, y) = \sum_{i=1}^n x_i y_i$ , denotes the standard inner product. A code  $C$  is **self-orthogonal** if it is included in its dual:  $C \subseteq C^\perp$ . A code is **even** if all its codewords have even weight. All self-orthogonal codes are even but not conversely. Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

### 2.2 Rings

Following [10], we define a ring on two generators  $a, b$  by its relations

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

Thus,  $I$  has characteristic two, and consists of four elements  $I = \{0, a, b, c\}$ , with  $c = a + b$ . The addition table is immediate from these definitions

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The multiplication table is as follows.

×	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

From this table, we infer that this ring is commutative, and without an identity element for the multiplication. It is local with maximal ideal  $J =$

$\{0, b\}$ , and residue field  $\mathbb{F}_2 = \{0, 1\}$ , the finite field of order 2. Thus we have a ***b*-adic decomposition** as follows. Every element  $i \in I$  can be written

$$i = as + bt,$$

where  $s, t \in \mathbb{F}_2$  and where we have defined a natural **action** of  $\mathbb{F}_2$  on  $I$  by the rule  $r0 = 0r = 0$  and  $r1 = 1r = r$  for all  $r \in I$ . Thus  $a = 1a, c = 1c$  and  $c = a1 + b1$ . Note that for all  $r \in I$ , this action is “distributive” in the sense that  $r(s \oplus t) = rs + rt$ , where  $\oplus$  denotes the addition in  $\mathbb{F}_2$ . On occasion we will use the **inner product notation**  $(x, r)$  for  $x \in \mathbb{F}_2^n, r \in I^n$  to denote

$$(x, r) = \sum_{i=1}^n x_i r_i = \sum_{x_i=1} r_i.$$

Denote by  $\alpha : I \rightarrow I/J \simeq \mathbb{F}_2$  the **map of reduction modulo  $J$** . Thus  $\alpha(0) = \alpha(b) = 0$ , and  $\alpha(a) = \alpha(c) = 1$ . This map is extended in the natural way in a map from  $I^n$  to  $\mathbb{F}_2^n$ .

### 2.3 Modules

A **linear  $I$ -code  $C$**  of length  $n$  is an  $I$ -submodule of  $I^n$ . It can be described as the  $I$ -span of the rows of a **generator matrix**. An **additive code** of length  $n$  over  $\mathbb{F}_4$  is an additive subgroup of  $\mathbb{F}_4^n$ . It is an  $\mathbb{F}_2$  vector space with  $4^k$  elements for some  $k \leq n$  (here  $2k$  is an integer, but  $k$  may be half-integral). Using a **generator matrix  $G$** , such a code can be cast as the  $\mathbb{F}_2$ -span of its rows. To every linear  $I$ -code  $C$  is attached an additive  $\mathbb{F}_4$ -code  $\phi(C)$  by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow 1, c \rightarrow \omega^2,$$

where  $\mathbb{F}_4 = \mathbb{F}_2[\omega]$ , extended naturally to  $\mathbb{F}_4^n$ .

We use the Magma notation

$$[< 0, 1 >, \dots, < i, A_i >, \dots, < n, A_n >]$$

for the **weight distribution** of a quaternary code, where  $A_i$  is the number of codewords of weight  $i$ . Two  $I$ -codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other. The **automorphism group**  $Aut(C)$  of an  $I$ -code  $C$  is the group of column permutations that leave  $C$  invariant.

## 2.4 Duality

Define an **inner product** on  $I^n$  by  $(x, y) = \sum_{i=1}^n x_i y_i$ .

The **dual**  $C^\perp$  of  $C$  is the module defined by

$$C^\perp = \{y \in I^n \mid \forall x \in C, (x, y) = 0\}.$$

Thus the dual of a module is a module. A code is **self-dual** if it is equal to its dual.

**Remark 1** *The repetition code of length 2 is defined by  $R_2 := \{00, aa, bb, cc\}$ . Its dual is  $R_2^\perp = \{00, aa, bb, cc, 0b, b0, ac, ca\}$ , a supercode of  $R_2$  of size 8. In length one, we have  $J^\perp = I$ .*

Remark 1 shows that the product of the sizes of a code and its dual is not always  $4^n$ . A code  $C$  of length  $n$  is **nice** if  $|C||C^\perp| = 4^n$ .

A code  $C$  is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

Clearly,  $C$  is **self-orthogonal** if and only if  $C \subseteq C^\perp$ .

A code of length  $n$  is **quasi self-dual** if it is self-orthogonal and of size  $2^n$ .

Following a terminology from [8], a quasi self-dual code over  $I$  with all weights even is called a **Type IV** code.

**Remark 2** *The repetition code of length 2 is quasi self-dual over  $I$  and is of Type IV. This shows, by taking direct sums of codes, that Type IV codes over  $I$  exist for all even lengths. We see that  $J$  is a quasi self-dual code over  $I$ . This shows, again by taking direct sums, that QSD codes exist for all integer lengths.*

We introduce the notion of **Quasi Type IV** code (QT4) as a QSD code with an even torsion code. Every Type IV code is quasi Type IV but not conversely as the next example shows. The motivation will appear in the mass formula section.

**Example 1** The code with three generators  $\begin{pmatrix} a & b & a & b \\ 0 & b & b & 0 \\ b & 0 & 0 & b \end{pmatrix}$  is QSD but not Type IV as the sum of first and second row has odd weight. But its torsion code with generator matrix  $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$  is an even code.

### 3 Structure of linear codes

Let  $C$  be a code of length  $n$  over  $I$ . With this code we associate two binary codes of length  $n$  :

1. the **residue code** defined by  $res(C) = \{\alpha(y) \mid y \in C\}$ ,
2. the **torsion code** defined by  $tor(C) = \{x \in \mathbb{F}_2^n \mid bx \in C\}$ .

It can be checked that for all  $x \in I^n$ , we have  $Tr(\phi(x)) = \alpha(x)$ , and thus  $res(C) = Tr(\phi(C))$ , where  $\forall z \in \mathbb{F}_4, Tr(z) = z + z^2$ . Similarly, we see that  $tor(C)$  is the subfield subcode of  $\phi(C)$ . Denote by  $\alpha_C$  the restriction of  $\alpha$  to  $C$ . We see that  $tor(C)b = \text{Ker}(\alpha_C)$ , and that  $res(C) = \text{Im}(\alpha_C)$ . By the first isomorphism theorem applied to the map  $\alpha_C$ , we see that  $|C| = |res(C)||tor(C)|$ . There is a relationship between these two codes.

**Lemma 1** *If  $C$  is an  $I$ -linear code then  $res(C) \subseteq tor(C)$ .*

**Proof.** Write an arbitrary codeword in  $b$ -adic decomposition form as  $ax + by$ , with  $x, y$  binary vectors. Since  $\alpha(ax + by) = x$ , we have  $x \in res(C)$ . Note that, by definition of the residue code, any  $x \in res(C)$  arises in that way. Multiplying the codeword  $ax + by$  by  $a$ , we see that  $bx \in C$ , implying  $x \in tor(C)$ . ■

We let  $k_1 = \dim(res(C))$ , and  $k_2 = \dim(tor(C)) - k_1$ , a nonnegative quantity by Lemma 1, and say that  $C$  is of **type**  $(k_1, k_2)$ . It can be seen that  $C$  is free as an  $I$ -module if and only if  $k_2 = 0$ . Further, by a previous observation,  $|C| = |res(C)||tor(C)| = 2^{2k_1+k_2}$ . We give a characterization of the generator matrix of a linear code as a function of these invariants.

**Theorem 1** *Assume  $C$  is an  $I$ -linear code of length  $n$  and type  $(k_1, k_2)$ . Then, up to a permutation of columns, a generator matrix  $G$  of  $C$  is of the form*

$$G = \begin{pmatrix} aI_{k_1} & aX & Y \\ 0 & bI_{k_2} & bZ \end{pmatrix},$$

where  $I_j$  denotes the identity matrix of order  $j$ , the matrix  $Y$  has entries in  $I$ , and  $X, Z$  are binary matrices.

**Proof.** Write the generator matrices of  $\text{res}(C)$  and  $\text{tor}(C)$  as  $G_1 = (I_{k_1} \ X \ \alpha(Y))$  and  $G_2 = \begin{pmatrix} I_{k_1} & X & \alpha(Y) \\ 0 & I_{k_2} & Z \end{pmatrix}$ , respectively, with  $Y$  a matrix of suitable dimensions with entries in  $I$ , and  $X, Z$  are binary matrices. By the first isomorphism theorem applied to the map  $\alpha_C$ , the matrix  $G$  can be written on the form  $\begin{pmatrix} R \\ bT \end{pmatrix}$ , where

$$\text{tor}(C) = \text{res}(C) \oplus \langle T \rangle,$$

and  $\alpha(R) = G_1$ . By taking linear combinations with rows of  $T$ , the matrix  $R$  can be written in the above form. ■

**Theorem 2** *Assume  $C$  is a nice  $I$ -linear code of length  $n$  and type  $(k_1, k_2)$ . Then a parity check matrix  $H$  of  $C$  consistent with Theorem 1 is of the form*

$$H = \begin{pmatrix} Y^t + aZ^tX^t & aZ^t & aI_{n-k_1-k_2} \\ bX^t & bI_{k_2} & 0 \end{pmatrix},$$

where  $I_j$  denotes the identity matrix of order  $j$ , and  $X, Y, Z$  are as in Theorem 1. In particular  $C^\perp$  is of type  $(n - k_1 - k_2, k_2)$ . If, furthermore  $C$  is self-dual, then  $n = 2k_1 + k_2$ .

**Proof.** A direct calculation shows that  $HG^t = 0$ . This shows that, as  $I$ -modules,  $\langle H \rangle = I^{n-k_1}H \subseteq C^\perp$ . Equality follows by size comparison upon noticing that  $\langle H \rangle$  has type  $(n - k_1 - k_2, k_2)$ , and upon observing that by the niceness hypothesis we have  $|C||C^\perp| = 4^n$ . The last assertion follows by unicity of the type  $k_1 = n - k_1 - k_2$ . ■

To prepare for the derivation of a mass formula, we develop a method similar to that of [11]. We define a map  $F$  from  $\text{res}(C)$  to  $\mathbb{F}_2^n/\text{tor}(C)$  by the formula

$$F(x) = \{y \in \mathbb{F}_2^n \mid ax + by \in C\}.$$



Since  $C$  is  $I$ -linear, it follows that  $F$  is  $\mathbb{F}_2$ -linear. Writing  $Y = aY_1 + bY_2$ , we see that the information given by  $F$  is equivalent to that given by  $Y_2$ . We write  $\phi(C) = (\text{res}(C), \text{tor}(C), F)$ .

The following Theorem is immediate, but essential.

**Theorem 3** *Keep the above notation. The map  $\phi$  is a one to one correspondence between  $I$ -linear codes  $C$  and triplets  $(C_1, C_2, F)$  such that  $C_1, C_2$  are binary codes satisfying  $C_1 \subseteq C_2$  and such that the map  $F$  is  $\mathbb{F}_2$ -linear.*

**Proof.** Given  $\phi(C) = (C_1, C_2, F)$ , the code  $C$  is uniquely recovered as

$$C = \bigcup_{x \in C_1} \bigcup_{y \in F(x)} ax + by.$$

■

## 4 Construction of QSD codes

We call the next construction of  $I$ -codes from binary codes the **multilevel construction**.

**Theorem 4** *Let  $C_1$  be a self-orthogonal binary code of length  $n$ , and let  $C_2$  be a binary code of length  $n$ , where  $C_1 \subseteq C_2$ . The code  $C$  defined by the relation*

$$C = aC_1 + bC_2,$$

*is a self-orthogonal code. Its residue code is  $C_1$  and its torsion code is  $C_2$ . If, furthermore,  $|C_1||C_2| = 2^n$ , it is QSD.*

**Proof.** The code  $C$  is closed under addition, by linearity of  $C_1$  and  $C_2$ . Let us check it is closed under multiplication by a scalar of  $I$ . Note that  $aC = a^2C_2 + abC_2 = bC_1 \subseteq bC_2 \subseteq C$ . Likewise, we see that  $cC = caC_1 + cbC_2 = bC_1 \in C$ . Because  $bI = \{0\}$ , we get  $bC = baC_1 + b^2C_2 = \{0\} \subseteq C$ .

Thus  $C$  is  $I$ -linear.

For all  $x, x'$  in  $C_1$  and  $y, y'$  in  $C_2$  we have the inner products

$$(ax + cy, ax' + cy') = a^2(x, x') + ab(x, y) + ba(y, x') + b^2(y, y') = b(x, x') = 0$$

since  $x \in C_1$  which is self orthogonal. Thus  $C$  is self-orthogonal. The residue and torsion codes are immediately derived from the definitions. Since  $|C| = |C_1||C_2|$ , the last assertion follows.

■

Note that for given residue and torsion codes many QSD codes may exist. As an example, the code  $C_1$  generated by the matrix  $\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}$  and the code  $C_2$  generated by the matrix  $\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$  have same residue and torsion codes.

## 5 Mass formulas

### 5.1 QSD codes

Define  $N_I(n, k)$  as the number of QSD  $I$ -codes of length  $n$  and type  $(k, n - 2k)$ . Let  $\Phi(N, K)$  denote the number of self-orthogonal binary codes having parameters  $[N, K]$ . The following explicit formula for this quantity can be found in [13, p.1]. It is valid for  $k \geq 1$ .

$$\Phi(N, K) = \begin{cases} \frac{\prod_{j=1}^K (2^{N+1-2j} - 1)}{\prod_{j=1}^K (2^j - 1)} & \text{if } n \geq 3 \text{ is odd,} \\ \frac{(2^{N-K} - 1) \prod_{j=1}^{K-1} (2^{N-2j} - 1)}{\prod_{j=1}^K (2^j - 1)} & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

We need the following counting lemma. For information on  $q$ -binomial coefficients we refer the reader to [14, p. 443].

**Lemma 2** *The number  $E(q, n, r, s)$  of subspaces of dimension  $r$  of  $\mathbb{F}_q^n$  containing a given subspace of dimension  $s < r$  is  $\binom{n-s}{r-s}_q$ , where  $\binom{i}{j}_q$  denotes the  $q$ -binomial coefficient defined for integers  $i \geq j$  by*

$$\binom{i}{j}_q = \frac{(q^i - 1)(q^{i-1} - 1) \cdots (q^{i-j+1} - 1)}{(q^j - 1)(q^{j-1} - 1) \cdots (q - 1)}.$$

**Proof.** Follows by Lemma 3 of [6] with  $i = j = s$  and  $k = r$  for  $q = 2$ . ■

**Lemma 3** *The number  $EE(n, r, s)$  of even codes of dimension  $r$  of  $\mathbb{F}_2^n$  containing a given subspace of dimension  $s < r$  is  $\binom{n-1-s}{r-s}_2$ .*

**Proof.** Follows by Lemma 3 of [6] with  $i = j = s$  and  $k = r$  for  $q = 2$ . Here  $C$  in the notation of [6] is the even weight hyperplane of dimension

$n - 1$ . ■

**Theorem 5** *For all lengths  $n$  and type  $(k, n - 2k)$ , with  $k > 0$  we have*

$$N_I(n, k) = \Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}.$$

**Proof.** Let  $C$  be a QSD code of length  $n$  and type  $(k, n - 2k)$ . We reason in terms of the map  $\phi$ . Note the following three independent counts.

1. There are  $\Phi(n, k)$  self-orthogonal  $[n, k]$  codes that can be used as  $res(C)$ .
2. The residue code is contained in  $E(2, n, n - k, k) = \binom{n - k}{n - 2k}_2$  possible torsion codes by Lemma 2.
3. The additive map  $F$  is arbitrary from  $res(C)$  to  $\mathbb{F}_2^n / tor(C)$ . By the QSD hypothesis, these two vector spaces have dimension  $k$ .

The result follows by multiplying the three factors together. ■

The following mass formula follows by the usual counting technique.

**Corollary 1** *For given length  $n$  and type  $(k, n - 2k)$ , with  $k > 0$ , we have*

$$\sum_C \frac{1}{|Aut(C)|} = \frac{\Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}}{n!},$$

where  $C$  runs over distinct representatives of equivalence classes under column permutations of QSD codes of length  $n$  and type  $(k, n - 2k)$ .

## 5.2 Quasi Type IV codes

Define  $N_4(n, k)$  as the number of QT4  $I$ -codes of length  $n$  and type  $(k, n - 2k)$ . The analogue of Lemma 2 is as follows.

**Lemma 4** *The number  $EE(n, r, s)$  of even codes of dimension  $r$  of  $\mathbb{F}_2^n$  containing a given subspace of dimension  $s < r$  is  $\binom{n - 1 - s}{r - s}_2$ .*

**Proof.** Follows by Lemma 3 of [6] with  $i = j = s$  and  $k = r$  for  $q = 2$ . Here  $C$  is the even weight hyperplane of dimension  $n - 1$ . ■

Using this Lemma the analogue of Theorem 5 for QT4 codes can be shown easily.

**Theorem 6** For all lengths  $n$  and type  $(k, n - 2k)$ , with  $k > 0$  we have

$$N_4(n, k) = \Phi(n, k) \binom{n-1-k}{n-2k}_2 2^{k^2}.$$

The following mass formula follows by the usual counting technique.

**Corollary 2** For given length  $n$ , and type  $(k, n - 2k)$ , with  $k > 0$ , we have

$$\sum_C \frac{1}{|Aut(C)|} = \frac{\Phi(n, k) \binom{n-1-k}{n-2k}_2 2^{k^2}}{n!},$$

where  $C$  runs over distinct representatives of equivalence classes under column permutations of  $QT_4$  codes of length  $n$  and type  $(k, n - 2k)$ .

## 6 Short length classification ( $n < 4$ )

In the following, we classify, up to coordinate permutation, QSD codes. We construct QSD codes by means of the multilevel construction, and of the characterization of Theorem 1, in short lengths ( $n < 4$ ) and rely on the mass formula to know the maximum number of equivalence classes.

For  $n = 1$  there is just one QSD code with generator matrix  $(b)$ .

### 6.1 $n = 2$

There are two codes of length 2. Both are Type IV codes and their weight distribution is  $[< 0, 1 >, < 2, 3 >]$ . One is generated by  $(a \ a)$  and the other one by  $(a \ c)$ .

### 6.2 $n = 3$

There are eighteen distinct QSD codes of length 3 among which six codes are not permutation equivalent. All these codes have an automorphism group of order 2. The following four codes have a minimum distance equal to 1. One code has generator matrix

$$\begin{pmatrix} a & a & b \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution [ $\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 3, 4 \rangle$ ].

One code has generator matrix

$$\begin{pmatrix} a & a & 0 \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution [ $\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 5 \rangle$ ].

The last two codes have weight distribution [ $\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle$ ]. Their generator matrices are

$$\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 & c \\ 0 & b & 0 \end{pmatrix}.$$

Two codes have minimum distance 2 and weight distribution [ $\langle 0, 1 \rangle, \langle 2, 5 \rangle, \langle 3, 2 \rangle$ ]. Their generator matrices are

$$\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$$

and

$$\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}.$$

These two codes are quasi Type IV codes with, as torsion code, a binary code of parameters  $[3, 2, 2]$  and generator matrix  $\begin{pmatrix} b & 0 & b \\ 0 & b & b \end{pmatrix}$ . Note that we obtain two quasi Type IV codes, which is consistent with Corollary 2.

## 7 Conclusion

In this article, we have studied quasi self-dual codes over the non-unital commutative ring  $I$  of order four. The existence of codes that are not nice precludes any attempt to derive a general MacWilliams formula, since this would imply a relation between the size of a code and that of its dual. We have thus introduced QSD codes as an alternative to the concept of self-dual

codes. For similar reasons, we have introduced the notion of quasi Type IV codes, in view of the difficulty of studying Type IV codes. We have derived a mass formula for QSD codes and for QT4 codes to classify them under coordinate permutation equivalence.

The main open problem arising from this work would be to derive a mass formula for Type IV codes. This would require a characterization of these codes akin to what exists over  $E$  [1], or over unital rings of Type IV [8]. It would be interesting to derive a classification under a notion of equivalence based on the symmetry of the alphabet between  $a$  and  $c$ . It might be a simpler classification than the classification under column permutation of the present paper.

On the computational side, pushing the classification of QSD codes beyond length 3, is certainly a worthy project, and might lead to the discovery of new additive  $\mathbb{F}_4$ -codes in lengths  $> 3$ .

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