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► **To cite this version:**

Adel Alahmadi, Alaa Altassan, Widyan Basaffar, Alexis Bonnecaze, Hatoon Shoaib, et al.. Quasi type IV codes over a non-unital ring. *Applicable Algebra in Engineering, Communication and Computing*, 2021, 10.1007/s00200-021-00488-6 . hal-03173734

**HAL Id: hal-03173734**

**<https://hal.science/hal-03173734>**

Submitted on 18 Mar 2021

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## Quasi type IV codes over a non-unital ring

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Received: 28 October 2019 / Accepted: 4 January 2021

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### Abstract

There is a local ring  $I$  of order 4, without identity for the multiplication, defined by generators and relations as

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

We give a natural map between linear codes over  $I$  and additive codes over  $\mathbb{F}_4$ , that allows for efficient computations. We study the algebraic structure of linear codes over this non-unital local ring, their generator and parity-check matrices. A canonical form for these matrices is given in the case of so-called nice codes. By analogy with  $\mathbb{Z}_4$ -codes, we define residue and torsion codes attached to a linear  $I$ -code. We introduce the notion of quasi self-dual codes (QSD) over  $I$ , and Type IV  $I$ -codes, that is, QSD codes all codewords of which have even Hamming weight. This is the natural analogue of Type IV codes over the field  $\mathbb{F}_4$ . Further, we define quasi Type IV codes over  $I$  as those QSD codes with an even torsion code. We give a mass formula for QSD codes, and another for quasi Type IV codes, and classify both types of codes, up to coordinate permutation equivalence, in short lengths.

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**Keywords** Rings · Codes · Additive  $\mathbb{F}_4$ -codes · Mass formulas · Type IV codes

**Mathematics Subject Classification** Primary 94 B05 · Secondary 16 A10

## 1 Introduction

While rings have been used as alphabets in Coding Theory for more than thirty years [14], it is only recently that non-unital rings have been used to that effect [1]. In that paper, the authors introduce the notion of Type IV codes over the ring  $E$  in the terminology of [8, 13], in analogy with Type IV codes over unital rings of order four [7].

In this paper we study codes over another non-unital commutative ring of order four, denoted by  $I$  in the classification of [8, 13]. We call  $I$ -code of length  $n$  any  $I$ -submodule of  $I^n$ . Such a code can be thought of as an additive  $\mathbb{F}_4$ -code in the sense of [3] with an extra module structure. This allows us to use the additive code package of Magma [12] in computations, and might lead in the future to the discovery of new additive  $\mathbb{F}_4$ -codes. In the ring  $I$  the notion of self-dual code must be adapted since the usual relation between the size of the code and that of its dual [15], does not hold in general. We thus introduce, in Section 2.4, following [1], the notion of quasi self-dual code (QSD), that is of an  $I$ -code of length  $n$  which is both self-orthogonal and of size  $2^n$ . As with other local rings of order 4, residue and torsion codes play a fundamental role in the structure of QSD codes. Thus, in Sect. 3, we derive a canonical form for the generator matrix of any  $I$ -code, based on the residue and torsion codes. However, an  $I$ -code cannot be determined uniquely by its residue and torsion codes. But it is completely characterized by the triplet formed by these two codes and a map between the residue code and the cosets of the torsion code. This is similar to the technique used to derive mass formulas for self-dual codes over rings [2, 5, 9]. In Sect. 4 we give a construction of QSD codes from a pair of binary codes; we call this the multilevel construction in analogy with a similar construction over  $\mathbb{Z}_4$  [4]. In Sect. 5, we derive a mass formula for QSD codes with the objective of classifying these codes. The method of calculation of this formula leads us to introduce the notion of quasi Type IV codes: QSD codes whose torsion code is even, and to derive a mass formula for them. Finally, we give a classification of QSD codes up to length 3 in Sect. 6, and some concluding remarks in Sect. 7.

## 2 Background material

### 2.1 Binary codes

Denote by  $wt(x)$  the **(Hamming) weight** of  $x \in \mathbb{F}_2^n$ , or, in other words, the number of indices  $i$  where  $x_i \neq 0$ . A linear code is a subspace of  $\mathbb{F}_2^n$ , viewed as an  $\mathbb{F}_2$ -vector space. The dual of a binary linear code  $C$  is denoted by  $C^\perp$  and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \mid \forall x \in C, (x, y) = 0\},$$

where  $(x, y) = \sum_{i=1}^n x_i y_i$ , denotes the standard inner product. A code  $C$  is **self-orthogonal** if it is included in its dual:  $C \subseteq C^\perp$ . A code is **even** if all its codewords have even weight. All self-orthogonal codes are even but not conversely. Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

### 2.2 Rings

Following [8], we define a ring on two generators  $a, b$  by its relations

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

Thus,  $I$  has characteristic two, and consists of four elements  $I = \{0, a, b, c\}$ , with  $c = a + b$ . The addition table is immediate from these definitions

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The multiplication table is as follows.

×	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

From this table, we infer that this ring is commutative, and without an identity element for the multiplication. It is local with maximal ideal  $J = \{0, b\}$ , and residue field  $\mathbb{F}_2 = \{0, 1\}$ , the finite field of order 2. Thus we have a  **$b$ -adic decomposition** as follows. Every element  $i \in I$  can be written

$$i = as + bt,$$

where  $s, t \in \mathbb{F}_2$  and where we have defined a natural **action** of  $\mathbb{F}_2$  on  $I$  by the rule  $r0 = 0r = 0$  and  $r1 = 1r = r$  for all  $r \in I$ . Thus  $a = 1a, b = 1b$  and  $c = a1 + b1 = c1$ . Note that for all  $r \in I$ , this action is “distributive” in the sense that  $r(s \oplus t) = rs + rt$ , where  $\oplus$  denotes the addition in  $\mathbb{F}_2$ . On occasion we will use the **inner product notation**  $(x, r)$  for  $x \in \mathbb{F}_2^n, r \in I^n$  to denote

$$(x, r) = \sum_{i=1}^n x_i r_i = \sum_{x_i=1} r_i.$$

Denote by  $\alpha : I \rightarrow I/J \simeq \mathbb{F}_2$  the **map of reduction modulo  $J$** . Thus  $\alpha(0) = \alpha(b) = 0$ , and  $\alpha(a) = \alpha(c) = 1$ . This map is extended in the natural way in a map from  $I^n$  to  $\mathbb{F}_2^n$ .

### 2.3 Modules

A **linear  $I$ -code  $C$**  of length  $n$  is an  $I$ -submodule of  $I^n$ . It can be described as the  $I$ -span of the rows of a **generator matrix**. An  **$\mathbb{F}_4$ -code of length  $n$**  is just any non-empty subset of  $\mathbb{F}_4^n$ . The notion and notation of **Hamming weight** over  $\mathbb{F}_4$  is the same as over  $\mathbb{F}_2$ . An **additive code** of length  $n$  over  $\mathbb{F}_4$  is an additive subgroup of  $\mathbb{F}_4^n$ . It is an  $\mathbb{F}_2$  vector space with  $4^k$  elements for some  $k \leq n$  (here  $2k$  is an integer, but  $k$  may be half-integral). Using a **generator matrix  $G$** , such a code can be cast as the  $\mathbb{F}_2$ -span of its rows. To every linear  $I$ -code  $C$  is attached an additive  $\mathbb{F}_4$ -code  $\phi(C)$  by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow 1, c \rightarrow \omega^2,$$

where  $\mathbb{F}_4 = \mathbb{F}_2[\omega]$ , extended naturally to  $\mathbb{F}_4^n$ . Trivially, the Hamming weight of  $\phi(x)$  is  $wt(x)$ .

We use the Magma notation

$$[\langle 0, 1 \rangle, \dots, \langle i, A_i \rangle, \dots, \langle n, A_n \rangle]$$

for the **weight distribution** of an additive  $\mathbb{F}_4$ -code code, where  $A_i$  is the number of codewords of weight  $i$ . Two  $I$ -codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other (and, consequently, the  $\phi$ -image of one on the  $\phi$ -image of the other). The **automorphism group  $Aut(C)$**  of an  $I$ -code  $C$  is the group of column permutations that leave  $C$  invariant.

### 2.4 Duality

Define an **inner product** on  $I^n$ , by writing  $(x, y) = \sum_{i=1}^n x_i y_i$  for all  $x, y \in I^n$ .

The **dual  $C^\perp$**  of a linear  $I$ -code  $C$  is the module defined by

$$C^\perp = \{y \in I^n \mid \forall x \in C, (x, y) = 0\}.$$

Thus the dual of a module is a module. A code is **self-dual** if it is equal to its dual.

**Remark 1** The **repetition code** of length 2 is the linear  $I$ -code defined by  $R_2 := \{00, aa, bb, cc\}$ . Its dual is  $R_2^\perp = \{00, aa, bb, cc, 0b, b0, ac, ca\}$ , a supercode of  $R_2$  of size 8. In length one, we have  $J^\perp = I$ .

Remark 1 shows that the product of the sizes of a code and its dual is not always  $4^n$ . A code  $C$  of length  $n$  is **nice** if  $|C||C^\perp| = 4^n$ . A linear  $I$ -code  $C$  is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

Clearly,  $C$  is **self-orthogonal** if and only if  $C \subseteq C^\perp$ .

A code of length  $n$  is **quasi self-dual** (or QSD for short) if it is self-orthogonal and of size  $2^n$ . Note that the direct sum of two QSD codes is QSD.

Following a terminology from [7], a quasi self-dual code over  $I$  with all weights even is called a **Type IV** code. This notion is also stable by taking direct sums.

**Remark 2** The repetition code of length 2 is quasi self-dual over  $I$  and is of Type IV. This shows, by taking direct sums of codes, that Type IV codes over  $I$  exist for all even lengths. We see that  $J$  is a quasi self-dual code over  $I$ . This shows, again by taking direct sums, that QSD codes exist for all integer lengths.

We introduce the notion of **Quasi Type IV** code (QT4) as a QSD code with an even torsion code (see next section for a definition of torsion code). Every Type IV code is quasi Type IV but not conversely as the next example shows. The motivation will appear in the mass formula section.

**Example 1** The code with three generators  $\begin{pmatrix} a & b & a & b \\ 0 & b & b & 0 \\ b & 0 & 0 & b \end{pmatrix}$  is QSD but not Type IV as the sum of first and second row has odd weight. But its torsion code with generator matrix  $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$  is an even code.

### 3 Structure of linear codes

Let  $C$  be a code of length  $n$  over  $I$ . With this code we associate two binary codes of length  $n$ :

1. the **residue code** defined by  $res(C) = \{ \alpha(y) \mid y \in C \}$ ,
2. the **torsion code** defined by  $tor(C) = \{ x \in \mathbb{F}_2^n \mid bx \in C \}$ .

Recall that the **Trace** function of  $\mathbb{F}_4$  down to  $\mathbb{F}_2$  is defined by  $\forall z \in \mathbb{F}_4, Tr(z) = z + z^2$ . In other words

$$Tr(0) = Tr(1) = 0; Tr(\omega) = Tr(\omega^2) = 1.$$

By distinguishing four cases and induction on  $n$  it can be checked that for all  $x \in I^n$ , we have  $Tr(\phi(x)) = \alpha(x)$ , and thus  $res(C) = Tr(\phi(C))$ . Similarly, we see that  $tor(C)$  is the so-called **subfield subcode** of  $\phi(C)$ , that is  $\phi(C) \cap \mathbb{F}_2^n$ . Denote by  $\alpha_C$  the restriction of  $\alpha$  to  $C$ . We see that  $tor(C)b = Ker(\alpha_C)$ , and that  $res(C) = Im(\alpha_C)$ . By the first isomorphism theorem applied to the map  $\alpha_C$ , we see that  $|C| = |res(C)||tor(C)|$ . There is a relationship between these two codes.

**Lemma 1** If  $C$  is a linear  $I$ -code then  $res(C) \subseteq tor(C)$ .

**Proof** Write an arbitrary codeword  $z$  in  $b$ -adic decomposition form as  $z = ax + by$ , with  $x, y$  binary vectors. Since  $\alpha(ax + by) = x$ , we have  $x \in res(C)$ . Note that, by

definition of the residue code, any  $x \in \text{res}(C)$  arises in that way. Multiplying the codeword  $ax + by$  by  $a$ , we see that  $bx \in C$ , implying  $x \in \text{tor}(C)$ .  $\square$

We let  $k_1 = \dim(\text{res}(C))$ , and  $k_2 = \dim(\text{tor}(C)) - k_1$ , a nonnegative quantity by Lemma 1, and say that  $C$  is of **type**  $(k_1, k_2)$ . It can be seen that  $C$  is free as an  $I$ -module if and only if  $k_2 = 0$ . Further, by a previous observation,  $|C| = |\text{res}(C)||\text{tor}(C)| = 2^{2k_1+k_2}$ . We give a characterization of the generator matrix of a linear code as a function of these invariants.

**Theorem 1** Assume  $C$  is an  $I$ -linear code of length  $n$  and type  $(k_1, k_2)$ . Then, up to a permutation of columns, there exists a generator matrix  $G$  of  $C$  that is of the form

$$G = \begin{pmatrix} aI_{k_1} & aX & Y \\ 0 & bI_{k_2} & bZ \end{pmatrix},$$

where  $I_j$  denotes the identity matrix of order  $j$ , the matrix  $Y$  has entries in  $I$ , and  $X, Z$  are binary matrices.

**Proof** Write the generator matrices of  $\text{res}(C)$  and  $\text{tor}(C)$  as  $G_1 = (I_{k_1} \ X \ \alpha(Y))$  and  $G_2 = \begin{pmatrix} I_{k_1} & X & \alpha(Y) \\ 0 & I_{k_2} & Z \end{pmatrix}$ , respectively, with  $Y$  a matrix of suitable dimensions with entries in  $I$ , and  $X, Z$  are binary matrices. By the first isomorphism theorem applied to the map  $\alpha_C$ , the matrix  $G$  can be written on the form  $\begin{pmatrix} R \\ bT \end{pmatrix}$ , where

$$\text{tor}(C) = \text{res}(C) \oplus \langle T \rangle,$$

and  $\alpha(R) = G_1$ . By taking linear combinations with rows of  $T$ , the matrix  $R$  can be written in the above form.  $\square$

**Theorem 2** Assume  $C$  is a nice  $I$ -linear code of length  $n$  and type  $(k_1, k_2)$ . Then a parity check matrix  $H$  of  $C$  consistent with Theorem 1 is of the form

$$H = \begin{pmatrix} Y^t + aZ^tX^t & aZ^t & aI_{n-k_1-k_2} \\ bX^t & bI_{k_2} & 0 \end{pmatrix},$$

where  $I_j$  denotes the identity matrix of order  $j$ , and  $X, Y, Z$  are as in Theorem 1. In particular  $C^\perp$  is of type  $(n - k_1 - k_2, k_2)$ . If, furthermore  $C$  is self-dual, then  $n = 2k_1 + k_2$ .

**Proof** A direct calculation shows that  $HG^t = 0$ . This shows that, as  $I$ -modules,  $\langle H \rangle = I^{n-k_1}H \subseteq C^\perp$ . Equality follows by size comparison upon noticing that  $\langle H \rangle$  has type  $(n - k_1 - k_2, k_2)$ , and upon observing that by the niceness hypothesis we have  $|C||C^\perp| = 4^n$ . The last assertion follows by unicity of the type  $k_1 = n - k_1 - k_2$ .  $\square$

To prepare for the derivation of a mass formula, we develop a method similar to that of [9]. We define a map  $F$  from  $\text{res}(C)$  to  $\mathbb{F}_2^n/\text{tor}(C)$  by the formula

$$F(x) = \{y \in \mathbb{F}_2^n \mid ax + by \in C\}.$$

Since  $C$  is  $I$ -linear, it follows that  $F$  is  $\mathbb{F}_2$ -linear. We write  $\psi(C) = (res(C), tor(C), F)$ . The following Theorem is immediate, but essential.

**Theorem 3** *Keep the above notation. The map  $\psi$  is a one to one correspondence between linear  $I$ -codes  $C$  of length  $n$ , and triplets  $(C_1, C_2, F)$  such that  $C_1, C_2$  are binary codes of length  $n$ , satisfying  $C_1 \subseteq C_2$  and such that the map  $F$  is  $\mathbb{F}_2$ -linear from  $C_1$  to  $\mathbb{F}_2^n/C_2$ .*

**Proof** Let  $C$  be an arbitrary linear  $I$ -code. Let  $\psi(C) = (C_1, C_2, F)$ , be given. Writing an arbitrary codeword of  $C$  as  $ax + by$ , with  $x, y$  binary vectors of length  $n$ , we see that

- $\alpha(ax + by) = x$ , implying  $x \in C_1$ ;
- $y \in F(x)$ , by definition of  $F$ .

From there it follows that the code  $C$  is uniquely determined as

$$C = \bigcup_{x \in C_1} \bigcup_{y \in F(x)} ax + by.$$

□

#### 4 Construction of QSD codes

We call the next construction of  $I$ -codes from binary codes the **multilevel construction**.

**Theorem 4** *Let  $C_1$  be a self-orthogonal binary code of length  $n$ , and let  $C_2$  be a binary code of length  $n$ , where  $C_1 \subseteq C_2$ . The code  $C$  defined by the relation*

$$C = aC_1 + bC_2,$$

*is a self-orthogonal code. Its residue code is  $C_1$  and its torsion code is  $C_2$ . If, furthermore,  $|C_1||C_2| = 2^n$ , then  $C$  is QSD.*

**Proof** The code  $C$  is closed under addition, by linearity of  $C_1$  and  $C_2$ . Let us check it is closed under multiplication by a scalar of  $I$ . Note that  $aC = a^2C_1 + abC_2 = bC_1 \subseteq bC_2 \subseteq C$ . Likewise, we see that  $cC = caC_1 + cbC_2 = bC_1 \in C$ . Because  $bI = \{0\}$ , we get  $bC = baC_1 + b^2C_2 = \{0\} \subseteq C$ .

Thus  $C$  is  $I$ -linear.

For all  $x, x'$  in  $C_1$  and  $y, y'$  in  $C_2$  we have the inner products

$$(ax + by, ax' + by') = a^2(x, x') + ab(x, y') + ba(y, x') + b^2(y, y') = b(x, x') = 0$$

since  $x \in C_1$  which is self orthogonal. Thus  $C$  is self-orthogonal. The residue and torsion codes are immediately derived from the definitions. Since  $|C| = |C_1||C_2|$ , the last assertion follows.  $\square$

Note that for given residue and torsion codes many QSD codes may exist. As an example, the code  $C_1$  generated by the matrix  $\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}$  and the code  $C_2$  generated by the matrix  $\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$  have same residue and torsion codes.

## 5 Mass formulas

### 5.1 QSD codes

Define  $N_I(n, k)$  as the number of QSD  $I$ -codes of length  $n$  and type  $(k, n - 2k)$ . Let  $\Phi(N, K)$  denote the number of self-orthogonal binary codes having parameters  $[N, K]$ . The following explicit formula for this quantity can be found in [10, p.1]. It is valid for  $K \geq 1$ .

$$\Phi(N, K) = \begin{cases} \frac{\prod_{j=1}^K (2^{N+1-2j}-1)}{\prod_{j=1}^K (2^j-1)} & \text{if } n \geq 3 \text{ is odd,} \\ \frac{(2^{N-K}-1) \prod_{j=1}^{K-1} (2^{N-2j}-1)}{\prod_{j=1}^K (2^j-1)} & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

We need the following counting lemma. For information on  $q$ -binomial coefficients we refer the reader to [11, p. 443].

**Lemma 2** *The number  $E(q, n, r, s)$  of subspaces of dimension  $r$  of  $\mathbb{F}_q^n$  containing a given subspace of dimension  $s < r$  is  $\binom{n-s}{r-s}_q$ , where  $\binom{i}{j}_q$  denotes the  $q$ -binomial coefficient defined for integers  $i \geq j$  by*

$$\binom{i}{j}_q = \frac{(q^i - 1)(q^{i-1} - 1) \dots (q^{i-j+1} - 1)}{(q^j - 1)(q^{j-1} - 1) \dots (q - 1)}.$$

**Proof** Follows by Lemma 3 of [6] with  $i = j = s$  and  $k = r$ .  $\square$

We also need the following simple Lemma.

**Lemma 3** *If  $C$  is a self-orthogonal  $I$ -code, then its residue  $res(C)$  is a binary self-orthogonal code.*

**Proof** By linearity of the map  $\alpha$  it can be seen that  $\forall x, y \in C$ , we have

$$(\alpha(x), \alpha(y)) = \alpha((x, y)) = \alpha(0) = 0.$$

The result follows by  $\text{res}(C) = \text{Im}(\alpha_C)$ . □

We can now count the QSD codes of given length and residue dimension.

**Theorem 5** *For all lengths  $n$  and type  $(k, n - 2k)$ , with  $k > 0$  we have*

$$N_I(n, k) = \Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}.$$

**Proof** Let  $C$  be a QSD code of length  $n$  and type  $(k, n - 2k)$ . We reason in terms of the map  $\psi$ . Note the following three independent counts.

1. By Lemma 3, we know that  $\text{res}(C)$  is self-orthogonal. By definition of  $\Phi$ , there are  $\Phi(n, k)$  self-orthogonal binary  $[n, k]$  codes that can be used as  $\text{res}(C)$ .
2. The residue code is contained in  $E(2, n, n - k, k) = \binom{n - k}{n - 2k}_2$  possible torsion codes by Lemma 2 for  $r = k, s = n - k$ , and  $q = 2$ .
3. The additive map  $F$  is arbitrary from  $\text{res}(C)$  to  $\mathbb{F}_2^n / \text{tor}(C)$ . By the QSD hypothesis, these two vector spaces have dimension  $k$ . Hence, by considering the matrix of  $F$ , we see that there are  $2^{k^2}$  choices for  $F$ .

The result follows by multiplying the three factors together. □

The following mass formula follows by the usual counting technique.

**Corollary 1** *For given length  $n$  and type  $(k, n - 2k)$ , with  $k > 0$ , we have*

$$\sum_C \frac{1}{|\text{Aut}(C)|} = \frac{\Phi(n, k) \binom{n - k}{n - 2k}_2 2^{k^2}}{n!},$$

where  $C$  runs over distinct representatives of equivalence classes under column permutations of QSD codes of length  $n$  and type  $(k, n - 2k)$ .

### 5.2 Quasi type IV codes

Define  $N_4(n, k)$  as the number of QT4  $I$ -codes of length  $n$  and type  $(k, n - 2k)$ . The analogue of Lemma 2 is as follows.

**Lemma 4** *The number  $EE(n, r, s)$  of even codes of dimension  $r$  of  $\mathbb{F}_2^n$  containing a given subspace of dimension  $s < r$  is  $\binom{n - 1 - s}{r - s}_2$ .*

**Proof** Follows by Lemma 3 of [6] with  $i = j = s$  and  $k = r$  for  $q = 2$ . Here  $C$  in the notation of [6] is the binary code of length  $n$ , and dimension  $n - 1$ , consisting of all even Hamming weight vectors. □

Using this Lemma the following analogue of Theorem 5 for QT4 codes can be shown. The easy proof is omitted.

**Theorem 6** For all lengths  $n$  and type  $(k, n - 2k)$ , with  $k > 0$  we have

$$N_4(n, k) = \Phi(n, k) \binom{n-1-k}{n-2k}_2 2^{k^2}.$$

The following mass formula follows by the usual counting technique.

**Corollary 2** For given length  $n$ , and type  $(k, n - 2k)$ , with  $k > 0$ , we have

$$\sum_C \frac{1}{|\text{Aut}(C)|} = \frac{\Phi(n, k) \binom{n-1-k}{n-2k}_2 2^{k^2}}{n!},$$

where  $C$  runs over distinct representatives of equivalence classes under column permutations of QT4 codes of length  $n$  and type  $(k, n - 2k)$ .

## 6 Short length classification ( $n < 4$ )

In the following, we classify, up to coordinate permutation, QSD codes. We construct QSD codes by means of the multilevel construction, and of the characterization of Theorem 1, in short lengths ( $n < 4$ ) and rely on the mass formula to know the maximum number of equivalence classes.

For  $n = 1$  there is just one QSD code with generator matrix  $(b)$ .

### 6.1 $n = 2$

There are two codes of length 2. Both are Type IV codes and their weight distribution is  $[\langle 0, 1 \rangle, \langle 2, 3 \rangle]$ . One is generated by  $(a \ a)$  and the other one by  $(a \ c)$ .

### 6.2 $n = 3$

There are exactly six codes that are not permutation equivalent. All these codes have an automorphism group of order 2. The following four codes have a minimum distance equal to 1.

- One code has generator matrix

$$\begin{pmatrix} a & a & b \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution  $[\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 3, 4 \rangle]$ .

- One code has generator matrix

$$\begin{pmatrix} a & a & 0 \\ 0 & b & 0 \end{pmatrix}$$

and weight distribution [ $\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 5 \rangle$ ].

- The last two codes amongst these four share the same weight distribution [ $\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle$ ]. Their generator matrices are

$$\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 & c \\ 0 & b & 0 \end{pmatrix}.$$

Two codes have minimum distance 2 and weight distribution [ $\langle 0, 1 \rangle, \langle 2, 5 \rangle, \langle 3, 2 \rangle$ ]. Their generator matrices are

$$\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix}$$

and

$$\begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix}.$$

These two codes are quasi Type IV codes with, as torsion code, a binary code of parameters [3, 2, 2] and generator matrix  $\begin{pmatrix} b & 0 & b \\ 0 & b & b \end{pmatrix}$ . Note that we obtain two quasi Type IV codes, which is consistent with Corollary 1.

## 7 Conclusion

In this article, we have studied quasi self-dual codes over the non-unital commutative ring  $I$  of order four. The existence of codes that are not nice precludes any attempt to derive a general MacWilliams formula, since this would imply a relation between the size of a code and that of its dual. We have thus introduced QSD codes as an alternative to the concept of self-dual codes. For similar reasons, we have introduced the notion of quasi Type IV codes, in view of the difficulty of studying Type IV codes. We have derived a mass formula for QSD codes and for QT4 codes to classify them under coordinate permutation equivalence.

The main open problem arising from this work would be to derive a mass formula for Type IV codes. This would require a characterization of these codes akin to what exists over  $E$  [1], or over unital rings of Type IV [7]. It would be interesting to derive a classification under a notion of equivalence based on the symmetry of the

alphabet between  $a$  and  $c$ . It might be a simpler classification than the classification under column permutation of the present paper.

On the computational side, pushing the classification of QSD codes beyond length 3, is certainly a worthy project, and might lead to the discovery of new additive  $\mathbb{F}_4$ -codes in lengths  $> 3$ .

**Acknowledgements** This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant number KEP-PhD-29-130-40. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

## References

1. Alahmadi, A., Bonnetcaze, A., Shoaib, H., Altassan, A., Bassafar, W., Solé, P.: Type IV codes over a non-unital ring, submitted
2. Balmaceda, J.M.L., Betty, R.A.L., Nemenzo, F.: Mass formula for self-dual codes over  $\mathbb{Z}_{p^2}$ . *Discrete Math.* **308**, 2984–3002 (2008)
3. Calderbank, A.R., Rains, E.M., Sloane, N.J.A.: Quantum error correction via codes over  $GF(4)$ . *IEEE Trans. Inf. Theory* **44**, 1369–1387 (1998)
4. Conway, J.H., Sloane, N.J.A.: Self-dual codes over the integers modulo four. *J. Comb. Theory Ser. A* **62**, 30–45 (1993)
5. Choi, W.: Mass formula of self-dual codes over Galois rings  $GR(p^2, 2)$ . *Korean J. Math.* **24**(4), 751–764 (2016)
6. Delsarte, P.: Association schemes and t-designs in regular semi-lattices. *J. Comb. Theory* **20**, 230–243 (1976)
7. Dougherty, S.T., Gaborit, P., Harada, M., Munemasa, A., Solé, P.: Type IV self-dual codes over rings. *IEEE Trans. Inf. Theory* **45**(7), 2345–2360 (1999)
8. Fine, B.: Classification of finite rings of order  $p^2$ . *Math. Mag.* **66**(4), 248–252 (1993)
9. Gaborit, P.: Mass formulas for self-dual codes over  $\mathbb{Z}_4$  and  $\mathbb{F}_q + u_{\mathbb{F}_q}$  rings. *IEEE Trans. Inf. Theory* **42**, 1222–1228 (1996)
10. Hou, X.D.: On the number of inequivalent binary self-orthogonal codes. *Trans. Inf. Theory* **53**, 2459–2479 (2007)
11. MacWilliams, F.J., Sloane, N.J.A.: *The Theory of Error-Correcting Codes*. North-Holland, Amsterdam (1977)
12. <http://magma.maths.usyd.edu.au/magma/>
13. Raghavendran, R.: A class of finite rings. *Compos. Math.* **21**, 195–229 (1969)
14. Shi, M., Alahmadi, A., Solé, P.: *Codes and Rings: Theory and Practice*. Academic Press, Cambridge (2017)
15. Wood, Jay A.: Duality for modules over finite rings and applications to coding theory. *Am. J. Math.* **121**(3), 555–575 (1999)

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