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From adherence to slip in nanofluidics: a mathematical justification based on a drop of viscosity. The 3d case

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March 18, 2021

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1 Introduction

1.1 Context and goals

Nanofluidics is the study of the flow of materials at the nanoscale [5]. This is the scale of flow in Carbon Nanotubes which are physical systems of great interest nowadays [28, 41]. At this scale, flows present some striking features, such as the capacity for the material to slip much more easily than one would expect [4, 39, 40]. The origin of this slip is the subject of current debate in the physics community [23, 33].

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In classical fluid mechanics, the interaction between the fluid and the walls of a bounded domain is usually modeled *via* appropriate boundary conditions. Probably the three most used boundary conditions are (let us note u the velocity field of the flow)

No-slip (or adherence): imposing that u vanishes at the wall.

Slip: imposing that the normal component of u vanishes and that the tangential part of the normal stress is proportional to the tangential part of the velocity field. The inverse of the proportionality factor has the dimension of a length called the slip length.

Perfect slip: imposing that the normal component of u vanishes and that the tangential part of the normal stress also vanishes.

Let us point that, although one could expect the averaging procedure usually applied to obtain macroscopic flow equations to fail at the nanoscale, Stokes equations remain surprisingly efficient in nanofluidics [27, 32].

From a mathematical perspective, one successful strategy initiated in the late 1990's [2, 17, 29] to explain the occurrence of adherence or partial slip on solid walls, consists in modeling micro-asperities on the surface and analyzing their effect on the flow by an homogenisation process, imposing only a mild non penetration boundary condition on the rugous wall, *i.e.* that the normal component of the fluid velocity vanishes. This so-called "rugosity effect" has been studied quite extensively in the last decades, which has led to a rather complete description of the asymptotic effect of rough patterns on viscous flow [11, 12, 22, 25, 30].

In this paper, we consider a completely different interpretation of the apparent slip length measured in nanoscopic devices, proposed in [34], where the author postulates that the source of this slip arises from a "depletion layer with reduced viscosity near the wall". This hypothesis is supported by experimental evidence [36] and Molecular Dynamics simulations [31] bringing out that the viscosity drops near the wall of the nanotube. In [34], the flow is modeled as a Stokes flow in an infinite cylindrical pipe, with no-slip boundary condition, but two viscosities: a "bulk" one at the center of the pipe and a "wall" one near the walls which is smaller than the bulk one. In this model, the fluid is supposed to adhere at the wall. Yet, by solving the equation in this simple geometry, the author was able to describe the resulting flow as if it had an effective slip length and a constant viscosity equal to the one in the bulk, computing this length in terms of the viscosity drop and the sizes of the depletion layer and tube radius. From this result we started wondering: is there a general mathematical framework to study the passing from a model with varying viscosity and no-slip to a model with constant viscosity and slip?

The question is to link two models with different equations in the same domain and different boundary conditions. Many different problems of this type are (or have been) studied with the help of asymptotic analysis in PDE: one expresses the parameters of one of the model as function of small parameter ε and by studying the behaviour of the solution as ε goes to 0, one proves that the solution converges to the second model. The now classical problem that might be the most closely related to ours, and that we already mentioned, is the rugosity effect. In that case, one attempts to link a model posed on a family of domains depending on a parameter ε and "converging" in some sense with the perfect slip boundary condition on each domain, and to explain how one obtains a positive slip length, or no-slip on the limit domain. In a certain sense, the problem that we propose to address is the opposite: whereas rugosity aims at explaining how a fluid can "slow down" because of the wall, our problem is to explain and justify mathematically the "speeding up" of the fluid caused by a drop of viscosity near the wall.

The problem we consider is strongly related to the so-called reinforcement problems introduced by Sanchez-Palencia in [38], where an elastic medium is reinforced by the adjunction of

a thin layer of very strong material. From a mathematical point of view, such models give rise to singular perturbation problems, where the modulus of ellipticity of the operator tends to zero in the thin layer of extra material, as the layer shrinks.

Brézis, Caffarelli and Friedman solved the interior and boundary reinforcement problems for elliptic equations, in the case of Dirichlet boundary conditions on a C^2 boundary and using strong solutions in [10]. A few years later, geometric measure theory and Gamma-convergence were successfully applied to boundary reinforcement problems (see for instance Acerbi and Buttazzo [1], Buttazzo and Kohn [15], Buttazzo, Dal Maso and Mosco [14]).

In a recent paper [7], we have proposed a different approach based on a rescaling of the solution in the depletion layer, in the spirit of the unfolding method [3, 16, 21], and on the construction of a relevant sequence of test functions that are able to capture the asymptotic behaviour of the problem in the boundary layer associated with the region of low viscosity.

In this paper, we adapt this approach to treat the case of a three-dimensional Stokes system. Starting from the natural energy bound associated with the problem, we obtain compactness on the rescaled velocity and pressure fields in the low viscosity layer by adapting arguments from [18], and derive the Reynolds equation for the rescaled pressure. Then, we use a sequence of well-adapted test functions, whose behaviour in the vicinity of the wall is also determined by a Reynolds equation. This procedure allows us to determine the boundary condition satisfied by the limit velocity field, which depends, as expected, on the ratio between the value of the viscosity and the thickness of the depletion layer.

1.2 The model

The model we will study is as follows. Throughout the paper, we consider a spatial domain which is periodic with period $(0, 1)^2$ in the horizontal direction. More precisely, let $\mathbf{T}^2 = (\mathbf{R}/\mathbf{Z})^2$ be the two-dimensional torus and define $\Omega = \mathbf{T}^2 \times (0, 1)$. We note $x = (x', x_3) \in \Omega$ the macroscopic variable, of horizontal component $x' = (x_1, x_2)$. Analogously, any vector field u defined on Ω will be decomposed as $u = (u', u_3)$ with $u_3 = u \cdot e_3$ and $u' = u - u_3 e_3$, where (e_1, e_2, e_3) is the canonical basis of \mathbf{R}^3 . We denote by

- $\Gamma^t = \mathbf{T}^2 \times \{1\}$ the upper boundary of Ω ,
- $\Gamma^b = \mathbf{T}^2 \times \{0\}$ the lower boundary,

The depletion layer will thus be located around Γ^t and we will model it to have a typical size of ε . More precisely, let us introduce a function $d : \mathbf{T}^2 \rightarrow \mathbf{R}$ such that

$$d \in W^{1,\infty}(\mathbf{T}^2) \quad \text{and} \quad d > 0 \text{ in } \mathbf{T}^2, \quad (1)$$

and the function γ_ε defined on \mathbf{T}^2 by

$$\gamma_\varepsilon(x') = 1 - \varepsilon d(x'). \quad (2)$$

We denote by Γ_ε the graph of γ_ε , defined by

$$\Gamma_\varepsilon = \{(x', \gamma_\varepsilon(x')), x' \in \mathbf{T}^2\}. \quad (3)$$

In our setting, the depletion layer is defined as

$$B_\varepsilon = \{(x', x_3) \in \Omega, \gamma_\varepsilon(x') < x_3 < 1\} \subset \Omega.$$

To simplify notation we will note

$$\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon}.$$

To the macroscopic variable $x = (x', x_3) \in B_\varepsilon$, we associate the microscopic variable $y = (y', y_3)$ defined by

$$y' = x', \quad y_3 = \frac{1 - x_3}{\varepsilon}. \quad (4)$$

The depletion layer is then described in microscopic variable by

$$B_\varepsilon = \{(y', 1 - \varepsilon y_3), (y', y_3) \in \omega\},$$

where ω is defined by

$$\omega = \{(y', y_3) \in \mathbf{T}^2 \times (0, 1), 0 < y_3 < d(y')\}.$$

We denote by

- $\widehat{\Gamma}^t = \{(y', d(y')), y' \in \mathbf{T}^2\}$ the upper boundary of ω ,
- $\widehat{\Gamma}^b = \{(y', 0), y' \in \mathbf{T}^2\}$ the lower boundary,
- $\widehat{\Gamma}^\ell = \partial\Omega \setminus (\widehat{\Gamma}^t \cup \widehat{\Gamma}^b)$ the lateral boundary.

Let u_ε be the velocity of the fluid and p_ε the pressure, and let f be a right-hand side defined on Ω . The viscosity μ_ε is defined by

$$\mu_\varepsilon(x', x_3) = \mathbf{1}_{\Omega_\varepsilon}(x', x_3) + \varepsilon^\alpha \mathbf{1}_{B_\varepsilon}(x', x_3) \quad (5)$$

where $\mathbf{1}_{\Omega_\varepsilon}$ and $\mathbf{1}_{B_\varepsilon}$ stand for the indicator functions of Ω_ε and B_ε respectively. We will work with Stokes equation

$$-\operatorname{div}(2\mu_\varepsilon D(u_\varepsilon)) + \nabla p_\varepsilon = f \quad \text{in } \Omega \quad (6)$$

$$\operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega \quad (7)$$

In equation (6), $D(u_\varepsilon)$ stands for the symmetric part of the Jacobian matrix of u_ε , defined by

$$[D(u_\varepsilon)]_{i,j} = \frac{1}{2}(\partial_j u_{\varepsilon,i} + \partial_i u_{\varepsilon,j}) \quad 1 \leq i, j \leq 3.$$

As usual this system is supplemented with boundary conditions. For the sake of simplicity we have chosen to put no-slip boundary condition at the bottom Γ^b . So in the sequel we will always have

$$u_\varepsilon = 0 \quad \text{on } \Gamma^b, \quad (8)$$

as well as periodic boundary conditions on Γ^ℓ .

Finally our main concern is with the boundary condition on Γ^t . Since our aim is to determine if the drop of viscosity occurring in the depletion layer B_ε (see formula (5)) may increase slippage in the limit $\varepsilon \rightarrow 0$, we consider the least favorable situation where no-slip is also imposed on Γ^t , that is

$$u_\varepsilon = 0 \quad \text{on } \Gamma^t. \quad (9)$$

1.3 Functional spaces and useful inequalities

We denote by V_D the space

$$V_D = \{v \in H_0^1(\Omega, \mathbf{R}^3), \operatorname{div} v = 0 \text{ a.e. in } \Omega\},$$

which is the space where we define solutions of problem (6)–(9). We will also use the space V_N where the prescribed boundary condition on Γ^t is the non penetration:

$$V_N = \{v \in H^1(\Omega, \mathbf{R}^3), \operatorname{div} v = 0 \text{ a.e. in } \Omega, v = 0 \text{ on } \Gamma^b, v_3 = 0 \text{ on } \Gamma^t\}.$$

In the above definitions, to lighten the notation, we do not use any special symbol to emphasize that in this context, the Sobolev spaces $H^1(\Omega, \mathbf{R}^3)$ and $H_0^1(\Omega, \mathbf{R}^3)$ contain functions that are $(0, 1)^2$ -periodic with respect to the x' variable. Throughout the paper, any function of variable $x = (x', x_3)$ or $y = (y', y_3)$ will be tacitly periodic in the horizontal variable.

Let us state some well-known inequalities that will be often used in the sequel, where C stands for a nonnegative constant independent on ε .

Poincaré inequalities Since the family Ω_ε is uniformly bounded in the vertical direction, there holds for any $\varepsilon > 0$ and $v \in H^1(\Omega_\varepsilon)$ such that $v = 0$ on Γ^b ,

$$\int_{\Omega_\varepsilon} |v|^2 \leq C \int_{\Omega_\varepsilon} |\nabla v|^2. \quad (10)$$

Finally, since the domains B_ε have thickness ε , for any $\varepsilon > 0$ and $v \in H^1(B_\varepsilon)$ such that $v = 0$ on Γ^t ,

$$\int_{B_\varepsilon} |v|^2 \leq C\varepsilon^2 \int_{B_\varepsilon} |\nabla v|^2. \quad (11)$$

Korn inequalities In V_D , there holds Korn inequality

$$\forall v \in V_D \quad \int_{\Omega} |\nabla v|^2 \leq 2 \int_{\Omega} |D(v)|^2 \quad (12)$$

where ∇v is the Jacobian matrix of v . This is indeed the classical Korn inequality in $H_0^1(\Omega, \mathbf{R}^3)$ for any bounded domain (see for instance [8, Remark IV.7.3]).

In Ω_ε , we will apply the following version of Korn inequality, which is proved in [13] in a very similar geometric setting. Let $L > 0$ such that

$$\forall \varepsilon > 0 \quad \forall x' \in [0, 1]^2 \quad |\nabla \gamma_\varepsilon(x')| \leq L.$$

There exists a constant C_K (depending solely on L) such that for any function $v \in H^1(\Omega_\varepsilon, \mathbf{R}^3)$ satisfying $v = 0$ on Γ^b ,

$$\int_{\Omega_\varepsilon} |\nabla v|^2 \leq C_K \int_{\Omega_\varepsilon} |D(v)|^2. \quad (13)$$

The fact that the constant C_K in Korn inequality (13) depends only on the Lipschitz constant of γ_ε was first remarked by Nitsche [35].

As a consequence, the following uniform Korn inequality is also valid in B_ε : there exists a constant C_K (depending solely on L) such that for any function $v \in H^1(B_\varepsilon, \mathbf{R}^3)$ satisfying $v = 0$ on Γ^t ,

$$\int_{B_\varepsilon} |\nabla v|^2 \leq C_K \int_{B_\varepsilon} |D(v)|^2. \quad (14)$$

Indeed, extending by zero such function v for $y_3 > 1$ and replacing variable x_3 by $1 - x_3$, we can apply (13) to deduce (14).

2 Main results

Existence and uniqueness of u_ε From standard theory on Stokes equation, because we have the global Korn inequality (12), we can rely on the following theorem:

Proposition 1. *Let f be in $L^2(\Omega, \mathbf{R}^3)$. There exists a unique function u_ε in V_D solution of the following problem*

$$\begin{cases} -\operatorname{div}(2\mu_\varepsilon D(u_\varepsilon)) + \nabla p_\varepsilon = f & \text{in } \Omega \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \Gamma^t \cup \Gamma^b \end{cases} \quad (15)$$

which means that for all $\phi \in V_D$,

$$\int_{\Omega_\varepsilon} 2D(u_\varepsilon) : D(\phi) + \varepsilon^\alpha \int_{B_\varepsilon} 2D(u_\varepsilon) : D(\phi) = \int_{\Omega} f \cdot \phi. \quad (16)$$

Energy bound We say that a family of solutions $u_\varepsilon \in V_D$ to (16) satisfies the *energy bound* if there exists a constant $C > 0$ such that

$$\int_{\Omega_\varepsilon} |D(u_\varepsilon)|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |D(u_\varepsilon)|^2 \leq C. \quad (17)$$

Proposition 2. *Let f be in $L^2(\Omega, \mathbf{R}^3)$. Then for any $0 < \alpha \leq 2$, the weak solution u_ε to (15) satisfies the energy bound (17).*

Proof of Proposition 2.

Let $f \in L^2(\Omega)$ and $u_\varepsilon \in H_0^1(\Omega, \mathbf{R}^3)$ be the weak solution to (15). Combining Poincaré inequality (21) with Korn inequality (13), and (11) with (14), there holds

$$\begin{aligned} \int_{\Omega_\varepsilon} |u_\varepsilon|^2 dx &\leq C \int_{\Omega_\varepsilon} |D(u_\varepsilon)|^2 dx, \\ \int_{B_\varepsilon} |u_\varepsilon|^2 dx &\leq C\varepsilon^2 \int_{B_\varepsilon} |D(u_\varepsilon)|^2 dx. \end{aligned}$$

Summing the previous inequalities and testing against $\phi = u_\varepsilon$ in the weak formulation (16), we deduce that for $0 < \alpha \leq 2$,

$$\begin{aligned} \int_{\Omega} |u_\varepsilon|^2 dx &\leq C \left(\int_{\Omega_\varepsilon} |D(u_\varepsilon)|^2 dx + \varepsilon^2 \int_{B_\varepsilon} |D(u_\varepsilon)|^2 dx \right) \\ &\leq C \left(\int_{\Omega_\varepsilon} |D(u_\varepsilon)|^2 dx + \varepsilon^\alpha \int_{B_\varepsilon} |D(u_\varepsilon)|^2 dx \right) \\ &\leq C \int_{\Omega} f \cdot u_\varepsilon dx. \end{aligned}$$

Hence, by Cauchy-Schwarz inequality, $\int_{\Omega} |u_\varepsilon|^2 dx \leq C$, which in turn implies the energy bound (17).

Now we are in a position to state the main result of the paper. To express the boundary condition that we obtain on Γ^t , we need some extra notation: for a vector field ξ defined on Γ^t , we denote by $[\xi]_{tan}$ its tangential part, defined by $[\xi]_{tan} := \xi - (\xi \cdot e_3)e_3$.

Theorem 1. *Let $f \in L^2(\Omega)$, $\alpha \in (0, 2]$ and let $(u_\varepsilon)_\varepsilon$ be the family of solutions obtained from Proposition 2. Then u_ε converges weakly in $L^2(\Omega)$ toward a function \bar{u} such that*

if $0 < \alpha < 1$, \bar{u} is the variational solution to the problem:

$$\begin{cases} -\operatorname{div}(2D(\bar{u})) + \nabla p = f & \text{in } \Omega \\ \operatorname{div} \bar{u} = 0 & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \Gamma^t \cup \Gamma^b \end{cases} \quad (18)$$

if $\alpha = 1$, \bar{u} is the variational solution to the problem:

$$\begin{cases} -\operatorname{div}(2D(\bar{u})) + \nabla p = f & \text{in } \Omega \\ \operatorname{div} \bar{u} = 0 & \text{in } \Omega \\ \bar{u}_3 = 0, \quad 2[D(\bar{u})e_3]_{tan} + \frac{1}{d(x')} \bar{u}' = 0 & \text{on } \Gamma^t \\ \bar{u} = 0 & \text{on } \Gamma^b \end{cases} \quad (19)$$

if $1 < \alpha \leq 2$, \bar{u} is the variational solution to the problem:

$$\begin{cases} -\operatorname{div}(2D(\bar{u})) + \nabla p = f & \text{in } \Omega \\ \operatorname{div} \bar{u} = 0 & \text{in } \Omega \\ \bar{u}_3 = 0, \quad 2[D(\bar{u})e_3]_{tan} = 0 & \text{on } \Gamma^t \\ \bar{u} = 0 & \text{on } \Gamma^b \end{cases} \quad (20)$$

3 Compactness results on a sequence (u_ε) satisfying the energy bound (17)

In this section, we consider a general sequence $(u_\varepsilon) \in V_D$ satisfying the energy bound (17), and gather compactness results that we will use in the proof of Theorem 1.

3.1 Modifying u_ε in B_ε

Since u_ε vanishes on Γ^b , by Poincaré inequality, there exists a constant $C > 0$ such that

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^2 \leq C \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2. \quad (21)$$

In this inequality, the constant C can be chosen independent of ε since the domains $\Omega \setminus B_\varepsilon$ are uniformly bounded in the x_3 direction. Combining (21) with the energy bound (17) and the uniform Korn inequality (13), we obtain the existence of a constant $C > 0$ such that

$$\forall \varepsilon > 0 \quad \int_{\Omega_\varepsilon} |u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \leq C. \quad (22)$$

Since the family of functions γ_ε defined by (2) is uniformly Lipschitz, there exists a family of linear extension operators

$$E_\varepsilon : H^1(\Omega_\varepsilon, \mathbf{R}^3) \rightarrow H^1(\Omega, \mathbf{R}^3)$$

such that for any $\varepsilon > 0$,

$$\forall w \in H^1(\Omega_\varepsilon, \mathbf{R}^3) \quad E_\varepsilon(w)|_{\Omega_\varepsilon} = w$$

and a constant $C > 0$ such that

$$\forall \varepsilon > 0 \quad \|E_\varepsilon\|_{\mathcal{L}(H^1(\Omega_\varepsilon, \mathbf{R}^3), H^1(\Omega, \mathbf{R}^3))} \leq C$$

(see for instance [20]). Throughout the paper, we will note \bar{u}_ε the function in $H^1(\Omega, \mathbf{R}^3)$ defined by

$$\bar{u}_\varepsilon = E_\varepsilon(u_\varepsilon). \quad (23)$$

3.2 Existence of a limit $\bar{u} \in V_N$

Combining the previous bound with (22), we get that the sequence $(\bar{u}_\varepsilon)_\varepsilon$ is bounded in $H^1(\Omega, \mathbf{R}^3)$. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exists a function \bar{u} in $H^1(\Omega, \mathbf{R}^3)$ such that, up to extracting a subsequence, \bar{u}_ε converges to \bar{u} weakly in $H^1(\Omega, \mathbf{R}^3)$ and strongly in $L^2(\Omega, \mathbf{R}^3)$.

Let us show that \bar{u} is at least in V_N . The following Lemma will be useful several times:

Lemma 1. *Let Λ be Γ^t or Γ^b . Denote by $T_\Lambda : H^1(\Omega, \mathbf{R}^3) \rightarrow H^{1/2}(\Lambda, \mathbf{R}^3)$ the trace operator. Then $T_\Lambda \bar{u}_\varepsilon$ converges strongly to $T_\Lambda \bar{u}$ in $L^2(\Lambda, \mathbf{R}^3)$.*

Indeed, T_Λ is linear and strongly continuous and since \bar{u}_ε converges weakly to \bar{u} in $H^1(\Omega, \mathbf{R}^3)$ then $T_\Lambda \bar{u}_\varepsilon$ converges weakly to $T_\Lambda \bar{u}$ in $H^{1/2}(\Lambda, \mathbf{R}^3)$ (see, for instance, [9, Theorem III.9]). But since the embedding from $H^{1/2}(\Lambda)$ into $L^2(\Lambda)$ is compact, then $T_\Lambda \bar{u}_\varepsilon$ converges strongly to $T_\Lambda \bar{u}$ in $L^2(\Lambda, \mathbf{R}^3)$ (see for instance [37, Theorem VI.11]).

In the case $\Lambda = \Gamma^b$, since we know that $T_\Lambda \bar{u}_\varepsilon = 0$ for all ε , and that $T_\Lambda \bar{u}_\varepsilon$ converges weakly to $T_\Lambda \bar{u}$ in $H^{1/2}(\Lambda, \mathbf{R}^3)$, we get that $T_\Lambda \bar{u} = 0$ in $H^{1/2}(\Lambda, \mathbf{R}^3)$. Hence, $\bar{u} = 0$ on Γ^b .

Now let us establish that $\operatorname{div} \bar{u} = 0$ a.e. in Ω . Let $\zeta \in C_c^\infty(\Omega, \mathbf{R}^3)$. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, ζ is supported in Ω_ε . Using integration by part, the convergence of \bar{u}_ε to \bar{u} in $L^2(\Omega, \mathbf{R}^3)$, the fact that \bar{u}_ε and u_ε coincide in Ω_ε and the incompressibility condition $\operatorname{div} u_\varepsilon = 0$ in Ω , we obtain

$$\begin{aligned} \int_{\Omega} (\operatorname{div} \bar{u}) \zeta &= - \int_{\Omega} \bar{u} \cdot \nabla \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \left(- \int_{\Omega} \bar{u}_\varepsilon \cdot \nabla \zeta \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left(- \int_{\Omega_\varepsilon} \bar{u}_\varepsilon \cdot \nabla \zeta \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left(- \int_{\Omega_\varepsilon} u_\varepsilon \cdot \nabla \zeta \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega_\varepsilon} (\operatorname{div} u_\varepsilon) \zeta \right) = 0. \end{aligned}$$

3.3 Case $0 < \alpha < 1$: obtention of the boundary condition $\bar{u} = 0$ on Γ^t

By the energy bound (17) and Korn inequality (14), there holds

$$\int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \leq C \varepsilon^{-\alpha}.$$

We claim that if $0 < \alpha < 1$, the limit \bar{u} actually satisfies the same boundary condition

$$\bar{u} = 0 \quad \text{on } \Gamma^t,$$

in other words, $\bar{u} \in V_D$. The distinction between the cases $\alpha < 1$ and $\alpha \geq 1$ comes from the following estimate of the L^2 -norm of the trace of u_ε on Γ_ε (defined by (3)):

$$\int_{\Gamma_\varepsilon} |u_\varepsilon|^2 \leq C \varepsilon^{1-\alpha}. \tag{24}$$

The previous inequality can be obtained using the energy bound, the boundary condition $u_\varepsilon(x', 1) = 0$ for a.e. $x' \in (0, 1)^2$ and integrating on vertical lines $\{x'\} \times (1 - \varepsilon d(x'), 1)$, as

follows.

$$\begin{aligned}
\int_{\Gamma_\varepsilon} |u_\varepsilon|^2 &= \int_{(0,1)^2} |u_\varepsilon(x', 1 - \varepsilon d(x'))|^2 \sqrt{1 + \varepsilon^2 |\nabla d(x')|^2} \, dx' \\
&\leq \sqrt{1 + \varepsilon^2 \|\nabla d\|_\infty^2} \int_{(0,1)^2} |u_\varepsilon(x', 1 - \varepsilon d(x'))|^2 \, dx' \\
&\leq \sqrt{1 + \varepsilon^2 \|\nabla d\|_\infty^2} \int_{(0,1)^2} \left(\int_{1-\varepsilon d(x')}^1 |\partial_3 u_\varepsilon(x', s)| \, ds \right)^2 \, dx' \\
&\leq \sqrt{1 + \varepsilon^2 \|\nabla d\|_\infty^2} \varepsilon \|d\|_\infty \int_{(0,1)^2} \int_{1-\varepsilon d(x')}^1 |\partial_3 u_\varepsilon(x', s)|^2 \, ds \, dx' \\
&\leq C \varepsilon \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \leq C \varepsilon^{1-\alpha}.
\end{aligned}$$

Applying a similar argument to the function \bar{u}_ε , which is not vanishing on Γ^t , we obtain the following trace inequality:

$$\int_{\Gamma^t} |\bar{u}_\varepsilon|^2 \leq C \left(\varepsilon \int_{B_\varepsilon} |\nabla \bar{u}_\varepsilon|^2 + \int_{\Gamma_\varepsilon} |\bar{u}_\varepsilon|^2 \right).$$

Since \bar{u}_ε is bounded in $H^1(\Omega)$, and \bar{u}_ε and u_ε have the same trace on Γ_ε , (24) implies that

$$\int_{\Gamma^t} |\bar{u}_\varepsilon|^2 \leq C (\varepsilon + \varepsilon^{1-\alpha}).$$

Hence, if $0 < \alpha < 1$, $\lim_{\varepsilon \rightarrow 0} \int_{\Gamma^t} |\bar{u}_\varepsilon|^2 = 0$.

Combining this information with Lemma 1, we get that \bar{u} vanishes on Γ^t . Since we have already proven that \bar{u} is in V_N , this proves that $\bar{u} \in V_D$ if $0 < \alpha < 1$.

3.4 Case $\alpha \geq 1$: construction of the rescaled function v_ε

In the case $\alpha \geq 1$, we need to do a finer analysis of the behaviour of u_ε in B_ε . In order to work only with free divergence vector field, we introduce a rescaled function v_ε , depending on the micro-variable $y = (y', y_3) \in \omega$ and defined by $v_\varepsilon(y) = (v'_\varepsilon(y), v_{\varepsilon,3}(y))$ with

$$v'_\varepsilon(y) = \varepsilon^{(\alpha+3)/2} u'_\varepsilon(y', 1 - \varepsilon y_3), \quad v_{\varepsilon,3}(y) = -\varepsilon^{(\alpha+1)/2} u_{\varepsilon,3}(y', 1 - \varepsilon y_3) \quad \text{for a.e. } y \in \omega. \quad (25)$$

The derivatives of v_ε are given by

$$\begin{aligned}
\nabla_{y'} v'_\varepsilon(y) &= \varepsilon^{(\alpha+3)/2} \nabla_{x'} u'_\varepsilon(y', 1 - \varepsilon y_3), & \partial_{y_3} v'_\varepsilon(y) &= -\varepsilon^{(\alpha+5)/2} \partial_{x_3} u'_\varepsilon(y', 1 - \varepsilon y_3), \\
\nabla_{y'} v_{\varepsilon,3}(y) &= -\varepsilon^{(\alpha+1)/2} \nabla_{x'} u_{\varepsilon,3}(y', 1 - \varepsilon y_3), & \partial_{y_3} v_{\varepsilon,3}(y) &= \varepsilon^{(\alpha+3)/2} \partial_{x_3} u_{\varepsilon,3}(y', 1 - \varepsilon y_3).
\end{aligned}$$

Using the change of variable (4) and the energy bound, we obtain the following estimates:

$$\int_\omega |\nabla_{y'} v_{\varepsilon,3}|^2 \, dy = \varepsilon^\alpha \int_{B_\varepsilon} |\nabla_{x'} u_{\varepsilon,3}|^2 \, dx \leq C, \quad (26)$$

$$\int_\omega \left(|\nabla_{y'} v'_\varepsilon|^2 + |\partial_{y_3} v_{\varepsilon,3}|^2 \right) \, dy = \varepsilon^{\alpha+2} \int_{B_\varepsilon} \left(|\nabla_{x'} u'_\varepsilon|^2 + |\partial_{x_3} u_{\varepsilon,3}|^2 \right) \, dx \leq C \varepsilon^2, \quad (27)$$

$$\int_\omega |\partial_{y_3} v'_\varepsilon|^2 \, dy = \varepsilon^{\alpha+4} \int_{B_\varepsilon} |\partial_{x_3} u'_\varepsilon|^2 \, dx \leq C \varepsilon^4. \quad (28)$$

Since u_ε vanishes on Γ^t , there holds $v_\varepsilon(y', 0) = 0$ for a.e. $y' \in (0, 1)^2$. In particular, by Poincaré inequality, there exists a constant $C > 0$ such that

$$\forall \varepsilon > 0 \quad \int_\omega |v_\varepsilon|^2 \, dy \leq C \int_\omega |D_y v_\varepsilon|^2 \, dy.$$

As a result, the previous estimates yield that v_ε is uniformly bounded in $H^1(\omega, \mathbf{R}^3)$. Using similar arguments as for the sequence \bar{u}_ε , we deduce that, up to extraction, v_ε converges weakly in $H^1(\omega, \mathbf{R}^3)$ and strongly in $L^2(\omega, \mathbf{R}^3)$ to a function $v \in H^1(\omega, \mathbf{R}^3)$ such that

$$\operatorname{div} v = 0 \quad \text{in } \omega \quad \text{and} \quad v = 0 \quad \text{on } y_3 = 0.$$

Moreover, by weak lower semicontinuity of the L^2 norm in ω , estimates (27) and (28) imply that $\nabla_{y'} v'$ and $\partial_{y_3} v$ vanish in ω . In particular, v_3 depends only on y' and v' is a constant, which is necessarily equal to zero due to the boundary condition on $y_3 = 0$.

Let us prove that $v_3(y') = 0$ for a.e. $y' \in (0, 1)^2$, so that v vanishes. Since $v_{\varepsilon,3}$ converges weakly to v_3 in $H^1(\omega)$, by continuity of the trace, $v_{\varepsilon,3}(y', 0)$ converges to $v_3(y', 0)$ in $L^2((0, 1)^2)$. But by definition (25) and the Dirichlet boundary condition satisfied by u_ε on Γ^t , $v_{\varepsilon,3}(y', 0) = 0$ a.e. in $(0, 1)^2$. Hence, $v_3 = 0$ on $y_3 = 0$. Since v_3 is independent on y_3 , v_3 vanishes in ω .

Using that $\frac{1}{\varepsilon} \nabla_{y'} v'_\varepsilon$, $\frac{1}{\varepsilon} \partial_{y_3} v_{\varepsilon,3}$ and $\frac{1}{\varepsilon^2} \partial_{y_3} v'_\varepsilon$ are bounded in $L^2(\omega, \mathbf{R}^{2 \times 2})$, $L^2(\omega)$ and $L^2(\omega, \mathbf{R}^2)$ respectively, we finally obtain the existence of functions $M \in L^2(\omega, \mathbf{R}^{2 \times 2})$, $m_3 \in L^2(\omega)$ and $m' \in L^2(\omega, \mathbf{R}^2)$ such that up to extraction,

$$\begin{aligned} \frac{1}{\varepsilon} \nabla_{y'} v'_\varepsilon &\rightharpoonup M \quad \text{weakly in } L^2(\omega, \mathbf{R}^{2 \times 2}), \\ \frac{1}{\varepsilon} \partial_{y_3} v_{\varepsilon,3} &\rightharpoonup m_3 \quad \text{weakly in } L^2(\omega), \\ \frac{1}{\varepsilon^2} \partial_{y_3} v'_\varepsilon &\rightharpoonup m' \quad \text{weakly in } L^2(\omega, \mathbf{R}^2). \end{aligned}$$

In order to obtain more information on the previous limits, we introduce estimates in the space $H^1(0, 1; L^2((0, 1)^2))$. To lighten the presentation, we start with the case of a flat interface before giving the result in the general case.

Case of a flat interface: $d(x') \equiv 1$.

Estimates in $H^1(0, 1; L^2((0, 1)^2))$. We refer to [19, Chapter 1] for the main properties of vector valued Sobolev spaces.

Take $d(x') \equiv 1$ for a start (so $\omega = \mathbf{T}^2 \times (0, 1)$). We claim that $\frac{1}{\varepsilon} v_{\varepsilon,3}$ is bounded in $H^1(0, 1; L^2((0, 1)^2))$. Indeed, there holds $\int_\omega |\frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3}|^2 \, dy \leq C$ and $\frac{1}{\varepsilon} v_{\varepsilon,3}(y', 0) = 0$ a.e. in $(0, 1)^2$, so by integration on vertical lines:

$$\int_\omega |\frac{1}{\varepsilon} v_{\varepsilon,3}(y)|^2 \, dy \leq \int_\omega |\frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3}(y)|^2 \, dy \leq C.$$

Hence, up to extraction, there exists $w_3 \in H^1(0, 1; L^2((0, 1)^2))$ such that $\frac{1}{\varepsilon} v_{\varepsilon,3} \rightharpoonup w_3$ weakly in $H^1(0, 1; L^2((0, 1)^2))$, so that

$$\frac{1}{\varepsilon} v_{\varepsilon,3} \rightharpoonup w_3 \quad \text{and} \quad \frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3} \rightharpoonup \partial_3 w_3 \quad \text{weakly in } L^2(\omega). \quad (29)$$

In particular, $m_3 = \partial_3 w_3$. Also, by continuity of the trace operator from $H^1(0, 1; L^2((0, 1)^2)) \rightarrow L^2(\{0\} \times (0, 1)^2)$, $w_3 = 0$ on $y_3 = 0$. Finally, by continuity of the injection of $H^1(0, 1; L^2((0, 1)^2))$ in $L^\infty(0, 1; L^2((0, 1)^2))$, there exists a constant $C > 0$ such that for any $\varepsilon > 0$,

$$\left\| \frac{1}{\varepsilon} v_{\varepsilon,3} \right\|_{L^\infty(0,1;L^2((0,1)^2))} \leq C \left\| \frac{1}{\varepsilon} v_{\varepsilon,3} \right\|_{H^1(0,1;L^2((0,1)^2))} \leq C.$$

By an analogous argument, for $j = 1, 2$, there exist $w_j \in H^1(0, 1; L^2((0, 1)^2))$ such that $\frac{1}{\varepsilon^2} v_{\varepsilon,j} \rightharpoonup w_j$ weakly in $H^1(0, 1; L^2((0, 1)^2))$, in other words

$$\frac{1}{\varepsilon^2} v_{\varepsilon,j} \rightharpoonup w_j \quad \text{and} \quad \frac{1}{\varepsilon^2} \partial_3 v_{\varepsilon,j} \rightharpoonup \partial_3 w_j \quad \text{weakly in } L^2(\omega) \quad (30)$$

which implies that $m' = \partial_3 w'$, and $w' = 0$ on $y_3 = 0$. Besides,

$$\left\| \frac{1}{\varepsilon^2} v'_\varepsilon \right\|_{L^\infty(0,1;L^2((0,1)^2, \mathbf{R}^2))} \leq C.$$

Use of the free divergence condition on v_ε By the convergences (29) and (30), there holds

$$\frac{1}{\varepsilon^2} \operatorname{div}_{y'} v'_\varepsilon \rightharpoonup \operatorname{div}_{y'} w' \quad \text{and} \quad \frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3} \rightharpoonup \partial_3 w_3 \quad \text{weakly in } L^2(\omega).$$

Since $\operatorname{div}_y v_\varepsilon = 0$ in ω , we have for any function $\eta \in L^2(\omega)$,

$$0 = \int_\omega \left(\varepsilon \frac{1}{\varepsilon^2} \operatorname{div}_{y'} v'_\varepsilon + \frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3} \right) \eta(y) \, dy.$$

Passing to the limit in the previous relation yields $\int_\omega \partial_3 w_3 \eta = 0$, so $\partial_3 w_3 = 0$ a.e. in ω and w_3 is a function of y' only. Since we already established that w_3 vanishes on $y_3 = 0$, we conclude that $w_3 = 0$ a.e. in ω .

In the critical case $\alpha = 1$, coming back to the definition of $v_{\varepsilon,3}$ (25), we get that the sequence of functions $y \in \omega \mapsto u_{\varepsilon,3}(y', 1 - \varepsilon y_3)$ converges weakly to zero in $H^1(0, 1; L^2((0, 1)^2))$. Now, by continuity of the trace on $y_3 = 1$, we deduce that $y' \in (0, 1)^2 \mapsto u_{\varepsilon,3}(y', 1 - \varepsilon)$ converges weakly to zero in $L^2((0, 1)^2)$. By definition of \bar{u}_ε , this also holds for the sequence of functions $y' \in (0, 1)^2 \mapsto \bar{u}_{\varepsilon,3}(y', 1 - \varepsilon)$.

General case ($W^{1,\infty}$ interface). In the general case of a function $d(x')$, we can set

$$\tilde{v}_\varepsilon(z) = v_\varepsilon(z', d(z')z_3) \quad \text{for } z \in (0, 1)^3, \quad (31)$$

and notice that $\frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3}$ remains bounded in $H^1(0, 1; L^2((0, 1)^2))$. Hence, there exists \tilde{w}_3 in $H^1(0, 1; L^2((0, 1)^2))$ such that $\frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3} \rightharpoonup \tilde{w}_3$ weakly in $H^1(0, 1; L^2((0, 1)^2))$, yielding

$$\frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3} \rightharpoonup \tilde{w}_3 \quad \text{and} \quad \frac{1}{\varepsilon} \partial_3 \tilde{v}_{\varepsilon,3} \rightharpoonup \partial_3 \tilde{w}_3 \quad \text{weakly in } L^2((0, 1)^2 \times (0, 1)).$$

Moreover, $\tilde{w}_3 = 0$ on $z_3 = 0$. Now defining $w_3(y) = \tilde{w}_3(y', \frac{y_3}{d(y')})$ for $y \in \omega$, it is easy to check that w_3 and $\partial_3 w_3$ are in $L^2(\omega)$ and satisfy (29). For instance, taking $\eta \in C_c^1(\omega)$ and setting $(y', y_3) = (z', d(z')z_3)$, there holds

$$\begin{aligned} \int_\omega \frac{1}{\varepsilon} v_{\varepsilon,3}(y) \eta(y) \, dy &= \int_{(0,1)^3} \frac{1}{\varepsilon} v_{\varepsilon,3}(z', d(z')z_3) \eta(z', d(z')z_3) \, dz \\ &= \int_{(0,1)^3} \frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3}(z) \eta(z', d(z')z_3) \, dz. \end{aligned}$$

Since the function $z \mapsto \eta(z', d(z')z_3) d(z')$ is in $C_c^1((0, 1)^3)$, we can pass to limit by weak convergence of $\frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3}$ in $L^2((0, 1)^3)$, and obtain by definition of w_3

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\omega} \frac{1}{\varepsilon} v_{\varepsilon,3}(y) \eta(y) \, dy &= \int_{(0,1)^3} \tilde{w}_3(z) \eta(z', d(z')z_3) d(z') \, dz \\ &= \int_{\omega} w_3(y) \eta(y) \, dy. \end{aligned}$$

Also, $w_3 = 0$ on $y_3 = 0$.

By a similar argument, we can prove the existence of $\tilde{w}' \in H^1(0, 1; L^2((0, 1)^2))$ such that $\frac{1}{\varepsilon^2} \tilde{v}'_{\varepsilon}$ converges to \tilde{w}' weakly in $H^1(0, 1; L^2((0, 1)^2))$, and $\tilde{w}' = 0$ on $z_3 = 0$. Setting $w'(y) = \tilde{w}'(y', \frac{y_3}{d(y)})$, we deduce (30) (which implies that $m' = \partial_3 w'$) and that $w' = 0$ on $y_3 = 0$.

Using that $\operatorname{div}_y v_{\varepsilon} = 0$ in ω and the convergences (29) and (30), we conclude as in the case of a flat interface that w_3 is independent on y_3 , hence $w_3 = 0$ in ω . As a result, \tilde{w} vanishes in $(0, 1)^3$, so the sequence of functions $z' \in (0, 1)^2 \mapsto \frac{1}{\varepsilon} \tilde{v}_{\varepsilon,3}(z', 1)$ converges weakly to 0 in $L^2((0, 1)^2)$.

In the critical case $\alpha = 1$, coming back to the definitions of $v_{\varepsilon,3}$ (25) and $\tilde{v}_{\varepsilon,3}$ (31), we get that the sequence of functions $z \in (0, 1)^3 \mapsto u_{\varepsilon,3}(z', 1 - \varepsilon z_3 d(z'))$ converges weakly to zero in $H^1(0, 1; L^2((0, 1)^2))$. Now, by continuity of the trace on $z_3 = 1$, we deduce that $y' \in (0, 1)^2 \mapsto u_{\varepsilon,3}(y', 1 - \varepsilon d(y'))$ converges weakly to zero in $L^2((0, 1)^2)$. By definition of \bar{u}_{ε} , this also holds for the sequence of functions $y' \in (0, 1)^2 \mapsto \bar{u}_{\varepsilon,3}(y', 1 - \varepsilon d(y'))$.

Now we are in position to prove that the sequence $y' \in (0, 1)^2 \mapsto \bar{u}_{\varepsilon,3}(y', 1)$ converges weakly to 0 in $L^2((0, 1)^2)$, which implies that $\bar{u}_3 = 0$ on Γ^t .

Let $\eta = \eta(y') \in L^2((0, 1)^2)$. Integrating on vertical lines, we obtain

$$\begin{aligned} &\int_{(0,1)^2} \bar{u}_{\varepsilon,3}(y', 1) \eta(y') \, dy' \\ &= \int_{(0,1)^2} \left(\int_{1-\varepsilon d(y')}^1 \partial_3 \bar{u}_{\varepsilon,3}(y', y_3) \, dy_3 \right) \eta(y') \, dy' + \int_{(0,1)^2} \bar{u}_{\varepsilon,3}(y', 1 - \varepsilon d(y')) \eta(y') \, dy'. \end{aligned}$$

By the previous result, $\lim_{\varepsilon \rightarrow 0} \int_{(0,1)^2} \bar{u}_{\varepsilon,3}(y', 1 - \varepsilon d(y')) \eta(y') \, dy' = 0$. The other term can be estimated as follows:

$$\begin{aligned} \left| \int_{(0,1)^2} \left(\int_{1-\varepsilon d(y')}^1 \partial_3 \bar{u}_{\varepsilon,3}(y', y_3) \, dy_3 \right) \eta(y') \, dy' \right| &= \left| \int_{B_{\varepsilon}} \partial_3 \bar{u}_{\varepsilon,3}(y) \eta(y') \, dy \right| \\ &\leq \left(\int_{B_{\varepsilon}} |\partial_3 \bar{u}_{\varepsilon,3}|^2 \right)^{1/2} \left(\varepsilon \|d\|_{\infty} \int_{(0,1)^2} |\eta(y')|^2 \, dy' \right)^{1/2}. \end{aligned} \tag{32}$$

Since $\bar{u}_{\varepsilon,3}$ is uniformly bounded in $H^1(\Omega)$, we can pass to the limit and obtain

$$\lim_{\varepsilon \rightarrow 0} \left[\int_{(0,1)^2} \left(\int_{1-\varepsilon d(y')}^1 \partial_3 \bar{u}_{\varepsilon,3}(y', y_3) \, dy_3 \right) \eta(y') \, dy' \right] = 0.$$

We conclude that $\lim_{\varepsilon \rightarrow 0} \int_{(0,1)^2} \bar{u}_{\varepsilon,3}(y', 1) \eta(y') \, dy' = 0$.

Convergence in $H^2(0, 1; H^{-1}((0, 1)^2))$. Consider the flat configuration $d(x') \equiv 1$. We claim that there exists $w_3^{(2)} \in H^2(0, 1; H^{-1}((0, 1)^2))$ such that

$$\frac{1}{\varepsilon^2} v_{\varepsilon,3} \rightharpoonup w_3^{(2)} \quad \text{weakly in } H^2(0, 1; H^{-1}((0, 1)^2)). \tag{33}$$

It is a consequence of the fact that

$$\operatorname{div}_{y'} \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right) \rightharpoonup \operatorname{div}_{y'} w' \quad \text{weakly in } H^1(0, 1; H^{-1}((0, 1)^2)), \quad (34)$$

which can be proved as follows. Let $\phi \in C^\infty([0, 1], H_0^1((0, 1)^2))$. Using the incompressibility condition on v_ε , there holds

$$\begin{aligned} \int_0^1 \left\langle \partial_3(\operatorname{div}_{y'} \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right)), \partial_3 \phi \right\rangle_{H^{-1}((0,1)^2) \times H_0^1((0,1)^2)} dy_3 &= \int_0^1 \left\langle \operatorname{div}_{y'} \left(\partial_3 \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right) \right), \partial_3 \phi \right\rangle_{H^{-1} \times H_0^1} dy_3 \\ &= - \int_0^1 \int_{(0,1)^2} \partial_3 \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right) \cdot \nabla_{y'} (\partial_3 \phi) dy' dy_3 \\ &= - \int_\omega \partial_3 \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right) \cdot \nabla_{y'} (\partial_3 \phi) dy \end{aligned}$$

and the last integral converges to

$$\begin{aligned} - \int_\omega \partial_3 w' \cdot \nabla_{y'} (\partial_3 \phi) dy &= \int_0^1 \left\langle \operatorname{div}_{y'} (\partial_3 w'), \partial_3 \phi \right\rangle_{H^{-1} \times H_0^1} dy_3 \\ &= \int_0^1 \left\langle \partial_3(\operatorname{div}_{y'} w'), \partial_3 \phi \right\rangle_{H^{-1} \times H_0^1} dy_3. \end{aligned}$$

Also,

$$\begin{aligned} \int_0^1 \left\langle \operatorname{div}_{y'} w', \phi \right\rangle_{H^{-1} \times H_0^1} dy_3 &= - \int_\omega w' \cdot \nabla_{y'} \phi dy \\ &= \lim_{\varepsilon \rightarrow 0} \left(- \int_\omega \frac{1}{\varepsilon^2} v'_\varepsilon \cdot \nabla_{y'} \phi dy \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^1 \left\langle \operatorname{div}_{y'} \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right), \phi \right\rangle_{H^{-1} \times H_0^1} dy_3. \end{aligned}$$

(34) is proved.

Now, since $\operatorname{div}_y v_\varepsilon = 0$ in ω , there holds

$$\frac{1}{\varepsilon^2} \partial_3 v_{\varepsilon,3} = -\operatorname{div}_{y'} \left(\frac{1}{\varepsilon^2} v'_\varepsilon \right),$$

so that $\frac{1}{\varepsilon^2} \partial_3 v_{\varepsilon,3}$ is weakly convergent in $H^1(0, 1; H^{-1}((0, 1)^2))$. Since $\frac{1}{\varepsilon^2} v_{\varepsilon,3} = 0$ on $y_3 = 0$, it implies that $\frac{1}{\varepsilon^2} v_{\varepsilon,3}$ is bounded in $H^2(0, 1; H^{-1}((0, 1)^2))$, hence (33).

In the general case of a $W^{1,\infty}$ interface, we claim that there exists $\tilde{w}_3^{(2)}$ in $H^2(0, 1; H^{-1}((0, 1)^2))$ such that, up to extraction,

$$\frac{1}{\varepsilon^2} \tilde{v}_\varepsilon \cdot (-\nabla_{y'} d(z'), 1) \rightharpoonup \tilde{w}_3^{(2)} \quad \text{weakly in } H^2(0, 1; H^{-1}((0, 1)^2)), \quad (35)$$

where \tilde{v}_ε is defined by (31). This results from the fact that $\operatorname{div}_{z'} \tilde{v}'_\varepsilon$ can be expressed by

$$\begin{aligned} (\operatorname{div}_{z'} \tilde{v}'_\varepsilon)(z) &= (\operatorname{div}_{y'} v'_\varepsilon)(z', d(z')z_3) + \nabla_{y'} d(z') \cdot \partial_{y_3} v'_\varepsilon(z', d(z')z_3) \\ &= -\partial_{y_3} v_{\varepsilon,3}(z', d(z')z_3) + \nabla_{y'} d(z') \cdot \partial_{y_3} v'_\varepsilon(z', d(z')z_3) \\ &= \frac{1}{d(z')} \left(-\partial_{z_3} \tilde{v}_{\varepsilon,3}(z) + \nabla_{y'} d(z') \cdot \partial_{z_3} \tilde{v}'_\varepsilon(z) \right) \end{aligned}$$

so that

$$\partial_{z_3} \left(\frac{1}{\varepsilon^2} \tilde{v}_\varepsilon \cdot (-\nabla_{y'} d, 1) \right) = -\operatorname{div}_{z'} \left(\frac{1}{\varepsilon^2} \tilde{v}'_\varepsilon \right) \quad \text{a.e. in } (0, 1)^3.$$

By the same argument as above, the right hand side of this equality is bounded in $H^1(0, 1; H^{-1}((0, 1)^2))$, and since $\frac{1}{\varepsilon^2} \tilde{v}_\varepsilon$ vanishes on $z_3 = 0$, we deduce the claim (35).

Step towards Reynolds equation We claim that for any $\eta \in H^1((0,1)^2)$, the limit w' satisfies

$$\int_{(0,1)^2} \left(\int_0^{d(y')} w'(y', y_3) dy_3 \right) \cdot \nabla_{y'} \eta(y') dy' = 0. \quad (36)$$

By density, it is enough to prove (36) for $\eta = \eta(y') \in C^1([0,1]^2)$. Since $\operatorname{div} v_\varepsilon = 0$ in ω , using integration by parts, there holds

$$0 = \frac{1}{\varepsilon^2} \int_\omega (\operatorname{div} v_\varepsilon) \eta(y') dy = - \int_\omega \frac{1}{\varepsilon^2} v'_\varepsilon \cdot \nabla_{y'} \eta(y') dy.$$

Using (30), we can pass in the limit in the previous relation and obtain $\int_\omega w' \cdot \nabla_{y'} \eta = 0$, which by Fubini theorem can be rephrased as (36).

Boundary condition for w' on $y_3 = d(y')$, in the critical case $\alpha = 1$. To obtain the boundary condition satisfied by w' on $y_3 = d(y')$, we go back to the definition (25) and of \tilde{v}'_ε and \tilde{w}' , and observe that, in the critical case $\alpha = 1$,

$$\frac{1}{\varepsilon^2} \tilde{v}'_\varepsilon(z) = u'_\varepsilon(z', 1 - \varepsilon d(z') z_3) \quad \text{for a.e. } z \in (0,1)^3.$$

Hence, by weak convergence of $\frac{1}{\varepsilon^2} \tilde{v}'_\varepsilon$ to \tilde{w}' in $H^1(0,1; L^2((0,1)^2))$, and continuity of the trace on $z_3 = 1$, the sequence of function $y' \in (0,1)^2 \mapsto u'_\varepsilon(y', 1 - \varepsilon d(y'))$ converges weakly to $w'(y', d(y'))$ in $L^2((0,1)^2)$.

Taking $\eta' = \eta'(y') \in L^2((0,1)^2, \mathbf{R}^2)$, integrating on vertical lines, we obtain

$$\begin{aligned} & \int_{(0,1)^2} \bar{u}'_\varepsilon(y', 1) \cdot \eta'(y') dy' \\ &= \int_{(0,1)^2} \left(\int_{1-\varepsilon d(y')}^1 \partial_3 \bar{u}'_\varepsilon(y', y_3) dy_3 \right) \cdot \eta'(y') dy' + \int_{(0,1)^2} \bar{u}'_\varepsilon(y', 1 - \varepsilon d(y')) \cdot \eta'(y') dy'. \end{aligned}$$

Writing a similar inequality as (32), there holds

$$\lim_{\varepsilon \rightarrow 0} \left[\int_{(0,1)^2} \left(\int_{1-\varepsilon d(y')}^1 \partial_3 \bar{u}'_\varepsilon(y', y_3) dy_3 \right) \cdot \eta'(y') dy' \right] = 0.$$

Since $\lim_{\varepsilon \rightarrow 0} \int_{(0,1)^2} \bar{u}'_\varepsilon(y', 1) \cdot \eta'(y') dy' = \int_{(0,1)^2} \bar{u}'(y', 1) \cdot \eta'(y') dy'$, we can pass to the limit in the former inequality to obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{(0,1)^2} \bar{u}'_\varepsilon(y', 1 - \varepsilon d(y')) \cdot \eta'(y') dy' = \int_{(0,1)^2} \bar{u}'(y', 1) \cdot \eta'(y') dy'.$$

Since \bar{u}'_ε and u'_ε coincide on $y_3 = d(y')$, by uniqueness of the weak limit in $L^2((0,1)^2)$, this proves that

$$w'(y', d(y')) = \bar{u}'(y', 1) \quad \text{for a.e. } y' \in (0,1)^2. \quad (37)$$

Obtention of the boundary condition $\bar{u}_3 = 0$ on Γ^t in the supercritical case $1 < \alpha \leq 2$
Taking the trace on $z_3 = 1$ in (35), and coming back to variable y , we get that the sequence of functions

$$y' \in (0,1)^2 \mapsto -\frac{1}{\varepsilon^2} v'_\varepsilon(y', d(y')) \cdot \nabla_{y'} d(y') + \frac{1}{\varepsilon^2} v_{\varepsilon,3}(y', d(y'))$$

is weakly convergent in $H^{-1}((0,1)^2)$. Since the sequence

$$y' \in (0,1)^2 \mapsto \frac{1}{\varepsilon^2} v'_\varepsilon(y', d(y'))$$

is bounded in $L^2((0, 1)^2)$ (as the trace on $z_3 = 1$ of the function $\frac{1}{\varepsilon^2} \tilde{v}'_\varepsilon$, which is bounded in $H^1(0, 1; L^2((0, 1)^2))$), this implies that the sequence $\frac{1}{\varepsilon^2} v_{\varepsilon, 3}(y', d(y'))$ is also bounded in $H^{-1}((0, 1)^2)$. By definition (25), this means that the sequence

$$y' \in (0, 1)^2 \mapsto \varepsilon^{(\alpha-3)/2} u_{\varepsilon, 3}(y', d(y'))$$

is bounded in $H^{-1}((0, 1)^2)$. As a result, for any $\alpha < 3$, $u_{\varepsilon, 3}(\cdot, d(\cdot))$ converges to 0 in $H^{-1}((0, 1)^2)$, and since this sequence is in $L^2((0, 1)^2)$, it implies that for any $\eta \in C_c^1((0, 1)^2)$,

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{(0, 1)^2} u_{\varepsilon, 3}(y', d(y')) \eta(y') \, dy' \right) = 0.$$

From this step, we conclude as in the critical case that $\bar{u}_3 = 0$ on Γ^t .

4 Derivation of the Reynolds equation

4.1 Mixed formulation of the problem (15)

The mixed formulation of the problem reads:

$$\mathcal{A}_\varepsilon(u_\varepsilon, \phi) - \mathcal{B}(p_\varepsilon, \phi) = \mathcal{L}(\phi) \quad \forall \phi \in H_0^1(\Omega, \mathbf{R}^3), \quad (38)$$

$$\mathcal{B}(q, u_\varepsilon) = 0 \quad \forall q \in L^2(\Omega). \quad (39)$$

where :

- $\mathcal{A}_\varepsilon : H_0^1(\Omega, \mathbf{R}^3) \times H_0^1(\Omega, \mathbf{R}^3) \rightarrow \mathbf{R}$ is defined by

$$\mathcal{A}_\varepsilon(u, \phi) = \int_{\Omega} 2\mu_\varepsilon D(u) : D(\phi)$$

- $\mathcal{B} : L^2(\Omega) \times H_0^1(\Omega, \mathbf{R}^3) \rightarrow \mathbf{R}$ is defined by

$$\mathcal{B}(p, \phi) = \int_{\Omega} p \operatorname{div} \phi$$

- $\mathcal{L} : H_0^1(\Omega, \mathbf{R}^3) \rightarrow \mathbf{R}$ is defined by

$$\mathcal{L}(\phi) = \int_{\Omega} f \cdot \phi$$

Fix $\varepsilon > 0$. \mathcal{A}_ε and \mathcal{B} are continuous bilinear forms on $H_0^1(\Omega, \mathbf{R}^3) \times H_0^1(\Omega, \mathbf{R}^3)$ and $L^2(\Omega) \times H_0^1(\Omega, \mathbf{R}^3)$ respectively, and \mathcal{L} is a continuous linear form on $H_0^1(\Omega, \mathbf{R}^3)$. Moreover, by Korn inequalities (13) and (14), there holds

$$\mathcal{A}_\varepsilon(\phi, \phi) \geq \frac{C}{\varepsilon^\alpha} \int_{\Omega} |\nabla \phi|^2$$

where $C > 0$ is a constant, so \mathcal{A}_ε is coercive on $H_0^1(\Omega, \mathbf{R}^3)$.

By classical results (see for instance [26, paragraph 4.1]), the existence and uniqueness of a solution $(u_\varepsilon, p_\varepsilon) \in V_D \times L_0^2(\Omega)$ to the mixed formulation (38)-(39) can be established provided that the following inf-sup condition is fulfilled:

$$\exists \delta > 0, \text{ such that } \inf_{q \in L_0^2(\Omega) \setminus \{0\}} \sup_{v \in H_0^1(\Omega, \mathbf{R}^3) \setminus \{0\}} \frac{\mathcal{B}(q, v)}{\|q\|_{L_0^2(\Omega)} \|v\|_{H_0^1(\Omega, \mathbf{R}^3)}} \geq \delta. \quad (40)$$

By [24, Theorem III.3.1] (following the construction in [6]), for any $q \in L^2_0(\Omega)$, there exists a constant $C > 0$ and $w \in H^1_0(\Omega, \mathbf{R}^3)$ such that

$$\operatorname{div} w = q \quad \text{in } \Omega \quad \text{and} \quad \|w\|_{H^1_0(\Omega, \mathbf{R}^3)} \leq C \|q\|_{L^2(\Omega)}. \quad (41)$$

The inf-sup condition (40) is a direct consequence of that statement. Indeed, let $q \in L^2(\Omega) \setminus \{0\}$ and $w \in H^1_0(\Omega, \mathbf{R}^3)$ satisfying (41). Then

$$\begin{aligned} \sup_{v \in H^1_0(\Omega, \mathbf{R}^3) \setminus \{0\}} \frac{\mathcal{B}(q, v)}{\|q\|_{L^2_0(\Omega)} \|v\|_{H^1_0(\Omega, \mathbf{R}^3)}} &\geq \frac{\mathcal{B}(q, w)}{\|q\|_{L^2_0(\Omega)} \|w\|_{H^1_0(\Omega, \mathbf{R}^3)}} \\ &\geq \frac{\mathcal{B}(q, w)}{C \|q\|_{L^2_0(\Omega)}^2} \geq \frac{1}{C}. \end{aligned} \quad \text{Bogov}$$

Estimates on the pressure. In order to estimate $\|p_\varepsilon\|_{L^2(\Omega)}$, we consider a function $m \in L^2(\Omega)$ and the associated function $w \in H^1_0(\Omega, \mathbf{R}^3)$ such that $\operatorname{div} w = m - \int_\Omega m$ (in this case Ω is of volume 1) and $\|w\|_{H^1_0(\Omega, \mathbf{R}^3)} \leq C \|m - \int_\Omega m\|_{L^2(\Omega)}$. Using that $\int_\Omega p_\varepsilon = 0$, we can write

$$\begin{aligned} \int_\Omega p_\varepsilon m &= \int_\Omega p_\varepsilon \left(m - \int_\Omega m \right) \\ &= \int_\Omega p_\varepsilon \operatorname{div} w \\ &= \int_{\Omega_\varepsilon} 2 D(u_\varepsilon) : D(w) + \int_{B_\varepsilon} 2\varepsilon^\alpha D(u_\varepsilon) : D(w) - \int_\Omega f \cdot w. \end{aligned}$$

Hence, by Cauchy-Schwarz inequality, Poncaré inequality and Korn inequality (12), there exists a constant $C > 0$ such that

$$\left| \int_\Omega p_\varepsilon m \right| \leq C \left(\|D(u_\varepsilon)\|_{L^2(\Omega, \mathbf{R}^3)} + \|f\|_{L^2(\Omega, \mathbf{R}^3)} \right) \|w\|_{H^1_0(\Omega, \mathbf{R}^3)}.$$

Using the energy bound and that $\|m - \int_\Omega m\|_{L^2(\Omega)} \leq \|m\|_{L^2(\Omega)}$, we conclude that

$$\|p_\varepsilon\|_{L^2(\Omega)} \leq C(1 + \varepsilon^{-\alpha})^{1/2} \leq C\varepsilon^{-\alpha/2}.$$

We can also introduce a different pressure \bar{p}_ε defined in Ω , such that p_ε and \bar{p}_ε differ by a constant, and that $\int_{\Omega_\varepsilon} \bar{p}_\varepsilon = 0$ for any $\varepsilon > 0$. Now fix $\varepsilon > 0$ and consider $\bar{m} \in L^2(\Omega_\varepsilon)$ and $\bar{w} \in H^1_0(\Omega_\varepsilon, \mathbf{R}^3)$ such that $\operatorname{div} \bar{w} = \bar{m} - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \bar{m}$ in Ω_ε and $\|\nabla \bar{w}\|_{L^2(\Omega_\varepsilon, \mathbf{R}^{3 \times 3})} \leq C \|\bar{m} - \int_{\Omega_\varepsilon} \bar{m}\|_{L^2(\Omega_\varepsilon)}$ (the fact that C is independent on ε is proven in [13, Proposition 5.2]). Extending \bar{p}_ε and \bar{w} by zero in B_ε , there holds

$$\begin{aligned} \int_{\Omega_\varepsilon} \bar{p}_\varepsilon \bar{m} &= \int_{\Omega_\varepsilon} \bar{p}_\varepsilon \left(\bar{m} - \int_{\Omega_\varepsilon} \bar{m} \right) \\ &= \int_{\Omega_\varepsilon} \bar{p}_\varepsilon \operatorname{div} \bar{w} \\ &= \int_{\Omega_\varepsilon} 2 D(u_\varepsilon) : D(\bar{w}) - \int_{\Omega_\varepsilon} f \cdot \bar{w}. \end{aligned}$$

so that by similar arguments as above,

$$\|\bar{p}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C(\|D(u_\varepsilon)\|_{L^2(\Omega_\varepsilon, \mathbf{R}^{3 \times 3})} + \|f\|_{L^2(\Omega, \mathbf{R}^3)}).$$

Using the energy bound, we get that \bar{p}_ε is uniformly bounded in $L^2(\Omega)$, hence there exists $\bar{p} \in L^2(\Omega)$ such that, up to extraction,

$$\bar{p}_\varepsilon \rightharpoonup \bar{p} \quad \text{weakly in } L^2(\Omega).$$

Rescaled pressure in B_ε Define $q_\varepsilon : \omega \rightarrow \mathbf{R}$ by

$$q_\varepsilon(y) = \varepsilon^{(\alpha+1)/2} p_\varepsilon(y', 1 - \varepsilon y_3) \quad \text{for a.e. } y \in \omega. \quad (42)$$

Then,

$$\begin{aligned} \int_\omega |q_\varepsilon(y)|^2 dy &= \frac{1}{\varepsilon} \int_{B_\varepsilon} \varepsilon^{\alpha+1} |p_\varepsilon(x)|^2 dx \\ &= \varepsilon^\alpha \int_{B_\varepsilon} |p_\varepsilon|^2 \leq C. \end{aligned}$$

In particular, there exists $q \in L^2(\omega)$ such that, up to extraction,

$$q_\varepsilon \rightharpoonup q \quad \text{weakly in } L^2(\omega). \quad (43)$$

4.2 Limit equation in the depletion layer

Let us start with the critical case $\alpha = 1$. Let $\widehat{\phi} \in H_0^1(\omega, \mathbf{R}^3)$ and define $\phi_\varepsilon \in H_0^1(B_\varepsilon, \mathbf{R}^3)$ by

$$\phi'_\varepsilon(x) = \widehat{\phi}'\left(x', \frac{1-x_3}{\varepsilon}\right) \quad \text{and} \quad \phi_{\varepsilon,3}(x) = -\varepsilon \widehat{\phi}_3\left(x', \frac{1-x_3}{\varepsilon}\right).$$

We also have

$$u'_\varepsilon(x) = \frac{1}{\varepsilon^2} v'_\varepsilon\left(x', \frac{1-x_3}{\varepsilon}\right) \quad \text{and} \quad u_{\varepsilon,3}(x) = -\frac{1}{\varepsilon} v_{\varepsilon,3}\left(x', \frac{1-x_3}{\varepsilon}\right)$$

and

$$p_\varepsilon(x) = \frac{1}{\varepsilon} q_\varepsilon\left(x', \frac{1-x_3}{\varepsilon}\right).$$

Setting $\phi_\varepsilon = 0$ in Ω_ε and testing against ϕ_ε in the mixed formulation (38), we obtain the relation

$$2\varepsilon \int_{B_\varepsilon} D(u_\varepsilon) : D(\phi_\varepsilon) - \int_{B_\varepsilon} p_\varepsilon \operatorname{div} \phi_\varepsilon = \int_{B_\varepsilon} f \cdot \phi_\varepsilon.$$

Writing

$$\begin{aligned} D(u_\varepsilon) &= \begin{pmatrix} \frac{1}{\varepsilon^2} D_{y'}(v'_\varepsilon) & -\frac{1}{2} \left(\frac{1}{\varepsilon^3} \partial_{y_3} v'_\varepsilon + \frac{1}{\varepsilon} \nabla_{y'} v_{\varepsilon,3} \right) \\ -\frac{1}{2} \left(\frac{1}{\varepsilon^3} \partial_{y_3} v'_\varepsilon + \frac{1}{\varepsilon} \nabla_{y'} v_{\varepsilon,3} \right) & \frac{1}{\varepsilon^2} \partial_{y_3} v_{\varepsilon,3} \end{pmatrix} \\ D(\phi_\varepsilon) &= \begin{pmatrix} D_{y'}(\widehat{\phi}') & -\frac{1}{2} \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' + \varepsilon \nabla_{y'} \widehat{\phi}_3 \right) \\ -\frac{1}{2} \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' + \varepsilon \nabla_{y'} \widehat{\phi}_3 \right) & \partial_{y_3} \widehat{\phi}_3 \end{pmatrix} \end{aligned}$$

and using the change of variables $x' = y', x_3 = 1 - \varepsilon y_3$, we obtain

$$\begin{aligned} &\int_\omega \frac{1}{\varepsilon^2} \partial_3 v'_\varepsilon \cdot \partial_3 \widehat{\phi}' - \int_\omega q_\varepsilon \operatorname{div}_y \widehat{\phi} \\ &= \varepsilon \int_\omega f'(y', 1 - \varepsilon y_3) \cdot \widehat{\phi}'(y) dy - \varepsilon^2 \int_\omega f_3(y', 1 - \varepsilon y_3) \widehat{\phi}_3(y) dy \\ &\quad - 2\varepsilon^2 \int_\omega \frac{1}{\varepsilon^2} D_{y'}(v'_\varepsilon) : D_{y'}(\widehat{\phi}') - 2\varepsilon \int_\omega \frac{1}{\varepsilon} \partial_3 v_{\varepsilon,3} \partial_3 \widehat{\phi}_3 - \varepsilon^3 \int_\omega \frac{1}{\varepsilon} \nabla_{y'} v_{\varepsilon,3} \cdot \nabla_{y'} \widehat{\phi}_3 \\ &\quad - \varepsilon^2 \int_\omega \frac{1}{\varepsilon^2} \partial_3 v'_\varepsilon \cdot \nabla_{y'} \widehat{\phi}_3 - \varepsilon \int_\omega \frac{1}{\varepsilon} \nabla_{y'} v_{\varepsilon,3} \cdot \partial_3 \widehat{\phi}' \end{aligned} \quad (44)$$

For $1 \leq j \leq 3$, using Cauchy-Schwarz inequality and Poincaré inequality in ω , we obtain the estimate

$$\begin{aligned} \int_{\omega} f_j(y', 1 - \varepsilon y_3) \widehat{\phi}(y) \, dy &= \frac{1}{\varepsilon} \int_{B_\varepsilon} f_j(x) \widehat{\phi}_j(x', \frac{1-x_3}{\varepsilon}) \, dx \\ &\leq \frac{1}{\varepsilon} \left(\int_{B_\varepsilon} f_j(x)^2 \, dx \right)^{1/2} \left(\int_{B_\varepsilon} \widehat{\phi}_j(x', \frac{1-x_3}{\varepsilon})^2 \, dx \right)^{1/2} \\ &\leq \frac{1}{\varepsilon} \left(\int_{\Omega} f_j(x)^2 \, dx \right)^{1/2} \left(\varepsilon \int_{\omega} \phi_j(y)^2 \, dy \right)^{1/2} \\ &\leq C \varepsilon^{-1/2} \|f\|_{L^2(\Omega, \mathbf{R}^3)} \|\widehat{\phi}\|_{H_0^1(\omega, \mathbf{R}^3)}. \end{aligned}$$

In particular,

$$\begin{aligned} \left| \varepsilon \int_{\omega} f'(y', 1 - \varepsilon y_3) \cdot \widehat{\phi}'(y) \, dy \right| &\leq C \varepsilon^{1/2} \|f\|_{L^2(\Omega, \mathbf{R}^3)} \|\widehat{\phi}\|_{H_0^1(\omega, \mathbf{R}^3)}, \\ \left| \varepsilon^2 \int_{\omega} f_3(y', 1 - \varepsilon y_3) \widehat{\phi}_3(y) \, dy \right| &\leq C \varepsilon^{3/2} \|f\|_{L^2(\Omega, \mathbf{R}^3)} \|\widehat{\phi}\|_{H_0^1(\omega, \mathbf{R}^3)}. \end{aligned}$$

Using the convergences established in subsections 3.4 and 4, we can pass to the limit in (44) to obtain

$$\int_{\omega} \partial_3 w' \cdot \partial_3 \widehat{\phi} - \int_{\omega} q \operatorname{div}_y \widehat{\phi} = 0.$$

Taking $\widehat{\phi} = (0, \widehat{\phi}_3)$, we deduce $\int_{\omega} -q \partial_3 \widehat{\phi}_3 = 0$, so that $\partial_3 q = 0$ a.e. in ω and q depends only on y' . Now, taking $\widehat{\phi} = (\widehat{\phi}', 0)$, we get $\int_{\omega} \partial_3 w' \cdot \partial_3 \widehat{\phi}' - \int_{\omega} q \operatorname{div}_{y'} \widehat{\phi}' = 0$, hence

$$-\partial_{3,3}^2 w' + \nabla_{y'} q = 0 \quad \text{in } H^{-1}(\omega, \mathbf{R}^2). \quad (45)$$

Since $w' = 0$ on $y_3 = 0$, integrating the previous equation in the y_3 variable yields the existence of a function $c'(y')$ such that

$$w'(y) = \frac{y_3^2}{2} \nabla_{y'} q(y') + y_3 c'(y'). \quad (46)$$

Coming back to expression (46), we get that

$$c'(y') = \frac{\bar{u}'(y', 1)}{d(y')} - \frac{d(y')}{2} \nabla_{y'} q(y') \quad (47)$$

hence

$$\begin{aligned} w'(y) &= \frac{y_3^2}{2} \nabla_{y'} q(y') + y_3 \left(\frac{\bar{u}'(y', 1)}{d(y')} - \frac{d(y')}{2} \nabla_{y'} q(y') \right) \\ &= -\frac{y_3}{2} (d(y') - y_3) \nabla_{y'} q + \frac{y_3}{d(y')} \bar{u}'(y', 1). \end{aligned}$$

Integrating the previous expression on $y_3 \in (0, d(y'))$, we obtain

$$\int_0^{d(y')} w'(y) \, dy_3 = \frac{1}{2} \left(-\frac{d(y')^3}{6} \nabla_{y'} q + d(y') \bar{u}'(y', 1) \right)$$

and plugging this expression in (36), we get the Reynolds equation on q :

$$\forall \eta \in H^1(\mathbf{T}^2) \quad \int_{(0,1)^2} \frac{d(y')^3}{6} \nabla_{y'} q \cdot \nabla_{y'} \eta \, dy' = \int_{(0,1)^2} d(y') \bar{u}'(y', 1) \cdot \nabla_{y'} \eta \, dy' \quad (48)$$

Let Q be the following space:

$$Q = \left\{ r \in H^1(\mathbf{T}^2), \int_{(0,1)^2} r \, dy' = 0 \right\}.$$

By Poincaré-Wirtinger inequality, Q is a Hilbert space for the scalar product $(r_1, r_2) = (\int_{(0,1)^2} \nabla_{y'} r_1 \cdot \nabla_{y'} r_2 \, dy')^{1/2}$, so by standard elliptic theory, there exists a unique solution $q_0 \in Q$ to problem (48), which depends linearly on \bar{u} . Moreover, a function $q_1 \in H^1(\mathbf{T}^2)$ is solution to (48) if and only if $q_1 - \int_{(0,1)^2} q_1 = q_0$. Hence, there exists a constant C_q such that

$$q = q_0 + C_q. \quad (49)$$

As a result, setting $\mathcal{R}(\bar{u}) = \nabla_{y'} q$, $\mathcal{R}(\bar{u})$ is in $L^2(\mathbf{T}^2, \mathbf{R}^2)$ and satisfies

$$\forall \eta \in H^1(\mathbf{T}^2) \quad \int_{(0,1)^2} \frac{d(y')^3}{6} \mathcal{R}(\bar{u}) \cdot \nabla_{y'} \eta \, dy' = \int_{(0,1)^2} d(y') \bar{u}'(y', 1) \cdot \nabla_{y'} \eta \, dy'.$$

In general, for any vector field $v \in H^1(\Omega, \mathbf{R}^3)$, there exists one unique function $r_0 \in Q$ such that

$$\forall \eta \in H^1(\mathbf{T}^2) \quad \int_{(0,1)^2} \frac{d(y')^3}{6} \nabla_{y'} r_0 \cdot \nabla_{y'} \eta \, dy' = \int_{(0,1)^2} d(y') v'(y', 1) \cdot \nabla_{y'} \eta \, dy'$$

and defining $\mathcal{R}(v) = \nabla_{y'} r_0$ in $L^2((0, 1)^2, \mathbf{R}^2)$, there holds

$$\forall \eta \in H^1(\mathbf{T}^2) \quad \int_{(0,1)^2} \frac{d(y')^3}{6} \mathcal{R}(v) \cdot \nabla_{y'} \eta \, dy' = \int_{(0,1)^2} d(y') v'(y', 1) \cdot \nabla_{y'} \eta \, dy', \quad (50)$$

and the operator $\mathcal{R} : H^1(\Omega, \mathbf{R}^3) \rightarrow L^2(\mathbf{T}^2, \mathbf{R}^2)$ is linear and continuous. Indeed, setting $\nabla_{y'} \eta = \mathcal{R}(v)$ in (50) yields

$$\frac{1}{6} \int_{(0,1)^2} |d(y')^2 \mathcal{R}(v)|^2 \frac{1}{d(y')} \, dy' = \int_{(0,1)^2} v'(y', 1) \cdot (d(y')^2 \mathcal{R}(v)) \frac{1}{d(y')} \, dy',$$

and using Hölder inequality in $L^2(\mathbf{T}^2, \frac{1}{d(y')} \, dy')$, we obtain the estimate

$$\int_{(0,1)^2} d(y')^3 |\mathcal{R}(v)|^2 \, dy' \leq 36 \int_{(0,1)^2} \frac{1}{d(y')} |v'(y', 1)|^2 \, dy'. \quad (51)$$

Using (1) and the continuity of the trace operator $H^1(\Omega) \rightarrow L^2(\Gamma^t)$, this implies in particular the existence of a constant $C > 0$, depending only on d , such that $\|\mathcal{R}(v)\|_{L^2(\mathbf{T}^2, \mathbf{R}^2)} \leq C \|v\|_{H^1(\Omega, \mathbf{R}^3)}$.

5 Identification of the limit problem

Using the elements gathered in the previous sections, we are able to characterize \bar{u} as the solution to one of the limit problems (18), (19) or (20).

5.1 Subcritical case $0 < \alpha < 1$

Since we have already established that $\bar{u} = 0$ on Γ^t , it remains to prove that for any $\phi \in V_D$,

$$\int_{\Omega} 2D(\bar{u}) : D(\phi) = \int_{\Omega} f \cdot \phi. \quad (52)$$

By density, we may assume that $\phi(x', x_3) = 0$ for $x_3 > 1 - \delta$, where δ is small. But then for any $0 < \varepsilon < \delta/\|d\|_\infty$, the variational formulation (16) reduces to

$$\int_{\Omega_\varepsilon} 2D(u_\varepsilon) : D(\phi) = \int_{\Omega} f \cdot \phi,$$

which can be rephrased as

$$\int_{\Omega} 2D(\bar{u}_\varepsilon) : D(\phi) = \int_{\Omega} f \cdot \phi$$

since ϕ is supported in Ω_ε . (52) follows by weak convergence of \bar{u}_ε to \bar{u} in $H_0^1(\Omega)$.

5.2 Limit problem in the case $\alpha \geq 1$

Let $\phi \in H^1(\Omega, \mathbf{R}^3)$ such that $\operatorname{div} \phi = 0$ in Ω and satisfying the boundary conditions

$$\phi_3 = 0 \quad \text{on } \Gamma^t, \quad \phi = 0 \quad \text{on } \Gamma^b, \quad (53)$$

and $\widehat{\phi}' \in H^1(\omega, \mathbf{R}^2)$ such that

$$\begin{cases} \widehat{\phi}'(y', 0) = -\phi'(y', 1) \\ \widehat{\phi}'(y', d(y')) = 0 \end{cases} \quad \text{for } y' \in (0, 1)^2. \quad (54)$$

We test in the mixed formulation (38) against the test function $\phi_\varepsilon \in H_0^1(\Omega, \mathbf{R}^3)$ defined by

$$\begin{cases} \phi'_\varepsilon(x) = \phi'(x) + \mathbf{1}_{B_\varepsilon}(x) \widehat{\phi}'(x', \frac{1-x_3}{\varepsilon}) \\ \phi_{\varepsilon,3}(x) = \phi_3(x) \end{cases} \quad \text{for } x \in \Omega. \quad (55)$$

Recall that all functions are periodic of period $(0, 1)^2$ in the horizontal direction.

Notice that conditions (53)-(54) guarantee that ϕ_ε is in $H_0^1(\Omega, \mathbf{R}^3)$. The definition of function $\widehat{\phi}'$ will be made precise later.

Testing against ϕ_ε in (38) yields

$$\begin{aligned} & 2 \int_{\Omega_\varepsilon} D(u_\varepsilon) : D(\phi) + 2\varepsilon^\alpha \int_{B_\varepsilon} D(u_\varepsilon) : D(\phi) \\ & + 2\varepsilon^\alpha \int_{B_\varepsilon} \left[\nabla_{x'}(u'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') - \frac{1}{2}(\partial_{x_3} u'_\varepsilon + \nabla_{x'} u_{\varepsilon,3}) \cdot \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}'\right) \right] - \int_{B_\varepsilon} p_\varepsilon \operatorname{div}_{y'} \widehat{\phi}' \\ & = \int_{\Omega} f \cdot \phi + \int_{B_\varepsilon} f' \cdot \widehat{\phi}' \end{aligned} \quad (56)$$

where functions u_ε, ϕ, f and their derivatives are computed at x , function $\widehat{\phi}$ and its derivatives with respect to y are computed at $(x', \frac{1-x_3}{\varepsilon})$, and all integrals are performed with respect to x .

In order to clarify the presentation, we pass to the limit in (56) in two steps. In step 1, we deal with the integrals involving ϕ and we leave the integrals involving $\widehat{\phi}$ for step 2.

Step 1.

- Using function \bar{u}_ε defined by (23), which coincides with u_ε in Ω_ε and is uniformly bounded in $H^1(\Omega)$, we can write

$$2 \int_{\Omega_\varepsilon} D(u_\varepsilon) : D(\phi) = 2 \int_{\Omega} D(\bar{u}_\varepsilon) : D(\phi) - 2 \int_{B_\varepsilon} D(\bar{u}_\varepsilon) : D(\phi).$$

Since \bar{u}_ε converges weakly to \bar{u} in $H^1(\Omega, \mathbf{R}^3)$, there holds immediately

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} D(\bar{u}_\varepsilon) : D(\phi) \right) = \int_{\Omega} D(\bar{u}) : D(\phi).$$

By Hölder inequality,

$$\left| \int_{B_\varepsilon} D(\bar{u}_\varepsilon) : D(\phi) \right| \leq \left(\int_{B_\varepsilon} |D(\bar{u}_\varepsilon)|^2 \right)^{1/2} \left(\int_{B_\varepsilon} |D(\bar{\phi})|^2 \right)^{1/2}.$$

Writing $\int_{B_\varepsilon} |D(\bar{\phi})|^2 = \int_{\Omega} \mathbf{1}_{B_\varepsilon} |D(\bar{\phi})|^2$ and using Lebesgue dominated convergence theorem, we obtain $\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |D(\bar{\phi})|^2 = 0$. Since \bar{u}_ε is uniformly bounded in $H^1(\Omega, \mathbf{R}^3)$, we deduce that $\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} D(\bar{u}_\varepsilon) : D(\phi) = 0$, and hence

$$\lim_{\varepsilon \rightarrow 0} \left(2 \int_{\Omega_\varepsilon} D(u_\varepsilon) : D(\phi) \right) = 2 \int_{\Omega} D(\bar{u}) : D(\phi).$$

- Using Young inequality and the energy bound (17), there holds

$$\begin{aligned} \left| 2\varepsilon^\alpha \int_{B_\varepsilon} D(u_\varepsilon) : D(\phi) \right| &\leq \varepsilon^\alpha \left(\varepsilon^{\alpha/2} \int_{B_\varepsilon} |D(u_\varepsilon)|^2 + \varepsilon^{-\alpha/2} \int_{B_\varepsilon} |D(\phi)|^2 \right) \\ &\leq \varepsilon^{\alpha/2} \left(\varepsilon^\alpha \int_{B_\varepsilon} |D(u_\varepsilon)|^2 + \int_{\Omega} |D(\phi)|^2 \right) \\ &\leq (C + \|D(\phi)\|_{L^2(\Omega, \mathbf{R}^{3 \times 3})}^2) \varepsilon^{\alpha/2} \end{aligned}$$

so that

$$\lim_{\varepsilon \rightarrow 0} \left(2\varepsilon^\alpha \int_{B_\varepsilon} D(u_\varepsilon) : D(\phi) = 0 \right).$$

Step 2. We use the change of variables (4) to turn integrals in $x \in B_\varepsilon$ into integrals in $y \in \omega$.

- Using the rescaled function v_ε defined by (25), we have

$$\begin{aligned} &2\varepsilon^\alpha \int_{B_\varepsilon} \left[\nabla_{x'}(u'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') - \frac{1}{2}(\partial_{x_3} u'_\varepsilon + \nabla_{x'} u_{\varepsilon,3}) \cdot \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' \right) \right] \\ &= 2\varepsilon^{\alpha+1} \int_{\omega} \left[\frac{1}{\varepsilon^2} \nabla_{y'}(v'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') + \frac{1}{2} \left(\frac{1}{\varepsilon^3} \partial_{y_3} v'_\varepsilon + \frac{1}{\varepsilon} \nabla_{y'} v_{\varepsilon,3} \right) \cdot \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' \right) \right] \\ &= 2\varepsilon^{\alpha-1} \int_{\omega} \left[\nabla_{y'}(v'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') + \frac{1}{2\varepsilon^2} \partial_{y_3} v'_\varepsilon \cdot \partial_{y_3} \widehat{\phi}' + \frac{1}{2} \nabla_{y'} v_{\varepsilon,3} \cdot \partial_{y_3} \widehat{\phi}' \right]. \end{aligned}$$

By (27), $\nabla_{y'}(v'_\varepsilon)$ converges strongly to 0 in $L^2(\omega, \mathbf{R}^{2 \times 2})$, so that

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\omega} \nabla_{y'}(v'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') \right) = 0.$$

By the weak convergence (30), there holds

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\omega} \frac{1}{\varepsilon^2} \partial_{y_3} v'_\varepsilon \cdot \partial_{y_3} \widehat{\phi}' \right) = \int_{\omega} \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}'.$$

Finally, since $v_{\varepsilon,3}$ converges weakly to 0 in $H^1(\omega)$,

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\omega} \nabla_{y'} v_{\varepsilon,3} \cdot \partial_{y_3} \widehat{\phi}' \right) = 0.$$

As a consequence,

– in the critical case $\alpha = 1$,

$$\lim_{\varepsilon \rightarrow 0} \left(2\varepsilon^\alpha \int_{B_\varepsilon} \left[\nabla_{x'}(u'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') - \frac{1}{2}(\partial_{x_3} u'_\varepsilon + \nabla_{x'} u_{\varepsilon,3}) \cdot \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' \right) \right] \right) = \int_\omega \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}'; \quad (57)$$

– in the super-critical case $\alpha > 1$,

$$\lim_{\varepsilon \rightarrow 0} \left(2\varepsilon^\alpha \int_{B_\varepsilon} \left[\nabla_{x'}(u'_\varepsilon) : \nabla_{y'}(\widehat{\phi}') - \frac{1}{2}(\partial_{x_3} u'_\varepsilon + \nabla_{x'} u_{\varepsilon,3}) \cdot \left(\frac{1}{\varepsilon} \partial_{y_3} \widehat{\phi}' \right) \right] \right) = 0. \quad (58)$$

- Using the rescaled pressure q_ε defined by (42), there holds

$$\int_{B_\varepsilon} p_\varepsilon \operatorname{div}_{y'} \widehat{\phi}' = \int_\omega q_\varepsilon \operatorname{div}_{y'} \widehat{\phi}'$$

and by the weak convergence (43), we get

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{B_\varepsilon} p_\varepsilon \operatorname{div}_{y'} \widehat{\phi}' \right) = \int_\omega q \operatorname{div}_{y'} \widehat{\phi}'.$$

- By Lebesgue dominated convergence theorem, writing $\int_{B_\varepsilon} f' \cdot \widehat{\phi}' = \int_\Omega \mathbf{1}_{B_\varepsilon} f' \cdot \widehat{\phi}'$, it is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{B_\varepsilon} f' \cdot \widehat{\phi}' \right) = 0.$$

Gathering the previous results, we can pass to the limit in (56) and obtain:

- in the super-critical case $\alpha > 1$,

$$2 \int_\Omega D(\bar{u}) : D(\phi) = \int_\Omega f \cdot \phi. \quad (59)$$

This proves that \bar{u} is the variational solution to (20);

- in the critical case $\alpha = 1$,

$$2 \int_\Omega D(\bar{u}) : D(\phi) + \int_\omega \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}' - \int_\omega q \operatorname{div}_{y'} \widehat{\phi}' = \int_\Omega f \cdot \phi. \quad (60)$$

We will choose a particular profile for the function $\widehat{\phi}'$ in order to express the integrals

$$\int_\omega \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}' - \int_\omega q \operatorname{div}_{y'} \widehat{\phi}'$$

as a boundary integral on Γ^t involving \bar{u} and ϕ .

Construction of the function $\widehat{\phi}'$ By density, one can assume that ϕ is in $C^1(\bar{\Omega}, \mathbf{R}^2)$. Then we define $\widehat{\phi}' \in H^1(\omega, \mathbf{R}^2)$ by

$$\widehat{\phi}'(y) = - \left(1 - \frac{y_3}{d(y')} \right) \phi'(y', 1) + \frac{1}{2} (d(y') - y_3) y_3 \nabla_{y'} r(y'), \quad y \in \omega,$$

with $\nabla_{y'} r = \mathcal{R}(\phi)$, where the operator \mathcal{R} is defined by (50). $\widehat{\phi}'$ is periodic with respect to y' and satisfies (54), as well as the analogous condition to (36):

$$\int_{(0,1)^2} \left(\int_0^{d(y')} \widehat{\phi}'(y', y_3) dy_3 \right) \cdot \nabla_{y'} \eta(y') dy' = 0 \quad \text{for any } \eta \in H^1((0,1)^2). \quad (61)$$

Indeed, a direct computation yields

$$\int_0^{d(y')} \widehat{\phi}'(y', y_3) \, dy_3 = \frac{1}{2} \left(-d(y')\phi'(y', 1) + \frac{d(y')^3}{6} \nabla_{y'} r \right),$$

hence (61) results from the very definition of $R(\phi)$. This property will allow us to simplify the expression of the boundary layer terms $\int_{\omega} \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}' - \int_{\omega} q \operatorname{div}_{y'} \widehat{\phi}'$ in (60).

First,

$$\int_{\omega} \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}' = \int_{\omega} \frac{1}{d(y')} \partial_{y_3} w'(y) \cdot \phi'(y', 1) \, dy + \frac{1}{2} \int_{\omega} (d(y') - 2y_3) \nabla_{y'} r \, dy.$$

By Fubini theorem and using that $w' = 0$ on $y_3 = 0$ and condition (37), we get

$$\begin{aligned} \int_{\omega} \frac{1}{d(y')} \partial_{y_3} w'(y) \cdot \phi'(y', 1) \, dy &= \int_{y' \in (0,1)^2} \frac{1}{d(y')} \left(\int_0^{d(y')} \partial_{y_3} w'(y', y_3) \, dy_3 \right) \cdot \phi'(y', 1) \, dy' \\ &= \int_{(0,1)^2} \frac{1}{d(y')} (w'(y', d(y')) - w'(y', 0)) \cdot \phi'(y', 1) \, dy' \\ &= \int_{(0,1)^2} \frac{1}{d(y')} \bar{u}'(y', 1) \cdot \phi'(y', 1) \, dy'. \end{aligned}$$

Also, by Fubini theorem, noticing that $\int_0^d (d - 2s) \, ds = 0$ for any $d > 0$, we obtain

$$\begin{aligned} \int_{\omega} (d(y') - 2y_3) \nabla_{y'} r \, dy &= \int_{(0,1)^2} \left(\int_0^{d(y')} (d(y') - 2y_3) \, dy_3 \right) \nabla_{y'} r \, dy' \\ &= 0. \end{aligned}$$

Finally, since q satisfies the Reynolds equation (48), q is in $H^1((0, 1)^2)$ so by Green formula,

$$\begin{aligned} - \int_{\omega} q \operatorname{div}_{y'} \widehat{\phi}' &= \int_{\omega} \nabla_{y'} q \cdot \widehat{\phi}' \\ &= \int_{(0,1)^2} \nabla_{y'} q \cdot \left(\int_0^{d(y')} \widehat{\phi}'(y) \, dy_3 \right) \, dy'. \end{aligned}$$

Taking $\eta = q$ in (61), we conclude that the previous integral vanishes. Gathering the previous computations, we obtain the expression

$$\int_{\omega} \partial_{y_3} w' \cdot \partial_{y_3} \widehat{\phi}' - \int_{\omega} q \operatorname{div}_{y'} \widehat{\phi}' = \int_{(0,1)^2} \frac{1}{d(y')} \bar{u}'(y', 1) \cdot \phi'(y', 1) \, dy'.$$

Plugging the previous expression in (60), we conclude that \bar{u} is the variational solution to (19).

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