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# Solvability of the variable-viscosity fluid-porous flows coupled with an optimal stress jump interface condition

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## Abstract

In this study, we prove the well-posedness of both the variable-viscosity Stokes/Darcy-Brinkman and Stokes/Darcy coupled problems governing the viscous flow in fluid-porous systems. The coupling is made by a recent optimal stress jump interface condition associated to velocity continuity at the bottom surface of a transition layer inside the porous region. Indeed, this original condition has been shown to be physically relevant for multi-directional flows and optimal by minimizing the loss of flow rate in the viscous boundary layer. Besides, its inherent tensorial form ensures to handle arbitrary flow directions with anisotropic effects of the microstructure. The analysis of the transmission problem is carried out by introducing a unified mixed variational framework with no Lagrange multiplier at the interface. Moreover, the analysis of variable-viscosity fluid-porous flows seems new in the literature.

*Keywords:* Fluid-porous flow, Stokes/Darcy-Brinkman model, Stokes/Darcy model, Variable-viscosity, Optimal stress jump condition, Reduction of flow-rate loss, Well-posedness analysis

*2020 MSC:* 35A01, 35A02, 35J50, 35J57, 35Q35, 65N30, 76D07, 76S05

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## 1. Introduction

The P.D.E.'s governing the incompressible viscous creeping flow in fluid-porous systems  $\Omega$  (see Figure 1), *i.e.* the Stokes and Darcy equations, are not of the same order. In order to solve the coupling in the transmission problem, a pioneering approach dates back from Brinkman [1, 2] who early introduces the Darcy-Brinkman equation and the notion of effective viscosity of a porous medium. Some authors have then followed and extended such a single-domain continuum modelling, *i.e.* assuming the continuity of velocity and stress vectors between the fluid domain  $\Omega_f$  and the porous one  $\Omega_p$ , e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Following the pioneering work of Beavers and Joseph [13] introducing the two-domain approach, many authors use velocity slip interface conditions associated with continuity of the

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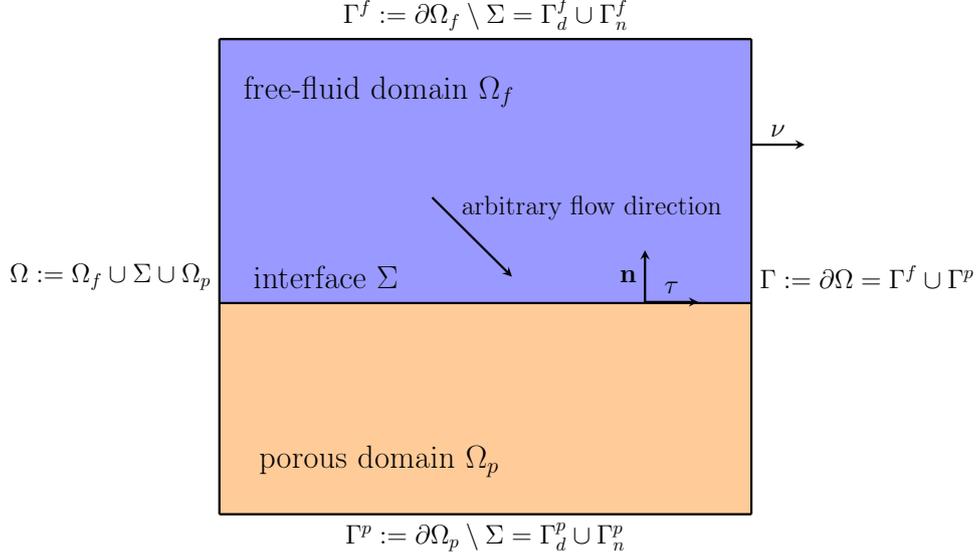


Figure 1: General configuration of the flow in a fluid-porous domain  $\Omega$  at a macroscale length  $L$ .

normal velocity and stress vector to couple the Stokes/Darcy or Navier-Stokes/Darcy problems. These conditions are extended *ad-hoc* for the multi-dimensional case from the 1-D flow inherited from the heuristic conditions of Beavers-Joseph-Jones [13, 14]; see also the comments in [15, 16, 17]. With usual notations precised in the next section 2, the set of multi-dimensional extended Beavers-Joseph-Jones conditions applied at the fluid-porous interface  $\Sigma = \Sigma_t$  (originally chosen at the top surface  $\Sigma_t$  of the transition layer and tangent to the upper solid inclusions) reads:

$$\begin{cases} [\mathbf{v} \cdot \mathbf{n}]_{\Sigma} = 0 \\ \boldsymbol{\tau}_j \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T)_{\Sigma} \cdot \mathbf{n} = \frac{\alpha_{bj}}{\sqrt{K_p}} [\mathbf{v} \cdot \boldsymbol{\tau}_j]_{\Sigma}, \quad \text{for } j = 1, 2 \\ \mathbf{n} \cdot [\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}]_{\Sigma} = 0 \end{cases} \quad \text{on } \Sigma = \Sigma_t, \quad (1)$$

where the Cauchy stress vector  $\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}$  is defined as:  $\boldsymbol{\sigma}^f(\mathbf{v}, p) \cdot \mathbf{n} := \mu (\nabla \mathbf{v} + \nabla \mathbf{v}^T)^f \cdot \mathbf{n} - p^f \mathbf{n}$  in the free-fluid region  $\Omega_f$  and  $\boldsymbol{\sigma}^p(\mathbf{v}, p) \cdot \mathbf{n} := -p^p \mathbf{n}$  in the porous bulk  $\Omega_p$ . Besides, the scalar parameter  $\alpha_{bj} > 0$  denotes the dimensionless velocity-slip coefficient that should be estimated by experimental data or averaged pore-scale simulations. The jump quantity  $[\cdot]_{\Sigma}$  on  $\Sigma$  is oriented by the unit normal vector  $\mathbf{n}$  on  $\Sigma$  (directed arbitrarily outwards of the porous region  $\Omega_p$ ). The couple of vectors  $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$  denotes a local orthonormal basis of tangential vectors on the surface  $\Sigma$ . However for the mathematical and numerical analysis, many authors then consider the following simplified Beavers-Joseph-Saffman form [18]. In this approximate condition, the velocity jump is no more explicitly included by neglecting the porous slip velocity with respect

to the fluid slip velocity, as derived later by [19, 20] for the one-dimensional flow:

$$\begin{cases} \llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket_{\Sigma} = 0 \\ \boldsymbol{\tau}_j \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T)_{\Sigma} \cdot \mathbf{n} = \frac{\alpha_{bj}}{\sqrt{K_p}} \mathbf{v}_{\Sigma}^f \cdot \boldsymbol{\tau}_j, \quad \text{for } j = 1, 2 \\ \mathbf{n} \cdot \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_{\Sigma} = 0 \end{cases} \quad \text{on } \Sigma = \Sigma_t, \quad (2)$$

We refer to [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] and many others for the numerous related numerical methods and analyses using the set (2). Only a few authors, e.g. [40, 41, 42, 43], consider the full Beavers-Joseph-Jones conditions (1) and the solvability is proved in [44, 45] with no restriction on the size of the slip coefficient  $\alpha_{bj} \geq 0$ . Moreover, the generalization below of (1) that also includes a jump of the stress vector is derived by asymptotic analysis in [46]:

$$\begin{cases} \llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket_{\Sigma} = 0, \\ \boldsymbol{\tau}_j \cdot (\boldsymbol{\sigma}_v^f(\mathbf{v}) \cdot \mathbf{n}_{\Sigma}) = \frac{\mu}{\sqrt{K_p}} \alpha_{\Sigma} \llbracket \mathbf{v} \cdot \boldsymbol{\tau}_j \rrbracket_{\Sigma}, \quad \text{for } j = 1, 2 \\ \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_{\Sigma} = \frac{\mu}{\sqrt{K_p}} \boldsymbol{\beta}_{\Sigma} \cdot \mathbf{v}_{\Sigma}^f \end{cases} \quad \text{on } \Sigma = \Sigma_t, \quad (3)$$

where  $\boldsymbol{\beta}_{\Sigma}$  denotes the symmetric and positive semi-definite friction tensor on  $\Sigma$ . Using such interface conditions (3), the boundary layer is theoretically calculated by WKB expansions of which the convergence is proved in [47]. Moreover, the resulting Stokes/Darcy-Brinkman and Stokes/Darcy coupled problems are proved to be well-posed with no restriction on the size of the data in [44, 45].

The present analysis is carried out for a different original set of interface conditions (20), *i.e.* velocity continuity and stress vector jump at the bottom surface  $\Sigma_b$  of a suitable transition layer inside the porous layer, as recently derived in [46] by asymptotic modeling. This condition is shown to be optimal to minimize the loss of flow rate in [48]; see an example in Appendix B. Moreover, the further analysis considers the variable-viscosity (and density) flow in view of a potential coupling with a convective heat transfer. Up to our knowledge, this seems new in the literature in the context of fluid-porous flows.

The paper is organized as follows. The next Section 2 describes the flow models and the set of boundary and interface conditions. In Section 3, the solvability of the variable-viscosity Stokes/Darcy-Brinkman coupled flow is proved in Theorem 1. In Section 4, the main result of well-posedness of the variable-viscosity Stokes/Darcy coupled flow is finally proved in Theorem 2 for the existence of solution and its Corollary 2 for the uniqueness.

## 2. Coupled fluid-porous flow models

### 2.1. Notations and definitions

Let  $\Omega \subset \mathbb{R}^d$  (for  $d \leq 3$ ) be an open bounded and connected set with Lipschitz-continuous boundary  $\Gamma := \partial\Omega$  and  $\boldsymbol{\nu}$  be the outward unit normal vector on  $\Gamma$ ; see Figure 1. The domain  $\Omega$  is composed of two disjoint connected subdomains, the fluid domain  $\Omega_f$  and the porous domain  $\Omega_p$ , each one with a Lipschitz-continuous boundary  $\partial\Omega_f$  and  $\partial\Omega_p$ , respectively. They are separated by a Lipschitz-continuous surface  $\Sigma \subset \mathbb{R}^{d-1}$  such that:  $\Omega = \Omega_f \cup \Sigma \cup \Omega_p$  with  $\Sigma = \partial\Omega_f \cap \partial\Omega_p$

( $\text{meas}(\Sigma) > 0$ ),  $\Gamma^f := \partial\Omega_f \cap \partial\Omega$  and  $\Gamma^p := \partial\Omega_p \cap \partial\Omega$ . Let  $\mathbf{n}$  be the unit normal vector on the interface  $\Sigma$  arbitrarily directed from  $\Omega_p$  to  $\Omega_f$ , and let the set  $\{\boldsymbol{\tau}_j\}$  for  $1 \leq j \leq d-1$ , be a local orthonormal basis of vectors on the tangent plane to  $\Sigma$ , the unit vector  $\boldsymbol{\tau}$  being any of these tangential vectors. We use the standard definitions and properties of the Lebesgue and Sobolev spaces, e.g. [49, 50]. In particular,  $\|\cdot\|_{s,\Omega}$  denotes the usual norm or semi-norm  $|\cdot|_{s,\Omega}$  of the Sobolev spaces  $H^s(\Omega) := W^{s,2}(\Omega)$  for  $\Omega = \Omega_f$  or  $\Omega_p$  and  $\langle \cdot, \cdot \rangle_{-1,\Omega}$  denotes the duality pairing between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . We also define the (real) Hilbert spaces below endowed with their respective usual inner products and associated norms [51, 52]:

$$\begin{aligned} \mathbf{H}_{div}(\Omega) &:= \{ \mathbf{u} \in L^2(\Omega)^d; \nabla \cdot \mathbf{u} \in L^2(\Omega) \}, \\ L_0^2(\Omega) &:= \left\{ q \in L^2(\Omega); \int_{\Omega} q \, dx = 0 \right\}. \end{aligned} \quad (4)$$

The space  $L_0^2(\Omega)$  is equipped with the  $L^2(\Omega)$  norm  $\|\cdot\|_{0,\Omega}$  and  $\mathbf{H}_{div}(\Omega)$  is equipped with the graph norm defined by:

$$\|\mathbf{u}\|_{\mathbf{H}_{div}(\Omega)}^2 := \|\mathbf{u}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{u}\|_{0,\Omega}^2 \quad \text{for all } \mathbf{u} \in \mathbf{H}_{div}(\Omega). \quad (5)$$

Since the Lipschitz surface  $\Sigma$  (with  $\text{meas}(\Sigma) > 0$ ) is not closed,  $\langle \cdot, \cdot \rangle_{-1/2,\Sigma}$  denotes the duality pairing between  $\tilde{H}^{1/2}(\Sigma) := H_{00}^{1/2}(\Sigma)$  and its topological dual space  $\tilde{H}^{-1/2}(\Sigma)$  that is a distribution space on  $\Sigma$ . We refer to [53, 54] for more background information on the space  $H_{00}^{1/2}(\Sigma)$  and its dual, and to [22, 37] for the context of fluid-porous flows; see also an illustrative related counterexample provided in [55, Exercise 2.24]. Following [37],  $\tilde{H}^{1/2}(\Sigma) = H_{00}^{1/2}(\Sigma)$  is the space of traces of all functions of  $H_{0,\partial\Omega_f \setminus \Sigma}^1$  (i.e. vanishing on  $\partial\Omega_f \setminus \Sigma$ ) or of  $H_{0,\partial\Omega_p \setminus \Sigma}^1$ . The spaces  $H^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$  and  $\tilde{H}^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$  are respectively equipped with the semi-norms and norms defined as follows:

$$|u|_{H^{1/2}(\Sigma)}^2 := \int_{\Sigma} \int_{\Sigma} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^d} \, d\mathbf{x} d\mathbf{y}, \quad \|u\|_{H^{1/2}(\Sigma)}^2 := \|u\|_{0,\Sigma}^2 + |u|_{H^{1/2}(\Sigma)}^2, \quad (6)$$

$$|u|_{\tilde{H}^{1/2}(\Sigma)}^2 := |u|_{H^{1/2}(\Sigma)}^2 + \int_{\Sigma} \frac{|u(\mathbf{x})|^2}{d(\mathbf{x}, \partial\Sigma)} \, d\mathbf{x}, \quad \|u\|_{\tilde{H}^{1/2}(\Sigma)}^2 := \|u\|_{0,\Sigma}^2 + |u|_{\tilde{H}^{1/2}(\Sigma)}^2, \quad (7)$$

where  $d(\mathbf{x}, \partial\Sigma) := \inf_{\mathbf{y} \in \partial\Sigma} |\mathbf{x} - \mathbf{y}|$  denotes the distance from any point  $\mathbf{x} \in \Sigma$  to the border  $\partial\Sigma$ . Since  $\Sigma$  is a bounded Lipschitz surface, the latter distance function belongs to  $W^{1,\infty}(\Sigma)$ . Thus, the continuous imbedding  $\tilde{H}^{1/2}(\Sigma) \hookrightarrow H^{1/2}(\Sigma)$  holds. However, the norms defined in (6) and (7) are not equivalent except when  $\Sigma$  is a closed surface (or curve for  $d = 2$ ). Besides, for any function  $\tilde{u} \in \tilde{H}^{1/2}(\Sigma)$ , its extension by zero to the whole boundary  $\partial\Omega_f$  and denoted by  $u$  satisfies  $u \in H^{1/2}(\partial\Omega_f)$  and for some  $c(\Sigma, \partial\Omega_f) > 0^1$ :

$$\|u\|_{1/2,\partial\Omega_f} \leq c(\Sigma, \partial\Omega_f) \|\tilde{u}\|_{\tilde{H}^{1/2}(\Sigma)}, \quad \text{for all } \tilde{u} \in \tilde{H}^{1/2}(\Sigma). \quad (8)$$

Moreover, the normal trace  $\mathbf{u} \cdot \mathbf{n}$  on the border  $\partial\Omega_f$  of any function  $\mathbf{u} \in \mathbf{H}_{div}(\Omega_f)$  belongs to  $H^{-1/2}(\partial\Omega_f)$  and the normal trace linear operator:  $\mathbf{H}_{div}(\Omega_f) \mapsto H^{-1/2}(\partial\Omega_f)$  is continuous and surjective with [51, Theorem 2.5 and Corollary 2.8] (and similarly on  $\partial\Omega_p$ ). The same result

<sup>1</sup>Without any more precision,  $c > 0$  or  $C > 0$  will denote a positive generic quantity depending only of the data.

holds on a part of  $\partial\Omega_f$  if this part is a closed surface. But on the surface  $\Sigma$  that is not closed, then we have  $\mathbf{u} \cdot \mathbf{n} \in \tilde{H}^{-1/2}(\Sigma) := \tilde{H}^{1/2}(\Sigma)'$ .

For any quantity  $\psi$  defined all over  $\Omega_f \cup \Omega_p$ , the restrictions on  $\Omega_f$  or  $\Omega_p$  are respectively denoted by  $\psi^f := \psi|_{\Omega_f}$  and  $\psi^p := \psi|_{\Omega_p}$ . For a function  $\psi$  having a jump on  $\Sigma$ , let  $\psi^- := \psi_\Sigma^p$  and  $\psi^+ := \psi_\Sigma^f$ <sup>2</sup> be the traces of  $\psi^p$  and  $\psi^f$  on each side of  $\Sigma$  (at least defined in a weak sense), respectively. Following the general framework introduced in [56], let us choose as reduced variables at the interface  $\Sigma$ , the jump of traces of  $\psi$  on  $\Sigma$  oriented by  $\mathbf{n}$  and the arithmetic mean of traces of  $\psi$  defined by:

$$\begin{aligned} \llbracket \psi \rrbracket_\Sigma &:= \psi^+ - \psi^- = (\psi^f - \psi^p)_\Sigma, \\ \bar{\psi}_\Sigma &:= \frac{1}{2}(\psi^+ + \psi^-) = \frac{1}{2}(\psi^f + \psi^p)_\Sigma. \end{aligned} \quad (9)$$

## 2.2. Creeping flow in the porous medium: variable-viscosity Darcy's model

The single-phase incompressible creeping flow in the saturated porous bulk  $\Omega_p$  is described by the Darcy law (see e.g. [57]), here considered with variable viscosity and density:

$$\begin{cases} \nabla \cdot \mathbf{v} = q_m & \text{in } \Omega_p, \\ \mu \mathbf{K}_p^{-1} \mathbf{v} + \nabla p = \rho \mathbf{f} & \text{in } \Omega_p, \end{cases} \quad (10)$$

where  $\mu > 0$  is the dynamic viscosity of the fluid with  $\mu \in L^\infty(\Omega)$ ,  $\rho > 0$  its mass density with  $\rho \in L^\infty(\Omega)$ . With the porosity  $\phi_p$  (volume fraction of fluid pores,  $0 < \phi_p < 1$ ), the intrinsic permeability tensor  $\mathbf{K}_p(\phi_p)$  of the porous region  $\Omega_p$  (a symmetric and uniformly positive definite bounded matrix in  $L^\infty(\Omega_p)^{d \times d}$ ) can be given by a porosity-permeability correlation or experimental data. For example, we can use the porosity-permeability correlation  $K_p(\phi_p)$  of Kozeny-Carman that has been calibrated for many random packed beds of spherical grains of variable sizes [58, 57, 59]. Besides in (10),  $\mathbf{v}$  denotes the filtration velocity defined as the superficial average over a representative unit volume and  $p$  is the pressure defined as the intrinsic average. The external force per mass unit  $\mathbf{f} \in L^2(\Omega)^d$  (e.g. gravitational acceleration), is included in the right-hand side, whereas  $q_m \in L^2(\Omega)$  denotes a given mass source or sink term. With (10), the Cauchy stress tensor in  $\Omega_p$  reduces to the pressure term with no viscous stress,  $\mathbf{I}$  being the unit tensor:

$$\boldsymbol{\sigma}^p(\mathbf{v}, p) := -p^p \mathbf{I} \quad \text{in } \Omega_p. \quad (11)$$

The Darcy number Da is classically introduced as a dimensionless parameter to characterize the flow in the porous medium:

$$\text{Da} := \frac{K_p}{L^2}, \quad (12)$$

where  $K_p := \|\mathbf{K}_p\|_\infty$  and  $L$  is the macroscopic length scale.

On the external boundary of the porous medium, *i.e.*  $\Gamma^p = \partial\Omega_p \setminus \Sigma$ , mixed Dirichlet/Neumann boundary conditions are assigned with zero normal flux on the part  $\Gamma_d^p$  and a null traction on  $\Gamma_n^p$  which can be interpreted with (11) as an homogeneous Dirichlet condition for the pressure  $p = 0$  on  $\Gamma_n^p$ :

$$\begin{cases} \mathbf{v}^p \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_d^p, \\ \boldsymbol{\sigma}^p(\mathbf{v}, p) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_n^p, \end{cases} \quad (13)$$

where  $\Gamma^p := \Gamma_d^p \cup \Gamma_n^p$  with  $\Gamma_d^p \cap \Gamma_n^p = \emptyset$  and  $\text{meas}(\Gamma_d^p) > 0$ .

<sup>2</sup>The upper or lower index of  $\psi_\Sigma^f, \psi_\Sigma^p$  will be sometimes omitted when there is no possible confusion.

### 2.3. Creeping flow in the porous medium: variable-viscosity Darcy-Brinkman's model

The incompressible creeping flow in the saturated porous medium  $\Omega_p$  can be also governed by the Darcy-Brinkman equations [1, 2, 7], here considered with variable viscosity and density:

$$\begin{cases} \nabla \cdot \mathbf{v} = q_m & \text{in } \Omega_p, \\ -\nabla \cdot \left( \frac{\mu}{\phi_p} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \right) + \mu \mathbf{K}_p^{-1} \mathbf{v} + \nabla p = \rho \mathbf{f} & \text{in } \Omega_p, \end{cases} \quad (14)$$

where  $\mu > 0$  is the dynamic viscosity of the fluid with  $\mu \in L^\infty(\Omega)$ ,  $\rho > 0$  its mass density with  $\rho \in L^\infty(\Omega)$ . With the porosity  $\phi_p$  (volume fraction of fluid pores,  $0 < \phi_p < 1$ ), the intrinsic permeability tensor  $\mathbf{K}_p(\phi_p)$  of the porous region  $\Omega_p$  (a symmetric and uniformly positive definite bounded matrix in  $L^\infty(\Omega_p)^{d \times d}$ ) can be given by a porosity-permeability correlation or experimental data. For example, we can use the porosity-permeability correlation  $K_p(\phi_p)$  of Kozeny-Carman that has been calibrated for many random packed beds of spherical grains of variable sizes [58, 57, 59]. Besides in (10),  $\mathbf{v}$  denotes the filtration velocity defined as the superficial average over a representative unit volume and  $p$  is the pressure defined as the intrinsic average. The external force per mass unit  $\mathbf{f} \in L^2(\Omega)^d$  (e.g. gravitational acceleration), is included in the right-hand side. With (14), the Cauchy stress tensor in  $\Omega_p$  reads:

$$\begin{cases} \boldsymbol{\sigma}^p(\mathbf{v}, p) := \boldsymbol{\sigma}_v^p(\mathbf{v}) - p \mathbf{I} & \text{where } \boldsymbol{\sigma}_v^p(\mathbf{v}) := 2 \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) \\ \text{with } \mathbf{D}(\mathbf{v}) := \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T), & \end{cases} \quad \text{in } \Omega_p, \quad (15)$$

where  $\boldsymbol{\sigma}_v^p(\mathbf{v})$  is the viscous stress tensor and  $\mathbf{D}(\mathbf{v})$  is the strain rate tensor (symmetric part of  $\nabla \mathbf{v}$ ).

On the external boundary of the porous medium, *i.e.*  $\Gamma^p = \partial\Omega_p \setminus \Sigma$ , mixed Dirichlet/Neumann boundary conditions are assigned with a null velocity on the part  $\Gamma_d^p$  and a given traction  $\mathbf{g} \in \tilde{H}^{-1/2}(\Gamma_n^p)^d$  on  $\Gamma_n^p$ :

$$\begin{cases} \mathbf{v}^p = 0 & \text{on } \Gamma_d^p, \\ \boldsymbol{\sigma}^p(\mathbf{v}, p) \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } \Gamma_n^p, \end{cases} \quad (16)$$

where  $\Gamma^p := \Gamma_d^p \cup \Gamma_n^p$  with  $\Gamma_d^p \cap \Gamma_n^p = \emptyset$  and  $\text{meas}(\Gamma_d^p) > 0$ .

### 2.4. Flow in the free-fluid region: variable-viscosity Stokes model

The incompressible viscous flow in the pure fluid domain  $\Omega_f$  is here governed by the Stokes model with a variable viscosity and density:

$$\begin{cases} \nabla \cdot \mathbf{v} = q_m & \text{in } \Omega_f, \\ -\nabla \cdot (\mu (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) + \nabla p = \rho \mathbf{f} & \text{in } \Omega_f, \end{cases} \quad (17)$$

For writing Eqs. (17) and (10) with the stress formulation, the Cauchy stress tensor  $\boldsymbol{\sigma}(\mathbf{v}, p)$  for a Newtonian fluid in  $\Omega_f$  associated with the stress vector  $\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}$  on the surface  $\Sigma$ :

$$\begin{cases} \boldsymbol{\sigma}^f(\mathbf{v}, p) := \boldsymbol{\sigma}_v^f(\mathbf{v}) - p \mathbf{I} & \text{where } \boldsymbol{\sigma}_v^f(\mathbf{v}) := 2 \mu^f \mathbf{D}(\mathbf{v}) \\ \text{with } \mathbf{D}(\mathbf{v}) := \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T), & \end{cases} \quad \text{in } \Omega_f, \quad (18)$$

where  $\sigma_b^f(\mathbf{v})$  is the viscous stress tensor and  $\mathbf{D}(\mathbf{v})$  is the strain rate tensor (symmetric part of  $\nabla \mathbf{v}$ ).

On the external boundary of the fluid region, *i.e.*  $\Gamma^f = \partial\Omega_f \setminus \Sigma$ , mixed Dirichlet/Neumann boundary conditions are applied with null velocity on the part  $\Gamma_d^f$  and a given stress vector  $\mathbf{h} \in \tilde{H}^{-1/2}(\Gamma_n^f)^d$  on  $\Gamma_n^f$ :

$$\begin{cases} \mathbf{v}^f = 0 & \text{on } \Gamma_d^f, \\ \boldsymbol{\sigma}^f(\mathbf{v}, p) \cdot \boldsymbol{\nu} = \mathbf{h} & \text{on } \Gamma_n^f. \end{cases} \quad (19)$$

where  $\Gamma^f := \Gamma_d^f \cup \Gamma_n^f$  with  $\Gamma_d^f \cap \Gamma_n^f = \emptyset$  and  $\text{meas}(\Gamma_d^f) > 0$ ,  $\text{meas}(\Gamma_n^f) > 0$ .

### 2.5. Fluid-porous stress jump interface conditions

The coupling of the fluid-porous flow (17, 10) requires additional interface conditions on  $\Sigma$ . Here, we deal with a new setting issued from recent advances on fluid-porous flows carried out in [46, 48] with an asymptotic modeling and analysis to couple the Stokes problem with either the Darcy-Brinkman or Darcy models. Hence, the following set of stress jump interface conditions associated to velocity continuity is derived in [46, Section III] (when  $\Sigma$  is non-centered inside the inter-region; see also the summary supplied in [60, Eqs. (18) with (30) and Remark 3]) to couple the Stokes and Darcy models at the bottom surface  $\Sigma = \Sigma_b$  of a thin transition layer between the fluid and porous regions:

$$\begin{cases} \llbracket \mathbf{v} \rrbracket_\Sigma = 0, & \text{i.e. } \mathbf{v}_\Sigma^f = \mathbf{v}_\Sigma^p := \mathbf{v}_\Sigma \\ \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma = \frac{\mu_\Sigma^p}{\sqrt{K_p}} \boldsymbol{\beta}_\Sigma \mathbf{v}_\Sigma - \mathbf{f}_\Sigma & \text{on } \Sigma = \Sigma_b, \end{cases} \quad (20)$$

where  $K_p := \|\mathbf{K}_p\|_\infty$  (or any permeability reference) and  $\mathbf{f}_\Sigma \in L^2(\Sigma)^d$  is a given external surfacic force on  $\Sigma$  ( $\mathbf{f}_\Sigma \in \tilde{H}^{-1/2}(\Sigma)^d$  is also admissible). The stress jump friction tensor  $\boldsymbol{\beta}_\Sigma$  denotes a uniformly positive semi-definite bounded matrix (possibly symmetric) with thus  $\boldsymbol{\beta}_\Sigma \in L^\infty(\Sigma)^{d \times d}$ .

Compared to the usual velocity slip conditions with no normal stress jump, extended *ad-hoc* from the 1-D Beavers-Joseph [13] or simplified Beavers-Joseph-Saffman [18, 19] conditions used by almost all authors, *e.g.* [22, 28, 29, 37], the jump interface set (20) is shown in [48] to be optimal to reduce the loss of flow rate in the viscous boundary layer of the porous medium. Moreover, a calibration procedure is proposed in [48] to determine the optimal location of  $\Sigma_b$ , *i.e.* the thickness  $d$  from the top surface  $\Sigma_t$  of the transition layer (where Beavers-Joseph's conditions of velocity slip are applied), together with the friction tensor  $\boldsymbol{\beta}_\Sigma$ . A typical example for the benchmark of Poiseuille's channel flow through fluid-porous layers is provided in Appendix B.

However with the regularity of the data in  $\Omega_p$  chosen here, we have  $\mathbf{v}^p \in \mathbf{H}_{div}(\Omega_p)$  and thus the normal trace  $\mathbf{v}^p \cdot \mathbf{n}$  on  $\Sigma$  can be defined in a weak sense, but there is no guarantee to define the tangential trace  $\mathbf{v}^p \wedge \mathbf{n}$  on  $\Sigma$  since the curl  $\nabla \times \mathbf{v}^p$  does not *a priori* belong to  $L^2(\Omega_p)^d$  (unless the tensor  $\mathbf{K}_p$  and the viscosity  $\mu$  are constant). Therefore with less smooth data as it is the case in practical applications, we shall also consider the following admissible modified version of (20) to couple the Stokes and Darcy flows:

$$\begin{cases} \llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket_\Sigma = 0, & \mathbf{v}_\Sigma := \mathbf{v}_\Sigma^f \\ \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma = \frac{\mu_\Sigma^p}{\sqrt{K_p}} \boldsymbol{\beta}_\Sigma \mathbf{v}_\Sigma - \mathbf{f}_\Sigma & \text{on } \Sigma = \Sigma_b. \end{cases} \quad (21)$$

Under the above form, the stress jump interface conditions (20) can be viewed from a variational point of view as a dual version of the extended Beavers-Joseph-Saffman velocity jump conditions (2). Unlike the latter conditions (2) or (1) where the velocity slip coefficient  $\alpha_\Sigma$  is a scalar parameter only (see the derivation of this equation in [46]), the intrinsic vector form of the stress jump conditions (20) or (21), where  $\beta_\Sigma$  is a symmetric and positive semi-definite tensor, is likely to actually handle multi-dimensional cases with arbitrary flow directions and anisotropic effects of the microstructure.

### 3. Solvability of the coupled Stokes/Darcy-Brinkman fluid-porous flow

The set of interface conditions (20) being formulated in terms of velocity and stress vectors, it is more suitable to write the fluid-porous model (10, 17) in the following divergential form with the Cauchy stress formulation using (15, 18) for  $\mathbf{f} \in L^2(\Omega)^d$  and  $q_m \in L^2(\Omega)$ :

$$\begin{cases} \nabla \cdot \mathbf{v} = q_m & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma}^p(\mathbf{v}, p) + \mu^p \mathbf{K}_p^{-1} \mathbf{v}^p = \rho \mathbf{f} & \text{in } \Omega_p, \\ -\nabla \cdot \boldsymbol{\sigma}^f(\mathbf{v}, p) = \rho \mathbf{f} & \text{in } \Omega_f. \end{cases} \quad (22)$$

The interest of the stress formulation (22) is also to deal with the case of variable viscosity  $\mu \in L^\infty(\Omega)$  when for example, the flow has to be coupled with a convective heat transfer governed by an advection-diffusion equation for the temperature, thus with a temperature-dependent density  $\rho \in L^\infty(\Omega)$  too. Then to give a sense to (20), we assume that  $\mu \in M$  defined by:

$$M := \{ \mu \in L^\infty(\Omega); \mu^p := \mu|_{\Omega_p} \in H^1(\Omega_p), \mu^f := \mu|_{\Omega_f} \in H^1(\Omega_f) \}, \quad (23)$$

Hence, the Sobolev continuous imbedding yields:  $\mu_\Sigma^p \in H^{1/2}(\Sigma) \hookrightarrow L^4(\Sigma)$  (for  $d \leq 3$ ), although  $\mu_\Sigma^p \in L^2(\Sigma)$  is sufficient here. We have also the natural assumptions of boundedness issued from physical properties and there exists constants  $\phi_m > 0$ ,  $\mu_M \geq \mu_m > 0$  and  $k_M \geq k_m > 0$  such that:

$$\begin{cases} \phi_p \in ]0, 1[; \quad 0 < \phi_m \leq \phi_p(x) < 1 & \text{a.e. } x \in \Omega, \\ \mu \in M; \quad 0 < \mu_m \leq \mu(x) \leq \mu_M & \text{a.e. } x \in \Omega, \\ \beta_\Sigma \in L^\infty(\Sigma)^{d \times d}; \quad \beta_\infty := \|\beta_\Sigma\|_{L^\infty(\Sigma)^{d \times d}}, \\ \mathbf{K}_p^{-1} \in L^\infty(\Omega_p)^{d \times d}; \quad k_\infty := \|\mathbf{K}_p^{-1}\|_{L^\infty(\Omega_p)^{d \times d}}, \\ k_m |\mathbf{y}|^2 \leq (\mathbf{K}_p^{-1}(x) \mathbf{y}) \cdot \mathbf{y} \leq k_M |\mathbf{y}|^2, \quad \forall \mathbf{y} \in \mathbb{R}^d & \text{a.e. } x \in \Omega_p, \\ \rho \in L^\infty(\Omega); \quad \rho_\infty := \|\rho\|_{L^\infty(\Omega)}, \end{cases} \quad (24)$$

where  $|\cdot|$  denotes the Euclidean vector norm in  $\mathbb{R}^d$ .

Then for the velocity and pressure solutions spaces and test function spaces, we define the following (real) Hilbert spaces endowed with their natural respective inner products and associated norms:

$$\begin{aligned} H_{0, \Gamma_d^f}^1(\Omega_f)^d &:= \left\{ \mathbf{u} \in H^1(\Omega_f)^d; \mathbf{u} = 0 \text{ on } \Gamma_d^f \right\}, \\ \mathbf{W} &:= H_{0, \Gamma_d^f \cup \Gamma_d^p}^1(\Omega)^d = \left\{ \mathbf{w} \in H^1(\Omega)^d; \mathbf{w} = 0 \text{ on } \Gamma_d^f \cup \Gamma_d^p \right\}, \\ Q &:= L^2(\Omega), \end{aligned} \quad (25)$$

with the Hilbertian norms (and naturally associated inner products) in  $\mathbf{W}$  (inherited from  $H^1(\Omega)^d$ ) and  $Q$  defined by:

$$\begin{aligned} \|\mathbf{w}\|_{\mathbf{W}}^2 &:= \|\mathbf{w}\|_{0,\Omega}^2 + \|\nabla \mathbf{w}\|_{0,\Omega}^2 = \|\mathbf{w}\|_{1,\Omega}^2, & \text{for all } \mathbf{w} \in \mathbf{W}, \\ \|q\|_Q &:= \|q\|_{0,\Omega}, & \text{for all } q \in Q. \end{aligned} \quad (26)$$

Hence,  $\mathbf{W}$  is a Hilbert space as a closed subspace of  $H^1(\Omega)^d$ , and we have thus the continuous imbedding  $\mathbf{W} \hookrightarrow H^1(\Omega)^d$ . Then for all  $\mathbf{v} \in \mathbf{W}$ , we have continuity of the traces on  $\Sigma$ :  $[\mathbf{v}]_{\Sigma} = 0$  and thus  $\mathbf{v}_{\Sigma} := \mathbf{v}_{\Sigma}^p = \mathbf{v}_{\Sigma}^f \in \tilde{H}^{1/2}(\Sigma)^d \hookrightarrow H^{1/2}(\Sigma)^d$ . Moreover, we have by Sobolev imbedding for all  $\mathbf{v} \in \mathbf{W}$ :  $\mathbf{v}_{\Sigma} \in \tilde{H}^{1/2}(\Sigma)^d \hookrightarrow L^4(\Sigma)^d$  (for  $d \leq 3$ ) with the related continuity inequality of the imbedding.

Let us now introduce two bilinear forms  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$  and two linear functionals  $\ell(\cdot)$ ,  $g(\cdot)$ :

$$a(\cdot, \cdot) : \mathbf{W} \times \mathbf{W} \mapsto \mathbb{R}, \quad b(\cdot, \cdot) : \mathbf{W} \times Q \mapsto \mathbb{R}, \quad \ell(\cdot) : \mathbf{W} \mapsto \mathbb{R}, \quad g(\cdot) : Q \mapsto \mathbb{R}, \quad (27)$$

respectively defined for all  $\mathbf{v}, \mathbf{w} \in \mathbf{W}$  and  $p \in Q$  by:

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}) &:= 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx + 2 \int_{\Omega_p} \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx \\ &\quad + \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx + \int_{\Sigma} \frac{\mu_{\Sigma}^p}{\sqrt{K_p}} (\boldsymbol{\beta}_{\Sigma} \mathbf{v}_{\Sigma}) \cdot \mathbf{w}_{\Sigma} \, ds, \end{aligned} \quad (28)$$

$$b(\mathbf{w}, p) := - \int_{\Omega_f} p \nabla \cdot \mathbf{w} \, dx - \int_{\Omega_p} p \nabla \cdot \mathbf{w} \, dx = - \int_{\Omega} p \nabla \cdot \mathbf{w} \, dx, \quad (29)$$

$$\ell(\mathbf{w}) := \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{w} \, dx + \int_{\Sigma} \mathbf{f}_{\Sigma} \cdot \mathbf{w}_{\Sigma} \, ds + \langle \mathbf{g}, \mathbf{w} \rangle_{-1/2, \Gamma_n^p} + \langle \mathbf{h}, \mathbf{w} \rangle_{-1/2, \Gamma_n^f}, \quad (30)$$

$$g(p) := - \int_{\Omega_f} q_m p \, dx - \int_{\Omega_p} q_m p \, dx = - \int_{\Omega} q_m p \, dx, \quad (31)$$

where  $\langle \cdot, \cdot \rangle_{-1/2, \Gamma_n^p}$  denotes the duality pairing between the spaces  $\tilde{H}^{-1/2}(\Gamma_n^p)$  and  $\tilde{H}^{1/2}(\Gamma_n^p)$  and similarly on  $\Gamma_n^f$ .

Let us first prove some preliminary results.

**Lemma 1 (Equivalent forms of the Stokes/Darcy-Brinkman transmission problem).** *For all data as mentioned above, the boundary-value problem (22, 16, 19, 20) with (15, 18) assuming solutions  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  is equivalent to the following mixed weak problem:*

$$\left\{ \begin{array}{l} \text{find a pair } (\mathbf{v}, p) \in \mathbf{W} \times Q \text{ such that:} \\ a(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = \ell(\mathbf{w}), \quad \text{for all } \mathbf{w} \in \mathbf{W}, \\ b(\mathbf{v}, q) = g(q), \quad \text{for all } q \in Q = L^2(\Omega), \end{array} \right. \quad (32)$$

with the definitions (28–31).

**PROOF.** Firstly, let  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  be a solution to the transmission problem (22, 16, 19, 20). Since  $\mathbf{f} \in L^2(\Omega)^d$ , we have with (15, 18) and (22):  $\boldsymbol{\sigma}^f(\mathbf{v}, p) \in \mathbf{H}_{div}(\Omega_f)^d$  and  $\boldsymbol{\sigma}^p(\mathbf{v}, p) \in \mathbf{H}_{div}(\Omega_p)^d$ . With the normal trace operator  $\mathbf{H}_{div}(\Omega) \mapsto H^{-1/2}(\partial\Omega)$  (cf. [51, Theorem 2.5]) applied in the domains  $\Omega_f$  and  $\Omega_p$ , we have thus:  $\boldsymbol{\sigma}^f(\mathbf{v}, p) \cdot \boldsymbol{\nu} \in H^{-1/2}(\partial\Omega_f)^d$  and

$\boldsymbol{\sigma}^p(\mathbf{v}, p) \cdot \boldsymbol{\nu} \in H^{-1/2}(\partial\Omega_p)^d$ . Using now Green's formulas, the  $L^2(\Omega_f)$  scalar product by any test function  $\mathbf{w} \in \mathbf{W}$  of the momentum equation in  $\Omega_f$  yields from (22) with (18):

$$\begin{aligned} 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx - \int_{\Omega_f} p \nabla \cdot \mathbf{w} \, dx - \langle \boldsymbol{\sigma}^f(\mathbf{v}, p) \cdot \boldsymbol{\nu}, \mathbf{w} \rangle_{-1/2, \partial\Omega_f} \\ = \int_{\Omega_f} \rho \mathbf{f} \cdot \mathbf{w} \, dx, \end{aligned} \quad (33)$$

where the first term is obtained using the equality:  $2 \mathbf{D}(\mathbf{v}) : \nabla \mathbf{w} = \mathbf{D}(\mathbf{v}) : (\nabla \mathbf{w} + \nabla \mathbf{w}^T)$  that holds because of the symmetry of the tensor  $\mathbf{D}(\mathbf{v})$ . Since  $\mathbf{w}^f = 0$  on  $\Gamma_d^f$ , the duality pairing on  $\partial\Omega_f \setminus \Gamma_d^f = \Gamma_n^f \cup \Sigma$  in (33) holds between  $\tilde{H}^{1/2}(\Gamma_n^f \cup \Sigma)^d := H_{00}^{1/2}(\Gamma_n^f \cup \Sigma)^d$  and its dual space  $\tilde{H}^{-1/2}(\Gamma_n^f \cup \Sigma)^d := (\tilde{H}^{1/2}(\Gamma_n^f \cup \Sigma)^d)'$ ; see [54]. Then using  $\mathbf{w}^f = 0$  on  $\Gamma_d^f$  and incorporating in (33) the stress boundary condition from (19), we get since  $\mathbf{h} \in \tilde{H}^{-1/2}(\Gamma_n^f)^d$ :

$$\begin{aligned} 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx - \int_{\Omega_f} p \nabla \cdot \mathbf{w} \, dx + \langle \boldsymbol{\sigma}^f(\mathbf{v}, p) \cdot \mathbf{n}, \mathbf{w}_\Sigma \rangle_{-1/2, \Sigma} \\ = \int_{\Omega_f} \rho \mathbf{f} \cdot \mathbf{w} \, dx + \langle \mathbf{h}, \mathbf{w} \rangle_{-1/2, \Gamma_n^f}. \end{aligned} \quad (34)$$

By doing similarly in  $\Omega_p$  from (22, 15, 16), it yields with  $\mathbf{w}^p = 0$  on  $\Gamma_d^p$ :

$$\begin{aligned} 2 \int_{\Omega_p} \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx - \int_{\Omega_p} p \nabla \cdot \mathbf{w} \, dx + \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx \\ - \langle \boldsymbol{\sigma}^p(\mathbf{v}, p) \cdot \mathbf{n}, \mathbf{w}_\Sigma \rangle_{-1/2, \Sigma} = \int_{\Omega_p} \rho \mathbf{f} \cdot \mathbf{w} \, dx + \langle \mathbf{g}, \mathbf{w} \rangle_{-1/2, \Gamma_n^p}. \end{aligned} \quad (35)$$

Now by summing (34) and (35), it turns out that for all  $\mathbf{w} \in \mathbf{W}$ :

$$\begin{aligned} 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx + 2 \int_{\Omega_p} \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx + \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx \\ - \int_{\Omega} p \nabla \cdot \mathbf{w} \, dx + \langle \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma, \mathbf{w}_\Sigma \rangle_{-1/2, \Sigma} \\ = \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{w} \, dx + \langle \mathbf{g}, \mathbf{w} \rangle_{-1/2, \Gamma_n^p} + \langle \mathbf{h}, \mathbf{w} \rangle_{-1/2, \Gamma_n^f}. \end{aligned} \quad (36)$$

For all  $\mu \in M$ , the Sobolev continuous imbedding yields:  $\mu_\Sigma^p \in H^{1/2}(\Sigma) \hookrightarrow L^4(\Sigma)$  (for  $d \leq 3$ ). Then from (20), the Hölder inequality implies that  $\llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma \in L^2(\Sigma)^d$ . Moreover, we have also the functional setting below with continuous imbeddings by identifying  $L^2(\Sigma)^d$  with its dual space:

$$\tilde{H}^{1/2}(\Sigma)^d \hookrightarrow L^2(\Sigma)^d \hookrightarrow \tilde{H}^{-1/2}(\Sigma)^d. \quad (37)$$

Hence, by including in (36) the stress jump condition (20) with  $\llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma \in L^2(\Sigma)^d$ , we finally get with the definitions (28–30):

$$a(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = \ell(\mathbf{w}), \quad \text{for all } \mathbf{w} \in \mathbf{W}. \quad (38)$$

Then, taking the  $L^2$  scalar product by any test function  $q \in Q = L^2(\Omega)$  of the mass conservation equation in (22) gives with (29, 31):

$$b(\mathbf{v}, q) = g(q), \quad \text{for all } q \in Q. \quad (39)$$

Therefore, (38) and (39) yield that the pair  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  solves the weak problem (32).

Conversely, let  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  be a solution to the weak problem (32). In particular from the definition of  $\mathbf{W}$ ,  $\mathbf{v}$  satisfies  $\nabla \cdot \mathbf{v} \in L^2(\Omega)$  and the homogeneous boundary conditions of (16, 19) on  $\Gamma_d^p$  and  $\Gamma_d^f$ , respectively. With the properties of Lebesgue's integral, it is clear by taking for example any smooth and compactly supported test function  $q \in \mathcal{C}_c^\infty(\Omega)$  that (39) is equivalent to  $\nabla \cdot \mathbf{v} = q_m$  in  $\Omega$ , and thus  $\mathbf{v}$  satisfies the mass conservation equation in (22). By choosing in (38) any smooth and compactly supported test function  $\mathbf{w} \in \mathcal{C}_c^\infty(\Omega_f)^d$ , we get after integration by part:

$$\langle \nabla \cdot (-2\mu^f \mathbf{D}(\mathbf{v}) + \nabla p), \mathbf{w} \rangle_{-1, \Omega_f} = \int_{\Omega_f} \rho \mathbf{f} \cdot \mathbf{w} \, dx, \quad \text{for all } \mathbf{w} \in \mathcal{C}_c^\infty(\Omega_f)^d, \quad (40)$$

where  $\langle \cdot, \cdot \rangle_{-1, \Omega_f}$  is the duality pairing between  $H^{-1}(\Omega_f)^d$  and  $H_0^1(\Omega_f)^d$ . Since  $\mathcal{C}_c^\infty(\Omega_f)^d$  is dense in  $H_0^1(\Omega_f)^d$ , (40) yields:

$$-2\nabla \cdot (\mu^f \mathbf{D}(\mathbf{v})) + \nabla p = \rho \mathbf{f} \quad \text{in } \Omega_f. \quad (41)$$

Doing similarly in  $\Omega_p$ , we get:

$$-2\nabla \cdot \left( \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) \right) \mu^p \mathbf{K}_p^{-1} \mathbf{v} + \nabla p = \rho \mathbf{f} \quad \text{in } \Omega_p. \quad (42)$$

Hence, it turns out with (41) and (42) that the pair  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  is a solution to the set of governing equations (22). The velocity continuity  $\mathbf{v}^f = \mathbf{v}^p$  on  $\Sigma$  is obviously included in the definition of the space  $\mathbf{W}$ . By taking now in (38) any test function  $\mathbf{w} \in H_0^1(\Omega)^d$  and comparing with (36), it yields:

$$\langle \llbracket \boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n} \rrbracket_\Sigma, \mathbf{w}_\Sigma \rangle_{-1/2, \Sigma} = \int_\Sigma \frac{\mu_\Sigma^p}{\sqrt{K_p}} (\boldsymbol{\beta}_\Sigma \mathbf{v}_\Sigma) \cdot \mathbf{w}_\Sigma \, ds - \int_\Sigma \mathbf{f}_\Sigma \cdot \mathbf{w}_\Sigma \, ds, \quad (43)$$

$$\forall \mathbf{w} \in H_0^1(\Omega)^d.$$

Since the trace space of  $\mathbf{w}$  on  $\Sigma$  is large enough, *i.e.* the linear trace operator  $H_0^1(\Omega)^d \mapsto \tilde{H}^{1/2}(\Sigma)^d$  is surjective as shown at the end of the proof of Lemma 6, Eq. (43) implies that the stress jump transmission condition in (20) is recovered on  $\Sigma$ . Similarly as above, by taking in (38) any test function  $\mathbf{w} \in H_{0, \Gamma_d^f}^1(\Omega_f)^d$  or  $\mathbf{w} \in H_{0, \Gamma_d^p}^1(\Omega_p)^d$ , the traction boundary conditions in (19, 16) are also recovered on  $\Gamma_n^f$  and  $\Gamma_n^p$ , respectively. Finally, we have shown that the pair  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  solves the boundary-value problem (22, 16, 19, 20).  $\square$

**Lemma 2 (Continuity of  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$ ,  $\ell(\cdot)$  and  $g(\cdot)$ ).** *For all data as mentioned above, there exists  $C_a > 0$ ,  $C_b > 0$ ,  $C_\ell > 0$  and  $C_g > 0$  depending only of the data such that the following continuity inequalities hold:*

$$|a(\mathbf{v}, \mathbf{w})| \leq C_a \|\mathbf{v}\|_{\mathbf{W}} \|\mathbf{w}\|_{\mathbf{W}}, \quad \text{for all } (\mathbf{v}, \mathbf{w}) \in \mathbf{W} \times \mathbf{W} \quad (44)$$

$$|b(\mathbf{w}, q)| \leq C_b \|\mathbf{w}\|_{\mathbf{W}} \|q\|_Q, \quad \text{for all } (\mathbf{w}, q) \in \mathbf{W} \times Q \quad (45)$$

$$|\ell(\mathbf{w})| \leq C_\ell \|\mathbf{w}\|_{\mathbf{W}}, \quad \text{for all } \mathbf{w} \in \mathbf{W} \quad (46)$$

$$|g(q)| \leq C_g \|q\|_Q, \quad \text{for all } q \in Q. \quad (47)$$

PROOF. For all  $\mu \in M$  and  $(\mathbf{v}, \mathbf{w}) \in \mathbf{W} \times \mathbf{W}$ , the Sobolev continuous imbedding:  $H^{1/2}(\Sigma) \hookrightarrow L^4(\Sigma)$  (for  $d \leq 3$ ), the Hölder inequality with the trace inequalities in  $H^1(\Omega_p)$  and  $H^1(\Omega_f)^d$  imply that the last term in (28) verifies the continuity bound:

$$\left| \int_{\Sigma} \frac{\mu_{\Sigma}^p}{\sqrt{K_p}} (\boldsymbol{\beta}_{\Sigma} \mathbf{v}_{\Sigma}) \cdot \mathbf{w}_{\Sigma} \, ds \right| \leq c(\Omega_f, \Omega_p, \Sigma, K_p) \|\boldsymbol{\beta}_{\Sigma}\|_{\infty} \|\mu^p\|_{1, \Omega_p} \|\mathbf{v}\|_{1, \Omega_f} \|\mathbf{w}\|_{1, \Omega_f}. \quad (48)$$

With the Cauchy-Schwarz inequality in  $\Omega_f$  and  $\Omega_p$  for the other terms of  $a(\mathbf{v}, \mathbf{w})$  in (28) and the bounds from (24), we have for all  $\mathbf{v}, \mathbf{w} \in \mathbf{W}$ :

$$\begin{aligned} \left| 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx \right| &\leq 2 \mu_M \|\mathbf{D}(\mathbf{v})\|_{0, \Omega_f} \|\mathbf{D}(\mathbf{w})\|_{0, \Omega_f} \\ &\leq 2 \mu_M \|\nabla \mathbf{v}\|_{0, \Omega_f} \|\nabla \mathbf{w}\|_{0, \Omega_f} \\ \left| 2 \int_{\Omega_p} \frac{\mu^p}{\phi_p} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx \right| &\leq 2 \mu_M \phi_m^{-1} \|\nabla \mathbf{v}\|_{0, \Omega_p} \|\nabla \mathbf{w}\|_{0, \Omega_p} \\ \left| \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx \right| &\leq \mu_M \|\mathbf{K}_p^{-1}\|_{\infty} \|\mathbf{v}\|_{0, \Omega_p} \|\mathbf{w}\|_{0, \Omega_p}. \end{aligned} \quad (49)$$

Thus, using (48) and (49), we get (44) with  $C_a > 0$  depending only on the data.

By applying the Cauchy-Schwarz inequality in (29), it is straightforward to obtain (45) with  $C_b = 2$  since for all  $\mathbf{w} \in \mathbf{W}$ ,  $q \in Q$ :

$$|b(\mathbf{w}, q)| \leq \|\nabla \cdot \mathbf{w}\|_{0, \Omega} \|q\|_{0, \Omega} \leq 2 \|\nabla \mathbf{w}\|_{0, \Omega} \|q\|_{0, \Omega}, \quad (50)$$

because usual calculations show that:  $\|\nabla \cdot \mathbf{w}\|_{0, \Omega}^2 \leq 3 \|\nabla \mathbf{w}\|_{0, \Omega}^2$ , for all  $\mathbf{w} \in H^1(\Omega)^d$ .

Using the Cauchy-Schwarz inequality from (30) and the duality inequalities combined with trace inequalities in  $H^1(\Omega_p)^d$  and  $H^1(\Omega_f)^d$ , we get for all  $\mathbf{w} \in \mathbf{W}$ :

$$\begin{aligned} |\ell(\mathbf{w})| &\leq \|\rho\|_{\infty} \|\mathbf{f}\|_{0, \Omega} \|\mathbf{w}\|_{0, \Omega} + c(\Omega_f, \Sigma) \|\mathbf{f}_{\Sigma}\|_{0, \Sigma} \|\mathbf{w}\|_{1, \Omega_f} \\ &\quad + c(\Omega_p, \Gamma_n^p) \|\mathbf{g}\|_{\tilde{H}^{-1/2}(\Gamma_n^p)^d} \|\mathbf{w}\|_{1, \Omega_p} + c(\Omega_f, \Gamma_n^f) \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d} \|\mathbf{w}\|_{1, \Omega_f}, \end{aligned} \quad (51)$$

from which (46) is obtained with some  $C_{\ell} > 0$ . The last continuity inequality (47) holds true immediately with  $C_g = \|q_m\|_{0, \Omega}$  from (31).  $\square$

**Lemma 3 (Coercivity of  $a(\cdot, \cdot)$  on  $\mathbf{W}$ ).** *For all data as mentioned above, the bilinear form  $a(\cdot, \cdot)$  is coercive on  $\mathbf{W}$ , i.e. there exists  $\alpha_a(\Omega_f, \Omega_p, \Gamma_d^f, \Gamma_d^p, \mu_m) > 0$  such that:*

$$a(\mathbf{w}, \mathbf{w}) \geq \alpha_a \|\mathbf{w}\|_{\mathbf{W}}^2, \quad \text{for all } \mathbf{w} \in \mathbf{W}. \quad (52)$$

PROOF. The last term of  $a(\mathbf{w}, \mathbf{w})$  from (28) being non-negative, we have using (24):

$$a(\mathbf{w}, \mathbf{w}) \geq 2 \mu_m \|\mathbf{D}(\mathbf{w})\|_{0, \Omega_f}^2 + 2 \mu_m \|\mathbf{D}(\mathbf{w})\|_{0, \Omega_p}^2 + \mu_m k_m \|\mathbf{w}\|_{0, \Omega_p}^2, \quad \forall \mathbf{w} \in \mathbf{W}. \quad (53)$$

Then, we recall Poincaré-Friedrichs' inequality in  $H_{0, \Gamma_d^f}^1(\Omega_f)^d$  (e.g. [52, Proposition III.2.38]),  $\Omega_f$  being a bounded and connected Lipschitz domain of  $\mathbb{R}^d$ :

$$\|\mathbf{v}\|_{0, \Omega_f} \leq C_P^f(\Omega_f, \Gamma_d^f) \|\nabla \mathbf{v}\|_{0, \Omega_f}, \quad \text{for all } \mathbf{v} \in H_{0, \Gamma_d^f}^1(\Omega_f)^d, \quad (54)$$

and the following Korn inequality in  $H_{0,\Gamma_d^f}^1(\Omega_f)^d$  as a consequence of Korn's second inequality in  $H^1(\Omega_f)^d$  (see e.g. [52, Lemma IV.7.6]) combined with (54):

$$\|\mathbf{D}(\mathbf{v})\|_{0,\Omega_f}^2 \geq C_K^f(\Omega_f, \Gamma_d^f) \|\mathbf{v}\|_{1,\Omega_f}^2, \quad \text{for all } \mathbf{v} \in H_{0,\Gamma_d^f}^1(\Omega_f)^d. \quad (55)$$

Using (55) and similar Korn's inequality in  $H_{0,\Gamma_d^p}^1(\Omega_p)^d$  on the domain  $\Omega_p$  in (53), we get:

$$\begin{aligned} a(\mathbf{w}, \mathbf{w}) &\geq 2\mu_m \min(C_K^f, C_K^p) \left( \|\mathbf{w}\|_{1,\Omega_f}^2 + \|\mathbf{w}\|_{1,\Omega_p}^2 \right) \\ &\geq 2\mu_m \min(C_K^f, C_K^p) \|\mathbf{w}\|_{\mathbf{W}}^2, \quad \text{for all } \mathbf{w} \in \mathbf{W}. \end{aligned} \quad (56)$$

Hence with  $\alpha_a(\Omega_f, \Omega_p, \Gamma_d^f, \Gamma_d^p, \mu_m) := 2\mu_m \min(C_K^f, C_K^p) > 0$ , the coercivity inequality (52) holds.  $\square$

Then, we prove the following result of well-posedness with no restriction on the natural size of the data.

**Theorem 1 (Solvability of variable-viscosity Stokes/Darcy-Brinkman flow coupled with (20)).**

*Under the assumptions (24), let us consider any data  $\rho \in L^\infty(\Omega)$ ,  $\mu \in M$ ,  $\mathbf{f} \in L^2(\Omega)^d$ ,  $q_m \in L^2(\Omega)$ ,  $\mathbf{f}_\Sigma \in L^2(\Sigma)^d$ ,  $\mathbf{g} \in \tilde{H}^{-1/2}(\Gamma_n^p)^d$ ,  $\mathbf{h} \in \tilde{H}^{-1/2}(\Gamma_n^f)^d$ , any bounded, symmetric and uniformly positive definite tensor  $\mathbf{K}_p \in L^\infty(\Omega_p)^{d \times d}$  and any bounded uniformly positive semi-definite tensor  $\beta_\Sigma \in L^\infty(\Sigma)^{d \times d}$ .*

*Then, there exists a unique solution  $(\mathbf{v}, p) \in \mathbf{W} \times Q$  to the mixed weak problem (32), i.e. also to the Stokes/Darcy-Brinkman fluid-porous flow (22, 16, 19) coupled with the stress jump interface conditions (20) on  $\Sigma$ . In addition, the solution satisfies the following energy estimate:*

$$\|\mathbf{v}\|_{\mathbf{W}} + \|p\|_Q \leq C(\alpha_a, \beta_b, \|a\|, \|b\|) (\|\ell\|_{\mathbf{W}'} + \|q_m\|_{0,\Omega}). \quad (57)$$

PROOF. By using the above Lemmas 1, 2, 3 and applying [51, Theorem 4.1 & Corollary 4.1] resulting from the Banach-Nečas-Babuška theory [61, 62], it remains to prove that  $b(\cdot, \cdot)$  satisfies the inf-sup condition, i.e. there exists  $\beta_b > 0$  depending only on the data such that:

$$\sup_{\mathbf{w} \in \mathbf{W}; \mathbf{w} \neq 0} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{\mathbf{W}}} \geq \beta_b \|q\|_Q, \quad \text{for all } q \in Q. \quad (58)$$

This stems from Nečas' Theorem in [63], e.g. [52, Theorem IV.3.1], stating that for any  $q_0 \in L_0^2(\Omega)$ , there exists a function  $\mathbf{u}_0 \in H_0^1(\Omega)^d$  such that with some  $C_N(\Omega) > 0$ :

$$\nabla \cdot \mathbf{u}_0 = q_0 \quad \text{in } \Omega \quad \text{and} \quad \|\mathbf{u}_0\|_{1,\Omega} \leq C_N(\Omega) \|q_0\|_{0,\Omega}. \quad (59)$$

By considering now any function  $q \in Q = L^2(\Omega)$  and denoting by  $m(q)$  its mean all over the domain  $\Omega$ :

$$m(q) := \frac{1}{|\Omega|} \int_{\Omega} q \, dx \quad \text{where } |\Omega| := \text{meas}(\Omega), \quad (60)$$

then we have  $q_0 := q - m(q) \in L_0^2(\Omega)$  and thus (59) holds true. Following [52, Section IV.7.1] to get a lifting of the mass source term  $q$ , let us now define the vector field  $\tilde{\mathbf{u}} \in \mathbf{W}$  by:

$$\tilde{\mathbf{u}} := \frac{m(q)}{d} \mathbf{x} \quad \text{for all } \mathbf{x} := (x_1, \dots, x_d)^T \in \Omega \subset \mathbb{R}^d, \quad \text{with } \mathbf{u} = 0 \quad \text{on } \Gamma_d^f \cup \Gamma_d^p, \quad (61)$$

that verifies:

$$\nabla \cdot \tilde{\mathbf{u}} = m(q) \quad \text{and} \quad \nabla \tilde{\mathbf{u}} = \frac{m(q)}{d} \mathbf{I} \quad \text{in } \Omega. \quad (62)$$

Thus, using the Cauchy-Schwarz inequality to bound  $m(q)$ , there exists  $\tilde{C}(\Omega, d) > 0$  such that:

$$\|\tilde{\mathbf{u}}\|_{1,\Omega} \leq \tilde{C}(\Omega, d) \|q\|_{0,\Omega}. \quad (63)$$

Therefore, we have constructed a function  $\mathbf{u} := (\mathbf{u}_0 + \tilde{\mathbf{u}}) \in \mathbf{W}$  such that, using (59) and (63), we have with some  $C(\Omega, d) > 0$ :

$$\nabla \cdot \mathbf{u} = q_0 + m(q) = q \quad \text{in } \Omega \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{W}} = \|\mathbf{u}\|_{1,\Omega} \leq C(\Omega, d) \|q\|_{0,\Omega}, \quad (64)$$

i.e. that the Divergence operator  $\nabla \cdot : \mathbf{W} \subset H^1(\Omega)^d \mapsto Q = L^2(\Omega)$  admits a right-inverse. Now taking  $\mathbf{w} = -\mathbf{u}$  as a particular candidate for a lower bound in (58), we get with (64):

$$\sup_{\mathbf{w} \in \mathbf{W}; \mathbf{w} \neq 0} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{\mathbf{W}}} \geq \frac{-b(\mathbf{u}, q)}{\|\mathbf{u}\|_{\mathbf{W}}} = \frac{\|q\|_{0,\Omega}^2}{\|\mathbf{u}\|_{\mathbf{W}}} \geq \frac{1}{C(\Omega, d)} \|q\|_Q, \quad (65)$$

from which the inf-sup condition (58) holds with  $\beta_b(\Omega, d) = 1/C(\Omega, d)$  since  $q$  was chosen arbitrary in  $L^2(\Omega)$  in the above construction.

Then, the energy estimate (57) follows with usual arguments.  $\square$

**Remark 1 (Generalizations).** Replacing the assumption of  $\mathbf{f}_\Sigma$  given in  $L^2(\Sigma)^d$  by a data  $\mathbf{f}_\Sigma \in \tilde{H}^{-1/2}(\Sigma)^d$  is straightforward throughout the analysis, which can be required for the multi-physics coupling where complex phenomena occur at the interface  $\Sigma$ . Moreover, the case with non-homogeneous velocity boundary conditions on  $\Gamma_d^f$  or  $\Gamma_d^p$  is admissible too. These extensions are not considered here for the sake of brevity.

#### 4. Solvability of the coupled Stokes/Darcy fluid-porous flow

As carried out in [44, 45], the strategy to prove the well-posed coupling of Stokes/Darcy flows uses a regularization with a vanishing effective viscosity  $\tilde{\mu}^p = \varepsilon > 0$  in the porous domain  $\Omega_p$ , and then passes to the limit when  $\varepsilon \rightarrow 0$ .

Let us define the following (real) Hilbert spaces endowed with their natural respective inner products and associated norms:

$$\begin{aligned} \mathbf{X} &:= \{ \mathbf{u} \in \mathbf{H}_{div}(\Omega); \mathbf{u}^f \in H^1(\Omega_f)^d \}, \\ \mathbf{V} &:= \left\{ \mathbf{v} \in \mathbf{X}; \mathbf{v}^f \in H_{0,\Gamma_d^f}^1(\Omega_f)^d, \mathbf{v}_\Sigma := \mathbf{v}_\Sigma^f, \mathbf{v}^p \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_d^p \right\}, \\ \mathbf{V}_0 &:= \{ \mathbf{v} \in \mathbf{V}; \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \}, \end{aligned} \quad (66)$$

with the Hilbertian norms (and naturally associated inner products) in  $\mathbf{X}, \mathbf{V}$  (inherited from  $\mathbf{X}$ ) defined by:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{X}}^2 &:= \|\mathbf{u}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{u}\|_{0,\Omega}^2 + \|\nabla \mathbf{u}^f\|_{0,\Omega_f}^2, & \text{for all } \mathbf{u} \in \mathbf{X}, \\ \|\mathbf{v}\|_{\mathbf{V}}^2 &:= \|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{v}\|_{0,\Omega}^2 + \|\nabla \mathbf{v}^f\|_{0,\Omega_f}^2, & \text{for all } \mathbf{v} \in \mathbf{V}, \\ \|\mathbf{v}\|_{\mathbf{V}_0}^2 &:= \|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla \mathbf{v}^f\|_{0,\Omega_f}^2, & \text{for all } \mathbf{v} \in \mathbf{V}_0. \end{aligned} \quad (67)$$

Hence,  $\mathbf{V}$  is a Hilbert space as a closed subspace of the Hilbert space  $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ ; see Propositions 1 and 2 in Appendix A. Besides, we have the following continuous imbeddings:  $\mathbf{W} \hookrightarrow \mathbf{V} \hookrightarrow \mathbf{X}$  and  $\mathbf{W}$  is dense in  $\mathbf{V}$  because  $H^1(\Omega_p)^d$  is dense in  $\mathbf{H}_{div}(\Omega_p)$ . Then for all  $\mathbf{v} \in \mathbf{V}$ , we have:  $[[\mathbf{v} \cdot \mathbf{n}]_{\Sigma}] = 0$  on  $\Sigma$  from Lemma 6 in Appendix A and thus  $\mathbf{v} \cdot \mathbf{n} := \mathbf{v}^p \cdot \mathbf{n} = \mathbf{v}^f \cdot \mathbf{n} \in \tilde{H}^{1/2}(\Sigma) \hookrightarrow H^{1/2}(\Sigma)$  since  $\mathbf{v}^f \cdot \mathbf{n} \in \tilde{H}^{1/2}(\Sigma)$ . Moreover, we have by Sobolev imbedding for all  $\mathbf{v} \in \mathbf{V}$ :  $\mathbf{v}_{\Sigma} := \mathbf{v}_{\Sigma}^f \in H^{1/2}(\Sigma)^d \hookrightarrow L^4(\Sigma)^d$  (for  $d \leq 3$ ) with the related continuity inequality of the imbedding.

#### 4.1. Solvability of the regularized Stokes/Darcy problem

In the regularized procedure, we consider for all  $\varepsilon > 0$  the Stokes/Darcy-Brinkman problem with no velocity jump on  $\Sigma$  as follows still with natural bounds of the data (24) and for  $\mathbf{f} \in L^2(\Omega)^d$  and  $q_m \in L^2(\Omega)$ :

$$\begin{cases} \nabla \cdot \mathbf{v}_{\varepsilon} = q_m & \text{in } \Omega, \\ -2\varepsilon \nabla \cdot \mathbf{D}(\mathbf{v}_{\varepsilon}^p) + \mu^p \mathbf{K}_p^{-1} \mathbf{v}_{\varepsilon}^p + \nabla p_{\varepsilon}^p = \rho \mathbf{f} & \text{in } \Omega_p, \\ -2\nabla \cdot (\mu^f \mathbf{D}(\mathbf{v}_{\varepsilon}^f)) + \nabla p_{\varepsilon}^f = \rho \mathbf{f} & \text{in } \Omega_f, \end{cases} \quad (68)$$

still supplemented with the boundary conditions (16) on  $\Gamma^p$  and 19 on  $\Gamma^f$  and coupled with the stress jump interface conditions below on  $\Sigma$  with the definitions (9) and  $\mathbf{f}_{\Sigma} \in L^2(\Sigma)^d$ :

$$\begin{cases} [[\mathbf{v}_{\varepsilon}]_{\Sigma}] = 0 & \text{i.e. } \mathbf{v}_{\varepsilon\Sigma}^f = \mathbf{v}_{\varepsilon\Sigma}^p := \mathbf{v}_{\varepsilon\Sigma} \\ [[\boldsymbol{\sigma}(\mathbf{v}_{\varepsilon}, p_{\varepsilon}) \cdot \mathbf{n}]_{\Sigma}] = \frac{\mu_{\Sigma}^p}{\sqrt{K_p}} \beta_{\Sigma} \mathbf{v}_{\varepsilon\Sigma} - \mathbf{f}_{\Sigma} & \text{on } \Sigma. \end{cases} \quad (69)$$

Let us now consider the bilinear form  $a_{\varepsilon}(\cdot, \cdot)$  below, the other forms in (27, 29–31) remaining unchanged:

$$a_{\varepsilon}(\cdot, \cdot) : \mathbf{W} \times \mathbf{W} \mapsto \mathbb{R}, \quad (70)$$

defined for all  $\mathbf{v}, \mathbf{w} \in \mathbf{W}$  by:

$$\begin{aligned} a_{\varepsilon}(\mathbf{v}, \mathbf{w}) &:= 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx + 2\varepsilon \int_{\Omega_p} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx \\ &+ \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx + \int_{\Sigma} \frac{\mu_{\Sigma}^p}{\sqrt{K_p}} (\beta_{\Sigma} \mathbf{v}_{\Sigma}) \cdot \mathbf{w}_{\Sigma} \, ds. \end{aligned} \quad (71)$$

Then we have the following solvability result as a consequence of Theorem 1.

**Corollary 1 (Solvability of the regularized Stokes/Darcy flow (68) coupled with (69)).** *Under the assumptions of Theorem 1 and for all  $\varepsilon > 0$ , there exists a unique solution  $(\mathbf{v}_{\varepsilon}, p_{\varepsilon}) \in \mathbf{W} \times Q$  to the coupled Stokes/Darcy-Brinkman fluid-porous flow (68, 16, 19) supplemented with the stress jump interface conditions (69) that is also equivalent to the following mixed weak problem:*

$$\begin{cases} \text{find a pair } (\mathbf{v}_{\varepsilon}, p_{\varepsilon}) \in \mathbf{W} \times Q \text{ such that:} \\ a_{\varepsilon}(\mathbf{v}_{\varepsilon}, \mathbf{w}) + b(\mathbf{w}, p_{\varepsilon}) = \ell(\mathbf{w}), & \text{for all } \mathbf{w} \in \mathbf{W}, \\ b(\mathbf{v}_{\varepsilon}, q) = g(q), & \text{for all } q \in Q = L^2(\Omega), \end{cases} \quad (72)$$

with the definitions (71, 29–31).

PROOF. The result directly stems from Lemmas 1, 2, 3 and Theorem 1.  $\square$

#### 4.2. Solvability of the Stokes/Darcy problem

Now to recover the Stokes/Darcy problem from the study in the previous section 4.1, it is necessary to pass to the limit in the mixed weak formulation (72) when  $\varepsilon$  goes to zero. This requires uniform energy estimates proved further in Lemma 4. Henceforth to get (13) with (11), we consider an homogeneous Dirichlet boundary condition for the pressure  $p = 0$  on  $\Gamma_n^p$ , i.e.  $\mathbf{g} = 0$ . Besides in addition to (24), we assume the following bounds on the data on  $\Sigma$  with  $\mu_m > 0$  and  $\beta_M \geq \beta_m > 0$  to be compatible with the coupling of Stokes/Darcy flows:

$$\begin{cases} \mu \in M; & \mu_\Sigma^p(s) \geq \mu_m > 0 & \text{a.e. } s \in \Sigma, \\ \beta_\Sigma \in L^\infty(\Sigma)^{d \times d}; & \beta_\infty := \|\beta_\Sigma\|_{L^\infty(\Sigma)^{d \times d}}, & \\ & \beta_m |\mathbf{y}|^2 \leq (\beta_\Sigma(s) \mathbf{y}) \cdot \mathbf{y} \leq \beta_M |\mathbf{y}|^2, \forall \mathbf{y} \in \mathbb{R}^d & \text{a.e. } s \in \Sigma. \end{cases} \quad (73)$$

**Lemma 4 (Uniform energy estimates for the weak problem (72) with vanishing viscosity).** *Under the assumptions of Corollary 1, with (73) and  $\mathbf{g} = 0$ , the unique solution  $(\mathbf{v}_\varepsilon, p_\varepsilon) \in \mathbf{W} \times Q$  of the mixed weak problem (72) satisfies the following uniform energy estimate for all  $\varepsilon > 0$  such that  $\varepsilon \leq 1$ :*

$$\begin{aligned} & \frac{\mu_m}{2} C_K^f \|\mathbf{v}_\varepsilon^f\|_{1, \Omega_f}^2 + \varepsilon C_K^p \|\mathbf{v}_\varepsilon^p\|_{1, \Omega_p}^2 + \frac{\mu_m}{2} k_m \|\mathbf{v}_\varepsilon^p\|_{0, \Omega_p}^2 + \frac{\mu_m}{2\sqrt{K_p}} \beta_m \|\mathbf{v}_{\varepsilon\Sigma}\|_{0, \Sigma}^2 \\ & \quad + \frac{\beta_b^2}{2} \|p_\varepsilon\|_{0, \Omega}^2 \\ & \leq C \left( \|q_m\|_{0, \Omega}^2 + \rho_\infty^2 \|\mathbf{f}\|_{0, \Omega}^2 + \|\mathbf{f}_\Sigma\|_{0, \Sigma}^2 + \beta_\infty^2 \|\mu\|_{1, \Omega_p}^2 + \|\mathbf{h}\|_{\dot{H}^{-1/2}(\Gamma_n^f)}^2 \right), \end{aligned} \quad (74)$$

with some  $C(\Omega_f, \Omega_p, \Sigma, \Gamma_d^f, \Gamma_d^p, \Gamma_n^f, K_p, \mu_M, \mu_m, \beta_m, k_m, k_\infty, \beta_b) > 0$  depending only on the data and  $C_K^f(\Omega_f, \Gamma_d^f)$ ,  $C_K^p(\Omega_p, \Gamma_d^p) > 0$  coming from Korn's inequalities in  $\Omega_f$  (55) and  $\Omega_p$ , respectively.

**PROOF.** By taking  $w = \mathbf{v}_\varepsilon$  and  $q = p_\varepsilon$  in the weak problem (72) and subtracting the two equations, we have:

$$a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) = \ell(\mathbf{v}_\varepsilon) - g(p_\varepsilon) \leq |\ell(\mathbf{v}_\varepsilon)| + |g(p_\varepsilon)|. \quad (75)$$

More precisely with the definition (71),  $a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon)$  reads:

$$\begin{aligned} a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) &= 2 \int_{\Omega_f} \mu^f |\mathbf{D}(\mathbf{v}_\varepsilon)|^2 dx + 2\varepsilon \int_{\Omega_p} |\mathbf{D}(\mathbf{v}_\varepsilon)|^2 dx + \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}_\varepsilon) \cdot \mathbf{v}_\varepsilon dx \\ & \quad + \int_{\Sigma} \frac{\mu_\Sigma^p}{\sqrt{K_p}} (\beta_\Sigma \mathbf{v}_{\varepsilon\Sigma}) \cdot \mathbf{v}_{\varepsilon\Sigma} ds, \end{aligned} \quad (76)$$

that can be lower bounded as follows using Korn's inequalities (55) on  $\Omega_f$  and  $\Omega_p$  and the bounds (24,73):

$$\begin{aligned} a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) &\geq 2\mu_m C_K^f(\Omega_f, \Gamma_d^f) \|\mathbf{v}_\varepsilon\|_{1, \Omega_f}^2 + 2\varepsilon C_K^p(\Omega_p, \Gamma_d^p) \|\mathbf{v}_\varepsilon\|_{1, \Omega_p}^2 \\ & \quad + \mu_m k_m \|\mathbf{v}_\varepsilon\|_{0, \Omega_p}^2 + \frac{\mu_m}{\sqrt{K_p}} \beta_m \|\mathbf{v}_{\varepsilon\Sigma}\|_{0, \Sigma}^2. \end{aligned} \quad (77)$$

Then with the definitions (30,31) and  $\mathbf{g} = 0$ , the right-hand side terms in (75) are upper bounded as below using arguments already invoked above and the inequality  $ab \leq (a^2 + b^2)/2$ :

$$\begin{aligned}
|\ell(\mathbf{v}_\varepsilon)| &\leq \rho_\infty \|\mathbf{f}\|_{0,\Omega} \|\mathbf{v}_\varepsilon\|_{0,\Omega} + \|\mathbf{f}_\Sigma\|_{0,\Sigma} \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma} + \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d} \|\mathbf{v}_\varepsilon\|_{\tilde{H}^{1/2}(\Gamma_n^f)^d} \\
&\leq \rho_\infty \|\mathbf{f}\|_{0,\Omega} (\|\mathbf{v}_\varepsilon\|_{0,\Omega_p} + \|\mathbf{v}_\varepsilon\|_{1,\Omega_f}) + \|\mathbf{f}_\Sigma\|_{0,\Sigma} \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma} \\
&\quad + c(\Omega_f, \Gamma_n^f) \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d} \|\mathbf{v}_\varepsilon\|_{1,\Omega_f} \\
&\leq \mu_m C_K^f \|\mathbf{v}_\varepsilon\|_{1,\Omega_f}^2 + \frac{\mu_m}{4} k_m \|\mathbf{v}_\varepsilon\|_{0,\Omega_p}^2 + \frac{1}{4} \frac{\mu_m}{\sqrt{K_p}} \beta_m \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma}^2 \\
&\quad + C(\Omega_f, \Gamma_n^f, \mu_m, k_m, K_p, \beta_m) \left( \rho_\infty^2 \|\mathbf{f}\|_{0,\Omega}^2 + \|\mathbf{f}_\Sigma\|_{0,\Sigma}^2 + \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d}^2 \right).
\end{aligned} \tag{78}$$

$$\begin{aligned}
|g(p_\varepsilon)| &\leq \|q_m\|_{0,\Omega} \|p_\varepsilon\|_{0,\Omega} \\
&\leq \frac{1}{2} \beta_b^2 \|p_\varepsilon\|_{0,\Omega}^2 + \frac{1}{2\beta_b^2} \|q_m\|_{0,\Omega}^2.
\end{aligned} \tag{79}$$

Besides, by estimating the pressure  $p_\varepsilon$  with the inf-sup condition (58), we have using (72):

$$\beta_b \|p_\varepsilon\|_{0,\Omega} \leq \sup_{\mathbf{w} \in \mathbf{W}; \mathbf{w} \neq 0} \frac{b(\mathbf{w}, p_\varepsilon)}{\|\mathbf{w}\|_{\mathbf{W}}} = \sup_{\mathbf{w} \in \mathbf{W}; \mathbf{w} \neq 0} \frac{|\ell(\mathbf{w}) - a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{W}}}, \tag{80}$$

and the bounds of the two terms in the right-hand side follow with the definitions (71,30) and  $\mathbf{g} = 0$ :

$$\begin{aligned}
|\ell(\mathbf{w})| &\leq \rho_\infty \|\mathbf{f}\|_{0,\Omega} \|\mathbf{w}\|_{0,\Omega} + \|\mathbf{f}_\Sigma\|_{0,\Sigma} \|\mathbf{w}_\Sigma\|_{0,\Sigma} + \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d} \|\mathbf{w}\|_{\tilde{H}^{1/2}(\Gamma_n^f)^d} \\
&\leq \left( \rho_\infty \|\mathbf{f}\|_{0,\Omega} + c(\Omega_f, \Sigma) \|\mathbf{f}_\Sigma\|_{0,\Sigma} + c(\Omega_f, \Gamma_n^f) \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d} \right) \|\mathbf{w}\|_{\mathbf{W}},
\end{aligned} \tag{81}$$

$$\begin{aligned}
|a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{w})| &\leq 2\mu_M \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_f} \|\nabla \mathbf{w}\|_{0,\Omega_f} + 2\varepsilon \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_p} \|\nabla \mathbf{w}\|_{0,\Omega_p} \\
&\quad + \mu_M k_\infty \|\mathbf{v}_\varepsilon\|_{0,\Omega_p} \|\mathbf{w}\|_{0,\Omega_p} + c(\Omega_f, \Omega_p, \Sigma, K_p) \beta_\infty \|\mu\|_{1,\Omega_p} \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma} \|\mathbf{w}\|_{\mathbf{W}} \\
&\leq (2\mu_M \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_f} + 2\varepsilon \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_p} + \mu_M k_\infty \|\mathbf{v}_\varepsilon\|_{0,\Omega_p}) \|\mathbf{w}\|_{\mathbf{W}} \\
&\quad + c(\Omega_f, \Omega_p, \Sigma, K_p) \beta_\infty \|\mu\|_{1,\Omega_p} \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma} \|\mathbf{w}\|_{\mathbf{W}}.
\end{aligned} \tag{82}$$

By gathering (81,82) in (80), it gives:

$$\begin{aligned}
\beta_b \|p_\varepsilon\|_{0,\Omega} &\leq 2\mu_M \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_f} + 2\varepsilon \|\nabla \mathbf{v}_\varepsilon\|_{0,\Omega_p} + \mu_M k_\infty \|\mathbf{v}_\varepsilon\|_{0,\Omega_p} \\
&\quad + c(\Omega_f, \Omega_p, \Sigma, K_p) \beta_\infty \|\mu\|_{1,\Omega_p} \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma} \\
&\quad + \rho_\infty \|\mathbf{f}\|_{0,\Omega} + c(\Omega_f, \Sigma) \|\mathbf{f}_\Sigma\|_{0,\Sigma} + c(\Omega_f, \Gamma_n^f) \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d}.
\end{aligned} \tag{83}$$

Now from (83) at the power 2 and Young's inequality, we get:

$$\begin{aligned}
\beta_b^2 \|p_\varepsilon\|_{0,\Omega}^2 &\leq \frac{\mu_m}{2} C_K^f \|\mathbf{v}_\varepsilon\|_{1,\Omega_f}^2 + \varepsilon C_K^p \|\mathbf{v}_\varepsilon\|_{1,\Omega_p}^2 + \frac{\mu_m}{4} k_m \|\mathbf{v}_\varepsilon\|_{0,\Omega_p}^2 \\
&\quad + \frac{1}{4} \frac{\mu_m}{\sqrt{K_p}} \beta_m \|\mathbf{v}_{\varepsilon\Sigma}\|_{0,\Sigma}^2 \\
&\quad + C \left( \rho_\infty^2 \|\mathbf{f}\|_{0,\Omega}^2 + \|\mathbf{f}_\Sigma\|_{0,\Sigma}^2 + \beta_\infty^2 \|\mu\|_{1,\Omega_p}^2 + \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)^d}^2 \right),
\end{aligned} \tag{84}$$

with some  $C(\Omega_f, \Omega_p, \Sigma, \Gamma_d^f, \Gamma_n^f, K_p, \mu_M, \mu_m, k_m, k_\infty) > 0$ .

Then, by summing the inequalities (75) and (84) taking account of (77) and the estimates (78,79), it yields after absorption of some terms in the left-hand side and for all  $0 < \varepsilon \leq 1$ :

$$\begin{aligned} & \frac{\mu_m}{2} C_K^f \|\mathbf{v}_\varepsilon^f\|_{1, \Omega_f}^2 + \varepsilon C_K^p \|\mathbf{v}_\varepsilon^p\|_{1, \Omega_p}^2 + \frac{\mu_m}{2} k_m \|\mathbf{v}_\varepsilon^p\|_{0, \Omega_p}^2 + \frac{\mu_m}{2\sqrt{K_p}} \beta_m \|\mathbf{v}_{\varepsilon\Sigma}\|_{0, \Sigma}^2 \\ & \quad + \frac{\beta_b^2}{2} \|p_\varepsilon\|_{0, \Omega}^2 \\ & \leq C \left( \|q_m\|_{0, \Omega}^2 + \rho_\infty^2 \|\mathbf{f}\|_{0, \Omega}^2 + \|\mathbf{f}_\Sigma\|_{0, \Sigma}^2 + \beta_\infty^2 \|\mu\|_{1, \Omega_p}^2 + \|\mathbf{h}\|_{\tilde{H}^{-1/2}(\Gamma_n^f)}^2 \right), \end{aligned} \quad (85)$$

with some  $C(\Omega_f, \Omega_p, \Sigma, \Gamma_d^f, \Gamma_n^f, K_p, \mu_M, \mu_m, \beta_m, k_m, k_\infty, \beta_b) > 0$  depending only on the data, which finally proves (74).  $\square$

**Lemma 5 (Weak limits in the problem (72) with vanishing viscosity).** *Under the assumptions of Lemma 4, there exists  $\mathbf{v} \in \mathbf{V}$ ,  $\chi^p \in L^2(\Omega_p)^{d \times d}$  and  $p \in Q$  such that, up to a subsequence, the solution sequence  $(\mathbf{v}_\varepsilon, p_\varepsilon)_{\varepsilon > 0} \in \mathbf{W} \times Q$  of the mixed weak problem (72) satisfies the following weak limits when  $\varepsilon \rightarrow 0$ :*

- i)  $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$  weakly in  $\mathbf{V}$  and  $\mathbf{v}_\varepsilon^f \rightarrow \mathbf{v}^f$  strongly in  $L^2(\Omega_f)^d$ ,
- ii)  $\sqrt{\varepsilon} \nabla \mathbf{v}_\varepsilon^p \rightharpoonup \chi^p$  weakly in  $L^2(\Omega_p)^{d \times d}$ ,
- iii)  $\mathbf{v}_{\varepsilon\Sigma}^p = \mathbf{v}_{\varepsilon\Sigma} = \mathbf{v}_{\varepsilon\Sigma}^f \rightharpoonup \mathbf{v}_\Sigma := \mathbf{v}_\Sigma^f$  weakly in  $\tilde{H}^{1/2}(\Sigma)^d$  and strongly in  $L^2(\Sigma)^d$ ,
- iv)  $\mathbf{v}_{\varepsilon\Sigma}^p \cdot \mathbf{n} = \mathbf{v}_{\varepsilon\Sigma} \cdot \mathbf{n} = \mathbf{v}_{\varepsilon\Sigma}^f \cdot \mathbf{n} \rightharpoonup \mathbf{v}_\Sigma^p \cdot \mathbf{n} = \mathbf{v}_\Sigma \cdot \mathbf{n} = \mathbf{v}_\Sigma^f \cdot \mathbf{n}$  weakly in  $\tilde{H}^{1/2}(\Sigma)$  and strongly in  $L^2(\Sigma)$ ,
- v)  $p_\varepsilon \rightharpoonup p$  weakly in  $L^2(\Omega)$ .

**PROOF.** From the uniform bounds in (74) or (85) of Lemma 4, the Hilbert spaces being reflexive, it results the existence of the following weak limits by the Banach-Alaoglu theorem on compact subsets for weak topologies and the compact injections theorem of Rellich (*i.e.* here the continuous injection  $H^1(\Omega_f)^d \hookrightarrow L^2(\Omega_f)^d$  is compact), e.g. [50, Theorems III.15 & III.16] and [50, Theorem IX.16], respectively. More precisely, there exists  $\mathbf{v}^f \in H^1(\Omega_f)^d$ ,  $\mathbf{v}^p \in L^2(\Omega_p)^d$ ,  $\chi^p \in L^2(\Omega_p)^{d \times d}$ ,  $\mathbf{v}_\Sigma \in L^2(\Sigma)^d$  and  $p \in L^2(\Omega)$  such that the weak limits below hold up to subsequences when  $\varepsilon \rightarrow 0$ :

- $\mathbf{v}_\varepsilon^f \rightharpoonup \mathbf{v}^f$  weakly in  $H^1(\Omega_f)^d$  and  $\mathbf{v}_\varepsilon^f \rightarrow \mathbf{v}^f$  strongly in  $L^2(\Omega_f)^d$ ,
- $\mathbf{v}_\varepsilon^p \rightharpoonup \mathbf{v}^p$  weakly in  $L^2(\Omega_p)^d$ ,
- $\sqrt{\varepsilon} \nabla \mathbf{v}_\varepsilon^p \rightharpoonup \chi^p$  weakly in  $L^2(\Omega_p)^{d \times d}$ ,
- $\mathbf{v}_{\varepsilon\Sigma} \rightharpoonup \mathbf{v}_\Sigma$  weakly in  $L^2(\Sigma)^d$ ,
- $p_\varepsilon \rightharpoonup p$  weakly in  $L^2(\Omega)$ .

So, the items ii) and v) hold true.

This also allows us to define a vector field  $\mathbf{v} \in L^2(\Omega)^d$  by its restrictions per subdomain  $\mathbf{v}^f \in H^1(\Omega_f)^d$  and  $\mathbf{v}^p \in L^2(\Omega_p)^d$ . Besides with (68) or (72), we have  $\nabla \cdot \mathbf{v}_\varepsilon = q_m$ , so bounded in  $L^2(\Omega)$  independently of  $\varepsilon$ . Thus,  $\nabla \cdot \mathbf{v}_\varepsilon$  has a weak limit in  $L^2(\Omega)$  which is nothing else than  $\nabla \cdot \mathbf{v}$  because the derivation operator is continuous in the sense of distributions and because the limit is unique. Then, it turns out that  $\mathbf{v}$  belongs to  $\mathbf{H}_{div}(\Omega)$  with  $\nabla \cdot \mathbf{v}_\varepsilon \rightharpoonup \nabla \cdot \mathbf{v}$  weakly

in  $L^2(\Omega)$ . Moreover with the linear continuous normal trace operator  $\mathbf{H}_{div}(\Omega) \mapsto \tilde{H}^{-1/2}(\Sigma)$ ,  $\mathbf{v}_\varepsilon \in \mathbf{W}$  satisfies Green's formula below in  $\Omega_p$  (and similarly in  $\Omega_f$ ):

$$\int_{\Omega_p} \mathbf{v}_\varepsilon^p \cdot \nabla \phi \, dx + \int_{\Omega_p} \phi \nabla \cdot \mathbf{v}_\varepsilon^p \, dx = \langle \mathbf{v}_{\varepsilon\Sigma}^p \cdot \mathbf{n}, \phi \rangle_{-1/2, \Sigma} \quad \text{for all } \phi \in H_{0, \Gamma^p}^1(\Omega_p). \quad (86)$$

Passing to the limit in (86) when  $\varepsilon \rightarrow 0$  with the weak limits gives:

$$\int_{\Omega_p} \mathbf{v}^p \cdot \nabla \phi \, dx + \int_{\Omega_p} \phi \nabla \cdot \mathbf{v}^p \, dx = \langle \mathbf{v}_\Sigma^p \cdot \mathbf{n}, \phi \rangle_{-1/2, \Sigma} \quad \text{for all } \phi \in H_{0, \Gamma^p}^1(\Omega_p), \quad (87)$$

and similarly for  $\mathbf{v}^f$  in  $\Omega_f$ . So, we conclude that  $\mathbf{v}_{\varepsilon\Sigma}^p \cdot \mathbf{n} \rightharpoonup \mathbf{v}_\Sigma^p \cdot \mathbf{n} = \mathbf{v}_\Sigma \cdot \mathbf{n}$  by uniqueness of the limit and using Lemma 6, we conclude that  $\mathbf{v}_\Sigma^p \cdot \mathbf{n} = \mathbf{v}_\Sigma \cdot \mathbf{n} = \mathbf{v}_\Sigma^f \cdot \mathbf{n}$  since  $\mathbf{v} \in \mathbf{H}_{div}(\Omega)$ . By proceeding similarly with Green's formula for the trace  $\mathbf{v}_{\varepsilon\Sigma}^f \in \tilde{H}^{1/2}(\Sigma)^d$  of  $\mathbf{v}_\varepsilon^f \in H^1(\Omega_f)^d$  on  $\Sigma$ , we get that:  $\mathbf{v}_{\varepsilon\Sigma}^f = \mathbf{v}_{\varepsilon\Sigma} = \mathbf{v}_{\varepsilon\Sigma}^f \rightharpoonup \mathbf{v}_\Sigma := \mathbf{v}_\Sigma^f$  weakly in  $\tilde{H}^{1/2}(\Sigma)^d$ . Hence from what precedes, the items *iii*) and *iv*) both hold true.

Finally, by doing once again similarly as above with the normal trace  $\mathbf{v}_\varepsilon^p \cdot \boldsymbol{\nu} = 0$  on  $\Gamma_d^p$  and the trace  $\mathbf{v}_\varepsilon^f = 0$  on  $\Gamma_d^f$ , we get that these homogeneous boundary conditions are preserved at the limit, *i.e.*  $\mathbf{v}^p \cdot \boldsymbol{\nu} = 0$  on  $\Gamma_d^p$  and  $\mathbf{v}^f = 0$  on  $\Gamma_d^f$ , because  $\mathbf{v}^p \in \mathbf{H}_{div}(\Omega_p)$  and  $\mathbf{v}^f \in H^1(\Omega_f)^d$ . Therefore with all the previous considerations, it turns out that  $\mathbf{v}$  belongs to  $\mathbf{V}$  and that the first item *i*) finally holds true, which concludes the proof.  $\square$

Using the fact that  $\varepsilon \nabla \mathbf{v}_\varepsilon^p \rightharpoonup 0$  weakly in  $L^2(\Omega_p)^{d \times d}$  with the item *ii*), it is an easy matter to show that the weak convergence stated in the above Lemma 5 is actually strong. However, the proof will be left to the reader for the sake of shortness since the strong convergence is not necessary further.

Now, since the Hilbert space  $\mathbf{V}$  defined in (66) satisfies  $\mathbf{W} \hookrightarrow \mathbf{V}$  and that the definition of the bilinear form  $b(\cdot, \cdot)$  given in (29) still makes sense in  $\mathbf{V} \times Q$ , let us still denote by  $b(\cdot, \cdot)$  its continuous extension to  $\mathbf{V} \times Q \mapsto \mathbb{R}$ . In addition, we define the bilinear form  $a_{sd}(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \mapsto \mathbb{R}$  and extend the linear continuous form  $\ell(\cdot)$  to a linear continuous functional  $\mathbf{V} \mapsto \mathbb{R}$  with  $\mathbf{g} = 0$  (still denoted by  $\ell(\cdot)$ ) by:

$$a_{sd}(\mathbf{v}, \mathbf{w}) := 2 \int_{\Omega_f} \mu^f \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}) \, dx + \int_{\Omega_p} \mu^p (\mathbf{K}_p^{-1} \mathbf{v}) \cdot \mathbf{w} \, dx \quad (88)$$

$$+ \int_{\Sigma} \frac{\mu_\Sigma^p}{\sqrt{K_p}} (\boldsymbol{\beta}_\Sigma \mathbf{v}_\Sigma^f) \cdot \mathbf{w}_\Sigma^f \, ds, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{V},$$

$$\ell(\mathbf{w}) := \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{w} \, dx + \int_{\Sigma} \mathbf{f}_\Sigma \cdot \mathbf{w}_\Sigma^f \, ds + \langle \mathbf{h}, \mathbf{w} \rangle_{-1/2, \Gamma_n^f}, \quad \forall \mathbf{w} \in \mathbf{V}, \quad (89)$$

the linear form  $g(\cdot)$  in (31) being unchanged. The above extensions of  $b(\cdot, q)$  and  $\ell(\cdot)$  can be actually extended by continuity using the direct consequence [50, Corollary I.2] of the analytic version of Hahn-Banach's Theorem. Following Lemma 2, it is also an easy matter to show that the bilinear form  $a_{sd}(\cdot, \cdot)$  is continuous on  $\mathbf{V} \times \mathbf{V}$ .

Then, the main result of solvability of the variable-viscosity Stokes/Darcy problem can be proved below with no restriction on the size of the data.

**Theorem 2 (Solvability of variable-viscosity Stokes/Darcy flow coupled with (21)).** *Under the assumptions (24,73), let us consider any data  $\rho \in L^\infty(\Omega)$ ,  $\mu \in M$ ,  $\mathbf{f} \in L^2(\Omega)^d$ ,  $q_m \in L^2(\Omega)$ ,  $\mathbf{f}_\Sigma \in L^2(\Sigma)^d$ ,  $\mathbf{h} \in \tilde{H}^{-1/2}(\Gamma_n^f)^d$ , any bounded, symmetric and uniformly positive definite tensor  $\mathbf{K}_p \in L^\infty(\Omega_p)^{d \times d}$  and any bounded uniformly positive semi-definite tensor  $\beta_\Sigma \in L^\infty(\Sigma)^{d \times d}$ .*

*Then, there exists at least one solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  to the mixed Petrov-Galerkin weak problem below:*

$$\left\{ \begin{array}{ll} \text{find a pair } (\mathbf{v}, p) \in \mathbf{V} \times Q \text{ such that:} \\ a_{sd}(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = \ell(\mathbf{w}), & \text{for all } \mathbf{w} \in \mathbf{W}, \\ b(\mathbf{v}, q) = g(q), & \text{for all } q \in Q = L^2(\Omega), \end{array} \right. \quad (90)$$

with the definitions (88,29,89,31).

Moreover, this solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  also solves the Stokes/Darcy fluid-porous flow (22,11,13,19) coupled with the modified stress jump interface conditions (21) on  $\Sigma$ . Therefore, the regularity of the data in the Darcy equation (10) implies that  $\nabla p^p \in L^2(\Omega_p)^d$  and thus  $p^p \in H^1(\Omega_p)$  and since  $\mathbf{g} = 0$  on  $\Gamma_n^p$ , we have finally with (11):  $p^p \in H_{0, \Gamma_n^p}^1(\Omega_p)$ .

**PROOF.** Using the results of Lemma 5, it suffices to pass to the limit when  $\varepsilon \rightarrow 0$  in the two linear equations of the weak problem (72), where only the weak limit of  $(\mathbf{v}_\varepsilon, p_\varepsilon)$  is required, by noticing that:

$$\lim_{\varepsilon \rightarrow 0} a_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{w}) = a_{sd}(\mathbf{v}, \mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{W}, \quad (91)$$

since  $\sqrt{\varepsilon} \mathbf{v}_\varepsilon^p$  remains bounded independently of  $\varepsilon$  and thus  $\sqrt{\varepsilon} \nabla \mathbf{v}_\varepsilon^p \rightharpoonup \boldsymbol{\chi}^p$  and  $\varepsilon \nabla \mathbf{v}_\varepsilon^p \rightharpoonup 0$  weakly in  $L^2(\Omega_p)^{d \times d}$ . Hence, it turns out that the limit  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  of  $(\mathbf{v}_\varepsilon, p_\varepsilon)$  solves the mixed weak problem (90).

Then, by adapting the proof of Lemma 1 made for  $\mathbf{v} \in \mathbf{W}$  to the present case with  $\mathbf{v} \in \mathbf{V}$ , it results the equivalence between the mixed weak problem (90) and the Stokes/Darcy flow (22,11,13,19) coupled with the modified stress jump interface conditions (21) on  $\Sigma$ , assuming existence of solutions  $(\mathbf{v}, p) \in \mathbf{V} \times Q$ . Therefore, that ends the proof.  $\square$

**Remark 2 (On the continuity of the tangential velocity in (20) on  $\Sigma$ ).** *It stems from Theorem 2 that any solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  of the Stokes/Darcy problem (90) satisfies the modified stress jump interface conditions (21), thus with continuity of the normal velocity on  $\Sigma$ . Indeed, without any more regularity for the velocity field  $\mathbf{v}^p$  in  $\Omega_p$ , it is not allowed to argue that  $\mathbf{v} \wedge \mathbf{n}_\Sigma := \mathbf{v}^f \wedge \mathbf{n}_\Sigma \in \tilde{H}^{1/2}(\Sigma)^d$  is the trace of the tangential component of  $\mathbf{v}^p$  on  $\Sigma$ . However, as soon as the regularity of the Darcy problem 10 is sufficient to define the latter tangential trace  $\mathbf{v}^p \wedge \mathbf{n}_\Sigma$  on  $\Sigma$  (in addition to the normal trace  $\mathbf{v}^p \cdot \mathbf{n}_\Sigma$ ), at least in a weak sense, e.g. with  $\mathbf{v}^p \wedge \mathbf{n}_\Sigma \in \tilde{H}^{-1/2}(\Sigma)^d$  if  $\mathbf{v}^p \in \mathbf{H}_{rot}^q(\Omega_p) := \{\mathbf{u} \in L^2(\Omega_p)^d; \nabla \times \mathbf{u} \in L^q(\Omega_p)^d, 1 \leq q \leq 2\}$  by density of the continuous imbedding  $H^1(\Omega_p)^d \hookrightarrow \mathbf{H}_{rot}^q(\Omega_p)$ , then the results of Lemma 5 ensure the velocity continuity  $\mathbf{v}_\Sigma^p = \mathbf{v}_\Sigma = \mathbf{v}_\Sigma^f \in \tilde{H}^{1/2}(\Sigma)^d$  (by uniqueness of the limit) and the original interface conditions (20) are satisfied on  $\Sigma$ .*

**Corollary 2 (Equivalent problem to (90) and uniqueness of the solution).** *Under the assumptions of Theorem 2, any solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  of (90) also solves the following mixed problem and obviously vice versa since  $\mathbf{W} \hookrightarrow \mathbf{V}$ :*

$$\left\{ \begin{array}{ll} \text{find a pair } (\mathbf{v}, p) \in \mathbf{V} \times Q \text{ such that:} \\ a_{sd}(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = \ell(\mathbf{w}), & \text{for all } \mathbf{w} \in \mathbf{V}, \\ b(\mathbf{v}, q) = g(q), & \text{for all } q \in Q = L^2(\Omega), \end{array} \right. \quad (92)$$

with the definitions (88,29,89,31).

Then, the problem (92) has a unique solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  that is thus the solution of the Stokes/Darcy flow (22,11,13,19) supplemented with the modified stress jump interface conditions (21). In addition, this solution satisfies the following energy estimate:

$$\|\mathbf{v}\|_{\mathbf{V}} + \|p\|_Q \leq C(\alpha_a, \beta_b, \|a\|, \|b\|) (\|\ell\|_{\mathbf{V}'} + \|q_m\|_{0,\Omega}). \quad (93)$$

PROOF. Let  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  be any solution of (90). Since the continuous injection  $\mathbf{W} \hookrightarrow \mathbf{V}$  is dense, for all  $\mathbf{w} \in \mathbf{V}$  there exists a sequence  $(\mathbf{w}_k)_{k \in \mathbb{N}} \subset \mathbf{W}$  such that  $\|\mathbf{w} - \mathbf{w}_k\|_{\mathbf{V}} \rightarrow 0$  when  $k \rightarrow +\infty$ . Then using  $\mathbf{w} = \mathbf{w}_k + (\mathbf{w} - \mathbf{w}_k)$ , we have since  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  solves (90) and using the continuity properties:

$$\begin{aligned} |a_{sd}(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) - \ell(\mathbf{w})| &\leq |a_{sd}(\mathbf{v}, \mathbf{w}_k) + b(\mathbf{w}_k, p) - \ell(\mathbf{w}_k)| \\ &\quad + |a_{sd}(\mathbf{v}, \mathbf{w} - \mathbf{w}_k)| + |b(\mathbf{w} - \mathbf{w}_k, p)| + |\ell(\mathbf{w} - \mathbf{w}_k)| \\ &\leq (\|a_{sd}\| \|\mathbf{v}\|_{\mathbf{V}} + \|b\| \|p\|_Q + \|\ell\|_{\mathbf{V}'}) \|\mathbf{w} - \mathbf{w}_k\|_{\mathbf{V}} \\ &\quad \rightarrow_{k \rightarrow \infty} 0. \end{aligned} \quad (94)$$

With (90), the second equation in (92) obviously holds too, and thus  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  also solves (92). The reciprocal holds true immediately because  $\mathbf{W} \hookrightarrow \mathbf{V}$ . Therefore, the weak problems (90) and (92) are equivalent.

Let us now observe that the bilinear and linear forms involved in (92) are continuous. Moreover, it easily results from the proof of Lemma 3 that the bilinear form  $a_{sd}(\cdot, \cdot)$  is coercive on the Hilbert space  $\mathbf{V}_0$  defined in (66) that is the kernel or null space of the form  $b(\cdot, \cdot)$ . Then it results from the theory of abstract mixed weak problem [51, Theorem 4.1 & Corollary 4.1] that the weak problem (92) is well-posed if and only if the bilinear form  $b(\cdot, \cdot)$  satisfies the inf-sup condition below:

$$\sup_{\mathbf{w} \in \mathbf{V}; \mathbf{w} \neq 0} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{\mathbf{V}}} \geq \beta_b \|q\|_Q, \quad \text{for all } q \in Q. \quad (95)$$

Now, it suffices to notice that the construction made in the proof of Theorem 1 to show the inf-sup condition of  $b(\cdot, \cdot)$  between the spaces  $\mathbf{W}$  and  $Q$  also works here between  $\mathbf{V}$  and  $Q$  because the injection  $\mathbf{W} \hookrightarrow \mathbf{V}$  is continuous. Hence, the inf-sup condition (95) also holds and the mixed weak problem (92) has a unique solution that is also the solution  $(\mathbf{v}, p) \in \mathbf{V} \times Q$  of (90). This concludes the proof.  $\square$

**Remark 3 (Concluding observation).** *It turns out finally that Theorem 2 and its Corollary 2 provide an independent proof of a generalized version of the stationary counterpart of Theorem 5.1 in [45]. Indeed, the present analysis is carried out for more suitable fluid-porous configurations where the interface  $\Sigma$  is not a closed surface in view of practical applications and for a different context including more general boundary conditions, a non-zero mass source term and the case of variable-viscosity flow. Moreover, the technique of proof is different: in [45], the analysis is made via Leray's weak formulations with divergence-free test functions and recovering the pressure with De Rham's Theorem. Here, the analysis is performed with mixed weak formulations that are thus ready for many numerical discretization methods.*

## 5. Conclusion

In this study, the well-posed coupling of the Stokes/Darcy fluid-porous flows with variable viscosity is solved using original interface conditions from [46, 48] where the normal velocity

field is continuous at a suitable interface  $\Sigma = \Sigma_b$  inside the porous medium on which an optimal stress jump condition is applied. Moreover, the variable-viscosity Stokes/Darcy-Brinkman flow coupled with velocity continuity and stress jump interface conditions is also proved to be well-posed. The present approach is unusual and not found in the literature because the Stokes and Darcy partial differential equations are not of the same order and thus, almost all authors use *ad-hoc* extended velocity slip conditions inherited from [13, 18]. However, the proposed mixed variational framework, and then also the further numerical discretization, is simpler than with Beavers-Joseph's or Beavers-Joseph-Saffman's conditions because no Lagrange multiplier is required at the interface.

## Appendix A. Auxiliary results

**Lemma 6 (Equivalence of normal velocity continuity on  $\Sigma$ ).** *Let  $\mathbf{v} \in L^2(\Omega)^d$  such that  $\nabla \cdot \mathbf{v}^f \in L^2(\Omega_f)$  in  $\Omega_f$  and  $\nabla \cdot \mathbf{v}^p \in L^2(\Omega_p)$  in  $\Omega_p$ . Then  $\nabla \cdot \mathbf{v} \in L^2(\Omega)$  over the whole domain  $\Omega$  if and only if  $\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket_\Sigma := (\mathbf{v}^f \cdot \mathbf{n} - \mathbf{v}^p \cdot \mathbf{n})|_\Sigma = 0$  on  $\Sigma$ .*

PROOF. By the normal trace theorem [51, Theorem 2.5] in  $\mathbf{H}_{div}(\Omega_f)$  or  $\mathbf{H}_{div}(\Omega_p)$ ,  $\mathbf{v}^f$  or  $\mathbf{v}^p$  admits a normal trace  $\mathbf{v}^f \cdot \boldsymbol{\nu} \in H^{-1/2}(\partial\Omega_f)$  on  $\partial\Omega_f$  or  $\mathbf{v}^p \cdot \boldsymbol{\nu} \in H^{-1/2}(\partial\Omega_p)$  on  $\partial\Omega_p$ , respectively. Besides from Section 2.1, we have on  $\Sigma$ :  $\mathbf{v}^f \cdot \mathbf{n} \in \tilde{H}^{-1/2}(\Sigma)$  and  $\mathbf{v}^p \cdot \mathbf{n} \in \tilde{H}^{-1/2}(\Sigma)$ . With  $\mathbf{v} \in L^2(\Omega)^d$ , we have a priori  $\nabla \cdot \mathbf{v} \in H^{-1}(\Omega)$ . Then for all  $\phi \in H_0^1(\Omega)$ , using successively Green's formula in  $\Omega$ ,  $\Omega_f$  and  $\Omega_p$ , we get:

$$\begin{aligned} \langle \nabla \cdot \mathbf{v}, \phi \rangle_{-1, \Omega} &= - \int_{\Omega} \mathbf{v} \cdot \nabla \phi \, dx = - \int_{\Omega_f} \mathbf{v}^f \cdot \nabla \phi \, dx - \int_{\Omega_p} \mathbf{v}^p \cdot \nabla \phi \, dx \\ &= \int_{\Omega_f} \nabla \cdot \mathbf{v}^f \phi \, dx + \int_{\Omega_p} \nabla \cdot \mathbf{v}^p \phi \, dx \\ &\quad + \langle \mathbf{v}^f \cdot \mathbf{n}|_\Sigma, \phi|_\Sigma \rangle_{-1/2, \Sigma} - \langle \mathbf{v}^p \cdot \mathbf{n}|_\Sigma, \phi|_\Sigma \rangle_{-1/2, \Sigma} \\ &= \int_{\Omega} \nabla \cdot \mathbf{v} \phi \, dx + \langle \llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket_\Sigma, \phi|_\Sigma \rangle_{-1/2, \Sigma}, \end{aligned} \tag{A.1}$$

for all  $\phi \in H_0^1(\Omega)$ ,

where  $\langle \cdot, \cdot \rangle_{-1/2, \Sigma}$  denotes the duality pairing between  $\tilde{H}^{-1/2}(\Sigma)$  and  $\tilde{H}^{1/2}(\Sigma)$ . Now, let us observe that any function  $\psi \in \tilde{H}^{1/2}(\Sigma) := H_{00}^{1/2}(\Sigma) \hookrightarrow H^{1/2}(\Sigma)$  can be extended in  $\Omega$  by a function  $\phi \in H_0^1(\Omega)$ . Indeed, it suffices to take the solution  $\phi \in H_0^1(\Omega)$  to the Dirichlet transmission problem below:

$$\begin{cases} \Delta \phi = 0 & \text{in } \Omega_f \cup \Omega_p, \\ \phi = 0 & \text{on } \Gamma := \partial\Omega, \\ \phi = \psi & \text{on } \Sigma := \partial\Omega_f \cap \partial\Omega_p, \end{cases} \tag{A.2}$$

which states that the linear trace operator  $H_0^1(\Omega) \mapsto \tilde{H}^{1/2}(\Sigma)$  is surjective. Then, the previous equality A.1 gives the desired equivalence result.  $\square$

**Proposition 1 (Hilbert space  $\mathbf{X}$  in (66)).** *The vector space  $\mathbf{X}$  endowed with the inner product  $(\cdot, \cdot)_{\mathbf{X}}$  naturally associated with the pre-Hilbertian norm  $\|\cdot\|_{\mathbf{X}}$  defined in (67) is a Hilbert space.*

PROOF. Let us consider any Cauchy sequence  $(\mathbf{u}_k)_{k \in \mathbb{N}}$  in  $\mathbf{X}$ . With the definition of the norm  $\|\cdot\|_{\mathbf{X}}$  in (67), this implies that both  $\|\mathbf{u}_k\|_{\mathbf{H}_{div}(\Omega)}$  and  $\|\mathbf{u}_k^f\|_{1, \Omega_f}$  (where  $(\mathbf{u}_k^f) := (\mathbf{u}_k)|_{\Omega_f}$ ) satisfy Cauchy's property. Since then the spaces  $\mathbf{H}_{div}(\Omega)$  and  $H^1(\Omega_f)^d$  are complete, the sequence  $(\mathbf{u}_k)$  has a limit  $\mathbf{u} \in \mathbf{H}_{div}(\Omega)$  (of course also in  $L^2(\Omega)^d$ ) and  $(\mathbf{u}_k^f)$  has a limit  $\tilde{\mathbf{u}}^f \in H^1(\Omega_f)^d$  (also in  $L^2(\Omega_f)^d$ ) when  $k \rightarrow +\infty$ . By uniqueness of the limit in  $L^2(\Omega_f)^d$ , we get  $\tilde{\mathbf{u}}^f = \mathbf{u}^f$ . Hence, we conclude that the Cauchy sequence  $(\mathbf{u}_k)$  converges to  $\mathbf{u} \in \mathbf{X}$  and the space  $\mathbf{X}$  is complete.  $\square$

**Proposition 2 (Hilbert space  $\mathbf{V}$  in (66)).** *The vector space  $\mathbf{V}$  endowed with the inner product  $(\cdot, \cdot)_{\mathbf{V}}$  naturally associated with the norm  $\|\cdot\|_{\mathbf{V}}$  (inherited from  $\|\cdot\|_{\mathbf{X}}$ ) defined in (67) is a Hilbert space as a closed subspace of  $\mathbf{X}$ .*

PROOF. Let us consider any sequence  $(\mathbf{v}_k)_{k \in \mathbb{N}}$  in  $\mathbf{V}$  that converges to some  $\mathbf{v} \in \mathbf{X}$ , i.e.  $\|\mathbf{v}_k - \mathbf{v}\|_{\mathbf{X}} \rightarrow 0$  when  $k \rightarrow +\infty$ . This implies that:  $\|\mathbf{v}_k - \mathbf{v}\|_{\mathbf{H}_{div}(\Omega)} \rightarrow 0$  and  $\|\mathbf{v}_k^f - \mathbf{v}^f\|_{1, \Omega_f} \rightarrow 0$ . Then, with the continuity of the linear operator of normal trace on  $\Gamma_d^p$ , we have:

$$\|\mathbf{v}_k^p \cdot \boldsymbol{\nu} - \mathbf{v}^p \cdot \boldsymbol{\nu}\|_{\tilde{H}^{-1/2}(\Gamma_d^p)} \leq c(\Omega_p, \Gamma_d^p) \|\mathbf{v}_k^p - \mathbf{v}^p\|_{\mathbf{H}_{div}(\Omega_p)} \rightarrow 0 \quad \text{when } k \rightarrow +\infty. \quad (\text{A.3})$$

Hence we get that  $\mathbf{v} \in \mathbf{X}$  also satisfies  $\mathbf{v}^p \cdot \boldsymbol{\nu} = 0$  on  $\Gamma_d^p$  because  $\mathbf{v}_k^p \cdot \boldsymbol{\nu} = 0$  on  $\Gamma_d^p$  with  $\mathbf{v}_k \in \mathbf{V}$  for all  $k \in \mathbb{N}$ . Besides, from the inequality of trace's continuity in  $H^1(\Omega_f)^d$ , we have since  $\tilde{H}^{1/2}(\Sigma)^d \hookrightarrow H^{1/2}(\Sigma)^d$ :

$$\|\mathbf{v}_k^f - \mathbf{v}^f\|_{\tilde{H}^{1/2}(\Sigma)^d} \leq c(\Omega_f, \Sigma) \|\mathbf{v}_k^f - \mathbf{v}^f\|_{1, \Omega_f} \rightarrow 0 \quad \text{when } k \rightarrow +\infty, \quad (\text{A.4})$$

and thus  $\mathbf{v}_k^f|_{\Sigma} \rightarrow \mathbf{v}^f|_{\Sigma}$  in  $\tilde{H}^{1/2}(\Sigma)^d$ . Since we have also  $\mathbf{v}_k^f = 0$  on  $\Gamma_d^f$ , we get similarly with the trace inequality now applied on  $\Gamma_d^f$  that  $\mathbf{v}^f = 0$  on  $\Gamma_d^f$ . Finally, we have shown that  $\mathbf{v} \in \mathbf{V}$  holds and thus  $\mathbf{V}$  is closed in  $\mathbf{X}$  which ends the proof.  $\square$

## Appendix B. Pressure-driven Poiseuille fluid-porous channel flow

The benchmark problem of pressure-driven 1-D Poiseuille channel flow through a free-fluid layer of height  $2H$  superposed over a semi-infinite isotropic homogeneous porous one is studied with:  $\mathbf{f} = 0$ ,  $q_m = 0$ ,  $\mathbf{f}_\Sigma = 0$  and a no-slip velocity condition at the upper wall (at  $z = 2H$ ). The reference solution of the single-domain continuum model is governed by the generalized Darcy-Brinkman equations with porosity smoothly varying in the viscous boundary layer (see [5, 7, 8, 64, 47]) from  $\phi = 1$  (pure fluid) on  $\Sigma_t$  at  $z = 0$  to  $\phi = \phi_p$  in the porous bulk. The permeability profile is given by the porosity-permeability correlation  $K(\phi)$  of Kozeny-Carman that has been calibrated for many random packed beds of spherical grains of variable sizes [57]. So, the reference solution is computed by a second-order accurate (in the  $L^2$ -norm) finite-volume method. We refer to [48] for more details. Then, the Beavers-Joseph's velocity slip condition (1) at  $\Sigma_t$  ( $Z := z/H = 0$ ) [13] and the optimal stress jump condition (20) at  $\Sigma_b$  ( $Z = -\delta^*$ ) [46] are calibrated in a dimensionless setting ( $L = H$  being the macroscale length) using the reference slip fluid velocity  $U_s := u_s/V$  on  $\Sigma_t$  and the Darcy filtration velocity  $U_D := u_D/V = \text{Da}$ . The analytical calibration with  $U(Z = 2) = 0$  yields:

$$\beta_\Sigma = \frac{1 - \frac{U_s}{2} + \delta^*}{\sqrt{\text{Da}}} \approx \frac{1}{\sqrt{\text{Da}}} \quad \text{on } \Sigma = \Sigma_b \quad \text{at } Z = -\delta^*, \quad (\text{B.1})$$

where the thickness  $\delta^* = d/H$  satisfies:  $\delta^* \lesssim U_s$  in the present dimensionless setting. Typical streamwise velocity profiles are shown in Figures B.2 (over the free-fluid layer) and B.3 (zoom in the viscous boundary layer inside the porous medium). The comparative performance to reduce the loss of flow rate in the viscous boundary layer (here of thickness  $\delta_B = 5\ell/H$  where  $\ell$  is the size of the unit cell representative of the microstructure) inside the porous medium [47] are supplied in Table B.1. This shows that the optimal stress condition (20) largely outperforms Beavers-Joseph's one (1). Moreover, it is remarkable that the reduced friction coefficient satisfies:  $\beta_\Sigma \sqrt{\text{Da}} \approx 1$  when  $H$  is large enough such that  $2\delta^* - U_s \ll 2$ , here  $H = 100\ell$  (often far more in practice). In addition, the comparison with the Stokes/Darcy-Brinkman model coupled with (1) at  $\Sigma_t$  ( $Z := z/H = 0$ ) or with the optimal stress jump condition (20) at  $\Sigma_b$  ( $Z = -\delta^*$ ) is shown in Figure B.4 (zoom in the viscous boundary layer). For this porosity  $\phi_p = 0.75$ , the former gives a relative loss of flow rate  $\text{er}^{bl} = 98.1\%$  whereas the latter yields  $\text{er}^{bl} = 8.88\%$  and is thus the best solution. The increase of performance between the Stokes/Darcy-Brinkman and Stokes/Darcy models is clearly better for larger porosities  $\phi_p \geq 0.85$ ; see [48].

$\phi_p$ & $U_D = \text{Da}$ ( $10^{-7}$ )	$U_s$ ( $10^{-2}$ )	$\delta^*$ ( $10^{-2}$ )	SD at $\Sigma_t$ ( $Z = 0$ )	SD at $\Sigma_b$ ( $Z = -\delta^*$ )
0.25 & 0.196	1.2974	1.2974	100 %	9.01 %
0.50 & 2.693	1.4408	1.4408	100 %	9.27 %
0.75 & 22.90	1.7254	1.7251	99.9 %	9.80 %
0.95 & 397.0	2.6625	2.6586	99.5 %	11.9 %

Table B.1: Comparative performance of the relative loss of flow rate  $\text{er}^{bl}$  (Relative error in  $L^1$ -norm in the viscous boundary layer) in the Poiseuille channel flow for the Stokes/Darcy (SD) model coupled with Beavers-Joseph's velocity slip condition (1) at  $\Sigma_t$  [13] or optimal stress jump condition (20) at  $\Sigma_b$  [46]:  $H = 100\ell$  and  $\delta_B = 5\ell/H - U_m \simeq 0.51$  at  $Z_m \simeq 0.99$  with  $\text{er}_m^f \leq 2 \cdot 10^{-7}$  and  $\text{er}^f \leq 4 \cdot 10^{-12}$  (parabolic profiles superposed in the free-fluid layer).

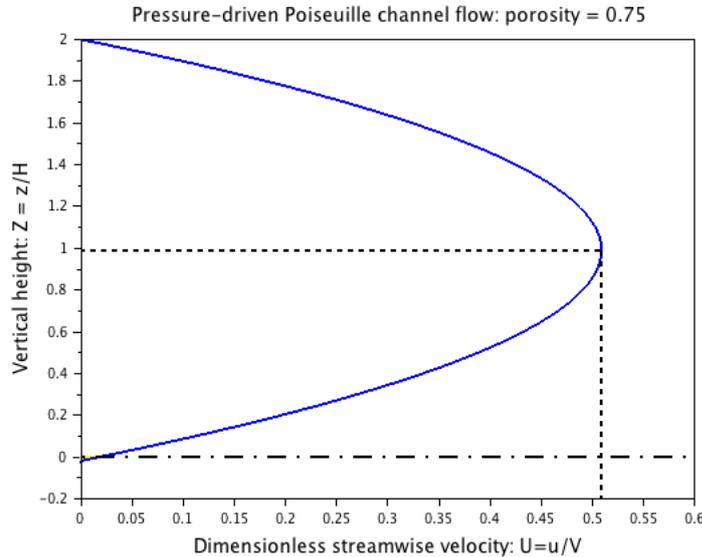


Figure B.2: Comparison of streamwise velocity solutions for the Stokes/Darcy model in the Poiseuille channel flow with  $H = 100 \ell$ ,  $\delta_B = 5 \ell/H$ ,  $\phi_p = 0.75$  and  $Da = 2.29 \cdot 10^{-6}$ : general view with  $U_m = 0.509$  at  $Z_m = 0.992$ ,  $U_D = Da$  and all solutions superposed in the free-fluid layer (same caption as in figure B.3).

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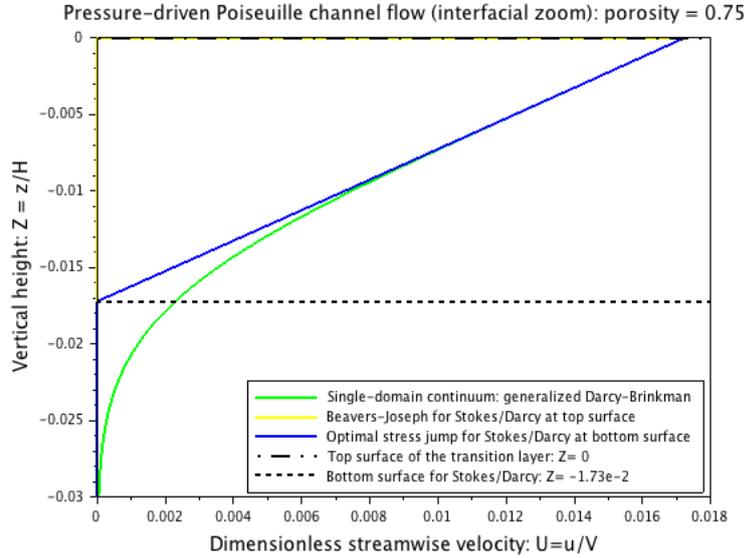


Figure B.3: Comparison of streamwise velocity solutions for the Stokes/Darcy model of the Poiseuille channel flow with  $H = 100 \ell$ ,  $\delta_B = 5 \ell/H$ ,  $\phi_p = 0.75$ ,  $Da = 2.29 \cdot 10^{-6}$ : free-fluid layer with maximum  $U_m = 0.509$  at  $Z_m = 0.992$  – interfacial viscous boundary layer with  $U_D = Da = 2.29 \cdot 10^{-6}$ ,  $U_s = 1.7254 \cdot 10^{-2}$  and  $\delta^* = 1.7251 \cdot 10^{-2}$ .

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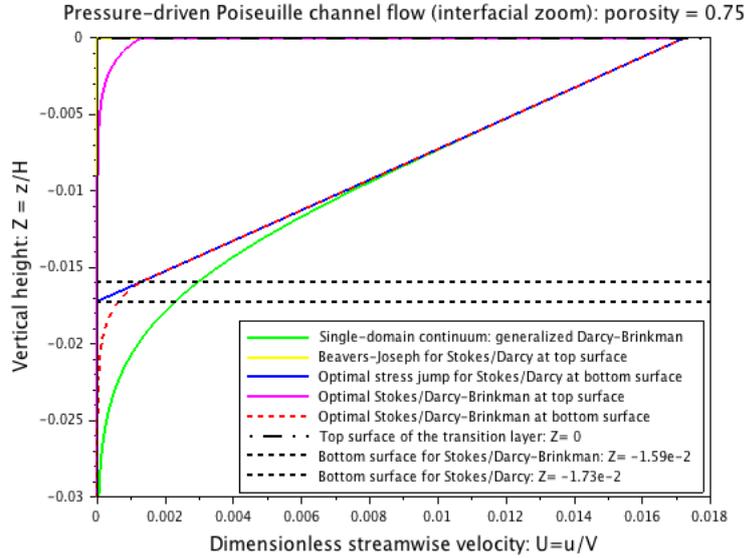


Figure B.4: Comparison of streamwise velocity solutions for the Stokes/Darcy and Stokes/Darcy-Brinkman models of the Poiseuille channel flow with  $H = 100 \ell$ ,  $\delta_B = 5 \ell/H$ ,  $\phi_p = 0.75$ ,  $Da = 2.29 \cdot 10^{-6}$ : free-fluid layer with maximum  $U_m = 0.509$  at  $Z_m = 0.992$  – interfacial viscous boundary layer with  $U_D = Da = 2.29 \cdot 10^{-6}$ ,  $U_s = 1.7254 \cdot 10^{-2}$ ,  $\delta^* = 1.7251 \cdot 10^{-2}$  and  $\delta^* = 1.5942 \cdot 10^{-2}$ .

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