Constructive exact control of semilinear 1D heat equations

Jérôme Lemoine * Arnaud Münch†

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Abstract

The exact distributed controllability of the semilinear heat equation $\partial_t y - \Delta y + g(y) = f\,1_\omega$ posed over multi-dimensional and bounded domains, assuming that $g \in C^1(\mathbb{R})$ satisfies the growth condition $\limsup_{r\to\infty} g(r)/(|r| \ln^{3/2}|r|) = 0$ has been obtained by Fernández-Cara and Zuazua in 2000. The proof based on a non constructive fixed point arguments makes use of precise estimates of the observability constant for a linearized heat equation. In the one dimensional setting, assuming that g' does not grow faster than $\beta \ln^{3/2}|r|$ at infinity for $\beta > 0$ small enough and that g' is uniformly Hölder continuous on $\mathbb{R}$ with exponent $p \in [0,1]$, we design a constructive proof yielding an explicit sequence converging to a controlled solution for the semilinear equation, at least with order $1+p$ after a finite number of iterations.

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1 Introduction

Let $\Omega = (0,1)$, $\omega \subset\subset \Omega$ be any non-empty open set and let $T > 0$. We set $Q_T = \Omega \times (0,T)$, $q_T = \omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. We are concerned with the null controllability problem for the following semilinear heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y + g(y) = f 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \ \text{ on } \ \Sigma_T, \quad y(\cdot,0) = u_0 \ \text{ in } \ \Omega, \end{cases} \tag{1}$$

where $u_0 \in H_0^1(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function. We assume moreover that the nonlinear function $g : \mathbb{R} \mapsto \mathbb{R}$ is, at least, locally Lipschitz-continuous and, following [17], that g satisfies

$$|g'(r)| \leq C(1 + |r|^5) \quad \forall r \in \mathbb{R}. \tag{2}$$

Under this condition, (1) possesses exactly one local in time solution. Moreover, we recall in accordance with the results in [7, Section 5] that under the growth condition

$$|g(r)| \leq C(1 + |r| \ln(1 + |r|)) \quad \forall r \in \mathbb{R}, \tag{3}$$

the solutions to (1) are globally defined in $[0,T]$ and one has

$$y \in C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)). \tag{4}$$

Without a growth condition of the kind (3), the solutions to (1) can blow up before $t = T$; in general, the blow-up time depends on g and the size of $\|u_0\|_{L^2(\Omega)}$ (see [20]).

*Laboratoire de mathématiques Blaise Pascal, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 3, place Vasarely, 63178 Aubière, France. e-mail: jerome.lemoine@uca.fr.

†Laboratoire de mathématiques Blaise Pascal, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 3, place Vasarely, 63178 Aubière, France. e-mail: arnaud.munch@uca.fr. Funding by the French government research program "Investissements d'Avenir" through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25) (Corresponding author).

System (1) is said to be *exactly controllable to trajectories* at time T if, for any $u_0 \in L^2(\Omega)$ and any globally defined bounded trajectory $y^\star \in C^0([0,T]; L^2(\Omega))$ (corresponding to data $u_0^\star \in L^2(\Omega)$ and $f^\star \in L^2(q_T)$), there exist controls $f \in L^2(q_T)$ and associated states y that are again globally defined in $[0,T]$ and satisfy (4) and

$$y(x,T) = y^\star(x,T), \quad x \in \Omega. \tag{5}$$

The uniform controllability strongly depends on the nonlinearity g. Assuming a growth condition on the nonlinearity g at infinity, this problem has been solved by Fernández-Cara and Zuazua in [17, Theorem 1.2] (which also covers the multi-dimensional case for which $\Omega \subset \mathbb{R}^d$ is a bounded connected open set with Lipschitz boundary).

Theorem 1. *[17] Let $T > 0$ be given. Assume that* (1) *admits at least one solution $y^\star$, globally defined in $[0,T]$ and bounded in Q_T. Assume that $g : \mathbb{R} \mapsto \mathbb{R}$ is C^1 and satisfies* (2) *and*

$(\mathbf{H_1})$ $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r| \ln^{3/2}|r|} = 0$

then (1) *is controllable in time T.*

Therefore, if $|g(r)|$ does not grow at infinity faster than $|r|\ln^p(1+|r|)$ for any $p < 3/2$, then (1) is controllable. We also mention [2] which gives the same result assuming additional sign condition on g, namely $g(r)r \geq -C(1+r^2)$ for all $r \in \mathbb{R}$ and some $C > 0$. On the contrary, if g is too "super-linear" at infinity, precisely, if $p > 2$, then for some initial data, the control cannot compensate the blow-up phenomenon occurring in $\Omega\backslash\overline{\omega}$ (see [17, Theorem 1.1]). The problem remains open when g behaves at infinity like $|r|\ln^p(1+|r|)$ with $3/2 \leq p \leq 2$. We mention however the recent work of Le Balc'h [21] where uniform controllability results in large time are obtained for $p \leq 2$ assuming additional sign conditions on g, notably that $g(r) > 0$ for $r > 0$ or $g(r) < 0$ for $r < 0$, a condition not satisfied for $g(r) = -r\ \ln^p(1+|r|)$. Eventually, in the case $p > 2$ for which the blow-up phenomenon can not be compensated by means of control, we also mention [9] in the one dimensional case where a positive boundary controllability result is proved for steady-state initial and final data and for T large enough.

In the sequel, for simplicity, we shall assume that $g(0) = 0$ and that $f^\star \equiv 0, u_0^\star \equiv 0$ so that $y^\star$ is the null trajectory. The proof given in [17] is based on a fixed point method, initially introduced in [31] for a one dimensional wave equation. Precisely, it is shown that the operator $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y := \Lambda(z)$ is a null controlled solution of the linear boundary value problem

$$\begin{cases} \partial_t y - \partial_{xx} y + y\,\tilde{g}(z) = f 1_\omega & \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 & \text{in} \quad \Omega \end{cases}, \qquad \tilde{g}(r) := \begin{cases} g(r)/r & r \neq 0 \\ g'(0) & r = 0 \end{cases} \tag{6}$$

maps a closed ball $B(0,M) \subset L^\infty(Q_T)$ into itself, for some $M > 0$. The Kakutani's theorem then provides the existence of at least one fixed point for the operator Λ, which is also a controlled solution for (1). The control of minimal $L^\infty(q_T)$ norm is considered in [17]. This allows, including in the multi-dimensional case, to obtain controlled solutions in $L^\infty(Q_T)$.

The main goal of this work is to determine an approximation of the controllability problem associated with (1), that is to construct an explicit sequence $(f_k)_{k\in\mathbb{N}}$ converging strongly toward a null control for (1). A natural strategy is to take advantage of the method used in [17, 21] and consider, for any element $y_0 \in L^\infty(Q_T)$, the Picard iterations defined by $y_{k+1} = \Lambda(y_k)$, $k \geq 0$ associated with the operator Λ. The resulting sequence of controls $(f_k)_{k\in\mathbb{N}}$ is so that $f_{k+1} \in L^2(q_T)$ is a null control for y_{k+1} solution of

$$\begin{cases} \partial_t y_{k+1} - \partial_{xx} y_{k+1} + y_{k+1}\,\widetilde{g}(y_k) = f_{k+1} 1_\omega & \text{in} \quad Q_T, \\ y_{k+1} = 0 \ \text{ on } \ \Sigma_T, \quad y_{k+1}(\cdot,0) = u_0 \ \text{ in } \ \Omega. \end{cases} \tag{7}$$

Numerical experiments reported in [13] exhibit the non convergence of the sequences $(y_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ for some initial conditions large enough. This phenomenon is related to the fact that the operator Λ is in general not contracting, including the cases for which $\widetilde{g}$ is globally Lipschitz. We also refer to [4, 5] where this strategy is implemented.

In the one-dimensional case, a least-squares type approach, based on the minimization over $Z := L^2((T-t)^{-1}; Q_T)$ of the functional $\mathcal{R} : Z \to \mathbb{R}^+$ defined by $\mathcal{R}(z) := \|z - \Lambda(z)\|_Z^2$ has been introduced and analyzed in [13]. Assuming that $\widetilde{g} \in C^1(\mathbb{R})$ and $\widetilde{g}' \in L^\infty(\mathbb{R})$, it is proved that $\mathcal{R} \in C^1(Z; \mathbb{R}^+)$ and satisfies, for some constant $C > 0$, the following inequality

$$(1 - C\|\widetilde{g}'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^\infty(\Omega)})\sqrt{2\mathcal{R}(z)} \leq \|\mathcal{R}'(z)\|_{L^2(Q_T)} \quad \forall z \in L^2(Q_T).$$

This implies that if $\|\widetilde{g}'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^\infty(\Omega)}$ is small enough, then any critical point for $\mathcal{R}$ is a fixed point for Λ (see [13, Proposition 3.2]). Under such smallness assumption, numerical experiments reported in [13] display the convergence of gradient based minimizing sequences for $\mathcal{R}$ and a better behavior than the Picard iterates. The analysis of convergence is however not performed. As is usual for nonlinear problems and also considered in [13], we may employ a Newton type method to find a zero of the mapping $\widetilde{F} : Y \mapsto W$ defined by

$$\widetilde{F}(y, f) = (\partial_t y - \partial_{xx} y + g(y) - f 1_\omega, y(\cdot\,, 0) - u_0) \quad \forall (y, f) \in Y \tag{8}$$

where the Hilbert space Y and W are defined as follows

$$Y := \Big\{ (y, f) : \rho y \in L^2(Q_T),\ \rho_0(\partial_t y - \partial_{xx} y) \in L^2(Q_T), y = 0 \text{ on } \Sigma_T,\ \rho_0 f \in L^2(q_T) \Big\}$$

and $W := L^2(\rho_0; Q_T) \times L^2(\Omega)$ for some appropriates weights ρ and ρ_0 (defined in the next section). Here $L^2(\rho_0; Q_T)$ stands for $\{z : \rho_0 z \in L^2(Q_T)\}$. It is shown (see [13, Section 3.3]) that, if $g \in C^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$, then $\widetilde{F} \in C^1(Y; W)$ allowing to derive the Newton iterative sequence: given (y_0, f_0) in Y, define $(y_k, f_k)_{k\in\mathbb{N}}$ in $Y^{\mathbb{N}}$ as follows $(y_{k+1}, f_{k+1}) = (y_k, f_k) - (Y_k, F_k)$ where F_k is a control for Y_k solution of

$$\begin{cases} \partial_t Y_k - \partial_{xx} Y_k + g'(y_k)\, Y_k = F_k\, 1_\omega + \partial_t y_k - \partial_{xx} y_k + g(y_k) - f_k 1_\omega & \text{in} \quad Q_T, \\ Y_k = 0 \quad \text{on} \quad \Sigma_T, \quad Y_k(\cdot, 0) = u_0 - y_k(\cdot, 0) \quad \text{in} \quad \Omega \end{cases} \tag{9}$$

such that $Y_k(\cdot, T) = -y_k(\cdot, T)$ in Ω. Numerical experiments in [13] exhibit the lack of convergence of the Newton method for large enough initial condition, for which the solution y is not close enough to the zero trajectory.

The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see the monography [8] and references therein). However, as far as we know, few are concerned with the approximation of exact controls for nonlinear partial differential equations, and the construction of convergent control approximations for nonlinear equations remains a challenge.

In this article, given any initial data $u_0 \in H_0^1(0, 1)$, we design an algorithm providing a sequence $(y_k)_{k\in\mathbb{N}}$ converging to a controlled solution for (1), under assumptions on g that are slightly stronger than $(\mathbf{H_1})$. Our main result can be formulated as follows.

Theorem 2. *Let $T > 0$ be given. Assume that $g \in C^1(\mathbb{R})$ satisfies $g(0) = 0$ and the growth condition*

$(\mathbf{H_1'})$ $\quad \exists \alpha > 0,$ *s.t.* $|g'(r)| \leq \alpha + \beta^\star \ln^{3/2}(1 + |r|), \quad \forall r \in \mathbb{R}$

for some $\beta^\star = \beta^\star(\Omega, T) > 0$ small enough and

$(\overline{\mathbf{H}}_\mathbf{p})$ $\quad \exists p \in [0, 1]$ *such that* $\sup_{\substack{a,b\in\mathbb{R} \\ a\neq b}} \frac{|g'(a) - g'(b)|}{|a-b|^p} < +\infty.$

Then, for any $u_0 \in H_0^1(\Omega)$, one can construct a sequence $(y_k, f_k)_{k\in\mathbb{N}}$ converging strongly to a controlled pair for (1). Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

This result is achieved (in the spirit of [27] devoted to a linear case) by minimizing the non convex functional

$$E_s(y,f) := \frac{1}{2}\|\rho_0(s)(\partial_t y - \partial_{xx} y + g(y) - f\,1_\omega)\|^2_{L^2(Q_T)} \tag{10}$$

over a convex space $\mathcal{A}(s)$ which incorporates the initial and controllability requirement and where ρ_0 denotes a Carleman type weight parametrized by $s \geq 1$ and which blows as $t \to T^-$. The least-squares functional E_s measures how much a pair (y,f) in $\mathcal{A}(s)$ is close to a controlled solution for (1). The controllability for (1) is reflected by the fact that the global minimum of the nonnegative functional E_s is zero, over all pairs (y,f) solutions of (1) and satisfying the controllability requirement at the final time.

The paper is organized as follows. In Section 2, we derive a controllability result for a linearized heat equation with potential in $L^\infty(Q_T)$ and source term in $L^2(Q_T)$. Then, in Section 3, we define the least-squares functional E_s and the corresponding non convex optimization problem (30) over the Hilbert space $\mathcal{A}(s)$. We show that E_s is Gateaux-differentiable over $\mathcal{A}(s)$ and that any critical point (y,f) for E_s for which $g'(y)$ belongs to $L^\infty(Q_T)$ is also a zero of E_s (see Proposition 2). This is done by introducing a pair (Y^1,F^1) for $E_s(y,f)$ for which $E_s'(y,f)\cdot(Y^1,F^1)$ is proportional to $E_s(y,f)$. Then, in Section 4, assuming that the nonlinear function g is such that g' is uniformly Hölder continuous with exponent p, for some $p \in [0,1]$, we determine a minimizing sequence based on (Y^1,F^1) which converges strongly for s large enough to a controlled pair for the semilinear heat equation (1) (see Theorem 4). Section 5 gathers several remarks on the approach: we notably emphasize that this least-squares approach coincides with the damped Newton method one may use to find a zero of the mapping $\widetilde{F}$ mentioned above; this explains the super-linear convergence stated in Theorem 2. We also discuss some other linearizations of the system (1) and show that the analysis extends to the exact controllability to trajectories. We conclude in Section 6 with some perspectives, notably the extension of our analysis to the multi-dimensional case.

As far as we know, the method introduced and analyzed in this work is the first one providing an explicit, algorithmic construction of exact controls for semilinear heat equations with non Lipschitz nonlinearity. It extends the study [22] where it is assumed that $g' \in L^\infty(\mathbb{R})$ (i.e. $\beta^\star = 0$ in $(\mathbf{H_1'})$) which allows to obtain directly a uniform bound of the observability constant. The stronger case $\beta^\star > 0$ small considered here requires a refined analysis similar to the one recently developed by the authors in [3, 29] for the wave equation. The parabolic case is however much more intricate (than the hyperbolic one) as it makes appear Carleman type weights depending on the controlled solution. These works devoted to controllability problems take their roots in the works [23, 24] concerned with the approximation of solution of Navier-Stokes type problem, through least-squares methods: they refine the analysis performed in [25, 26] inspired from the seminal contribution [6].

Notations. Throughout, we denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(\mathbb{R})$, by $(\cdot,\cdot)_X$ the scalar product of X (if X is a Hilbert space) and by $\langle\cdot,\cdot\rangle_{X,Y}$ the duality product between X and Y.

Given any $p \in [0,1]$, we introduce for any $g \in C^1(\mathbb{R})$ the following hypothesis :

$(\overline{\mathbf{H}}_\mathbf{p})$ $\quad [g']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a)-g'(b)|}{|a-b|^p} < +\infty$

meaning, for $p \in (0,1]$, that g' is uniformly Hölder continuous with exponent p. For $p = 0$, by extension, we set $[g']_0 := 2\|g'\|_\infty$. In particular, g satisfies $(\overline{\mathbf{H}}_\mathbf{0})$ if and only if $g \in \mathcal{C}^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$, and g satisfies $(\overline{\mathbf{H}}_\mathbf{1})$ if and only if g' is Lipschitz continuous (in this case, g' is almost everywhere differentiable and $g'' \in L^\infty(\mathbb{R})$), and we have $[g']_1 \leq \|g''\|_\infty$.

We also denote by C a positive constant depending only on Ω, ω and T that may vary from lines to lines.

2 A controllability result for a linearized heat equation with potential and right hand side

This section is devoted to a controllability result for a linear heat equation with potential in $L^\infty(Q_T)$ and right hand side in $L^2(\rho_0(s); Q_T)$ for a precise weight $\rho_0(s)$ parametrized by $s \in \mathbb{R}_+^\star$ defined in the sequel. More precisely we are interested in the existence of a control v such that the solution z of

$$\begin{cases} \partial_t z - \partial_{xx} z + Az = v1_\omega + B & \text{in} \quad Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, 0) = z_0 \text{ in } \Omega \end{cases} \tag{11}$$

satisfies

$$z(\cdot, T) = 0 \text{ in } \Omega. \tag{12}$$

We follow the usual strategy of [18] to construct a solution of the null controllability problem, using Carleman type estimates. Instead of using the classical estimates of [18], we use the one in [1] for which it is easier to deal with non zero initial data as the weight function does not blow up as $t \to 0$.

2.1 Carleman estimate

For any $s \geq 1$, we consider the weight functions $\rho(s) = \rho(x,t,s)$, $\rho_0(s) = \rho_0(x,t,s)$ and $\rho_1(s) = \rho_1(x,t,s)$ which are continuous, strictly positives and in $L^\infty(Q_{T-\delta})$ for any $\delta > 0$. Precisely, we use the weights introduced in [1]: $\rho_0(s) = \xi^{-3/2}\rho(s)$, $\rho_1(s) = \xi^{-1}\rho(s)$, $\rho_2(s) = \xi^{-1/2}\rho(s)$ where $\rho(s)$ and ξ are defined, for all $s \geq 1$ and $\lambda \geq 1$, as follows

$$\rho(x,t,s) = \exp(s\varphi(x,t)), \quad \xi(x,t) = \theta(t)\exp(\lambda\widehat{\psi}(x)) \tag{13}$$

where $\theta \in \mathcal{C}^2([0,T))$ is defined such that, noting $\mu = s\lambda^2 e^{2\lambda}$ and $0 < T_1 < \min(\frac{1}{4}, \frac{3T}{8})$,

$$\theta(t) = \begin{cases} 1 + \left(1 - \dfrac{4t}{T}\right)^\mu & \forall t \in [0, T/4] \\ 1 \quad \forall t \in [T/4, T - 2T_1] \\ \theta \text{ is increasing on } [T - 2T_1, T - T_1], \\ \dfrac{1}{T-t} \quad \forall t \in [T - T_1, T) \end{cases} \tag{14}$$

and $\varphi \in \mathcal{C}^1([0,T))$ is defined by

$$\varphi(x,t) = \theta(t)\big(\lambda\exp(12\lambda) - \exp(\lambda\widehat{\psi}(x))\big) \tag{15}$$

with $\widehat{\psi} = \widetilde{\psi} + 6$, where $\widetilde{\psi} \in \mathcal{C}^1(\overline{\Omega})$ satisfies $\widetilde{\psi} \in (0,1)$ in Ω, $\widetilde{\psi} = 0$ on $\partial\Omega$ and $|\partial_x\widetilde{\psi}(x)| > 0$ in $\overline{\Omega\backslash\omega}$. We emphasize that the weights blow up at $t \to T^-$ but not as $t \to 0^+$.

Remark 1. *We shall use in the sequel that $1 < \rho_0 \leq \rho_1 \leq \rho$ in Q_T. Indeed, since $\xi \geq 1$, $\rho_0 \leq \rho_1 \leq \rho$. Moreover, for all $(x,t) \in Q_T$ and $\lambda \geq 1$, we check that $\varphi(x,t) \geq \frac{3}{2}\xi(x,t)$ and thus, since $s \geq 1$ and $\xi(x,t) \geq 1$, we get*

$$\rho_0(x,t,s) = \xi^{-3/2}(x,t)\rho(x,t,s) \geq \xi^{-3/2}(x,t)\exp\left(\frac{3}{2}s\xi(x,t)\right) \geq e^{3/2s} \quad \forall (x,t) \in Q_T.$$

The controllability property for the linear system (11) is based on the following Carleman estimate.

Lemma 1. *Let $P_0 := \{q \in C^2(\overline{Q_T}) \ : \ q = 0 \text{ on } \Sigma_T\}$. There exist $\lambda_0 \geq 1$ and $s_0 \geq 1$ such for all $\lambda \geq \lambda_0$ and for all $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$, the following Carleman estimate holds*

$$\begin{aligned}\int_\Omega \rho^{-2}(0,s)|\partial_x p(0)|^2 + s^2\lambda^3 e^{14\lambda}\int_\Omega \rho^{-2}(0,s)|p(0)|^2 + s\lambda^2\int_{Q_T}\rho_2^{-2}(s)|\partial_x p|^2 + s^3\lambda^4\int_{Q_T}\rho_0^{-2}(s)|p|^2 \\ \leq C\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \partial_{xx}p + Ap|^2 + Cs^3\lambda^4\int_{q_T}\rho_0^{-2}(s)|p|^2, \quad \forall p \in P_0.\end{aligned} \tag{16}$$

Proof. This estimate is deduced from the one obtained in [1, Theorem 2.5] devoted to the case $A \equiv 0$: there exist $\lambda_1 \geq 1$ and $s_0 \geq 1$ such that for every smooth function z on $\overline{Q_T}$ satisfying $z = 0$ on Σ_T and for all $s \geq s_0$ and $\lambda \geq \lambda_1$

$$\begin{aligned}\int_\Omega \rho^{-2}(0,s)|\partial_x p(0)|^2 + s^2\lambda^3 e^{14\lambda}\int_\Omega \rho^{-2}(0,s)|p(0)|^2 + s\lambda^2\int_{Q_T}\rho_2^{-2}(s)|\partial_x p|^2 + s^3\lambda^4\int_{Q_T}\rho_0^{-2}(s)|p|^2 \\ \leq C\int_{Q_T}\rho^{-2}(s)|\partial_t p + \partial_{xx}p|^2 + Cs^3\lambda^4\int_{q_T}\rho_0^{-2}(s)|p|^2.\end{aligned}$$

Writing

$$\begin{aligned}\int_{Q_T}\rho^{-2}(s)|\partial_t p + \partial_{xx}p|^2 &\leq 2\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \partial_{xx}p + Ap|^2 + 2\int_{Q_T}\rho^{-2}(s)|Ap|^2 \\ &\leq 2\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \partial_{xx}p + Ap|^2 + 2\|A\|^2_{L^\infty(Q_T)}\int_{Q_T}\rho^{-2}(s)|p|^2\end{aligned}$$

we infer, since $\rho_0 \leq \rho$, that

$$\begin{aligned}&\int_\Omega \rho^{-2}(0,s)|\partial_x p(0)|^2 + s^2\lambda^3 e^{14\lambda}\int_\Omega \rho^{-2}(0,s)|p(0)|^2 + s\lambda^2\int_{Q_T}\rho_2^{-2}(s)|\partial_x p|^2 + s^3\lambda^4\int_{Q_T}\rho_0^{-2}(s)|p|^2 \\ &\leq C\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \partial_{xx}p + Ap|^2 + C\|A\|^2_{L^\infty(Q_T)}\int_{Q_T}\rho_0^{-2}(s)|p|^2 + Cs^3\lambda^4\int_{q_T}\rho_0^{-2}(s)|p|^2.\end{aligned}$$

Taking $\lambda \geq \lambda_0 = \max(\lambda_1, (2C)^{1/4})$ and $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$ leads to (16). $\square$

In the sequel we assume that $\lambda = \lambda_0$ and denote by C any constant depending only on Ω, ω, λ_0 and T.

We then define and check that the bilinear form

$$(p,q)_P := \int_{Q_T}\rho^{-2}(s)L_A^\star p\, L_A^\star q + s^3\lambda_0^4\int_{q_T}\rho_0^{-2}(s)p\,q$$

where $L_A^\star q := -\partial_t q - \partial_{xx}q + Aq$ for all $q \in P_0$ is a scalar product on P_0 (see [14]). The completion P of P_0 for the norm $\|\cdot\|_P$ associated with this scalar product is a Hilbert space. By density arguments, (16) remains true for all $p \in P$, that is, for $\lambda = \lambda_0$,

$$\int_\Omega \rho^{-2}(0,s)|\partial_x p(0)|^2 + s^2\lambda_0^3 e^{14\lambda_0}\int_\Omega \rho^{-2}(0,s)|p(0)|^2 + s\lambda_0^2\int_{Q_T}\rho_2^{-2}(s)|\partial_x p|^2 + s^3\lambda_0^4\int_{Q_T}\rho_0^{-2}(s)|p|^2 \leq C\|p\|_P^2 \tag{17}$$

for all $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$.

Remark 2. *We denote by P (instead of P_A) the completion of P_0 for the norm $\|\cdot\|_P$ since P does not depend on A (see [13, Lemma 3.1]).*

Lemma 2. *Let $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$. There exists a unique solution $p \in P$ of*

$$(p,q)_P = \int_\Omega z_0 q(0) + \int_{Q_T} Bq, \quad \forall q \in P. \tag{18}$$

This solution satisfies the following estimate (with $c := \|\varphi(\cdot,0)\|_{L^\infty(\Omega)}$)

$$\|p\|_P \leq Cs^{-3/2}\big(\|\rho_0(s)\,B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{L^2(\Omega)}\big). \tag{19}$$

Proof. The linear map $L_1 : P \to \mathbb{R}$, $q \mapsto \int_{Q_T} Bq$ is continuous. Indeed, for all $q \in P$

$$\Big|\int_{Q_T} Bq\Big| \leq \Big(\int_{Q_T} |\rho_0(s)B|^2\Big)^{1/2}\Big(\int_{Q_T} |\rho_0^{-1}(s)q|^2\Big)^{1/2}$$

and since from the Carleman estimate (17) we have $\Big(\int_{Q_T} |\rho_0^{-1}(s)q|^2\Big)^{1/2} \leq Cs^{-3/2}\|q\|_P$, therefore

$$|L_1(q)| = \Big|\int_{Q_T} Bq\Big| \leq Cs^{-3/2}\|\rho_0(s)B\|_{L^2(Q_T)}\|q\|_P.$$

Thus L_1 is continuous.

From (17) we deduce that the linear map $L_2 : P \to \mathbb{R}$, $q \mapsto \int_\Omega z_0 q(0)$ is continuous. Indeed, noting $c := \|\varphi(\cdot,0)\|_\infty$ and using $s \geq 1$, we obtain for all $q \in P$ that:

$$\begin{aligned}|L_2(q)| &= s^{-3/2}e^{cs}\|z_0\|_{L^2(\Omega)} s^{3/2}e^{-cs}\|q(0)\|_{L^2(\Omega)}\\ &\leq s^{-3/2}e^{cs}\|z_0\|_{L^2(\Omega)} s^{3/2}\|q(0)e^{-s\varphi(x,0)}\|_{L^2(\Omega)} = s^{-3/2}e^{cs}\|z_0\|_{L^2(\Omega)} s^{3/2}\|\rho^{-1}(0,s)q(0)\|_{L^2(\Omega)}\\ &\leq Cs^{-3/2}e^{cs}\|z_0\|_{L^2(\Omega)}\|q\|_P.\end{aligned}$$

Using Riesz's theorem, we conclude that there exists exactly one solution $p \in P$ of (18) and this solution satisfies (19). $\square$

2.2 Application to controllability

We now show how Lemma 1 and Lemma 2 imply a controllability result for the linear system 11. This part is mainly classical and follows closely [18] but we present it with some details as it is an essential part of our arguments developed in Section 3. The main result giving a precise bound of a control pair in term of the potential and right hand side is as follows.

Theorem 3. *Assume $A \in L^\infty(Q_T)$, $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$, $B \in L^2(\rho_0(s), Q_T)$ and $z_0 \in L^2(\Omega)$. Then there exists a control $v \in L^2(\rho_0(s); q_T)$ such that the weak solution z of (11) satisfies (12).*

Moreover, the unique control v which minimizes together with the corresponding solution z the functional $J : L^2(\rho(s); Q_T) \times L^2(\rho_0(s); q_T) \to \mathbb{R}^+$ defined by $J(z,v) := \frac{1}{2}\|\rho(s)\,z\|^2_{L^2(Q_T)} + \frac{s^{-3}\lambda_0^{-4}}{2}\|\rho_0(s)\,v\|^2_{L^2(q_T)}$ satisfies the following estimates

$$\|\rho(s)\,z\|_{L^2(Q_T)} + s^{-3/2}\lambda_0^{-2}\|\rho_0(s)\,v\|_{L^2(q_T)} \leq Cs^{-3/2}\big(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{L^2(\Omega)}\big) \tag{20}$$

with $c := \|\varphi(\cdot,0)\|_{L^\infty(\Omega)}$ and

$$\|\rho_1(s)z\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_1(s)\partial_x z\|_{L^2(Q_T)} \leq C_1(s,A)\big(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{L^2(\Omega)}\big) \tag{21}$$

where

$$C_1(s,A) := Cs^{-1/2}(1 + \|A\|^{1/2}_{L^\infty(Q_T)}).$$

Moreover, if $z_0 \in H^1_0(\Omega)$ then $z \in L^\infty(Q_T)$ and

$$\|z\|_{L^\infty(Q_T)} \leq Ce^{-\frac{3}{2}s}(1 + \|A\|_{L^\infty(Q_T)})\big(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{H^1_0(\Omega)}\big). \tag{22}$$

We refer to [16, 18] for estimates of the null control of minimal $L^2(q_T)$-norm (corresponding to $\rho_0 \equiv 1$ and $\rho = 0$) in the case $B \equiv 0$, refined later on in [12, 17]. For simplicity, we divide the proof of Theorem 3 in several technical lemma.

We introduce the convex set

$$\mathcal{C}(z_0, T) := \Big\{ (z, v) : \rho(s)z \in L^2(Q_T),\ \rho_0(s)v \in L^2(q_T),\ (z, v) \text{ solves (11)-(12) in the transposition sense} \Big\}.$$

Let us remark that (z, v) solves (11)-(12) in the transposition sense if (z, v) is solution of

$$\int_{Q_T} zL_A^\star q = \int_{q_T} vq + \int_\Omega z_0 q(0) + \int_{Q_T} Bq, \quad \forall q \in P. \tag{23}$$

Therefore if $(z, v) \in \mathcal{C}(z_0, T)$, then since $v \in L^2(q_T)$ and $B \in L^2(Q_T)$, z coincides with the unique weak solution of (11) associated with v. We can now claim that $\mathcal{C}(z_0, T)$ is non empty.

Lemma 3. *Let $s \geq \max(\|A\|_{L^\infty(Q_T)}^{2/3}, s_0)$, $p \in P$ the unique solution of (18) and (z, v) be defined by*

$$z = \rho^{-2}(s)L_A^\star p \quad \text{and} \quad v = -s^3\lambda_0^4\rho_0^{-2}(s)p|_{q_T}. \tag{24}$$

Then $(z, v) \in \mathcal{C}(z_0, T)$ and satisfies the estimate (20).

Proof. From the definition of P, $\rho(s)z \in L^2(Q_T)$ and $\rho_0(s)v \in L^2(q_T)$ and therefore, since $\rho \geq \rho_0 > 1$, $z \in L^2(Q_T)$ and $v \in L^2(q_T)$. In view of (18), (z, v) is solution of (23) and satisfies (20) that is, z is the solution of (11)-(12) associated with v in the transposition sense. Thus $(z, v) \in \mathcal{C}(z_0, T)$. □

Let us now consider the following extremal problem, introduced by Fursikov and Imanuvilov in [18]

$$\begin{cases} \text{Minimize } J_s(z, v) = \dfrac{1}{2}\displaystyle\int_{Q_T} \rho^2(s)|z|^2 + \dfrac{s^{-3}\lambda_0^{-4}}{2}\int_{q_T} \rho_0^2(s)|v|^2 \\ \text{Subject to } (z, v) \in \mathcal{C}(z_0, T). \end{cases} \tag{25}$$

Then $(z, v) \mapsto J_s(z, v)$ is strictly convex and continuous on $L^2(\rho(s); Q_T) \times L^2(\rho_0(s); q_T)$. Therefore (25) possesses at most a solution in $\mathcal{C}(z_0, T)$. More precisely we have :

Lemma 4. *Let $s \geq \max(\|A\|_{L^\infty(Q_T)}^{2/3}, s_0)$. Then $(z, v) \in \mathcal{C}(z_0, T)$ defined in Lemma 3 is the unique solution of (25).*

Proof. Let $(y, w) \in \mathcal{C}(z_0, T)$. Since J_s is convex and differentiable on $L^2(\rho(s); Q_T) \times L^2(\rho_0(s); q_T)$ we have:

$$\begin{aligned} J_s(y, w) &\geq J_s(z, v) + \int_{Q_T} \rho^2(s)z(y - z) + s^{-3}\lambda_0^{-4}\int_{q_T} \rho_0^2(s)v(w - v) \\ &= J_s(z, v) + \int_{Q_T} L_A^\star p(y - z) - \int_{q_T} p(w - v) = J_s(z, v) \end{aligned}$$

y being the solution of (11) associated with w in the transposition sense. □

Proof of Theorem 3. Lemma 4 gives the existence of a control $v \in L^2(\rho_0(s); q_T)$ such that the corresponding solution z of (11) satisfies (12). Moreover, this control is the unique control which minimizes together with the corresponding solution z the functional J_s and satisfies (20). To finish the proof of Theorem 3, it suffices to prove that (z, v) satisfies the estimate (21) and if $z_0 \in H_0^1(\Omega)$, then $z \in L^\infty(Q_T)$ and satisfies the estimate (22).

Multiplying (11) by $\rho_1^2(s)z$ and integrating by part we obtain

$$\frac{1}{2}\int_\Omega (\partial_t|z|^2)\,\rho_1^2(s) + \int_\Omega \rho_1^2(s)|\partial_x z|^2 + 2\int_\Omega \rho_1(s)z\partial_x\rho_1(s)\cdot\partial_x z + \int_\Omega \rho_1^2(s)Azz = \int_\omega v\rho_1^2(s)z + \int_\Omega B\rho_1^2(s)z. \tag{26}$$

But $\int_\Omega (\partial_t |z|^2)\rho_1^2(s) = \partial_t \int_\Omega |z|^2\rho_1^2(s) - 2\int_\Omega |z|^2\rho_1(s)\partial_t\rho_1(s)$ and $\partial_t\rho_1(s) = -\frac{\partial_t\theta}{\theta}\rho_1(s) + s\frac{\partial_t\theta}{\theta}\varphi\rho_1(s)$. Moreover, from the definition of θ and φ we have:

$$\Big|\frac{\partial_t\theta}{\theta}(t)\Big| \le \begin{cases} Cs & \forall t\in[0,T/4] \\ 0 & \forall t\in[T/4, T-2T_1] \\ \theta & \forall t\in[T-T_1, T) \end{cases}$$

and $\left|\frac{\partial_t\theta}{\theta}(t)\right| \le C \quad \forall t\in[T-2T_1, T-T_1]$ since $\dfrac{\partial_t\theta}{\theta}(T-2T_1) = 0$, $\dfrac{\partial_t\theta}{\theta}(T-T_1) = \frac{1}{T_1}$ and θ is $\mathcal{C}^2$. Since $\theta \le \xi$ and $s\ge 1$, on $[0,T)$:

$$\Big|\frac{\partial_t\theta}{\theta}\Big| \le Cs\xi.$$

From the definition of φ, $\varphi \le C\theta \le C\xi$ and thus on $[0,T)$:

$$\Big|\frac{\partial_t\theta}{\theta}\varphi\Big| \le Cs\xi^2.$$

Thus, since $s\ge 1$, $\xi\ge 1$ and $\rho(s) = \xi\rho_1(s)$, on $[0,T)$:

$$-\int_\Omega |z|^2\rho_1(s)\partial_t\rho_1(s) = \int_\Omega \frac{\partial_t\theta}{\theta}|\rho_1(s)z|^2 - s\int_\Omega \frac{\partial_t\theta}{\theta}\varphi|\rho_1(s)z|^2 \le Cs^2\int_\Omega \rho^2(s)|z|^2.$$

On the other hand

$$\partial_x\rho_1(s) = \partial_x(\xi^{-1}\rho(s)) = \partial_x(\xi^{-1})\rho(s) + \xi^{-1}\partial_x\rho(s) = -\partial_x\widehat{\psi}\big(\lambda_0\xi^{-1}\rho(s) + s\lambda_0\rho(s)\big) = -\partial_x\widehat{\psi}\lambda_0\rho(s)\big(\xi^{-1}+s\big)$$

and thus, since $\xi\ge 1$ and $s\ge 1$, we write $(\xi^{-1}+s)\le 2s$ and

$$\begin{aligned}\Big|\int_\Omega \rho_1(s)z\partial_x\rho_1(s)\cdot\partial_x z\Big| &\le 2\|\partial_x\widehat{\psi}\|_{L^\infty(\Omega)}\lambda_0\int_\Omega s|\rho(s)z|\,|\rho_1(s)\partial_x z| \\ &\le Cs^2\int_\Omega |\rho(s)z|^2 + \frac{1}{2}\int_\Omega |\rho_1(s)\partial_x z|^2.\end{aligned}$$

We also have, since $\rho(s) = \xi\rho_1(s)$ and $\xi\ge 1$ the estimate, $\left|\int_\Omega \rho_1^2(s)Azz\right| \le C\|A\|_{L^\infty(Q_T)}\|\rho(s)z\|^2_{L^2(\Omega)}$. Finally, since $\rho_1^2(s) = \xi^{-1/2}\rho_0(s)\rho(s)$ and $\xi^{-1/2}\le 1$, we infer that

$$\Big|\int_\omega v\rho_1^2(s)z\Big| \le \Big|\int_\omega \rho_0(s)v\xi^{-1/2}\rho(s)z\Big| \le \Big(\int_\omega |\rho_0(s)v|^2\Big)^{1/2}\|\rho(s)z\|_{L^2(\Omega)}$$

and $|\int_\Omega B\rho_1^2(s)z| \le \|\rho_0(s)B\|_{L^2(\Omega)}\|\rho(s)z\|_{L^2(\Omega)}$. Thus (26) implies that

$$\begin{aligned}\partial_t\int_\Omega \rho_1^2(s)|z|^2 + \int_\Omega \rho_1^2(s)|\partial_x z|^2 \le & C\big(s^2 + \|A\|_{L^\infty(Q_T)}\big)\|\rho(s)z\|^2_{L^2(\Omega)} \\ & + \Big(\Big(\int_\omega |\rho_0(s)v|^2\Big)^{1/2} + \|\rho_0(s)B\|_{L^2(\Omega)}\Big)\|\rho(s)z\|_{L^2(\Omega)}\end{aligned}$$

and therefore for all $t\in[0,T)$, since $\|\rho_1(s,0)z_0\|^2_{L^2(\Omega)} \le e^{cs}\|z_0\|^2_{L^2(\Omega)}$, we get

$$\begin{aligned}\Big(\int_\Omega \rho_1^2(s)|z|^2\Big)(t) + \int_{Q_t}\rho_1^2(s)|\partial_x z|^2 \le & C\big(s^2 + \|A\|_{L^\infty(Q_T)}\big)\|\rho(s)z\|^2_{L^2(Q_T)} \\ & + \big(\|\rho_0(s)v\|_{L^2(q_T)} + \|\rho_0(s)B\|_{L^2(Q_T)}\big)\|\rho(s)z\|_{L^2(Q_T)} + e^{cs}\|z_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Using (20), we obtain since $s\ge 1$ that for all $t\in[0,T)$:

$$\Big(\int_\Omega \rho_1^2(s)|z|^2\Big)(t) + \int_{Q_t}\rho_1^2(s)|\partial_x z|^2 \le Cs^{-1}\big(1 + \|A\|_{L^\infty(Q_T)}\big)\big(\|\rho_0(s)B\|^2_{L^2(Q_T)} + e^{cs}\|z_0\|^2_{L^2(\Omega)}\big) \tag{27}$$

which gives (21).

If now $z_0 \in H^1_0(\Omega)$, then since z is a weak solution of (11) associated with v, maximal regularity in L^2 and Sobolev embeddings give that $z \in L^2(0,T;H^2(\Omega))$, $\partial_t z \in L^2(Q_T)$ and therefore $z \in L^\infty(Q_T)$. Moreover, multiplying (11) by $\partial_{xx}z$ and integrating by part we obtain

$$\frac{1}{2}\partial_t \int_\Omega |\partial_x z|^2 + \int_\Omega |\partial_{xx}z|^2 \leq \| -Az + v1_\omega + B\|_{L^2(\Omega)} \|\partial_{xx}z\|_{L^2(\Omega)}$$

and thus, since $\rho(s) \geq \rho_0(s) \geq e^{\frac{3}{2}s}$:

$$\begin{aligned}\partial_t \int_\Omega |\partial_x z|^2 &\leq \int_\Omega | -Az + v1_\omega + B|^2 \leq 3\big(\int_\Omega |Az|^2 + \int_\Omega |v1_\omega|^2 + \int_\Omega |B|^2\big)\\ &\leq 3e^{-3s}\Big(\|A\|^2_{L^\infty(Q_T)}\|\rho(s)z\|^2_{L^2(\Omega)} + \|\rho_0(s)v1_\omega\|^2_{L^2(\Omega)} + \|\rho_0(s)B\|^2_{L^2(\Omega)}\Big)\end{aligned}$$

which gives, a.e in $t \in (0,T)$

$$\int_\Omega |\partial_x z|^2(t) \leq 3e^{-3s}\Big(\|A\|^2_{L^\infty(Q_T)}\|\rho(s)z\|^2_{L^2(Q_T)} + \|\rho_0(s)v\|^2_{L^2(q_T)} + \|\rho_0(s)B\|^2_{L^2(Q_T)}\Big) + \|z_0\|^2_{H^1_0(\Omega)}.$$

Since $z(\cdot,t) \in H^1_0(\Omega)$, a.e in $t \in (0,T)$, $z(x,t) = \int_0^x \partial_x z(r,t)dr \leq \|\partial_x z\|_2(t)$ and thus, using (20), since $s \geq 1$:

$$\begin{aligned}\|z\|_{L^\infty(Q_T)} &\leq \|\partial_x z\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq \sqrt{3}e^{-\frac{3}{2}s}\Big(\|A\|_{L^\infty(Q_T)}\|\rho(s)z\|_{L^2(Q_T)} + \|\rho_0(s)v\|_{L^2(q_T)} + \|\rho_0(s)B\|_{L^2(Q_T)}\Big) + \|z_0\|_{H^1_0(\Omega)}\\ &\leq Ce^{-\frac{3}{2}s}(1 + \|A\|_{L^\infty(Q_T)})\big(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{H^1_0(\Omega)}\big)\end{aligned}$$

that is (22). □

Remark 3. *Remark that $c = \|\varphi(\cdot,0)\|_{L^\infty(\Omega)} > 3/2$ so that the bound (22) of $\|z\|_{L^\infty(Q_T)}$ is not uniform with respect to the parameter $s \geq 1$.*

3 The least-squares method

In this section, we assume that the nonlinear function g satisfies the hypothesis $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0,1]$ and that

$(\mathbf{H_2})$ There exists $\alpha \geq 0$ and $\beta > 0$ such that $|g'(r)| \leq \alpha + \beta \ln^{3/2}(1+|r|)$ for every r in $\mathbb{R}$.

We introduce the notation

$$\psi(r) := \alpha + \beta \ln^{3/2}(1+|r|), \quad \forall r \in \mathbb{R}. \tag{28}$$

We also assume that $g(0) = 0$ leading in particular to the estimate $|g(r)| \leq |r|(\alpha + \beta \ln^{3/2}(1+|r|))$ for every $r \in \mathbb{R}$. The case $p = 0$ in $(\overline{\mathbf{H}}_\mathbf{p})$ corresponds to $\beta = 0$ and $\alpha = \|g'\|_\infty$ and thus $\psi(r) \leq \|g'\|_\infty$ for every $r \in \mathbb{R}$. Remark that $(\mathbf{H_2})$ implies (2) and $(\mathbf{H_1})$.

3.1 The least-squares method

We introduce, for all $s \geq s_0$, the vector space $\mathcal{A}_0(s)$

$$\begin{aligned}\mathcal{A}_0(s) := \Big\{&(y,f) : \rho(s)\,y \in L^2(Q_T),\ \rho_0(s)f \in L^2(q_T),\\ &\rho_0(s)(\partial_t y - \partial_{xx}y) \in L^2(Q_T),\ y(\cdot,0) = 0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T\Big\}\end{aligned}$$

where $\rho(s)$, $\rho_1(s)$ and $\rho_0(s)$ are defined in (13). Note that $\mathcal{A}_0(s)$ endowed with the scalar product

$$\begin{aligned}\big((y,f),(\overline{y},\overline{f})\big)_{\mathcal{A}_0(s)} := \big(\rho(s)y,\rho(s)\overline{y}\big)_{L^2(Q_T)} + \big(\rho_0(s)f,\rho_0(s)\overline{f}\big)_{L^2(q_T)}\\ + \big(\rho_0(s)(\partial_t y-\partial_{xx}y),\rho_0(s)(\partial_t\overline{y}-\partial_{xx}\overline{y})\big)_{L^2(Q_T)}\end{aligned}$$

is a Hilbert space. The corresponding norm is $\|(y,f)\|_{\mathcal{A}_0(s)} := \sqrt{((y,f),(y,f))_{\mathcal{A}_0(s)}}$. We also consider the affine space

$$\begin{aligned}\mathcal{A}(s) := \Big\{(y,f) : \rho(s)\,y \in L^2(Q_T),\ \rho_0(s)f\in L^2(q_T),\\ \rho_0(s)(\partial_t y-\partial_{xx}y)\in L^2(Q_T),\ y(\cdot,0)=u_0 \text{ in } \Omega,\ y=0 \text{ on } \Sigma_T\Big\}\end{aligned} \tag{29}$$

so that we can write $\mathcal{A}(s) = (\overline{y},\overline{f}) + \mathcal{A}_0(s)$ for any element $(\overline{y},\overline{f})\in\mathcal{A}(s)$. We endow $\mathcal{A}(s)$ with the metric $d_{\mathcal{A}(s)}$ induced by the norm of $\mathcal{A}_0(s)$, that is $d_{\mathcal{A}(s)}\big((y_1,f_1),(y_2,f_2)\big) = \|(y_1-y_2,f_1-f_2)\|_{\mathcal{A}_0(s)}$, for all $(y_1,f_1)\in\mathcal{A}(s)$ and $(y_2,f_2)\in\mathcal{A}(s)$. Clearly, if $(y,f)\in\mathcal{A}(s)$, then $y\in C([0,T];L^2(\Omega))$ and since $\rho(s)\,y\in L^2(Q_T)$, then $y(\cdot,T)=0$. The null controllability requirement is therefore incorporated in the spaces $\mathcal{A}_0(s)$ and $\mathcal{A}(s)$.

Remark 4. *For any $(y,f)\in\mathcal{A}(s)$, since $\rho_0(s)\geq 1$ (see Remark 1), we get that $\partial_t y-\partial_{xx}y\in L^2(Q_T)$; since $u_0\in H_0^1(\Omega)$, standard arguments imply that $y\in L^\infty(Q_T)$ with*

$$\|y\|_{L^\infty(Q_T)} \leq C\big(\|u_0\|_{H_0^1(\Omega)} + \|\partial_t y-\partial_{xx}y\|_{L^2(Q_T)}\big).$$

In particular, for any $(y,f)\in\mathcal{A}_0(s)$, $\|y\|_{L^\infty(Q_T)}\leq Ce^{-\frac{3}{2}s}\|(y,f)\|_{\mathcal{A}_0(s)}$ for some $C>0$ independent of s.

For any fixed $(y_0,f_0)\in\mathcal{A}(s)$ and $s\geq 0$, we can now consider the following non convex extremal problem:

$$\inf_{(y,f)\in\mathcal{A}_0(s)} E_s(y_0+y,f_0+f) \tag{30}$$

where the least-squares functional $E_s:\mathcal{A}(s)\to\mathbb{R}$ is defined as follows

$$E_s(y,f) := \frac{1}{2}\bigg\|\rho_0(s)\Big(\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega\Big)\bigg\|^2_{L^2(Q_T)}. \tag{31}$$

We check that $\rho_0(s)g(y)\in L^2(Q_T)$ for any $(y,f)\in\mathcal{A}(s)$ so that E_s is well-defined. Precisely, using that $|g(r)|\leq|r|\big(\alpha+\beta\ln^{3/2}(1+|r|)\big)=|r|\psi(r)$ for every r and that $\rho_0\leq\rho$, we write

$$\begin{aligned}\|\rho_0(s)g(y)\|_{L^2(Q_T)} &\leq \|\rho_0(s)|y|\psi(y)\|_{L^2(Q_T)}\\ &\leq \psi\big(\|y\|_{L^\infty(Q_T)}\big)\|\rho(s)y\|_{L^2(Q_T)} \leq \psi\big(\|y\|_{L^\infty(Q_T)}\big)\|(y,f)\|_{\mathcal{A}(s)}.\end{aligned} \tag{32}$$

Any pair $(y,f)\in\mathcal{A}$ for which $E_s(y,f)$ vanishes is a controlled pair of (1), and conversely. In this sense, the functional E_s is a so-called error functional which measures the deviation of (y,f) from being a solution of the underlying nonlinear equation. Moreover, although the hypothesis $(\mathbf{H_2})$ is stronger than $(\mathbf{H_1})$, Theorem 1 proved in [17] does not imply the existence of zero of E_s in $\mathcal{A}(s)$, since controls of minimal $L^\infty(q_T)$ norm are considered in [17]. Nevertheless, our constructive approach will show that, for s large enough, the extremal problem (30) admits solutions $(y,f)\in\mathcal{A}(s)$ for which E_s vanishes.

We also emphasize that the $L^2(Q_T)$ norm in E_s indicates that we are looking for regular weak solutions of the parabolic equation (1). We refer to [22] devoted to the case $g'\in L^\infty(\mathbb{R})$ and the multidimensional case where the $L^2(0,T;H^{-1}(\Omega))$ norm is considered leading to weaker solutions.

A practical way of taking a functional to its minimum is through some use of its derivative. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the convexity of the functional E_s. However, for nonlinear equation like (1), one cannot expect this property to hold for the functional E_s. Nevertheless, we are going to construct a minimizing sequence which always converge to a zero of E_s. To do so, we introduce the following definition.

Definition 1. *For any s large enough and $(y,f)\in\mathcal{A}(s)$, we define the unique pair $(Y^1,F^1)\in\mathcal{A}_0(s)$ solution of*

$$\begin{cases}\partial_t Y^1-\partial_{xx}Y^1+g'(y)Y^1=F^1 1_\omega+\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega & \text{in} \quad Q_T,\\ Y^1=0 \text{ on } \Sigma_T, \quad Y^1(\cdot,0)=0 \text{ in } \Omega\end{cases}\tag{33}$$

and which minimizes the functional J_s defined in Theorem 3. In the sequel, it is called the minimal controlled pair.

The next proposition shows that there do exists some (Y^1,F^1) in $\mathcal{A}_0(s)$. We emphasize that F^1 is a null control for the solution Y^1. Preliminary, we prove the following result.

Lemma 5. *There exists $(y,f)\in L^2(Q_T)\times L^2(q_T)$ such that $(y,f)\in\mathcal{A}(s)$ for all $s\ge 0$.*

Proof. Let $y^\star$ be the solution of

$$\begin{cases}\partial_t y^\star-\partial_{xx}y^\star=0 & \text{in} \quad Q_T,\\ y^\star=0 \text{ on } \Sigma_T, \quad y^\star(\cdot,0)=u_0\in H^1_0(\Omega) \text{ in } \Omega,\end{cases}$$

so that $y^\star\in L^2(0,T;H^2(\Omega))$ and $\partial_t y^\star\in L^2(0,T;L^2(\Omega))$. Let now any function $\phi\in C^\infty([0,T])$, $0\le\phi\le 1$ such that $\phi(0)=1$ and $\phi\equiv 0$ in $[T/2,T]$. Then, we easily check that the pair $(y,0)$ with $y:=\phi\,y^\star$ belongs to $\mathcal{A}(s)$ for any $s\ge 0$. $\square$

Proposition 1. *Let $(y,f)\in\mathcal{A}(s)$ with $s\ge\max\big(\|g'(y)\|^{2/3}_{L^\infty(Q_T)},s_0\big)$. There exists a minimal controlled pair $(Y^1,F^1)\in\mathcal{A}_0(s)$ solution of (33). It satisfies the estimate:*

$$\|(Y^1,F^1)\|_{\mathcal{A}_0(s)}\le C\sqrt{E_s(y,f)}\tag{34}$$

for some $C>0$.

Proof. For all $(y,f)\in\mathcal{A}(s)$, $\rho_0(s)(\partial_t y-\partial_{xx}y+g(y)-f1_\omega)\in L^2(Q_T)$. The existence of a null control F^1 is therefore given by Proposition 3. Choosing the control F^1 which minimizes together with the corresponding solution Y^1 the functional J_s defined in Theorem 3, we get from (20)-(21) the following estimate (recall that $Y^1(\cdot,0)=0$) :

$$\begin{aligned}\|\rho(s)\,Y^1\|_{L^2(Q_T)}+s^{-3/2}\lambda_0^{-2}\|\rho_0(s)F^1\|_{L^2(q_T)}&\le Cs^{-3/2}\|\rho_0(s)(\partial_t y-\partial_{xx}y+g(y)-f1_\omega)\|_{L^2(Q_T)}\\&\le Cs^{-3/2}\sqrt{E_s(y,f)}.\end{aligned}\tag{35}$$

Eventually, from the equation solved by Y^1,

$$\begin{aligned}&\|\rho_0(s)(\partial_t Y^1-\partial_{xx}Y^1)\|_{L^2(Q_T)}\\&\qquad\le\|\rho_0(s)F^1\|_{L^2(q_T)}+\|\rho_0(s)g'(y)Y^1\|_{L^2(Q_T)}+\|\rho_0(s)(\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega)\|_{L^2(Q_T)}\\&\qquad\le\|\rho_0(s)F^1\|_{L^2(q_T)}+\|\rho_0(s)g'(y)Y^1\|_{L^2(Q_T)}+\sqrt{2E_s(y,f)}.\end{aligned}$$

But, since $\rho_0(s)\le\rho(s)$, using (35), we have

$$\|\rho_0(s)g'(y)Y^1\|^2_{L^2(Q_T)}\le\|g'(y)\|^2_{L^\infty(Q_T)}\|\rho(s)Y^1\|^2_{L^2(Q_T)}\le Cs^{-3}\|g'(y)\|^2_{L^\infty(Q_T)}E_s(y,f)\tag{36}$$

thus

$$\|\rho_0(s)(\partial_t Y^1-\partial_{xx}Y^1)\|_{L^2(Q_T)}\le C\big(1+s^{-3/2}\|g'(y)\|_{L^\infty(Q_T)}\big)\sqrt{E_s(y,f)}\tag{37}$$

which proves that (Y^1, F^1) belongs to $\mathcal{A}_0(s)$. Eventually,

$$\begin{aligned}
\|(Y^1,F^1)\|^2_{\mathcal{A}_0(s)} &= \|\rho(s)Y^1\|^2_{L^2(Q_T)} + \|\rho_0(s)F^1 1_\omega\|^2_{L^2(Q_T)} + \|\rho_0(s)(\partial_t Y^1 - \partial_{xx}Y^1)\|^2_{L^2(Q_T)}\\
&\leq \|\rho(s)Y^1\|^2_{L^2(Q_T)} + 4\|\rho_0(s)F^1 1_\omega\|^2_{L^2(Q_T)} + 3\|\rho_0(s)g'(y)Y^1\|^2_{L^2(Q_T)} + 3(\sqrt{2E_s(y,f)})^2\\
&\leq C\big(s^{-3}E_s(y,f) + s^{-3}\|g'(y)\|^2_{L^\infty(Q_T)}E_s(y,f) + E_s(y,f)\big)\\
&\leq CE_s(y,f)(1 + s^{-3} + s^{-3}\|g'(y)\|^2_{L^\infty(Q_T)}).
\end{aligned}$$

Using that $s \geq \max(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0) \geq 1$, we get $s^{-3} \leq 1$ and $s^{-3}\|g'(y)\|^2_{L^\infty(Q_T)} \leq 1$ leading to the result. □

Remark 5. *From (33), we observe that $z := y - Y^1 \in L^2(\rho(s); Q_T)$ is a null controlled solution satisfying*

$$\begin{cases} \partial_t z - \partial_{xx} z + g'(y)z = (f - F^1)1_\omega + g'(y)y - g(y) \quad \text{in} \quad Q_T,\\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot,0) = u_0 \text{ in } \Omega \end{cases} \tag{38}$$

by the control $(f - F^1) \in L^2(\rho_0(s); q_T)$.

Remark 6. *We emphasize that the presence of a right hand side term in (33), namely $\partial_t y - \partial_{xx}y + g(y) - f\,1_\omega$, forces us to introduce the non trivial weights $\rho_0(s)$, $\rho_1(s)$ and $\rho(s)$ in the space $\mathcal{A}(s)$. This can be seen in the equality* (18)*: since $\rho_0^{-1}(s)q$ belongs to $L^2(Q_T)$ for all $q \in P$, we need to impose that $\rho_0(s)B \in L^2(Q_T)$ with here $B = \partial_t y - \partial_{xx}y + g(y) - f\,1_\omega$. Working with the linearized equation* (6) *(introduced in [17]) which does not make appear any right hand side, we may avoid the introduction of Carleman type weights. Actually, [17] considers controls of minimal $L^\infty(q_T)$ norm. Introduction of weights allows however the characterization* (18)*, which is very convenient at the practical level. We refer to [14] where this is discussed at length.*

We also emphasize that we have considered bounded weights at the initial time $t = 0$ because of the constraints "$\rho(s)y \in L^2(Q_T)$" and "$y(\cdot,0) = u_0$ in Ω" appearing in the set $\mathcal{A}(s)$.

3.2 Main properties of the functional E_s

The interest of the minimal controlled pair $(Y^1, F^1) \in \mathcal{A}_0(s)$ lies in the following result.

Proposition 2. *For any $(y,f) \in \mathcal{A}(s)$ and $s \geq \max\big(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\big)$, let $(Y^1, F^1) \in \mathcal{A}_0(s)$ defined in Definition 1. Then the derivative of E_s at the point $(y,f) \in \mathcal{A}(s)$ along the direction (Y^1, F^1) given by $E'_s(y,f)\cdot(Y^1,F^1) := \lim_{\eta\to 0,\eta\neq 0} \frac{E_s((y,f)+\eta(Y^1,F^1)) - E_s(y,f)}{\eta}$ satisfies*

$$E'_s(y,f)\cdot(Y^1,F^1) = 2E_s(y,f). \tag{39}$$

Proof. We preliminary check that E_s is for all $(Y,F) \in \mathcal{A}_0(s)$, E_s is Gâteaux-differentiable at the point $(y,f) \in \mathcal{A}(s)$ along the direction $(Y,F) \in \mathcal{A}_0(s)$. For all $\lambda \in \mathbb{R}$, simple computations lead to the equality

$$E_s(y + \lambda Y, f + \lambda F) = E_s(y,f) + \lambda E'_s(y,f)\cdot(Y,F) + h\big(s,(y,f),\lambda(Y,F)\big)$$

with

$$E'_s(y,f)\cdot(Y,F) = \Big(\rho_0(s)(\partial_t y - \partial_{xx}y + g(y) - f\,1_\omega), \rho_0(s)(\partial_t Y - \partial_{xx}Y + g'(y)Y - F\,1_\omega)\Big)_{L^2(Q_T)} \tag{40}$$

and

$$\begin{aligned}h(s,(y,f),\lambda(Y,F)) :=&\lambda\Big(\rho_0(s)(\partial_t Y-\partial_{xx}Y+g'(y)Y-F\,1_\omega),\rho_0(s)l(y,\lambda Y)\Big)_{L^2(Q_T)}\\&+\frac{\lambda^2}{2}\|\rho_0(s)(\partial_t Y-\partial_{xx}Y+g'(y)Y-F\,1_\omega)\|^2_{L^2(Q_T)}\\&+\Big(\rho_0(s)(\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega),\rho_0(s)l(y,\lambda Y)\Big)_{L^2(Q_T)}\\&+\frac{1}{2}\|\rho_0(s)l(y,\lambda Y)\|^2_{L^2(Q_T)}\end{aligned}$$

where l is defined by

$$l(y,Y):=g(y+Y)-g(y)-g'(y)Y. \tag{41}$$

The application $(Y,F)\to E'_s(y,f)\cdot(Y,F)$ is linear and continuous from $\mathcal{A}_0(s)$ to $\mathbb{R}$ as it satisfies

$$\begin{aligned}&|E'_s(y,f)\cdot(Y,F)|\\&\quad\le\|\rho_0(s)(\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega)\|_{L^2(Q_T)}\|\rho_0(s)(\partial_t Y-\partial_{xx}Y+g'(y)Y-F\,1_\omega)\|_{L^2(Q_T)}\\&\quad\le\sqrt{2E_s(y,f)}\Big(\|\rho_0(s)(\partial_t Y-\partial_{xx}Y)\|_{L^2(Q_T)}+\|\rho_0(s)F\|_{L^2(q_T)}+\|\rho_0(s)g'(y)Y\|_{L^2(Q_T)}\Big)\\&\quad\le\sqrt{2E_s(y,f)}\Big(\|\rho_0(s)(\partial_t Y-\partial_{xx}Y)\|_{L^2(Q_T)}+\|\rho_0(s)F\|_{L^2(q_T)}+\|g'(y)\|_{L^\infty(Q_T)}\|\rho(s)Y\|_{L^2(Q_T)}\Big)\\&\quad\le C\big(1+\|g'(y)\|_{L^\infty(Q_T)}\big)\sqrt{E_s(y,f)}\|(Y,F)\|_{\mathcal{A}_0(s)}.\end{aligned} \tag{42}$$

Similarly, for all $\lambda\in\mathbb{R}^\star$

$$\begin{aligned}\Big|\frac{1}{\lambda}h\big(s,(y,f),\lambda(Y,F)\big)\Big|\le&\Big(|\lambda|\|\rho_0(s)(\partial_t Y-\partial_{xx}Y+g'(y)Y-F\,1_\omega)\|_{L^2(Q_T)}+\sqrt{2E_s(y,f)}\\&+\frac{1}{2}\|\rho_0(s)l(y,\lambda Y)\|_{L^2(Q_T)}\Big)\frac{1}{|\lambda|}\|\rho_0(s)l(y,\lambda Y)\|_{L^2(Q_T)}\\&+\frac{|\lambda|}{2}\|\rho_0(s)(\partial_t Y-\partial_{xx}Y+g'(y)Y-F\,1_\omega)\|^2_{L^2(Q_T)}.\end{aligned}$$

Since $g'\in\mathcal{C}(\mathbb{R})$ we have, a.e in Q_T : $\Big|\frac{1}{\lambda}l(y,\lambda Y)\Big|=\Big|\frac{g(y+\lambda Y)-g(y)}{\lambda}-g'(y)Y\Big|\to 0$ as $\lambda\to 0$ and, since $Y\in L^\infty(Q_T)$ and $y\in L^\infty(Q_T)$, a.e in Q_T

$$\Big|\frac{1}{\lambda}l(y,\lambda Y)\Big|=\Big|\frac{g(y+\lambda Y)-g(y)}{\lambda}-g'(y)Y\Big|\le\big(\sup_{\theta\in[0,1]}\|g'(y+\theta Y)\|_{L^\infty(Q_T)}+\|g'(y)\|_{L^\infty(Q_T)}\big)|Y|$$

and therefore (recalling that $\rho_0\le\rho$)

$$\begin{aligned}\frac{1}{|\lambda|}\|\rho_0(s)l(y,\lambda Y)\|_{L^2(Q_T)}&\le\big(\sup_{\theta\in[0,1]}\|g'(y+\theta Y)\|_{L^\infty(Q_T)}+\|g'(y)\|_{L^\infty(Q_T)}\big)\|\rho_0 Y\|_{L^2(Q_T)}\\&\le\big(\sup_{\theta\in[0,1]}\|g'(y+\theta Y)\|_{L^\infty(Q_T)}+\|g'(y)\|_{L^\infty(Q_T)}\big)\|\rho Y\|_{L^2(Q_T)}.\end{aligned}$$

It then follows from the Lebesgue dominated convergence theorem that $\frac{1}{\lambda}\|\rho_0(s)l(y,\lambda Y)\|_{L^2(Q_T)}\to 0$ as $\lambda\to 0$ and therefore that $h(s,(y,f),\lambda(Y,F))=o(\lambda)$. Thus the functional E_s is Gâteaux-differentiable at the point $(y,f)\in\mathcal{A}(s)$ along the direction $(Y,F)\in\mathcal{A}_0(s)$. Eventually, the equality (39) follows from the relation (33) satisfied by the pair (Y^1,F^1). $\square$

Remark that from the equality (40), the derivative $E'_s(y,f)$ is independent of (Y,F). We can then define the norm $\|E'_s(y,f)\|_{\mathcal{A}'_0(s)}:=\sup_{(Y,F)\in\mathcal{A}_0(s),(Y,F)\neq(0,0)}\frac{E'_s(y,f)\cdot(Y,F)}{\|(Y,F)\|_{\mathcal{A}_0(s)}}$ associated to $\mathcal{A}'_0(s)$, the set of the linear and continuous applications from $\mathcal{A}_0(s)$ to $\mathbb{R}$.

Combining (34) and (39), we deduce the following estimates of $E_s(y,f)$ in term of the norm of $E'_s(y,f)$.

Proposition 3. *For any $(y, f) \in \mathcal{A}(s)$ and $s \geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$, the inequalities hold true*

$$\frac{1}{C\big(1 + \|g'(y)\|_{L^\infty(Q_T)}\big)}\|E_s'(y,f)\|_{\mathcal{A}_0'(s)} \leq \sqrt{E_s(y,f)} \leq C\|E_s'(y,f)\|_{\mathcal{A}_0'(s)}$$

for some $C > 0$.

Proof. (39) rewrites $E_s(y, f) = \frac{1}{2}E_s'(y,f)\cdot(Y^1,F^1)$ where $(Y^1,F^1) \in \mathcal{A}_0(s)$ is solution of (33) and therefore, with (34)

$$\begin{aligned} E_s(y,f) &\leq \frac{1}{2}\|E_s'(y,f)\|_{\mathcal{A}_0'(s)}\|(Y^1,F^1)\|_{\mathcal{A}_0(s)} \\ &\leq C\|E_s'(y,f)\|_{\mathcal{A}_0'(s)}\sqrt{E_s(y,f)}. \end{aligned}$$

On the other hand, using (42), for all $(Y,F) \in \mathcal{A}_0(s)$:

$$|E_s'(y,f)\cdot(Y,F)| \leq C\big(1 + \|g'(y)\|_{L^\infty(Q_T)}\big)\sqrt{E_s(y,f)}\|(Y,F)\|_{\mathcal{A}_0(s)}$$

leading to the left inequality. $\square$

In particular, any *critical* point $(y, f) \in \mathcal{A}(s)$ for E_s (i.e. for which $E_s'(y, f)$ vanishes) is a zero for E_s, a pair solution of the controllability problem. In other words, any sequence $(y_k, f_k)_{k\in\mathbb{N}}$ of $\mathcal{A}(s)$ satisfying $\|E_s'(y_k, f_k)\|_{\mathcal{A}_0'(s)} \to 0$ as $k \to \infty$ and for which $(\|g'(y_k)\|_\infty)_{k\in\mathbb{N}}$ is bounded is such that $E_s(y_k, f_k) \to 0$ as $k \to \infty$. We insist that this property does not imply the convexity of the functional $E(s)$ (nor *a fortiori* the strict convexity of E_s, which actually does not hold here in view of the multiple zeros for E_s) but show that a minimizing sequence for E_s can not be stuck in a local minimum. Our least-squares algorithm, designed in the next section, is based on that property.

Eventually, the left inequality indicates that the functional E_s is flat around its zero set. As a consequence, gradient based minimizing sequences for E_s are inefficient as they usually achieve a low rate of convergence (we refer to [27] and also [25] devoted to the Navier-Stokes equation where this phenomenon is observed).

We end this section with the following crucial estimate.

Lemma 6. *Assume that g satisfies $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0, 1]$. For any $(y, f) \in \mathcal{A}(s)$ with $s \geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$, let $(Y^1,F^1) \in \mathcal{A}_0(s)$ be given in Definition 1 associated with (y, f). For any $\lambda \in \mathbb{R}_+$ the following estimate holds*

$$\sqrt{E_s\big((y,f) - \lambda(Y^1,F^1)\big)} \leq \sqrt{E_s(y,f)}\left(|1-\lambda| + \lambda^{p+1}c_1(s)\sqrt{E_s(y,f)}^{\,p}\right) \tag{43}$$

with

$$c_1(s) := \frac{C^{1+p}}{1+p}s^{-3/2}e^{-\frac{3p}{2}s}[g']_p. \tag{44}$$

Proof. The expansion of $E_s\big((y, f) - \lambda(Y^1,F^1)\big)$ together with Definition 1 of (Y^1,F^1) leads to

$$\begin{aligned} &2E_s\big((y,f) - \lambda(Y^1,F^1)\big) \\ &= \Big\|\rho_0(s)\big(\partial_t y - \partial_{xx}y + g(y) - f\,1_\omega\big) - \lambda\rho_0(s)\big(\partial_t Y^1 - \partial_{xx}Y^1 + g'(y)Y^1 - F\,1_\omega\big) + \rho_0(s)l(y, -\lambda Y^1)\Big\|^2_{L^2(Q_T)} \\ &= \Big\|\rho_0(s)(1-\lambda)\big(\partial_t y - \partial_{xx}y + g(y) - f\,1_\omega\big) + \rho_0(s)l(y, -\lambda Y^1)\Big\|^2_{L^2(Q_T)} \\ &\leq \Big(\big\|\rho_0(s)(1-\lambda)\big(\partial_t y - \partial_{xx}y + g(y) - f\,1_\omega\big)\big\|_{L^2(Q_T)} + \big\|\rho_0(s)l(y, -\lambda Y^1)\big\|_{L^2(Q_T)}\Big)^2 \\ &\leq 2\left(|1-\lambda|\sqrt{E_s(y,f)} + \big\|\rho_0(s)l(y, -\lambda Y^1)\big\|_{L^2(Q_T)}\right)^2 \end{aligned} \tag{45}$$

where l is defined in (41).

For any $(x,y)\in\mathbb{R}^2$, $y\neq 0$, and $\lambda\in\mathbb{R}$, we write $g(x+\lambda y)-g(x)=\int_0^\lambda yg'(x+\xi y)d\xi$ leading to

$$\begin{aligned}|g(x+\lambda y)-g(x)-\lambda g'(x)y| &\leq \int_0^\lambda |y||g'(x+\xi y)-g'(x)|d\xi\\ &\leq \int_0^\lambda |y|^{1+p}|\xi|^p\frac{|g'(x+\xi y)-g'(x)|}{|\xi y|^p}d\xi\\ &\leq [g']_p|y|^{1+p}\frac{|\lambda|^{1+p}}{1+p}.\end{aligned}$$

It follows that

$$|l(y,\lambda Y^1)|=\Big|g(y+\lambda Y^1)-g(y)-\lambda g'(y)Y^1\Big|\leq [g']_p|Y^1|^{1+p}\frac{|\lambda|^{p+1}}{1+p}$$

and thus, since $Y^1\in L^\infty(Q_T)$ (see Remark 4) and $\rho_0(s)\leq\rho(s)$:

$$\begin{aligned}\|\rho_0(s)l(y,-\lambda Y^1)\|_{L^2(Q_T)} &\leq [g']_p\frac{|\lambda|^{p+1}}{1+p}\|\rho_0(s)|Y^1|^{1+p}\|_{L^2(Q_T)}\\ &\leq [g']_p\frac{|\lambda|^{p+1}}{1+p}\|Y^1\|^p_{L^\infty(Q_T)}\|\rho(s)Y^1\|_{L^2(Q_T)}.\end{aligned}\tag{46}$$

Eventually, Remark 4 and estimates (34) and (35) in term of $\sqrt{E_s(y,f)}$ imply

$$\|Y^1\|^p_{L^\infty(Q_T)}\|\rho(s)Y^1\|_{L^2(Q_T)}\leq Cs^{-3/2}e^{-\frac{3p}{2}s}C^pE_s(y,f)^{\frac{p+1}{2}}.$$

Combining this estimate with (45) and (46), we get (43). $\square$

4 Convergence of the least-squares method

We now examine, for s large enough, the convergence of a minimizing sequence $(y_k,f_k)\in\mathcal{A}(s)$ for E_s. In this respect, we observe from the equality (39) that, for any $(y,f)\in\mathcal{A}(s)$, $-(Y^1,F^1)$ given in Definition 1, is a descent direction for the functional E_s at the point (y,f), as soon as $s\geq\max(\|g'(y)\|^{2/3}_{L^\infty(Q_T)},s_0)$. Therefore, we can define at least formally, for any fixed $m\geq 1$, a minimizing sequence $(y_k,f_k)_{k\in\mathbb{N}}\in\mathcal{A}(s)$ as follows:

$$\begin{cases}(y_0,f_0)\in\mathcal{A}(s),\\ (y_{k+1},f_{k+1})=(y_k,f_k)-\lambda_k(Y^1_k,F^1_k),\quad k\geq 0,\\ \lambda_k=\text{argmin}_{\lambda\in[0,m]}E_s\big((y_k,f_k)-\lambda(Y^1_k,F^1_k)\big)\end{cases}\tag{47}$$

where $(Y^1_k,F^1_k)\in\mathcal{A}_0(s)$ is the minimal controlled pair solution of

$$\begin{cases}\partial_tY^1_k-\partial_{xx}Y^1_k+g'(y_k)Y^1_k=F^1_k1_\omega+\partial_ty_k-\partial_{xx}y_k+g(y_k)-f_k1_\omega\quad\text{in}\quad Q_T,\\ Y^1_k=0\ \text{ on }\ \Sigma_T,\quad Y^1_k(\cdot,0)=0\ \text{ in }\ \Omega\end{cases}\tag{48}$$

associated with $(y_k,f_k)\in\mathcal{A}(s)$. In particular, the pair (Y^1_k,F^1_k) vanishes when $E_s(y_k,f_k)$ vanishes. The real number $m\geq 1$ is arbitrarily fixed and is introduced in order to keep the sequence $(\lambda_k)_{k\in\mathbb{N}}$ bounded.

We highlight that, in order to give a meaning to (47), we need to prove that we can choose the parameter s independent of k, that is $s\geq\max\big(\|g'(y_k)\|^{2/3}_{L^\infty(Q_T)},s_0\big)$ for all $k\in\mathbb{N}$. In this respect, it suffices to prove that there exists $M>0$ such that $\|y_k\|_{L^\infty(Q_T)}\leq M$ for every $k\in\mathbb{N}$. Under $(\mathbf{H_2})$, this implies that $\|g'(y_k)\|_{L^\infty(Q_T)}\leq\psi(M)$ for every $k\in\mathbb{N}$, where ψ is defined in (28).

In the next section, assuming the parameter s large enough, we prove the existence of such M by induction making use of the properties of the functional E_s proven in Section 3.2.

4.1 Boundedness of the sequence $(y_k)_{k\in\mathbb{N}}$ in $L^\infty(Q_T)$

The boundedness is obtained after several technical results. We first prove an estimate of $\|y_{n+1}\|_{L^\infty(Q_T)}$ assuming a bound of $\|y_k\|_{L^\infty(Q_T)}$ for $0\le k\le n$ and s large enough.

Proposition 4. *Assume that g satisfies $(\mathbf{H_2})$ and $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p\in[0,1]$. Let $M>0$ large enough and $s\ge \max\big(C(p)\psi(M)^{2/3},s_0\big)$ with $C(p)=1$ if $p\in(0,1]$, and $C(0)=(2C)^{3/2}$. Let $(y_0,f_0)\in\mathcal{A}(s)$ such that $M\ge\|y_0\|_{L^\infty(Q_T)}$. Assume that, for some $n\ge 0$, $(y_k,f_k)_{0\le k\le n}$ defined from (47) satisfies $\|y_k\|_{L^\infty(Q_T)}\le M$ for every $0\le k\le n$. Then*

$$\|y_{n+1}\|_{L^\infty(Q_T)} \le \|y_0\|_{L^\infty(Q_T)}+Cm\max\Big(\frac{p+1}{p}\sqrt{E_s(y_0,f_0)},\frac{(1+p)^{\frac{1}{p}+1}}{p}c_1^{1/p}(s)E_s(y_0,f_0)\Big) \tag{49}$$

if $p\in(0,1]$ and

$$\|y_{n+1}\|_{L^\infty(Q_T)}\le\|y_0\|_{L^\infty(Q_T)}+Cm\frac{\sqrt{E_s(y_0,f_0)}}{1-c_1(s)} \tag{50}$$

if $p=0$. Recall that $c_1(s)$ is defined in (44).

We point out that the existence of $(y_0,f_0)\in\mathcal{A}(s)$ follows from Lemma 5.

Proof. The inequality $\|y_n\|_{L^\infty(Q_T)}\le M$ implies that $\|g'(y_n)\|^{2/3}_{L^\infty(Q_T)}\le\psi(M)^{2/3}$ and then $s\ge\max\big(\psi(M)^{2/3},s_0\big)\ge\max\big(\|g'(y_n)\|^{2/3}_{L^\infty(Q_T)},s_0\big)$. Proposition 1 allows to construct the pair $(Y_n^1,F_n^1)\in\mathcal{A}_0(s)$ solution of (33). Then, (47) allows to define (y_{n+1},f_{n+1}). Estimate (43) implies that

$$\sqrt{E_s\big((y_k,f_k)-\lambda(Y_k^1,F_k^1)\big)}\le\sqrt{E_s(y_k,f_k)}\Big(|1-\lambda|+\lambda^{p+1}c_1(s)\sqrt{E_s(y_k,f_k)}^{\,p}\Big)$$

and then

$$\sqrt{E_s(y_{k+1},f_{k+1})}\le\sqrt{E_s(y_k,f_k)}\min_{\lambda\in[0,m]}p_k(s,\lambda) \tag{51}$$

with

$$p_k(s,\lambda):=|1-\lambda|+\lambda^{p+1}c_1(s)E_s(y_k,f_k)^{p/2},\quad\forall\lambda\in\mathbb{R},\ \forall s>0. \tag{52}$$

Since $\big(E_s(y_k,f_k)\big)_{0\le k\le n}$ decreases, $\big(p_k(s,\lambda)\big)_{0\le k\le n}$ decreases for all λ (p_k do not depend on k if $p=0$) and thus, defining $p_k(s,\widetilde{\lambda_k}):=\min_{\lambda\in[0,m]}p_k(s,\lambda)$, $\big(p_k(s,\widetilde{\lambda_k})\big)_{0\le k\le n}$ decreases as well. (51) then implies, for all $0\le k\le n-1$, that

$$\sqrt{E_s(y_{k+1},f_{k+1})}\le\sqrt{E_s(y_k,f_k)}p_k(s,\widetilde{\lambda_k})\le\sqrt{E_s(y_k,f_k)}p_0(s,\widetilde{\lambda_0}). \tag{53}$$

First case : $p\in(0,1]$. We prove that

$$\sum_{k=0}^n\sqrt{E_s(y_k,f_k)}\le\max\Big(\frac{p+1}{p}\sqrt{E_s(y_0,f_0)},\frac{(1+p)^{\frac{1}{p}+1}}{p}c_2(s)E_s(y_0,f_0)\Big) \tag{54}$$

where $c_2(s):=c_1^{1/p}(s)$. Since $\partial_\lambda p_0(s,0)=-1$, $p_0(s,\widetilde{\lambda_0})<p_0(s,0)=1$ we deduce from (53) that :

$$\sum_{k=0}^n\sqrt{E_s(y_k,f_k)}\le\sqrt{E_s(y_0,f_0)}\frac{1-p_0(s,\widetilde{\lambda_0})^{n+1}}{1-p_0(s,\widetilde{\lambda_0})}\le\frac{\sqrt{E_s(y_0,f_0)}}{1-p_0(s,\widetilde{\lambda_0})}. \tag{55}$$

If $c_2(s)\sqrt{E_s(y_0,f_0)}<\frac{1}{(p+1)^{1/p}}$, we check that $p_0(s,\widetilde{\lambda_0})\le p_0(s,1)=c_1(s)\sqrt{E_s(y_0,f_0)}^{\,p}\le\frac{1}{p+1}$ and thus

$$\frac{\sqrt{E_s(y_0,f_0)}}{1-p_0(s,\widetilde{\lambda_0})}\le\frac{p+1}{p}\sqrt{E_s(y_0,f_0)}.$$

If $c_2(s)\sqrt{E_s(y_0,f_0)} \geq \frac{1}{(p+1)^{1/p}}$, then for all $\lambda \in [0,1]$, $\partial_\lambda p_0(s,\lambda) = -1 + (p+1)\lambda^p c_1(s) E_s(y_0,f_0)^{p/2}$ and thus $\partial_\lambda p_0(s,\lambda) = 0$ if and only if $\lambda = \frac{1}{(p+1)^{1/p} c_2(s)\sqrt{E_s(y_0,f_0)}}$ leading to

$$p_0(s,\widetilde{\lambda_0}) = 1 - \frac{p}{(1+p)^{\frac{1}{p}+1}} \frac{1}{c_2(s)\sqrt{E_s(y_0,f_0)}}$$

and

$$\frac{\sqrt{E_s(y_0,f_0)}}{1-p_0(s,\widetilde{\lambda_0})} \leq \frac{(1+p)^{\frac{1}{p}+1}}{p} c_2(s) E_s(y_0,f_0).$$

(55) then leads to (54). Then (47) implies that $y_{n+1} = y_0 - \sum_{k=0}^n \lambda_k Y_k^1$ and thus, using (34)

$$\begin{aligned}\|y_{n+1}\|_{L^\infty(Q_T)} &\leq \|y_0\|_{L^\infty(Q_T)} + m\sum_{k=0}^n \|Y_k^1\|_{L^\infty(Q_T)} \leq \|y_0\|_{L^\infty(Q_T)} + Cm\sum_{k=0}^n \|(Y_k^1,F_k^1)\|_{\mathcal{A}_0(s)} \\ &\leq \|y_0\|_{L^\infty(Q_T)} + Cm\sum_{k=0}^n \sqrt{E_s(y_k,f_k)}\end{aligned}$$

which gives (49), using (54).

Second case : $p = 0$. Recall that for $p=0$, $\psi(r) = \|g'\|_\infty$ for every $r \in \mathbb{R}$. Then simply $p_k(s,\lambda) = |1-\lambda| + \lambda c_1(s)$ for all k and

$$p_0(s,\widetilde{\lambda_0}) = \min_{\lambda\in[0,m]} p_0(s,\lambda) = \min_{\lambda\in[0,1]} p_0(s,\lambda) = p_0(s,1) = c_1(s)$$

with $c_1(s) = Cs^{-3/2}[g']_0 = 2Cs^{-3/2}\|g'\|_\infty = 2Cs^{-3/2}\psi(M)$. Taking s large enough, precisely $s > \max((2C)^{1/3}\psi(M)^{2/3}, s_0)$, we obtain that $c_1(s) < 1$. We then have for all $0 \leq k \leq n-1$ that

$$\sqrt{E_s(y_{k+1},f_{k+1})} \leq \sqrt{E_s(y_k,f_k)}\, c_1(s)$$

and thus

$$\sum_{k=0}^n \sqrt{E_s(y_k,f_k)} \leq \sqrt{E_s(y_0,f_0)} \frac{1-c_1(s)^{n+1}}{1-c_1(s)} \leq \frac{\sqrt{E_s(y_0,f_0)}}{1-c_1(s)}. \tag{56}$$

Proceeding as before, we get (50). □

We now intend to choose s such that $\|y_{n+1}\|_{L^\infty(Q_T)} \leq M$. In view of estimates (49) and (50), we need to this end an estimate of $E_s(y_0,f_0) = \frac{1}{2}\|\rho_0(s)(\partial_t y_0 - \partial_{xx} y_0 + g(y_0) - f_0 1_\omega)\|^2_{L^2(Q_T)}$ in terms of s. Since the weight $\rho(s)$ does not belong to $L^2(Q_T)$, such estimate is not straightforward for any $(y_0,f_0) \in \mathcal{A}(s)$. We select the pair $(y_0,f_0) \in \mathcal{A}(s)$ solution of the linear problem, i.e. $g \equiv 0$ in (1).

Lemma 7. *Assume that g satisfies $(\mathbf{H_2})$. For any $s \geq s_0$, let $(y_0,f_0) \in \mathcal{A}(s)$ be the solution of the extremal problem (25) in the linear case for which $g \equiv 0$. Then,*

$$\sqrt{E_s(y_0,f_0)} \leq \Big(\alpha + \beta\big(c^{3/2} + \ln^{3/2}(1 + C\|u_0\|_{H_0^1(\Omega)})\big)\Big) e^{cs}\|u_0\|_{H_0^1(\Omega)}. \tag{57}$$

with $c = \|\varphi(\cdot,0)\|_{L^\infty(\Omega)}$.

Proof. Estimate (20) of Proposition 3 with $A=0, B=0$ and $z_0 = u_0$ leads to

$$\|\rho(s)y_0\|_{L^2(Q_T)} + s^{-3/2}\lambda_0^{-2}\|\rho_0(s)f_0\|_{L^2(q_T)} \leq Cs^{-3/2}e^{cs}\|u_0\|_{L^2(\Omega)} \tag{58}$$

while (22) leads, since $s \geq 1$ and $\rho_0 \geq 1$, to

$$\|y_0\|_{L^\infty(Q_T)} \leq Ce^{-\frac{3}{2}s}e^{cs}\|u_0\|_{H_0^1(\Omega)} \leq Ce^{cs}\|u_0\|_{H_0^1(\Omega)}. \tag{59}$$

It follows that, using (32), since $\rho_0 \le \rho$ and $s \ge 1$

$$
\begin{aligned}
\sqrt{E_s(y_0,f_0)} &= \frac{1}{\sqrt{2}}\|\rho_0(s)g(y_0)\|_{L^2(Q_T)} \le \frac{1}{\sqrt{2}}\psi(\|y_0\|_{L^\infty(Q_T)})\|\rho(s)y_0\|_{L^2(Q_T)}\\
&\le C\psi(Ce^{cs}\|u_0\|_{H_0^1(\Omega)})s^{-3/2}e^{cs}\|u_0\|_{H_0^1(\Omega)}\\
&\le C\big(\alpha+\beta\ln^{3/2}(1+Ce^{sc}\|u_0\|_{H_0^1(\Omega)})\big)s^{-3/2}e^{cs}\|u_0\|_{H_0^1(\Omega)}\\
&\le C\big(\alpha+\beta\ln^{3/2}(e^{sc}(1+C\|u_0\|_{H_0^1(\Omega)}))\big)s^{-3/2}e^{cs}\|u_0\|_{H_0^1(\Omega)}\\
&\le C\Big(\alpha+\beta\big((sc)^{3/2}+\ln^{3/2}(1+C\|u_0\|_{H_0^1(\Omega)})\big)\Big)s^{-3/2}e^{cs}\|u_0\|_{H_0^1(\Omega)}\\
&\le C\Big(\alpha+\beta\big(c^{3/2}+\ln^{3/2}(1+C\|u_0\|_{H_0^1(\Omega)})\big)\Big)e^{cs}\|u_0\|_{H_0^1(\Omega)}.
\end{aligned}
$$

□

We are now in position to prove to following result.

Proposition 5. *Assume that g satisfies $(\mathbf{H_2})$ and $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0,1]$. Assume moreover that $2cC(p)\beta^{2/3} < 1$ and let (y_0, f_0) be the controlled pair given by Lemma 7. There exists $M_0 > 0$ such that, if we have constructed from (47) the pairs $(y_k, f_k)_{0\le k\le n} \in \mathcal{A}(s)$ with $s = \max(C(p)\psi(M_0)^{2/3}, s_0)$ satisfying $\|y_k\|_{L^\infty(Q_T)} \le M_0$ for all $0 \le k \le n$, then the pair (y_{n+1}, f_{n+1}) constructed from (47) also belongs to $\mathcal{A}(s)$ and satisfies*

$$\|y_{n+1}\|_{L^\infty(Q_T)} \le M_0.$$

Proof. The inequality $(a+b)^{2/3} \le a^{2/3} + b^{2/3}$ for all $a, b \ge 0$ allows to write, for all $M > 0$:

$$\psi(M)^{2/3} = (\alpha + \beta\ln^{3/2}(1+M))^{2/3} \le \alpha^{2/3} + \beta^{2/3}\ln(1+M).$$

Assume that for some M large enough, $\|y_k\|_{L^\infty(Q_T)} \le M$. Estimate (59) with $s = \max(C(p)\psi(M)^{2/3}, s_0)$ then leads to

$$
\begin{aligned}
\|y_0\|_{L^\infty(Q_T)} &\le C\|u_0\|_{H_0^1(\Omega)}e^{cs} \le C\|u_0\|_{H_0^1(\Omega)}e^{c(s_0+C(p)\psi(M)^{2/3})} \le Ce^{cs_0}\|u_0\|_{H_0^1(\Omega)}e^{cC(p)\psi(M)^{2/3}}\\
&\le Ce^{cs_0}\|u_0\|_{H_0^1(\Omega)}e^{cC(p)\big(\alpha^{2/3}+\beta^{2/3}\ln(1+M)\big)}\\
&\le Ce^{cs_0}\|u_0\|_{H_0^1(\Omega)}e^{C(p)\alpha^{2/3}}e^{cC(p)\beta^{2/3}\ln(1+M)}\\
&\le Ce^{cs_0}\|u_0\|_{H_0^1(\Omega)}e^{C(p)\alpha^{2/3}}(1+M)^{cC(p)\beta^{2/3}}\\
&\le c(\alpha,u_0)(1+M)^{cC(p)\beta^{2/3}}.
\end{aligned}
\tag{60}
$$

Similarly, this estimate of e^{cs} and (57) lead to

$$
\begin{aligned}
\sqrt{E_s(y_0,f_0)} &\le \Big(\alpha+\beta\big(c^{3/2}+\ln^{3/2}(1+C\|u_0\|_{H_0^1(\Omega)})\big)\Big)e^{cs_0}e^{C(p)\alpha^{2/3}}(1+M)^{cC(p)\beta^{2/3}}\|u_0\|_{H_0^1(\Omega)}\\
&\le c(\alpha,\beta,u_0)(1+M)^{cC(p)\beta^{2/3}}.
\end{aligned}
\tag{61}
$$

First case : $p \in (0,1]$. Since $s \ge 1$, the constant $c_1(s)$ defined in (44) satisfies $c_1(s) \le \frac{C^{1+p}}{1+p}[g']_p$. Therefore, by combining (49), (60) and (61), we get

$$
\begin{aligned}
\|y_{n+1}\|_{L^\infty(Q_T)} \le\, & c(\alpha,u_0)(1+M)^{cC(p)\beta^{2/3}}\\
&+Cm\max\Big(\frac{1}{p}c(\alpha,\beta,u_0)(1+M)^{cC(p)\beta^{2/3}},\\
&\qquad\qquad \frac{(1+p)}{p}[g']_p^{1/p}C^{(1+p)/p}c^2(\alpha,\beta,u_0)(1+M)^{2cC(p)\beta^{2/3}}\Big)\\
\le\, & C(p,\alpha,u_0,[g']_p,\beta)(1+M)^{2cC(p)\beta^{2/3}}.
\end{aligned}
$$

Now, if β is small enough so that $2cC(p)\beta^{2/3} < 1$, the real M_0 defined as follows

$$M_0 := \inf\left\{M > 0 \mid C(p,\alpha,u_0,[g']_p,\beta)(1+M)^{2cC(p)\beta^{2/3}} \leq M\right\} \tag{62}$$

exists and is independent of n. Consequently, taking $M = M_0$ and $s = \max(C(p)\psi(M_0)^{2/3}, s_0)$ lead to $\|y_{n+1}\|_{L^\infty(Q_T)} \leq M_0$.

Second case : $p = 0$. In this case, $c_1(s) = Cs^{-3/2}[g']_0 = 2Cs^{-3/2}\|g'\|_\infty < 1$ for s large enough. By combining (50), (60) and (61), we get

$$\begin{aligned}\|y_{n+1}\|_{L^\infty(Q_T)} &\leq c(\alpha,u_0)(1+M)^{cC(0)\beta^{2/3}} + \frac{1}{1-c_1(s)}c(\alpha,\beta,u_0)(1+M)^{cC(0)\beta^{2/3}}. \\ &\leq C(\alpha,u_0,[g']_0,\beta)(1+M)^{cC(0)\beta^{2/3}}\end{aligned}$$

and we conclude as in the previous case with

$$M_0 := \inf\left\{M > 0 \mid C(\alpha,u_0,[g']_0,\beta)(1+M)^{cC(0)\beta^{2/3}} \leq M\right\}. \tag{63}$$

□

By induction, we can now conclude to the boundedness of the sequence $(y_k)_{k\in\mathbb{N}}$ for the norm $L^\infty(Q_T)$.

Proposition 6. *Assume that g satisfies $(\mathbf{H_2})$ and $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0,1]$. Assume moreover that $2cC(p)\beta^{2/3} < 1$. Let M_0 be given by (62) for $p \in (0,1]$, by (63) for $p = 0$ and $s = \max(C(p)\psi(M_0)^{2/3}, s_0)$. Let $(y_0, f_0) \in \mathcal{A}(s)$ be the solution of the extremal problem (25) in the linear situation for which $g \equiv 0$. Then the sequence $(y_k, f_k)_{k\in\mathbb{N}} \in \mathcal{A}(s)$ defined by (47) satisfies*

$$\|y_k\|_{L^\infty(Q_T)} \leq M_0, \quad \forall k \in \mathbb{N}.$$

Proof. From the construction of M_0, $\|y_0\|_{L^\infty(Q_T)} < M_0$. Therefore, the uniform boundedness of the sequence $(y_k)_{k\in\mathbb{N}}$ follows by induction from Proposition 5. □

4.2 Convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ and proof of Theorem 2

Once we know that we can choose the parameter s large enough but independent of k, we may prove the decay to zero of the sequence $(E_s(y_k, f_k))_{k\in\mathbb{N}}$.

Proposition 7. *Assume that g satisfies $(\mathbf{H_2})$ and $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0,1]$. Assume moreover that $2cC(p)\beta^{2/3} < 1$. Let M_0 be given by (62) for $p \in (0,1]$, by (63) for $p = 0$ and $s = \max(C(p)\psi(M_0)^{2/3}, s_0)$. Let $(y_0, f_0) \in \mathcal{A}(s)$ be the solution of the extremal problem (25) in the linear situation for which $g \equiv 0$. Let $(y_k, f_k)_{k\in\mathbb{N}} \in \mathcal{A}(s)$ be the sequence defined by (47). Then, the sequence $(E_s(y_k, f_k))_{k\in\mathbb{N}} \to 0$ tends to 0 as $k \to \infty$. The convergence is at least linear, and is at least of order $1+p$ after a finite number of iterations.*

Proof. **First case :** $p \in (0,1]$. From the definition of p_k given in (52) we have $p_k(\widetilde{\lambda}_k) := \min_{\lambda\in[0,m]} p_k(\lambda) \leq p_k(1) = c_1(s)E_s(y_k, f_k)^{p/2}$ and thus

$$c_2(s)\sqrt{E_s(y_{k+1}, f_{k+1})} \leq \left(c_2(s)\sqrt{E_s(y_k, f_k)}\right)^{1+p}, \quad c_2(s) := c_1^{1/p}(s). \tag{64}$$

Thus, if $c_2(s)\sqrt{E_s(y_0, f_0)} < \frac{1}{(p+1)^{1/p}}$, then $c_2(s)\sqrt{E_s(y_k, f_k)} \to 0$ as $k \to \infty$ with a rate $1+p$. On the other hand, if $c_2(s)\sqrt{E_s(y_0, f_0)} \geq \frac{1}{(p+1)^{1/p}}$, then we check that $I := \{k \in \mathbb{N},\ c_2(s)\sqrt{E_s(y_k, f_k)} \geq \frac{1}{(p+1)^{1/p}}\}$ is a finite subset of $\mathbb{N}$. Indeed, for all $k \in I$ and for all $\lambda \in [0,1]$: $\partial_\lambda p_k(s,\lambda) = -1+(p+1)\lambda^p c_1(s) E_s(y_k,f_k)^{p/2}$ and thus $\partial_\lambda p_k(s,\lambda) = 0$ if and only if $\lambda = \frac{1}{(p+1)^{1/p} c_2(s)\sqrt{E_s(y_k,f_k)}}$, which gives

$$
\begin{aligned}
p_k(s,\widetilde{\lambda_k}) = \min_{\lambda\in[0,m]} p_k(\lambda) = \min_{\lambda\in[0,1]} p_k(\lambda) &= p_k\Big(\frac{1}{(1+p)^{1/p} c_2(s)\sqrt{E_s(y_k,f_k)}}\Big) \\
&= 1 - \frac{p}{(1+p)^{\frac{1}{p}+1}} \frac{1}{c_2(s)\sqrt{E_s(y_k,f_k)}}
\end{aligned}
$$

and thus

$$
\begin{aligned}
c_2(s)\sqrt{E_s(y_{k+1},f_{k+1})} &\leq \Big(1 - \frac{p}{(1+p)^{\frac{1}{p}+1}} \frac{1}{c_2(s)\sqrt{E_s(y_k,f_k)}}\Big) c_2(s)\sqrt{E_s(y_k,f_k)} \\
&= c_2(s)\sqrt{E_s(y_k,f_k)} - \frac{p}{(1+p)^{\frac{1}{p}+1}}.
\end{aligned}
\tag{65}
$$

This inequality implies that the sequence $\big(c_2(s)\sqrt{E_s(y_k,f_k)}\big)_{k\in I}$ strictly decreases and at at least linearly writing that

$$
\begin{aligned}
\frac{c_2(s)\sqrt{E_s(y_{k+1},f_{k+1})}}{c_2(s)\sqrt{E_s(y_k,f_k)}} &\leq 1 - \frac{p}{(1+p)^{\frac{1}{p}+1} c_2(s)\sqrt{E_s(y_k,f_k)}} \\
&\leq 1 - \frac{p}{(1+p)^{\frac{1}{p}+1} c_2(s)\sqrt{E_s(y_0,f_0)}} < 1 \quad \forall k \in I.
\end{aligned}
\tag{66}
$$

It follows that there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$, $c_2(s)\sqrt{E_s(y_k,f_k)} < \frac{1}{(p+1)^{1/p}}$, that is I is a finite subset of $\mathbb{N}$. Moreover, from $k = k_0+1$, arguing as in the previous case, the convergence to zero of $c_2(s)\sqrt{E_s(y_k,f_k)}$ is as least of order $1+p$. The index k_0 is defined in Remark 8 below.

Second case : $p = 0$. Then for all $k \in \mathbb{N}$, since $c_1(s) < 1$, $p_k(s,\widetilde{\lambda_k}) = c_1(s)$ (since $\widetilde{\lambda_k} = 1$) and therefore

$$
\sqrt{E_s(y_{k+1},f_{k+1})} \leq c_1(s)\sqrt{E_s(y_k,f_k)} \leq c_1(s)^{k+1}\sqrt{E_s(y_0,f_0)}. \tag{67}
$$

Thus $\sqrt{E_s(y_k,f_k)} \to 0$ as $k \to \infty$. □

We can now prove the convergence of the sequence $(y_k,f_k)_{k\in\mathbb{N}}$ announced in Theorem 2. Precisely, we have the following main result.

Theorem 4. *Assume that g satisfies* $(\mathbf{H_2})$ *and* $(\overline{\mathbf{H}}_\mathbf{p})$ *for some $p \in [0,1]$. Assume moreover that β is small enough so that*

$$
2cC(p)\beta^{2/3} < 1
$$

with $c = \|\varphi(\cdot,0)\|_{L^\infty(\Omega)}$. Let M_0 be given by (62) for $p \in (0,1]$, by (63) for $p = 0$ and let $s = \max(C(p)\psi(M_0)^{2/3}, s_0)$. Let $(y_0,f_0) \in \mathcal{A}(s)$ be the solution of the extremal problem (25) *in the linear situation for which $g \equiv 0$ and let $(y_k,f_k)_{k\in\mathbb{N}}$ be the sequence defined by (47). Then, $(y_k,f_k)_{k\in\mathbb{N}} \to (y,f)$ in $\mathcal{A}(s)$ where f is a null control for y solution of* (1)*. The convergence is at least linear, and is at least of order $1+p$ after a finite number of iterations.*

Proof. For all $k \in \mathbb{N}$, let $F_k = -\sum_{n=0}^k \lambda_n F_n^1$ and $Y_k = \sum_{n=0}^k \lambda_n Y_n^1$. Let us prove that $(Y_k,F_k)_{k\in\mathbb{N}}$ converges in $\mathcal{A}_0(s)$, i.e. that the series $\sum \lambda_n(F_n^1,Y_n^1)$ converges in $\mathcal{A}_0(s)$. Using that $\|(Y_k^1,F_k^1)\|_{\mathcal{A}_0(s)} \leq C(M_0)\sqrt{E_s(y_k,f_k)}$ for all $k \in \mathbb{N}$ (see (34)), we write, using (54) and (55) :

$$
\sum_{n=0}^k \lambda_n \|(Y_n^1,F_n^1)\|_{\mathcal{A}_0(s)} \leq m\sum_{n=0}^k \|(Y_n^1,F_n^1)\|_{\mathcal{A}_0(s)} \leq C(M_0)\sum_{n=0}^k \sqrt{E_s(y_n,f_n)} \leq \sqrt{E_s(y_0,f_0)}\frac{C(M_0)}{1-p_0(\widetilde{\lambda_0})}.
$$

We deduce that the series $\sum_n \lambda_n(Y_n^1,F_n^1)$ is normally convergent and so convergent. Consequently, there exists $(Y,F) \in \mathcal{A}_0(s)$ such that $(Y_k,F_k)_{k\in\mathbb{N}}$ converges to (Y,F) in $\mathcal{A}_0(s)$.

Denoting $y = y_0 + Y$ and $f = f_0 + F$, we then have that $(y_k, f_k)_{k\in\mathbb{N}} = (y_0 + Y_k, f_0 + F_k)_{k\in\mathbb{N}}$ converges to (y, f) in $\mathcal{A}(s)$.

It suffices now to verify that the limit (y, f) satisfies $E_s(y, f) = 0$. Using that (Y_k^1, F_k^1) goes to zero in $\mathcal{A}_0(s)$ as $k \to \infty$, we pass to the limit in (48) and get that $(y, f) \in \mathcal{A}(s)$ solves (1), that is $E_s(y, f) = 0$. Moreover, we have

$$\|(y,f) - (y_k, f_k)\|_{\mathcal{A}_0(s)} \leq C(M_0)\sqrt{E_s(y_k, f_k)}, \quad \forall k > 0 \tag{68}$$

which implies, using Proposition 7, the announced order of convergence. Precisely,

$$\begin{aligned}
\|(y,f) - (y_k, f_k)\|_{\mathcal{A}_0(s)} &= \Big\| \sum_{p=k+1}^{\infty} \lambda_p (Y_p^1, F_p^1)\Big\|_{\mathcal{A}_0(s)} \leq m \sum_{p=k+1}^{\infty} \|(Y_p^1, F_p^1)\|_{\mathcal{A}_0(s)}\\
&\leq m\, C(M_0) \sum_{p=k+1}^{\infty} \sqrt{E_s(y_p, f_p)}\\
&\leq m\, C(M_0) \sum_{p=k+1}^{\infty} p_0(\widetilde{\lambda}_0)^{p-k} \sqrt{E_s(y_k, f_k)}\\
&\leq m\, C(M_0) \frac{p_0(\widetilde{\lambda}_0)}{1 - p_0(\widetilde{\lambda}_0)} \sqrt{E_s(y_k, f_k)}.
\end{aligned}$$

□

We emphasize, in view of the non uniqueness of the zeros of E_s, that an estimate (similar to (68)) of the form $\|(y,f) - (\overline{y}, \overline{f})\|_{\mathcal{A}_0(s)} \leq C(M_0)\sqrt{E_s(\overline{y}, \overline{f})}$ does not hold for all $(\overline{y}, \overline{f}) \in \mathcal{A}(s)$. We also mention the fact that the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ and its limit (y, f) are uniquely determined from the initial guess (y_0, f_0) and from our criterion of selection of the pair (Y_k^1, F_k^1) for every k. In other words, the solution (y, f) is unique up to the element (y_0, f_0) and the functional J_s.

We also have the following convergence of the optimal sequence $(\lambda_k)_{k\in\mathbb{N}}$.

Lemma 8. *Under hypotheses of Theorem (4) with $p \in (0, 1]$, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ defined in (47) converges to 1 as $k \to \infty$.*

Proof. If $p \in (0, 1]$, in view of (45) we have, as long as $E_s(y_k, f_k) > 0$, since $\lambda_k \in [0, m]$

$$\begin{aligned}
(1 - \lambda_k)^2 &= \frac{E_s(y_{k+1}, f_{k+1})}{E_s(y_k, f_k)} - (1 - \lambda_k) \frac{\big(\rho_0(s)\big(\partial_t y_k + \Delta y_k + g(y_k) - f_k 1_\omega\big), \rho_0(s) l(y_k, -\lambda_k Y_k^1)\big)_{L^2(Q_T)}}{E_s(y_k, f_k)}\\
&\qquad - \frac{\big\|\rho_0(s) l(y_k, -\lambda_k Y_k^1)\big\|^2_{L^2(Q_T)}}{2E_s(y_k, f_k)}\\
&\leq \frac{E_s(y_{k+1}, f_{k+1})}{E_s(y_k, f_k)} - (1 - \lambda_k) \frac{\big(\rho_0(s)\big(\partial_t y_k + \Delta y_k + g(y_k) - f_k 1_\omega\big), \rho_0(s) l(y_k, -\lambda_k Y_k^1)\big)_{L^2(Q_T)}}{E_s(y_k, f_k)}\\
&\leq \frac{E_s(y_{k+1}, f_{k+1})}{E_s(y_k, f_k)} + \sqrt{2}m \frac{\sqrt{E_s(y_k, f_k)}\|\rho_0(s) l(y_k, -\lambda_k Y_k^1)\|_{L^2(Q_T)}}{E(s, y_k, f_k)}\\
&\leq \frac{E_s(y_{k+1}, f_{k+1})}{E_s(y_k, f_k)} + \sqrt{2}m \frac{\|\rho_0(s) l(y_k, -\lambda_k Y_k^1)\|_{L^2(Q_T)}}{\sqrt{E_s(y_k, f_k)}}.
\end{aligned}$$

But, from (46), (34) and Remark 4, we infer that

$$\begin{aligned}
\|\rho_0(s) l(y_k, -\lambda_k Y_k^1)\|_{L^2(Q_T)} &\leq [g']_p \frac{\lambda_k^{p+1}}{p+1} \|Y\|^p_{L^\infty(Q_T)} \|\rho(s) Y\|_{L^2(Q_T)}\\
&\leq \lambda_k^{p+1} \frac{C s^{-3/2} e^{-\frac{3p}{2}s}}{p+1} [g']_p E_s(y_k, f_k)^{\frac{p+1}{2}}\\
&\leq C(s) m^{p+1} [g']_p E_s(y_k, f_k)^{\frac{p+1}{2}}
\end{aligned}$$

and thus
$$(1-\lambda_k)^2 \le \frac{E_s(y_{k+1},f_{k+1})}{E_s(y_k,f_k)} + m^{p+2}C(s)[g']_p(E_s(y_k,f_k))^{p/2}.$$
Consequently, since $E_s(y_k,f_k)\to 0$ and $\frac{E_s(y_{k+1},f_{k+1})}{E_s(y_k,f_k)}\to 0$, we deduce that $(1-\lambda_k)^2\to 0$ as $k\to\infty$. $\square$

If $p=0$ and if s is large enough, then $c_1(s)<1$ and $\widetilde{\lambda_k}=1$ for every $k\in\mathbb{N}$ leading to the decay of $(E_s(y_k,f_k))_{k\in\mathbb{N}}$ to 0 (see (67)). Moreover, estimate (43) implies that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k=1$ for every k also leads to the decay $(E_s(y_k,f_k))_{k\in\mathbb{N}}$ with an order at least linear. Whether or not this constant sequence if the optimal one (as defined in (47)) is an open question.

Remark 7. *In Theorem 4, the sequence $(y_k,f_k)_{k\in\mathbb{N}}$ is initialized with the solution of minimal norm corresponding to $g\equiv 0$. This natural choice in practice leads to a precise estimate of $\sqrt{E_s(y_0,f_0)}$ with respect to the parameter s. Many other pairs are available such as for instance the pair $(y_0,f_0)=(y,0)=(\phi\, y^\star,0)$ constructed in Lemma 5 since it leads to the following estimate in term of s:*
$$\begin{aligned}
\sqrt{E_s(\phi y^\star,0)} &= \frac{1}{\sqrt{2}}\|\rho_0\big(\partial_t(\phi y^\star)-\Delta(\phi y^\star)+g(\phi y^\star)\big)\|_{L^2(Q_T)} = \frac{1}{\sqrt{2}}\|\rho_0(s)\big(\partial_t\phi\, y^\star+g(\phi y^\star)\big)\|_{L^2(Q_T)}\\
&\le \frac{T}{2}\|\rho_0(s)\|_{L^\infty(Q_{T/2})}\|\partial_t\phi\|_{L^\infty(Q_{T/2})}\|y^\star\|_{L^\infty(Q_{T/2})}+\|\rho_0(s)g(\phi y^\star))\|_{L^2(Q_T)}\\
&\le C(T)\bigg(\|\rho_0(s)\|_{L^\infty(Q_{T/2})}\|u_0\|_{H^1(\Omega)}+\psi(\|\phi y^\star\|_{L^\infty(Q_{T/2})})\|\rho(s)\phi y^\star\|_{L^\infty(Q_{T/2})}\bigg)\\
&\le C(T)\|\rho(s)\|_{L^\infty(Q_{T/2})}\|u_0\|_{H^1(\Omega)}\bigg(1+\psi(\|u_0\|_{L^\infty(\Omega)})\bigg)\\
&\le C(T)\|u_0\|_{H^1(\Omega)}\bigg(1+\alpha+\beta\ln^{3/2}(1+\|u_0\|_{H^1(\Omega)})\bigg)e^{s\|\varphi\|_{L^\infty(Q_{T/2})}}.
\end{aligned}$$

Remark 8. *As stated in Theorem 4, the convergence is at least of order $1+p$ after a number k_0 of iterations. Using (65), k_0 is given by*
$$k_0=\bigg\lfloor\frac{1+p}{p}\Big((1+p)^{1/p}c_2(s)\sqrt{E_s(y_0,f_0)}-1\Big)\bigg\rfloor+1, \tag{69}$$
(where $\lfloor\cdot\rfloor$ is the integer part) if $(1+p)^{1/p}c_2(s)\sqrt{E_s(y_0,f_0)}-1>0$, and $k_0=1$ otherwise.

5 Comments

Several comments are in order.

Asymptotic condition. The asymptotic condition $(\mathbf{H_2})$ on g' is slightly stronger than the asymptotic condition $(\mathbf{H_1})$ made in [17]: this is due to our linearization of (1) which involves $r\to g'(r)$ while the linearization (6) in [17] involves $r\to g(r)/r$. There exist cases covered by Theorem 1 in which exact controllability for (1) is true but that are not covered by Theorem 4. Note however that the example $g(r)=a+br+cr\ln^{3/2}(1+|r|)$, for any $a,b\in\mathbb{R}$ and for any $c>0$ small enough (which is somehow the limit case in Theorem 1) satisfies $(\mathbf{H_2})$ as well as $(\overline{\mathbf{H}}_\mathbf{p})$ for any $p\in[0,1]$.

While Theorem 1 was established in [17] by a nonconstructive fixed point argument, we obtain here, in turn, a new proof of the exact controllability of semilinear one dimensional heat equations, which is moreover constructive, with an algorithm that converges unconditionally, with order at least $1+p$.

Minimization functional. The estimate (34) is a key point in the convergence analysis and is independent of the choice of the functional J_s defined in Proposition 3 in order to select a pair (Y^1, F^1) in $\mathcal{A}_0(s)$. Thus, we may consider other weighted functionals, for instance the choice $J_s(y,f) = \frac{1}{2}\|\rho_0(s) f\|^2_{L^2(q_T)}$ discussed in [28].

Link with Newton method. If we introduce $F : \mathcal{A}(s) \to L^2(Q_T)$ by $F(y,f) := \rho_0^{-1}(s)(\partial_t y - \partial_{xx} y + g(y) - f\,1_\omega)$, we get that $E_s(y,f) = \frac{1}{2}\|F(y,f)\|^2_{L^2(Q_T)}$ and check that, for $\lambda_k = 1$ for every k, the algorithm (47) coincides with the Newton algorithm associated with the mapping F. This explains the super-linear convergence in Theorem 4. The optimization of the parameter λ_k is crucial here as it allows to get a global convergence result. It leads to so-called damped Newton method (for F) (we refer to [10, Chapter 8]). As far as we know, the analysis of damped type Newton methods for partial differential equations has deserved very few attention in the literature. We mention [23, 30] in the context of fluid mechanics.

A variant. To simplify, let us take $\lambda_k = 1$ for every k, as in the standard Newton method. Then, for each $k \in \mathbb{N}$, the optimal pair $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is such that the element (y_{k+1}, f_{k+1}) minimizes over $\mathcal{A}(s)$ the functional $(z,v) \to J_s(z - y_k, v - f_k)$. Alternatively, we may select the pair (Y_k^1, F_k^1) so that the element (y_{k+1}, f_{k+1}) minimizes the functional $(z,v) \to J(z,v)$. This leads to the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} \partial_t y_{k+1} - \partial_{xx} y_{k+1} + g'(y_k) y_{k+1} = f_{k+1} 1_\omega + g'(y_k) y_k - g(y_k) & \text{in } Q_T, \\ y_k = 0 \text{ on } \Sigma_T, \quad y_{k+1}(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{70}$$

In this case, for every $k \in \mathbb{N}$, (y_k, f_k) is a controlled pair for a linearized heat equation, while, in the case of the algorithm (47), (y_k, f_k) is a sum of controlled pairs (Y_j^1, F_j^1) for $0 \le j \le k$. This analysis of this variant used in [13] is apparently less straightforward.

Local controllability when removing the growth condition $(\mathbf{H_2})$. As in [3, 29] devoted to the wave equation, we may expect to remove the growth condition $(\mathbf{H_2})$ on g' if the initial value $E_s(y_0, f_0)$ is small enough. For s fixed, in view of Lemma 7, this is notably true if $g(0) = 0$ and if the norm $\|u_0\|_{H_0^1(\Omega)}$ of the initial data to be controlled is small enough. This would allow to recover the local controllability of the heat equation (usually obtained by an inverse mapping theorem, see [18, chapter 1]) and would be in agreement with the usual convergence of the standard Newton method. In the parabolic case considered here, the proof is however open, since in order to prove the convergence of $(E_s(y_k, y_k))_{k\in\mathbb{N}}$ to zero, for some s large enough independent of k, we need to prove that the sequence $(\|y_k\|_{L^\infty(Q_T)})_{k\in\mathbb{N}}$ is bounded. This is in contrast with [3, 29] where the control of minimal $L^2(Q_T)$ independently of any parameter s.

Weakening of the condition $(\overline{\mathbf{H}}_\mathbf{p})$. Given any $p \in [0,1]$, we introduce for any $g \in C^1(\mathbb{R})$ the following hypothesis :

$(\overline{\mathbf{H}}'_\mathbf{p})$ There exist $\overline{\alpha}, \overline{\beta}, \gamma \in \mathbb{R}^+$ such that $|g'(a) - g'(b)| \le |a-b|^p\big(\overline{\alpha} + \overline{\beta}(|a|^\gamma + |b|^\gamma)\big), \quad \forall a, b \in \mathbb{R}$

which coincides with $(\overline{\mathbf{H}}_\mathbf{p})$ if $\gamma = 0$ for $\overline{\alpha} + 2\overline{\beta} = [g']_p$. If $\gamma \in (0,1)$ is small enough and related to the constant β appearing in the growth condition $(\mathbf{H_2})$, Theorem 4 still holds if $(\overline{\mathbf{H}}_\mathbf{p})$ is replaced by the weaker hypothesis $(\overline{\mathbf{H}}'_\mathbf{p})$.

Influence of the parameter s and a simpler linearization. Taking s large enough in the case $p = 0$ (corresponding to $g' \in L^\infty(\mathbb{R})$) allows to ensure that the coefficient $c_1(s)$ (see (44)) is strictly less than one, and then to prove the strong convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$. This highlights the influence of the parameter s appearing in the Carleman weights ρ, ρ_0 and ρ_1. Actually, in this case, a similar convergence can be obtained by considering a simpler linearization of the system (1). For any

$s \geq s_0$ and $z \in L^2(\rho(s); Q_T)$, we define the controlled pair $(y, f) \in \mathcal{A}(s)$ solution of

$$\begin{cases} \partial_t y - \partial_{xx} y = f 1_\omega - g(z) & \text{in} \quad Q_T, \\ y = 0 \ \text{ on } \ \Sigma_T, \quad y(\cdot, 0) = u_0 \ \text{ in } \ \Omega, \end{cases}$$

and which minimizes the weighted cost J_s. If $g(0) = 0$ and g is globally Lipschitz, then $\rho_0(s) g(z) \in L^2(Q_T)$ and Theorem 3 implies $\|\rho(s)\, y\|_{L^2(Q_T)} \leq C s^{-3/2} \big(\|\rho_0(s) g(z)\|_{L^2(Q_T)} + e^{cs} \|u_0\|_{L^2(\Omega)}\big)$. This allows to define the operator $K : L^2(\rho(s); Q_T) \to L^2(\rho(s); Q_T)$ by $y := K(z)$. From Lemma 3, for any $z_i \in L^2(\rho(s); Q_T)$, $i = 1, 2$, $y_i := K(z_i)$ is given by $y_i = \rho^{-2}(s) L_0^\star p_i$ where $p_i \in P_0$ solves

$$(p_i, q)_P = \int_\Omega u_0 q(0) - \int_{Q_T} g(z_i) q, \quad \forall q \in P_0.$$

Taking $q := p_1 - p_2$, we then get

$$\begin{aligned} \|p_1 - p_2\|_P^2 &\leq \int_{Q_T} |g(z_1) - g(z_2)||p_1 - p_2| \\ &\leq \|\rho_0(s)(g(z_1) - g(z_2))\|_{L^2(Q_T)} \|\rho_0^{-1}(s)(p_1 - p_2)\|_{L^2(Q_T)} \\ &\leq \|g'\|_\infty \|\rho_0(s)(z_1 - z_2)\|_{L^2(Q_T)} \|\rho_0^{-1}(s)(p_1 - p_2)\|_{L^2(Q_T)}. \end{aligned}$$

Using that $\rho_0 \leq \rho$ and Lemma 1, we obtain

$$\big\|\rho(s)\big(K(z_1) - K(z_2)\big)\big\|_{L^2(Q_T)} \leq \|g'\|_\infty C \lambda_0^4 s^{-3/2} \|\rho(s)(z_1 - z_2)\|_{L^2(Q_T)}, \quad \forall z_1, z_2 \in L^2(\rho(s); Q_T)$$

and conclude that, if $s > \max(s_0, (C\lambda_0^4 \|g'\|_\infty)^{2/3})$, then the operator K is contracting. This allows to infer the convergence of the sequence $(y_k)_{k \in \mathbb{N}}$ defined by $y_{k+1} = K(y_k)$, $k \geq 0$ for any $y_0 \in L^2(\rho(s); Q_T)$ to a controlled solution of (1). In order to replace the assumption $g' \in L^\infty(\mathbb{R})$ by $(\mathbf{H_1})$, one needs to show some compactness properties for K, which is an open question.

The linearization (6) **associated with the weighted cost J_s.** Similarly, one can wonder if a parameter s large enough may lead to a contracting property for the operator Λ introduced in [17] and leading to the linearization (6). For any $\beta > 0$, we introduce the hypothesis

$(\mathbf{H_1'})$ $\limsup_{|r| \to \infty} \frac{|g(r)|}{|r| \ln^{3/2} |r|} \leq \beta$

similar to $(\mathbf{H_1})$. Then, as in [17] (where controls of minimal $L^\infty(Q_T)$ norm are considered), the linearization (6) also leads to a compactness property when associated with the cost J_s.

Proposition 8. *Assume that g satisfies $(\mathbf{H_1'})$ with $c\beta^{2/3} < 1$ with $c = \|\varphi(\cdot, 0)\|_{L^\infty(\Omega)}$. Let $z \in L^\infty(Q_T)$ and $s = \max(\|\widetilde{g}(z)\|^{2/3}_{L^\infty(Q_T)}, s_0)$. Let $(y, f) \in \mathcal{A}(s)$ the minimizer of the functional J_s and solution of*

$$\begin{cases} \partial_t y - \partial_{xx} y + y\, \widetilde{g}(z) = f 1_\omega & \text{in} \quad Q_T, \\ y = 0 \ \text{ on } \ \Sigma_T, \quad y(\cdot, 0) = u_0 \ \text{ in } \ \Omega. \end{cases}$$

There exists $M > 0$ such that if $\|z\|_{L^\infty(Q_T)} \leq M$ and $s := \max(\psi(M)^{2/3}, s_0)$ with $\psi(r) = \alpha + \beta \ln^{3/2}(1 + |r|)$, then $\|y\|_{L^\infty(Q_T)} \leq M$. We note $\Lambda_s : L^\infty(Q_T) \to L^\infty(Q_T)$ such that $y = \Lambda_s(z)$.

Proof. $(\mathbf{H_1'})$ implies that $|\widetilde{g}(r)| \leq \psi(r) := \alpha + \beta \ln^{3/2}(1 + |r|)$ for all $r \in \mathbb{R}$. Consequently, $\|\widetilde{g}(z)\|_{L^\infty(Q_T)} \leq \psi(\|z\|_{L^\infty(Q_T)}) = \alpha + \beta \ln^{3/2}(1 + \|z\|_{L^\infty(Q_T)})$ leading to $c\|\widetilde{g}(z)\|^{2/3}_{L^\infty(Q_T)} \leq c\alpha^{2/3} + c\beta^{2/3} \ln(1 + \|z\|_{L^\infty(Q_T)})$ and then to $e^{cs} \leq e^{c(s_0 + \alpha^{2/3})}(1 + \|z\|_{L^\infty(Q_T)})^{c\beta^{2/3}}$. Estimate (20) then implies

$$\|\rho(s)\, y\|_{L^2(Q_T)} + s^{-3/2} \lambda_0^{-2} \|\rho_0(s)\, f\|_{L^2(q_T)} \leq e^{c(s_0 + \alpha^{2/3})}(1 + \|z\|_{L^\infty(Q_T)})^{c\beta^{2/3}} \|u_0\|_{L^2(\Omega)}$$

and in particular, since $\rho(s)\geq\rho_0(s)\geq e^{\frac{3}{2}s}\geq 1$ and $s^{-3/2}e^{\frac{3}{2}s}\geq 1$ that

$$\|y\|_{L^2(Q_T)}+\lambda_0^{-2}\|f\|_{L^2(q_T)}\leq e^{c(s_0+\alpha^{2/3})}(1+\|z\|_{L^\infty(Q_T)})^{c\beta^{2/3}}\|u_0\|_{L^2(\Omega)}. \tag{71}$$

Moreover, if $u_0\in H_0^1(\Omega)$, standard estimate for the heat equation reads as

$$\|y\|_{L^\infty(Q_T)}\leq C\biggl(\|\widetilde{g}(z)\|_{L^\infty(Q_T)}\|y\|_{L^2(Q_T)}+\|f\|_{L^2(q_T)}+\|u_0\|_{L^2(\Omega)}\biggr)$$

which combined with (71) leads to

$$\|y\|_{L^\infty(Q_T)}\leq C\biggl(\bigl(1+\alpha+\beta\ln^{3/2}(1+\|z\|_{L^\infty(Q_T)})\bigr)e^{c(s_0+\alpha^{2/3})}(1+\|z\|_{L^\infty(Q_T)})^{c\beta^{2/3}}\|u_0\|_{L^2(\Omega)}+\|u_0\|_{L^2(\Omega)}\biggr).$$

It follows that if $c\beta^{2/3}<1$, then there exists an $M>0$ depending on $\|u_0\|_{L^2(\Omega)},\alpha,s_0,\lambda_0,\Omega,T$ such that $\|z\|_{L^\infty(Q_T)}\leq M$ implies $\|y\|_{L^\infty(Q_T)}\leq M$ since

$$\frac{C\biggl((1+\alpha+\beta\ln^{3/2}(1+M))e^{c(s_0+\alpha^{2/3})}(1+M)^{c\beta^{2/3}}\|u_0\|_{L^2(\Omega)}+\|u_0\|_{L^2(\Omega)}\biggr)}{M}\longrightarrow 0^+,\quad \text{as}\quad M\to\infty.$$

□

Let now $z_i\in L^2(\rho(s);Q_T)$ and $y_i:=\Lambda_\rho(z_i)$ for $i=1,2$. Then, using the estimates of Theorem 3 and the characterization of Lemma 2, we can proved, for all $s\geq\max(\psi(M)^{2/3},s_0)$ that

$$\|\rho(s)\bigl(\Lambda_s(z_1)-\Lambda_s(z_2)\bigr)\|_{L^2(Q_T)}\leq C\|\widetilde{g}'\|_{L^\infty(0,M)}\|u_0\|_{L^2(\Omega)}\|\zeta\rho^{-1}(s)\|_{L^\infty(Q_T)}s^{-3/2}e^{cs}\|\rho(s)(z_1-z_2)\|_{L^2(Q_T)} \tag{72}$$

for some $C=C(\Omega,T)$ and $\zeta=(T-t)^{-1/2}$. The existence of a parameter s large enough for which the operator Λ_s enjoys a contracting property remains however an open issue.

Controllability to trajectories The least-squares approach developed here extends to the case of the exact controllability to trajectories, defined in the introduction and studied in [17]. Let $y^\star$ be a bounded trajectory associated with the data $u_0^\star\in H_0^1(\Omega)$ and $f^\star\in L^2(q_T)$ solution of

$$\begin{cases}\partial_t y^\star-\partial_{xx}y^\star+g(y^\star)=f^\star 1_\omega & \text{in}\quad Q_T,\\ y^\star=0\ \text{ on }\ \Sigma_T,\quad y^\star(\cdot,0)=u_0^\star\ \text{ in }\ \Omega.\end{cases} \tag{73}$$

We aim to find a control $f\in L^2(q_T)$ such that y satisfies (1) and (5). We write that $z:=y-y^\star$ and $v:=f-f^\star$ solve

$$\begin{cases}\partial_t z-\partial_{xx}z+G(x,t,z)=v1_\omega & \text{in}\quad Q_T,\\ z=0\ \text{ on }\ \Sigma_T,\quad z(\cdot,0)=u_0-u_0^\star\ \text{ in }\ \Omega,\end{cases} \tag{74}$$

with for all $r\in\mathbb{R}$, $G(x,t,r):=g(y^\star(x,t)+r)-g(y^\star(x,t))$. The function satisfies $G(\cdot,\cdot,0)=0$, is C^1 with respect to the variable r if g is C^1 and satisfies the growth condition $(\mathbf{H_2})$ if g satisfies $(\mathbf{H_2})$ as well. Precisely, using that $(a+b)^{3/2}\leq\sqrt{2}(a^{3/2}+b^{3/2})$ for all $a,b\geq 0$, we infer that, for all $(x,t)\in Q_T$,

$$\begin{aligned}|\partial_r G'(x,t,r)|=|g'(y^\star(x,t)+r)|&\leq\alpha+\beta\ln^{3/2}(1+|y^\star(x,t)+r|)\\&\leq\alpha+\beta\ln^{3/2}\biggl((1+|y^\star|)(1+\frac{|z|}{1+|y^\star|})\biggr)\\&\leq\alpha+\sqrt{2}\beta\ln^{3/2}(1+|y^\star|)+\sqrt{2}\beta\ln^{3/2}\biggl(\frac{1+|r|}{1+|y^\star|}\biggr)\\&\leq\alpha'+\beta'\ln^{3/2}(1+|r|),\quad\forall r\in\mathbb{R}\end{aligned}$$

with $\alpha'=\alpha+\sqrt{2}\beta\ln^{3/2}(1+\|y^\star\|_{L^\infty(Q_T)})$ and $\beta'=\sqrt{2}\beta$. Consequently, Theorem 4 applied to system (74) allows to define a sequence $(z_k,v_k)_{k\in\mathbb{N}}$ which converges toward a controlled pair for (74). This implies the convergence of the sequence $(z_k+y^\star,v_k+f^\star)_{k\in\mathbb{N}}$ toward a solution of (1) and (5).

Extension to the multi-dimensional case The extension of our arguments to dimensions greater than one (as considered in [17]) remains to be done. We have used the fact that the controlled solution of the linearized problem (11) belongs to $L^\infty(Q_T)$. This allows to give a meaning to $\|g'(y)\|_{L^\infty(Q_T)}$ and therefore to use the Carleman estimate of Lemma 1 with potential A in $L^\infty(Q_T)$. In the multi-dimensional setting, due to the right hand side B in $L^2(Q_T)$, the controlled solution of (11) is not in general in $L^\infty(Q_T)$, including for controls in $L^\infty(q_T)$ but only in $L^\infty(0,T;L^p(\Omega))$ for some $p<\infty$. The extension to the multi-dimensional case therefore requires a refined Carleman estimate dealing with potentials in $L^\infty(0,T;L^q(\Omega))$ for some $q<\infty$ related to p (so as to give a meaning to $\|g'(y)\|_{L^\infty(0,T;L^q(\Omega))}$ for $y\in L^\infty(0,T;L^p(\Omega))$) and very likely a stronger condition than $(\mathbf{H_2})$ of the form

$(\mathbf{H_2'})$ There exists $\alpha\geq 0$ and $\beta>0$ such that $|g'(r)|\leq \alpha+\beta\ln^m(1+|r|)$ for every r in $\mathbb{R}$

for some $m=m(q)<3/2$ so as to ensure the convergence of the minimizing sequence for E_s. This is notably the situation for the wave equation where a stronger growth condition is made on g' in [3] devoted to the multi-dimensional case than in [29] devoted to the one dimensional case.

6 Conclusions

Exact controllability of (1) has been established in [17], under a growth condition on g, by means of a Kakutani fixed point argument which is not constructive. Under a slightly stronger growth condition and under the additional assumption that g' is uniformly Hölder continuous with exponent $p\in[0,1]$, we have designed, in the one dimensional setting, an explicit algorithm and proved its convergence to a controlled solution of (1). Moreover, the convergence is super-linear of order greater than or equal to $1+p$ after a finite number of iterations. In turn, our approach gives a new and constructive proof of the exact controllability of (1), which is, at least in the one-dimensional setting, simpler that in [17] where refined L^1 Carleman estimates are employed. In fact, in the one-dimensional setting, we can achieve the power $3/2$ appearing in the growth condition since the controlled sequence $(y_k)_{k\in\mathbb{N}}$ belongs to the space $L^\infty(Q_T)$. This is in general no longer true in the multidimensional setting in view of the $L^2(Q_T)$ right hand side term in (48), even with $L^\infty(q_T)$ controls considered in [17] leading to an interesting open question.

We also emphasize that the method is general and may be applied to any other equations or systems for which a precise observability estimate for the linearized problem is available. Such estimates are usually obtained by the way of Carleman estimates as initially done in the monography of Imanuvilov-Fursikov [18], extended later to a very large number of systems and situations. For instance, the method can be extended to the case of boundary controls. This remains however to be done. Moreover, the introduction of the Carleman type weights, which blow up at the final time and which depends on several parameters (itself related to the controlled solution), makes the analysis quite intricate. From this point of view, the case of hyperbolic equations (considered in [29, 3]) is simpler. Whether or not an appropriate choice of these parameters may lead directly to some contracting properties for some fixed point operator is also an open and interesting issue. Eventually, it would be also interesting to address other types of linearity involving notably the gradient of the solution (see [11]): we mention notably the Burgers equation and the Navier-Stokes system, formally solved numerically from a controllability viewpoint in [19, Part 1] and in [15] respectively.

References

[1] Mehdi Badra, Sylvain Ervedoza, and Sergio Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(2):529–574, 2016.

[2] Viorel Barbu. Exact controllability of the superlinear heat equation. *Appl. Math. Optim.*, 42(1):73–89, 2000.

[3] Arthur Bottois, Jérôme Lemoine, and Arnaud Münch. Constructive exact control of semilinear multi-dimensional wave equations. *Preprint. arXiv:2101.06446.*

[4] Franck Boyer. On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems. In *CANUM 2012, 41e Congrès National d'Analyse Numérique*, volume 41 of *ESAIM Proc.*, pages 15–58. EDP Sci., Les Ulis, 2013.

[5] Franck Boyer and Jérôme Le Rousseau. Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(5):1035–1078, 2014.

[6] M. O. Bristeau, O. Pironneau, R. Glowinski, J. Periaux, and P. Perrier. On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element methods. I. Least square formulations and conjugate gradient. *Comput. Methods Appl. Mech. Engrg.*, 17(18)(part, part 3):619–657, 1979.

[7] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[8] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[9] Jean-Michel Coron and Emmanuel Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.

[10] Peter Deuflhard. *Newton methods for nonlinear problems*, volume 35 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2011. Affine invariance and adaptive algorithms, First softcover printing of the 2006 corrected printing.

[11] A. Doubova, E. Fernández-Cara, M. González-Burgos, and E. Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.*, 41(3):798–819, 2002.

[12] Thomas Duyckaerts, Xu Zhang, and Enrique Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1):1–41, 2008.

[13] Enrique Fernández-Cara and Arnaud Münch. Numerical null controllability of semi-linear 1-D heat equations: fixed point, least squares and Newton methods. *Math. Control Relat. Fields*, 2(3):217–246, 2012.

[14] Enrique Fernández-Cara and Arnaud Münch. Strong convergent approximations of null controls for the 1D heat equation. *SeMA J.*, 61:49–78, 2013.

[15] Enrique Fernández-Cara, Arnaud Münch, and Diego A. Souza. On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations. *J. Sci. Comput.*, 70(2):819–858, 2017.

[16] Enrique Fernández-Cara and Enrique Zuazua. The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equations*, 5(4-6):465–514, 2000.

[17] Enrique Fernández-Cara and Enrique Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5):583–616, 2000.

[18] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[19] Roland Glowinski, Jacques-Louis Lions, and Jiwen He. *Exact and approximate controllability for distributed parameter systems*, volume 117 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. A numerical approach.

[20] A. A. Lacey. Global blow-up of a nonlinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 104(1-2):161–167, 1986.

[21] Kévin Le Balc'h. Global null-controllability and nonnegative-controllability of slightly superlinear heat equations. *J. Math. Pures Appl. (9)*, 135:103–139, 2020.

[22] Jérôme Lemoine, Irene Marín-Gayte, and Arnaud Münch. Approximation of null controls for semilinear heat equations using a least-squares approach. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. 63, 28, 2021.

[23] Jérôme Lemoine and Arnaud Münch. A fully space-time least-squares method for the unsteady Navier-Stokes system. *Preprint. arXiv:1909.05034. In revision in J. of Mathematical Fluid Mechanics.*

[24] Jérôme Lemoine and Arnaud Münch. Resolution of the implicit Euler scheme for the Navier-Stokes equation through a least-squares method. *Numer. Math.*, 147(2):349–391, 2021.

[25] Jérôme Lemoine, Arnaud Münch, and Pablo Pedregal. Analysis of continuous H^{-1}-least-squares approaches for the steady Navier-Stokes system. *Appl. Math. Optim.*, 83(1):461–488, 2021.

[26] Arnaud Münch. A least-squares formulation for the approximation of controls for the Stokes system. *Math. Control Signals Systems*, 27(1):49–75, 2015.

[27] Arnaud Münch and Pablo Pedregal. Numerical null controllability of the heat equation through a least squares and variational approach. *European J. Appl. Math.*, 25(3):277–306, 2014.

[28] Arnaud Münch and Diego A. Souza. A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation. *Adv. Comput. Math.*, 42(1):85–125, 2016.

[29] Arnaud Münch and Emmanuel Trélat. Constructive exact control of semilinear 1d wave equations by a least-squares approach. *Preprint. arXiv:2011.08462.*

[30] Pierre Saramito. A damped Newton algorithm for computing viscoplastic fluid flows. *J. Non-Newton. Fluid Mech.*, 238:6–15, 2016.

[31] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(1):109–129, 1993.