

Cohomology groups for spaces of 12-fold tilings

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Abstract

We consider tilings of the plane with 12-fold symmetry obtained by the cut and projection method. We compute their cohomology groups using the techniques introduced in [5]. To do this we completely describe the window, the orbits of lines under the group action and the orbits of 0-singularities. The complete family of generalized 12-fold tilings can be described using 2-parameters and it presents a surprisingly rich cohomological structure. To put this finding into perspective, one should compare our results with the cohomology of the generalized 5-fold tilings (more commonly known as generalized Penrose tilings). In this case the tilings form a 1-parameter family, which fits in simply one of two types of cohomology.

1 Introduction

This paper deals with tilings of the plane \mathbb{R}^2 . Given a tiling, the group of translations of the plane acts on it, allowing us to associate to each tiling a space, the hull of the tiling, defined as the closure of the orbit of the tiling under the group action. This space, which has been thoroughly studied for the last 20 years, has many interesting properties. We suggest [11] for a good introduction.

Cohomology has been an essential tool of algebraic topology. It is a topological invariant that associates groups (or a more complex ring structure) to spaces and can be used, among other things, to tell spaces apart. The computation of the cohomology of the hull of a tiling emerged at the beginning of the 2000's. It has been successfully used to obtain dynamical results on tilings, see for examples [8, 9] or [2, 12]. In this article we will compute the cohomology of some tiling spaces, enhancing our understanding of them and providing the community with some complete calculations.

There are several classes of tilings with nice properties. Worth mentioning are the class of substitution tilings and the class of cut and project tilings.

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For the former one, there is a well developed technique that allows to compute the cohomology groups of these tilings [1, 7]. For the latter ones, in spite of the general methods described by Gähler, Hunton and Kellendonk to compute these groups [5], very few examples have been treated in detail. Some examples can be found in [6, 1]. However, most of these examples are also substitution tilings; thus, the theoretical method concerning the cut and project tilings has never been optimized to be applied in actual computations.

In this article we are going to focus on a two parameter family of cut and projection tiling spaces: the generalized 12-fold tilings. The hull of these tilings will be denoted by $\Omega_{E_{12}^\gamma}$ with $\gamma \in \mathbb{R}^2$ (the notation and construction method are further explained in Sections 1.1 and 1.2). Our goal is double: first, we want to obtain a deeper understanding of this space of tilings using cohomology, and second we want to exploit the techniques in [5] to carry out the computations. Our computational efforts crystallize in the following theorem.

Theorem 1. *The cohomology groups of $\Omega_{E_{12}^\gamma}$ satisfy:*

1. *For all parameters γ , all the cohomology groups are torsion free.*
2. *For all parameters γ , $H^0(\Omega_{E_{12}^\gamma}) = \mathbb{Z}$.*
3. *The rank of the group $H^1(\Omega_{E_{12}^\gamma})$ depends on γ and takes values in the set $\{7, 10, 13, 16, 19, 22, 25\}$. The precise values of γ associated to the different ranks are explicit in Figure 8.*
4. *The rank of the group $H^2(\Omega_{E_{12}^\gamma})$ depends on γ . Moreover, if the rank of $H^1(\Omega_{E_{12}^\gamma}) \leq 13$, then the rank of $H^2(\Omega_{E_{12}^\gamma})$ is presented in Propositions 32 to 39.*

As mentioned before, the proof of this theorem is based on the techniques described in [5], which we briefly summarize in the next subsection. A detailed plan of the proof is given in Subsection 1.4.

In this article, the computation of the groups $H^1(\Omega_{E_{12}^\gamma})$ is completely explicit; on the other hand, the results concerning the groups $H^2(\Omega_{E_{12}^\gamma})$ are computer assisted if $\gamma \notin \mathbb{Z}[\sqrt{3}] \times \mathbb{Z}[\sqrt{3}]$. We have not explicitly described all the possible ranks of the second cohomology groups as a function of the parameter γ . However, the information gathered in Proposition 42, particularly in Tables 1 to 4, makes it very easy to compute the rank of $H^2(\Omega_{E_{12}^\gamma})$ for any fixed value of $\gamma \in \mathbb{R}^2$. For a ‘generic’ value of γ we propose the following conjecture, supported by computer calculations.

Conjecture 2. *The maximal cohomology attained among all the generalized 12-fold tilings is*

$$H^0(\Omega_{E_{12}^\gamma}) = \mathbb{Z}, \quad H^1(\Omega_{E_{12}^\gamma}) = \mathbb{Z}^{25} \quad \text{and} \quad H^2(\Omega_{E_{12}^\gamma}) = \mathbb{Z}^{564}.$$

1.1 Background on cut-and-projection tilings cohomology

Given a 2-plane E in the vector space \mathbb{R}^n containing no integer line (a line directed by an element of \mathbb{Z}^n) we want to study the tiling T of the plane E obtained by the cut and projection method. We recall this construction: to start with we need to consider the following orthogonal decomposition

$$\mathbb{R}^n = E \oplus E^\perp.$$

Let π be the projection on E and π^\perp the projection on E^\perp . We are now ready to describe the tiling T we want to study: Its vertices are the images by π of the points in $\mathbb{Z}^n \cap (E + [0, 1]^n)$. The tiles of the tiling are rhombi obtained by the projection π of the 2-faces of the complex \mathbb{Z}^n which are inside the “band” $E + [0, 1]^n$. Let us denote Ω_E the hull of this tiling. In our case it is the closure of T under the action of the group of translations of the plane E .

Remark that we can also consider an affine plane $\mathcal{E} = E + \gamma$ for some vector γ . We can associate a tiling to this affine plane by considering the band $\mathcal{E} + [0, 1]^n$ and doing the same construction. We obtain a tiling of $E + \gamma$ and can define its hull, Ω_E^γ . Obviously if γ is in \mathbb{Z}^n or in E we have the same hull. Otherwise the tilings can be different. This is in particular the case for Penrose tilings, where one obtains for a generic value γ what is known under the name of a “generalized Penrose tiling”.

During the last decade, starting with the work of [1] and [4], a lot of effort has been put into understanding the topology of Ω_E . We are particularly interested in computing the cohomology groups with integer coefficients, $H^k(\Omega_E, \mathbb{Z})$ for $k = 0, 1, 2$, of the hull of the tiling. A general description of these cohomology groups is given by Gähler-Hunton-Kellendonk in [5]. In order to state their results, we need to introduce quite a bit of notation and establish some conventions, which we will follow throughout the paper.

- The group $\pi^\perp(\mathbb{Z}^n)$ will be called Γ . The closure of this group is the product of a free abelian group and a continuous group since it has no torsion part: $\bar{\Gamma} \sim \mathbb{Z}^k \times \mathbb{R}^\ell$. We call Δ the discrete part of $\bar{\Gamma}$.
- Denote by F the vector space generated by $\Delta \subset E^\perp$. We will decompose the space E^\perp as $E^\perp = F \oplus F^\perp$, with F^\perp parallel to $\bar{\Gamma}$, and denote by Δ_0 the stabilizer of F^\perp under the group Γ .
- Assuming that $\ell = \dim F^\perp = 2$, we let P be the collection of planes parallel to F^\perp , defined as

$$P = \bigcup_{\delta \in \Delta} F^\perp + \delta.$$

- The polytope obtained by projecting the cube $[0, 1]^n$ onto E^\perp will be called the window and is denoted W . We need to assume that the intersection between W and P is a union of polygons, segments and points. The boundaries of the polygons and the segments are contained in lines directed by the vectors f_1, \dots, f_n .

- We consider the action of Γ on the lines described in the last point. The set of orbits is denoted I_1 and each orbit is called a 1-singularity. The cardinality of the set I_1 will be denoted $L_1 = |I_1|$.
- The intersection of any two 1-singularities will be called a 0-singularity. The set of all orbits of 0-singularities is denoted I_0 . The cardinality of the set I_0 will be denoted $L_0 = |I_0|$.
- Let $\Gamma^i \leq \Gamma$ be the stabilizer under the action of Γ of the vector space spanned by f_i . Every 1-singularity $\alpha \in I_1$ is generated by some direction f_i . We denote by L_0^α the set of orbits of 0-singular points on α under the action of Γ^i , and we set $\Gamma^\alpha := \Gamma^i$, if the 1-singularity α is in direction f_i .
- Since $\Gamma^\alpha \subset \Delta_0$ for all $\alpha \in I_1$, we can define the natural inclusion map

$$\beta : \bigoplus_{\alpha \in I_1} \Lambda^2 \Gamma^\alpha \rightarrow \Lambda^2 \Delta_0, \quad (1)$$

where $\Lambda^2 A$ denotes the exterior product of two copies of the free abelian group A . In our case, each group Γ^α is of rank two, thus generated by two vectors, and $\Lambda^2 \Gamma^\alpha$ represents the exterior product of the two generators.

- We define the numbers R and e as:

$$R = \text{rk } \beta \quad \text{and} \quad e = -L_0 + \sum_{\alpha \in I_1} L_0^\alpha.$$

e will turn out to be the Euler characteristic of the tiling space Ω_E .

With all this notation and conventions in place, we are now ready to state two general results about the cohomology groups $H^*(\Omega_E, \mathbb{Z})$.

Firstly, we note (compare [5, Thm. 2.10]) that the cohomology is finitely generated, iff there exists a natural number ν such that $\text{rk } \Delta_0 = \nu \dim F^\perp$ (or, equivalently, $d = (\nu - 1) \dim F^\perp$), and $\nu = \text{rk } \Gamma^\alpha$ for all $\alpha \in I_1$. In our case, these conditions are met for $\nu = 2$, and the cohomologies of all dodecagonal tilings considered here are finitely generated.

Secondly, with the notation introduced above, the cohomology groups can now be expressed explicitly.

Theorem 3 (Thm. 5.3 in [5]). *The free abelian part of $H^k(\Omega_E, \mathbb{Z})$ is given by the following isomorphisms:*

$H^0(\Omega_E, \mathbb{Z})$	$H^1(\Omega_E, \mathbb{Z})$	$H^2(\Omega_E, \mathbb{Z})$
\mathbb{Z}	\mathbb{Z}^{4+L_1-R}	\mathbb{Z}^{3+L_1+e-R}

The torsion part of $H^2(\Omega_E, \mathbb{Z})$ is isomorphic to the torsion in $\text{coker } \beta := \Lambda^2 \Delta_0 / \langle \Lambda^2 \Gamma^\alpha \rangle_{\alpha \in I_1}$, whereas the other cohomology groups are torsion free.

We note that $R = \text{rk } \beta = \text{rk} \langle \Lambda^2 \Gamma^\alpha \rangle_{\alpha \in I_1}$ depends only on the directions of the 1-singularities α , not on their positions, and the same holds true for $\text{coker } \beta$. Hence, these quantities are the same for all dodecagonal tilings considered here. As we shall see, we have $R = 3$, and the torsion of $\text{coker } \beta$ vanishes, so that the cohomology of all dodecagonal tilings is free.

1.2 Background on n -fold tilings

In this section we recall the definition of the n -fold tiling and summarize some known results on its cohomology groups. Following the description in the previous section, to produce a tiling by the cut and projection method we need to start with a 2-plane in a vector space. In this case the plane we intend to tile, E_n , is defined as follows:

- If $n = 2p + 1$, then E_n is the 2-plane in \mathbb{R}^{2p+1} generated by

$$\begin{pmatrix} 1 \\ \cos \frac{2\pi}{2p+1} \\ \vdots \\ \cos \frac{4p\pi}{2p+1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \sin \frac{2\pi}{2p+1} \\ \vdots \\ \sin \frac{4p\pi}{2p+1} \end{pmatrix}.$$

- If $n = 2p$, then E_n is the 2-plane in \mathbb{R}^p generated by

$$\begin{pmatrix} 1 \\ \cos \frac{\pi}{p} \\ \vdots \\ \cos \frac{(p-1)\pi}{p} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \sin \frac{\pi}{p} \\ \vdots \\ \sin \frac{(p-1)\pi}{p} \end{pmatrix}.$$

The plane E_n yields a decomposition of the ambient space as $\mathbb{R}^k = E_n \oplus E_n^\perp$, where the value k depends on the parity of n . Furthermore, the space E_n^\perp decomposes as $E_n^\perp = F_n \oplus F_n^\perp$ where F_n is the vector space spanned by the discrete part of $\pi^\perp(\mathbb{Z}^k) \subset E_n^\perp$. The space E_n^\perp allows to define a family of “related” tilings by considering the translate planes $E_n + \gamma$, $\gamma \in E_n^\perp$. If $\gamma \in F_n^\perp$, then the tiling of the plane E_n and of $E_n + \gamma$ coincide, so the interesting cases occur when the translation vector γ belongs to F_n . Moreover, if $\gamma, \gamma' \in F_n$ and $\gamma - \gamma' \in \Delta$, then the tilings of $E_n + \gamma$ and $E_n + \gamma'$ coincide. Notice that translating E_n by γ and looking at integer points in the band is the same as keeping E_n and looking at points of the form $-\gamma + \mathbb{Z}^n$ in the band. Finally, remark that all the results on the cohomology groups only depend on the geometry of W and the action of Γ on the lines supporting W . We intersect W by planes $F_n^\perp + \delta$ and try to compute numbers involved in Theorem 3. Here if we translate E by γ we will study these tilings by cutting the window W , which does not depend on the parameter γ , with translates of the family of planes P of the form

$$P^\gamma = \bigcup_{\delta \in \Delta} F_n^\perp + \delta + \gamma, \text{ with } \gamma \in F_n.$$

For each of these tilings, we are interested in the cohomology of their respective hulls, $\Omega_{E_n^\gamma}$. We refer to [3] for further references on these tilings.

The following statement resumes all the known values of the cohomology groups of the (undecorated) n -fold tilings (cf. [5, Table 5.1]). These results had mostly been announced without proof already in [6], but that paper contains some mistakes in the cohomology of the generalized Penrose tilings, which were corrected later by Kalugin [10].

Theorem 4. *For small values of n , the n -fold tilings satisfy:*

1. *If $n = 8$, these tilings are also known as Ammann–Beenker tilings. In this case $F = 0$. The cohomology groups with integer coefficients of the 8-fold tilings are:*

$H^0(\Omega_{E_8})$	$H^1(\Omega_{E_8})$	$H^2(\Omega_{E_8})$
\mathbb{Z}	\mathbb{Z}^5	\mathbb{Z}^9

2. *If $n = 5$, these tilings are also known as (Generalized) Penrose tilings. In this case F is of dimension one and we can understand the translation parameter γ as a real number. The cohomology groups with integer coefficients of the 5-fold tilings depend on whether γ is in $\mathbb{Z}[\varphi]$ or not, where φ is the golden mean. The groups are as follows:*

γ	$H^0(\Omega_{E_5^\gamma})$	$H^1(\Omega_{E_5^\gamma})$	$H^2(\Omega_{E_5^\gamma})$
$\mathbb{Z}[\varphi]$	\mathbb{Z}	\mathbb{Z}^5	\mathbb{Z}^8
$\mathbb{R} \setminus \mathbb{Z}[\varphi]$	\mathbb{Z}	\mathbb{Z}^{10}	\mathbb{Z}^{34}

3. *If $n = 12$, then F is of dimension two. For all $\gamma \in \Delta$, it holds $\Omega_{E_{12}^\gamma} = \Omega_{E_{12}}$. In this case the cohomology groups with integer coefficients of the 12-fold tilings are:*

$H^0(\Omega_{E_{12}})$	$H^1(\Omega_{E_{12}})$	$H^2(\Omega_{E_{12}})$
\mathbb{Z}	\mathbb{Z}^7	\mathbb{Z}^{28}

4. *If $n = 7$, the cohomology groups with integer coefficients of the 7-fold tilings are not finitely generated.*

1.3 Generalized Penrose tilings vs. 12-fold tilings

To give some perspective to the results in this paper, we will make a fast comparison between the computation of the cohomology groups of the generalized Penrose tilings and the 12-fold tilings.

In the Penrose case, we start from the space \mathbb{R}^5 and we consider the cut and project method onto a two dimensional plane. The main character in the construction, the polytope W , is in this case a 3-dimensional rhombic-icosahedron with vertices in five parallel planes depicted in Figure 1. The intersection of

these planes and W is a series of points and polygons. The boundaries of the polygons are contained in lines directed by 5 different vectors. To compute the cohomology of the standard Penrose tiling, we need to study the orbits of these lines under the action of a certain group Γ .

In the classic 12-fold tiling, we obtain a 4-dimensional polytope W , described in Proposition 12. Its vertices are contained in 16 parallel planes. This time the boundaries of the polygons determined by these vertices are contained in lines directed by 6 different vectors (see Figure 3).

The main difference between these two settings though is not simply that there is one more direction in the 12-fold tiling. The main difference is apparent when we consider the *generalized* tilings. Indeed, to compute the cohomology of the generalized tilings we need to shift the planes cutting the polytope. In the Penrose case, the shifting occurs along the vertical axis of the rhombic-icosahedron (see Figure 1), and it can be encoded by the real number γ in the statement of Theorem 4.2. When we shift slightly the planes cutting the Penrose rhombic-icosahedron we obtain some polygons with 10 sides (contained still in 5 directions). Now, depending on whether or not the translation parameter γ is in $\mathbb{Z}[\varphi]$ (where φ is the golden mean) the parallel opposite lines on the decagon will belong or not to the same orbit under the action of Γ .

In the generalized 12-fold tiling, the situation is much richer. The shifting of the planes that cut the polytope is encoded by a 2-dimensional parameter $\gamma = (\gamma_1, \gamma_2)$. In Figure 6 one can see the number of sides that a polygon cut by the shifted planes will have. The 10 lines obtained in the generalized Penrose become 24 in the generalized 12-fold tiling (see Proposition 24). Each one of the original 6 directions can have at most 4 representatives in different orbits. The game is now to understand when, depending on the value of γ , these representatives are in the same orbit under the action of Γ . The result, surprisingly complicated, is the content of Proposition 27 and Figure 8. This information is enough to compute completely the first cohomology group of the 12-fold generalized tilings. When trying to compute the second cohomology groups, the intricacy grows further as explored in Section 4.

1.4 Organization of the paper

The proof of the main theorem of this paper is a long and involved computation of the quantities e, L_1 and R which will allow us, via Theorem 3, to compute the cohomology groups of the 12-fold tilings (as discussed after that theorem, these groups are all free). Roughly, the steps we will follow are:

1. We intersect W with P^γ : we obtain several polygons. We compute the equations of the lines which support the edges of the polygons. The case $\gamma \in \Delta$ is presented in Section 2.2 while the general case can be found in Section 3.1.
2. For each line α obtained in the preceding point, we consider its orbit under the action of Γ . The number of orbits is the quantity L_1 . The

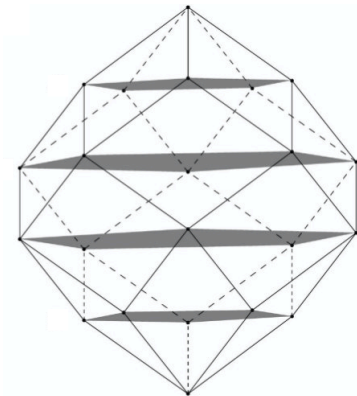


Figure 1: Window for the Penrose tilings

relevant computations can be found in Section 3.2 and culminate with Proposition 27 and Figure 8.

The lines α are in one of 6 possible directions. The smallest value of L_1 is then six, with just one Γ -orbit per direction; and the largest is $L_1 = 24$, corresponding to the case of four Γ -orbits per direction.

3. We compute the stabilizer Γ^α of each line α and obtain the quantity R , which is the rank of $\langle \Lambda^2 \Gamma^\alpha \rangle_{\alpha \in I_1}$. This computation takes place in Section 3.3 and allows us to compute the first cohomology groups for the 12-fold tiling in Proposition 30.
4. We count the number of intersections, up to the action, between a 1-singularity α and the other 1-singularities. This quantity is L_0^α . We have not completed the computation of L_0^α for all the possible values of the parameter γ . The study breaks down into too many subcases which did not seem worth looking into in full detail. However:

- In Section 4.1 we present a complete calculation of the second cohomology group when $\gamma = 0$. In this case we have actually computed the values L_0 and L_0^α (Lemma 31), which is all that is needed to determine the cohomology groups of $\Omega_{E_{12}}$. These groups were known (Theorem 4.3) and are restated in Proposition 32.

The case $\gamma = 0$ is the simplest analyzed and corresponds to the case of $L_1 = 6$ described above.

- In Section 4.2 we have further computed the second cohomology groups, and thus the full cohomology, for all the cases in which under the action of Γ there are at most two representatives per line, that is, the cases $L_1 \leq 12$. This computation is computer assisted and we collect the results in Propositions 32 to 39.
- Finally, in Section 4.3 we analyze the general case and summarize the results in Section 4.4. We consider the largest set of 1-singularities, which has 4 orbits per direction and thus 24 1-singularities. We intersect each of these with the translates by the action of Γ of the 20 non parallel 1-singularities. This yields a complete list of 0-singularities in each 1-singularity. Our results are collected in Proposition 42, which provides an upper bound on the quantity L_0^α . The complete lists of 0-singularities are displayed in Tables 1 to 4.

2 The 12-fold tilings

The 12-fold tiling space corresponding to the plane E_{12}^γ with $\gamma \in \Delta$ was introduced by Socolar in [13], where it was shown to be also a substitution tiling. Thus, the classical method exposed in [11] allows us to compute its cohomology groups. Our goal now is to extend this computation to all parameters $\gamma \in F_{12}$. Since we will be only concerned with the case $n = 12$, from now on we will denote simply by E, F etc. the spaces E_{12}, F_{12} etc. We start by establishing some preferred bases for the spaces involved in the decompositions

$$\mathbb{R}^6 = E \oplus E^\perp \text{ and } E^\perp = F^\perp \oplus F.$$

The plane E will have the fixed orthonormal basis $\{u, v\}$ given by

$$u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{3}/2 \\ 1/2 \\ 0 \\ -1/2 \\ -\sqrt{3}/2 \end{pmatrix}, v = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \\ 1 \\ \sqrt{3}/2 \\ 1/2 \end{pmatrix}.$$

The plane F will have the following fixed orthogonal basis:

$$A = \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, B = \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

The basis $\{A, B\}$ is not orthonormal, but it has been chosen to facilitate the computations. Finally, the plane F^\perp will be chosen to be the algebraic conjugate

of E and we will denote $\{u', v'\}$ its orthonormal basis. (Recall that to obtain u', v' we simply need to replace $\sqrt{3}$ by its opposite in the definition of u and v .)

The projection π^\perp onto E^\perp is described in the following lemma. The simple proof is left to the reader.

Lemma 5. *The image of the canonical basis of \mathbb{R}^6 by the projection π^\perp onto $F^\perp \oplus F$ defines six vectors $g_i, i = 1, \dots, 6$. We have $g_i = f_i + \delta_i$ with $f_i \in F^\perp, \delta_i \in F$ and:*

$$\begin{cases} f_1 = (\sqrt{3}/3, 0) \\ f_2 = (-1/2, \sqrt{3}/6) \\ f_3 = (\sqrt{3}/6, -1/2) \\ f_4 = (0, \sqrt{3}/3) \\ f_5 = (-\sqrt{3}/6, -1/2) \\ f_6 = (1/2, \sqrt{3}/6) \end{cases} \quad \text{and} \quad \begin{cases} \delta_1 = (1, 0) \\ \delta_2 = (0, 1) \\ \delta_3 = (-1, 0) \\ \delta_4 = (0, -1) \\ \delta_5 = (1, 0) \\ \delta_6 = (0, 1). \end{cases}$$

The vectors f_i are expressed in the basis $\{u', v'\}$ and the δ_i in the basis $\{A, B\}$.

2.1 Algebra

In this section we collect some technical lemmas that will be useful in the forthcoming computations. Many of them are straightforward and the proofs are left to the reader.

Lemma 6. *The vectors $f_i \in F^\perp, i = 1, \dots, 6$, satisfy the following two linear relations over \mathbb{Z} :*

$$f_5 = f_3 - f_1 \quad \text{and} \quad f_6 = f_4 - f_2.$$

Moreover, we have

$$\begin{cases} f_1 - f_5 = \sqrt{3}f_6 \\ f_2 - f_6 = -\sqrt{3}f_1 \\ f_i + f_{i+2} = -\sqrt{3}f_{i+1} \quad \forall i = 1, \dots, 6 \end{cases}$$

Corollary 7. *The following group isomorphisms hold*

$$\pi^\perp(\mathbb{Z}^6) \sim \mathbb{Z}A \oplus \mathbb{Z}B \oplus \langle f_1, \dots, f_6 \rangle_{\mathbb{Z}} \quad \text{and} \quad \overline{\pi^\perp(\mathbb{Z}^6)} \sim \mathbb{Z}^2 \oplus \mathbb{R}^2.$$

Proof. Every element in $\pi^\perp(\mathbb{Z}^6)$ is of the form $\sum n_i g_i$. By Lemma 5, we can rewrite this expression as

$$\sum_{i=1}^6 n_i f_i + \sum_{i=1}^6 n_i \delta_i = \sum_{i=1}^6 n_i f_i + (n_1 - n_3 + n_5)A + (n_2 - n_4 + n_6)B.$$

It is straightforward that \mathbb{Z}^2 is isomorphic to the abelian group generated by A, B , while Lemma 6 implies that the closure of the set $\{\sum_{i=1}^6 n_i f_i : n_i \in \mathbb{Z}\}$ is isomorphic to \mathbb{R}^2 . \square

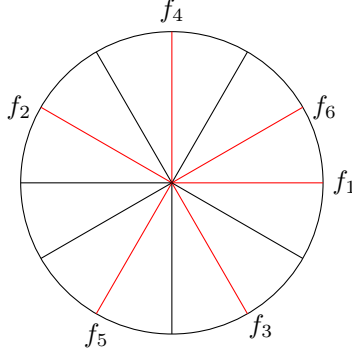


Figure 2: The vectors f_1, \dots, f_6 in F^\perp and the directions of the 12th roots of unity in \mathbb{C} .

Corollary 8. *If we identify the plane F^\perp with \mathbb{C} , then the vectors f_i correspond to some roots of the equation $z^{12} = 3^{-6}$. The correspondence, illustrated in Figure 2, can be written as follows, where x denotes the complex number $e^{i\frac{\pi}{6}}$:*

$$f_i = \frac{1}{\sqrt{3}}x^{5(i-1)}, \quad i = 1, \dots, 6. \quad (2)$$

Lemma 9. *The complex number $x = e^{i\frac{\pi}{6}}$ is a simple root of the polynomial $X^2 - \sqrt{3}X + 1$. Moreover, every element of $\mathbb{Z}[x]$ can be expressed as a polynomial in x of degree at most one with coefficients in $\mathbb{Z}[\sqrt{3}]$. The precise expressions are collected in the following chart for $k \in \mathbb{N}$:*

1	x	x^2	x^3	x^4	x^5	x^6	x^{6+k}
1	x	$x\sqrt{3} - 1$	$2x - \sqrt{3}$	$\sqrt{3}x - 2$	$x - \sqrt{3}$	-1	$-x^k$

Notation. Since the group $\mathbb{Z}[\sqrt{3}]$ will play a major role in this article, for simplicity we will denote it G .

When trying to determine the cohomology groups of the 12-fold tilings, we will need to understand when certain \mathbb{R} -linear combinations of powers of x belong to $\mathbb{Z}[x]$. In some sense, the next lemma generalizes the equation $\sqrt{3}f_6 = f_1 - f_5$.

Lemma 10. *Let $\alpha, \beta \in \mathbb{R}$ and $i < j$ two integers in $[0, 5]$.*

1. $\alpha x^i \in \mathbb{Z}[x]$ holds if and only if α belongs to G .

2. $\alpha x^i + \beta x^j \in \mathbb{Z}[x]$ holds if and only if

$$\begin{cases} j-i = 1, 5 \text{ and } \alpha, \beta \in G. \\ j-i = 3 \text{ and } \begin{cases} \beta = \frac{1}{2}(\beta_1 + \beta_2\sqrt{3}), \alpha = \frac{1}{2}(\alpha_1 + \alpha_2\sqrt{3}), \alpha_i, \beta_i \in \mathbb{Z}, \\ \alpha_1 - 3\beta_2, \alpha_2 - \beta_1 \in 2\mathbb{Z}. \end{cases} \\ j-i = 2 \text{ and } \begin{cases} \beta = \frac{1}{\sqrt{3}}(\beta_1 + \beta_2\sqrt{3}), \alpha = \frac{1}{\sqrt{3}}(\alpha_1 + \alpha_2\sqrt{3}), \alpha_i, \beta_i \in \mathbb{Z}, \\ \alpha_1 - \beta_1 \in 3\mathbb{Z}. \end{cases} \\ j-i = 4 \text{ and } \begin{cases} \beta = \frac{1}{\sqrt{3}}(\beta_1 + \beta_2\sqrt{3}), \alpha = \frac{1}{\sqrt{3}}(\alpha_1 + \alpha_2\sqrt{3}), \alpha_i, \beta_i \in \mathbb{Z}, \\ \alpha_1 - 2\beta_1 \in 3\mathbb{Z}. \end{cases} \end{cases}$$

Proof. We start with the first claim. By Lemma 9 it is clear that if α belongs to $\mathbb{Z}[\sqrt{3}]$ then αx^i belongs to $\mathbb{Z}[x]$. We prove the other implication. By assumption, $\alpha x^i = \sum_j n_j x^j$, $n_j \in \mathbb{Z}$, which implies $\alpha = x^{-i} \sum_j n_j x^j$. Again by Lemma 9, we know we can write $\alpha = ax + b$, with $a, b \in G$. Since $\{x, 1\}$ is a basis of \mathbb{C} as an \mathbb{R} vector space, we conclude that $\alpha = b \in G$.

We now proceed to argue the second claim in the statement. We need to understand which conditions on α, β imply $\alpha x^i + \beta x^j \in \mathbb{Z}[x]$. Since x^i is invertible in $\mathbb{Z}[x]$, this last expression is equivalent to $\alpha + \beta x^{j-i} \in \mathbb{Z}[x]$, which, by Lemma 9 can be rewritten as $\alpha + \beta x^{j-i} = ax + b$, $a, b \in G$. Thus, depending on the value of $j-i$ and again by Lemma 9, we are looking to one of the following equations

$$\begin{cases} \beta x + \alpha = ax + b & j-i = 1. \\ \beta\sqrt{3}x - \beta + \alpha = ax + b & j-i = 2. \\ 2\beta x - \beta\sqrt{3} + \alpha = ax + b & j-i = 3. \\ \beta\sqrt{3}x - 2\beta + \alpha = ax + b & j-i = 4. \\ \beta x - \beta\sqrt{3} + \alpha = ax + b & j-i = 5. \end{cases}$$

Now, since $\{x, 1\}$ is an \mathbb{R} basis for \mathbb{C} , we deduce

- If $j-i = 1$: $\beta = a$, $\alpha = b \iff \beta, \alpha \in G$.
- If $j-i = 5$: $\beta = a$, $-\beta\sqrt{3} + \alpha = b \iff \beta, \alpha \in G$.
- If $j-i = 3$:

$$\begin{aligned} 2\beta = a, -\beta\sqrt{3} + \alpha = b &\iff \exists \beta_1, \beta_2, \alpha_1, \alpha_2 \in \mathbb{Z}, \beta = \frac{1}{2}(\beta_1 + \beta_2\sqrt{3}), \\ \alpha = \frac{1}{2}(\alpha_1 + \alpha_2\sqrt{3}) \text{ and } b = \frac{\alpha_1 - 3\beta_2}{2} - \sqrt{3}\frac{\alpha_2 - \beta_1}{2} &\iff \\ \beta, \alpha \in \frac{1}{2}G, \alpha_1 - 3\beta_2, \alpha_2 - \beta_1 \in 2\mathbb{Z}. & \end{aligned}$$

For the two remaining cases, an analogous computation to the preceding one, and using the same notations, yields

- If $j-i = 2$: $\beta\sqrt{3} = a$, $-\beta + \alpha = b \iff \alpha, \beta \in \frac{1}{\sqrt{3}}G$ and $\alpha_1 - \beta_1 \in 3\mathbb{Z}$.

- If $j-1=4$: $\beta\sqrt{3}=a$, $-2\beta+\alpha=b \iff \alpha, \beta \in \frac{1}{\sqrt{3}}G$ and $\alpha_1-2\beta_1 \in 3\mathbb{Z}$.

□

We end this section with a last technical lemma which will not be needed until the reader is confronted with Figure 9.

Lemma 11. *Let $S := \{(\gamma_1, \gamma_2) \notin \frac{1}{2\sqrt{3}}G \times \frac{1}{2}G\}$ and consider its two subsets:*

$$A = \{(\gamma_1, \gamma_2) \in S \mid 2\gamma_1\sqrt{3} + 2\gamma_2 \in G\} \text{ and } B = \{(\gamma_1, \gamma_2) \in S \mid \gamma_1\sqrt{3} + 2\gamma_2 \in G\}.$$

Moreover, interchanging the roles of γ_1 and γ_2 , we obtain the analogous sets $S' := \{(\gamma_1, \gamma_2) \notin \frac{1}{2}G \times \frac{1}{2\sqrt{3}}G\}$,

$$D = \{(\gamma_1, \gamma_2) \in S' \mid 2\gamma_2\sqrt{3} + 2\gamma_1 \in G\} \text{ and } E = \{(\gamma_1, \gamma_2) \in S' \mid \gamma_2\sqrt{3} + 2\gamma_1 \in G\}.$$

Finally, set $C = S \setminus (A \cup B)$ and $F = S' \setminus (D \cup E)$. Then,

1. $A \cap B = D \cap E = \emptyset$, which implies $S = A \sqcup B \sqcup C$ and $S' = D \sqcup E \sqcup F$.

2. $B \cap S' \subset F$ and $E \cap S \subset C$.

3. $A \cap D$ is the set of couples (γ_1, γ_2) subject to the following conditions:

(a) $(\gamma_1, \gamma_2) \notin \frac{1}{2\sqrt{3}}(G \times G)$.

(b) $(\gamma_1, \gamma_2) \in \frac{1}{4}(G \times G)$, i.e. $4\gamma_1 = a + b\sqrt{3}$, $4\gamma_2 = c + d\sqrt{3}$, $a, b, c, d \in \mathbb{Z}$.

(c) The integers a, b, c, d satisfy one of the following parity conditions:

i. a, d even and c, b odd.

ii. a, d odd and c, b even.

iii. a, b, c, d odd.

Proof. By the symmetry in the statement, it suffices to show the first claim for the sets A, B : if $(x, y) \in A \cap B$ then $2x\sqrt{3} + 2y$ and $x\sqrt{3} + 2y$ are in G . Subtracting these two expressions we obtain that $x\sqrt{3} \in G$ which contradicts $(x, y) \in S$. We conclude $A \cap B = \emptyset$ and by an analogous argument $D \cap E = \emptyset$.

Now, we proceed to show that $B \cap S' \subset F$ and $E \cap S \subset C$. This amounts to show that $B \cap E = B \cap D = A \cap E = \emptyset$.

- $B \cap E = \emptyset$: if there was an element $(x, y) \in B \cap E \subset S \cap S'$, then $x, y \notin \frac{1}{2\sqrt{3}}G$. However, since $(x, y) \in B \cap E$ we arrive to the following contradiction.

$$\begin{cases} x\sqrt{3} + 2y \in G \\ 2x + y\sqrt{3} \in G \end{cases} \Rightarrow \begin{cases} 6x + 4\sqrt{3}y \in G \\ 6x + 3\sqrt{3}y \in G \end{cases} \Rightarrow \sqrt{3}y \in G.$$

- $B \cap D = \emptyset$: if there was an element $(x, y) \in B \cap D \subset S \cap S'$, then $x, y \notin \frac{1}{2\sqrt{3}}G$. However, since $(x, y) \in B \cap D$ we arrive to the following contradiction.

$$\begin{cases} x\sqrt{3} + 2y \in G \\ 2x + 2y\sqrt{3} \in G \end{cases} \Rightarrow \begin{cases} 3x + 2\sqrt{3}y \in G \\ 2x + 2\sqrt{3}y \in G \end{cases} \Rightarrow x \in G.$$

- $A \cap E = \emptyset$: the analysis in this case is completely analogous to $B \cap D = \emptyset$ swapping the two variables.

Finally, we will completely determine the set $A \cap D \subset S \cap S'$. If $(x, y) \in A \cap D \subset S \cap S'$, then $x, y \notin \frac{1}{2\sqrt{3}}G$ and the following conditions need to be satisfied:

$$\begin{cases} 2x\sqrt{3} + 2y \in G \\ 2x + 2y\sqrt{3} \in G \end{cases} \Rightarrow \begin{cases} 2x\sqrt{3} + 2y \in G \\ 2x\sqrt{3} + 6y \in G \end{cases} \Rightarrow 4y \in G.$$

By the symmetry of the variables we must also have $4x \in G$. So, $(x, y) \in \frac{1}{4}(G \times G)$ and there exist $a, b, c, d \in \mathbb{Z}$ such that $4x = a + b\sqrt{3}$ and $4y = c + d\sqrt{3}$. Now, if $(x, y) \in A$ we have:

$$2\sqrt{3}\frac{1}{4}(a + b\sqrt{3}) + 2\frac{1}{4}(c + d\sqrt{3}) = \frac{3b + c}{2} + \frac{a + d}{2}\sqrt{3} \in G = \mathbb{Z}[\sqrt{3}].$$

For this last condition to hold we need $a + d, 3b + c \in 2\mathbb{Z}$. Analyzing the constraints imposed by $(x, y) \in D$, we obtain $c + b, 3d + a \in 2\mathbb{Z}$. It follows that a and d and that c and b have the same parity. Moreover, since $x, y \notin \frac{1}{2\sqrt{3}}G$, a and b cannot be both even and likewise c and d . The parity conditions in the statement follow. □

2.2 Description of the window

In this section we describe the polytope $W = \pi^\perp([0, 1]^6)$ and its intersection with the family of planes P . By definition, the window W is the convex hull of the $2^6 = 64$ points $\sum_{i=1}^6 n_i g_i$, where each $g_i \in E^\perp$ is the projection of the i^{th} element of the canonical basis of \mathbb{R}^6 and n_i is either 0 or 1. We end this section establishing a convention regarding the description of the three dimensional faces of W , which will be thoroughly studied in the next section. The main result of this part is the following proposition.

Proposition 12.

- *The polytope $W \subset E^\perp = F^\perp \oplus F$ has 52 vertices, 132 edges, 120 faces of dimension two and 40 faces of dimension three.*
- *All edges have the same length.*

- The vertices are distributed in 16 affine planes parallel to F^\perp . Each of these planes intersects
 - F in a point (see Figure 3); and
 - W in either a point, a triangle or a hexagon.

The proof of this proposition will be splitted in several lemmas.

Lemma 13. *The intersection of the window W and the family of translates of F^\perp given by $P = \bigcup_{\delta \in \Delta} F^\perp + \delta$ consists of points, triangles and hexagons.*

Proof. Let us denote by P_δ the affine plane $F^\perp + \delta$. Throughout this proof we will follow the conventions established in Sections 1.1, 1.2, and 2.

By Corollary 7, Δ is isomorphic to \mathbb{Z}^2 with generators $\{A, B\}$. So, the family P consists of all translates of F^\perp by an integer linear combination of A and B . The vertices of W are of the form

$$n_1 g_1 + \cdots + n_6 g_6 = n_1 f_1 + \cdots + n_6 f_6 + (n_1 - n_3 + n_5)A + (n_2 - n_4 + n_6)B, \quad (3)$$

where $n_i \in \{0, 1\}$. Since the expressions $n_1 + n_5 - n_3$ and $n_2 - n_4 + n_6$ take only 4 different values, the numbers $\{-1, 0, 1, 2\}$, it follows that the vertices of W project onto F onto 16 different points of the lattice Δ . The first step will be to understand how many of these vertices are projected onto each of these 16 lattice points. To this end, in the next array we collect in how many different ways the above mentioned numbers are obtained:

$n_i + n_{i+4} - n_{i+2}$	-1	0	1	2
$i = 1$	1	3	3	1
$i = 2$	1	3	3	1

It follows that on the plane $P_{(n_1+n_5-n_3, n_2-n_4+n_6)}$ the number of vertices of W is at most the product of the corresponding entries on the second and third lines in the above array. (Notice that not all the 64 projections of the vertices of the unit cube in \mathbb{R}^6 need to be vertices of W .) After a straightforward computation, we obtain:

- In each of $P_{(-1,-1)}, P_{(-1,2)}, P_{(2,-1)}, P_{(2,2)}$ there is one single vertex of W .
- In each of $P_{(-1,0)}, P_{(-1,1)}, P_{(0,-1)}, P_{(0,2)}, P_{(1,-1)}, P_{(1,2)}, P_{(2,0)}, P_{(2,1)}$ there are at most 3 vertices of W . It follows that the intersection of W with any of these planes is the convex hull of the 3 points obtained projecting the canonical basis of \mathbb{R}^6 . These points turn out to not be aligned in any of the 8 planes, so in this case the intersection of W and these planes is a collection of 8 triangles.
- In each of $P_{(0,0)}, P_{(0,1)}, P_{(1,0)}, P_{(1,1)}$ there are at most 9 vertices of W . We list the 9 points in each plane and compute their convex hull. It turns out that in each case the convex hull is a hexagon drawn in Figure 4. In each of the 4 planes, the vertices of the corresponding hexagon in cyclic order are given by:

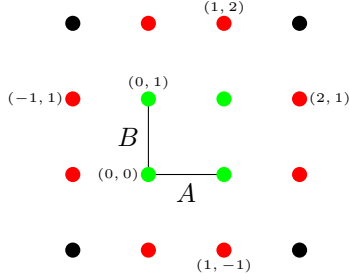


Figure 3: The polytope $W \subset E^\perp = F \cup F^\perp$ has its vertices on 16 planes parallel to F^\perp . The projection of each of these planes onto F , with fixed basis $\{A, B\}$, is a point. In this figure the number of vertices of W in each of the different 16 planes is encoded by the following color code: Black=isolated point, red=triangle, green=hexagon.

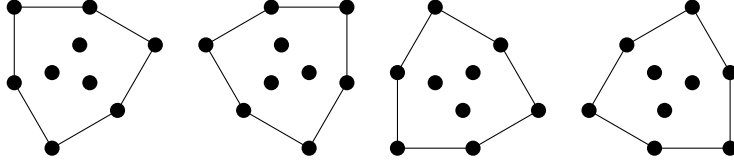


Figure 4: The four hexagons which appear in the planes $P_{(0,0)}, P_{(1,0)}, P_{(0,1)}, P_{(1,1)}$.

– Hexagon in the plane $P_{(0,0)}$:

$$\boxed{f_2 + f_4 \quad f_4 + f_6 \quad f_1 + f_3 + f_4 + f_6 \quad f_1 + f_3 \quad f_5 + f_3 \quad f_2 + f_3 + f_4 + f_5}$$

– Hexagon in the plane $P_{(1,0)}$:

$$\boxed{f_5 + f_2 + f_4 \quad f_1 + f_2 + f_4 \quad f_1 + f_4 + f_6 \quad f_1 + f_5 + f_3 + f_4 + f_6 \quad f_1 + f_5 + f_3 \quad f_5}$$

– Hexagon in the plane $P_{(0,1)}$:

$$\boxed{f_2 + f_4 + f_6 \quad f_2 \quad f_3 + f_2 + f_5 \quad f_6 + f_5 + f_3 \quad f_1 + f_6 + f_3 \quad f_1 + f_6 + f_3 + f_2 + f_4}$$

– Hexagon in the plane $P_{(1,1)}$:

$$\boxed{f_1 + f_6 \quad f_1 + f_2 + f_4 + f_6 \quad f_2 + f_4 + f_5 + f_6 \quad f_2 + f_5 \quad f_1 + f_2 + f_3 + f_5 \quad f_1 + f_3 + f_5 + f_6}$$

□

From this last lemma we deduce several results on the structure of W . We state them here without proof. They are all the result of simple observations or direct computations. In what follows, we will refer to the points of intersection between W and the planes $P_{(-1,-1)}, P_{(-1,2)}, P_{(2,-1)}, P_{(2,2)}$ as *isolated points* or *isolated vertices*.

Corollary 14. *The edges of the polygons in the statement of Lemma 13 are parallel to the directions $f_i \subset F^\perp, i = 1, \dots, 6$.*

Corollary 15. *The 1-dimensional complex of W , that is, the set of projections of the edges of the unit cube in \mathbb{R}^6 onto $F^\perp \oplus F$, has the following properties:*

- *No two vertices in one of the planes P_δ are connected by an edge.*
- *Every isolated vertex has valency 6.*
- *Every vertex in a triangle has valency 5. The 5 edges join the vertex in the triangle with 2 vertices on the hexagons, with 2 vertices on the closest triangle, and with the closest isolated vertex.*
- *In every hexagon, vertices of valency 6 alternate with vertices of valency 4. The latter vertices connect to vertices in the planes at distance 1, the former ones exhibit the same pattern plus two edges connecting to vertices in other hexagons.*

The following corollary is the last one in this section. In it we introduce the notion of a **cube**, which is our shortcut notation for a “3-dimensional face of the polytope W ”. These cubes will play an essential role in this article and the next subsection is devoted to their explicit description.

Corollary 16. *The window W has 40 faces of dimension 3. Every three dimensional face has the combinatorics of a cube and comprises 8 vertices. We will refer to these faces as “cubes”. Their disposition with respect to the vertices of W is as follows.*

- *Vertices of valency 6 belong to 8 cubes.*
- *Vertices of valency 5 belong to 6 cubes.*
- *Vertices of valency 4 belong to 4 cubes.*

In what follows we will need a more precise description of these cubes. Notice that, in order to completely determine a cube it suffices to know one of its vertices and the three edges of the cube intersecting at it. A vertex of a cube of W is a point in $F^\perp \oplus F$ and we will express it as a linear combination of the vectors g_1, \dots, g_6 . To describe the edges of the cube we start by remarking the following: in the plane F there is a “rightwards” direction, given by the vector A and an “upwards” direction, given by the vector B (see Figure 3). From equation (3) it follows that an edge of a cube directed by

- the vector g_1 or g_5 : is horizontal and rightwards when projected onto F .
- the vector g_2 or g_6 : is vertical and upwards when projected onto F .
- the vector g_3 : is horizontal and leftwards when projected onto F .
- the vector g_4 : is vertical and downwards when projected onto F .

Therefore, it is enough to recall the projections onto F^\perp , given by vectors f_i , of the edges of the cube we want to describe, since it is immediate from this information and the above remarks how to recover the cube. Furthermore, instead of the f_i , since we are only interested at this point in the directions of the edges, we can use the complex vector $x = e^{\frac{i\pi}{6}}$ and its multiples $x^i, i = 0, \dots, 11$ to describe the edges of a cube (cf. Equation (2)). With this notation the following holds:

- the vectors x^i with i even represent edges of the cube which are horizontal in F . More specifically, x^i with $i = 0, 4, 8$ are edges of cubes that project onto rightwards vectors on F while x^i with $i = 2, 6, 10$ project onto leftwards vectors.
- the vectors x^i with i odd represent edges of the cube which are vertical in F . More specifically, x^i with $i = 1, 5, 9$ are edges of cubes that project onto upwards vectors on F while x^i with $i = 3, 7, 11$ project onto downwards vectors.

Convention 17. *In the rest of the paper: A cube is given by a vertex g and three vectors x^i, x^j, x^k which support the edges of the cubes containing the vertex. We denote the cube by*

$$\{g, x^i, x^j, x^k\}.$$

Example 18. *As an example, the set $\{g_3 + g_4, x^1, x^5, x^9\}$ determines a cube in W with a vertex in $g_3 + g_4 \in P_{(-1,-1)}$ and 3 edges which go upwards in the plane F and thus connect the vertex $g_3 + g_4$ with 3 vertices in the plane $P_{(-1,0)}$. Each of these vertices is itself connected to 2 vertices in the plane $P_{(-1,1)}$ with edges that can be again encoded by the vectors x^1, x^5, x^9 . Finally, the 3 vertices in $P_{(-1,1)}$ (each of which connects to 2 vertices in $P_{(-1,0)}$) are also connected with the single vertex in $P_{(-1,2)}$. This is a complete description of the 6 vertices and 12 edges of a cube in W .*

2.3 Cubes

With the notation developed above, we now proceed to describe all the cubes in W . The details of the computation will not be needed in the rest of the paper; however, the last results of this section Proposition 21 and Corollary 22, summarizing the findings, are essential.

According to Corollary 15 each of the 4 isolated vertices has valency 6 and from Corollary 16 we learn that these 6 edges belong to 8 different cubes. After an easy computation we find out that if we fix any one of the isolated vertices, precisely two of the 8 cubes which contain it contain also another isolated vertex. It follows that the number of different cubes containing isolated vertices is $8 + 7 + 7 + 6 = 28$. The combinatorics of these cubes is as follows:

- The cubes containing two isolated vertices contain also the 6 vertices of the two triangles on a line (or column) in Figure 3. There are 4 such cubes. Following our convention, the two vertical cubes are given by the sets

$\{g_3 + g_4, x^1, x^5, x^9\}$ and $\{g_1 + g_4 + g_5, x^1, x^5, x^9\}$ while the two horizontal ones correspond to the sets $\{g_3 + g_4, x^0, x^4, x^8\}$ and $\{g_2 + g_3 + g_6, x^0, x^4, x^8\}$.

- The remaining cubes containing one isolated vertex, which are 24 in number, have all two edges going in the same direction on F and a third edge going in the perpendicular direction. The complete list follows:

- Cubes containing $v_1 = g_3 + g_4$ (bottom left isolated vertex):

$$\{v_1, x^i, x^{i+3}, x^{i+4}\}, i = 1, 5, 9 \quad \text{and} \quad \{v_1, x^i, x^{i+1}, x^{i+4}\}, i = 0, 4, 8.$$

- Cubes containing $v_2 = g_2 + g_3 + g_6$ (top left isolated vertex):

$$\{v_2, x^i, x^{i+1}, x^{i+4}\}, i = 3, 7, 11 \quad \text{and} \quad \{v_2, x^i, x^{i+3}, x^{i+4}\}, i = 0, 4, 8.$$

- Cubes containing $v_3 = g_1 + g_2 + g_5 + g_6$ (top right isolated vertex):

$$\{v_3, x^i, x^{i+3}, x^{i+4}\}, i = 3, 7, 11 \quad \text{and} \quad \{v_3, x^i, x^{i+1}, x^{i+4}\}, i = 2, 6, 10.$$

- Cubes containing $v_4 = g_1 + g_4 + g_5$ (bottom right isolated vertex):

$$\{v_4, x^i, x^{i+1}, x^{i+4}\}, i = 1, 5, 9 \quad \text{and} \quad \{v_4, x^i, x^{i+3}, x^{i+4}\}, i = 2, 6, 10.$$

Example 19. *If we focus on the isolated vertex in the plane $P_{(-1,-1)}$ it belongs to 8 cubes which, when projected to F , can be described as: a cube developing completely to the right, with a vertex in the isolated point in $P_{(2,-1)}$; a cube developing completely vertically with a vertex in the isolated point $P_{(-1,2)}$; 3 cubes that have two edges towards the right direction, with their furthest vertices in the planes $P_{(1,-1)}$ and $P_{(0,-1)}$ and one edge upwards; and finally 3 cubes with two upward edges, leaving the furthest vertices in this direction on the planes $P_{(-1,1)}$ and $P_{(0,1)}$, and one rightward edge. The cubes around the other 3 isolated points have a completely analogous symmetric configuration.*

The cubes which contain no isolated vertex have vertices on the triangles and hexagons described in Lemma 13. Moreover, by Corollaries 15 and 16 we know that each vertex on a triangle has valency 5 and thus it belongs to 6 different cubes. From the description above we have that 5 of these cubes contain an isolated vertex. The cube that we are missing connects the vertex in the triangle to two vertices in the closest hexagon and to one vertex in the closest triangle. There is a total of 12 such cubes, 3 for each pair of nearby triangles in Figure 3. We will not give a description as precise as the ones above for these cubes since for our purposes the precise vertex on the triangle that these cubes contain will be irrelevant. The information we need from these cubes is the following:

- Each vertex v_1, v_2, v_3 in the triangle in the plane $P_{(-1,0)}$ belongs to a cube that contains no isolated vertex of W . These three cubes are given by $\{v_j, x^i, x^{i+5}, x^{i+4}\}, i = 0, 4, 8.$

- Each vertex v_1, v_2, v_3 in the triangle in the plane $P_{(0,-1)}$ belongs to a cube that contains no isolated vertex of W . These three cubes are given by $\{v_j, x^i, x^{i-1}, x^{i+4}\}$, $i = 1, 5, 9$.
- Each vertex v_1, v_2, v_3 in the triangle in the plane $P_{(2,1)}$ belongs to a cube that contains no isolated vertex of W . These three cubes are given by $\{v_j, x^i, x^{i+5}, x^{i+4}\}$, $i = 2, 6, 10$.
- Each vertex v_1, v_2, v_3 in the triangle in the plane $P_{(1,2)}$ belongs to a cube that contains no isolated vertex of W . These three cubes are given by $\{v_j, x^i, x^{i-1}, x^{i+4}\}$, $i = 3, 7, 11$.

From this description we learn that there are two different types of cubes in the polytope W :

Definition 20.

- Standard cubes, which when projected onto F have two edges in the same direction and a third edge in the perpendicular one;
- Long cubes, whose 3 defining edges project onto the same direction on F .

These two types of cubes can be horizontal or vertical, depending on which is their longest direction when projected onto F , see Figure 5.

We summarize our findings in the next proposition.

Proposition 21. *The polytope W has 40 cubes.*

- 2 vertical and 2 horizontal long cubes each containing two isolated points.
- 24 standard cubes each containing precisely one isolated point.
- 12 standard cubes containing only points in the triangles and hexagons.

Corollary 22. *The sets of directions of the edges of the 40 cubes in W are:*

- the long cubes yield the sets $\{x^0, x^2, x^4\}$ and $\{x^1, x^3, x^5\}$.
- The 24 standard cubes with one isolated point yield

$$\{x^i, x^{i+4}, x^{i+1}\} \quad \text{and} \quad \{x^i, x^{i+4}, x^{i+3}\} \quad \text{with} \quad i = 0, \dots, 11.$$
- The 12 remaining standard cubes yield

$$\{x^i, x^{i+4}, x^{i+5}\}, i = 0, 2, 4, 6, 8, 10 \quad \text{and} \quad \{x^i, x^{i+4}, x^{i-1}\}, i = 1, 3, 5, 7, 9, 11.$$

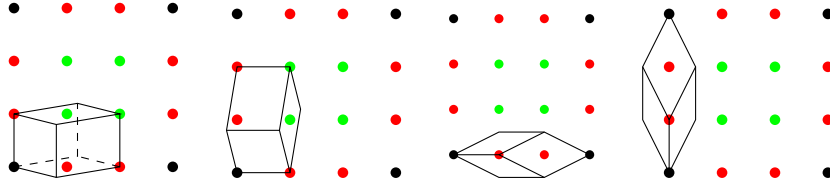


Figure 5: standard horizontal, standard vertical, long horizontal and long vertical cubes. Remark that all vertices in the long cubes are projected onto the grid points in a horizontal or vertical line: one vertex on each black dot and three vertices on each red dot.

3 Cubes, lines of intersection and $H^1(\Omega_{E_{12}^\gamma})$

3.1 Intersection of a cube and a plane

We want to apply the method described in Section 1.4 to compute the cohomology groups of the generalized 12-fold tilings. The first step is to cut the polytope W by the families of parallel planes $P^\gamma = \bigcup_{\delta \in \Delta} F^\perp + \gamma + \delta$ with $\gamma = (\gamma_1, \gamma_2) \in F$ (expressed with respect to the fixed orthogonal basis $\{A, B\}$). Due to the periodicity of Δ , restricting γ_1, γ_2 to vary in the range $[0, 1) \times [0, 1)$ is enough to obtain all the different families P^γ . The intersection of W with P^γ when $\gamma = (0, 0)$ has been treated in Proposition 12 and we now proceed to study the intersection when $\gamma \neq (0, 0)$. To this end, we start noticing that a plane intersects W if and only if it intersects one of its 3 dimensional faces. Indeed, since the 4-dimensional polytope W has a boundary consisting of 3-dimensional faces and it sits in the 4-dimensional space $F^\perp \oplus F$, a plane cannot intersect it without intersecting its boundary. (Notice that a plane could intersect W without intersecting an edge or a 2 dimensional face.) Thus, our next goal is to intersect the 40 cubes described in Proposition 21 with the planes in the families $P^\gamma = \bigcup_{\delta \in \Delta} E_2 + \gamma + \delta$ and $\gamma \in (0, 1) \times (0, 1)$.

Inside the family of planes P^γ very few of them do actually intersect the window W . Indeed, it is evident from Figure 6 that if $\gamma_i \neq 0$ for $i = 1, 2$ then there are precisely 9 planes in P^γ with non trivial intersection: the ones that project into the squares delimited by the colored dots in the figure. On the other hand, if $\gamma \neq (0, 0)$ but $\gamma_1 = 0$ or $\gamma_2 = 0$, then there are 12 such planes. If we want to individuate one plane in the family P^γ we will use the same notation as in the preceding section, that is, P_δ^γ is defined as the plane $F^\perp + \gamma + \delta$. Notice that in what follows, when we refer to a cube we will always consider it ‘parametrized’ as in the explicit list of the 40 cubes which constitute the boundary of W described in Section 2.3.

Lemma 23. *Consider a plane $P_\delta^\gamma = F^\perp + \gamma + \delta$ intersecting a cube of W .*

- *If the cube is standard and horizontal, then the intersection is a segment directed by the difference of the two horizontal vectors in the cube.*

- If the cube is standard and vertical, then the intersection is a segment directed by the difference of the two vertical vectors.
- If the cube is long, then P_δ^γ is orthogonal to a diagonal of the cube and thus $P_\delta^\gamma \cap W$ is either a triangle or a hexagon.

Proof. An arbitrary point p in a cube determined by the tuple $\{v, x^i, x^j, x^k\}$ has coordinates in $E^\perp = F^\perp \oplus F$ given by $p = v + \alpha_1 g_r + \alpha_2 g_s + \alpha_3 g_t$ where $\alpha_\ell \in [0, 1]$ and g_r, g_s and g_t are the projections of the basis vectors in \mathbb{R}^6 onto $F^\perp \oplus F$ or their opposites which are determined from x^i, x^j and x^k as previously explained. We are interested in the projection onto F of this point, which we will denote $\pi_F(p) := \pi^\perp(p) \cap F$.

If the cube is horizontal, then two of the exponents defining it will be even, say they are the two first ones, and therefore $\pi_F(p) = \pi_F(v) + (\alpha_1 + \alpha_2, \alpha_3)$. On the other hand, if the cube were vertical, we might assume that again the two first exponents defining it are odd and therefore in this case $\pi_F(p) = \pi_F(v) + (\alpha_3, \alpha_1 + \alpha_2)$. It follows that if $p \in P_\delta^\gamma$ then,

- in the horizontal case we have $\pi_F(v) + (\alpha_1 + \alpha_2, \alpha_3) = (\delta_1 + \gamma_1, \delta_2 + \gamma_2)$;
- in the vertical case $\pi_F(v) + (\alpha_3, \alpha_1 + \alpha_2) = (\delta_1 + \gamma_1, \delta_2 + \gamma_2)$ must hold.

So, for a point in a horizontal cube to be also in P_δ^γ its α_3 component, the vertical one in this case, is completely determined by the above equality while α_1 and α_2 are subject to the linear condition $\alpha_1 + \alpha_2 = \text{some constant}$. As α_1 and α_2 vary in $[0, 1]$, this condition determines a segment in P_δ^γ . Indeed, the projection of p onto this plane can be written as $\pi_{P_\delta^\gamma}(v) + \alpha_1 f_r + \alpha_2 f_s + \alpha_3 f_t$ and the above conditions found on the α_i 's allow us to rewrite the expression as:

$$\begin{aligned} & \pi_{P_\delta^\gamma}(v) + (\delta_1 + \gamma_1 - \pi_F(v)_1) f_r - \alpha_2 f_r + \alpha_2 f_s + (\delta_2 + \gamma_2 - \pi_F(v)_2) f_t \\ &= \pi_{P_\delta^\gamma}(v) + (\delta_1 + \gamma_1 - \pi_F(v)_1) f_r + \alpha_2 (f_s - f_r) + (\delta_2 + \gamma_2 - \pi_F(v)_2) f_t \end{aligned} \quad (4)$$

Notice that in this last expressions all parameters are fixed except for α_2 and as it varies it defines the segment of intersection between P_δ^γ and the cube $\{v, x^i, x^j, x^k\}$. As claimed in the statement, this segment is directed by the vector $f_s - f_r$. This vector is a multiple of $x^i - x^j$, the difference of the two even powers of x defining the cube. The argument is completely analogous in the vertical case. See Figure 7.

We now proceed to examine the intersection of P_δ^γ with a long cube. For the sake of concreteness we will assume that the cube is horizontal, which means that it is directed by three even powers of x (the vertical case is completely analogous with three odd powers instead). Just as before, a point in this cube is given by $p = v + \alpha_1 g_r + \alpha_2 g_s + \alpha_3 g_t$ and this time its projection onto F reads $\pi_F(p) = \pi_F(v) + (\alpha_1 + \alpha_2 + \alpha_3, 0)$. If we assume $p \in P_\delta^\gamma$ we obtain the equality $\pi_F(v) + (\alpha_1 + \alpha_2 + \alpha_3, 0) = (\delta_1 + \gamma_1, \delta_2 + \gamma_2)$. This means that the plane and the cube are in the same affine space of dimension 3. In this space

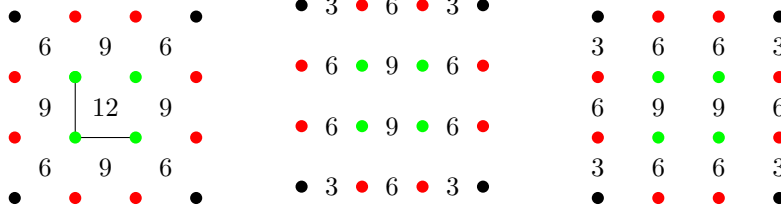


Figure 6: Each of these colored grid points represent the projection on F of a plane F^\perp containing some vertices of the polytope W . Recall that the plane F has basis $\{A, B\}$ highlighted on the left-most grid. The numbers in these grids represent the number of cubes in W that the planes $P^{(\gamma_1, \gamma_2)}$ meet. The first grid collects the numbers for $\gamma_1, \gamma_2 \neq 0$, the second for $\gamma_2 = 0$ and the last one for $\gamma_1 = 0$.

the equation of the plane is $\alpha_1 + \alpha_2 + \alpha_3 = \delta_1 + \gamma_1 - \pi_F(v)_1$ and the cube is $[0, 1]^3$. Thus the plane is orthogonal to a diagonal of the cube. The intersection is a polygon, more precisely, a triangle or a hexagon depending on the value of $\delta_1 + \gamma_1 - \pi_F(v)_1$. See Figure 7. \square

With the information we have gathered about the planes P^γ and the cubes in W we could now describe every segment in their intersection. Notice that, for γ with $\gamma_i \neq 0$ for $i = 1, 2$, this would imply to explicit 72 segments. Indeed, we would be looking at 9 different planes in the family P^γ that effectively intersect the cubes in W and the number of cubes intersecting each plane varies depending on where in F , see Figure 6, these planes are projected to. For example in the center square a plane intersects 12 cubes of W whereas in a corner square it meets only 6 cubes.

Thankfully, for the computation we are interested in, we do not need to go into this level of detail. In fact, the segments of intersection define lines in the space $E^\perp = F^\perp \oplus F$ and we are interested in them only up to the action of $\Gamma = \pi(\mathbb{Z}^6)$. In Lemma 23 we have described these segments/lines as points of the form $v \pm \alpha_1 g_i \pm \alpha_2 g_j \pm \alpha_3 g_k$ where v is a vertex of a cube and the α_i 's are real numbers in the interval $[0, 1]$ subject to some constraints. Since v is a vertex of W it is also an element of the group Γ and therefore by a translation of vector $-v$ we might simply consider the expression $\alpha_1(\pm g_i) + \alpha_2(\pm g_j) + \alpha_3(\pm g_k)$. Geometrically, this amounts to look at all the lines we are interested in together in the single plane $P_{(0,0)}^\gamma = P_0^\gamma$. At this point, since all the information about these lines on the F component is summarized in the point γ , we might look at their expressions in the plane P_0^γ , which we recall is parallel to F^\perp , obtaining the expressions $\alpha_1(\pm f_i) + \alpha_2(\pm f_j) + \alpha_3(\pm f_k)$. Our description of the cubes in W is in terms of the powers of the complex vector $x = e^{\frac{i\pi}{6}}$ instead of the vectors f_i . In this language, the lines we are interested in are of the form

$$\alpha_1\left(\frac{1}{\sqrt{3}}x^r\right) + \alpha_2\left(\frac{1}{\sqrt{3}}x^s\right) + \alpha_3\left(\frac{1}{\sqrt{3}}x^t\right)$$

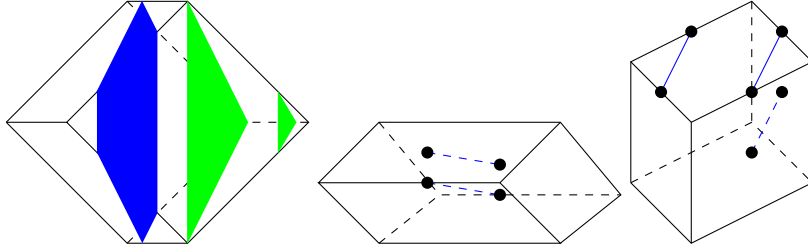


Figure 7: The left figure represents the intersection between a plane and a long cube. The plane and the cube live in the same 3-dimensional space and the intersection is either a hexagon (pink) or a triangle (green). The two other figures depict the intersection of the planes and the standard cubes. Each plane intersects a given cube in one line whose direction is determined by the cube. In the figure dotted lines are meant to be inside the cubes and solid ones on the faces.

with the necessary constraints on the parameters α_i .

As explained in Section 1.4, to compute the cohomology groups of the hull of the tiling, we need to know the quantity L_1 , which is the number of orbits of lines in the intersection between P^γ and W under the action of Γ . With all the previous conventions in place, we are now ready to give a convenient description of these lines which will be handy to later compute L_1 . We will use the standard convention denoting by $A + \lambda u$ the line of direction u passing through the point A with $\lambda \in \mathbb{R}$.

Proposition 24. *The intersection of the family of planes P^γ with the cubes in W yields a family of segments and polygons. Each of these segments and each side of a polygon belongs to a line that, when translated to the plane P_0^γ via a vector in $\Delta \subset F$, can be described by one of the following 36 equations:*

$$\pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{cases} x^i \\ x^{i+2} \\ x^{i+4} \end{cases} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} \right) + \lambda x^i \quad \text{for } i \in \{0, 2, 4\}$$

$$\pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{cases} x^{i+2} \\ x^{i+4} \\ x^{i+6} \end{cases} + \frac{\gamma_1}{\sqrt{3}} x^{i+1} \right) + \lambda x^i \quad \text{for } i \in \{1, 3, 5\}.$$

Proof. The list of the 40 cubes which constitute the boundary of the window is explicit in the discussion before Proposition 21. Since we are working modulo the translations of Δ we might ignore the vertices of the cubes and use only the information in Corollary 22.

We start considering standard cubes, which are all of the form $\{x^i, x^{i+4}, x^k\}$ for $i = 1, \dots, 11$ and k as in Corollary 22. By Lemma 23 the intersection of a standard horizontal cube and a single plane in P^γ is a segment as in Equation (4)

directed by the vector $x^{i+4} - x^i$, which is a multiple of the vector x^{i-1} . If the cube were vertical, the roles of γ_1 and γ_2 are swapped. For the sake of clarity, let us consider horizontal cubes. Since we are now working modulo the action of Δ and are interested in the whole line defined by the segment, the relevant part of the expression (4) which we need reads

$$\gamma_1 f_r + \alpha_2 (f_s - f_r) + \gamma_2 f_t \quad \text{where } \alpha_2 \in \mathbb{R}.$$

This expression, rewritten in terms of the powers of x turns into

$$\gamma_1 \frac{1}{\sqrt{3}} x^i + \gamma_2 \frac{1}{\sqrt{3}} x^k + \alpha_2 \frac{1}{\sqrt{3}} x^{i-1}.$$

Finally, rewriting $\alpha_2 \frac{1}{\sqrt{3}}$ as $\lambda \in \mathbb{R}$ we obtain the general expression of the lines we are looking for. To compile the list in the statement of the proposition we simply need to write down all the different lines obtained by this procedure from the standard cubes in Corollary 22. To ease future computations we have adopted the convention that the lines will be written in such a way that the direction is given by the i -th power of x yielding the expressions:

$$\begin{aligned} \text{horizontal cubes : } & \gamma_1 \frac{1}{\sqrt{3}} x^{i+1} + \gamma_2 \frac{1}{\sqrt{3}} x^k + \lambda x^i \text{ for } \begin{cases} i \text{ odd mod } 12 \\ k = i + 2, i + 4, i + 6 \end{cases} \\ \text{vertical cubes : } & \gamma_2 \frac{1}{\sqrt{3}} x^{i+1} + \gamma_1 \frac{1}{\sqrt{3}} x^k + \lambda x^i \text{ for } \begin{cases} i \text{ even mod } 12 \\ k = i, i + 2, i + 4 \end{cases} \end{aligned}$$

Note that we can make the notation a bit more compact taking into account that $x^i = -x^{i+6 \bmod 12}$. Adjusting the signs of γ_1 and γ_2 we obtain the first point in the statement of the proposition. Indeed, for $\gamma_1, \gamma_2 \neq 0$ all and only standard cubes are intersected by the planes in P^γ since the long cubes are projected onto lines where one γ_i is constantly equal to zero (see Figure 5).

We now consider the case of $\gamma = (0, \gamma_2)$. The intersection of any plane in P^γ with a standard horizontal cube will be either reduced to a point or a segment as described in Lemma 23. Furthermore, the intersection with a standard vertical cube will be a segment as described in Lemma 23. The reader can “visualize” this in Figure 5: for $\gamma = (0, \gamma_2)$ the planes intersecting standard horizontal cubes might find their intersection on an edge of the cube or in the “middle”, while the intersection with a vertical cube for these values of γ happens on a “lateral face”. We obtain thus all the lines in the first point of this proposition with $\gamma_2 = 0$. Furthermore, for these values of γ the intersection with the two vertical long cubes is non trivial. By Lemma 23 the intersection consists of triangles or hexagons perpendicular to the diagonal of the cube. Since we are interested on the lines defined by the sides of the polygons only up to the action of Δ , it is enough to consider the intersection with just one of the two cubes. We choose $\{g_3 + g_4, x^1, x^5, x^9\}$. The three planes in the family $P^{(0, \gamma_2)}$ which intersect this long vertical cube are $P_{(-1, j)}^{(0, \gamma_2)}$ with $j = -1, 0, 1$.

If $j = -1$ we obtain that the intersection is a triangle whose edges, when considered modulo Δ , that is, translated to the plane P_0^γ , can be described as

$$\gamma_2 \frac{1}{\sqrt{3}} x^i + \lambda x^{i-1} \quad \text{for } i = 1, 5, 9 \text{ and } \lambda \in \mathbb{R}.$$

(Recall that the vector $x^{i+4} - x^i$ is a multiple of x^{i-1} .) Following the convention of writing the direction of the line as the i -th power we obtain the lines

$$\gamma_2 \frac{1}{\sqrt{3}} x^{i+1} + \lambda x^i \quad \text{for } i = 0, 4, 8 \text{ and } \lambda \in \mathbb{R}.$$

Now, if $j = 0$ the intersection is a hexagon with parallel opposite sides. The lines they define have the same directions as the ones obtained for $j = -1$ and modulo Δ they coincide. Indeed, expressing the equations of these lines as $A + \lambda v$ we have that A is a point of coordinates $g_3 + g_4$ plus some integer multiples of some basis vectors g_i (yielding the coordinates of another vertex of the cube) plus $\gamma_2 \frac{1}{\sqrt{3}} x^i$ with $i = 1, 5$ or 9 ; and the vector v is given by a difference of the form $x^{i+4} - x^i$. A completely analogous argument can be made to justify that modulo Δ we do not obtain new lines when considering the case $j = 1$. Finally, notice that the 3 lines we have found for $j = -1$ coincide with lines obtained from the intersection with standard vertical cubes so they have been already been accounted for.

The case of the long horizontal cubes is completely analogous to the one of the vertical ones, we only need to swap the roles of γ_1 and γ_2 and of the even and odd powers of x .

Finally, the intersection $P^\gamma \cap W$ when $\gamma = (0, 0)$ was studied in Lemma 13. The lines we obtain in this case are supported by the sides of the triangles and hexagons found in this intersection. According to Corollary 14 these lines have all direction x^i for some $i = 0, \dots, 5$. Moreover, since they all go through vertices of W , it is evident that they can be translated to lines going through $(0, 0) \in P_0^0$ via an element in Δ . \square

3.2 Orbits of lines

Now that we have a complete list of lines explicit in Proposition 24, we need to count the number of orbits of these lines under the action of $\Gamma = \Delta \oplus \langle f_1, \dots, f_6 \rangle_{\mathbb{Z}}$ (see Corollary 7). From Proposition 24 we know that the lines appearing in $W \cap P^\gamma$ have direction x^i with $i = 0, \dots, 5$ and, depending on the value of γ , for each fixed direction x^{i_0} there are a certain number of lines when we consider them translated to the plane P_0^γ via an element in Δ . Since we are working in the plane P_0^γ the action of Γ is simplified to the action of $\Delta_0 := \langle f_1, \dots, f_6 \rangle_{\mathbb{Z}}$. For computational reasons, it is better to use the powers of x than the f_i , so we remind the reader that Equation (2) tells us that $\Delta_0 \sim \frac{1}{\sqrt{3}} \mathbb{Z}[x]$. The number of orbits will depend on the value of γ , however we can do a first overall simplification.

Lemma 25. *The two lines in each of the following accolades*

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^{i+2} \\ x^{i+4} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i, -\left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^{i+2} \\ x^{i+4} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1}\right) + \lambda x^i, \quad i \in \{0, 2, 4\}$$

$$\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{i+2} \\ x^{i+4} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{i+1} + \lambda x^i, -\left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{i+2} \\ x^{i+4} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{i+1}\right) + \lambda x^i, \quad i \in \{1, 3, 5\}.$$

are in the same Δ_0 orbit.

Proof. Since the argument is completely analogous for all the cases, in order to ease the notation we will write the details of the proof only for the lines:

$$\frac{\gamma_1}{\sqrt{3}} x^{i+2} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i \quad \text{and} \quad \frac{\gamma_1}{\sqrt{3}} x^{i+4} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i \quad \text{for } i \in \{0, 2, 4\}.$$

The difference of two arbitrary points in these lines can be expressed as

$$\pm \frac{\gamma_1}{\sqrt{3}} (x^{i+2} - x^{i+4}) + \mu x^i, \quad \text{for some } \mu \in \mathbb{R}.$$

Now, it holds that $\pm(x^{i+2} - x^{i+4}) = \pm x^i$, and therefore

$$\pm \frac{\gamma_1}{\sqrt{3}} (x^{i+2} - x^{i+4}) + \mu x^i \in \Delta_0 \iff (\pm \frac{\gamma_1}{\sqrt{3}} + \mu) x^i \in \mathbb{Z}[x].$$

By Lemma 10 this last condition holds if and only if $\pm \frac{\gamma_1}{\sqrt{3}} + \mu \in G$, which is always satisfied for, say, $\mu = \mp \frac{\gamma_1}{\sqrt{3}}$ and thus the two lines we started with are in the same Δ_0 orbit. \square

Corollary 26. *The lines in Proposition 24 in different Δ_0 orbits are at most the following 24:*

$$\pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1}\right) + \lambda x^i \quad \text{for } i \in \{0, 2, 4\}$$

$$\pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{i+4} \\ x^{i+6} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{i+1}\right) + \lambda x^i \quad \text{for } i \in \{1, 3, 5\}.$$

Proposition 27. *The number L_1 of different orbits of lines under the action of Γ depends on $\gamma = (\gamma_1, \gamma_2)$ and belongs to the set $\{6, 9, 12, 15, 18, 21, 24\}$. The precise value depends on various constraints on γ and it can be found in Figure 8.*

When $L_1 = 24$ we have that the 1-singularities have 6 different directions and each direction has 4 representatives under the action of Γ . The other values of L_1 are obtained when some of the parallel representatives of the 1-singularities are identified under the action of Γ . The other extreme case, $L_1 = 6$, arrives when $\gamma_1, \gamma_2 \in G$ and there is only one orbit per direction.

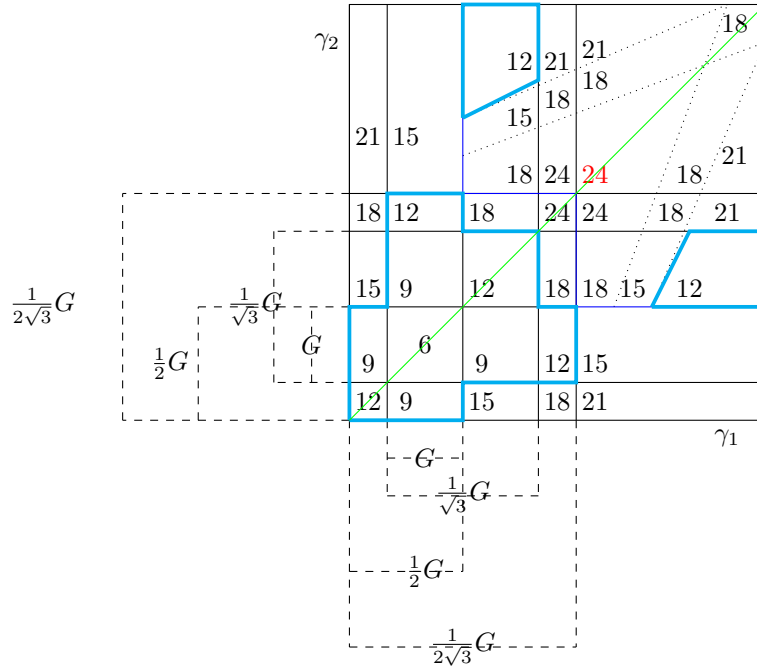


Figure 8: The number of 1-singularities depending on $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ (recall that $G = \mathbb{Z}[\sqrt{3}]$). The top right part of the chart is to be understood as the superposition of the regions labelled A, B, C, D, E and F in Figure 9. Moreover, adding 1 to each of the values on the chart we obtain the rank of $H^1(\Omega_{E_{\gamma_2}})$ (cf. Proposition 30). Finally, the regions enclosed in blue are the ones for which the second cohomology groups have been completely determined in Section 4.2

Proof. To start with, recall that we are working with the simplifying assumption that γ belongs to $[0, 1)^2$ (see beginning of Section 3.1). Moreover, the orbits of lines under the action of Γ will be computed via the action of the group Δ_0 on translates by Δ of the lines to the plane P_0^γ (see beginning of Section 3.2).

- If the value of $\gamma = (0, 0)$, then by Proposition 24 we know that we have a total of 6 lines, one for each direction x^i , and these lines are evidently in different Δ_0 orbits.

- If precisely one between γ_1 and γ_2 is zero then, by Proposition 24, there are 18 lines to be considered. We start considering the case $\gamma_1 = 0$. In this case there are six lines with 3 different directions x^i , $i \in \{0, 2, 4\}$ and 12 lines with direction x^i , $i \in \{1, 3, 5\}$. We want to understand under which conditions on γ_2 two parallel lines are on the same orbit under the action of Δ_0 . For the even directions, the equations of the lines under consideration are

$$\frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i \quad \text{and} \quad -\frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda' x^i$$

and they will be in the same Δ_0 orbit if and only if there exist $\lambda, \lambda' \in \mathbb{R}$ such that

$$\frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i - \left(-\frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda' x^i\right) \in \Delta_0.$$

Reorganizing we obtain that this condition is equivalent to

$$\gamma_2 \frac{2}{\sqrt{3}}x^{i+1} + x^i(\lambda - \lambda') \in \Delta_0 \quad \text{for some } \lambda, \lambda' \in \mathbb{R}.$$

Finally, relabelling adequately the real parameters, this last condition reads

$$2\gamma_2 x^{i+1} + \mu x^i \in \mathbb{Z}[x] \quad \text{for some } \mu \in \mathbb{R},$$

and by Lemma 10 it will be fulfilled if and only if $2\gamma_2, \mu \in G$.

For the odd directions, $i \in \{1, 3, 5\}$, the equations of the lines under consideration are

$$\frac{\gamma_2}{\sqrt{3}}x^{i+4} + \lambda_1 x^i, \quad -\frac{\gamma_2}{\sqrt{3}}x^{i+4} + \lambda_2 x^i, \quad \frac{\gamma_2}{\sqrt{3}}x^{i+6} + \lambda_3 x^i \quad \text{and} \quad -\frac{\gamma_2}{\sqrt{3}}x^{i+6} + \lambda_4 x^i.$$

Following the same arguments as before, we conclude that the two first lines will be in the same Δ_0 orbit if

$$2\gamma_2 x^{i+4} + \mu x^i \in \mathbb{Z}[x] \quad \text{for some } \mu \in \mathbb{R},$$

which holds, via Lemma 10, if $2\gamma_2 \in \frac{1}{\sqrt{3}}G$. Noticing that $x^{i+6} = -x^i$ we conclude that the last two lines are coincident and in fact one only line.

Finally we need to analyze the possibility of the lines passing through the points $\pm \frac{\gamma_2}{\sqrt{3}}x^{i+4}$ and $\frac{\gamma_2}{\sqrt{3}}x^i$ being in the same Δ_0 orbit. This will be the case if

$$\pm \frac{\gamma_2}{\sqrt{3}}x^{i+4} + x^i(\lambda - \lambda' - \frac{\gamma_2}{\sqrt{3}}) \in \Delta_0 \quad \text{for some } \lambda, \lambda' \in \mathbb{R},$$

or equivalently, if

$$\pm\gamma_2x^{i+4} + \mu x^i \in \mathbb{Z}[x] \quad \text{for some } \mu \in \mathbb{R}.$$

By Lemma 10 we conclude this is the case if $\gamma_2 \in \frac{1}{\sqrt{3}}G$. Summing up, since $\frac{1}{\sqrt{3}}G \subset \frac{1}{2\sqrt{3}}G$ we have:

- * For each even direction there is one Δ_0 orbit if $\gamma_2 \in \frac{1}{2}G$ and two if $\gamma_2 \notin \frac{1}{2}G$.
- * For each odd direction we can have:
 - three Δ_0 orbits if $\gamma_2 \notin \frac{1}{2\sqrt{3}}G$.
 - two Δ_0 orbits if $\gamma_2 \in \frac{1}{2\sqrt{3}}G \setminus \frac{1}{\sqrt{3}}G$ with representatives passing through the points $\frac{\gamma_2}{\sqrt{3}}x^{i+4}$ and $\frac{\gamma_2}{\sqrt{3}}x^i$.
 - one Δ_0 orbit if $\gamma_2 \in \frac{1}{\sqrt{3}}G$.

The total number of Δ_0 orbits is presented in Figure 9, where the right hand side has the even directions with values 3 and 6 and the left hand side the odd directions, with values 3,6 and 9. The quantity L_1 is computed by combining together the information on the lines with even and odd directions. This information can be read in Figure 8.

The arguments and calculations needed to arrive to the computation of the number of L_0 when $\gamma = (\gamma_1, 0)$ are analogous to the ones presented for the case $\gamma = (0, \gamma_2)$. Indeed, in the former case there will be 3 lines for each even direction since two of the ones in Corollary 26 will be coincident and the usage of Lemma 10 will yield the same conclusions. All the relevant results are again contained in Figures 8 and 9.

• We now proceed to analyze the case $\gamma_i \neq 0$ for $i = 1, 2$. To start with notice that a point in any of these lines is of the form $A + \lambda v$ where A is a point in P_0^γ , $\lambda \in \mathbb{R}$ and v is the direction of the line. If two lines ℓ and ℓ' are in the same Δ_0 -orbit we have that

$$A + \lambda v - (A' + \lambda'v) \in \Delta_0 \quad \text{for some } \lambda, \lambda' \in \mathbb{R},$$

which implies

$$(A - A') + (\lambda - \lambda')v \in \Delta_0,$$

and relabeling the real parameter it is equivalent to

$$A - A' + \mu v \in \Delta_0 \quad \text{for some } \mu \in \mathbb{R}.$$

The vector $v = x^i$ and the power i might be even or odd. We will write down the full details for the i even case and the results for the odd case will be given at the end, without details. For each even power there are 4 different lines and our aim is to understand the action of Δ_0 on them.

From Corollary 26, we know that for a fixed even direction x^{i_0} the points A, A' we need to consider are of the form

$$\pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{cases} x^{i_0} \\ x^{i_0+2} \end{cases} + \frac{\gamma_2}{\sqrt{3}} x^{i_0+1} \right)$$

and, depending on the choices of signs, the difference $A - A'$ can be written as

$$\text{Case 1. } \pm \frac{1}{\sqrt{3}} \gamma_1 (x^j - x^k) \quad j, k \in \{i_0, i_0 + 2\}, j \neq k, \text{ or}$$

$$\text{Case 2. } \pm \frac{1}{\sqrt{3}} (\gamma_1 (x^j + x^k) + 2\gamma_2 x^{i_0+1}) \quad j, k \in \{i_0, i_0 + 2\}.$$

We start considering the first of these two cases. Under these circumstances $A - A' + \mu x^{i_0} \in \Delta_0$ if and only if $\pm \gamma_1 (x^j - x^k) + \mu' x^{i_0} \in \mathbb{Z}[x]$ for some $\mu' \in \mathbb{R}$. Due to the \pm sign in front of this expression, we can assume that $j > k$, so $k = i_0$ and $j = i_0 + 2$, which implies $x^j - x^k = x^{i_0+4}$. It follows that in this first case the condition we need to determine on γ_1 is when does

$$\gamma_1 x^{i_0+4} + \mu' x^{i_0} \in \mathbb{Z}[x] \quad \text{for some } \mu' \in \mathbb{R}$$

hold. By Lemma 10 we conclude that this is the case if and only if

$$\sqrt{3}\gamma_1 \in G. \quad (5)$$

We move on to consider Case 2 above. Now $A - A' + \mu x^{i_0} \in \Delta_0$ if and only if $\gamma_1 (x^j + x^k) + 2\gamma_2 x^{i_0+1} + \mu' x^{i_0} \in \mathbb{Z}[x]$, for some $\mu' \in \mathbb{R}$. With the appropriate substitutions of $x^j + x^k$, depending on whether $j = k$ or $j \neq k$, what we need to understand is for which values of γ_i do the conditions

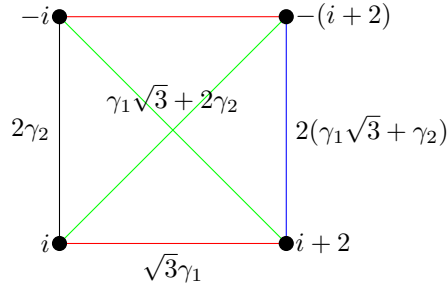
$$2(\gamma_1 x^{i_0} + \gamma_2 x^{i_0+1}) + \mu' x^{i_0} \in \mathbb{Z}[x] \iff (2\gamma_1 + \mu') x^{i_0} + 2\gamma_2 x^{i_0+1} \in \mathbb{Z}[x] \quad (6)$$

$$2(\gamma_1 x^{i_0+2} + \gamma_2 x^{i_0+1}) + \mu' x^{i_0} \in \mathbb{Z}[x] \iff 2(\gamma_1 x^2 + \gamma_2 x) + \mu' \in \mathbb{Z}[x] \quad (7)$$

$$\gamma_1 \sqrt{3} x^{i_0+1} + 2\gamma_2 x^{i_0+1} + \mu' x^{i_0} \in \mathbb{Z}[x] \iff (\gamma_1 \sqrt{3} + 2\gamma_2) x^{i_0+1} + \mu' x^{i_0} \in \mathbb{Z}[x] \quad (8)$$

With the help of Lemma 10 we conclude that condition (6) is fulfilled if and only if $\gamma_2 \in \frac{1}{2}G$, while condition (8) is fulfilled if and only if $\gamma_1 \sqrt{3} + 2\gamma_2 \in G$. Now, condition (7) is equivalent to $(2\sqrt{3}\gamma_1 + 2\gamma_2)x + \mu - \gamma_1 \in \mathbb{Z}[x]$. Thus we obtain the condition $2(\gamma_1 \sqrt{3} + \gamma_2) \in G$.

To help the reader visualize the results obtained so far, we propose the following figure:



After fixing one even direction, every vertex labelled $i, i+2, -i, -(i+2)$ in the figure represents one of the four parallel lines in this direction. The labelling corresponds to the power of x with coefficient γ_1 in the description of the line in Corollary 26. The conditions we have studied above identify parallel lines as follows:

- Condition 5 identifies lines labelled $\pm i$ and $\pm(i+2)$ if $\sqrt{3}\gamma_1 \in G$.
- Condition 6 identifies lines labelled i and $-i$ if $2\gamma_2 \in G$.
- Condition 7 identifies lines labelled $i+2$ and $-(i+2)$ if $2(\gamma_1\sqrt{3} + \gamma_2) \in G$.
- Condition 8 identifies lines labelled $\pm i$ and $\mp(i+2)$ if $\gamma_1\sqrt{3} + 2\gamma_2 \in G$.

To finish the analysis and obtain the number of orbits of lines under the action of Γ , we need to understand when do the above conditions arrive simultaneously. The results follow.

- If $\sqrt{3}\gamma_1 \in G$ and
 - $2\gamma_2 \in G$, then there is 1 orbit.
 - $2\gamma_2 \notin G$, then there are 2 orbits with representatives i and $-i$.
- If $\sqrt{3}\gamma_1 \in \frac{1}{2}G \setminus G$ and
 - $2\gamma_2 \in G$, then there are 2 orbits with representatives i and $i+2$.
 - $2\gamma_2 \notin G$, then there are 4 different orbits.
- If $\sqrt{3}\gamma_1 \notin \frac{1}{2}G$ and
 - $2\gamma_2 \in G$, then there are 3 orbits with representatives $i, \pm(i+2)$.
 - $2\gamma_2 \notin G$ and
 - (A) $2\gamma_1\sqrt{3} + 2\gamma_2 \in G$, then 3 orbits with repr. $\pm i$ and $i+2$.
 - (B) $\gamma_1\sqrt{3} + 2\gamma_2 \in G$, then 2 orbits with repr. i and $i+2$.
 - (C) $2\gamma_1\sqrt{3} + 2\gamma_2, \gamma_1\sqrt{3} + 2\gamma_2 \notin G$, then 4 orbits.

Since the above results do not depend on the even direction fixed, to obtain the total number of orbits for the even directions we need to simply multiply the above numbers by 3. The labelling A, B, C in the last case corresponds to the sets and notation in Lemma 11. The results up to here have been summarized in Figure 9.

Next, we must consider the case of lines directed by $v = x^i$ with i an odd number. Following the same arguments as in the even case, we obtain the analogous 4 conditions:

- Lines labelled $\pm(i+4)$ and $\pm(i+6)$ are in the same Δ_0 -orbit if and only if for some $\mu' \in \mathbb{R}$, it holds

$$-\gamma_2 x^{i_0+2} + \mu' x^{i_0} \in \mathbb{Z}[x] \iff \sqrt{3}\gamma_2 \in G$$

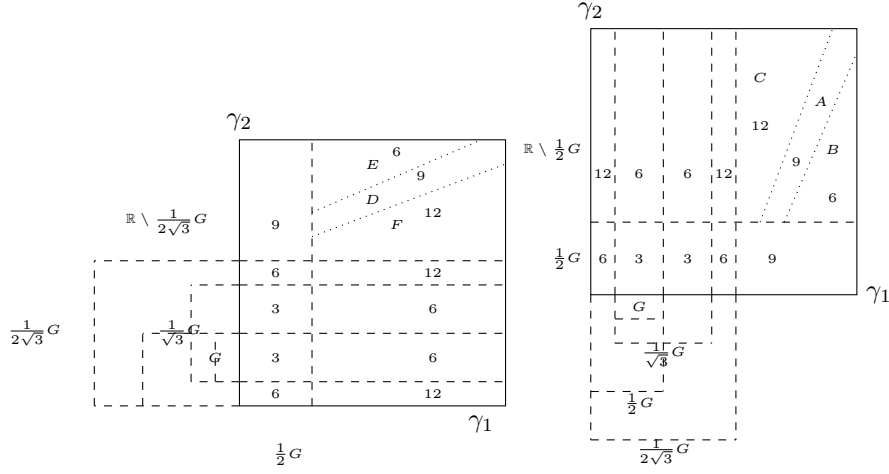


Figure 9: The number of different orbits of lines depending on (γ_1, γ_2) . On the left picture we consider lines directed by x^i with i odd. The right picture shows the result for i even.

- Lines labelled $i + 4$ and $-(i + 4)$ are in the same Δ_0 -orbit if and only if for some $\mu' \in \mathbb{R}$, it holds

$$2(\gamma_2 x^{i_0+4} + \gamma_1 x^{i_0+1}) + \mu' x^{i_0} \in \mathbb{Z}[x] \iff (2\sqrt{3}\gamma_2 + 2\gamma_1)x - (4\gamma_2 + \mu') \in \mathbb{Z}[x]$$

and this last condition is equivalent to $2(\sqrt{3}\gamma_2 + \gamma_1) \in G$. This should be compared with set D in Lemma 11.

- Lines labelled $i + 6$ and $-(i + 6)$ are in the same Δ_0 -orbit if and only if for some $\mu' \in \mathbb{R}$, it holds

$$2(-\gamma_2 x^{i_0} + \gamma_1 x^{i_0+1}) + \mu' x^{i_0} \in \mathbb{Z}[x] \iff 2\gamma_1 x^{i_0+1} + (-2\gamma_2 + \mu')x^{i_0} \in \mathbb{Z}[x]$$

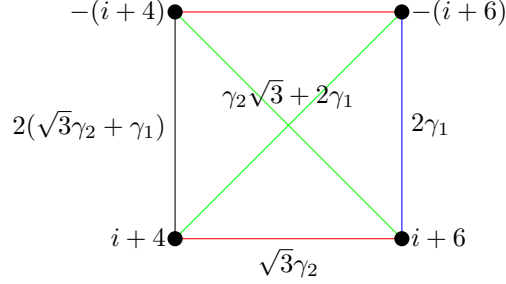
and this last condition is equivalent to $2\gamma_1 \in G$.

- Lines labelled $\pm(i + 4)$ and $\mp(i + 6)$ are in the same Δ_0 -orbit if and only if for some $\mu' \in \mathbb{R}$, it holds

$$\gamma_2 \sqrt{3} x^{i_0+5} + 2\gamma_1 x^{i_0+1} + \mu' x^{i_0} \in \mathbb{Z}[x] \iff (\gamma_2 \sqrt{3} + 2\gamma_1)x + (\mu' - 3\gamma_2) \in \mathbb{Z}[x]$$

and this last condition is equivalent to $\gamma_2 \sqrt{3} + 2\gamma_1 \in G$. This should be compared with set E in Lemma 11.

In the case of lines directed by an odd power of x the ‘square’ we obtain is:



A summary of our findings on the number of orbits of lines directed by odd powers of x is summarized in Figure 9.

Finally, to obtain L_1 , the total number of orbits of lines under the action of Γ , we need to combine the information on the lines directed by odd and even powers of x . The considerations in Lemma 11 together with the analysis carried out in this proof yield the final result, which we have encapsulated in Figure 8. \square

3.3 First cohomology group $H^1(\Omega_{E_{12}^\gamma})$

In order to compute the rank of the group $H^1(\Omega_{E_{12}^\gamma})$ via Theorem 3 we need one more ingredient: the value of R , which is the rank of the \mathbb{Z} -module generated by $\Lambda^2\Gamma^i, i = 1, \dots, n$. Recall that $\Gamma^i \leq \Gamma$ is the stabilizer under the action of Γ of the vector space spanned by f_i .

Lemma 28. *For all $i = 1, \dots, 6$, the rank 2 groups Γ^i are explicit in the following chart:*

Γ^1	Γ^2	Γ^3
$\langle f_1, f_6 - f_2 \rangle$	$\langle f_2, f_1 + f_3 \rangle$	$\langle f_3, f_2 + f_4 \rangle$
Γ^4	Γ^5	Γ^6
$\langle f_4, f_3 + f_5 \rangle$	$\langle f_5, f_4 + f_6 \rangle$	$\langle f_6, f_1 - f_5 \rangle$

Proof. Consider a line directed by the vector f_i . The group Γ^i is the subgroup $\{h \in \Gamma \mid f_i + h = \mu f_i, \mu \in \mathbb{R}\}$. Thus we need to solve an equation of the type

$$\lambda f_i = h = n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4 + n_5 f_5 + n_6 f_6,$$

where $\lambda \in \mathbb{R}$ and $(n_1, \dots, n_6) \in \mathbb{Z}^6$. Using Equation (2) the above expression can be rewritten in terms of roots of unity and our task is to understand which $\lambda \in \mathbb{R}$ can be written as $\lambda = \sum_k n_k x^k$. By Lemma 10 we deduce $\lambda \in \mathbb{Z}[\sqrt{3}]$. The integer combinations of the f_i which yield elements of the form $\mathbb{Z}[\sqrt{3}]f_i$ are collected in Lemma 6. The results in the table in the statement follow. \square

Lemma 29. *Let β be the map from Eq. (1). For all parameters γ we have $R = \text{rk } \beta = 3$, and $\text{coker } \beta \cong \mathbb{Z}^3$.*

Proof. By definition, the image of the map β depends only on the stabilizers Γ^α of the 1-singularities α , that is, on their directions f_i , and not on their positions. Let us denote $\Lambda_i = \Lambda^2 \Gamma^i$ for $i = 1, \dots, 6$, so that $\text{im } \beta$ is generated by the Λ_i . Using the explicit description of the groups Γ^i in Lemma 28 and the relations in Lemma 6, we compute the relevant exterior products:

$$\begin{cases} \Lambda_1 = f_1 \wedge f_6 - f_1 \wedge f_2 = -2f_1 \wedge f_2 + f_1 \wedge f_4, \\ \Lambda_2 = -f_1 \wedge f_2 + f_2 \wedge f_3, \\ \Lambda_3 = -f_2 \wedge f_3 + f_3 \wedge f_4, \\ \Lambda_4 = f_1 \wedge f_4 - 2f_3 \wedge f_4, \\ \Lambda_5 = 2f_5 \wedge f_4 - f_5 \wedge f_2 = f_1 \wedge f_2 - 2f_1 \wedge f_4 + f_2 \wedge f_3 + 2f_3 \wedge f_4, \\ \Lambda_6 = 2f_6 \wedge f_1 - f_6 \wedge f_3 = 2f_1 \wedge f_2 - 2f_1 \wedge f_4 + f_2 \wedge f_3 + f_3 \wedge f_4. \end{cases}$$

In order to compute the rank of the \mathbb{Z} -module $\text{im } \beta$ generated by the $\Lambda^2 \Gamma^i$, we represent each of these exterior products, with respect to the basis $f_i \wedge f_j$ given by the lexicographic order, as a column in the following matrix:

$$\begin{pmatrix} -2 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -2 & -2 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{pmatrix}.$$

The quantity R we seek is the rank of this matrix. It is evident that the first

three columns are linearly independent. We have
$$\begin{cases} \Lambda_4 = \Lambda_1 - 2\Lambda_2 - 2\Lambda_3, \\ \Lambda_5 = -2\Lambda_1 + 3\Lambda_2 + 2\Lambda_3, \\ \Lambda_6 = -2\Lambda_1 + 2\Lambda_2 \end{cases}$$

and therefore $R = 3$.

It is also easy to see that $-(\Lambda_2 + \Lambda_3)$, $\Lambda_1 - 2(\Lambda_2 + \Lambda_3)$ and $-\Lambda_3$ span a \mathbb{Z} -module of rank 3 with free complement in \mathbb{Z}^6 . As the latter equals $\Lambda^2 \Delta_0$ in our basis, $\text{coker } \beta$ is free of rank 3. \square

Proposition 30. *The rank of $H^1(\Omega_{E_{12}}^\gamma)$ depends on γ and equals the values depicted in Figure 8 plus 1.*

Proof. By Theorem 3 the rank of H^1 can be computed as $4 + L_1 - R$. The value of L_1 as a function of γ was computed in Lemma 27. Finally, by the previous lemma we know that for all γ we have $R = 3$ and the result follows. \square

4 Intersection of lines

We are now left with the task of computing the rank of the second cohomology groups. To this end we need to compute the quantity e in Theorem 3. It is

defined as a linear combination of the quantities L_0^α and L_0 which depend on γ . We recall the reader that the latter quantity is the cardinality of the set of all orbits of 0-singularities, while the former is the cardinality of the set of orbits of 0-singularities on the specific 1-singularity α .

The 0-singularities are obtained intersecting the representatives of orbits of lines studied in Section 3.2. From Proposition 27 we know that the number of orbits depends on the value of the parameter γ . In the next subsection we compute L_0^α and L_0 in the case $\gamma = (0, 0)$, which corresponds to having 6 different directions, or in other words, one orbit representative per direction. Following this, Subsection 4.2 describes all the cohomology groups of the generalized 12-fold tilings with the property of having at most two representatives of orbits of lines per direction. In the language of Proposition 27 this section deals with L_1 having value 6, 9 or 12. The corresponding values of γ are the ones in the regions enclosed by a blue line in Figure 8. These results come from a computed aided calculation. Finally, in the last subsection we address the general case. We have not obtained a complete description of the second cohomology groups in this case. However, again with a computed aided calculation, we present an example of the extremal case in Proposition 27 when we have 24 different line orbits to deal with.

4.1 First case: $\gamma = (0, 0)$

Lemma 31. *If $\gamma = (0, 0)$, then $L_0 = 14$ and for each 1-singularity α we have $L_0^\alpha = 6$.*

Proof. First of all we show that $L_0^\alpha = 6$ for each 1-singularity α . For a fixed line $\alpha = \mathbb{R}f_i$, with $i = 1, \dots, 6$, we are interested in the intersections of this line with the orbits of the other lines under the action of Γ . Using the bijection described in Equation 2 we can write the relevant equations in terms of powers of the root of unity x instead of the vectors f_i . The generic formula we have to deal with sets a point in a line of direction x^i , or rather points in a Γ^i -orbit, equal a point in a line of direction x^j . In symbols we have, for $i \neq j \in \{1, \dots, 6\}$:

$$\mu x^i + \sum n_k x^k = \lambda x^j, n_k \in \mathbb{Z}, \mu, \lambda \in \mathbb{R} \iff \lambda = \mu x^{i-j} + \sum_k n_k x^k.$$

It follows that λ is of the form $cx + d + \mu(ax + b)$ with $a, b, c, d \in G$. This gives us the conditions

$$\begin{cases} \lambda = d + \mu b \\ -c = \mu a \end{cases} \implies \lambda = d - \frac{b}{a}c. \quad (9)$$

Inside the infinite collection of points of intersection between a line of direction x^j and the translates by Γ of a line of direction x^i , determined in this setting by the possible values of λ , we need to count how many are in different orbits. That is, the points λ and λ' will be identified if $\lambda - \lambda' \in G$.

Since $x^{i-j} = a + xb$, by Lemma 9 when $i - j$ varies we obtain five possible values for $-\frac{b}{a}$ which are collected in the following set:

$$S = \left\{0, -\frac{1}{\sqrt{3}}, \frac{-\sqrt{3}}{2}, \frac{-2}{\sqrt{3}}, -\sqrt{3}\right\}.$$

Keeping the notation λ to refer to its class modulo G , from (9) we see that we need to understand how many different equivalence classes are there for numbers of the form $\lambda = \frac{b}{a}(c_1 + c_2\sqrt{3})$ where $c_1, c_2 \in \mathbb{Z}$ and $\frac{b}{a} \in S$. The complete list is the following

$$T = \left\{0, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right\},$$

from which we deduce that $L_0^\alpha = 6$ independently of the value of α , that is, of the direction of the 1-singularity.

We have identified on each line 6 different equivalence classes of points. However, some of these classes in different lines can be related by an element of Γ . For example, the class of 0 belongs to every 1-singularity so there are at most 31 different classes. The final quantity we need to compute, L_0 , is precisely this total number of classes of 0-singularities under the action of Γ .

Notice that the classes of 0-singularities in the line x^i which are in the same equivalence classes of 0-singularities in the line x^j only depend on the difference $i - j$. Indeed, we have to solve an equation of the form

$$\lambda_1 x^i - \lambda_2 x^j \in \Gamma \iff \lambda_1 x^i - \lambda_2 x^j = a + bx$$

with $a, b \in G$ and $\lambda_1, \lambda_2 \in T$, which is equivalent to $\lambda_1 x^{i-j} - \lambda_2 = a' + b'x$ for some $a', b' \in G$. With the help of Lemma 9 we see that the line of direction x^0 and the line directed by x^1 only share one equivalence class of 0-singularities, that of 0. So we have that $L_0 \geq 11$. Comparing the equivalence classes between the lines directed by x^0 and x^2 we obtain:

$$\lambda_1 x^2 - \lambda_2 = a + bx \iff \lambda_1(x\sqrt{3} - 1) - \lambda_2 = a + bx \implies \begin{cases} \lambda_1 \in \frac{1}{\sqrt{3}}G \\ \lambda_2 \in -\lambda_1 + G \end{cases}.$$

It follows that the class of $\frac{1}{\sqrt{3}}$ and the class of $\frac{2}{\sqrt{3}}$ are in the same Γ orbit when considered in lines directed by x^i and x^{i-2} . From here we conclude that $L_0 \geq 14$ since we have the eleven 0-singularities in different orbits from the lines directed by x^0 and x^1 plus three more in the line directed by x^2 .

An equivalent computation to the one shown above, comparing the lines directed by x^3 and x^0 and by x^3 and x^1 shows that there are no more orbits of 0-singularities and gives a complete description. Indeed, it turns out that the line directed by x^3 has four orbits in common with the one directed by x^0 ; while it has three orbits in common with the line directed by x^1 . Figure 10 summarizes the the result: on each line of direction x^i there are six marked points and points in different lines with the same color are in the same orbit under the action of Γ . We conclude that $L_0 = 14$. \square

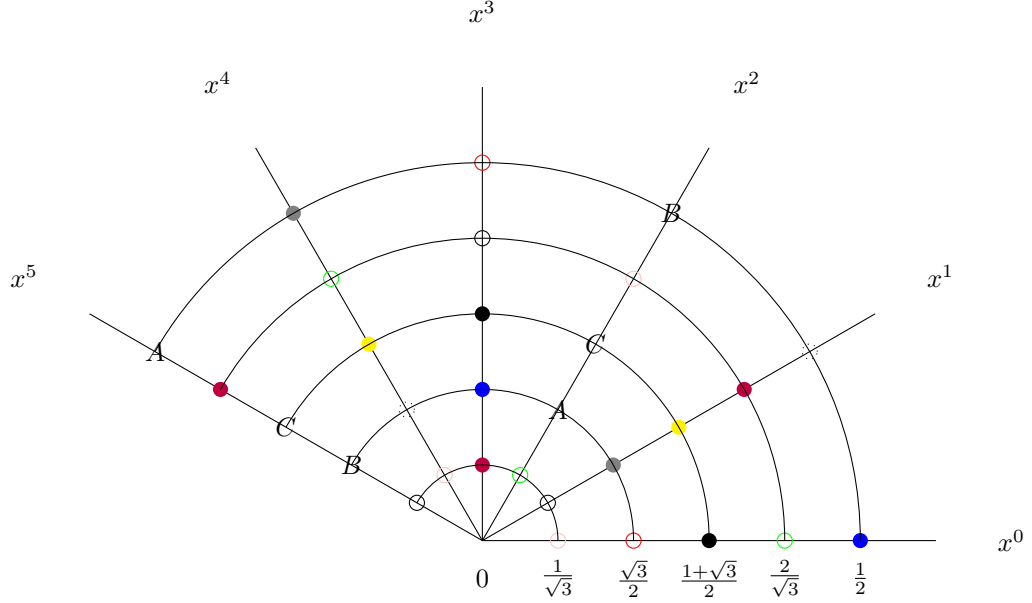


Figure 10: Orbits of points if $\gamma \in \Delta$

4.2 All cases with up to two lines per direction

In this section, we summarize the results of some of the simpler cases, namely all those with up to two 1-singularities per direction. For this, we need to determine all values of γ giving rise to these cases, and then determine for each of them the values of L_0 , L_1 , and L_0^α for all line orbits α . As $R = 3$ and coker β is free independently of γ , we have by Theorem 3:

$$H^2(\Omega_{E_{12}^\gamma}) = \mathbb{Z}^{L_1+e}, \quad H^1(\Omega_{E_{12}^\gamma}) = \mathbb{Z}^{1+L_1} \quad \text{and} \quad H^0(\Omega_{E_{12}^\gamma}) = \mathbb{Z},$$

where $e = -L_0 + \sum_{\alpha \in I_1} L_0^\alpha$.

For each value of γ involved, we have determined the combinatorics of line intersections with the help of a computer program. In order to be able to connect the number of orbits L_0^α of points on the individual lines α to the total number of point orbits L_0 , we have split the numbers L_0^α as $L_0^\alpha = \sum_p L_{0,p}^\alpha$, where $L_{0,p}^\alpha$ is the number of those orbits whose points are intersections of exactly p 1-singularities. Correcting for double counting, we obtain $L_0 = \sum_{\alpha \in I_1} L_{0,p}^\alpha/p$.

These values are tabulated in each case. In these tables, we list $L_{0,p}$ for each *type of line*. For example, the symbol $L_{0,p}^1$ indicates lines with combinatorics of “type 1”, the symbol $L_{0,p}^2$ lines with combinatorics of “type 2” etc. The superscripts do not stand for directions or any other property of the lines, but they collect lines with the same type of combinatorics. The column n in the table indicates the number of lines of a certain type. The column *dir* specifies whether lines of this type have even or odd directions, or both. In two further

lines in the tables, labelled $\sum L_{0,p}^\alpha$ and $L_{0,p}$, we give the sum of these data over all line types, and the corresponding data for the total orbits. The last column contains the sum over all values of p .

Case 1. We start analyzing $\gamma_1, \gamma_2 \in G$, which is equivalent to $\gamma = (0, 0)$. This is the only case with just one line in each of the six directions. The explicit computations of L_0 and L_0^α in this case were carried out in Section 4.1. The intersection combinatorics in the notation introduced in this section are given in the following table:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	6	e, o	3	2	0	0	1	6
$\sum L_{0,p}^\alpha$	6		18	12	0	0	6	36
$L_{0,p}$			9	4	0	0	1	14

We see that there is only one line type, containing one orbit with six-fold intersections, two orbits with three-fold intersections, and three orbits with two-fold intersections. These combinatorics should be compared with Figure 10: all lines, with even or odd direction, have the same combinatorics. There is precisely one orbit point, with representative the origin, which is a six-fold intersection. In each of the lines in Figure 10 there are 3 symbols that appear exactly two times in the figure, those are the orbits of two-fold intersections; and there are two symbols that appear exactly three times in the figure, those are the orbits of three-fold intersections. Putting everything together, we get the following end result:

Proposition 32 (cf. Theorem 4.3). *The ranks of the cohomology groups of the 12-fold tiling, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}})$	$\text{rk } H^1(\Omega_{E_{12}})$	$\text{rk } H^0(\Omega_{E_{12}})$
36	14	6	22	28	7	1

Case 2. We now determine the intersection combinatorics under the assumption that there is only one line in each even direction and two lines per odd direction. From the proof of Proposition 27 (see particularly Figure 9) we see that this situation can arise in two ways.

- The first possibility is $\gamma_1 \in G$ and $\gamma_2 \in \frac{1}{2}G \setminus G$. All γ -values in this set give the same intersection combinatorics, which are given in the following table:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	3	e	10	0	0	2	0	12
$L_{0,p}^2$	3	o	4	5	0	1	0	10
$L_{0,p}^3$	3	o	2	4	0	2	0	8
$\sum L_{0,p}^\alpha$	9		48	27	0	15	0	90
$L_{0,p}$			24	9	0	3	0	36

We see here that we have two line types in each odd direction, with different intersection combinatorics, but only one line type in even directions. Putting these data together, we obtain:

Proposition 33. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1 \in G$ and $\gamma_2 \in \frac{1}{2}G \setminus G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
90	36	9	54	63	10	1

- The second possibility occurs for $\gamma_1 \in \frac{1}{\sqrt{3}}G \setminus G$ and $\gamma_2 \in G$. For all γ -values in this set, we obtain:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	3	e	6	1	0	0	2	9
$L_{0,p}^2$	6	o	9	2	0	0	1	12
$\sum L_{0,p}^\alpha$	9		72	15	0	0	12	99
$L_{0,p}$			36	5	0	0	2	43

Here, for each odd direction, we have two lines of the same type. These two lines are related by symmetry - they are mirror images of each other. The cohomology of this case is slightly different:

Proposition 34. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1 \in \frac{1}{\sqrt{3}}G \setminus G$ and $\gamma_2 \in G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
99	43	9	56	65	10	1

Of course, there is also the possibility of one line per odd direction, and two lines per even direction, but this is completely analogous to the above two cases,

and we do not list them separately. This symmetry is reflected in Figure 8 by the symbolic green diagonal.

We now turn to the situation where we have two lines in each even and each odd direction.

Case 3. We start with the case where the two lines in even directions are mirror images of each other. There are then three subcases.

- In the first subcase, both γ_1 and γ_2 are contained in $\frac{1}{\sqrt{3}}G \setminus G$ (cf. proof of Proposition 27 and Figure 8), in which case also the lines in odd directions form mirror pairs. The corresponding γ -values then all lead to the same intersection combinatorics as follows:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	12	e, o	12	1	0	0	2	15
$\sum L_{0,p}^\alpha$	12		144	12	0	0	24	180
$L_{0,p}$			72	4	0	0	4	80

We see here that in this case, all even and odd lines are of the same type, and are in fact all symmetry equivalent. This intersection combinatorics results in the following cohomology:

Proposition 35. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1, \gamma_2 \in \frac{1}{\sqrt{3}}G \setminus G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
180	80	12	100	112	13	1

- In the second subcase, we have $\gamma_1 \in \frac{1}{\sqrt{3}}G \setminus G$, but now $\gamma_2 \notin \frac{1}{2\sqrt{3}}G$. More precisely, γ_2 depends on γ_1 , and must satisfy $2\gamma_1 + \sqrt{3}\gamma_2 \in G$, which corresponds to region E in Figure 9 (cf. proofs of Proposition 27 and Lemma 11). All γ -values satisfying the conditions in this second subcase lead to the following intersection combinatorics:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	6	e	12	0	0	3	0	15
$L_{0,p}^2$	6	o	10	3	0	2	0	15
$\sum L_{0,p}^\alpha$	12		132	18	0	30	0	180
$L_{0,p}$			66	6	0	6	0	78

Here, the lines in even and odd directions have different intersection combinatorics, but in each direction they form mirror pairs. This intersection combinatorics results in the following cohomology:

Proposition 36. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1 \in \frac{1}{\sqrt{3}}G \setminus G$, $\gamma_2 \notin \frac{1}{2\sqrt{3}}G$ and $2\gamma_1 + \sqrt{3}\gamma_2 \in G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
180	78	12	102	114	13	1

We note that this case also occurs with the roles of even and odd directions interchanged, which we do not list separately.

- In the third subcase, the two lines in an odd direction are not symmetry equivalent, which is the case if $\gamma_1 \in G$ and $\gamma_2 \in \frac{1}{2\sqrt{3}}G \setminus (\frac{1}{\sqrt{3}}G \cup \frac{1}{2}G)$ (again this characterization of the parameters comes from the proof of Proposition 27). For all these γ -values, we obtain:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	6	e	16	0	0	2	0	18
$L_{0,p}^2$	3	o	8	4	0	2	0	14
$L_{0,p}^3$	3	o	4	2	0	4	0	10
$\sum L_{0,p}^\alpha$	12		132	18	0	30	0	180
$L_{0,p}$			66	6	0	6	0	78

Here, we have two types of odd lines with different intersection combinatorics, but overall, this results in the same cohomology ranks as above:

Proposition 37. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1 \in G$ and $\gamma_2 \in \frac{1}{2\sqrt{3}}G \setminus (\frac{1}{\sqrt{3}}G \cup \frac{1}{2}G)$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
180	78	12	102	114	13	1

Of course, this latter case can also occur with the roles of even and odd directions interchanged, but we do not list it separately.

Case 4: Finally, we can have two line types in both even and odd directions, which occurs if both γ_1 and γ_2 are contained in $\frac{1}{2}G \setminus G$ (cf. proof of Proposition 27 and Figure 8). This is the first time in our analysis in which the G -action on the set of 0-singularities, that is points of intersection between representatives of lines, has more than one orbit. Indeed, the set of parameters $\frac{1}{2}G \setminus G$ splits in three G -cosets, so that altogether we need to analyze the behavior of γ -values (γ_1, γ_2) from nine different $(G \times G)$ -cosets. The analysis

of these 9 sets gives rise to only two different intersection combinatorics. The simpler combinatorics is obtained for the γ -values in the cosets with representatives $\{\frac{1}{2}(1, \sqrt{3}), \frac{1}{2}(\sqrt{3}, 1), \frac{1}{2}(1 + \sqrt{3}, 1 + \sqrt{3})\}$. These can be characterised as those cosets for which $\gamma_1 + \gamma_2 \in \frac{\sqrt{3}+1}{2}G$. The intersection combinatorics then is:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	12	e, o	6	4	0	0	2	12
$\sum L_{0,p}^\alpha$	12		72	48	0	0	24	144
$L_{0,p}$			36	16	0	0	4	56

Here, we have again a single type of lines, and the cohomology becomes:

Proposition 38. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1, \gamma_2 \in \frac{1}{2}G \setminus G$ and $\gamma_1 + \gamma_2 \in \frac{\sqrt{3}+1}{2}G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
144	56	12	88	100	13	1

For the remaining six cosets of γ -values, the combinatorics is:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	6	e, o	18	0	2	0	0	20
$L_{0,p}^2$	6	e, o	12	0	4	0	0	16
$\sum L_{0,p}^\alpha$	12		180	0	36	0	0	216
$L_{0,p}$			90	0	9	0	0	99

Here, we have two line types, each of which occurs both in even and odd directions. This results in the following cohomology:

Proposition 39. *The ranks of the cohomology groups of the generalized 12-fold tilings with parameter $\gamma = (\gamma_1, \gamma_2)$ satisfying $\gamma_1, \gamma_2 \in \frac{1}{2}G \setminus G$ and $\gamma_1 + \gamma_2 \notin \frac{\sqrt{3}+1}{2}G$, together with the quantities L_0^α, L_0, L_1 and e , are given by:*

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{12}^\gamma})$	$\text{rk } H^1(\Omega_{E_{12}^\gamma})$	$\text{rk } H^0(\Omega_{E_{12}^\gamma})$
216	99	12	117	129	13	1

4.3 General case: preliminary computations

We now turn our efforts into computing the numbers L_0 and L_0^α in terms of $\gamma = (\gamma_1, \gamma_2)$. The strategy is the same as the one described for the case $\gamma =$

(0,0): we start by intersecting a fixed line from the 24 in Corollary 26 with all the Γ -translates of the other 20 non-parallel lines. The computation and description of these intersections is the whole content of this subsection. The results obtained, which the reader might want to look at first, are collected and stated in the next section, concretely in Proposition 42.

Our first observation is that each family $(1, x^i)$ with $i > 0$ can be a basis of F^\perp (see Figure 2). For future reference, we collect in the following table the expressions of the powers of x in these basis (cf. Lemma 9):

	$(1, x^2)$	$(1, x^3)$	$(1, x^4)$	$(1, x^5)$
x	$\frac{1}{\sqrt{3}}(1 + x^2)$	$\frac{x^3}{2} + \frac{\sqrt{3}}{2}$	$\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}x^4$	$\sqrt{3} + x^5$
x^2		$\frac{1}{2} + x^2 \frac{\sqrt{3}}{2}$	$1 + x^4$	$2 + \sqrt{3}x^5$
x^3	$\frac{1}{\sqrt{3}}(2x^2 - 1)$		$\frac{\sqrt{3}}{3} + \frac{2}{\sqrt{3}}x^4$	$\sqrt{3} + 2x^5$
x^4	$x^2 - 1$	$-\frac{1}{2} + x^3 \frac{\sqrt{3}}{2}$		$1 + \sqrt{3}x^5$
x^5	$\frac{1}{\sqrt{3}}(x^2 - 2)$	$\frac{x^3}{2} - \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3} + \frac{1}{\sqrt{3}}x^4$	

Definition 40. In order to simplify the following results we define the sets $A_3 = \{0, \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\}$ and $A_4 = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\}$. The notation $x + A_3$ stands for the three values $x + y$ with $y \in A_3$.

We consider two generic lines given in Corollary 26 of directions x^i, x^k with $k > i$. There are three possible choices and we will deal with them separately: i, k are odd, i, k are even or i, k have different parity.

4.3.1 Two even lines

We pick two lines directed by x^i and compute their intersection with the translates by Γ of the four different lines directed by x^k when i, k are both even:

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^k \\ x^{k+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{k+1} \right) + \mu x^k + \sum n_p x^p.$$

We simplify by x^i and obtain

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x + \lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^{k-i} \\ x^{k+2-i} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{k+1-i} \right) + \mu x^{k-i} + \sum n_p x^{p-i}. \quad (10)$$

Recall that, by Lemma 9, the number $\sum n_p x^{p-i}$ is equal to $ax + b$ with $a, b \in G = \mathbb{Z}[\sqrt{3}]$. We consider the basis $(1, x^{k-i})$ for F^\perp and we compute the value of λ using the array at the beginning of the section. Notice that because of our choice of basis we can compute directly λ without worrying about μ . There are two subcases to consider:

- if $k - i = 2$, then subbing the information in (10) we obtain, considering only the first coordinate in the basis $(1, x^2)$:

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \sqrt{3}/3 + \lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \left(-\frac{\sqrt{3}}{3}\right) \right) + a \frac{\sqrt{3}}{3} + b,$$

which implies

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} - \frac{\gamma_2}{3} \right) + a \frac{\sqrt{3}}{3} + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \frac{\gamma_2}{3}.$$

The 8 families of values for λ , each corresponding to the intersection of one of the two lines directed by x^i with one of the four lines directed by x^k with $k - i = 2$, are collected in the following accolades. On the left hand side we have the intersection with the line passing through the point $\frac{\gamma_1}{\sqrt{3}}x^i + \frac{\gamma_2}{\sqrt{3}}x^{i+1}$ and on the right hand side passing through the point $\frac{\gamma_1}{\sqrt{3}}x^{i+2} + \frac{\gamma_2}{\sqrt{3}}x^{i+1}$.

$$\lambda = \begin{cases} -\frac{2\gamma_2}{3} + \frac{a\sqrt{3}}{3} - \frac{\gamma_1}{\sqrt{3}} \\ -\frac{2\gamma_2}{3} + \frac{a\sqrt{3}}{3} - \frac{2\gamma_1}{\sqrt{3}} \\ \frac{a\sqrt{3}}{3} - \frac{\gamma_1}{\sqrt{3}} \\ \frac{a\sqrt{3}}{3} \end{cases} \quad \text{and} \quad \lambda = \begin{cases} -\frac{2\gamma_2}{3} + \frac{a\sqrt{3}}{3} \\ -\frac{2\gamma_2}{3} + \frac{a\sqrt{3}}{3} - \frac{\gamma_1}{\sqrt{3}} \\ \frac{a\sqrt{3}}{3} \\ \frac{a\sqrt{3}}{3} + \frac{\gamma_1}{\sqrt{3}} \end{cases}$$

The final step is to consider the above values modulo G . Since $a, b \in G$, we can get rid of b in all of the above expressions and moreover, expressing a as $a_1 + a_2\sqrt{3}$ where $a_1, a_2 \in \mathbb{Z}$ we see that modulo G the numbers $\frac{a\sqrt{3}}{3}$ split in three different equivalence classes with representatives in $\{0, \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\}$. We conclude that the orbits of points on the two lines

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i$$

obtained by intersecting them with the 4 lines directed by x^{i+2} are determined by the following two sets of 12 values of λ .

- Intersection with $\frac{\gamma_1}{\sqrt{3}}x^i + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

$$\boxed{-\frac{\gamma_1}{\sqrt{3}} - \frac{2\gamma_2}{3} + A_3} \quad \boxed{-\frac{2\gamma_2}{3} - \frac{2\gamma_1}{\sqrt{3}} + A_3} \quad \boxed{-\frac{\gamma_1}{\sqrt{3}} + A_3} \quad \boxed{A_3} \quad (11)$$

- Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+2} + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

$$\boxed{-\frac{2\gamma_2}{3} + A_3} \quad \boxed{-\frac{2\gamma_2}{3} - \frac{\gamma_1}{\sqrt{3}} + A_3} \quad \boxed{A_3} \quad \boxed{\frac{\gamma_1}{\sqrt{3}} + A_3} \quad (12)$$

Remark 41. Before we analyze the next case, we remark that if instead of the two lines

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i,$$

we had considered the intersections with the two lines

$$- \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} \right) + \lambda x^i,$$

then we would have obtained the same values as in the above two tables with the opposite signs.

- if $k - i = 4$, we perform the same computations as in the preceding point. The details are left to the reader. The equations to consider, ignoring the parameter $b \in G$ and written in the basis $(1, x^4)$ are

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{2\gamma_2}{3} + \lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} - \frac{\gamma_2}{3} \right) + a \frac{2}{\sqrt{3}}.$$

These yield 4 different families of values for λ :

$$\lambda = \begin{cases} -\gamma_2 - \frac{\gamma_1}{\sqrt{3}} + \frac{2a}{\sqrt{3}} \\ -\gamma_2 - \frac{2\gamma_1}{\sqrt{3}} + \frac{2a}{\sqrt{3}} \\ -\frac{\gamma_2}{3} + \frac{2a}{\sqrt{3}} \\ -\frac{\gamma_2}{3} - \frac{\gamma_1}{\sqrt{3}} + \frac{2a}{\sqrt{3}} \end{cases}$$

Notice that in this case, the two parallel lines of direction x^i under consideration yield the same values of λ which, modulo de action of G , are collected in the following table:

$-\gamma_2 - \frac{\gamma_1}{\sqrt{3}} + A_3$	$-\gamma_2 - \frac{2\gamma_1}{\sqrt{3}} + A_3$	$-\frac{\gamma_2}{3} + A_3$	$-\frac{\gamma_2}{3} - \frac{\gamma_1}{\sqrt{3}} + A_3$
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(13)

4.3.2 Two odd lines

If i, k are both odd numbers then we obtain, as in the preceding case, the following equations from Corollary 26 (with $k > i$):

$$\pm \left(\gamma_2 \begin{Bmatrix} x^{k+4} \\ x^{k+6} \end{Bmatrix} + \gamma_1 x^{k+1} \right) + \mu \sqrt{3} x^k + \sqrt{3} \sum_p n_p x^p = \lambda \sqrt{3} x^i + \gamma_2 \begin{Bmatrix} x^{i+4} \\ x^{i+6} \end{Bmatrix} + \gamma_1 x^{i+1}.$$

We simplify, multiplying the equation by x^{-i} :

$$\pm \left(\gamma_2 \begin{Bmatrix} x^{k+4-i} \\ x^{k+6-i} \end{Bmatrix} + \gamma_1 x^{k+1-i} \right) + \mu \sqrt{3} x^{k-i} + \sqrt{3} \sum_p n_p x^{p-i} = \lambda \sqrt{3} + \gamma_2 \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} + \gamma_1 x,$$

and we substitute the expression $\sum_p n_p x^{p-i}$ by $ax + b$ using Lemma 9:

$$\pm \left(\gamma_2 \begin{Bmatrix} x^{k+4-i} \\ x^{k+6-i} \end{Bmatrix} + \gamma_1 x^{k+1-i} \right) + \mu \sqrt{3} x^{k-i} + \sqrt{3}(ax+b) = \lambda \sqrt{3} + \gamma_2 \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} + \gamma_1 x. \quad (14)$$

We now consider the above equation rewritten in the basis $(1, x^{k-i})$, which allows us to easily deduce the value of λ . To this end, we use the array at the beginning of the section. We split the analysis in two cases, depending on whether $k-i$ is 2 or 4. We deduce

- If $k-i = 2$, we use the basis $(1, x^2)$ and we further split the computations depending on the sign on the right hand side of (14):

– Considering Equation (14) with sign +:

$$\begin{aligned} \lambda \sqrt{3} + \gamma_2 \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} + \gamma_1 \frac{\sqrt{3}}{3} &= b\sqrt{3} + a - \frac{\gamma_1}{\sqrt{3}} + \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} && \iff \\ \lambda \sqrt{3} &= a + b\sqrt{3} - 2\gamma_1 \frac{\sqrt{3}}{3} + \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} - \gamma_2 \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} && \iff \\ \lambda \sqrt{3} &= a + b\sqrt{3} - 2\gamma_1 \frac{\sqrt{3}}{3} + \gamma_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} && \iff \\ \lambda &= b + \frac{a\sqrt{3}}{3} - \frac{2\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \end{aligned}$$

– Considering Equation (14) with sign -:

$$\begin{aligned} \lambda \sqrt{3} + \gamma_2 \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} + \gamma_1 \frac{\sqrt{3}}{3} &= b\sqrt{3} + a + \gamma_1 \frac{\sqrt{3}}{3} - \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} && \iff \\ \lambda \sqrt{3} &= b\sqrt{3} + a - \gamma_2 \begin{Bmatrix} -2 \\ -1 \end{Bmatrix} && \iff \\ \lambda &= b + a/\sqrt{3} - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} -2 \\ -1 \end{Bmatrix} \end{aligned}$$

Finally, we obtain the following 12 different values of λ modulo G . Remark that in this case the two parallel lines of direction x^i yield the same values of λ .

$$\boxed{\begin{array}{|c|c|c|c|} \hline \frac{-2\gamma_1}{3} + A_3 & -\frac{2\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3 & \frac{2\gamma_2}{\sqrt{3}} + A_3 & \frac{\gamma_2}{\sqrt{3}} + A_3 \\ \hline \end{array}} \quad (15)$$

- If $k - i = 4$, we use the basis $(1, x^4)$ in (14) and perform a computation analogous to the one in the preceding case:

$$\begin{aligned}
\lambda\sqrt{3} + \gamma_2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + \gamma_1 \frac{2}{\sqrt{3}} &= b\sqrt{3} + 2a \pm \left(-\gamma_1 \frac{\sqrt{3}}{3} + \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \right) && \iff \\
\lambda\sqrt{3} &= \begin{cases} b\sqrt{3} + 2a - \frac{\gamma_1}{\sqrt{3}} + \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} + \gamma_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \frac{2\gamma_1}{\sqrt{3}} \\ b\sqrt{3} + 2a + \frac{\gamma_1}{\sqrt{3}} - \gamma_2 \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} - \gamma_2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} - \frac{2\gamma_1}{\sqrt{3}} \end{cases} && \iff \\
\lambda = b + \frac{2a}{\sqrt{3}} - \gamma_1 + \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} &\text{ or } \lambda = b + \frac{2a}{\sqrt{3}} - \frac{\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}
\end{aligned}$$

We conclude that, for i odd, the orbits of points on the two lines

$$\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{i+4} \\ x^{i+6} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{i+1} + \lambda x^i$$

obtained by intersecting them with the 4 lines directed by x^{i+4} are determined by the following 2 sets of 12 values of λ .

- Intersection with $\frac{\gamma_2}{\sqrt{3}}x^{i+4} + \frac{\gamma_1}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

$$\boxed{-\gamma_1 + A_3 \quad -\frac{\gamma_1}{3} + A_3 \quad -\gamma_1 - \frac{\gamma_2}{\sqrt{3}} + A_3 \quad -\frac{\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3} \quad (16)$$

- Intersection with $\frac{\gamma_2}{\sqrt{3}}x^{i+6} + \frac{\gamma_1}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

$$\boxed{-\gamma_1 + A_3 \quad -\gamma_1 + \frac{\gamma_2}{\sqrt{3}} + A_3 \quad -\frac{\gamma_1}{3} + \frac{2\gamma_2}{\sqrt{3}} + A_3 \quad -\frac{\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3} \quad (17)$$

4.3.3 Two lines of different parity, k odd

If i, k are of different parity and k is odd, then we use Corollary 26 to understand the intersection of two lines of direction x^i with the translates of 4 lines of direction x^k . We obtain

$$\pm \left(\gamma_2 \begin{Bmatrix} x^{k+4} \\ x^{k+6} \end{Bmatrix} + \gamma_1 x^{k+1} \right) + \mu\sqrt{3}x^k + \sqrt{3} \sum_p n_p x^p = \lambda\sqrt{3}x^i + \gamma_1 \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \gamma_2 x^{i+1}. \quad (18)$$

We isolate λ and divide the whole expression by x^i ,

$$\lambda\sqrt{3} = \pm \left(\gamma_2 \begin{Bmatrix} x^{k-i+4} \\ x^{k-i+6} \end{Bmatrix} + \gamma_1 x^{k+1-i} \right) + \mu\sqrt{3}x^{k-i} + \sqrt{3} \sum_p n_p x^{p-i} - \gamma_1 \begin{Bmatrix} x^0 \\ x^2 \end{Bmatrix} - \gamma_2 x.$$

We use Lemma 9 to substitute the expression $\sum_p n_p x^{p-i}$:

$$\lambda\sqrt{3} = \pm \left(\gamma_2 \begin{Bmatrix} x^{k-i+4} \\ x^{k-i+6} \end{Bmatrix} + \gamma_1 x^{k+1-i} \right) + \mu\sqrt{3}x^{k-i} + \sqrt{3}(ax+b) - \gamma_1 \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} - \gamma_2 x.$$

We finally divide by $\sqrt{3}$ and obtain the general equation we will be dealing with in the case of i, k of different parity, k odd:

$$\lambda = \pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{k-i+4} \\ x^{k-i+6} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{k+1-i} \right) + \mu x^{k-i} + ax + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} x. \quad (19)$$

The expression $k-i$ can take any of the following values $\{-3, -1, 1, 3, 5\}$. Notice that since $x^n = -x^{n+6}$, the cases -3 and 3 and -1 and 5 will yield the same sets of values of λ . Indeed, in Equation 19, when written with respect to the bases $(1, x^3)$ or $(1, x^{-3})$ the only change is an overall sign change in the first parenthesis, which leaves unchanged the set of possible values of λ . The same occurs when comparing the values $k-i \in \{-1, 5\}$ with respect to the bases $(1, x^{-1})$ or $(1, x^5)$. It follows that we will have a complete answer by studying the cases $k-i = -3, -1, 1$. To simplify the notation we will denote by $\Re[\cdot]$ the linear map that returns the first coordinate of an arbitrary expression on the basis $(1, x^{k-i})$.

- If $k-i = -3$, then Equation (19) yields, ignoring the monomial with coefficient μ ,

$$\lambda = \Re \left[\pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x \\ x^3 \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{-2} \right) + ax + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} x \right].$$

Now, ignoring the parameter b , since ultimately we are interested in λ up to the action of G , and using the chart at the beginning of this section we obtain:

$$\lambda = \pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} + \frac{\gamma_1}{2\sqrt{3}} \right) + \frac{a\sqrt{3}}{2} - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ \frac{1}{2} \end{Bmatrix} - \frac{\gamma_2}{2}$$

We conclude that, for i even, the orbits of points on the two lines

$$\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^i \\ x^{i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{i+1} + \lambda x^i,$$

obtained by intersecting them with the translates of the 4 lines directed by $x^k = x^{i-3}$ are determined by the following 2 sets of 16 values of λ modulo G . To compile the tables we used the fact that $\frac{a\sqrt{3}}{2}$ with $a \in G$ can be expressed as

$$a \frac{\sqrt{3}}{2} = (a_1 + a_2\sqrt{3}) \frac{\sqrt{3}}{2} = \frac{1}{2}(a_1\sqrt{3} + 3a_2) \text{ with } a_1, a_2 \in \mathbb{Z}.$$

It follows that modulo G , the expression $\frac{a\sqrt{3}}{2}$ is equivalent to an element in the set $\{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\}$.

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^i + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

$-\frac{\gamma_1}{2\sqrt{3}} + A_4$	$-\gamma_2 - \frac{\gamma_1\sqrt{3}}{2} + A_4$	$-\frac{\gamma_2}{2} - \frac{\gamma_1}{2\sqrt{3}} + A_4$	$-\frac{\gamma_2}{2} - \frac{\gamma_1\sqrt{3}}{2} + A_4$
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(20)

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+2} + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of λ :

A_4	$-\gamma_2 - \frac{\gamma_1}{\sqrt{3}} + A_4$	$-\frac{\gamma_2}{2} - \frac{\gamma_1}{\sqrt{3}} + A_4$	$-\frac{\gamma_2}{2} + A_4$
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(21)

- If $k - i = 1$ we follow the same steps as in the preceding case to obtain:

$$\lambda = \Re \left[\pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^3 \\ x^5 \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} \right) + \mu x^{-1} + ax + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} x \right],$$

which turns into

$$\lambda = \pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} \sqrt{3} \\ 0 \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} \right) + \sqrt{3}a - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} - \gamma_2.$$

Notice that in this case, modulo de action of G , there is only one orbit of values of λ for each of the 4 lines of direction x^{i-1} . The values of λ obtained are:

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^i + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of:

$$\lambda \in \{0, -\gamma_2, -2\gamma_2 - \frac{2\gamma_1}{\sqrt{3}}, -\gamma_2 - \frac{2\gamma_1}{\sqrt{3}}\}. \quad (22)$$

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+2} + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of:

$$\lambda \in \{-\frac{\gamma_1}{\sqrt{3}}, -\gamma_2 - \frac{2\gamma_1}{\sqrt{3}}, -2\gamma_2 - \sqrt{3}\gamma_1, -\gamma_2 - \sqrt{3}\gamma_1\}. \quad (23)$$

- If $k - i = 1$, we follow the same steps as in the preceding cases to obtain:

$$\lambda = \Re \left[\pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^5 \\ x^6 \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^2 \right) + \mu x^1 + ax + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} x \right],$$

which turns into

$$\lambda = \pm \left(\frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} -\sqrt{3} \\ 0 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} \right) + b - \frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix},$$

yielding,

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^i + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of:

$$\lambda \in \left\{ -\frac{2\gamma_1}{\sqrt{3}}, -\frac{2\gamma_1}{\sqrt{3}} - \gamma_2, 0, \gamma_2 \right\}. \quad (24)$$

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+2} + \frac{\gamma_2}{\sqrt{3}}x^{i+1} + \lambda x^i$ for the values of:

$$\lambda \in \left\{ 0, -\gamma_2, \frac{2\gamma_1}{\sqrt{3}}, \frac{2\gamma_1}{\sqrt{3}} + \gamma_2 \right\}. \quad (25)$$

4.3.4 Two lines of different parity, k even

The last case we need to analyze is k, i of different parity and k even. So, once again we fix two lines directed by x^i and we intersect them with the four different lines directed by x^k . That is, we are looking for the values of λ in the following equation:

$$\pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^k \\ x^{k+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{k+1} \right) + \mu x^k + \sum_p n_p x^p = \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^{i+4} \\ x^{i+6} \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^{i+1} + \lambda x^i,$$

which we divide by x^i to obtain:

$$\pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x^{k-i} \\ x^{k-i+2} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^{k-i+1} \right) + \mu x^{k-i} + \sum_p n_p x^p = \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} + \frac{\gamma_1}{\sqrt{3}} x^1 + \lambda. \quad (26)$$

This time the expression $k-i$ can take any of the following values $\{-5, -3, -1, 1, 3\}$, and just like in the case k odd, i even, the analysis will be complete if we study the cases $k-i = -3, -1, 1$.

- If $k-i = -3$, then working with the basis $(1, x^{-3})$ we have:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} -x^3 \\ -x^5 \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} x^4 \right) - \mu x^{-3} + \sum_p n_p x^p - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} x^1$$

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -\Re(x^5) \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} \Re(x^4) \right) + \Re(ax+b) - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} \Re(x^4) \\ -1 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} \Re(x)$$

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{Bmatrix} + \frac{\gamma_2}{2\sqrt{3}} \right) + a \frac{\sqrt{3}}{2} + b - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} -\frac{1}{2} \\ -1 \end{Bmatrix} - \frac{\gamma_1}{2}$$

yielding

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+4} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{Bmatrix} + \frac{\gamma_2}{2\sqrt{3}} \right) + a\frac{\sqrt{3}}{2} + b + \frac{\gamma_2}{2\sqrt{3}} - \frac{\gamma_1}{2}$$

which, modulo de action of Γ , can be summarized as:

$$\boxed{-\frac{\gamma_1}{2} + \frac{\gamma_2}{\sqrt{3}} + A_4 \quad \frac{\gamma_2}{\sqrt{3}} + A_4 \quad -\gamma_1 + A_4 \quad -\frac{\gamma_1}{2} + A_4} \quad (27)$$

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+6} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{Bmatrix} + \frac{\gamma_2}{2\sqrt{3}} \right) + a\frac{\sqrt{3}}{2} + b + \frac{\gamma_2}{\sqrt{3}} - \frac{\gamma_1}{2}$$

which, modulo de action of Γ , can be summarized as:

$$\boxed{-\gamma_1 + \frac{\gamma_2}{2\sqrt{3}} + A_4 \quad -\frac{\gamma_1}{2} + \frac{\gamma_2}{2\sqrt{3}} + A_4 \quad -\frac{\gamma_1}{2} + \frac{\gamma_2\sqrt{3}}{2} + A_4 \quad \frac{\gamma_2\sqrt{3}}{2} + A_4} \quad (28)$$

• If $k - i = -1$, then remark that $x^{-i} = -x^{6-i}$ and we obtain

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} -x^5 \\ x \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) - \mu x^5 + \sum_p n_p x^p - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} x^1$$

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \Re(x) \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) + \Re(ax + b) - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} \Re(x^4) \\ -1 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} \Re(x)$$

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \Re(x) \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) + a\Re(x) + b - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} \Re(x^4) \\ -1 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} \Re(x)$$

which turns into

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \sqrt{3} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) + a\sqrt{3} + b - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}}\sqrt{3}$$

yielding

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+4} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \sqrt{3} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) + a\sqrt{3} + b - \frac{\gamma_2}{\sqrt{3}} - \gamma_1,$$

that is

$$\lambda \in \left\{ -2\gamma_1 - \frac{2\gamma_2}{\sqrt{3}}, -\gamma_1, -\gamma_1 - \frac{2\gamma_2}{\sqrt{3}}, 0 \right\}. \quad (29)$$

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+6} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ \sqrt{3} \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} \right) + a\sqrt{3} + b + \frac{\gamma_2}{\sqrt{3}} - \gamma_1,$$

that is,

$$\lambda \in \left\{ \frac{2\gamma_2}{\sqrt{3}}, -\gamma_1 + \frac{2\gamma_2}{\sqrt{3}}, -\gamma_1, -2\gamma_1 \right\}. \quad (30)$$

• If $k - i = 1$, then working with the basis $(1, x)$ and Lemma 9 we have:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} x \\ x^3 \end{Bmatrix} + \frac{\gamma_2}{\sqrt{3}} x^2 \right) + \mu x + \sum_p n_p x^p - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} x^4 \\ x^6 \end{Bmatrix} - \frac{\gamma_1}{\sqrt{3}} x^1$$

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -\sqrt{3} \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} \right) - \frac{\gamma_2}{\sqrt{3}} \begin{Bmatrix} -2 \\ -1 \end{Bmatrix}$$

yielding

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+4} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -\sqrt{3} \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} \right) + \frac{2\gamma_2}{\sqrt{3}}$$

that is,

$$\lambda \in \left\{ -\gamma_1 + \frac{\gamma_2}{\sqrt{3}}, \frac{\gamma_2}{\sqrt{3}}, \gamma_2\sqrt{3}, \gamma_2\sqrt{3} + \gamma_1 \right\}. \quad (31)$$

– Intersection with $\frac{\gamma_1}{\sqrt{3}}x^{i+1} + \frac{\gamma_2}{\sqrt{3}}x^{i+6} + \lambda x^i$ for the values of:

$$\lambda = \pm \left(\frac{\gamma_1}{\sqrt{3}} \begin{Bmatrix} 0 \\ -\sqrt{3} \end{Bmatrix} - \frac{\gamma_2}{\sqrt{3}} \right) + \frac{\gamma_2}{\sqrt{3}}$$

that is,

$$\lambda \in \left\{ -\gamma_1, 0, \frac{2\gamma_2}{\sqrt{3}}, \frac{2\gamma_2}{\sqrt{3}} + \gamma_1 \right\}. \quad (32)$$

4.4 General case: results

In the previous section we have computed for each line of direction x^i all the intersection points with each of the Γ -translates of the other 20 lines of different directions. Moreover, we have collected these points in orbits under the Γ -action. The first step in order to compute the numbers L_0 and L_0^α , needed to determine $H^2(\Omega_{E_{12}^\gamma})$, is to organize the information gathered so far, paying special attention to repetitions in the above tables. This is done in Proposition 42.

We have not computed explicitly the values L_0 and L_0^α for all different parameters γ . The number of subcases is too large to be considered of interest by itself. We finish our study with Proposition 42, which gives an upper bound on L_0^α and moreover it describes explicitly the representatives of the 0-singularities on each 1-singularity in terms of the parameter γ .

Before we present the general picture in the next proposition, we show here the intersection combinatorics, obtained with computer assistance, for the case $\gamma_1 = 1/7 + \sqrt{3}/11$, $\gamma_2 = 1/13 + \sqrt{3}/17$. For technical reasons, our program requires $\gamma_i \in \mathbb{Q}(\sqrt{3})$, but we conjecture these values are representatives of the ‘generic case’. Remark that the parameter (γ_1, γ_2) is taken in the region in Figure 8 labelled with a red 24. The intersection combinatorics for this case are:

	n	dir	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	tot
$L_{0,p}^1$	24	e, o	42	0	2	0	0	44
$\sum L_{0,p}^\alpha$	24		1008	0	48	0	0	1056
$L_{0,p}$			504	0	12	0	0	516

We see that we have here a single line type occurring in both even and odd directions, and that the value of L_0^α really attains the maximal value of 44 admitted by Proposition 42. It is worth pointing out that in this generic case, not all line intersections are generic in the sense that only two lines may intersect in a single point (notice the entry in the table under $p = 4$). The reason is that the positions of the singular lines do not move independently of each other when the γ -values vary. In this generic case, we obtain the following cohomology:

$\sum L_0^\alpha$	L_0	L_1	e	$\text{rk } H^2(\Omega_{E_{\gamma_2}})$	$\text{rk } H^1(\Omega_{E_{\gamma_2}})$	$\text{rk } H^0(\Omega_{E_{\gamma_2}})$
1056	516	24	540	564	25	1

We conjecture this to be the maximal cohomology attained among all the generalized 12-fold tilings studied in this paper.

Turning now our attention back to the general case, we present in the next proposition our findings on the quantity L_0^α . The statement refers to the following explicit list of the 1-singularities α :

- (a) $\alpha = \pm \frac{1}{\sqrt{3}}(\gamma_1 x^i + \gamma_2 x^{i+1}) + \lambda x^i$, $\lambda \in \mathbb{R}$, $i \in \{0, 2, 4\}$.
- (b) $\alpha = \pm \frac{1}{\sqrt{3}}(\gamma_1 x^{i+2} + \gamma_2 x^{i+1}) + \lambda x^i$, $\lambda \in \mathbb{R}$, $i \in \{0, 2, 4\}$.
- (c) $\alpha = \pm \frac{1}{\sqrt{3}}(\gamma_1 x^{i+1} + \gamma_2 x^{i+4}) + \lambda x^i$, $\lambda \in \mathbb{R}$ $i \in \{1, 3, 5\}$.
- (d) $\alpha = \pm \frac{1}{\sqrt{3}}(\gamma_1 x^{i+1} + \gamma_2 x^{i+6}) + \lambda x^i$, $\lambda \in \mathbb{R}$ $i \in \{1, 3, 5\}$.

Proposition 42. *All the 1-singularities α have an associated value L_0^α bounded by 44. Moreover, the values of λ identifying representatives of points of intersection between α (in the above (a), (b), (c) and (d) cases) and the translates of the 20 non-parallel 1-singularities are collected (respectively) in Table 1, Table 2, Table 3 and Table 4. (The values in the tables are for α with a positive $\frac{1}{\sqrt{3}}$ factor; if the factor is negative, one should consider minus the entries in the tables.)*

Proof. The proof of this statement is nothing but organizing the information collected in the previous computations. We fix a direction x^i and we look at the 4 parallel lines given by Corollary 26.

1. If i is even, then to understand the 0-singularities on:

- (a) $\alpha = \frac{1}{\sqrt{3}}(\gamma_1 x^i + \gamma_2 x^{i+1}) + \lambda x^i$, we collect the values of λ from the sets 11 and 13 (for the intersections with lines directed by x^{i+2} and x^{i+4} respectively) and from the sets 20, 22 and 24 (for the intersections with lines directed by x^{i+3} , x^{i+5} and x^{i+1} respectively).

Notice that the value $-\frac{2\gamma_1}{\sqrt{3}} - \gamma_2$ appears in the sets 13, 22 and 24; the value 0 is in both sets 22 and 24. The collection of values with no repetitions is the content of Table 1. Depending on the parameter γ , some of the values in the table could be in the same orbit under the action of Γ . We conclude then that the total number of different values in the table provides an upper bound for L_0^α , which in this case is easily checked to be 44.

If $\alpha = -\frac{1}{\sqrt{3}}(\gamma_1 x^i + \gamma_2 x^{i+1}) + \lambda x^i$ we use Remark 41 to conclude that the values we are interested in are minus the ones displayed in Table 1.

- (b) $\alpha = \frac{1}{\sqrt{3}}(\gamma_1 x^{i+2} + \gamma_2 x^{i+1}) + \lambda x^i$, we proceed just as before extracting the values from the sets 12, 13, 21, 23 and 25. The summary of all the values, with one repetition, is presented in Table 2.

Again, if we consider $\alpha = -\frac{1}{\sqrt{3}}(\gamma_1 x^{i+2} + \gamma_2 x^{i+1}) + \lambda x^i$, by Remark 41, we obtain minus the values in Table 2. This time, the upper bound for L_0^α we obtain is 44.

2. If i is odd, we need to discuss the 0-singularities on:

- (c) $\alpha = \frac{1}{\sqrt{3}}(\gamma_2 x^{i+4} + \gamma_1 x^{i+1}) + \lambda x^i$, for which we extract the values of λ from the sets 15, 16, 27, 29 and 31. Notice that in these sets the value $-\gamma_1$ appears three times, $-\gamma_1 + \frac{\gamma_2}{\sqrt{3}}$ twice and $\frac{\gamma_2}{\sqrt{3}}$ also twice. The collection of all these values with only one repetition is the content of Table 3. The upper bound for L_0^α in this case is found to be 44. The case of $\alpha = -\frac{1}{\sqrt{3}}(\gamma_2 x^{i+4} + \gamma_1 x^{i+1}) + \lambda x^i$ is dealt with via Remark 41 as in the previous cases.

(d) $\alpha = \frac{1}{\sqrt{3}}(\gamma_2 x^{i+6} + \gamma_1 x^{i+1}) + \lambda x^i$ we use the values of λ in sets 15, 17, 28, 30, and 32. This time the value $-\gamma_1$ appears three times and the values $-\gamma_1 + \frac{\gamma_2}{\sqrt{3}} + A_3$, $-\gamma_1 + \frac{2\gamma_2}{\sqrt{3}}$ and $\frac{2\gamma_2}{\sqrt{3}}$ each appears twice. The complete set of values with no repetitions is collected in Table 4. This time the bound obtained is $L_0^\alpha \leq 44$. The case $\alpha = -\frac{1}{\sqrt{3}}(\gamma_2 x^{i+6} + \gamma_1 x^{i+1}) + \lambda x^i$ is dealt with via Remark 41.

□

Remark 43.

- If we replace γ by 0 in the tables, the reader can remark that we obtain $L_0^\alpha = 6$ as in Lemma 31.
- For all γ we have $L_0 \leq (44 + 44) \cdot 2 \cdot 3 \cdot 2$. Of course in the case $\gamma = 0$ the same upper bound gives $6 \cdot 6 = 36$, while the actual result is 14. Thus we can imagine that L_0 is always strictly less than this value.

$-\frac{\gamma_1}{\sqrt{3}} + A_3$	$-\frac{\gamma_1}{\sqrt{3}} - \frac{2\gamma_2}{\sqrt{3}} + A_3$	$-\frac{2\gamma_1}{\sqrt{3}} - \frac{2\gamma_2}{3} + A_3$	A_3
$-\frac{\gamma_1}{\sqrt{3}} - \gamma_2 + A_3$	$-\gamma_2 - \frac{2\gamma_1}{\sqrt{3}} + A_3$	$-\frac{\gamma_2}{3} + A_3$	$-\frac{\gamma_2}{3} - \frac{\gamma_1}{\sqrt{3}} + A_3$
$-\frac{\gamma_1}{2\sqrt{3}} + A_4$	$-\gamma_2 - \frac{\gamma_1\sqrt{3}}{2} + A_4$	$-\frac{\gamma_2}{2} - \frac{\gamma_1}{2\sqrt{3}} + A_4$	$-\frac{\gamma_2}{2} - \frac{\gamma_1\sqrt{3}}{2} + A_4$
0		$-2\gamma_2 - \frac{2\gamma_1}{\sqrt{3}}$	
$-\frac{2\gamma_1}{\sqrt{3}}$			γ_2

Table 1: Each value of λ is of the form $a\gamma_1 + b\gamma_2 + c$. The value $a\gamma_1 + b\gamma_2$ identifies the intersection point between the two 1-singularities, while $c \in A_3 \cup A_4$ encodes the different representatives of the equivalence classes under the action of G . The data on the first 2 lines is from the sets 11, 13 respectively while the last three lines have values from the sets 20, 22 and 24.

$\frac{\gamma_1}{\sqrt{3}} + A_3$	$-\frac{2\gamma_2}{3} + A_3$	$-\frac{2\gamma_2}{3} - \frac{\gamma_1}{\sqrt{3}} + A_3$	A_3
$-\frac{\gamma_1}{\sqrt{3}} - \frac{\gamma_2}{3} + A_3$	$-\frac{2\gamma_1}{\sqrt{3}} - \gamma_2 + A_3$	$-\frac{\gamma_2}{3} + A_3$	$-\frac{\gamma_1}{\sqrt{3}} - \gamma_2 + A_3$
A_4	$-\frac{\gamma_1}{\sqrt{3}} - \gamma_2 + A_4$	$-\frac{\gamma_1}{\sqrt{3}} - \frac{\gamma_2}{2} + A_4$	$-\frac{\gamma_2}{2} + A_4$
$-\frac{\gamma_1}{\sqrt{3}}$		$-2\gamma_2 - \sqrt{3}\gamma_1$	$-\gamma_2 - \sqrt{3}\gamma_1$
		$\frac{2\gamma_1}{\sqrt{3}}$	$\frac{2\gamma_1}{\sqrt{3}} + \gamma_2$

Table 2: Each value of λ is of the form $a\gamma_1 + b\gamma_2 + c$. The value $a\gamma_1 + b\gamma_2$ identifies the intersection point between the two 1-singularities, while $c \in A_3 \cup A_4$, encodes the different representatives of the equivalence classes under the action of G . There is one value repeated in the table, namely $-\frac{\gamma_1}{\sqrt{3}} - \gamma_2$. The data on the first 3 lines is from Tables 12, 13 and 21 respectively while the last line has values from tables 23 and 25.

$-\frac{2\gamma_1}{3} + A_3$	$-\frac{2\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3$	$\frac{\gamma_2}{\sqrt{3}} + A_3$	$\frac{2\gamma_2}{\sqrt{3}} + A_3$
$-\gamma_1 + A_3$	$-\frac{\gamma_1}{3} + A_3$	$-\gamma_1 - \frac{\gamma_2}{\sqrt{3}} + A_3$	$-\frac{\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3$
$-\frac{\gamma_1}{2} + \frac{\gamma_2}{\sqrt{3}} + A_4$	$\frac{\gamma_2}{\sqrt{3}} + A_4$	$-\gamma_1 + A_4$	$-\frac{\gamma_1}{2} + A_4$
$-2\gamma_1 - \frac{2\gamma_2}{\sqrt{3}}$		$-\gamma_1 - \frac{2\gamma_2}{\sqrt{3}}$	0
		$\gamma_2\sqrt{3}$	$\gamma_2\sqrt{3} + \gamma_1$

Table 3: Each value of λ is of the form $a\gamma_1 + b\gamma_2 + c$. The value $a\gamma_1 + b\gamma_2$ identifies the intersection point between the two 1-singularities, while $c \in A_3 \cup A_4$, encodes the different representatives of the equivalence classes under the action of G . The data on the first 2 lines is from the sets 15, 16 respectively while the last three lines have values from the sets 27, 29 and 31. Notice the repetition of the value $-\gamma_1$.

$-\frac{2\gamma_1}{3} + A_3$	$-\frac{2\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3$	$\frac{\gamma_2}{\sqrt{3}} + A_3$	$\frac{2\gamma_2}{\sqrt{3}} + A_3$
$-\gamma_1 + A_3$	$-\gamma_1 + \frac{\gamma_2}{\sqrt{3}} + A_3$	$-\frac{\gamma_1}{3} + \frac{2\gamma_2}{\sqrt{3}} + A_3$	$-\frac{\gamma_1}{3} + \frac{\gamma_2}{\sqrt{3}} + A_3$
$-\gamma_1 + \frac{\gamma_2}{2\sqrt{3}} + A_4$	$-\frac{\gamma_1}{2} + \frac{\gamma_2}{2\sqrt{3}} + A_4$	$-\frac{\gamma_1}{2} + \frac{\gamma_2\sqrt{3}}{2} + A_4$	$\frac{\gamma_2\sqrt{3}}{2} + A_4$
$\frac{2\gamma_2}{\sqrt{3}}$			$-2\gamma_1$
	0		$\gamma_1 + \frac{2\gamma_2}{\sqrt{3}}$

Table 4: Each value of λ is of the form $a\gamma_1 + b\gamma_2 + c$. The value $a\gamma_1 + b\gamma_2$ identifies the intersection point between the two 1-singularities, while $c \in A_3 \cup A_4$, encodes the different representatives of the equivalence classes under the action of G . The data on the first 2 lines is from the sets 15, 17 respectively while the last lines have values from the sets 28, 30 and 32.

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