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Density of Binary Disc Packings: The 9 Compact Packings

Nicolas Bédaride * Thomas Fernique †

Abstract

A disc packing in the plane is *compact* if its contact graph is a triangulation. There are 9 values of r such that a compact packing by discs of radii 1 and r exists. We prove, for each of these 9 values, that the maximal density over all the packings by discs of radii 1 and r is reached for a compact packing (we give it as well as its density).

1 Introduction

A *disc packing* (or *circle packing*) is a set of interior-disjoint discs in the Euclidean plane. Its *density* δ is the proportion of the plane covered by the discs:

$$\delta := \limsup_{k \rightarrow \infty} \frac{\text{area of the square } [-k, k]^2 \text{ inside the discs}}{\text{area of the square } [-k, k]^2}.$$

A central issue in packing theory is to find the maximal density of disc packings.

If the discs have all the same radius, it was proven in [FT43] that the density is maximal for the *hexagonal compact packings*, where discs are centered on a suitably scaled triangular grid (see also [CW10] for a short proof).

For *binary* disc packings, *i.e.*, packings by discs of radii 1 and $r \in (0, 1)$ where both disc sizes appear, there are only seven values of r for which the maximal density is known [Hep00, Hep03, Ken04]. These values are specific algebraic numbers which allow a *compact packing*, that is, a packing whose *contact graph* (the graph which connects the centers of any two tangent discs) is a triangulation. In each of these seven cases, the maximal density turns out to be reached for a compact disc packing. Compact packings thus seem to be good candidates to maximize the density.

It was proven in [Ken06] that there are exactly 9 values of r which allow a binary compact packing by discs of radii 1 and r : the seven above mentioned, and two other ones (r_5 and r_9 in Fig. 1). In this paper, we show that for these two values the maximal density is also reached for a compact packing. We actually provide a self-contained proof for all the 9 values:

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Theorem 1 For each r_i allowing a binary compact packing by discs of radius 1 and r_i , the density of any packing by discs of radius 1 and r_i is less than or equal to the density δ_i of the periodic compact packing \mathcal{P}_i depicted in Fig. 1.

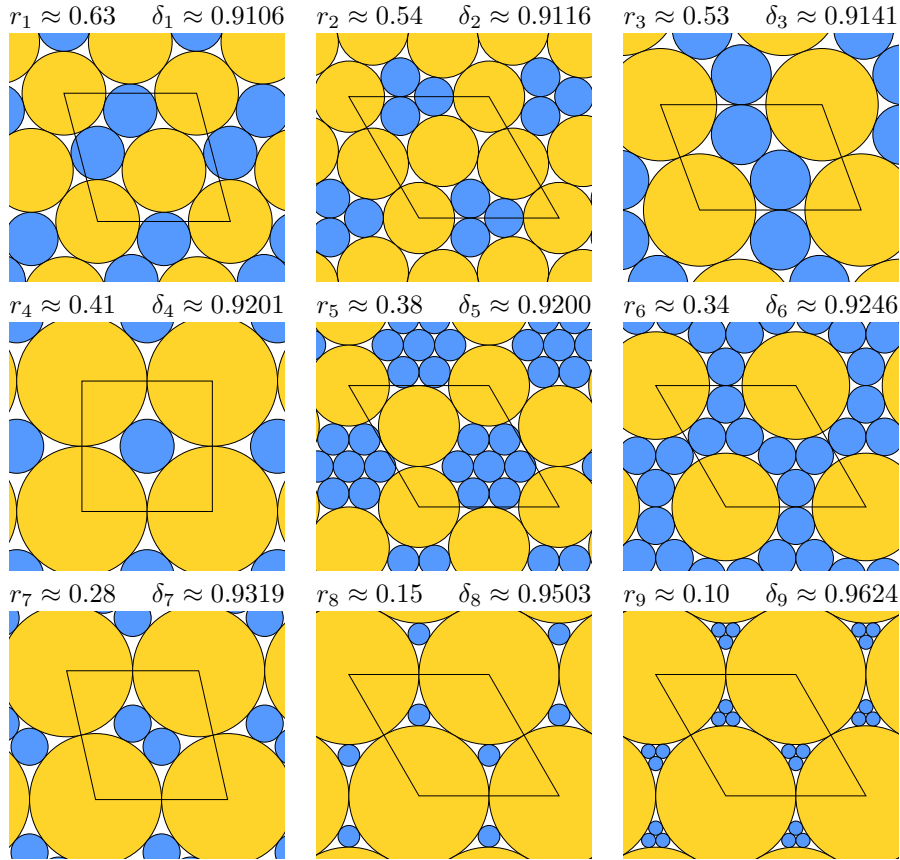


Figure 1: The packings $\mathcal{P}_1, \dots, \mathcal{P}_9$ which maximize the density. They are periodic, with the black parallelogram showing a fundamental domain (each picture is scaled so that its area is one quarter of the picture area). Numerical approximations by truncation of the ratio r_i of the disc radii and of the density δ_i of the packing are given (both r_i and δ_i/π are algebraic: their minimal polynomials are given in Appendix A).

Actually, we conjecture that this result extends for any number of disc radii, up to a "saturation hypothesis" which becomes necessary from three disc radii:

Conjecture 1 Consider the packings by n different given discs. If the densest compact packing is saturated, i.e., no further disc can be added, then the maximal density over all the packings is reached by a compact packing.

We do not expect the densest compact packing to be necessarily periodic, but we already know that it does when there are at most three different radii. Indeed, all the compact packings with three radii have been characterized [FHS] and the densest ones (among the compact packings) are all periodic. Note that only one of these compact packings has yet been proven to maximize the density among all the packings by discs with the three same radii [Fer]. Checking all the other cases could be the next step towards the above conjecture. However, all the mentioned proofs ([Hep00, Hep03, Ken04], Theorem 1 in this paper and [Fer]) rely on computations which involve the precise values of radii. Proving the result in its full generality (if it is true) would require a different approach. . .

2 Strategy

The strategy to prove Theorem 1 is inspired by the one in [Ken04]. Given a disc packing, we shall first decompose the plane by a specific triangulation \mathcal{T} of the disc centers (Sec. 4). Then, the *excess* $E(T)$ of a triangle $T \in \mathcal{T}$ is defined by

$$E(T) := \delta_i \times \text{area}(T) - \text{cov}(T),$$

where $\text{area}(T)$ is the area of T , $\text{cov}(T)$ is the area of T inside the discs centered on the vertices of T and δ_i is the density of the target packing (Fig. 1). A triangle with positive excess is thus less dense than the target density δ_i , and proving that the packing has overall density at most δ_i amounts to show

$$\sum_{T \in \mathcal{T}} E(T) \geq 0.$$

For this, we shall define over triangles a *potential* U which satisfies two inequalities. The first one, further referred to as *the global inequality*, involves all the triangles of \mathcal{T} :

$$\sum_{T \in \mathcal{T}} U(T) \geq 0. \tag{1}$$

The second one, further referred to as *the local inequality*, involves any triangle T which can appear in \mathcal{T} :

$$E(T) \geq U(T). \tag{2}$$

The result then trivially follows:

$$\sum_{T \in \mathcal{T}} E(T) \geq \sum_{T \in \mathcal{T}} U(T) \geq 0.$$

Since the global inequality for U is the same as for E , it seems we just made things worse by adding a second inequality. However, we shall choose U so that the global inequality is "not that global", *i.e.*, it follows from an inequality on a finite set of finite configurations. Namely, the potential of a triangle T will be the sum of *vertex potentials* $U_v(T)$ defined on each vertex $v \in T$ and *edge*

potentials $U_e(T)$ defined on each edge $e \in T$ such that, for any vertex v and edge e of any decomposition \mathcal{T} :

$$\sum_{T \in \mathcal{T} | v \in T} U_v(T) \geq 0 \tag{3}$$

$$\sum_{T \in \mathcal{T} | e \in T} U_e(T) \geq 0. \tag{4}$$

Inequality (3), which involves the triangles sharing a vertex, is proven in Section 5 (Prop. 1). Inequality (4), which involves pairs of triangles sharing an edge, is proven in Section 7 (Prop. 3).

The local inequality (2) has then to be proven for each triangle of the decomposition. We make two cases, depending whether the triangle is a so-called ε -tight triangle or not. The former case, considered in Section 8, is proven with elementary differential calculus (Prop. 4). The latter case, considered in Section 9, is proven with a computer by dichotomy (Prop. 5). Theorem 1 follows.

3 Computer use

Like for many results in packing theory (the most emblematic being probably the proof of Kepler's conjecture [Hal05]), the proof of Theorem 1 makes an important use of the computer.

The first use we will make of the computer is to check inequalities on real numbers. Since real numbers cannot be all exactly represented on a computer, we shall systemically use *interval arithmetic*. Let us briefly recall the basic principle (see, *e.g.*, [Tuc11] for a comprehensive introduction). Any real number is represented by an interval whose endpoints are representable floating-point numbers. For example, the constant π is represented by the software SageMath [Dev16] with its 53 bits of precision interval arithmetic by the closed interval

$$x = [3.14159265358979, 3.14159265358980].$$

Let us emphasize that the endpoints are not numerical approximation but real numbers representable on the computer. This is also the case for the values given in Tab. 2, 4, 5 and 6, which are obtained as lower or upper bounds of interval. Then, computations are performed so that "nothing gets lost", *i.e.*, the image by an n -ary real function f of intervals x_1, \dots, x_n must be an interval which contains at least all the real numbers $f(y_1, \dots, y_n)$ for $y_i \in x_i$. Exactly what this interval is depends on the actual software implementation of the function f . For example, for the above interval x , SageMath computes

$$\sin(x) = [-3.21624529935328 \times 10^{-16}, 1.22464679914736 \times 10^{-16}].$$

This computation does not prove that $\sin(\pi)$ is equal to zero, but it proves that it is quite small, *e.g.*, less than 10^{-15} . More generally, if a computation shows that the right endpoint of the interval which represents a real number A is less

than or equal to the left endpoint of the interval which represents a real number B , then it proves $A \leq B$. This is how we shall prove the inequalities (2), (3) and (4), using the interval arithmetic of the open-source software SageMath [Dev16].

The second use we will make of the computer is to check inequalities over intervals (namely products of intervals, in the proof of Prop. 5). We again use interval arithmetic for this, but instead of using an interval to represent an exact real number, we simply use the interval itself! In other words, to prove an inequality $f(x) \leq g(x)$ over an interval $[a, b]$, we compute $f(x)$ and $g(x)$ in interval arithmetic with $x = [a, b]$ and we check whether the right endpoints of the interval $f(x)$ is less than or equal to the left endpoints of the interval $g(x)$.

The last use we will make of the computer is to perform exhaustive search. This is done in a classic way in the proof of Prop. 1, where there is a finite number of cases to check, as well as in a less conventional way in the proof of Prop. 5, where we check the local inequality (2) on an infinite set of triangles. For that, we shall prove that the set of triangles is compact and break it down in finitely many sets which are sufficiently small so that interval arithmetic can be used (as explained above) to prove that *any* triangle in such a set satisfies the wanted inequality.

The complete code needed to verify all the results of this paper, less than 400 commented lines, can be found in the supplementary materials (`binary.sage`). The language is python, with some use of the open-source software SageMath [Dev16] (*e.g.* for interval arithmetic). Although quite slow, Python has the advantage of being easily readable and (currently) quite widespread. All the computations have been performed on our modest laptop, an Intel Core i5-7300U with 4 cores at 2.60GHz and 15,6 Go RAM.

4 Triangulation

Given a disc packing, define the *cell* of a disc as the set points of the plane which are closer to this disc than to any other (Fig. 2, left). These cells form a partition of the plane whose dual is a triangulation, referred to as the *FM-triangulation* of the packing (Fig. 2, right). Introduced in [FTM58] (see also [FT64]), FM-triangulations are also known as *additively weighted Delaunay triangulations*.

A triangle T which appears in the FM-triangulation of some disc packing is said to be *feasible*. It satisfies the two following properties [FTM58]:

1. There exists a disc, called the *support disc* of T , which is interior disjoint from the discs of the packing and tangent to each of the discs of T (it somehow extends the "empty disc property" of classic Delaunay triangulations). In particular, if a disc packing is *saturated*, *i.e.*, no further disc can be added, then any support disc is smaller than the smallest allowed disc (otherwise we could add it at the place of the support disc).
2. The disc sector delimited by any two edges of T never crosses the third edge of T (which may be false in classic Delaunay triangulations when the ratio of disc radius is greater than $\sqrt{2} - 1$).

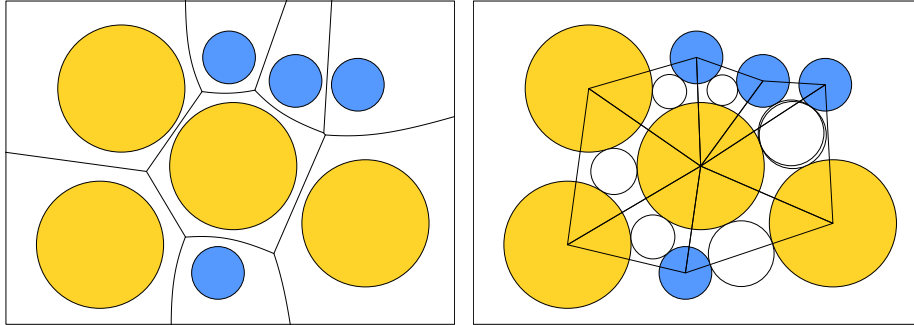


Figure 2: Some discs and their cells (left). The corresponding FM-triangulation with the support discs in white (right). Each support disc is centered on a vertex of the cell partition. It is tangent to the discs centered on the vertices of the triangle which corresponds in the dual to its center. Support discs may overlap but they do not overlap any disc of the packing.

A simple consequence we shall later rely on is that angles cannot be too small:

Lemma 1 *Let T be a triangle in a FM-triangulation of a saturated packing by discs of radius 1 and r . Let A , B and C denotes its vertices and x , y and z denotes the radii of the discs centered on these vertices. Then*

$$\sin \hat{A} \geq \min \left(\frac{y}{x + 2r + y}, \frac{z}{x + 2r + z} \right).$$

Proof. Assume that the edge AB is shorter than AC . On the one hand, the altitude of B is at least y because otherwise the disc sector defined by edges BA and BC would cross the edge AC . On the other hand, the length of the edge AB is at most $x + 2r + y$ because we can connect both A and B to the center of the support disc, which has radius at most r since the packing is saturated. This yields $\sin \hat{A} \geq \frac{y}{x + 2r + y}$. The same holds exchanging y and z if AC is shorter than AB , whence the claimed lower bound. \square

Following [Hep03], we call *tight* a triangle whose discs are mutually adjacent (Fig. 3). In particular, the FM-triangulation of any compact disc packing contains only tight triangles since there is a support disc in the hole between each three mutually adjacent discs.

Still following [Hep03], we call *stretched* a triangle with a small disc tangent to both the two other discs as well as to the line which passes through their centers (Fig. 4). They are not feasible in a saturated packing because their support disc have radius r (this would allow to add a small disc), but they can be arbitrarily approached near by feasible triangles. We shall see in Section 7 that stretched triangles are dangerous because they can be as dense as tight triangles. Indeed, two stretched triangles adjacent along their "stretched edge" (the edge tangent to one of the disc) can be recombined into two tight triangles by flipping this stretched edge.

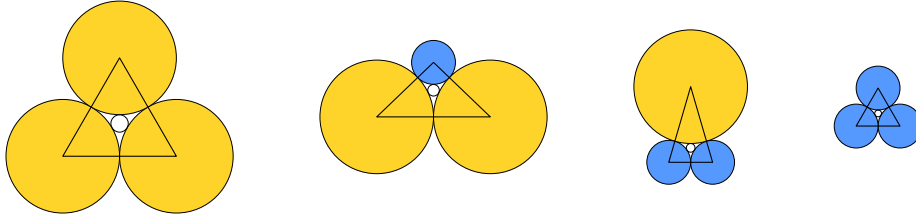


Figure 3: The four types of tight triangles, with their support disc (in white).

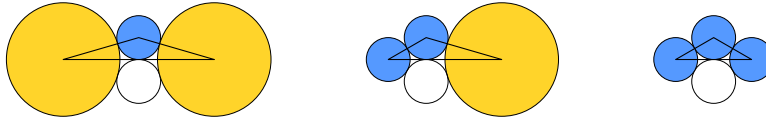


Figure 4: The three types of stretched triangles, with their support discs.

5 Global inequality for the vertex potential

We shall here define the vertex potential and show it satisfies Inequality (3).

5.1 Two constraints

Consider an FM-triangulation \mathcal{T} of a target packing. Summing over the triangles of \mathcal{T} yields

$$\sum_{T \in \mathcal{T}} \underbrace{E(T) - U(T)}_{\geq 0 \text{ by (2)}} = \underbrace{\sum_{T \in \mathcal{T}} E(T)}_{= 0 \text{ by definition of } E} - \underbrace{\sum_{T \in \mathcal{T}} U(T)}_{\geq 0 \text{ by (1)}}.$$

The three sums are thus equal to zero. In particular, one has $U(T) = E(T)$ for each type of tight triangle which appears in the target packing. Further, since the edge potential will be zero on any tight triangle T (Sec. 7), one has

$$\sum_{v \in T} U_v(T) = U(T).$$

Hence, summing over the vertices of \mathcal{T} yields:

$$\underbrace{\sum_{v \in \mathcal{T}} \sum_{T \in \mathcal{T} | v \in T} U_v(T)}_{\geq 0 \text{ by (3)}} = \sum_{T \in \mathcal{T}} \sum_{v \in T} U_v(T) = \sum_{T \in \mathcal{T}} U(T) = 0.$$

The inequality (3) must thus be an equality for any vertex of a target packing.

5.2 Vertex potential in tight triangles

For the sake of simplicity, we search for a vertex potential which depends only on the radii of the disc in the vertex and the radii of the two other discs in the

triangle. We denote by V_{abc} (or V_{cba}) the potential in the center of the disc of radius b in a tight triangle with discs of radius a , b and c . There are thus 6 quantities to be defined:

$$V_{111}, \quad V_{rrr}, \quad V_{r1r}, \quad V_{1rr}, \quad V_{1r1}, \quad V_{11r}.$$

The first of the two constraints seen Sec. 5.1 is $U(T) = E(T)$ for any tight triangle T of a target packing. For the sake of simplicity, we want $U(T) = E(T)$ for *any* tight triangle T . This yields four equations on the V_{abc} 's:

$$3V_{111} = E_{111}, \quad 3V_{rrr} = E_{rrr}, \quad V_{r1r} + 2V_{1rr} = E_{1rr}, \quad V_{1r1} + 2V_{11r} = E_{11r},$$

where E_{abc} is the excess of a tight triangle with discs of radius a , b and c . The second of the two constraints seen Sec. 5.1, namely equality in Inequality (3), yields an equation for each vertex of the target packings. Remarkably, there is only one equation for each radius of disc in each target packing, except for \mathcal{P}_5 where a small disc can be surrounded in two different ways. This latter case is a bit specific and will be dealt with in Sec. 5.4. Tab. 1 lists these equations.

i	Small disc	Large disc
1	$4V_{1r1} + 2V_{1rr}$	$6V_{11r} + V_{r1r}$
2	$V_{rrr} + 2V_{1r1} + V_{1rr}$	$4V_{11r} + 2V_{111} + V_{r1r}$
3	$4V_{1rr} + V_{1r1}$	$4V_{r1r} + 4V_{11r}$
4	$4V_{1r1}$	$8V_{11r}$
6	$V_{rrr} + 4V_{1rr}$	$12V_{r1r}$
7	$2V_{1rr} + 2V_{1r1}$	$8V_{11r} + 2V_{r1r}$
8	$3V_{1r1}$	$12V_{11r}$
9	$V_{rrr} + 2V_{1rr} + V_{1r1}$	$12V_{11r} + 6V_{r1r}$

Table 1: Quantities that must be zero to have equality in Inequality (3) around the center of a small or a large disc in the target packing.

We thus have six equations for each target packings (one in each of the four tight triangles and one around each of the two discs). They are actually not independent because the sum of the excess of tight triangles over the fundamental domain of each target packing is equal to zero. There is thus still one degree of freedom. We arbitrarily set $V_{r1r} := 0$, except for \mathcal{P}_6 where we set $V_{1r1} := 0$ because $V_{r1r} = 0$ is already enforced around a large disc. One checks that these 6 equations are independent and thus characterize the V_{abc} 's.

5.3 Vertex potential in any triangle

We are now in a position to define the vertex potential in any triangle. The idea is to modify the potential of a tight triangle depending on how much the triangle itself is deformed. Given a triangle T in a FM-triangulation of a disc packing, we denote by T^* the tight triangle obtained in contracting the edges until three discs become mutually tangent (such a triangle is always defined because if r_a ,

r_b and r_c are the radii of the discs, then the edge lengths are $r_a + r_b$, $r_b + r_c$ and $r_a + r_c$ and each of these length is greater than the sum of the two other ones).

Definition 1 Let v be a vertex of a triangle T . Let q be the radius of the disc of center v and x and y the radii of the two other discs of T . The vertex potential $U_v(T)$ of v is defined by

$$U_v(T) := V_{xqy} + m_q |\widehat{v}(T) - \widehat{v}(T^*)|,$$

where $m_q \geq 0$ depends only on q , and $\widehat{v}(T)$ and $\widehat{v}(T^*)$ denote the angle in v in T and T^* .

In particular, $U_v(T^*) = V_{xqy}$. The constant m_q controls the "deviation" in term of the angle changes between T and T^* . The point is to fix it so that the inequality (3) holds:

Proposition 1 Let $i \neq 5$ and v be a vertex of an FM-triangulation of a saturated packing by discs of radius 1 and r_i . Then, the sum of the vertex potentials of the triangles containing v is nonnegative provided that m_1 and m_r are bounded from below by the values given in Tab. 2.

i	m_1	m_r
1	0	0.0005
2	0.16	0.087
3	0	0.00028
4	0	0.0021
6	0.0091	0.0021
7	0	0.0011
8	0	0.002
9	0	0.002058

Table 2: Lower bounds on m_1 and m_r which ensure the vertex inequality (3) for any packing by discs of radius 1 and r_i .

Proof. Let v be a vertex of an FM-triangulation \mathcal{T} of a saturated packing by discs of radius 1 and r . Let q denote the radius of the disc of center v . Let T_1, \dots, T_k be the triangles of \mathcal{T} which contain v , ordered clockwise around v . We have:

$$\begin{aligned} \sum_{j=1}^k U_v(T_j) &= \sum_{j=1}^k U_v(T_j^*) + m_q \sum_{j=1}^k |\widehat{v}(T_j) - \widehat{v}(T_j^*)| \\ &\geq \sum_{j=1}^k U_v(T_j^*) + m_q \left| \sum_{j=1}^k \widehat{v}(T_j) - \sum_{j=1}^k \widehat{v}(T_j^*) \right|. \end{aligned}$$

Since the T_j 's surround v , $\sum_j \widehat{v}(T_j) = 2\pi$. If the coefficient of m_q is nonzero, then the inequality (3) is thus satisfied in v as soon as

$$m_q \geq -\frac{\sum_j U(T_j^*)}{\left|2\pi - \sum_j \widehat{v}(T_j^*)\right|}.$$

This lower bound depends only on the radii and order of the discs centered on the neighbors of v . There is only finitely many cases for each value of k , and the lower bounds on angles of Lemma 1 ensure that there is finitely many values of k (the largest one is $k = 80$, reached for $i = 9$ when there are only small discs around a large one¹). We can thus perform an exhaustive search on a computer to find a lower bound which holds for any v . We performed this exhaustive search (function `smallest_m` in `binary.sage`) using interval arithmetic as explained in Section 3: the lower bound on m_q given in Tab. 2 are the right endpoints of the computed intervals. To conclude, we also have to consider the case where m_q has a zero coefficient. This happens when the sum of the angles $\widehat{v}(T_j^*)$ is equal to 2π . We check this during the previous exhaustive search: if the computation yields for the coefficient of m_q an interval which contains zero, then we check whether $\sum_j U(T_j^*) \geq 0$. The computation shows that this always holds, except when v is surrounded in the same way as in the target packing. In this latter case, we get an interval which contains zero: this is the way we defined the vertex potential in tight triangles (namely, to satisfy the equations in Tab. 1) which ensures that the exact value is zero. \square

5.4 The case \mathcal{P}_5

We cannot proceed exactly the same way for \mathcal{P}_5 , because equality in Inequality (3) around the small disc surrounded by six other small discs would yield $V_{rrr} = 0$, which is incompatible with $3V_{rrr} = E_{rrr} > 0$. The potential in a vertex of a triangle with three small discs will depend on its neighborhood.

In an FM-triangulation of a packing by discs of radius 1 and $r = r_5$, let us call *singular* a small disc which is surrounded by two large discs and three small ones (in this order up to a cyclic permutation). The other discs are said to be *regular*. In particular, each regular small disc in the target packing \mathcal{P}_5 is surrounded by 6 singular discs (Fig. 1). We shall rely on the following simple lemma (proven in Fig. 5):

Lemma 2 *In an FM-triangulation of a packing by discs of radius 1 and r , there is at most two singular discs in a triangle with three discs of radius r .*

¹We can actually reduce further the number of cases to consider by bounding from below the angle of a triangle depending on the discs of this triangles. It is however only useful to speed up the search, because the cases that give the lower bound on m_q correspond to rather small values of k .

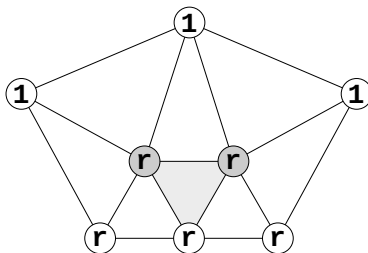


Figure 5: Consider a triangle rrr (shaded). If two vertices (shaded) are singular, then the third one has at least four neighbors of radius r , thus is regular.

In a tight triangle with small discs, we denote respectively by V'_{rrr} and V_{rrr} the potentials of singular and regular vertices. We set

$$V'_{rrr} := \frac{1}{2}E_{rrr}.$$

The above lemma ensures that the sum over the singular vertices of the triangle is at most E_{rrr} . The remaining potential (to sum up to E_{rrr} on the triangle) is shared equally among the $k \geq 1$ regular vertices. The value of V_{rrr} thus depends on the number of singular vertices in the triangle: it can be 0 , $\frac{1}{4}E_{rrr}$ or $\frac{1}{3}E_{rrr}$. In particular, it is always nonnegative. While checking Inequality (3) around regular vertices, we shall only assume $V_{rrr} \geq 0$. We can further proceed as for the other cases to define the V_{abc} 's, with V'_{rrr} instead of V_{rrr} and considering only singular vertices. Tab. 3 completes Tab. 1.

i	Small disc (singular)	Large disc
5	$V_{1r1} + 2V_{1rr} + 2V'_{rrr}$	$6V_{11r} + 3V_{r1r}$

Table 3: Quantities that must be zero to have equality in Inequality (3) around the center of a small or a large disc in in \mathcal{P}_5 .

We can then extend vertex potentials beyond tight triangles exactly as in Definition 1, since the regular or singular character of a small disc is defined for any triangle. Tab. 4 completes Tab. 2 to extend Proposition 1, which is proven in the same way, with the only difference being that in the exhaustive search through possible configurations around a vertex v , we simply use $V_{rrr} \geq 0$ if v is not singular (since knowing only the neighbors of v not always suffice to determine which of them are singular or regular).

i	m_1	m_r
5	0	0.048

Table 4: Lower bounds on m_1 and m_r which ensure the vertex inequality (3) for any packing by discs of radius 1 and r_5 .

6 Capping the potential

In Subsec. 5.2, we fixed the vertex potentials in order to have $E(T) = U(T)$ on the tight triangles. We then introduced, in Subsec. 5.3, a deviation controlled by the quantities m_1 and m_r to have Ineq. (3) around each vertex of any FM-triangulation of a saturated packing. More precisely, we found lower bounds on m_1 and m_r : any largest values would only make this latter inequality even more true. However, we shall keep in mind that we also have to eventually satisfy the local inequality (2), *i.e.*, $U(T) \leq E(T)$ for any triangle. With this in mind, it is best to fix m_1 and m_r as small as possible so as to minimize U . We can actually make U even smaller as follows.

Proposition 2 *Assume that, for any tight triangle T^* with a disc of radius q in v , one has*

$$m_q \geq -\frac{U_v(T^*)}{\widehat{v}(T^*)}. \quad (5)$$

Then the vertex inequality (3) still holds if we cap the vertex potential $U_v(T)$ by

$$z_q := -2\pi \min \frac{U_v(T^*)}{\widehat{v}(T^*)},$$

where the minimum is over the tight triangles T^ with a disc of radius q in v .*

Proof. Note that $z_q > 0$ since at least one of the $U_v(T^*)$'s is negative in order to have equality in Ineq. (3) for the target packing. We shall show the following lower bound on the vertex potential *per radian*:

$$\frac{U_v(T)}{\widehat{v}(T)} \geq \min \left(0, \frac{U_v(T^*)}{\widehat{v}(T^*)} \right).$$

The definition of z_q then ensures that, as soon as a vertex potential $U_v(T)$ is larger than z_q , the vertex potentials of all the other triangles sharing v are "not enough negative to be dangerous", namely, the sum of all the vertex potentials around v is nonnegative, that is, Ineq. (3) holds.

Let us prove the claimed lower bound per radian. It is trivial if $U_v(T^*) \geq 0$. Assume $U_v(T^*) \leq 0$. If $\widehat{v}(T) \geq \widehat{v}(T^*)$, then

$$\frac{U_v(T)}{\widehat{v}(T)} = \frac{U_v(T^*) + m_q(\widehat{v}(T) - \widehat{v}(T^*))}{\widehat{v}(T)} \geq \frac{U_v(T^*)}{\widehat{v}(T)} \geq \frac{U_v(T^*)}{\widehat{v}(T^*)}.$$

If $\widehat{v}(T) \leq \widehat{v}(T^*)$, then

$$\frac{U_v(T)}{\widehat{v}(T)} = \frac{U_v(T^*) + m_q(\widehat{v}(T^*) - \widehat{v}(T))}{\widehat{v}(T)} = \frac{U_v(T^*) + m_q\widehat{v}(T^*)}{\widehat{v}(T)} - 1.$$

Since $U_v(T^*) + m_q\widehat{v}(T^*) \geq 0$ by Ineq. (5), this is decreasing function of $\widehat{v}(T)$. It is thus bounded from below by its value for $\widehat{v}(T) = \widehat{v}(T^*)$, *i.e.*, $U_v(T^*)/\widehat{v}(T^*)$. \square

A computation shows that the values of m_r given in Tab. 2 and 4 satisfy the condition (5). This also holds for m_1 in cases c_2 and c_6 , but not in the other cases because m_1 is equal to zero. However, the value required to satisfy the condition (5) is quite small: a computation shows that 10^{-14} suffices for all the cases. We thus *modify* m_1 when its value in Tab. 2 and 4 is zero and increase it to 10^{-14} .

Since only the negative V_{abc} 's play a role in the definition of z_q , the value of V_{rrr} in the case \mathcal{P}_5 , which can range from 0 to $\frac{1}{2}E_{rrr}$ has no importance. The values Z_1 and Z_r listed in Tab. 5 are representable floating-point numbers which bound from above z_1 and z_r .

i	Z_1	Z_r
1	7.5×10^{-15}	0.00023
2	0.011	0.0046
3	1.8×10^{-14}	0.00025
4	5.0×10^{-15}	0.00096
5	1.3×10^{-14}	0.0076
6	0.0013	0.0016
7	9.2×10^{-15}	0.0011
8	8.1×10^{-15}	0.0012
9	2.333×10^{-14}	0.0008033

Table 5: Values of Z_1 and Z_r used to cap the vertex potentials.

7 Global inequality for the edge potential

A few randomized trials suggest that the vertex potential satisfy the local inequality (2) for triangles which are not too far from tight triangles. It however fails near stretched triangles, because the excess can become quite small. The typical situation is depicted in Fig. 6. The edge potential aims to fix this problem. The idea is that when a triangle T becomes stretched, its support disc overlaps an adjacent triangle T^* , imposing a void in T^* which increases $E(T^*)$ and may counterbalance the decrease of $E(T)$. We shall come back to this in Section 9. Here, we define the edge potential and prove that it satisfies Inequality (4).

Definition 2 *Let e be an edge of a triangle T . Let x and y be the radii of the discs centered on the endpoints of e . Denote by $d_e(T)$ the signed distance of the center X of the support disc of T to the edge e , which is positive if T and X are both on the same side of e , or negative otherwise. The edge potential $U_e(T)$ of e is defined by*

$$U_e(T) := \begin{cases} 0 & \text{if the edge } e \text{ is shorter than } l_{xy}, \\ q_{xy} \times d_e(T) & \text{otherwise,} \end{cases}$$

where $l_{xy} \geq 0$ and $q_{xy} \geq 0$ depend only on x and y .

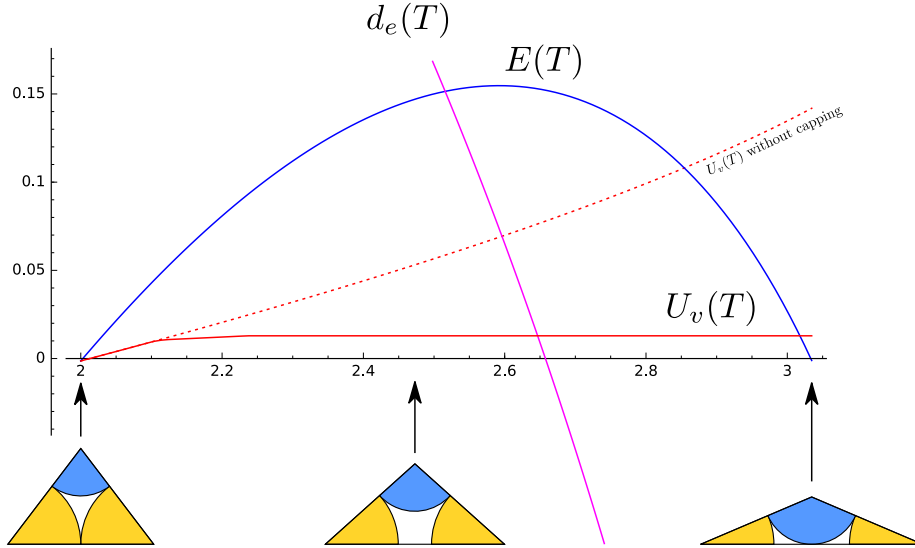


Figure 6: Starting from a tight triangle with two large disc and a small one (bottom left), the edge e between the two large discs is elongated until we get a stretched triangle (bottom right). The corresponding variations of the excess E , vertex potential U_v and signed distance $d_e(T)$ are depicted (top). For quasi-stretched triangle, the local inequality $E(T) \geq U_v(T)$ fails.

The constant l_{xy} is the threshold below which d_e has an effect and the coefficient q_{xy} controls the intensity of this effect. In contrast to the role of m_q for the vertex potential to satisfy Inequality (3) (Prop. 1), the values of l_e and q_e (and even the disc radii) do not play a role for the edge potential to satisfy Inequality (4):

Proposition 3 *If e is an edge of an FM-triangulation of a disc packing, then the sum of the edge potentials of the two triangles containing e is nonnegative.*

Proof. Consider an edge e shared by two triangles T and T^* of an FM-triangulation. We claim that $d_e(T) + d_e(T^*) \geq 0$. If each triangle and the center of its support disc are on the same side of e , then it holds because both $d_e(T)$ and $d_e(T^*)$ are nonnegative. Assume $d_e(T) \leq 0$, i.e., T and the center of its support disc are on either side of e . Denote by A and B the endpoints of e and by a and b the radii of the discs of center A and B (Fig. 7). The centers of the discs tangent to both discs of center A and B and radius a and b are the points M such that $\overline{AM} - a = \overline{BM} - b$, i.e., a branch of a hyperbola of foci A and B . This includes the centers X and X' of the support discs of T and T^* . In order to be tangent to the third disc of T^* , the support disc of T^* must have a center X' farther than X from the focal axis. Since the distances of X' and X to this axis are $-d_e(T)$ and $d_e(T^*)$, this indeed yields $d_e(T) + d_e(T^*) \geq 0$. This proves $U_e(T) + U_e(T^*) \geq 0$ if e has length at least l_{ab} . If e is shorter, both

$U_\varepsilon(T)$ and $U_\varepsilon(T^*)$ are zero and their sum is thus nonnegative. \square

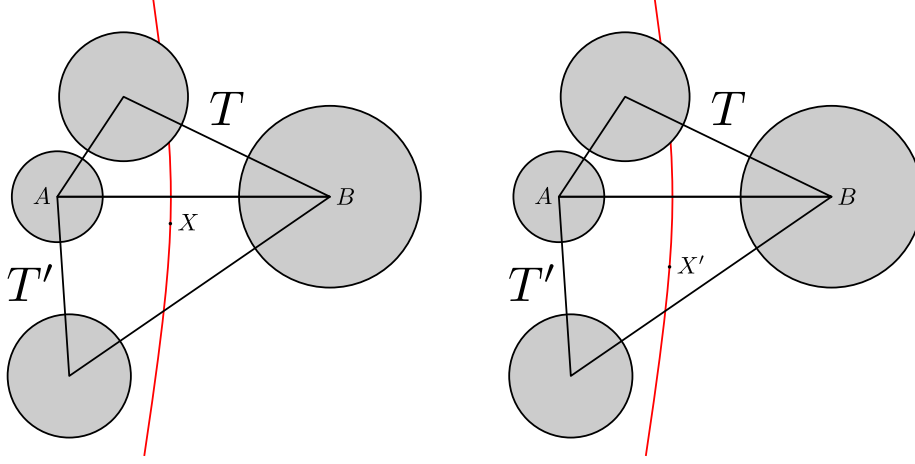


Figure 7: Comparatives positions of the centers of the support discs of two adjacent triangles.

8 Local inequality for ε -tight triangles

We prove the local inequality (2) in a neighborhood of tight triangles. A triangle is said to be ε -tight if its discs are pairwise at distance at most ε . Let T^* be a tight triangle with edge length x_1 , x_2 and x_3 and denote by \mathcal{T}_ε the set of ε -tight triangles with the same disc radii as T^* . On the one hand, the variation ΔE of the excess E between T^* and any triangle in \mathcal{T}_ε satisfies

$$\Delta E \geq \sum_{1 \leq i \leq 3} \min_{\mathcal{T}_\varepsilon} \frac{\partial E}{\partial x_i} \Delta x_i.$$

On the other hand, assuming that $x+y+\varepsilon$ is smaller than the smallest threshold l_{xy} below which the edge potential is zero (so that the potential U is simply the vertex potential, see Def. 1), the variation ΔU of the potential U between T^* and any triangle in \mathcal{T}_ε satisfies

$$\Delta U \leq \sum_{1 \leq i \leq 3} \max_{\mathcal{T}_\varepsilon} \frac{\partial U}{\partial x_i} \Delta x_i.$$

Since $E(T^*) = U(T^*)$ because of the way we define the vertex potential (first constraint, Subsec. 5.1), the local inequality $E(T) \geq U(T)$ holds over \mathcal{T}_ε for any ε such that

$$\min_{\mathcal{T}_\varepsilon} \frac{\partial E}{\partial x_i} \geq \max_{\mathcal{T}_\varepsilon} \frac{\partial U}{\partial x_i}.$$

We computed the formulas of the derivatives of E and U with SageMath². We then use interval arithmetic, once again, to compute the extremal values over \mathcal{T}_ε : each variable x_i is replaced by the interval $[r_j + r_k, r_j + r_k + \varepsilon]$, where r_j and r_k denote the radii of the discs centered on the endpoints of the edge of length x_i . A computation yields:

Proposition 4 *Let $i \in \{1, \dots, 9\}$. Take for m_1 and m_r the lower bound given in Tab. 2 or 4. Take for Z_1 and Z_r the values given in Tab. 5. Then, the local inequality $E(T) \geq U(T)$ holds for any ε -tight triangle of an FM-triangulation of a saturated packing by discs of radius 1 and r_i provided that ε satisfies the upper bound given in Tab. 6.*

i	ε
1	0.079
2	0.020
3	0.061
4	0.039
5	0.012
6	0.027
7	0.018
8	0.0049
9	0.001718

Table 6: Upper bounds on ε which ensure Inequality (2) for ε -tight triangles.

9 Local inequality for all the triangles

We explicitly define an edge potential such that the local inequality (2) holds for any triangle. Since the global inequality (1) result from Prop. 1 and 3, this will prove Theorem 1. The following lemma will be used to eliminate many non-feasible triangles:

Lemma 3 *If a triangle appears in an FM-triangulation of a saturated packing by discs of radius 1 and r , then*

- *the radius of its support disc is less than r ;*
- *its area is at least $\frac{1}{2}\pi r^2$;*
- *the altitude of any vertex is at least r .*

Proof.

²It can be easily do by hand since it mainly amounts to use the cosine theorem to express the angle of a triangle as a function of its edge length, but we are not particularly interested in the formulas.

- if the support disc has radius r or more, then we can add a small disc in the packing, in contradiction with the saturation hypothesis;
- the sectors defined by triangle edges of the discs centered in the triangle vertices are included in the triangle and their total area is at least half the area of a small disc;
- if the altitude of a vertex A is less than r , then the sector of the disc centered in A crosses the line going through the two opposite vertices B and C; it cannot cross it between B and C, but then the nearest vertex to A, say B, has a smaller altitude than A and the sector of the disc centered in B crosses the segment AC: contradiction (Fig. 8).

□

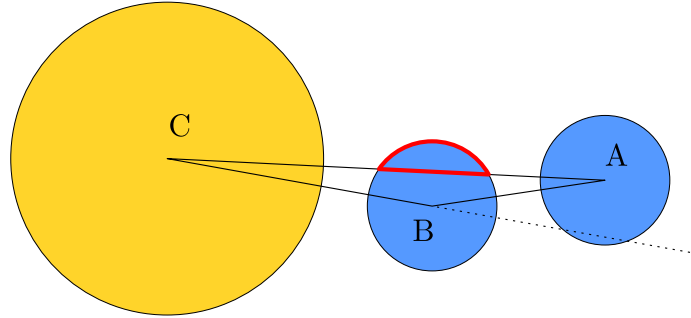


Figure 8: A triangle with a vertex of altitude less than r cannot be feasible.

Proposition 5 *Let $i \in \{1, \dots, 9\}$. Take for m_1 and m_r the lower bound given in Tab. 2 or 4. Take for Z_1 and Z_r the values given in Tab. 5. Take for ε the value given in Tab. 6. Take for l_{xy} and q_{xy} the values given in Tab. 7. Then, the local inequality $E(T) \geq U(T)$ holds for any triangle of an FM-triangulation of a saturated packing by discs of radius 1 and r_i .*

Proof. We shall check the inequality over all the possible triangles with the computer. For $x \leq y \leq z$ in $\{1, r\}$, any triangle with discs of radius x , y and z which appear in an FM-triangulation of a saturated packing has edge length in the compact set

$$[x + y, x + y + 2r] \times [x + z, x + z + 2r] \times [y + z, y + z + 2r].$$

Indeed, its support disc has radius at most r (saturation hypothesis) so that the center of a disc of radius q is at distance at most $q + r$ from the center of the support disc. We can thus compute $E(T)$ and $U(T)$ using these intervals for the edge lengths of T .

i	l_{11}	q_{11}	l_{1r}	q_{1r}	l_{rr}	q_{rr}
1	2.7	0.1	2.3	0.1	1.9	0.1
2	2.6	0.2	2.1	0.2	1.6	0.2
3	2.6	0.1	2.2	0.2	1.65	0.1
4	2.5	0.15	1.8	0.2	1.2	0.2
5	2.4	0.05	1.8	0.05	1.1	0.07
6	2.5	0.2	1.75	0.2	1.0	0.2
7	2.4	0.08	1.6	0.05	0.8	0.1
8	2.24	0.02	1.33	0.015	0.44	0.02
9	2.17	0.02	1.21787	0.015	0.285729	0.02

Table 7: Values l_{xy} and q_{xy} which define the edge potentials (see Appendix C for an explanation on how these values have been chosen).

Of course, since these intervals are quite large, we get for $E(T)$ and $U(T)$ large overlapping intervals which do not allow to conclude whether $E(T) \geq U(T)$ or not. We use dichotomy: while the intervals are too large to conclude, we halve them and check recursively on each of the 2^3 resulting compacts whether $E(T) \geq U(T)$ or not. If we get $E(T) \geq U(T)$ at some step, we stop the recursion. If we get $E(T) < U(T)$ at some step, we throw an error: the local inequality is not satisfied!

At each step, we also check whether Lemma 3 ensures that the triangle is not feasible, in which case we eliminate it and stop the recursion (the way we compute the radius of the support disc is detailed in Appendix B). Last, we also stop the recursion if we get an ε -tight triangle at some step, that is, if we get a subset of the compact

$$[x + y, x + y + \varepsilon] \times [x + z, x + z + \varepsilon] \times [y + z, y + z + \varepsilon].$$

Indeed, the local inequality is then already ensured by Prop. 4. This point is crucial and explains why we focused on ε -tight triangles in Section 8. Since $E(T) = U(T)$ for any tight triangle T , any non-empty interior compact which contains the point $(x + y, x + z, y + z)$, no matter how small it is, yields for $E(T)$ and $U(T)$ overlapping intervals which do not allow to decide whether $E(T) > U(T)$ or not: the recursion would last forever!

For each i , the whole process terminates without throwing any error. On our computer with our non-compiled python implementation, checking all the 9 cases take around 4 h 10 min (of which 3 h 40 min for $i = 9$ and 25 min for $i = 8$). Tab. 8 gives some statistics on the number of checked triangles. This proves the proposition. \square

A Radii and densities

Tab. 9 and 10 give minimal polynomials of the radii r_i 's and the reduced densities δ_i/π . The polynomials for the radii come from [Ken06]. The reduced densities

i	111	11r	1rr	rrr
1	1940	5587	7477	6000
2	5398	14757	20028	13336
3	1709	6574	5804	4880
4	1289	9738	15450	5041
5	1177	26741	65367	36758
6	1282	13644	27707	8891
7	785	24270	66760	5146
8	232	177542	1919744	14701
9	92	535837	19069730	19622

Table 8: Number of triangles of each type on which the local inequality (2) had to be checked. More than 86% of them are 1rr-triangles for $i = 9$: this is by far the hardest case.

are computed as follows. Consider the fundamental domain depicted in Fig. 1:

- compute the total area covered by discs divided by π : it is an algebraic number since the radii are algebraic;
- compute the area of the tight triangles in the fundamental domain, *e.g.*, using the Heron's formula: it is algebraic since so are the radii – hence the edge length;
- the quotient of both is thus an algebraic number whose minimal polynomial is the one given in Tab. 10.

1. $x^4 - 10x^2 - 8x + 9$,
2. $x^8 - 8x^7 - 44x^6 - 232x^5 - 482x^4 - 24x^3 + 388x^2 - 120x + 9$,
3. $8x^3 + 3x^2 - 2x - 1$,
4. $x^2 + 2x - 1$,
5. $9x^4 - 12x^3 - 26x^2 - 12x + 9$,
6. $x^4 - 28x^3 - 10x^2 + 4x + 1$,
7. $2x^2 + 3x - 1$,
8. $3x^2 + 6x - 1$,
9. $x^2 - 10x + 1$.

Table 9: Minimal polynomial of the radius r_i .

B Support disc

Consider an FM-triangle with sides of length a , b and c . Denote by α (resp. β and γ) the vertex opposite to the edge of length a (resp. b and c). Denote

1. $27x^4 + 112x^3 + 62x^2 + 72x - 29$,
2. $x^8 - 4590x^6 - 82440x^5 + 486999x^4 - 1938708x^3 + 2158839x^2 - 1312200x + 243081$,
3. $1024x^3 - 692x^2 + 448x - 97$,
4. $2x^2 - 4x + 1$
5. $944784x^4 - 3919104x^3 - 2191320x^2 - 1632960x + 757681$,
6. $144x^4 + 9216x^3 + 133224x^2 - 127104x + 25633$,
7. $4096x^4 + 2924x^2 - 289$,
8. $108x^2 + 288x - 97$,
9. $144x^4 - 4162200x^2 + 390625$.

Table 10: Minimal polynomial of the reduced density δ_i/π .

by r_a (resp. r_b and r_c) the radius of the disc of center α (resp. β and γ). We here explain how to get a formula that allows to compute the radius R of the support disc (which exists and is unique for an FM-triangle), even when edge lengths or disc radii are intervals.

Fix a Cartesian coordinate system with $\alpha = (0, 0)$ and $\beta = (c, 0)$. Denote by (u, v) the coordinates of γ , with $v > 0$. One has:

$$u = \frac{b^2 + c^2 - a^2}{2c} \quad \text{and} \quad v = \sqrt{b^2 - u^2}.$$

Denote by (x, y) the coordinates of the center of the support disc and by R its radius. The definition of the support disc yields three equations:

$$\begin{aligned} (R + r_a)^2 &= x^2 + y^2, \\ (R + r_b)^2 &= (b - x)^2 + y^2, \\ (R + r_c)^2 &= (u - x)^2 + (v - y)^2. \end{aligned}$$

Subtracting the second equation from the first yields an expression in R for x . Then, subtracting the third equation from the first and replacing x by its expression yields an expression in R for y . Last, replacing x and y by their expressions in the third equations yields a quadratic equation $AR^2 + BR + C = 0$, where A , B and C are complicated but explicit polynomials in a , b , c , r_a , r_b and r_c (they appear in the code of the function `radius` in `binary.sage`). The root we are interested in is the smallest positive one because we want the support disc to be interior-disjoint from the three discs³. Given exact values a , b , c , r_a , r_b and r_c , hence exact values of A , B and C , one can easily compute R :

$$R = \begin{cases} \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} & \text{if } A \neq 0, \\ -\frac{C}{B} & \text{if } A = 0. \end{cases}$$

³One can check that the discriminant of this quadratic polynomial is the product of the square of the area of the triangle and the terms $2(x - r_y - r_z)$ for each permutation (x, y, z) of (a, b, c) , hence always non-negative.

But what if A is an interval which contains zero (and not reduced to $\{0\}$)? The second above formula cannot be used, while the division by A in the first one yields $R = (-\infty, \infty)$. To get around that, we make two cases, depending whether the interval AC contains zero or not.

If AC does not contain zero, we simply use the first formula. More precisely, a short case study shows that the smallest positive root is

$$R = \frac{-B - \text{sign}(C)\sqrt{B^2 - 4AC}}{2A},$$

where $\text{sign}(C)$ denotes the sign of C , which is well defined since the interval C does not contain zero.

If AC contains zero, we shall use the fact that the roots of a polynomial are continuous in its coefficients. Namely, when A tends towards 0, one of the roots goes to infinity while the interesting one goes towards $-\frac{C}{B}$. This latter root is

$$R = \frac{-B + B\sqrt{1 - \frac{4AC}{B^2}}}{2A} = -\frac{C}{B} \frac{2B^2}{4AC} \left(1 - \sqrt{1 - \frac{4AC}{B^2}}\right).$$

With $x = \frac{4AC}{B^2}$ and $f(x) = \frac{2}{x}(1 - \sqrt{1-x})$, this can be written

$$R = -\frac{C}{B} f(x).$$

If we set $f(0) := 1$, then f becomes continuously derivable over $(-\infty, 1)$. The Taylor's theorem then ensures that for any real number $x < 1$, there exists a real number ξ between 0 and x such that

$$f(x) = 1 + xf'(\xi).$$

One checks that f' is positive and increasing over $(-\infty, 1)$. One computes

$$f'(0.78) \approx 0.9879 < 1.$$

If x is an interval which contains 0 and whose upper bound is at most 0.78, then

$$f(x) \subset 1 + x \times f'((-\infty, 0.78]) \subset 1 + x \times [0, 1] = 1 + x.$$

This yields the wanted interval around $-\frac{C}{B}$:

$$R \subset -\frac{C}{B} \left(1 + \frac{4AC}{B^2}\right).$$

The above formula still yields $R = (-\infty, \infty)$ if B contains 0 as well as A . Moreover, we assumed $x < 0.78$, which can be false if A , B and C are large interval: in such a case we have to use the first formula which also yields $R = (-\infty, \infty)$. Both cases however happens only when the intervals a , b and c have a quite large diameter, that is, in the very few first steps of the recursive local inequality checking.

C Parameters of the edge potential

Let us briefly explain how we chose the constants l_{xy} and q_{xy} in Prop. 5. The rule of thumb (which could perhaps be made rigorous) is that if the local inequality works for the triangles with only one pair of discs which are not tangent, then it seems to work for any triangle. We thus consider triangles T with circles of size x and y centered on the endpoints of an edge e and vary the length of e , as in Fig. 6. Then:

1. we choose l_{xy} close to the length for which $d_e(T)$ changes its sign;
2. we choose q_{xy} so that $U(T)$ is slightly less than $E(T)$ when T is stretched.

This is the way we defined the constants in Tab. 7.

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