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# SOME NORM INEQUALITIES FOR SOME POSITIVE BLOCK MATRICES

ANTOINE MHANNA<sup>1\*</sup>

ABSTRACT. We review Lin's inequality of numerical range widths and prove that, if  $M = [X_{i,j}]_{i,j=1}^n$  is positive semi-definite then  $\bigoplus_{i=1}^n (X_{i,i} - \sum_{j \neq i} X_{j,j}) \leq M^\tau \leq I_n \otimes \sum_{i=1}^n X_{i,i}$ , where  $M^\tau$  is the partial transpose of  $M$ . Some classical results are also discussed in terms of permutations.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $\mathbb{M}_n(\mathbb{M}_m)$  denote the space of  $n \times n$  complex matrices with entries in  $\mathbb{M}_m(\mathbb{C})$ . We write  $\mathbb{M}_n^H(\mathbb{M}_m)$  and  $\mathbb{M}_n^+(\mathbb{M}_m)$  for the set of Hermitian and positive semi-definite block matrices in  $\mathbb{M}_n(\mathbb{M}_m)$ , respectively. Let  $Im(X) := \frac{X - X^*}{2i}$  and  $Re(X) := \frac{X + X^*}{2}$  be the imaginary part and the real part of a matrix  $X$ , respectively. If  $W(X)$  denotes the field of values of  $X$ , then  $W(Re(X)) = \Re(W(X))$  and  $W(Im(X)) = \Im(W(X))$  see [6]. A positive semi-definite matrix  $A$  is denoted by  $A \geq 0$  and a norm  $\|\cdot\|$  over the space of matrices is a symmetric norm if  $\|UAV\| = \|A\|$  for all  $A$  and all unitaries  $U$  and  $V$ .

For a matrix  $M = [X_{i,j}] \in \mathbb{M}_n(\mathbb{M}_m)$  we denote  $M^\tau$  the matrix  $[X_{i,j}^*]$ , where each block is replaced by its adjoint. We define  $\text{Tr}_1(M) := \sum_{i=1}^n X_{i,i}$ ,  $\text{Tr}_2(M) := [\text{Tr}(X_{i,j})]$  for  $\text{Tr}_1(M)$  (resp.  $\text{Tr}_2(M)$ ) is an  $m \times m$  (resp. an  $n \times n$ ) complex matrix. A matrix  $M$  is positive partial transpose (P.P.T.) if  $M \geq 0$  and  $M^\tau \geq 0$ . The matrix  $I_m$  (or  $I$ ) is the identity matrix of dimension  $m$  and a 0 block-entry is an all zero submatrix. We finally denote  $\lambda_i(M)$  and  $\sigma_i(M)$ ,  $i = 1, \dots, n$  the eigenvalues and the singular values of  $M$ , respectively, arranged in decreasing order.

We need the following notations used later in the article. For  $X \in \mathbb{M}_m$ , the vertical width  $\omega_v(X)$  and the width  $\omega(X)$  of  $W(X)$  are the smallest possible real number  $v$  and  $w$  such that  $W(X)$  is contained in a vertical strip of width  $v$  and in a strip (rectangle) of width  $w$ , respectively. In

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particular,  $\omega(X) \leq \omega_v(X)$  and we may write

$$\min_{(a,b) \in \mathbb{R}^2 | W(X) \subseteq \{x \in \mathbb{C}; a \leq \Re(x) \leq b\}} |b - a| = \omega_v(X).$$

Notice that if the width of  $W(A)$  is  $\omega$ , then one can find a  $\theta \in [0, \pi]$  such that

$$rI_m \leq \operatorname{Re}(e^{i\theta}A) \leq (r + \omega)I_m$$

for some  $r \in \mathbb{R}$  and  $\omega = \omega_v(e^{i\theta}A)$ .

If  $S_i = \{M_k, 1 \leq k \leq i, M_k \in \mathbb{M}_m\}$  of cardinality  $i$ , we let  $\{S_i\}$  denote the set of matrices  $\{M_1 \pm \dots \pm M_i\}$  and an element in  $\{S_i\}$  is denoted by  $S_i^{[\sigma]}$ . We denote by  $\omega_v(\{S_i\})$  the  $2^{i-1}$ -tuple of  $\omega_v(S_i^{[\sigma]})$  for each  $S_i^{[\sigma]}$ .

Finally, if  $M := [X_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_m)$  with blocks  $X_{i,j}$  we write  $M = [{}_M[x_{k,l}^{i,j}]]$ , where  ${}_M[x_{k,l}^{i,j}]$  is the complex number belonging to block  $X_{i,j}$  on its line  $k$  and column  $l$ . For  $k, l: 1, \dots, m$  and  $i, j: 1, \dots, n$ , one has

$${}_M[x_{k,l}^{i,j}] = {}_M[x_{l,k}^{j,i}] \quad ; \quad {}_{M^T}[x_{k,l}^{i,j}] = \overline{{}_M[x_{l,k}^{i,j}]}.$$

The matrix  $\tilde{M}$  defined in  $\mathbb{M}_m(\mathbb{M}_n)$  by  ${}_{\tilde{M}}[x_{i,j}^{k,l}] = {}_M[x_{k,l}^{i,j}]$  is known to be congruent (by a permutation) to  $M$ , that is, there exist a permutation  $\mathcal{P}_M$  such that  $\mathcal{P}_M^T M \mathcal{P}_M = \tilde{M}$ ; see [3, Theorem 7] and [10, Theorem 2.4] with  $(\tilde{M})^\tau = (M^\tau)^T$ .

We sketch  $\mathcal{P}_M$  as follows: An index in  $[1, mn]$  is denoted by  $\iota$ , upon completing  $M$  by  $m \times m$  zero blocks or extending each block by  $0$ 's, we consider the completed matrix  $M_0 \in \mathbb{M}_r(\mathbb{M}_r)$  with  $\max(m, n) = r$ . Take the transposition  $T$  permuting for every  $s$ ,  $0 \leq s \leq r - 2$  and  $sr + 2 + s \leq \iota \leq sr + r$ ,  $\iota \leftrightarrow (\iota - sr - 1)r + 1 + s$ ,  $TM_0T = \tilde{M}_0$ . Applying block swapping (Proposition 1.1) one can show that there is some permutation  $P$  such that

$$P^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that  $M$  is invertible then so is  $\tilde{M}$ . Writing  $P$  by blocks as  $P = \begin{pmatrix} L & \star \\ \star & \star \end{pmatrix}$ , we see that  $L^T M L = \tilde{M}$ , which implies that  $L$  is invertible and thus a permutation matrix. By a continuity argument we can take  $\mathcal{P}_M = L$ ; see [5] and the references therein for further reading.

**Proposition 1.1.** *Let  $A = [A_{i,j}] \in \mathbb{M}_s(\mathbb{M}_p)$  and let  $B = [B_{i,j}] \in \mathbb{M}_s(\mathbb{M}_q)$  two complex matrices written by blocks. For  $C = [C_{i,j}] \in \mathbb{M}_s(\mathbb{M}_{p+q})$ , where  $C_{i,j} = \begin{pmatrix} A_{i,j} & 0 \\ 0 & B_{i,j} \end{pmatrix}$ , there is a permutation  $P$  such that  $P^T C P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ .*

*Proof.* A  $(u, v, r)$  block swap  $B_{uvr}$  is a permutation matrix (here of dimension  $s(p + q)$ ) of the form  $\begin{pmatrix} I_u & 0 & 0 & 0 \\ 0 & 0 & I_v & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ . Say  $s \geq 2$  for  $i = 1 \dots s - 1$ , take  $P = \prod_{i=1}^{s-1} B_{u_i v_i r_i}$ , where  $u_i = ip, v_i = iq$ , and  $r_i = p$ .  $\square$

## 2. MAIN RESULTS

If  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^H(\mathbb{M}_m)$  (not necessarily  $\geq 0$ ) with positive semi-definite diagonal blocks, then it is easy to see that  $|\lambda_n(M)| \leq \lambda_1(M)$  (to see this consider the eigenvalues of  $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ ). In particular if  $M \leq I \otimes S$ , then  $\|M\|_s \leq \|S\|_s$ , where  $\|\cdot\|_s$  is the spectral norm. This is not true for

$n > 2$  with  $n = 4$ , taking for example  $\begin{pmatrix} 0 & 3 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ , where  $\lambda_4 \approx -4.1$

and  $\lambda_1 \approx 3.22$ . In fact if  $M$  is an entry wise non-negative matrix we know by Perron-Frobenius theorem that  $|\lambda_n(M)| \leq \lambda_1(M)$ .

**2.1. Lin's Inequality.** In [1] the following norm inequality is proved:

**Theorem 2.1.** [1] *Let  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{M}_m)$ . Then for all symmetric norms*

$$\|M\| \leq \|A + B + \omega I_m\|,$$

where  $\omega$  is the width of a strip containing the numerical range of  $X$ .

The previous theorem has been generalized in [13] for  $2 \times 2$  block matrices. In [12] a generalization to any number of blocks was presented; here we refine Lin's idea mainly from Theorem's 2.1 proof (the base case): As  $M$  is positive,

we may write  $M = \begin{pmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{pmatrix} (X_1 \ X_2 \ \dots \ X_n)$  for some  $X_i \in \mathbb{M}_{m_n \times m}$  so

that  $M_{i,i} = X_i^* X_i$  and  $\|M\|_k = \left\| \sum_{i=1}^n X_i X_i^* \right\|_k$  ( $\|\cdot\|_k$  are Ky-Fan  $k$ -norms).

Identifying the  $n$ -tuple  $X$  to the set  $\{X_i, 1 \leq i \leq n\}$  we have (by a direct induction)  $\sum_{i=1}^n X_i X_i^* = \frac{1}{2^{n-1}} (\sum_{\sigma} (X^{[\sigma]})(X^{[\sigma]})^*)$ , applying Ky-Fan  $k$ -norms triangular inequality we get:

$$(2.1) \quad \|M\|_k = \frac{1}{2^{n-1}} \left\| \sum_{\sigma} (X^{[\sigma]})(X^{[\sigma]})^* \right\|_k \leq \frac{1}{2^{n-1}} \sum_{\sigma} \|(X^{[\sigma]})^*(X^{[\sigma]})\|_k.$$

**Theorem 2.2.** [12] *Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  and let  $S_i = \{X_{i,k}, i+1 \leq k \leq n\}$  for  $i = 1, \dots, n-1$ . Then for all symmetric norms*

$$\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + \sum_{i=1}^{n-1} v_i I_m \right\|,$$

where  $v_i$  is the average of  $\omega_v(\{S_i\})$ .

*Proof.* The proof follows from [12], or one can use (2.1), each term in the  $\sigma$  sum is a positive semi-definite matrix that can be written as:

$$\sum_{i=1}^n X_i^* X_i + \sum_{i=1}^{n-1} \pm Re(X_i^*(\pm X_{i+1} \cdots \pm X_n));$$

we can bound each term of the second sum by scalar identities and splitting the quantities we get the result by a counting argument for Ky-Fan  $k$ -norms; Ky-Fan dominance theorem completes the proof ([6, Sec 10.7]). The same last step can be seen in [1, Theorem 2.1 proof].  $\square$

**Corollary 2.3.** *Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ . Then for all symmetric norms*

$$\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + \sum_{i=2}^n v_i I_m \right\|,$$

where  $v_i$  ( $i = 2, \dots, n$ ) is the average of  $\omega_v(\{S_i\})$  and  $S_i = \{X_{k,i}, 1 \leq k \leq i-1\}$ .

*Proof.* Consider the matrix  $\mathcal{L}M\mathcal{L}$  in Theorem 2.2 with  $\mathcal{L}$  the anti-diagonal permutation matrix for blocks ( $\mathcal{L} = \begin{pmatrix} 0 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & 0 \end{pmatrix}$ ).  $\square$

**Corollary 2.4.** *Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  such that for  $i < j$ ,  $r_{i,j}I_m \leq Re(X_{i,j}) \leq r_{i,j}I_m + \delta_{i,j}I_m$ . Then  $\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + (\sum_{i < j} \delta_{i,j})I_m \right\|$  for all symmetric norms.*

*Proof.* The proof follows from Theorem 2.2, by writing  $X_{i,j} = Re(X_{i,j}) + iIm(X_{i,j})$  for  $i < j$  in the average expression and bounding the real parts we get the stated inequality.  $\square$

The case  $Re(X_{i,j}) = 0.I_m$  (skew-Hermitian blocks) is the main result of [14]. We also point out that in Theorem 2.1 of [12] the author replaced (by a miss-justification when  $n > 2$  as per our private communication) vertical widths by smallest widths  $\omega$  and proposed the same replacement in Corollary 2.4 as Question 2.5.

**2.2. Around Hiroshima majorization.** In this section, we reprove some classical results concerning matrices in  $\mathbb{M}_n^H(\mathbb{M}_m)$ , related to some permutations and P.P.T. matrices. The approach is close (dual) to that in [4].

**Corollary 2.5.** [4] *Let  $M \in \mathbb{M}_n^H(\mathbb{M}_m)$  such that  $M^\tau \leq Tr_2(M^\tau) \otimes I_m$ . Then there exist a permutation  $\mathcal{P}_M$  with:*

$$(2.2) \quad N = \mathcal{P}_M^T M^\tau \mathcal{P}_M \leq \mathcal{P}_M^T (Tr_2(M^\tau) \otimes I_m) \mathcal{P}_M = I_m \otimes Tr_1(N)$$

for  $N \in \mathbb{M}_m^H(\mathbb{M}_n)$ ,  $Tr_1(N) = Tr_2(M^\tau)$  is an  $n \times n$  matrix. Furthermore if  $M \geq 0$ , then  $N^\tau \geq 0$ .

**Remark 2.6.** Set  $M \in \mathbb{M}_n(\mathbb{M}_m)$ ,  $U$  unitary and  $U^*MU \in \mathbb{M}_n(\mathbb{M}_m)$ . Define the block  $m \times m$  permutation matrix  $P_M \in \mathbb{M}_n(\mathbb{M}_m)$  having zero or identity  $m \times m$  blocks; if  $U = P_M$  is a transposition block matrix, then it is easy to see that  $(P_M M P_M)^\tau = P_M M^\tau P_M$  and thus, upon composition, for any block  $m \times m$  permutation matrix  $P_M$ ,  $P_M(P_M^T M P_M)^\tau P_M^T = M^\tau$ .

Let  $M \in \mathbb{M}_n^H(\mathbb{M}_m)$  such that  $M^\tau \geq 0$  and  $U$  unitary, for  $U^*MU \in \mathbb{M}_n^H(\mathbb{M}_m)$  the matrix  $(U^*MU)^\tau$  may not be positive semi-definite in general;

consider  $M = U N U^* \in \mathbb{M}_3^+(\mathbb{M}_2)$  in Example 2.12 with  $U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ ,

however this is true if  $n = m$  and  $U = \mathcal{P}_M$ , this is also true for  $M \in \mathbb{M}_2^H(\mathbb{M}_m)$

and  $U^* = \begin{pmatrix} e^{i\zeta} \cos(\theta)I & \sin(\theta)I \\ -e^{i\zeta} \sin(\theta)I & \cos(\theta)I \end{pmatrix}$  ([13]).

From Quantum Physics theory we know that if  $M$  is P.P.T. then  $M \leq I_n \otimes \text{Tr}_1(M)$  and  $M \leq \text{Tr}_2(M) \otimes I_m$ ; see for example [9, 8] and more recently [2, 4]. Here we present a different proof as follows:

**Lemma 2.7.** If  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  then

$$\bigoplus_{i=1}^n (X_{i,i} - \sum_{j \neq i} X_{j,j}) \leq M^\tau \leq I_n \otimes \sum_{i=1}^n X_{i,i}.$$

*Proof.* We prove the right side inequality, as the lower bound has a similar proof, by noticing that  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0 \iff \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0$ . We write

$E = I_n \otimes \sum_{i=1}^n X_{i,i} - M^\tau$  as the sum of positive semi-definite matrices of the

form  $E_{k,j} = \begin{pmatrix} X_{j,j(k)} & -X_{k,j}^* \\ -X_{k,j} & X_{k,k(j)} \end{pmatrix}$ ; here the matrix  $E_{k,j}$  with  $k < j$  refers to

the  $mn \times mn$  block matrix having all blocks zero except:  $-X_{k,j}$  and  $-X_{k,j}^*$  taken on the same position as  $X_{k,j}$  and  $X_{k,j}^*$  in  $E$ , respectively,  $X_{j,j(k)}$  and  $X_{k,k(j)}$ , where  $X_{j,j(k)}$  is the block  $X_{j,j}$  on the  $k^{\text{th}}$  diagonal entry of  $E$  (as of

$E_{k,j}$ ). Since  $M \geq 0$ , up to the congruence  $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$  each  $E_{k,j}$  is a principal block submatrix of  $M$  and  $E = \sum_{k < j} E_{k,j} \geq 0$ .  $\square$

**Remark 2.8.** Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ , one can check following the previous proof that  $M \leq \bigoplus_{i=1}^n n X_{i,i}$ .

The last lemma and corollary imply the following:

**Corollary 2.9.** [4] If  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ , then  $M^\tau \leq \text{Tr}_2(M^\tau) \otimes I_m$ .

In [7]-Theorem 1- the following (Hiroshima majorization) is proved:

**Theorem 2.10.** [7] Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  such that  $M \leq \text{Tr}_2(M) \otimes I_m$ . Then for all  $k \in [1, mn]$ :

$$\sum_{i=1}^k \lambda_i(M) \leq \sum_{i=1}^k \lambda_i(\text{Tr}_2(M)).$$

**Corollary 2.11.** Let  $M = [X_{i,j}] \in \mathbb{M}_n^H(\mathbb{M}_m)$  such that

$$(2.3) \quad 0 \leq M \leq I_n \otimes \sum_{i=1}^n X_{i,i}.$$

Then for all  $k \in [1, m]$ :

$$\sum_{i=1}^k \sigma_i(M) \leq \sum_{i=1}^k \sigma_i\left(\sum_{i=1}^n X_{i,i}\right)$$

equivalently  $\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} \right\|$  for all symmetric norms.

Upon completing  $\sum_{i=1}^n X_{i,i}$  by zeros we can take  $k$  with  $1 \leq k \leq mn$  to get the last inequality from Ky-Fan dominance theorem. In [11], another direct proof of Corollary 2.11 is given with the fact that Corollary 2.11 and Theorem 2.10 are equivalent (see Corollary 2.5).

**Example 2.12.** It is well known that a matrix in  $\mathbb{M}_n^+(\mathbb{M}_m)$  ( $n > 2$ ) satisfying (2.3) need not be P.P.T. as for example  $N \in \mathbb{M}_3^+(\mathbb{M}_2) = \begin{pmatrix} 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ .

The next corollary can be proved from the double inequality of Lemma 2.7.

**Corollary 2.13.** If  $M \in \mathbb{M}_n^+(\mathbb{M}_m)$ , then  $\|M^\tau\|_s \leq \|\text{Tr}_1(M)\|_s$ .

The positive condition in Corollary 2.11 seems necessary as this example shows:

**Example 2.14.** We have  $M = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 4.5 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_2)$  and  $\sigma_1(M) +$

$$\sigma_2(M) = 4.5 + 3 > 7.4$$

For a matrix  $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  with all blocks diagonal, it is easily seen that  $D$  is P.P.T. as  $D^\tau = \bar{D}$  ( $\bar{D}$  is  $D$  with conjugate entries). Upon considering the rearrangement of  $D$  taking  $\mathcal{P}_D^T D \mathcal{P}_D$ , we see that each diagonal entry of  $\text{Tr}_1(D)$  is the sum of  $n$  eigenvalues of  $D$  which implies  $\|D\| \leq \|\text{Tr}_1(D)\|$  for all symmetric norms.

**Corollary 2.15.** [15] Let  $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ ; we have  $M \leq m \text{Tr}_2(M) \otimes I_m$ .

*Proof.* Take  $M_{\mathcal{P}_M} = \mathcal{P}_M^T M \mathcal{P}_M = [X'_{i,j}] \in \mathbb{M}_m^+(\mathbb{M}_n)$ , applying Remark 2.8  $M_{\mathcal{P}_M} \leq \bigoplus_{i=1}^m m X'_{i,i}$ , reversing the rearrangement  $M \leq mD$ , where  $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$  and  $D_{i,j}$  is the (main) diagonal of  $X_{i,j}$ . Since  $D$  is P.P.T. we get the result from Corollary 2.9.  $\square$

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