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Finding the jump rate for fastest decay in the Goldstein-Taylor model

Helge Dietert* Josephine Evans†

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Abstract

This paper is about the rate of convergence to equilibrium for hypocoercive linear kinetic equations. We look for the spatially dependent jump rate which yields the fastest decay rate of perturbations. For the Goldstein-Taylor model, we show (i) that for a locally optimal jump rate the spectral bound is determined by multiple, possibly degenerate, eigenvectors and (ii) that globally the fastest decay is obtained with a spatially homogeneous jump rate. Our proofs rely on a connection to damped wave equations and a relationship to the spectral theory of Schrödinger operators.

Keywords: Hypocoercivity; spatial weight; optimal control; Goldstein-Taylor model; wave equation

1 Introduction

A typical linear kinetic equation takes the form

$$\partial_t f + Tf = \sigma(x)\, C(f) \tag{1}$$

for a density $f = f(t, x, v)$ at time t over the phase space consisting of a spatial position x and a velocity v. Here T is a transport operator, σ is a spatial weight, and C is a collision operator driving the system to thermal equilibrium.

The theory of hypocoercivity, [26, 12], ensures, by a variety of proofs, that the equilibrium is reached with an exponential rate. The decay rate is limited by the spectral bound λ of $-T + \sigma C$. In the case of constant σ, the spectral bound λ typically depends on σ as indicated in Fig. 1, where we see two distinct regimes:

1. For small jump rates σ the spectral bound scales with $-\sigma$. Here the decay is limited by the thermalisation rate of the velocity variable so that a faster jump rate improves the spectral bound.

2. For bigger jump rates σ the spectral bound behaves like $-\sigma^{-1}$. In this regime, the decay rate is limited by the spatial diffusion. Here a faster decay rate means slower decay as the effective spatial transport decreases by the law of large numbers.

This motivates the main question of this research.

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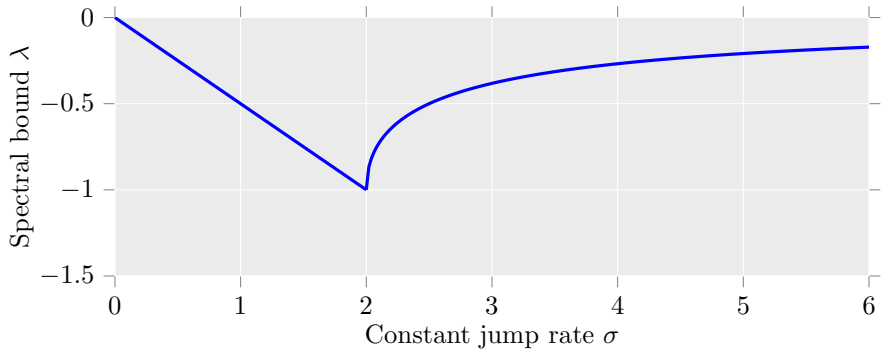

Figure 1: Typical spectral bound depending on the noise strength. For a given constant σ, we plot the spectral bound for typical models of (1).

Question 1. *Can we combine spatial regions of large and small jump rates in order to obtain a faster decay rate? More generally, what is the jump rate σ, among functions depending on x, which will give the smallest spectral bound, i.e. the fastest decay?*

There has been previous research on finding optimal or good bounds on the rate to equilibrium for a fixed jump rate σ, e.g. [2]. The goal of this work is to understand the dependence of the rate of convergence to equilibrium on σ by studying the optimal control problem of finding the choice of σ which maximises the decay rate. We believe this provides another direction to understand the precise dependence of the decay rate on the parameters in the equation. In our opinion, this approach has potential in a variety of other kinetic problems, e.g.

- For kinetic equations with a confining potential, one could investigate the interplay between jump rate and the confining potential.
- For equations posed on a domain with boundary, one could investigate, in a similar way, the dependence of the rate on the shape of the domain and boundary conditions. This could produce results similar to the celebrated Faber-Krahn inequality, that the spectral gap of the Dirichlet Laplacian is optimised, amongst all smooth convex subsets of $\mathbb{R}^d$, on balls [15, 21, 11].
- This problem is related to the control of nuclear reactors as for the radiative transfer equation σ is related to the presence or absence of control rods.
- In our perturbation result we show that the optimal σ must occur simultaneously with a degeneracy in the eigenspace associated to the spectral bound eigenvectors. We believe this might point to connections between the optimal σ and symmetries present in the equation. For example, when σ is constant, the equation is translation invariant which produces a doubling of eigenvalues. More generally, when σ possesses translation or reflection symmetries, this will result in multiple eigenvectors and a candidate for optimality.

Apart from applications in kinetic theory, Markov Chain Monte Carlo (MCMC) algorithms are a main motivation. In applications of Bayesian statistics, one needs to calculate the posterior distribution which is given up to a normalisation factor by

$$\mathrm{e}^{-\phi(x)}.$$

For high-dimensional problems, an explicit computation is prohibitively expensive and a common solution is to construct a stochastic process that converges to the sought distribution and to sample from that process. One such process is the diffusion process

$$\mathrm{d}X_t = -\nabla_x \phi(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t,$$

where W is Brownian noise.

This procedure can sometimes be slow, and Hamiltonian MCMC (HMCMC) has been developed as a way to increase the speed of convergence of these algorithms, see [13, 7] for a rigorous proof of the increase in speed and references within on HMCMC. The strategy of Hamiltonian Markov Chain Monte Carlo is to look at the related kinetic equation

$$\begin{cases} \mathrm{d}X_t = V_t \,\mathrm{d}t, \\ \mathrm{d}V_t = -\nabla_x \phi(X_t)\,\mathrm{d}t + \sigma(X_t)\,(\sqrt{2}\mathrm{d}W_t - V_t\,\mathrm{d}t), \end{cases}$$

which has the equilibrium distribution $M(v)\,\mathrm{e}^{-\phi(x)}$ for the velocity equilibrium $M(v)$ so that the sought distribution is obtained by the spatial distribution. The intuition is that the kinetic equation yields a faster transport of the distribution over large spatial distances. The previous analyses look at the case of constant σ. We ask the further question, whether the rate of convergence can be increased by making σ spatially dependent. This has been investigated numerically in statistics literature, for example in [17], where the authors propose a version of the Metropolis adjusted Langevin algorithm (MALA), which takes into account the geometry of ϕ.

A very simple model to study the exponential decay of kinetic equations is the one-dimensional Goldstein-Taylor model, which is still actively studied as a test case for hypocoercive results [4], and has been studied with σ depending on x in [6], relating it to the work by Lebeau [22]. It is a special case of BGK models with only two velocities ± 1. Setting $u = (t,x) = f(t,x,+1)$ and $v(t,x) = f(t,x,-1)$ to the respective spatial densities, the model writes

$$\begin{cases} \partial_t u + \partial_x u = \dfrac{\sigma(x)}{2}\,(v-u), \\ \partial_t v - \partial_x v = \dfrac{\sigma(x)}{2}\,(u-v), \end{cases} \tag{2}$$

where we consider the spatial variable x in the torus $\mathbb{T}$ with length 2π.

As used before [20, 6], the one-dimensional case has the special feature that the kinetic equation (2) is equivalent to a damped wave equation by considering

$$\rho(t,x) := \frac{u(t,x)+v(t,x)}{\sqrt{2}} \qquad \text{and} \qquad j(t,x) := \frac{u(t,x)-v(t,x)}{\sqrt{2}}. \tag{3}$$

Then the Goldstein-Taylor model (2) can be written as

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \partial_t j + \partial_x \rho = -\sigma(x)\, j. \end{cases} \tag{4}$$

In our results, we want to characterise the convergence towards the stationary state which is in the formulation (2) given by $u = v = \text{const}$ or in the formulation (4) by $\rho = \text{const}$ and $j = 0$. By the conservation of the mass $\int_{\mathbb{T}} \rho(x)\,\mathrm{d}x$, the limiting state can be identified and, using the linearity, it therefore suffices to study the perturbation from the limiting state.

Working in L^2, we therefore consider the evolution in the space

$$\mathrm{L}^2_p = \{(\rho,j) \in \mathrm{L}^2(\mathbb{T}) : \int_{\mathbb{T}} \rho(x)\,\mathrm{d}x = 0\} \tag{5}$$

with the natural norm given by

$$\|(\rho,j)\|_2^2 = \|\rho\|_2^2 + \|j\|_2^2.$$

The evolution in L^2_p is a semigroup with the generator A_σ (see Proposition 4 below). In Section 3 we characterise, as a first result, a possible optimal jump rate $\sigma = \sigma(x)$ by considering perturbations.

Theorem 2. *Suppose $\sigma \in \mathrm{L}^\infty(\mathbb{T})$ is such that the spectral bound of A_σ in L^2_p is locally minimised in L^∞. Then the spectral bound is not determined by a simple eigenvalue.*

The same wave equation (but with different boundary conditions) has been studied in Cox and Zuazua [9] with fixed ends. One aspect of their work is to characterise the eigenvalues along the real axis by the spectrum of Schrödinger operators, which allows them to find the spatial damping σ minimising the largest real eigenvalues. They cannot consider the full spectral bound. However, in our situation and with our aims in mind, we are able to go further by associating the equation to a different Schrödinger operator. With this newly associated Schrödinger operator, we can obtain a new bound on the spectral bound by looking at the second eigenvalue and exploiting the translation symmetry. This yields the following theorem (proved in Section 4):

Theorem 3. *For the Goldstein-Taylor model* (2)*, the smallest spectral bound in L^2_p is obtained with the constant $\sigma = 2$ giving the spectral bound -1.*

In the context of the wave equation, the corresponding question of the spatially dependent damping has been studied before and shows that the competing effects of the jump rate are more intricate than the result for a constant damping in Theorem 3 might suggest. It is noted by Castro and Cox [8] that an arbitrary large decay rate can be obtained in the case of fixed-ends with a spatialy dependent damping diverging towards the boundary. Taking the damping as an indicator function of a set ω and optimising the set ω, the competition between the effects is non-trivial and yields in general non-existence of optimal sets [19, 24]. It has been studied further in [25], where the results are phrased in terms of a related observability condition. The problem has also been formulated in terms of the overall energy [10] and from a numerical side the problem is also studied by e.g. [16, 3].

The study of the decay rate for the presented class of systems is a wide field ranging from works in kinetic theory [1, 2, 4] to the wave equation [22, 18].

As a first step, we formulate in Section 2 the spectral problem precisely and show that the spectral bound determines the decay rate of the L^2 norm. Furthermore, we find eigenvalues corresponding to the decay rate $\|\sigma\|_1/(4\pi)$ of the velocity distribution alone.

Proposition 4. *For non-negative $\sigma \in \mathrm{L}^\infty$, the closed linear operator A_σ defined by*

$$A_\sigma \begin{pmatrix} \rho \\ j \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & -\sigma(x) \end{pmatrix} \begin{pmatrix} \rho \\ j \end{pmatrix} \tag{6}$$

on L^2_p generates a contraction semigroup $(\mathrm{e}^{tA_\sigma})_t$ matching the evolution (4)*.*

For any $\epsilon > 0$ there exists an eigenvalue $\lambda \in \mathbb{C}$ with $|\Re\lambda + \|\sigma\|_1/(4\pi)| \le \epsilon$ and there are at most finitely many eigenvalues in the halfspace $\{\lambda \in \mathbb{C} : \Re\lambda \ge -\|\sigma\|_1/(4\pi) + \epsilon\}$.

If for $a > -\|\sigma\|_1/(4\pi)$ there exists no eigenvalue with $\Re\lambda \ge a$, then we have the growth bound

$$\|\mathrm{e}^{tA_\sigma}\| \lesssim \mathrm{e}^{ta} \qquad \forall t \ge 0. \tag{7}$$

Note that corresponding results for the wave equation with fixed ends have been shown in Cox and Zuazua [9].

The main implication of our study is that, for the Goldstein-Taylor model, the rate of convergence to equilibrium cannot be improved by making σ depend on x. We believe this suggests that the rate of convergence to equilibrium is unlikely to be increased, in kinetic models, by adding local oscillations to the jump rate σ. We note that this does not exclude the possibility that spatially dependent jump rates cannot improve the rate of convergence to equilibrium in the presence of more complex geometries (either a confining potential or a boundary that would break some of the symmetries in the model). In fact, it is proposed in [17] to vary σ on large scales in a way that is sympathetic to the confining function ϕ. Therefore, we close the introduction by the following open question:

Question 5. *Does there exist a linear kinetic equation posed on $\mathbb{R}^d$ for some d with a confining potential for which a strictly faster rate of convergence can be achieved by allowing the collision rate σ to depend on x than is achieved by constant σ?*

2 Semigroup and spectral problem

In this section, we prove Proposition 4 in two parts. First, we show that we have a well-defined semigroup for the evolution.

Proof of Proposition 4 (first part). The generator (6) formally gives the required PDE (4). Without the σ the solution is given explicitly by the characteristics defined by the transport and this explicit representation shows that it generates a semigroup. As $\sigma \in \mathrm{L}^\infty$, the contribution of σ in A_σ is a bounded perturbation so that it defines a semigroup.

The required mass conservation follows from the estimate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho(t,x)\,\mathrm{d}x = -\int_{\mathbb{T}} \partial_x j(t,x)\,\mathrm{d}x = 0$$

and the contraction property by the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \Big(|\rho|^2 + |j|^2\Big)\,\mathrm{d}x = -2\Re\int_{\mathbb{T}} \Big(\overline{\rho}\partial_x j + j(\partial_x\overline{\rho} + \sigma(x)\overline{j})\Big)\mathrm{d}x = -2\int_{\mathbb{T}} \sigma(x)\,|j|^2\,\mathrm{d}x \le 0.$$

This gives the first part of Proposition 4. □

In the spectral theory, the central object is the resolvent

$$R(\lambda, A_\sigma) := (\lambda - A_\sigma)^{-1}. \tag{8}$$

For $(a,b) \in \mathrm{L}^2_p$, the image $(\rho, j) = R(\lambda, A_\sigma)(a,b)$ is the solution to

$$(\lambda - A_\sigma)\begin{pmatrix}\rho\\ j\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} \tag{9}$$

as long it has a unique solution. This equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\rho\\ j\end{pmatrix} + \begin{pmatrix}0 & \sigma+\lambda\\ \lambda & 0\end{pmatrix}\begin{pmatrix}\rho\\ j\end{pmatrix} = \begin{pmatrix}b\\ a\end{pmatrix}. \tag{10}$$

Associated to (10), we define the solution operator for the homogeneous part and an operator $M(\sigma,\lambda)$ yielding the solvability condition.

Definition 6. *Fix $\sigma \in \mathrm{L}^\infty$ and $\lambda \in \mathbb{C}$. For $y \in \mathbb{T}$ and given $(\rho_{\mathrm{in}}, j_{\mathrm{in}})$ consider the linear ODE*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\rho(x)\\ j(x)\end{pmatrix} + \begin{pmatrix}0 & \sigma(x)+\lambda\\ \lambda & 0\end{pmatrix}\begin{pmatrix}\rho(x)\\ j(x)\end{pmatrix} = 0,\\ \begin{pmatrix}\rho(y)\\ j(y)\end{pmatrix} = \begin{pmatrix}\rho_{\mathrm{in}}\\ j_{\mathrm{in}}\end{pmatrix}\end{cases} \tag{11}$$

and define $\mathcal{S}^{y\to x}_{\sigma,\lambda}$ as solution operator so that $\mathcal{S}^{y\to x}_{\sigma,\lambda}((\rho_{\mathrm{in}}, j_{\mathrm{in}})) = (\rho(x), j(x))$. For the resolvent define the operator

$$M(\sigma,\lambda) = \mathrm{Id} - \mathcal{S}^{0\to 2\pi}_{\sigma,\lambda}. \tag{12}$$

By Duhamel's principle, a solution of the resolvent equation (10) can be expressed with the solution operator as

$$\begin{pmatrix}\rho(y)\\ j(y)\end{pmatrix} = \mathcal{S}^{0\to y}_{\sigma,\lambda}\begin{pmatrix}\rho_0\\ j_0\end{pmatrix} + \int_0^y \mathcal{S}^{x\to y}_{\sigma,\lambda}\begin{pmatrix}b(x)\\ a(x)\end{pmatrix}\mathrm{d}x \tag{13}$$

for constants ρ_0, j_0. The periodic boundary condition $\rho(2\pi) = \rho_0$ and $j(2\pi) = j_0$ imply that

$$M(\sigma,\lambda)\begin{pmatrix}\rho_0\\ j_0\end{pmatrix} = \int_0^{2\pi} \mathcal{S}^{x\to 2\pi}_{\sigma,\lambda}\begin{pmatrix}b(x)\\ a(x)\end{pmatrix}\mathrm{d}x, \tag{14}$$

which yields a unique solution if $M(\sigma,\lambda)$ is invertible. If not the kernel gives an eigenvalue solving

$$\begin{cases} \lambda\rho + \partial_x j = 0, \\ \partial_x \rho + (\lambda + \sigma(x))j = 0. \end{cases} \tag{15}$$

The characterisation by the operator M corresponds to the well-known shooting method for solving eigenvalue problems as done in [9]. For the further analysis, we note basic properties.

Lemma 7. *For a fixed $\sigma \in \mathrm{L}^\infty$ and $x, y \in \mathbb{T}$, the solution operator $\mathcal{S}^{x\to y}_{\sigma,\lambda}$ and $M(\sigma,\lambda)$ are analytic with respect to $\lambda \in \mathbb{C}$.*

Moreover, the solution operator is bounded as

$$\|\mathcal{S}^{x\to y}_{\sigma,\lambda}\| \le \exp\left(|\Re(\lambda)|\,|y-x| + \|\sigma\|_1/2\right)$$

and the solution operator $\mathcal{S}^{x\to y}_{\sigma,\lambda}$ and $M(\sigma,\lambda)$ are continuous with respect to σ in L^1 and the functions $t \mapsto \mathcal{S}^{x\to y}_{\sigma+t\eta,\lambda}$ and $t \mapsto M(\sigma+t\eta,\lambda)$ are differentiable in t when $\eta \in \mathrm{L}^1(\mathbb{T})$.

Proof. The existence and analyticity of the solution operator $\mathcal{S}^{x\to y}_{\sigma,\lambda}$ and $M(\sigma,\lambda)$ follow from standard ODE theory.

For the growth bound and the dependence with respect to $\|\sigma\|_1$, note the following a priori estimate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\left(|\rho|^2 + |j|^2\right) &= \Re\Big(\overline{\rho}\partial_x\rho + j\partial_x\overline{j}\Big) \\ &= -\Re\Big(\overline{\rho}(\sigma+\lambda)j - j\overline{\lambda}\overline{\rho}\Big) \\ &= -\Re\Big(\overline{\rho}j(\sigma+\lambda+\overline{\lambda})\Big) \\ &\le \Big[2|\Re(\lambda)| + \sigma(x)\Big]\frac{1}{2}\left(|\rho|^2 + |j|^2\right), \end{aligned}$$

which yields the result.

Similarly if we take two solutions (ρ_1, j_1) corresponding to jump rate σ_1 and (ρ_2, j_2) corresponding to jump rate σ_2 and with the same initial value (ρ_0, j_0), then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\left(|\rho_1-\rho_2|^2 + |j_1-j_2|^2\right) \le \Big[2|\Re(\lambda)| + \sigma_1(x) + \sigma_2(x)\Big]\frac{1}{2}\left(|\rho_1-\rho_2|^2 + |j_1-j_2|^2\right) + |\sigma_1(x)-\sigma_2(x)|\Re(j_1(\overline{\rho}_1-\overline{\rho}_2)).$$

Integrating this and using our previous bound, we get

$$\frac{1}{2}\left(|\rho_1-\rho_2|^2 + |j_1-j_2|^2\right) \le \|\sigma_1-\sigma_2\|_1 \mathrm{e}^{2\pi|\Re(\lambda)| + \|\sigma_1\|_1 + \|\sigma_2\|_1}\left(|\rho_0|^2 + |j_0|^2\right).$$

The continuity and differentiability properties follow from this. □

Hence we can precisely describe the resolvent.

Lemma 8. *The resolvent set of A_σ consists of $\lambda \in \mathbb{C}$ for which the matrix $M(\sigma,\lambda)$ is invertible and the resolvent is bounded in the operator norm as*

$$\|R(\lambda, A_\sigma)\| \lesssim \|M(\sigma,\lambda)^{-1}\|\, 2\pi \exp\Big(4\pi|\Re(\lambda)| + \|\sigma\|_1\Big).$$

Moreover, for every λ in the spectrum of A_σ, there exists at least one eigenvector.

Proof. The result follows directly from the representation (13). Using (14) we can write

$$\begin{pmatrix}\rho_0\\ j_0\end{pmatrix} = M(\sigma,\lambda)^{-1}\int_0^{2\pi}\mathcal{S}^{x\to 2\pi}_{\sigma,\lambda}\begin{pmatrix}b(x)\\ a(x)\end{pmatrix}\mathrm{d}x,$$

and use the bound from Lemma 7.

If $M(\sigma,\lambda)$ were invertible then we would have a bounded resolvent map, by the formula above, so λ would be in the resolvent set. Therefore if λ is in the spectrum of A_σ, then $M(\sigma,\lambda)$ will not be invertible and there exists an element in the kernel of $M(\sigma,\lambda)$ which yields an eigenvector. □

We now look at the asymptotic form of M as $|\Im\lambda| \to \infty$ over a finite range of $\Re\lambda$.

Lemma 9. *Fix a bounded interval $I \subset \mathbb{R}$ and $\sigma \in \mathrm{L}^\infty(\mathbb{T})$, then let*

$$M^\infty(\sigma,\lambda) := \begin{pmatrix} 1-\cosh\left(\lambda\left(2\pi+\frac{\|\sigma\|_1}{2\lambda}\right)\right) & \sinh\left(\lambda\left(2\pi+\frac{\|\sigma\|_1}{2\lambda}\right)\right) \\ \sinh\left(\lambda\left(2\pi+\frac{\|\sigma\|_1}{2\lambda}\right)\right) & 1-\cosh\left(\lambda\left(2\pi+\frac{\|\sigma\|_1}{2\lambda}\right)\right) \end{pmatrix}.$$

We have that $M(\sigma,\lambda) - M^\infty(\sigma,\lambda) \to 0$ uniformly over $\Re\lambda \in I$ as $|\Im\lambda| \to \infty$.

This matches [10, Theorem 5.1] where the result has been proven with explicit error bounds by a series solution. For being self-contained, we give another shorter proof.

Proof. The idea is like in the proof of the Riemann-Lebesgue lemma that we can approximate σ by a piecewise constant function $\tilde\sigma$, i.e. there exists $x_0 = 0 < x_1 < x_2 < \cdots < x_K = 2\pi$ and $\sigma_1, \sigma_2, \ldots, \sigma_K$ such that

$$\tilde\sigma(y) = \sigma_j \qquad \forall y \in [x_{j-1}, x_j).$$

For $\lambda \neq 0$ we find on each constant part the explicit solution

$$\mathcal{S}_{\tilde\sigma,\lambda}^{x_{j-1}\to x_j} = \begin{pmatrix} \cosh\left((x_j-x_{j-1})\sqrt{\lambda(\lambda+\sigma_j)}\right) & -\sqrt{\frac{\lambda+\sigma_j}{\lambda}}\sinh\left((x_j-x_{j-1})\sqrt{\lambda(\lambda+\sigma_j)}\right) \\ -\sqrt{\frac{\lambda}{\lambda+\sigma_j}}\sinh\left((x_j-x_{j-1})\sqrt{\lambda(\lambda+\sigma_j)}\right) & \cosh\left((x_j-x_{j-1})\sqrt{\lambda(\lambda+\sigma_j)}\right) \end{pmatrix}.$$

By considering Taylor series, we see that $|\sqrt{\lambda(\lambda+\sigma)} - \lambda(1+\sigma/\lambda)| \to 0$ and $\sqrt{(\lambda+\sigma)/\lambda} \to 1$ as $|\lambda| \to \infty$. Hence for $|\Im\lambda| \to \infty$ the whole matrix is uniformly asymptotically approaching

$$\tilde S_j := \begin{pmatrix} \cosh\left(\lambda\left((x_j-x_{j-1}) + \frac{(x_j-x_{j-1})\sigma_j}{2\lambda}\right)\right) & -\sinh\left(\lambda\left((x_j-x_{j-1}) + \frac{(x_j-x_{j-1})\sigma_j}{2\lambda}\right)\right) \\ -\sinh\left(\lambda\left((x_j-x_{j-1}) + \frac{(x_j-x_{j-1})\sigma_j}{2\lambda}\right)\right) & \cosh\left(\lambda\left((x_j-x_{j-1}) + \frac{(x_j-x_{j-1})\sigma_j}{2\lambda}\right)\right) \end{pmatrix}.$$

By the group property of the solution operator we find

$$\mathcal{S}_{\tilde\sigma,\lambda}^{0\to 2\pi} = \mathcal{S}_{\tilde\sigma,\lambda}^{x_{j-1}\to 2\pi} \circ \mathcal{S}_{\tilde\sigma,\lambda}^{x_{j-2}\to x_{j-1}} \circ \cdots \circ \mathcal{S}_{\tilde\sigma,\lambda}^{x_0\to x_1}$$

which therefore is uniformly asymptotically approaching

$$\tilde S_j \circ \tilde S_{j-1} \circ \cdots \circ \tilde S_1 = \begin{pmatrix} \cosh\left(\lambda\left(2\pi + \frac{\|\tilde\sigma\|_1}{2\lambda}\right)\right) & -\sinh\left(\lambda\left(2\pi + \frac{\|\tilde\sigma\|_1}{2\lambda}\right)\right) \\ -\sinh\left(\lambda\left(2\pi + \frac{\|\tilde\sigma\|_1}{2\lambda}\right)\right) & \cosh\left(\lambda\left(2\pi + \frac{\|\tilde\sigma\|_1}{2\lambda}\right)\right) \end{pmatrix}.$$

As we can approximate any function $\sigma \in \mathrm{L}^\infty$ arbitrary well by the piecewise constant function in L^1, the result follows from the stability of Lemma 7. □

We can now prove the remaining parts of Proposition 4.

Proof of Proposition 4 (remaining part). The asymptotic expression M^∞ of M in Lemma 9 is invertible except when $\lambda = -\|\sigma\|_1/(4\pi) + in$ for $n \in \mathbb{Z}$. As $\det M^\infty(\sigma,\lambda)$ is analytic and $M(\sigma,\lambda) - M^\infty(\sigma,\lambda) \to 0$ uniformly over $\Re\lambda \in [-\|\sigma\|_1/(4\pi) - \epsilon, -\|\sigma\|_1/(4\pi) + \epsilon]$ as $|\Im\lambda| \to \infty$, Rouché's theorem ensures that $\det M$ has a root close to the asymptotic root $\lambda = -\|\sigma\|_1/(4\pi) + in$ for $|n|$ large enough. In particular, this shows that there exists an eigenvalue $\lambda \in \mathbb{C}$ with $\Re\lambda \in [-\|\sigma\|_1/(4\pi) - \epsilon, -\|\sigma\|_1/(4\pi) + \epsilon]$.

As A_σ generates a contracting semigroup, there are no eigenvalues λ with $\Re\lambda > 0$. By the asymptotic expression there are no eigenvalues in $-\|\sigma\|_1/(4\pi) + \epsilon$ for large enough $|\Im\lambda|$. Therefore, the analyticity of M implies that there are at most finitely many eigenvalues in the remaining bounded region.

For the last part, assume a such that all eigenvalue λ of A_σ satisfy $\Re\lambda < a$. By the first part, this implies $a > -\|\sigma\|_1/(4\pi)$. For $\lambda \in \mathbb{C}$ with $\Re\lambda > 1$, recall that $(\mathrm{e}^{tA_\sigma})_t$ is a contraction semigroup, which yields the uniform bound $\|R(\lambda, A_\sigma)\| \leq 1$ on the resolvent. The asymptotic expression of Lemma 9 shows that $\|M(\sigma,\lambda)^{-1}\|$ is uniformly bounded for $\lambda \in \mathbb{C}$ with $\Re\lambda \in [a, 1]$ and $|\Im\lambda|$ for large enough. As there are no eigenvalue with real part bigger or equal to a and M is continuous, this shows

$$\sup_{\Re\lambda\in[-a,1]} \|M(\sigma, a + \mathrm{i}b)^{-1}\| < \infty.$$

Lemma 8 implies the same bound for the resolvent. As our function space L^2_p is a Hilbert space, we can thus apply Gearhart-Prüss-Greiner theorem [14, Thm 1.11 in Chapter V] to find the claimed growth bound. □

Remark 10. *An alternative for using the Gearhart-Prüss-Greiner theorem is an adaptation of the theory of positive semigroups as done in Bernard and Salvarani [5]. Cox and Zuazua [9] establish the growth bound by studying the eigenvector system in more detail.*

We close this section noting the eigenvalues for the case that σ is constant.

Lemma 11. *Suppose that σ is constant. Then the spectrum in L^2_p consists of*

$$-\frac{\|\sigma\|_1}{4\pi} \pm \sqrt{\frac{\|\sigma\|_1^2}{(4\pi)^2} - n^2} \qquad \text{for } n = 1, 2, 3, \ldots$$

and

$$-\frac{\|\sigma\|_1}{2\pi}.$$

Proof. In this setting, the spatial Fourier modes decouple and the result follows directly by solving the eigenvalue problem for each mode, where we exclude the stationary state as it is done in L^2_p. Alternatively, we can use the explicit expression of M for a constant σ as in the proof of Lemma 9. □

For a detailed study of the decay behaviour in the case of constant σ, we refer to Miclo and Monmarché [23] and Achleitner, Arnold, and Signorello [2].

3 Perturbation analysis

Given some $\sigma_0 \in \mathrm{L}^\infty$ and a perturbation $\eta \in \mathrm{L}^\infty$, we can compute how the eigenvalues of $A_{\sigma_0+\epsilon\eta}$ are changing for varying ϵ. The eigenvectors $(\rho_\epsilon, j_\epsilon)$ satisfy the equation

$$(\lambda_\epsilon - A_{\sigma_0+\epsilon\eta}) \begin{pmatrix} \rho_\epsilon \\ j_\epsilon \end{pmatrix} = 0$$

and formally taking the derivative with respect to ϵ yields

$$(\lambda'_\epsilon - A'_{\sigma_0+\epsilon\eta}) \begin{pmatrix} \rho_\epsilon \\ j_\epsilon \end{pmatrix} + (\lambda_\epsilon - A_{\sigma_0+\epsilon\eta}) \begin{pmatrix} \rho'_\epsilon \\ j'_\epsilon \end{pmatrix} = 0.$$

Testing with a suitable adjoint $(\rho^*_\epsilon, j^*_\epsilon)$ (made specific later), yields

$$\lambda'_\epsilon \left\langle \begin{pmatrix} \rho^*_\epsilon \\ j^*_\epsilon \end{pmatrix}, \begin{pmatrix} \rho_\epsilon \\ j_\epsilon \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \rho^*_\epsilon \\ j^*_\epsilon \end{pmatrix}, A'_{\sigma+\epsilon\eta} \begin{pmatrix} \rho_\epsilon \\ j_\epsilon \end{pmatrix} \right\rangle,$$

as the adjoint satisfies $(\overline{\lambda_\epsilon} - A^*_{\sigma_0+\epsilon\eta})(\rho^*_\epsilon, j^*_\epsilon) = 0$.

Hence, assuming that both inner products are non-zero, we find that formally λ_ϵ is differentiable with a non-zero derivative and thus by varying ϵ we can change the eigenvalue and improve the spectral gap if the spectral bound is determined by a single eigenvalue.

The key point for this perturbation analysis is to understand when the weight in front of λ'_ϵ can vanish. By using the formulation with the operator $M(\sigma,\lambda)$ from Definition 6, we show that for a simple eigenvalue this prefactor never vanishes.

As a first step, we sharpen the condition for the spectrum from Lemma 8.

Lemma 12. *Suppose λ_0 is separated point in the spectrum of A_σ, then λ_0 is determined by a simple eigenvalue if and only if* $\det(M(\sigma,\lambda))$ *has a simple root.*

Proof. Recall the formula (13) and (14) for the resolvent map. The inverse of M can be written as product of $\det(M)^{-1}$ and the adjugate. The spectral projection Π_{λ_0} to λ_0 is given by the contour integral

$$\left(\int_C R(\lambda, A_\sigma)\begin{pmatrix}a\\b\end{pmatrix}\mathrm{d}\lambda\right)(0) = \int_C \det(M(\sigma,\lambda))^{-1}\,\mathrm{adj}(M(\sigma,\lambda))\int_0^{2\pi}\mathcal{S}_{\sigma,\lambda}^{x\to 2\pi}\begin{pmatrix}b(x)\\a(x)\end{pmatrix}\mathrm{d}x\,\mathrm{d}\lambda,$$

where C is a simple curve separating λ_0 from the rest of the specturum. As both $\mathcal{S}_{\sigma,\lambda}^{x\to 2\pi}$ and $M(\sigma,\lambda)$ are analytic in λ, the integrand has poles only where $\det(M)^{-1}$ has poles. This integral can be computed using the Laurent series. This gives a one-dimensional image if and only if $\det(M)$ has a simple root. □

Remark 13. *The fact that we have only finitely many eigenvalues in any strip $a \le \Re(\lambda) \le b$ with $a > -\|\sigma\|_1/(4\pi)$ means that every eigenvalue is separated from the rest of the spectrum.*

This allows us to determine $M(\sigma,\lambda)$ for λ around a simple eigenvalue λ_0.

Lemma 14. *Suppose that $\lambda_0 \in \mathbb{C}$ is a simple eigenvalue. Then let (ρ_0, j_0) be in* $\ker M(\sigma,\lambda_0)$ *and normalised to $\|(\rho_0, j_0)\|_2 = 1$. Expressing the operator $M(\sigma,\lambda)$ in the basis*

$$V_1 = \begin{pmatrix}\rho_0\\ j_0\end{pmatrix} \qquad \text{and} \qquad V_2 = \begin{pmatrix}-\overline{j_0}\\ \overline{\rho_0}\end{pmatrix},$$

yields the matrix representation $\tilde{M}(\sigma,\lambda_0+\delta\lambda)$. Then around $\lambda = \lambda_0 + \delta\lambda$ it holds that

$$\tilde{M}(\sigma,\lambda_0+\delta\lambda) = \begin{pmatrix} O(\delta\lambda) & b + O(\delta\lambda)\\ c\,\delta\lambda + O((\delta\lambda)^2) & O(\delta\lambda)\end{pmatrix}$$

for small $\delta\lambda$ and constants $b, c \neq 0$.

Proof. By construction V_1 and V_2 form an orthonormal basis. Using the solution operator $\mathcal{S}_{\sigma,\lambda_0}^{\cdot\to\cdot}$ from Definition 6, define

$$\begin{pmatrix}\rho_1(x)\\ j_1(x)\end{pmatrix} = \mathcal{S}_{\sigma,\lambda_0}^{0\to x}\begin{pmatrix}\rho_0\\ j_0\end{pmatrix} \tag{16}$$

and

$$\begin{pmatrix}\rho_2(x)\\ j_2(x)\end{pmatrix} = \mathcal{S}_{\sigma,\lambda_0}^{0\to x}\begin{pmatrix}-\overline{j_0}\\ \overline{\rho_0}\end{pmatrix}. \tag{17}$$

The idea of the Wronskian is to consider $\rho_1(x)j_2(x) - \rho_2(x)j_1(x)$ and by (11) we find

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\rho_1(x)j_2(x) - \rho_2(x)j_1(x)\Big) = 0.$$

As $(\rho_0, j_0) \in \ker M(\sigma,\lambda_0)$, we have that $(\rho_1(2\pi), j_1(2\pi)) = (\rho_0, j_0)$. For V_2 we find with the Wronskian

$$\begin{aligned}\left\langle \begin{pmatrix}\rho_2(2\pi)\\ j_2(2\pi)\end{pmatrix}, V_2\right\rangle &= \rho_0\, j_2(2\pi) - j_0\,\rho_2(2\pi) = \rho_1(2\pi)\, j_2(2\pi) - j_1(2\pi)\,\rho_2(2\pi)\\ &= \rho_1(0)j_2(0) - \rho_2(0)j_1(0) = |\rho_0|^2 + |j_0|^2 = 1.\end{aligned}$$

Hence we have found that $\langle M(\sigma,\lambda_0)V_2, V_2\rangle = 0$. Further recalling that $V_1 \in \ker M(\sigma,\lambda_0)$ and that M and $\tilde{M}$ are analytic with respect to λ shows that

$$\tilde{M}(\sigma,\lambda_0+\delta\lambda) = \begin{pmatrix} O(\delta\lambda) & b + O(\delta\lambda) \\ c\,\delta\lambda + O((\delta\lambda)^2) & O(\delta\lambda) \end{pmatrix}$$

for some constants $b, c \in \mathbb{C}$. By Lemma 12, the determinant $M(\sigma,\lambda) = \tilde{M}(\sigma,\lambda)$ must have a single root at λ_0 so that $b, c \neq 0$. □

We now prove the perturbation result.

Proof of Theorem 2. We argue by contradiction and assume that for $\sigma \in \mathrm{L}^\infty$ the spectral bound is determined by the simple eigenvalue λ_0.

Around λ_0 express $M(\sigma,\lambda)$ in an adapted basis by $\tilde{M}$ as in Lemma 14. For a perturbation $\eta \in \mathrm{L}^\infty$, we then find for some additional constant d that

$$\tilde{M}(\sigma+\epsilon\eta,\lambda_0+\delta\lambda) = \begin{pmatrix} O(\epsilon,\delta\lambda) & b + O(\epsilon,\delta\lambda) \\ c\,\delta\lambda + d\,\epsilon + o((\epsilon,\delta\lambda)) & O(\epsilon,\delta\lambda) \end{pmatrix}.$$

By Duhamel's principle we have that

$$\mathcal{S}^{0\to x}_{\sigma+\epsilon\eta,\lambda_0} = \mathcal{S}^{0\to x}_{\sigma,\lambda_0} + \int_0^x \mathcal{S}^{y\to x}_{\sigma,\lambda_0}\,\epsilon\eta(y)\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\mathcal{S}^{0\to y}_{\sigma+\epsilon\eta,\lambda_0}\,\mathrm{d}y.$$

The bottom left hand term of the matrix $\tilde{M}$ is $\langle \mathcal{S}^{0\to 2\pi}_{\sigma+\epsilon\eta,\lambda_0} V_1, V_2\rangle$. Repeating the Wronskian argument as in the proof of Lemma 14, shows that

$$d = \int_0^{2\pi} \eta(y)\langle \mathcal{S}^{y\to 2\pi}_{\sigma,\lambda_0}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\mathcal{S}^{0\to x}_{\sigma,\lambda_0} V_1, V_2\rangle\,\mathrm{d}y = \int_0^{2\pi}\eta(y)\begin{pmatrix} -j_0 \\ \rho_0 \end{pmatrix}\cdot\mathcal{S}^{y\to 2\pi}_{\sigma,\lambda_0}\begin{pmatrix} -j_1(y) \\ 0 \end{pmatrix}\mathrm{d}y = \int_0^{2\pi} j_1(y)^2\,\eta(y)\,\mathrm{d}y$$

with $j_1(y)$ from (16)[1].

As the eigenvalue is the root of $\det\tilde{M}$, this implies that the eigenvalue behaves as

$$\lambda_0 - \frac{d}{c}\epsilon + o(\epsilon).$$

Hence, if we can find some η such that $\Re(d/c) \neq 0$, we can choose a small $\epsilon \in \mathbb{R}$ so that $\sigma + \epsilon\eta$ would have a smaller spectral bound. This shows the result if $\sigma + \epsilon\eta$ is a valid jump rate, i.e. non-negative.

As σ is a non-trivial jump rate, it is strictly positive in a subset I of positive measure. If $j_1(y)^2$ has not constant complex phase, we can always construct $\eta \in \mathrm{L}^\infty$ with support in I such that $\Re(d/c) \neq 0$ and thus $\sigma + \epsilon\eta$ yields a valid perturbation of the jump rate.

By the equation (11), j_1^2 can have a constant phase only if $\Re\lambda_0 = -\sigma/2$. In this case of a constant phase, either σ is constant and we are back to the special case of Lemma 11 or σ is not constant, where the only other possible value of σ is zero. In the later case, the eigenvalues with $\Re\lambda \approx -\|\sigma\|_1/(4\pi)$ of Proposition 4 show that λ_0 does not determine the spectral bound. □

4 Global optimum by associated Schrödinger equation

The eigenvector equation (15) for A_σ can be rewritten as the following second order equation

$$-\partial_x^2 j + \lambda(\lambda+\sigma)j = 0. \tag{18}$$

[1] One could also prove $c = \int_0^{2\pi}(j_1(y)^2 - \rho_1(y)^2)\,\mathrm{d}y$.

For a fixed $\lambda \in \mathbb{R}$, we then consider the Hamiltonian $H_{\sigma,\lambda}$ defined by

$$H_{\sigma,\lambda}\, j = -\partial_x^2 j + \lambda(\lambda + \sigma(x))\, j. \tag{19}$$

By the construction, our generator A_σ has an eigenvalue λ, if $H_{\sigma,\lambda}$ has a zero eigenvector.

By looking at how the spectrum changes with λ, similar to Cox and Zuazua [9], we show the existence of a slowly decaying eigenvector in the diffusive regime (when σ is large in L^1).

Proposition 15. *Suppose that $\|\sigma\|_1 > 4\pi$ and let*

$$\lambda_s = -\frac{\|\sigma\|_1}{4\pi} + \sqrt{\left(\frac{\|\sigma\|_1}{4\pi}\right)^2 - 1}.$$

Then there exists a $\lambda \in [\lambda_s, 0)$ such that λ is an eigenvalue of the generator A_σ from (6) *of the Goldstein-Taylor system* (4)*.*

Proof. The Hamiltonian $H_{\sigma,\lambda}$ from (19) is for $\lambda \in \mathbb{R}$ a self-adjoint operator and has real eigenvalues $\mu_1, \mu_2, \ldots$ (chosen in increasing order) converging to ∞.

For $\lambda = 0$, the Hamiltonian $H_{\sigma,\lambda}$ is just the Laplacian so that $\mu_1 = 0$ and $\mu_2 = 1$. By considering the perturbation around $\lambda = 0$, we find for a small enough ϵ that the first eigenvalues of $H_{\sigma,0-\epsilon}$ satisfy $\mu_1 < 0$ and $\mu_2 > 0$.

For the given σ, find a shift ϕ such that

$$\int_0^{2\pi} \sigma(x)\cos(2(x-\phi))\,\mathrm{d}x = 0.$$

Such a shift ϕ exists thanks to the intermediate value lemma as $\int_0^{2\pi} \sigma(x)\cos(2(x-\phi))\,\mathrm{d}x$ is a continuous function of ϕ and $\int_0^{2\pi} \sigma(x)\cos(2x)\,\mathrm{d}x = -\int_0^{2\pi} \sigma(x)\cos(2(x-\pi/2))\,\mathrm{d}x$.

Now we consider the two test functions

$$j_1(x) = 1$$

and

$$j_2(x) = \sin(x-\phi)$$

which are linearly independent.

For these test functions we find

$$\langle j_1, H_{\sigma,\lambda_s} j_1\rangle = \lambda_s(2\pi\lambda_s + \|\sigma\|_1) < 0$$

and

$$\langle j_2, H_{\sigma,\lambda_s} j_2\rangle = \frac{2\pi}{2}\left(1 + \lambda_s\left(\lambda_s + \frac{\|\sigma\|_1}{2\pi}\right)\right) \le 0.$$

Hence we must have $\mu_2(\lambda_s) \le 0$. As $\mu_2(0-\epsilon) > 0$ and the spectrum of $H_{\sigma,\lambda}$ varies continuously with respect to λ, this shows that there exist some $\bar{\lambda} \in [\lambda_s, -\epsilon)$ such that $\mu_2 = 0$ for $H_{\sigma,\bar{\lambda}}$. Therefore, $H_{\sigma,\bar{\lambda}}$ has a zero eigenvector and the previous discussion implies that then $\bar{\lambda}$ is an eigenvalue for the generator A_σ. □

We now conclude that the constant $\sigma = 2$ yields the fastest decay rate.

Proof of Theorem 3. By Lemma 11, the choice $\sigma = 2$ yields the spectral bound -1.

For another σ with $\|\sigma\|_1 \le 4\pi$, the bound from the velocity relaxation (7) in Proposition 4 shows that the spectral bound is greater than $-\|\sigma\|_1/4\pi \ge -1$.

For another σ with $\|\sigma\|_1 > 4\pi$, Proposition 15 implies that the spectral bound needs to be worse than $-\|\sigma\|_1/4\pi + \sqrt{(\|\sigma\|_1/4\pi)^2 - 1}$ which is in turn greater than -1 when $\|\sigma\|_1 > 4\pi$. □

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