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Mixed semi-continuous perturbation of time-dependent maximal monotone operators and subdifferentials.

Charles Castaing, Christiane Godet-Thobie, Soumia Saïdi and Manuel D.P. Monteiro Marques

Abstract. We are concerned in the present work with the existence of absolutely continuous solutions to a class of evolution problems governed by time-dependent maximal monotone operators $A(t)$ of the form $-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t)) + F(t, u(t))$, where the perturbation is a sum of a mixed semi-continuous compact set-valued map F and a single-valued map f . New variants dealing with a class of time-dependant subdifferential operators of the form $-\frac{du}{dt}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) + F(t, u(t))$ are also investigated. Some applications are given.

Mathematics Subject Classification (2010). 34A60, 34B15, 47H10, 49J52, 49J53, 35B10.

Keywords. maximal monotone operator, pseudo-distance, perturbation, subdifferential operator, evolution inclusion, mixed semi-continuity .

1. Introduction

Let H be a separable Hilbert space and $I := [0, T]$. Let $F : I \times H \rightrightarrows H$ be a mixed semi-continuous set-valued map and $f : I \times H \rightarrow H$ be a single-valued mapping. In this paper, we are mainly interested at first in the perturbed evolution problem

$$-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I, \quad (1.1)$$

governed by a time-dependent maximal monotone operator $A(t)$ in the vein of Kunze-Marques and Vladimirov's works [39, 51]. Here, the dependence $t \mapsto A(t)$ is of absolutely continuous variation, in the sense that there exists $\beta \in W^{1,2}(I, \mathbb{R})$ such that

$$\text{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)| \quad \forall t, s \in I,$$

where $\text{dis}(\cdot, \cdot)$ is the pseudo-distance between maximal monotone operators introduced by Vladimirov [51], cf. (2.3). Secondly we investigate the evolution problem with perturbation $G + F$

$$-\frac{du}{dt}(t) \in A(t)u(t) + G(t, u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I, \quad (1.2)$$

where $G : I \times H \rightrightarrows H$ is a set-valued map.

Tertio we are interested in the perturbed evolution problem

$$-\frac{du}{dt}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I, \quad (1.3)$$

governed by the subdifferential of a time-dependent proper lower semi-continuous convex function $\varphi(t, \cdot)$ of H into $[0, +\infty]$. There is a plethora of results dealing with convex weakly compact upper semi-continuous (or lower semi-continuous) perturbations for evolution inclusions involving maximal monotone (or m -accretive) operators, subdifferential operators, and sweeping processes. For first-order related problems, we refer the reader to [1, 13, 14, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 29, 31, 40, 47] and the references therein. Other corresponding results can be found in the case of a single-valued perturbation, we cite for instance [2, 3, 15, 23, 28, 35, 38, 43, 44, 45, 53].

About the problems under consideration, the set-valued map F is mixed semi-continuous in the sense of Tolstonogov [48], that is for each $t \in I$, at each $x \in H$ where $F(t, x)$ is convex, $F(t, \cdot)$ is upper semi-continuous, and whenever $F(t, x)$ is not convex, $F(t, \cdot)$ is lower semi-continuous on some neighbourhood of x . The main ingredient to study evolution problems with such perturbation is a selection theorem due to Tolstonogov [48] (see [49] for the infinite dimensional setting). There exist few results on mixed semi-continuous perturbation of sweeping process and maximal monotone operators in the literature, see [6, 7, 8, 9, 11, 30, 32, 33, 34]. So, the problems considered in this paper are new contributions on mixed semi-continuous perturbation of time-dependent maximal monotone operators and time-dependent subdifferential operators in the infinite dimensional framework. Perturbed differential inclusions involving time-dependent maximal monotone operators (resp. time-dependent subdifferential operators) by a single-valued map f (or a set-valued map F) in [3] (resp. [44, 45, 46]) lead naturally to consider general differential inclusions with the perturbations $f + F$ in (1.3) and $G + F$ in (1.2). Actually, the existence of solutions for such evolution problems remains a very well-active area of research. It is worth pointing out that the class of sweeping processes perturbed by a sum $F + f$, has been investigated [10], in the absolutely continuous framework. The author in [42] has developed several works on that topic, see also [41] in the bounded variation framework. Related results to evolution problems governed by time-dependent maximal monotone operators have been recently established, see [4, 6, 5].

The paper is organized as follows. In section 2, we introduce notation and the necessary background material. In section 3, we deal with mixed semi-continuous perturbation of evolution inclusions (1.1)-(1.2) and (1.3) making use of a selection theorem due to Tolstonogov [49] and Kakutani-Ky Fan fixed

point theorem. We give also a variant with a single-valued perturbation using Schauder fixed point theorem. Section 4 is devoted to study the existence and uniqueness of periodic solution.

2. Notation and Preliminaries

From now on $I := [0, T]$ ($T > 0$) is an interval of \mathbb{R} and H is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. We will denote by \overline{B}_H the closed unit ball of H and $\overline{B}_H[x, r]$ its closed ball of center $x \in H$ and radius $r > 0$. For any interval I of \mathbb{R} , we denote by $\mathcal{L}(I)$ (resp. $\mathcal{B}(H)$) the σ -algebra of measurable sets of I (resp. Borel σ -algebra of measurable sets of H).

On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \rightarrow H$, we consider the norm of uniform convergence on I $\|x\|_\infty = \sup_{t \in I} \|x(t)\|$. By $L^p_H(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \rightarrow H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \leq p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|$). By $W^{1,1}(I, H)$ and $W^{1,2}(I, H)$, we denote the spaces of absolutely continuous functions from I to H with derivatives in $L^1_H(I)$ and $L^2_H(I)$ respectively. A map $u : [0, T] \rightarrow H$ is absolutely continuous if there exists an integrable mapping v such that $u(t) = u_0 + \int_0^t v(s) ds$; in this case $\dot{u} = v$ a.e. on I . Let E be a Banach space and E' its topological dual, we denote by $\sigma(E, E')$ the weak topology on E .

We introduce in the following the definition and some properties of maximal monotone operators needed in the proofs of our results, we refer the reader to [12, 15, 52] for their basic theory and more details.

Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator. The domain, the range and the graph of A are the following sets

$$\begin{aligned} D(A) &= \{x \in H : Ax \neq \emptyset\}, \\ R(A) &= \{y \in H : \exists x \in D(A), y \in Ax\} = \cup\{Ax : x \in D(A)\}, \\ \text{gph}(A) &= \{(x, y) \in H \times H : x \in D(A), y \in Ax\}. \end{aligned}$$

We say that $A : D(A) \subset H \rightrightarrows H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $(x_i, y_i) \in \text{gph}(A)$, $i = 1, 2$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H stands for the identity mapping of H .

If A is a maximal monotone operator then, for every $x \in D(A)$, Ax is nonempty, closed and convex. So that, the projection of the origin into Ax , $A^0(x)$, exists and is unique.

For $\lambda > 0$, we define the resolvent and the Yosida approximation of A respectively by, $J_\lambda^A = (I_H + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda} (I_H - J_\lambda^A)$. These operators are both single-valued and defined on the whole space H , and we have

$$J_\lambda^A x \in D(A) \text{ and } A_\lambda(x) \in A(J_\lambda^A x), \text{ for every } x \in H. \quad (2.1)$$

$$\|A_\lambda(x)\| \leq \|A^0(x)\| \quad \forall x \in D(A). \tag{2.2}$$

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators, then we denote by $\text{dis}(A, B)$ (see [51]) the pseudo-distance between A and B defined by

$$\text{dis}(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + \|y\| + \|y'\|} : (x, y) \in \text{gph}(A), (x', y') \in \text{gph}(B) \right\}. \tag{2.3}$$

Clearly, $\text{dis}(A, B) \in [0, +\infty]$, $\text{dis}(A, B) = \text{dis}(B, A)$ and $\text{dis}(A, B) = 0$ iff $A = B$.

Let φ be a lower semi-continuous convex function from H into $\mathbb{R} \cup \{+\infty\}$ which is proper in the sense that its effective domain $\text{dom } \varphi$ defined by $\text{dom } \varphi := \{x \in H : \varphi(x) < +\infty\}$ is nonempty and, as usual, its Fenchel conjugate is defined by $\varphi^*(v) := \sup_{x \in H} [\langle v, x \rangle - \varphi(x)]$ and its Moreau regular-

ization by $\varphi_\lambda(x) = \inf \{ \frac{1}{2} \|x - y\|^2 + \varphi(y) \}$.

The subdifferential $\partial\varphi(x)$ of φ at $x \in \text{dom } \varphi$ is

$$\partial\varphi(x) = \{v \in H : \varphi(y) \geq \langle v, y - x \rangle + \varphi(x) \quad \forall y \in \text{dom } \varphi\},$$

and its effective domain is $D(\partial\varphi) = \{x \in H : \partial\varphi(x) \neq \emptyset\}$.

It is well known that if φ is a proper lower semi-continuous convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator.

Let $F : \Omega \rightrightarrows Y$ be a set-valued map from an open set Ω of a normed space X into a normed space Y . The set-valued map F is upper (resp. lower) semi-continuous at $x_0 \in \Omega$ if for every open set U of Y such that $F(x_0) \subset U$ (resp. $F(x_0) \cap U \neq \emptyset$) there exists a neighbourhood V of x_0 such that for all $x \in V$, we have $F(x) \subset U$ (resp. $F(x) \cap U \neq \emptyset$).

The set-valued map F is upper (resp. lower) semi-continuous if it is upper (resp. lower) semi-continuous at each point.

Let $F : T \rightrightarrows Y$ be a set-valued map from a measurable space (T, \mathcal{T}) into a metric space Y . The set-valued map F is measurable if for any open set U of Y , we have $F^{-1}(U) \in \mathcal{T}$, with $F^{-1}(U) = \{t \in T : F(t) \cap U \neq \emptyset\}$.

A function $f : I \times H \rightarrow \mathbb{R}$ is said to be a Carathéodory map if it is measurable with respect to the first variable and continuous with respect to the second one.

We refer to [25] for details concerning convex analysis and measurable set-valued-maps.

For the proof of our theorems we will need the following selection theorem (see Theorem 6.6. [49]).

Theorem 2.1. *Let Y be a separable Banach space and $J \subset \mathbb{R}$. Let $G : J \times Y \rightrightarrows Y$ be a set-valued map with compact values that satisfies*

- (j) G is $\mathcal{L}(J) \otimes \mathcal{B}(Y)$ -measurable;
- (jj) for every $t \in J$, at each $x \in Y$ such that $G(t, x)$ is convex, the set-valued map $G(t, \cdot)$ is upper semi-continuous on Y and whenever $G(t, x)$ is not convex, the set-valued map $G(t, \cdot)$ is lower semi-continuous on some neighbourhood of x ;

(jjj) there exists a function $g : J \times Y \rightarrow \mathbb{R}_+$ of Carathéodory type which is integrably bounded on bounded subsets of Y and which is such that

$$G(t, x) \cap \overline{B}_Y[0, g(t, x)] \neq \emptyset \quad \text{for any } x \in Y \text{ and a.e. } t \in J.$$

Then, for any $\varepsilon > 0$ and any compact set $K \subset C_Y(J)$, there is a nonempty closed convex set valued map $\Psi : K \rightrightarrows L_Y^1(J)$ which has a sequentially closed graph with respect to the norm of uniform convergence in K and the weak topology $\sigma(L_Y^1(J), L_Y^\infty(J))$ in $L_Y^1(J)$ and which is such that for any $x \in K$ and $h \in \Psi(x)$, one has for a.e. $t \in J$

$$\begin{aligned} h(t) &\in G(t, x(t)) \\ \|h(t)\| &\leq g(t, x(t)) + \varepsilon. \end{aligned}$$

We close this section by recalling the following lemma (Lemma A.5. [15]).

Lemma 2.1. Let $m \in L_{\mathbb{R}}^1([0, T])$ such that $m \geq 0$ a.e. on $[0, T]$ and let α be a positive constant. Consider a continuous function $\phi : [0, T] \rightarrow \mathbb{R}$ satisfying

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}\alpha^2 + \int_0^t m(s)\phi(s)ds \quad \text{for all } t \in [0, T].$$

Then, one has

$$|\phi(t)| \leq \alpha + \int_0^t m(s)ds \quad \text{for all } t \in [0, T].$$

3. Existence results

We are going to prove our main results under the following assumptions:

Let for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator such that

(H1) There exists a function $\beta \in W^{1,2}(I, \mathbb{R})$ which is non-negative on $[0, T[$ and non-decreasing with $\beta(T) < \infty$ and $\beta(0) = 0$ such that

$$\text{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)|, \quad \forall t, s \in I.$$

(H2) There exists a non-negative real number c such that

$$\|A^0(t, x)\| \leq c(1 + \|x\|) \quad \text{for } t \in I, x \in D(A(t)).$$

(H3) $t \mapsto J_\lambda(t)(x) = (I_H + \lambda A(t))^{-1}x$ is measurable for every $\lambda > 0$ and for every $x \in H$.

Let $f : I \times H \rightarrow H$ be a mapping such that

(i) $f(\cdot, x)$ is measurable on I for every $x \in H$;

(ii) $\|f(t, x) - f(t, y)\| \leq m\|x - y\|$ for all $(t, x, y) \in I \times H \times H$ for some positive constant m ;

(iii) $\|f(t, x)\| \leq m$ for all $(t, x) \in I \times H$.

or again

(ii) for every $\eta > 0$, there exists a non-negative function $\gamma_\eta(\cdot) \in L_{\mathbb{R}}^2(I)$ such that, for all $t \in I$ and for any $u, w \in \overline{B}_H[0, \eta]$

$$\|f(t, u) - f(t, w)\| \leq \gamma_\eta(t)\|u - w\|;$$

(iii) there exists a non-negative function $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $t \in I$ and for all $u \in H$, one has

$$\|f(t, u)\| \leq \beta(t)(1 + \|u\|).$$

Let $F : I \times H \rightrightarrows H$ be a compact set-valued map such that

- (j) F is $\mathcal{L}(I) \otimes \mathcal{B}(H)$ -measurable;
- (jj) for every $t \in I$, at each $x \in H$ such that $F(t, x)$ is convex, the set-valued map $F(t, \cdot)$ is upper semi-continuous and whenever $F(t, x)$ is not convex, $F(t, \cdot)$ is lower semi-continuous on some neighbourhood of x ;
- (jjj) $F(t, x) \cap \alpha(t)(1 + \|x\|)\overline{B}_H \neq \emptyset$ for all $(t, x) \in I \times H$ for some measurable function α with $0 < \alpha(t) < 1$ for all $t \in I$.

Let $\varphi : I \times H \rightarrow [0, +\infty]$ be a map satisfying:

- (H₁) For each $t \in I$, the function $x \mapsto \varphi(t, x)$ is proper lower semi-continuous and convex.
- (H₂) There exist a ρ -Lipschitz function $k : H \rightarrow \mathbb{R}_+$ and an absolutely continuous function $a : I \rightarrow \mathbb{R}$, with a non-negative derivative $\dot{a} \in L^2_{\mathbb{R}}(I)$, such that

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x)|a(t) - a(s)| \text{ for every } (t, s, x) \in I \times I \times H.$$

3.1. Mixed semi-continuous perturbation problem with time-dependent maximal monotone operators

For the sake of completeness we recall and summarize some important results [3, 4, 6].

Theorem 3.1. *Assume that for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying (H1)-(H2). Let $f : I \times H \rightarrow H$ be such that for every $x \in H$ the mapping $f(\cdot, x)$ belongs to $L^\infty_H(I)$, and that f satisfies the linear growth condition: there exists a non-negative real constant M such that $\|f(t, x)\| \leq M(1 + \|x\|)$ for $t \in I$, $x \in H$. Moreover, for every $R > 0$ there exists a non-negative real function $\alpha_R(\cdot) \in L^1_{\mathbb{R}}(I)$ such that*

$$\|f(t, x) - f(t, y)\| \leq \alpha_R(t)\|x - y\|, \quad \forall t \in I, \forall x, y \in \overline{B}_H[0, R].$$

Then, for all $u_0 \in D(A(0))$, the problem

$$-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t)) \text{ a.e. } t \in I,$$

has a unique absolutely continuous solution u that satisfies

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \text{ a.e. } t \in I,$$

for some real constant $K > 0$, which depends on $\|u_0\|, c, M, T$, and β .

Lemma 3.1. *Let for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ be a maximal monotone operator satisfying (H1)-(H2). Then, one has*

(J) For all $u_0 \in D(A(0))$ the problem

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) \text{ a.e. } t \in I \\ u(t) \in D(A(t)) \quad \forall t \in I, \\ u(0) = u_0, \end{cases}$$

has a unique absolutely continuous solution u with the property:

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \text{ a.e. } t \in I, \quad (3.1)$$

for some non-negative real constant K depending on $\|u_0\|$, c , T , and β .

($\mathcal{J}\mathcal{J}$) Assume in addition that (H3) is satisfied. Then, the composition operator $\mathcal{A} : D(\mathcal{A}) \subset L^2_H(I) \rightrightarrows L^2_H(I)$ defined by

$$\mathcal{A}u = \{v \in L^2_H(I) : v(t) \in A(t)u(t) \text{ a.e. } t \in I\}$$

for each $u \in D(\mathcal{A})$ where

$$D(\mathcal{A}) := \{u \in L^2_H(I) : \exists y \in L^2_H(I), (u(t), y(t)) \in \text{gph}(A(t)) \text{ a.e. } t \in I\},$$

is maximal monotone. Consequently, the graph of \mathcal{A} is strongly-weakly sequentially closed in $L^2_H(I) \times L^2_H(I)$.

Proof. (\mathcal{J}) follows from Theorem 3.1.

Taking into account the fact (\mathcal{J}), it is clear that $D(\mathcal{A})$ is nonempty and \mathcal{A} is well defined. On the other hand, it is easy to see that \mathcal{A} is monotone. For the maximality, we need to check that $R(I_{L^2_H(I)} + \lambda\mathcal{A}) = L^2_H(I)$ for each $\lambda > 0$. Let $g \in L^2_H(I)$. Then, from (H3), $t \mapsto v(t) := (I_H + \lambda A(t))^{-1}g(t) = J_\lambda(t)(g(t)) = g(t) - \lambda A_\lambda(t)(g(t))$ is measurable by noting that $J_\lambda(t)(\cdot)$ is non expansive and $A_\lambda(t)(\cdot)$ is $\frac{2}{\lambda}$ -Lipschitz. Set for each $t \in I$,

$$h(t) := \lambda A_\lambda(t)(g(t)) = \lambda(A_\lambda(t)(g(t)) - A_\lambda(t)(u(t)) + \lambda A_\lambda(t)(u(t)))$$

where u denotes the absolutely continuous solution of the unperturbed problem above. Then, h is measurable and $\|h(t)\| \leq 2\|g(t) - u(t)\| + \lambda\|A_\lambda(t)(u(t))\|$. So we deduce that $h \in L^2_H(I)$ because $g \in L^2_H(I)$ and $t \mapsto A_\lambda(t)(u(t)) \in L^\infty_H(I)$ using (2.2) and (H2). This proves that $v \in L^2_H(I)$ and $g \in v + \lambda\mathcal{A}v$, so that $R(I_{L^2_H(I)} + \lambda\mathcal{A}) = L^2_H(I)$. \square

In the light of the mentioned results above, we develop some new variants for time-dependent maximal monotone operators involving a perturbation $f + F$, where F is a mixed semi-continuous set-valued map.

Theorem 3.2. Let $I := [0, 1]$. Assume that for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator with $D(A(t))$ is ball-compact for every $t \in I$, satisfying (H1)-(H2)-(H3). Let $f : I \times H \rightarrow H$ be a mapping satisfying (i)-(ii)-(iii). Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj)-(jjj). Then, for all $u_0 \in D(A(0))$, the problem

$$-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t)) + F(t, u(t)) \text{ a.e. } t \in I,$$

has at least an absolutely continuous solution u .

Proof. Step 1 Let $u_0 \in D(A(0))$, $\varepsilon > 0$. Let $u : [0, 1] \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in A(t)u(t) \text{ a.e. } t \in I, \\ u(0) &= u_0 \in D(A(0)). \end{aligned}$$

The existence and uniqueness of such a solution follow from Lemma 3.1. Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \alpha(t)(1 + r(t)) + \varepsilon + m \quad \text{a.e. with } r(0) = \sup_{t \in [0,1]} \|u(t)\|.$$

Since $\dot{r} + \varepsilon \in L^\infty_{\mathbb{R}_+}(I) \subset L^1_{\mathbb{R}_+}(I)$, the set

$$M := \{h \in L^1_H(I) : \|h(t)\| \leq \dot{r}(t) + \varepsilon \text{ a.e.}\},$$

is clearly convex $\sigma(L^1_H(I), L^\infty_H(I))$ -compact.

Consider the problem of the form

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + f(t, u(t)) + h(t), & h \in M \quad \text{a.e. } t \in I, \\ u(0) = u_0 \in D(A(0)). \end{cases}$$

The existence and uniqueness of solutions follow from Theorem 3.1, by considering the perturbation defined by $f(t, x) + h(t)$. Denote by u_h , for any $h \in M$, the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

Using the monotonicity of $A(t)$, for all $t \in I$, one has the estimation

$$\frac{1}{2} \|u_h(t) - u(t)\|^2 \leq \int_0^t \|f(s, u_h(s)) + h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows

$$\|u_h(t) - u(t)\| \leq \int_0^t \|f(s, u_h(s)) + h(s)\| ds,$$

so that for all $t \in I$, one gets

$$\|u_h(t)\| \leq r(0) + \int_0^t [\dot{r}(s) + \varepsilon + m] ds = r(t) + \varepsilon t + mt \leq r(t) + m + \varepsilon. \quad (3.2)$$

Main fact: $L := \{u_h : h \in M\}$ is compact in $C_H(I)$.

The essential tools to prove it are the above estimate, the ball compactness assumption on $D(A(t))$, along with the maximal monotone property of the extension \mathcal{A} to $L^2_H(I)$ of $A(t)$ defined in Lemma 3.1. In view of the estimate in Theorem 3.1 with the perturbation $f + h$, one gets $\|\dot{u}_h(t)\| \leq K(1 + \dot{\beta}(t))$, for a.e. $t \in I$, where $K = K(\|u_0\|, c, I, m, \alpha(\cdot), \beta(\cdot))$ for all $h \in M$. Shortly $\|\dot{u}_h(t)\| \leq \gamma(t) := K(1 + \dot{\beta}(t))$ with $\gamma \in L^2_{\mathbb{R}_+}(I)$.

So, for any $h \in M$, \dot{u}_h is bounded in $L^2_H(I)$, that is,

$$S = \sup_{h \in M} \|\dot{u}_h\|_{L^2_H(I)} < +\infty. \quad (3.3)$$

It is clear that $L := \{u_h : h \in M\}$ is equicontinuous, Indeed, for any $s, t \in I$, one has

$$\begin{aligned} \|u_h(t) - u_h(s)\| &= \left\| \int_s^t \dot{u}_h(\tau) d\tau \right\| \\ &\leq (t-s)^{1/2} \left(\int_0^1 \|\dot{u}_h(\tau)\|^2 d\tau \right)^{1/2} \\ &\leq (t-s)^{1/2} \mathcal{S}. \end{aligned}$$

Moreover, by (3.2), one has $\sup_{h \in M} \|u_h(t)\| < +\infty$ for any $t \in I$. Since $u_h(t) \in D(A(t))$ for all $t \in I$ and for all $h \in M$, along with the ball-compactness of $D(A(t))$, for each $t \in I$, the set $\{u_h(t), h \in M\}$ is relatively compact in H . Thus, L is relatively compact in $\mathcal{C}_H(I)$.

It remains to show that L is compact in $\mathcal{C}_H(I)$. Let $(h_n)_n$ be a sequence in M and $(u_{h_n})_n$ be a sequence in L such that $u_{h_n} \rightarrow u \in \mathcal{C}_H(I)$. By weak compactness of M , we may assume that $h_n \rightarrow h \in M$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology.

Recall that u_{h_n} is the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t)u_{h_n}(t) + h_n(t) + f(t, u_{h_n}(t)), & h_n \in M, \text{ a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in D(A(0)). \end{cases}$$

By hypothesis on f , one writes $\|f(t, u_{h_n}(t))\| \leq m$ for all $t \in I$. Since $u_{h_n} \rightarrow u$ in $\mathcal{C}_H(I)$, along with the Lipschitz behavior of f with respect to its second variable yield (via the dominated convergence theorem) $f(\cdot, u_{h_n}(\cdot)) \rightarrow f(\cdot, u(\cdot))$ in $L_H^2(I)$. As a consequence, $f(\cdot, u_{h_n}(\cdot)) \rightarrow f(\cdot, u(\cdot))$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology. Furthermore, in view of (3.3), the sequence (\dot{u}_{h_n}) is bounded in $L_H^2(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{u}_{h_n})_n$ weakly converges in $L_H^2(I)$ to some element $z(\cdot)$.

For any integer n and any $y \in H$ and for $0 \leq s \leq t \leq 1$, relying on the absolute continuity of $(u_{h_n})_n$, one writes

$$\int_0^1 \langle y \mathbf{1}_{[s,t]}(\tau), \dot{u}_{h_n}(\tau) \rangle d\tau = \langle y, u_{h_n}(t) - u_{h_n}(s) \rangle.$$

Next, passing to the limit in the equality yields

$$\langle y, \int_s^t z(\tau) d\tau \rangle = \langle y, u(t) - u(s) \rangle.$$

Therefore, given any $s, t \in I$ with $s \leq t$, we get $\int_s^t z(\tau) d\tau = u(t) - u(s)$, and hence $u(\cdot)$ is absolutely continuous and z coincides almost everywhere in I with $\dot{u}(\cdot)$. Thus, $\dot{u} \in L_H^2(I)$ and $\dot{u}_{h_n} \rightarrow \dot{u}$ weakly in $L_H^2(I)$. As $\dot{u}_{h_n} + f(\cdot, u_{h_n}(\cdot)) + h_n(\cdot) \rightarrow \dot{u} + f(\cdot, u(\cdot)) + h(\cdot)$ weakly in $L_H^2(I)$, so that by the maximal monotone extension \mathcal{A} of $A(t)$ cf. Lemma 3.1, we get

$$-\dot{u}(t) \in A(t)u(t) + h(t) + f(t, u(t)), \quad h \in M, \text{ a.e. } t \in I.$$

Denote by u_h the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + h(t) + f(t, u_h(t)), & h \in M, \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

By uniqueness, one has $u = u_h$. Thus, the set L is compact in $\mathcal{C}_H(I)$.

Step 2 Now, given $\varepsilon > 0$, the compact set L in $\mathcal{C}_H(I)$ being constructed, we may apply Theorem 2.1 to the mixed semi-continuous set-valued map F by taking $g(t, x) = \alpha(t)(1 + \|x\|)$, $(t, x) \in I \times H$. Then, there is a nonempty closed convex set valued-map $\Psi : L \rightrightarrows L_H^1(I)$ whose graph is sequentially closed with respect to the topology of uniform convergence in L and the weak topology in $L_H^1(I)$ and such that, for any $x \in L$ and $y \in \Psi(x)$, for a.e. $t \in I$, one has

$$y(t) \in F(t, x(t)) \text{ and } \|y(t)\| \leq \alpha(t)(1 + \|x(t)\|) + \varepsilon.$$

This along with $u_h \in L$ and $y_h \in \Psi(u_h)$, one has for a.e. $t \in I$ $y_h(t) \in F(t, u_h(t))$ and $\|y_h(t)\| \leq \alpha(t)(1 + \|u_h(t)\|) + \varepsilon \leq \alpha(t)(1 + r(t) + \varepsilon + m) + \varepsilon = \alpha(t)(1 + r(t)) + \alpha(t)\varepsilon + \alpha(t)m + \varepsilon \leq \alpha(t)(1 + r(t)) + \varepsilon + m + \varepsilon = \dot{r}(t) + \varepsilon$;

taking into account the fact $\|u_h(t)\| \leq r(t) + \varepsilon + m$. It's easy to see that $y_h \in M$, hence $\Psi(u_h) \subset M$ for all $u_h \in L$. Thus, there is a non-empty convex weakly-compact set-valued map $\Psi : L \rightrightarrows L_H^1(I)$ with the following properties

- (I) $\Psi(x) \subset \{h \in M : h(t) \in F(t, x(t)) \text{ a.e.}\} \subset M$, for each $x \in L$,
- (II) the set-valued map Ψ has closed graph, in the sense, for any $y_n \in \Psi(x_n)$ such that (x_n) converging uniformly to $x \in L$ and $(y_n) \sigma(L_H^1(I), L_H^\infty(I))$ -converging in $L_H^1(I)$ to y , then, $y \in \Psi(x)$, equivalently, $\Psi : L \rightrightarrows M$ is upper semi-continuous from $L \subset \mathcal{C}_H(I)$ to M endowed with the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology.

Step 3 We finish the proof by using Kakutani-Ky Fan fixed point theorem.

For each $h \in M$, let us set $\Phi(h) = \Psi(u_h)$ where

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + h(t) & h \in M, \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

Then, it's clear that Φ is a convex weakly compact set-valued map from M to M . We claim that Φ has a fixed point h , i.e., $h \in \Phi(h) = \Psi(u_h)$, then, by (I) $h(t) \in F(t, u_h(t))$ a.e., one has

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)), \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e., thus proving that

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + F(t, u_h(t)) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)), \end{cases}$$

has at least a solution. For this purpose, by virtue of Kakutani-Ky Fan fixed point theorem, we need to prove that $\Phi : M \rightrightarrows M$ is upper semi-continuous with non-empty convex weakly compact values from M into itself, equivalently, the graph of Φ is sequentially closed in $M \times M$, for M equipped with

the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology.

Indeed, for each $n \in \mathbb{N}$, let $g_n \in \Phi(h_n) = \Psi(u_{h_n})$ with

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t)u_{h_n}(t) + f(t, u_{h_n}(t)) + h_n(t) & \text{a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in D(A(0)), \end{cases}$$

with $u_{h_n} \in L$ by our definition of L and $(g_n) \subset M$ which $\sigma(L_H^1(I), L_H^\infty(I))$ -converges to $g \in M$. Further, by compactness of L , $(u_{h_n})_n$ uniformly converges to u_h with

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

By the property (ll) of the set-valued map Ψ , one has $g \in \Psi(u_h) = \Phi(h)$. So, the graph of $\Phi : M \rightrightarrows M$ is closed, hence the convex weakly compact valued mapping Φ admits a fixed point by Kakutani-Ky Fan theorem, $h \in \Phi(h) = \Psi(u_h)$. By the property (l), we get $h(t) \in F(t, u_h(t))$ a.e. which implies

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)), \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e. just finishing the proof of the theorem. □

We end this subsection by the following theorem.

Theorem 3.3. *Let $I := [0, 1]$. Assume that for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator with $D(A(t))$ is ball-compact for every $t \in I$ satisfying (H1)-(H2)-(H3). Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj) and*

(j') $F(t, x) \cap \alpha(t)(1 + \|x\|)\overline{B}_H \neq \emptyset$ for all $(t, x) \in I \times H$ for some measurable function α with $0 < \alpha(t) < \frac{1}{2}$ for all $t \in I$.

Let $G : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj) and (j'j') $G(t, x) \cap m\overline{B}_H \neq \emptyset$, for all $(t, x) \in I \times H$, for some positive constant m with $0 < m < \frac{1}{2}$ for all $t \in I$.

Then, for all $u_0 \in D(A(0))$, the problem

$$-\frac{du}{dt}(t) \in A(t)u(t) + G(t, u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I,$$

has at least an absolutely continuous solution u .

Proof. Step 1 Let $u_0 \in D(A(0))$, $\varepsilon > 0$. Let $u : [0, 1] \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in A(t)u(t) \quad \text{a.e. } t \in I, \\ u(0) &= u_0 \in D(A(0)). \end{aligned}$$

The existence and uniqueness of such a solution follow from Lemma 3.1. Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \alpha(t)(1 + r(t)) + \varepsilon + m \quad \text{a.e. with } r(0) = \sup_{t \in [0, 1]} \|u(t)\|.$$

Let us consider the sets

$$\begin{aligned} M_1 &:= \{h \in L^1_H(I) : \|h(t)\| \leq m + \varepsilon/2 \text{ a.e.}\}, \\ M_2 &:= \{h \in L^1_H(I) : \|h(t)\| \leq \alpha(t)(1 + r(t)) + \varepsilon/2 \text{ a.e.}\}, \\ M &:= \{h \in L^1_H(I) : \|h(t)\| \leq \dot{r}(t) \text{ a.e.}\}, \end{aligned}$$

are clearly convex $\sigma(L^1_H(I), L^\infty_H(I))$ -compact with $M = M_1 + M_2$.

Now, denote by u_h , for any $h \in M$, the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

The existence and uniqueness of solutions follows from Theorem 3.1, by considering the perturbation $h(t)$. Using the monotonicity of $A(t)$, for all $t \in I$, one has the estimation

$$\frac{1}{2} \|u_h(t) - u(t)\|^2 \leq \int_0^t \|h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows

$$\|u_h(t) - u(t)\| \leq \int_0^t \|h(s)\| ds$$

so that for all $t \in I$, one gets

$$\|u_h(t)\| \leq r(0) + \int_0^t \dot{r}(s) ds = r(t). \tag{3.4}$$

Main fact: $L := \{u_h : h \in M\}$ is compact in $\mathcal{C}_H(I)$.

This follows essentially from the ball compactness assumption on $D(A(t))$, along with the maximal monotone property of the extension \mathcal{A} to $L^2_H(I)$ of $A(t)$ defined in Lemma 3.1. In view of estimate in Theorem 3.1 with perturbation h , one gets $\|\dot{u}_h(t)\| \leq K(1 + \dot{\beta}(t))$, for a.e. $t \in I$, where $K = K(\|u_0\|, c, I, \alpha, \varepsilon, m, \beta(\cdot))$ for all $h \in M$. Shortly $\|\dot{u}_h(t)\| \leq \gamma(t) := K(1 + \dot{\beta}(t))$ with $\gamma \in L^2_{\mathbb{R}_+}(I)$. So, for any $h \in M$, \dot{u}_h is bounded in $L^2_H(I)$, that is,

$$\mathcal{S} = \sup_{h \in M} \|\dot{u}_h\|_{L^2_H(I)} < +\infty. \tag{3.5}$$

It is clear that $L := \{u_h : h \in M\}$ is equicontinuous. Indeed, for any $s, t \in I$, one has

$$\begin{aligned} \|u_h(t) - u_h(s)\| &= \left\| \int_s^t \dot{u}_h(\tau) d\tau \right\| \\ &\leq (t - s)^{1/2} \left(\int_0^1 \|\dot{u}_h(\tau)\|^2 d\tau \right)^{1/2} \\ &\leq (t - s)^{1/2} \mathcal{S}. \end{aligned}$$

Moreover, by (3.4), one has $\sup_{h \in M} \|u_h(t)\| < +\infty$ for any $t \in I$. Since $u_h(t) \in D(A(t))$ for all $t \in I$ and for all $h \in M$, along with the ball-compactness of $D(A(t))$, for each $t \in I$, the set $\{u_h(t), h \in M\}$ is relatively compact in H . Thus, L is relatively compact in $\mathcal{C}_H(I)$.

It remains to show that L is compact in $\mathcal{C}_H(I)$. Let $(h_n)_n$ be a sequence in M and $(u_{h_n})_n$ be a sequence in L such that $u_{h_n} \rightarrow u \in \mathcal{C}_H(I)$. By weak compactness of M , we may assume that $h_n \rightarrow h \in M$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology.

Recall that u_{h_n} is the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t)u_{h_n}(t) + h_n(t), & h_n \in M, \text{ a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in D(A(0)). \end{cases}$$

Furthermore, in view of (3.5), the sequence (\dot{u}_{h_n}) is bounded in $L_H^2(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{u}_{h_n})_n$ weakly converges in $L_H^2(I)$ to some element $z(\cdot)$.

For any integer n and any $y \in H$ and for $0 \leq s \leq t \leq 1$, relying on the absolute continuity of $(u_{h_n})_n$, one writes

$$\int_0^1 \langle y \mathbf{1}_{[s,t]}(\tau), \dot{u}_{h_n}(\tau) \rangle d\tau = \langle y, u_{h_n}(t) - u_{h_n}(s) \rangle.$$

Next, passing to the limit in the equality yields

$$\langle y, \int_s^t z(\tau) d\tau \rangle = \langle y, u(t) - u(s) \rangle.$$

Therefore, given any $s, t \in I$ with $s \leq t$, we get $\int_s^t z(\tau) d\tau = u(t) - u(s)$, and hence $u(\cdot)$ is absolutely continuous and z coincides almost everywhere in I with $\dot{u}(\cdot)$. Thus, $\dot{u} \in L_H^2(I)$ and $\dot{u}_{h_n} \rightarrow \dot{u}$ weakly in $L_H^2(I)$. As $\dot{u}_{h_n} + h_n \rightarrow \dot{u} + h$ weakly in $L_H^2(I)$, so that by the maximal monotone extension \mathcal{A} of $A(t)$ cf. Lemma 3.1, we get

$$-\dot{u}(t) \in A(t)u(t) + h(t), \quad h \in M, \text{ a.e. } t \in I.$$

Denote by u_h the unique absolutely continuous solution to

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + h(t), & h \in M, \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)). \end{cases}$$

By uniqueness, one has $u = u_h$. Thus, the set L is compact in $\mathcal{C}_H(I)$.

Step 2 Now, given $\varepsilon > 0$, the compact set L in $\mathcal{C}_H(I)$ being constructed, we may apply Theorem 2.1 to the mixed semi-continuous set-valued map F by taking $g(t, x) = \alpha(t)(1 + \|x\|)$, $(t, x) \in I \times H$. Then, there is a nonempty closed convex set valued-map $\Psi_F : L \rightrightarrows L_H^1(I)$ whose graph is sequentially closed with respect to the topology of uniform convergence in L and the weak topology in $L_H^1(I)$ and such that, for any $x \in L$ and $y \in \Psi_F(x)$, for a.e. $t \in I$, one has $y(t) \in F(t, x(t))$ and $\|y(t)\| \leq \alpha(t)(1 + \|x(t)\|) + \varepsilon/2$. This along with $u_h \in L$ and $y_h \in \Psi_F(u_h)$, one has for a.e. $t \in I$ $y_h(t) \in F(t, u_h(t))$ and $\|y_h(t)\| \leq \alpha(t)(1 + \|u_h(t)\|) + \varepsilon \leq \alpha(t)(1 + r(t)) + \varepsilon/2$ taking into account the fact $\|u_h(t)\| \leq r(t)$. It's easy to see that $y_h \in M_2$, hence $\Psi_F(u_h) \subset M_2$ for all $u_h \in L$. Thus, there is a non-empty convex weakly-compact set-valued map $\Psi_F : L \rightrightarrows L_H^1(I)$ with the following properties

(k) $\Psi_F(x) \subset \{h \in M_2 : h(t) \in F(t, x(t))\} \subset M_2$ a.e., for each $x \in L$,

(kk) the set-valued map Ψ_F has closed graph, in the sense, for any $y_n \in \Psi(x_n)$

such that (x_n) converging uniformly to $x \in L$ and $(y_n) \sigma(L_H^1(I), L_H^\infty(I))$ -converging in $L_H^1(I)$ to y , then, $y \in \Psi_F(x)$, equivalently, $\Psi_F : L \rightrightarrows M_2$ is upper semi-continuous from $L \subset \mathcal{C}_H(I)$ to M_2 endowed with the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology.

Similarly, we may apply Theorem 2.1 to the mixed semi-continuous set-valued map G by taking $g(t, x) = m, (t, x) \in I \times H$. Then, there is a nonempty closed convex set valued-map $\Psi_G : L \rightrightarrows L_H^1(I)$ whose graph is sequentially closed with respect to the topology of uniform convergence in L and the weak topology in $L_H^1(I)$ and such that, for any $x \in L$ and $y \in \Psi_G(x)$, for a.e. $t \in I$, one has

$$y(t) \in G(t, x(t)) \text{ and } \|y(t)\| \leq m + \varepsilon/2.$$

This along with $u_h \in L$ and $y_h \in \Psi_G(u_h)$, yields for a.e. $t \in I$

$$y_h(t) \in G(t, u_h(t)) \text{ and } \|y_h(t)\| \leq m + \varepsilon/2.$$

It's easy to see that $y_h \in M_1$, hence $\Psi_G(u_h) \subset M_1$ for all $u_h \in L$. Thus, there is a non-empty convex weakly-compact set-valued map $\Psi_G : L \rightrightarrows L_H^1(I)$ with the following properties

(k') $\Psi_G(x) \subset \{h \in M_1 : h(t) \in G(t, x(t)) \subset M_1 \text{ a.e.}\}$, for each $x \in L$,
 ($k'k'$) the set-valued map Ψ_G has closed graph, in the following sense, for any $y_n \in \Psi_G(x_n)$ such that (x_n) converging uniformly to $x \in L$ and $(y_n) \sigma(L_H^1(I), L_H^\infty(I))$ converging in $L_H^1(I)$ to y , then, $y \in \Psi_G(x)$. So $\Psi_G : L \rightrightarrows M_1$ is upper semi-continuous from $L \subset \mathcal{C}_H(I)$ to M_1 endowed with the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology.

As consequence, let us set $\Psi(x) := \Psi_F(x) + \Psi_G(x), \forall x \in L$, then

(I) $\Psi : L \rightrightarrows L_H^1(I)$ is upper semi-continuous from $L \subset \mathcal{C}_H(I)$ to M endowed with the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology, equivalently its graph is sequentially closed with respect to the topology of uniform convergence in L and the weak topology in $L_H^1(I)$.

Step 3 We finish the proof by using Kakutani-Ky Fan fixed point theorem via the compactness of the solutions set L to the evolution inclusion given in Step 1. For each $h \in M$, let us set $\Phi(h) = \Psi(u_h) := \Psi_F(u_h) + \Psi_G(u_h)$ where u_h is the unique absolutely continuous solution to

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + h(t) & \text{a.e. } t \in I, \\ u(0) = u_0 \in D(A(0)). \end{cases}$$

Recall that $L := \{u_h : h \in M\}$ is compact in $\mathcal{C}_H(I)$. Then, it's clear that Φ is a convex weakly compact set-valued map from M to M . We claim that Φ has a fixed point h , i.e., $h \in \Phi(h) = \Psi(u_h) = \Psi_F(u_h) + \Psi_G(u_h)$, then, $h = h_1 + h_2$ with $h_1 \in \Psi_F(u_h), h_2 \in \Psi_G(u_h)$ so that $h_1(t) \in F(t, u_h(t))$ a.e. and $h_2(t) \in G(t, u_h(t))$ a.e. thus proving that

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + F(t, u_h(t)) + G(t, u_h(t)) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)), \end{cases}$$

has at least a solution u_h . For this purpose, by virtue of Kakutani-Ky Fan fixed point theorem, we need to prove that $\Phi : M \rightrightarrows M$ is upper semi-continuous with non-empty convex weakly compact values from M into itself, equivalently, the graph of Φ is sequentially compact in $M \times M$, for M equipped with the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology.

Indeed, for each $n \in \mathbb{N}$, let $f_n \in \Phi(h_n) = \Psi(u_{h_n}) = \Psi_F(u_{h_n}) + \Psi_G(u_{h_n})$ where

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t)u_{h_n}(t) + h_n(t) & \text{a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in D(A(0)), \end{cases}$$

with $u_{h_n} \in L$ by our definition of L and $(f_n) \subset M$ which $\sigma(L_H^1(I), L_H^\infty(I))$ -converges to $f \in M$. Further, by compactness of L , $(u_{h_n})_n$ uniformly converges to u_h with

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in D(A(0)), \end{cases}$$

As $f_n \in \Phi(h_n) = \Psi(u_{h_n})$ with $f_n \rightarrow f \in M$ weakly in $L_H^1(I)$ and $u_{h_n} \rightarrow u_h \in L$ uniformly, by the closed graph property (l) for Ψ , we get $f \in \Psi(u_h) = \Phi(h)$. Hence, the set-valued map $\Phi : M \rightrightarrows M$ has weakly compact graph, by Kakutani-Ky Fan theorem, there is $h \in \Phi(h) = \Psi(u_h)$. Then, $h = h_1 + h_2$ with $h_1 \in \Psi_F(u_h), h_2 \in \Psi_G(u_h)$ so that $h_1(t) \in F(t, u_h(t))$ a.e. and $h_2(t) \in G(t, u_h(t))$ a.e. thus proving that

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + F(t, u(t)) + G(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in D(A(0)), \end{cases}$$

has at least a solution u , just finishing the proof of the theorem. \square

3.2. Mixed semi-continuous perturbation problem with time-dependent sub-differential operators

For the sake of completeness we recall and summarize some needed results. Let us start with the existence and uniqueness result established in [43].

Theorem 3.4. *Let $\varphi : [0, T] \times H \rightarrow [0, +\infty]$ be a map satisfying (H_1) - (H_2) . Let $u_0 \in \text{dom}\varphi(0, \cdot)$. Then, the unperturbed differential inclusion*

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ on $[0, T]$. Moreover, $u(t) \in \text{dom}\varphi(t, \cdot)$ for all $t \in [0, T]$ and the function $t \mapsto \varphi(t, u(t))$ is absolutely continuous on $[0, T]$.

Now, we introduce the extension to $L_H^2(I)$ of subdifferential operators defined on H .

Denote by $A(t) := \partial\varphi(t, \cdot)$ the maximal monotone operator in H associated with $\partial\varphi(t, \cdot)$, $t \in I$ (φ satisfies conditions (H_1) - (H_2)). Let us consider the operator $\mathcal{A} : L_H^2(I) \rightrightarrows L_H^2(I)$ defined by

$$\mathcal{A}u = \{w \in L_H^2(I) : w(t) \in A(t)u(t) \text{ a.e.}\}.$$

Then, \mathcal{A} is well defined since by Theorem 3.4, the evolution inclusion

$$-\dot{u}(t) \in A(t)u(t) = \partial\varphi(t, u(t)) \text{ a.e. } t \in I, \quad u(0) \in \text{dom } \varphi(0, \cdot)$$

admits a unique absolutely continuous solution.

The operator \mathcal{A} enjoys the following property (see [43]).

Proposition 3.1. *If for any $t \in I$, $A(t) = \partial\varphi(t, \cdot)$ where φ satisfies conditions (H_1) - (H_2) then, \mathcal{A} is a maximal monotone operator.*

We also recall the following useful proposition.

Proposition 3.2. *Let $A(t) := \partial\varphi(t, \cdot)$, $\forall t \in I$ where φ satisfies conditions (H_1) - (H_2) . Let $(u_n)_n$ and $(w_n)_n$ be two sequences in $L^2_H(I)$ satisfying*

- (i) $w_n(t) \in A(t)u_n(t)$, $\forall n \in \mathbb{N}$ and for a.e. $t \in I$;*
- (ii) $(u_n)_n$ strongly converges to u in $L^2_H(I)$;*
- (iii) $(w_n)_n$ weakly converges to w in $L^2_H(I)$.*

Then, one has $w(t) \in A(t)u(t)$ a.e. $t \in I$.

Proof. By (i), one has $w_n \in \mathcal{A}u_n$ where \mathcal{A} is the maximal monotone operator associated with $A(t)$ given in Proposition 3.1. Since the graph of \mathcal{A} is sequentially strongly-weakly closed, by $w_n \in \mathcal{A}u_n$ and by (ii) and (iii), one deduces that $w \in \mathcal{A}u$, so that coming back to the definition of \mathcal{A} , one gets $w(t) \in A(t)u(t)$ a.e. $t \in I$. \square

The following proposition contains crucial estimates in the study of the evolution problem with perturbation $-\dot{u}(t) \in \partial\varphi(t, u(t)) + h(t)$ (see [44]).

Proposition 3.3. (a) *The unique absolutely continuous solution $u(\cdot)$ of the unperturbed problem satisfies*

$$\begin{aligned} \|\dot{u}\|_{L^2_H([0, T])} &\leq \frac{\rho}{2} \|\dot{a}\|_{L^2_{\mathbb{R}}([0, T])} + \\ &[\sqrt{T}k(0) \|\dot{a}\|_{L^2_{\mathbb{R}}([0, T])} + \frac{\rho^2}{4} \|\dot{a}\|_{L^2_{\mathbb{R}}([0, T])}^2 + \varphi(0, u_0) - \varphi(T, u(T))]^{\frac{1}{2}}. \end{aligned}$$

(b) *If $h \in L^2_H([0, T])$ and $u_0 \in \text{dom } \varphi(0, \cdot)$, then the following problem*

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + h(t) \text{ a.e. } t \in [0, T], \\ u(0) = u_0 \in \text{dom } \varphi(0, \cdot), \end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ that satisfies

$$\begin{aligned} \|\dot{u}\|_{L^2_H([0, T])} &\leq \frac{1}{2}(\rho + 1) \|\dot{a} + |h|\|_{L^2_{\mathbb{R}}([0, T])} + \|h\|_{L^2_H([0, T])} + \\ &[\sqrt{T}k(0) \|\dot{a} + |h|\|_{L^2_{\mathbb{R}}([0, T])} + \frac{(\rho + 1)^2}{4} \|\dot{a} + |h|\|_{L^2_{\mathbb{R}}([0, T])}^2 + \varphi(0, u_0) - \varphi(T, u(T))]^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

where $|h|$ is the function defined by $|h| : t \mapsto \|h(t)\|$ for all $t \in [0, T]$.

We recall also the existence and uniqueness result for the perturbed problem $-\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t))$ when $f(\cdot, \cdot)$ is separately measurable and Lipschitz continuous (see Theorem 4.1. [44]).

Theorem 3.5. *Let $I := [0, T]$. Assume that (H_1) - (H_2) hold. Let $f : I \times H \rightarrow H$ be a map satisfying (i)-(ii)-(iii). Then, for any $u_0 \in \text{dom}\varphi(0, \cdot)$, the following problem*

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \end{cases}$$

has one and only one absolutely continuous solution $u(\cdot)$ on I that satisfies

$$\int_0^T \|\dot{u}(t)\|^2 dt \leq d + \sigma \int_0^T \|f(t, u(t))\|^2 dt, \tag{3.7}$$

where

$$\begin{aligned} d &= (k^2(0) + 3(\rho + 1)^2) \int_0^T \dot{a}^2(t) dt + 2[T + \varphi(0, u_0)] \\ \sigma &= k^2(0) + 3(\rho + 1)^2 + 4. \end{aligned}$$

The following theorem deals with a perturbation $f + F$ with a mixed semi-continuous set-valued map F , extending a recent result given in [23] with $f = 0$.

Theorem 3.6. *Let $I := [0, 1]$. Assume that (H_1) - (H_2) hold and that $\text{dom}\varphi(t, \cdot)$ is ball-compact, for every $t \in I$. Let $f : I \times H \rightarrow H$ be a mapping satisfying (i)-(ii)-(iii). Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj)-(jjj).*

Then, for any $u_0 \in \text{dom}\varphi(0, \cdot)$, there is an absolutely continuous solution $u(\cdot)$ of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Proof. We will use some arguments of the proof of Theorem 3.2 with appropriate modifications. However this need a **careful** look.

Step 1 Let $u_0 \in \text{dom}\varphi(0, \cdot)$, $\varepsilon > 0$. Let $u : [0, 1] \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in \partial\varphi(t, u(t)) \text{ a.e. } t \in I, \\ u(0) &= u_0 \in \text{dom}\varphi(0, \cdot). \end{aligned}$$

The existence and uniqueness of such a solution follow from Theorem 3.4. Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \alpha(t)(1 + r(t)) + \varepsilon + m \quad \text{a.e. with } r(0) = \sup_{t \in [0, 1]} \|u(t)\|.$$

Since $\dot{r} + \varepsilon \in L^\infty_{\mathbb{R}_+}(I) \subset L^1_{\mathbb{R}_+}(I)$, the set

$$M := \{h \in L^1_H(I) : \|h(t)\| \leq \dot{r}(t) + \varepsilon \text{ a.e.}\},$$

is clearly convex $\sigma(L_H^1(I), L_H^\infty(I))$ -compact. For any $h \in M$, denote by u_h the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

The existence and uniqueness of such a solution is granted by Theorem 3.5, that is due to the particular choice of the perturbation f and the perturbation $h \in M$. In fact, to see that, let us set for any $h \in M$, $g(t, x) = f(t, x) + h(t)$ for all $(t, x) \in I \times H$ and check assumptions of Theorem 3.5. Clearly, the function g satisfies assumptions (i)-(ii). Moreover, for all $(t, x) \in I \times H$, one has

$$\|g(t, x)\| \leq m + \|h(t)\| \leq (m + \|h(t)\|)(1 + \|x\|),$$

which yields (iii) with $\beta(\cdot) := m + \dot{r}(\cdot) + \varepsilon \in L_{\mathbb{R}^+}^2(I)$. Using the monotonicity of $\partial\varphi(t, \cdot)$, for all $t \in I$, one has the estimation

$$\frac{1}{2}\|u_h(t) - u(t)\|^2 \leq \int_0^t \|f(s, u_h(s)) + h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows

$$\|u_h(t) - u(t)\| \leq \int_0^t \|f(s, u_h(s)) + h(s)\| ds,$$

so that for all $t \in I$, one gets

$$\|u_h(t)\| \leq r(0) + \int_0^t [\dot{r}(s) + \varepsilon + m] ds = r(t) + \varepsilon t + mt \leq r(t) + m + \varepsilon. \quad (3.8)$$

Now, we assert that

Main fact: $K := \{u_h : h \in M\}$ is compact in $\mathcal{C}_H(I)$.

This follows essentially from the above estimate, the ball compactness assumption on $\text{dom}\varphi(t, \cdot)$, along with the maximal monotone property of the extension \mathcal{A} to $L_H^2(I)$ defined in Proposition 3.1. In view of estimate (3.7) in Theorem 3.5 (with g instead of f), one gets

$$\int_0^1 \|\dot{u}_h(t)\|^2 dt \leq d + \sigma \int_0^1 \|f(t, u_h(t)) + h(t)\|^2 dt,$$

taking condition (iii) and the definition of M into account, yield

$$\int_0^1 \|\dot{u}_h(t)\|^2 dt \leq d + \sigma \int_0^1 (m + \dot{r}(t) + \varepsilon)^2 dt.$$

Since $\dot{r} + \varepsilon \in L_{\mathbb{R}^+}^\infty(I) \subset L_{\mathbb{R}^+}^2(I)$, then, the set $M := \{h \in L_H^1(I) : \|h(t)\| \leq \dot{r}(t) + \varepsilon \text{ a.e.}\}$ is convex $\sigma(L_H^1(I), L_H^\infty(I))$ -compact. From the estimate above, for any $h \in M$, \dot{u}_h is bounded in $L_H^2(I)$, that is,

$$\mathcal{S} = \sup_{h \in M} \|\dot{u}_h\|_{L_H^2(I)} < +\infty. \quad (3.9)$$

By the absolute continuity of u_h , $h \in M$, for any $s, t \in I$, one has

$$\begin{aligned} \|u_h(t) - u_h(s)\| &= \left\| \int_s^t \dot{u}_h(\tau) d\tau \right\| \\ &\leq (t-s)^{1/2} \left(\int_0^1 \|\dot{u}_h(\tau)\|^2 d\tau \right)^{1/2} \\ &\leq (t-s)^{1/2} \mathcal{S}. \end{aligned}$$

Then, the set K is equicontinuous. Moreover, by (3.8), one has $\sup_{h \in M} \|u_h(t)\| < +\infty$ for any $t \in I$. Therefore, K is bounded. Since $u_h(t) \in \text{dom} \varphi(t, \cdot)$ for all $t \in I$ and $h \in M$, along with the ball-compactness of $\text{dom} \varphi(t, \cdot)$, for each $t \in I$, the set $\{u_h(t), h \in M\}$ is relatively compact in H . Thus, K is relatively compact in $\mathcal{C}_H(I)$. It remains to show that K is compact in $\mathcal{C}_H(I)$. Let $(h_n)_n$ be a sequence in M and $(u_{h_n})_n$ be a sequence in K such that $u_{h_n} \rightarrow u \in \mathcal{C}_H(I)$. By weak compactness of M , we may assume that $h_n \rightarrow h \in M$ with respect to the $\sigma(L_H^1(I), L_H^\infty(I))$ -topology. Recall that u_{h_n} is the unique absolutely continuous solution to

$$\begin{cases} -\dot{u}_{h_n}(t) \in \partial\varphi(t, u_{h_n}(t)) + h_n(t) + f(t, u_{h_n}(t)) \text{ a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in \text{dom} \varphi(0, \cdot). \end{cases}$$

By (iii), one has $\|f(t, u_{h_n}(t))\| \leq m$ for all $t \in I$. Since $u_{h_n} \rightarrow u$ in $\mathcal{C}_H(I)$, along with the Lipschitz behavior of f with respect to its second variable yield (via the dominated convergence theorem) $f(\cdot, u_{h_n}(\cdot)) \rightarrow f(\cdot, u(\cdot))$ in $L_H^2(I)$. As a consequence, $f(\cdot, u_{h_n}(\cdot)) \rightarrow f(\cdot, u(\cdot))$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology. Furthermore, in view of (3.9), the sequence (\dot{u}_{h_n}) is bounded in $L_H^2(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{u}_{h_n})_n$ weakly converges in $L_H^2(I)$ to some element $z(\cdot)$. For any integer n and any $y \in H$ and for $0 \leq s \leq t \leq 1$, relying on the absolute continuity of $(u_{h_n})_n$, we can write

$$\int_0^1 \langle y \mathbf{1}_{[s,t]}(\tau), \dot{u}_{h_n}(\tau) \rangle d\tau = \langle y, u_{h_n}(t) - u_{h_n}(s) \rangle.$$

Next, passing to the limit in the equality yields

$$\langle y, \int_s^t z(\tau) d\tau \rangle = \langle y, u(t) - u(s) \rangle.$$

Therefore, given any $s, t \in I$ with $s \leq t$, we get $\int_s^t z(\tau) d\tau = u(t) - u(s)$, and hence $u(\cdot)$ is absolutely continuous and z coincides almost everywhere in I with $\dot{u}(\cdot)$. Thus, $\dot{u} \in L_H^2(I)$ and $\dot{u}_{h_n} \rightarrow \dot{u}$ weakly in $L_H^2(I)$ so that $\dot{u}_{h_n} + h_n + f(\cdot, u_{h_n}(\cdot)) \rightarrow \dot{u} + h + f(\cdot, u(\cdot))$ weakly in $L_H^2(I)$. Thanks to Proposition 3.2, one gets

$$-\dot{u}(t) \in \partial\varphi(t, u(t)) + h(t) + f(t, u(t)), \quad h \in M, \text{ a.e. } t \in I.$$

Denote by u_h the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t) + f(t, u_h(t)) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom} \varphi(0, \cdot). \end{cases}$$

By uniqueness, one has $u = u_h$. Thus, the set K is compact in $\mathcal{C}_H(I)$.

Step 2 Now, given $\varepsilon > 0$, the compact set K in $\mathcal{C}_H(I)$ being constructed, we may apply Theorem 2.1 to the mixed semi-continuous set-valued map F by taking $g(t, x) = \alpha(t)(1 + \|x\|)$, $(t, x) \in I \times H$. Then, there is a nonempty closed convex set valued-map $\Psi : K \rightrightarrows L^1_H(I)$ whose graph is sequentially closed with respect to the topology of uniform convergence in K and the weak topology in $L^1_H(I)$ and such that, for any $x \in K$ and $y \in \Psi(x)$, for a.e. $t \in I$, one has

$$y(t) \in F(t, x(t)) \text{ and } \|y(t)\| \leq g(t, x(t)) + \varepsilon = \alpha(t)(1 + \|x(t)\|) + \varepsilon.$$

This along with $u_h \in K$ and $y_h \in \Psi(u_h)$, one has for a.e. $t \in I$

$$y_h(t) \in F(t, u_h(t)) \text{ and}$$

$$\|y_h(t)\| \leq \alpha(t)(1 + \|u_h(t)\|) + \varepsilon \leq \alpha(t)(1 + r(t) + \varepsilon + m) + \varepsilon$$

$$= \alpha(t)(1 + r(t)) + \alpha(t)\varepsilon + \alpha(t)m + \varepsilon \leq \alpha(t)(1 + r(t)) + \varepsilon + m + \varepsilon = \dot{r}(t) + \varepsilon;$$

taking into account the fact $\|u_h(t)\| \leq r(t) + \varepsilon + m$. It's easy to see that $y_h \in M$, hence $\Psi(u_h) \subset M$ for all $u_h \in K$. Thus, there is a non-empty convex weakly-compact set-valued map $\Psi : K \rightrightarrows L^1_H(I)$ with the following properties

(I) $\Psi(x) \subset \{h \in M : h(t) \in F(t, x(t)) \text{ a.e.}\}$, for each $x \in K$,

(II) the set-valued map Ψ has closed graph, in the sense, for any $y_n \in \Psi(x_n)$ such that (x_n) converging uniformly to $x \in K$ and (y_n) $\sigma(L^1_H(I), L^\infty_H(I))$ -converging in $L^1_H(I)$ to y , then, $y \in \Psi(x)$, equivalently, $\Psi : K \rightrightarrows M$ is upper semi-continuous from $K \subset \mathcal{C}_H(I)$ to M endowed with the $\sigma(L^1_H(I), L^\infty_H(I))$ -topology.

Step 3 We finish the proof by using Kakutani-Ky Fan fixed point theorem.

For each $h \in M$, let us set $\Phi(h) = \Psi(u_h)$ where

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t), & h \in M, \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Then, it's clear that Φ is a convex weakly compact set-valued map from M to M . We claim that Φ has a fixed point h , i.e., $h \in \Phi(h) = \Psi(u_h)$, then, by (I) $h(t) \in F(t, u_h(t))$ a.e., we have

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e., thus proving that

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + F(t, u_h(t)) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

has at least a solution. For this purpose, by virtue of Kakutani-Ky Fan fixed point theorem, we need to prove that $\Phi : M \rightrightarrows M$ is upper semi-continuous with non-empty convex weakly compact values from M into itself, equivalently, the graph of Φ is sequentially compact in $M \times M$, for M equipped with the $\sigma(L^1_H(I), L^\infty_H(I))$ -topology.

Indeed, for each $n \in \mathbb{N}$, let $g_n \in \Phi(h_n) = \Psi(u_{h_n})$ with

$$\begin{cases} -\dot{u}_{h_n}(t) \in \partial\varphi(t, u_{h_n}(t)) + f(t, u_{h_n}(t)) + h_n(t) & \text{a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

with $u_{h_n} \in K$ by our definition of K and $(g_n) \subset M$ which $\sigma(L_H^1(I), L_H^\infty(I))$ -converges to $g \in M$. Further, by compactness of K , $(u_{h_n})_n$ uniformly converges to u_h with

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

By the property (ll) of the set-valued map Ψ , we have $g \in \Psi(u_h) = \Phi(h)$. So, the graph of $\Phi : M \rightrightarrows M$ is weakly compact, hence the convex weakly compact valued mapping Φ admits a fixed point by Kakutani-Ky Fan Theorem, $h \in \Phi(h) = \Psi(u_h)$. By the property (l), we get $h(t) \in F(t, u_h(t))$ a.e. which implies

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e. just finishing the proof of the theorem. □

Let us recall the following existence result for the perturbed problem $-\dot{u}(t) \in \partial\varphi(t, u(t)) + g(t, u(t))$ when the map $g(\cdot, \cdot)$ is of Carathéodory type, see Theorem 4.1. [45] (see Proposition 4.2. [45] for the estimate of the velocity).

Theorem 3.7. *Let $I = [0, T]$. Assume that (H_1) - (H_2) hold, and that $\text{dom}\varphi(t, \cdot)$ is ball-compact for every $t \in I$. Let $g : I \times H \rightarrow H$ be a Carathéodory map such that there exists a non-negative function $\beta(\cdot) \in L_{\mathbb{R}}^2(I)$ such that, for all $t \in I$ and for all $u \in H$, one has*

$$\|g(t, u)\| \leq \beta(t)(1 + \|u\|).$$

Then, for any $u_0 \in \text{dom}\varphi(0, \cdot)$, the following problem

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + g(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \end{cases}$$

has at least one absolutely continuous solution $u(\cdot)$ on I that satisfies

$$\int_0^T \|\dot{u}(t)\|^2 dt \leq d + \sigma \int_0^T \|g(t, u(t))\|^2 dt, \tag{3.10}$$

where

$$\begin{aligned} d &= (k^2(0) + 3(\rho + 1)^2) \int_0^T \dot{a}^2(t) dt + 2[T + \varphi(0, u_0)] \\ \sigma &= k^2(0) + 3(\rho + 1)^2 + 4. \end{aligned}$$

Now, we address a new variant when the evolution problem involves a perturbation $F + g$, with a Carathéodory map g .

Theorem 3.8. *Let $I := [0, 1]$. Assume that (H_1) - (H_2) hold, and that $\text{dom}\varphi(t, \cdot)$ is ball-compact for every $t \in I$. Let $g : I \times H \rightarrow H$ be a Carathéodory mapping satisfying $\|g(t, x)\| \leq m$ for all $(t, x) \in I \times H$, for a positive constant m . Suppose further that there exists a function $\gamma(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that*

$$\langle g(t, x_1) - g(t, x_2), x_1 - x_2 \rangle \geq \gamma(t)\|x_1 - x_2\|^2,$$

for all $x_1, x_2 \in H$ and $t \in I$.

Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j) - (jj) - (jjj) .

Then, for any $u_0 \in \text{dom}\varphi(0, \cdot)$, there is an absolutely continuous solution $u(\cdot)$ of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + F(t, u(t)) + g(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Proof. Step 1 We will use similar arguments as in the proof of Theorem 3.6, with appropriate modifications. This need a **careful** look.

Let $u_0 \in \text{dom}\varphi(0, \cdot)$, $\varepsilon > 0$. Let $u : [0, 1] \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in \partial\varphi(t, u(t)) \text{ a.e. } t \in I, \\ u(0) &= u_0 \in \text{dom}\varphi(0, \cdot). \end{aligned}$$

The existence and uniqueness of such a solution follow from Theorem 3.4. Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \alpha(t)(1 + r(t)) + \varepsilon + m \quad \text{a.e. with } r(0) = \sup_{t \in [0, 1]} \|u(t)\|.$$

Since $\dot{r} + \varepsilon \in L^\infty_{\mathbb{R}_+}(I) \subset L^1_{\mathbb{R}_+}(I)$, the set

$$M := \{h \in L^1_H(I) : \|h(t)\| \leq \dot{r}(t) + \varepsilon \text{ a.e.}\},$$

is clearly convex $\sigma(L^1_H(I), L^\infty_H(I))$ -compact.

Consider the problem of the form

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + g(t, u(t)) + h(t), & h \in M, \quad \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

The existence of solutions follows from Theorem 3.7, by considering the Caratheodory perturbation $g(t, x) + h(t)$. To prove uniqueness of the solution, let u_1 and u_2 be two solutions to the differential inclusion above. Then, one has for any $h \in M$, $-\dot{u}_1(t) - g(t, u_1(t)) - h(t) \in \partial\varphi(t, u_1(t))$ and $-\dot{u}_2(t) - g(t, u_2(t)) - h(t) \in \partial\varphi(t, u_2(t))$, for almost all $t \in I$. The monotone condition on g yields

$$\langle g(t, u_1(t)) - g(t, u_2(t)), u_1(t) - u_2(t) \rangle \geq \gamma(t)\|u_1(t) - u_2(t)\|^2.$$

Moreover, the monotone property of $\partial\varphi(t, \cdot)$ ensures that

$$\langle -\dot{u}_1(t) - g(t, u_1(t)) - (-\dot{u}_2(t) - g(t, u_2(t))), u_1(t) - u_2(t) \rangle \geq 0,$$

hence,

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \leq -\gamma(t)\|u_1(t) - u_2(t)\|^2,$$

that is,

$$\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq -\gamma(t) \|u_1(t) - u_2(t)\|^2 \text{ for a.e. } t \in I.$$

Remark that $\frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq 0$, since by assumption γ is a non-negative function. This along with the fact that $\|u_1(0) - u_2(0)\| = 0$, integrating, one gets $u_1 = u_2$. Thus, it results the uniqueness of the solution.

Now, denote by u_h , for any $h \in M$, the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + g(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Using the monotonicity of $\partial\varphi(t, \cdot)$, for all $t \in I$, one has the estimation

$$\frac{1}{2} \|u_h(t) - u(t)\|^2 \leq \int_0^t \|g(s, u_h(s)) + h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows

$$\|u_h(t) - u(t)\| \leq \int_0^t \|g(s, u_h(s)) + h(s)\| ds$$

so that for all $t \in I$, one gets

$$\|u_h(t)\| \leq r(0) + \int_0^t [\dot{r}(s) + \varepsilon + m] ds = r(t) + \varepsilon t + mt \leq r(t) + m + \varepsilon. \quad (3.11)$$

Main fact: $K := \{u_h : h \in M\}$ is compact in $\mathcal{C}_H(I)$.

This follows essentially from the above estimate, the ball compactness assumption on $\text{dom}\varphi(t, \cdot)$, along with the maximal monotone property of the extension \mathcal{A} to $L^2_H(I)$ defined in Proposition 3.1.

In view of estimate (3.10) in Theorem 3.7 (with $g + h$ instead of g), one gets

$$\int_0^1 \|\dot{u}_h(t)\|^2 dt \leq d + \sigma \int_0^1 \|g(t, u_h(t)) + h(t)\|^2 dt.$$

Then, since by assumption $\|g(t, x)\| \leq m$ for all $(t, x) \in I \times H$, along with the definition of M , yield

$$\int_0^1 \|\dot{u}_h(t)\|^2 dt \leq d + \sigma \int_0^1 (m + \dot{r}(t) + \varepsilon)^2 dt.$$

So, for any $h \in M$, \dot{u}_h is bounded in $L^2_H(I)$, that is,

$$\mathcal{S} = \sup_{h \in M} \|\dot{u}_h\|_{L^2_H(I)} < +\infty. \quad (3.12)$$

By the absolute continuity of u_h , $h \in M$, for any $s, t \in I$, one has

$$\begin{aligned} \|u_h(t) - u_h(s)\| &= \left\| \int_s^t \dot{u}_h(\tau) d\tau \right\| \\ &\leq (t - s)^{1/2} \left(\int_0^1 \|\dot{u}_h(\tau)\|^2 d\tau \right)^{1/2} \\ &\leq (t - s)^{1/2} \mathcal{S}. \end{aligned}$$

Then, the set K is equicontinuous. Moreover, by (3.11), one has for any $t \in I$, $\sup_{h \in M} \|u_h(t)\| < +\infty$. Since $u_h(t) \in \text{dom} \varphi(t, \cdot)$ for all $t \in I$ and $h \in M$, along

with the ball-compactness of $\text{dom} \varphi(t, \cdot)$, for each $t \in I$, the set $\{u_h(t), h \in M\}$ is relatively compact in H . Thus, K is relatively compact in $\mathcal{C}_H(I)$.

It remains to show that K is compact in $\mathcal{C}_H(I)$. Let $(h_n)_n$ be a sequence in M and $(u_{h_n})_n$ be a sequence in K such that $u_{h_n} \rightarrow u \in \mathcal{C}_H(I)$. By weak compactness of M , we may assume that $h_n \rightarrow h \in M$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology.

Recall that u_{h_n} is the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_{h_n}(t) \in \partial\varphi(t, u_{h_n}(t)) + h_n(t) + g(t, u_{h_n}(t)), & h_n \in M, \text{ a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

By hypothesis on g , one writes $\|g(t, u_{h_n}(t))\| \leq m$ for all $t \in I$. Since $u_{h_n} \rightarrow u$ in $\mathcal{C}_H(I)$ and g is continuous with respect to its second variable yield (via the dominated convergence theorem) $g(\cdot, u_{h_n}(\cdot)) \rightarrow g(\cdot, u(\cdot))$ in $L_H^2(I)$. As a consequence, $g(\cdot, u_{h_n}(\cdot)) \rightarrow g(\cdot, u(\cdot))$ with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology.

Furthermore, in view of (3.12), the sequence (\dot{u}_{h_n}) is bounded in $L_H^2(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{u}_{h_n})_n$ weakly converges in $L_H^2(I)$ to some element $z(\cdot)$.

For any integer n and any $y \in H$ and for $0 \leq s \leq t \leq 1$, relying on the absolute continuity of $(u_{h_n})_n$, one writes

$$\int_0^1 \langle y \mathbf{1}_{[s,t]}(\tau), \dot{u}_{h_n}(\tau) \rangle d\tau = \langle y, u_{h_n}(t) - u_{h_n}(s) \rangle.$$

Next, passing to the limit in the equality yields

$$\langle y, \int_s^t z(\tau) d\tau \rangle = \langle y, u(t) - u(s) \rangle.$$

Therefore, given any $s, t \in I$ with $s \leq t$, we get $\int_s^t z(\tau) d\tau = u(t) - u(s)$, and hence $u(\cdot)$ is absolutely continuous and z coincides almost everywhere in I with $\dot{u}(\cdot)$. Thus, $\dot{u} \in L_H^2(I)$ and $\dot{u}_{h_n} \rightarrow \dot{u}$ weakly in $L_H^2(I)$. Consequently $\dot{u}_{h_n} + h_n + g(\cdot, u_{h_n}(\cdot)) \rightarrow \dot{u} + h + g(\cdot, u(\cdot))$ weakly in $L_H^2(I)$. Thanks to Proposition 3.2, one gets

$$-\dot{u}(t) \in \partial\varphi(t, u(t)) + h(t) + g(t, u(t)) \text{ a.e. } t \in I.$$

Denote by u_h the unique absolutely continuous solution of

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t) + g(t, u_h(t)), & h \in M, \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

By uniqueness, one has $u = u_h$. Thus, the set K is compact in $\mathcal{C}_H(I)$.

Finally, we proceed as in **Step 2-Step 3** of Theorem 3.6 to get

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + g(t, u_h(t)) + h(t) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e. finishing the proof of the theorem. □

In the remainder of this section, we state a variant of the above results dealing with another class of time-dependent subdifferential operator. Let $\varphi : H \rightarrow \overline{\mathbb{R}}$ satisfying

H(φ) For each $t \in I$, the function $x \mapsto \varphi(t, x)$ is proper, lsc, and convex such that the following property holds:

for each $r \geq 0$ there exist functions $a_r \in W^{1,2}([0, 1], \mathbb{R})$, $b_r \in W^{1,1}([0, 1], \mathbb{R})$ such that for any $s, t \in [0, 1]$, $s \leq t$, and any $x \in \text{dom } \varphi(s, \cdot)$, $\|x\| \leq r$, there exists an element $y \in \text{dom } \varphi(t, \cdot)$ satisfying the inequalities

$$\|x - y\| \leq |a_r(t) - a_r(s)|(|\varphi(s, x)|^{1/2} + 1) \quad (3.13)$$

$$\varphi(t, y) \leq \varphi(s, x) + |b_r(t) - b_r(s)|(|\varphi(s, x)| + 1). \quad (3.14)$$

For the sake of completeness, it seems convenient to recall the following lemma from [36], [37].

Lemma 3.2. *Let $\varphi : H \rightarrow \overline{\mathbb{R}}$ satisfying **H**(φ). Then, one has*

(i) *there exists a constant $K_1 > 0$ such that*

$$\varphi_\lambda(t, x) \geq -K_1(\|x\| + 1), \quad t \in I, \lambda \in]0, 1], x \in H,$$

$$\|J_\lambda^t x\| = \|(I_H + \lambda \partial \varphi(t, \cdot))^{-1} x\| \leq K_1 + \|x\|, \quad t \in I, \lambda \in]0, 1], x \in H,$$

(ii) *for any function $x \in L^1_H(I)$, the function $t \mapsto \varphi(t, x(t))$ is measurable;*

(iii) *for any function $x \in L^1_H(I)$, the function $t \mapsto \varphi_\lambda(t, x(t))$, $\lambda \in]0, 1]$, is measurable.*

Recall also Lemma 2.2 [37].

Lemma 3.3. *Let $\varphi : H \rightarrow \overline{\mathbb{R}}$ satisfying **H**(φ), $\lambda \in]0, 1]$ and $x : I \rightarrow H$ be an absolutely continuous function. Then, the function $t \mapsto \varphi_\lambda(t, x(t))$ is a.e. differentiable, its derivative is integrable on I , and the following inequalities hold*

$$\varphi_\lambda(t, x(t)) - \varphi_\lambda(s, x(s)) \leq \int_s^t \frac{d}{d\tau} \varphi_\lambda(\tau, x(\tau)) d\tau, \quad s, t \in I, s \leq t,$$

$$\frac{d}{dt} \varphi_\lambda(t, x(t)) - \langle \partial \varphi_\lambda(t, x(t)), \dot{x}(t) \rangle \leq$$

$$|\dot{a}_r(t)| \|\partial \varphi_\lambda(t, x(t))\| (|\varphi_\lambda(t, x(t))|^{1/2} + 1) + |\dot{b}_r(t)| (|\varphi_\lambda(t, x(t))| + 1) \text{ a.e.,}$$

where

$$r \geq \sup\{\|J_\lambda^t(x(t))\|, t \in I, 0 < \lambda \leq 1\},$$

where $a_r(\cdot)$ and $b_r(\cdot)$ are the functions defined by (3.13) and (3.14).

Let us summarize a result from [36] as follows.

Theorem 3.9. *Let $\varphi : H \rightarrow \overline{\mathbb{R}}$ satisfying **H**(φ). Let $h \in L^2_H(I)$, $x_0 \in \text{dom } \varphi(0, \cdot)$ and $\lambda \in]0, 1]$. Then, the following inclusion*

$$\begin{cases} -\dot{x}(t) \in \partial \varphi(t, x(t)) + h(t) \text{ a.e., } h \in L^2_H(I), \\ x(0) = x_0 \in \text{dom } \varphi(0, \cdot); \end{cases} \quad (3.15)$$

admits a unique absolutely continuous solution $x : I \rightarrow H$, and the following equation

$$\begin{cases} -\dot{x}_\lambda(t) = \partial\varphi_\lambda(t, x_\lambda(t)) + h(t) \text{ a.e.}, & h \in L^2_H(I), \\ x_\lambda(0) = x_0 \in \text{dom}\varphi(0, \cdot); \end{cases}$$

admits a unique absolutely continuous solution $x_\lambda : I \rightarrow H$ for any $x(0) = x_\lambda(0) = x_0 \in \text{dom}\varphi(0, \cdot)$. Moreover, $\dot{x}(\cdot), \dot{x}_\lambda(\cdot) \in L^2_H(I)$ and there exists a constant M that depends only on $\|x_0\|$ and $\|h\|_{L^2_H}$ such that

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq M, \quad \|x_\lambda(\cdot)\|_\infty \leq M, \quad 0 < \lambda \leq 1, \\ \|\dot{x}(\cdot)\|_{L^2_H(I)} &\leq M, \quad \|\dot{x}_\lambda(\cdot)\|_{L^2_H} \leq M, \quad 0 < \lambda \leq 1, \\ |\varphi(t, x(t))| &\leq M, \quad |\varphi_\lambda(t, x_\lambda(t))| \leq M, \quad 0 < \lambda \leq 1. \end{aligned}$$

As $\lambda \downarrow 0$, one gets

$$\begin{aligned} x_\lambda(\cdot) &\rightarrow x(\cdot) \text{ in } \mathcal{C}_H(I), \\ \dot{x}_\lambda(\cdot) &\rightarrow \dot{x}(\cdot) \text{ in } L^2_H(I), \\ \varliminf_{\lambda \downarrow 0} \varphi_\lambda(t, x_\lambda(t)) &\geq \varphi(t, x(t)), \quad t \in I. \end{aligned}$$

We recall and summarize a compactness property (cf. Theorem 4.1 [50]) of the solution set to (3.15).

Theorem 3.10. *Assume that φ satisfies condition $H(\varphi)$ and that $\text{dom}\varphi(t, \cdot)$ is ball-compact, for every $t \in I$. Let $X : I \rightrightarrows H$ be measurable with convex and weakly compact values, and $X(t) \subset \beta(t)\overline{B}_H, \forall t \in I$, where $\beta \in L^2_{\mathbb{R}_+}(I)$. Let S^2_X denotes the set of all $L^2_H(I)$ -selections of X . Then, the set $\mathcal{X} := \{x_h : h \in S^2_X\}$ of absolutely continuous solutions to the evolution inclusion*

$$\begin{cases} -\dot{x}_h(t) \in \partial\varphi(t, x_h(t)) + h(t), & \text{a.e. } t \in I, \quad h \in S^2_X, \\ x_h(0) = x_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

is a compact subset of $\mathcal{C}_H(I)$.

Now, we are ready to handle the following variant concerning the class of differential inclusions involving the subdifferential of a map φ satisfying $\mathbf{H}(\varphi)$ with a mixed semi-continuous perturbation.

Theorem 3.11. *Let $I := [0, 1]$. Assume that φ satisfies condition $\mathbf{H}(\varphi)$ and that $\text{dom}\varphi(t, \cdot)$ is ball-compact, for every $t \in I$. Let $f : I \times H \rightarrow H$ be a mapping satisfying (i)-(ii)-(iii). Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj)-(jjj).*

Then, for any $u_0 \in \text{dom}\varphi(0, \cdot)$, there is an absolutely continuous solution $u(\cdot)$ to the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Proof. Step 1 Let $u_0 \in \text{dom}\varphi(0, \cdot)$, $\varepsilon > 0$. Let $u : [0, 1] \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in \partial\varphi(t, u(t)) \text{ a.e. } t \in I, \\ u(0) &= u_0 \in \text{dom}\varphi(0, \cdot). \end{aligned}$$

The existence and uniqueness of such a solution follow from Theorem 3.9. Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \alpha(t)(1 + r(t)) + \varepsilon + m \quad \text{a.e. with } r(0) = \sup_{t \in [0, 1]} \|u(t)\|.$$

Since $\dot{r} + \varepsilon \in L_{\mathbb{R}_+}^\infty(I) \subset L_{\mathbb{R}_+}^1(I)$, the set

$$M := \{h \in L_H^1(I) : \|h(t)\| \leq \dot{r}(t) + \varepsilon \text{ a.e.}\},$$

is clearly convex $\sigma(L_H^1(I), L_H^\infty(I))$ -compact. For any $h \in M$, denote by u_h the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

The existence and uniqueness of such a solution is granted by Theorem 3.9 (cf. also Theorem 3.5), that is due to the particular choice of the perturbation f and the perturbation $h \in M$. Let us set for any $h \in M$, $g(t, x) = f(t, x) + h(t)$ for all $(t, x) \in I \times H$. Clearly, the function g satisfies assumptions (i)-(ii). Moreover, for all $(t, x) \in I \times H$, one has

$$\|g(t, x)\| \leq m + \|h(t)\| \leq (m + \|h(t)\|)(1 + \|x\|),$$

which yields (iii) with $\beta(\cdot) := m + \dot{r}(\cdot) + \varepsilon \in L_{\mathbb{R}_+}^2(I)$.

Using the monotonicity of $\partial\varphi(t, \cdot)$, for all $t \in I$, one has the estimation

$$\frac{1}{2} \|u_h(t) - u(t)\|^2 \leq \int_0^t \|f(s, u_h(s)) + h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows

$$\|u_h(t) - u(t)\| \leq \int_0^t \|f(s, u_h(s)) + h(s)\| ds,$$

so that for all $t \in I$, one gets

$$\|u_h(t)\| \leq r(0) + \int_0^t [\dot{r}(s) + \varepsilon + m] ds = r(t) + \varepsilon t + mt \leq r(t) + m + \varepsilon. \quad (3.16)$$

Now, we assert that

Main fact: $K := \{u_h : h \in M\}$ is compact in $C_H(I)$. It follows from Theorem 3.12.

Finally, we proceed as in **Step 2-Step 3** of Theorem 3.6 to get

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + f(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

with $h(t) \in F(t, u_h(t))$ a.e. finishing the proof of the theorem. □

To finish the section we present an existence result of absolutely continuous solution to the inclusion $-\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t))$ when φ satisfies $\mathbf{H}(\varphi)$ and the perturbation f is Lipschitz with respect to $x \in H$ by using some ideas in [23] and the above tools. This result is quite useful in further applications.

Theorem 3.12. *Assume that φ satisfies condition $\mathbf{H}(\varphi)$ and that $\text{dom}\varphi(t, \cdot)$ is ball-compact, for every $t \in I$. Let $f : I \times H \rightarrow H$ be a map satisfying (i)-(ii)-(iii).*

Then there is a unique absolutely continuous solution to the inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + f(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

Proof. Step 1. We will follow some arguments used in [23].

Let $u_0 \in \text{dom}\varphi(0, \cdot)$. Let $u : I \rightarrow H$ be the unique absolutely continuous solution to

$$\begin{aligned} -\dot{u}(t) &\in \partial\varphi(t, u(t)) \text{ a.e. } t \in I, \\ u(0) &= u_0 \in \text{dom}\varphi(0, \cdot). \end{aligned}$$

Now, let $r : [0, 1] \rightarrow \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation

$$\dot{r}(t) = \beta(t)(1 + r(t)) \quad \text{a.e. with } r(0) = \sup_{t \in I} \|u(t)\|.$$

Since $\dot{r} \in L^2_{\mathbb{R}_+}(I)$, the set

$$K := \{h \in L^2_H(I) : \|h(t)\| \leq \dot{r}(t) \text{ a.e. } t \in I\},$$

is clearly convex $\sigma(L^2_H(I), L^2_H(I))$ -compact. For any $h \in K$ denote by u_h the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t) & \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Using the monotonicity of $\partial\varphi(t, \cdot)$ for all $t \in I$, one obtains the estimate

$$\frac{1}{2} \|u_h(t) - u(t)\|^2 \leq \int_0^t \|h(s)\| \|u_h(s) - u(s)\| ds.$$

Thanks to Lemma 2.1, it follows that

$$\|u_h(t) - u(t)\| \leq \int_0^t \|h(s)\| ds$$

so that

$$\|u_h(t)\| \leq r(0) + \int_0^t \dot{r}(s) ds = r(t). \tag{3.17}$$

Set $L := \{u_h : h \in K\}$. **Main fact** L is **compact** in $\mathcal{C}_H(I)$.

This follows from the above estimate and the compactness property Theorem 3.10.

Step 2. By construction, the set K is convex $\sigma(L^2_H(I), L^2_H(I))$ -compact (then

$\sigma(L_H^2(I), L_H^2(I))$ -closed), and by assumption (iii), for every $h \in K$, we note that

$$\|f(t, u_h(t))\| \leq \beta(t)(1 + \|u_h(t)\|) \leq \beta(t)(1 + r(t)).$$

Let us define the mapping Φ on K by

$$\Phi(h)(t) = f(t, u_h(t)) \text{ for all } t \in I,$$

where u_h is the unique absolutely continuous solution to the inclusion

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot). \end{cases}$$

Remark that $\Phi(h) \in K$. Then, $\Phi(K)$ is contained in the $\sigma(L_H^2(I), L_H^2(I))$ -compact set K . Clearly, if h is a fixed point of Φ ($h = \Phi(h)$), then u_h is an absolutely continuous solution to the inclusion under consideration, namely

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t) \text{ a.e. } t \in I, \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

with $h(t) = f(t, u_h(t))$ for all $t \in I$. Now let us prove that $\Phi : K \rightarrow K$ is continuous when K is endowed with the $\sigma(L_H^2(I), L_H^2(I))$ topology. Remark that K is convex $\sigma(L_H^2(I), L_H^2(I))$ -compact and H is separable so that K is convex $\sigma(L_H^2(I), L_H^2(I))$ -compact metrizable. It is enough to show that Φ is sequentially $\sigma(L_H^2(I), L_H^2(I))$ continuous on K . Let $(h_n) \subset \sigma(L_H^2(I), L_H^2(I))$ converges in $L_H^2(I)$ to $h \in K$. We have to prove that $(\Phi(h_n))$ weakly converges in $L_H^2(I)$ to $\Phi(h) \in K$. Recall that the set $L := \{u_h : h \in K\}$ of solutions to

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t), \text{ a.e. } t \in I, \ h \in K \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

is compact in $\mathcal{C}_H(I)$. Hence the solution u_{h_n} to

$$\begin{cases} -\dot{u}_{h_n}(t) \in \partial\varphi(t, u_{h_n}(t)) + h_n(t), \text{ a.e. } t \in I, \ h_n \in L \\ u_{h_n}(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

uniformly converges to the solution u_h to

$$\begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t), \text{ a.e. } t \in I, \ h \in K \\ u_h(0) = u_0 \in \text{dom}\varphi(0, \cdot) \end{cases}$$

It remains to check the required continuity. Let $g \in L_H^2(I)$. Then we have $|\langle g(t), f(t, u_h(t)) \rangle| \leq \|g(t)\|\alpha(t)(1 + \|u_h(t)\|) \leq \|g(t)\|\dot{r}(t)$ for any $h \in K$. Whence the measurable functions $\langle g(t), f(t, u_h(t)) \rangle$ are dominated by the integrable function $t \rightarrow \|g(t)\|\dot{r}(t)$. By construction we have for every $t \in I$ $\langle g(t), f(t, u_{h_n}(t)) \rangle \rightarrow \langle g(t), f(t, u_h(t)) \rangle$, since $f(t, \cdot)$ is continuous for every fixed t in I . As a consequence of Lebesgue's convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_0^1 \langle g(t), f(t, u_{h_n}(t)) \rangle dt = \int_0^1 \langle g(t), f(t, u_h(t)) \rangle dt.$$

So we conclude that $(\Phi(h_n))$ converges to $\Phi(h)$ on K with respect to the $\sigma(L_H^2(I), L_H^2(I))$ -topology. This means that $\Phi : K \rightarrow K$ is continuous on K endowed with the $\sigma(L_H^2(I), L_H^2(I))$ -topology. Applying Schauder fixed point theorem to the continuous mapping $\Phi : K \rightarrow K$ shows that Φ admits a

fixed point, $h = \Phi(h)$, thus proving the existence of an absolutely continuous solution to our inclusion. Uniqueness follows from the monotonicity of $\partial\varphi(t, \cdot)$ and the Lipschitz property of $f(t, \cdot)$. \square

4. Applications

We aim here to prove the existence of periodic solutions to our evolution problems.

Theorem 4.1. *Let $I := [0, 1]$. Assume that for every $t \in I$, $A(t) : D(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator with $D(A(t))$ is ball-compact for every $t \in I$, satisfying (H1)-(H2)-(H3). Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j)-(jj)-(jjj). Suppose further that: there exists a function $\gamma(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \gamma(t) \|x_1 - x_2\|^2,$$

for all $y_1 \in F(t, x_1)$, $y_2 \in F(t, x_2)$ and $t \in I$.

Suppose also that $D(A(0))$ is convex compact and $t \mapsto D(A(t))$ satisfies the periodic condition

$$D(A(0)) = D(A(1)).$$

Then, the differential inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = x_0 \in D(A(0)), \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ which is periodic, that is, $u(0) = u(1)$.

Proof. Let us check first uniqueness of the solution.

Let u_1 and u_2 be two solutions of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in D(A(0)), \end{cases} \quad (4.1)$$

whose existence is guaranteed by Theorem 3.2. Then, there exist two $L^2_H(I)$ -functions $h_1(\cdot)$ and $h_2(\cdot)$ such that for all $t \in I$

$$h_1(t) \in F(t, u_1(t)) \text{ and } h_2(t) \in F(t, u_2(t)),$$

and for almost all $t \in I$

$$-\dot{u}_1(t) - h_1(t) \in A(t)u_1(t) \text{ and } -\dot{u}_2(t) - h_2(t) \in A(t)u_2(t).$$

The monotone condition on F yields

$$\langle h_1(t) - h_2(t), u_1(t) - u_2(t) \rangle \geq \gamma(t) \|u_1(t) - u_2(t)\|^2.$$

Moreover, the monotone property of $A(t)$ ensures that

$$\langle -\dot{u}_1(t) - h_1(t) - (-\dot{u}_2(t) - h_2(t)), u_1(t) - u_2(t) \rangle \geq 0,$$

hence,

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \leq -\gamma(t) \|u_1(t) - u_2(t)\|^2,$$

that is,

$$\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq -\gamma(t) \|u_1(t) - u_2(t)\|^2 \text{ for a.e. } t \in I. \quad (4.2)$$

Remark that $\frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq 0$, since by assumption γ is a non-negative function. This along with the fact that $\|u_1(0) - u_2(0)\| = 0$, integrating, one gets $u_1 = u_2$. Thus, it follows the uniqueness of the solution for the problem (4.1).

Now we check the periodic property of the solution.

Let $a, b \in D(A(0)) = D(A(1))$ and denote by u_a (resp. u_b) the unique solution of the problem (4.1) with the initial condition a (resp. b) in $D(A(0)) = D(A(1))$. Then, we take in inequality (4.2), $u_1 = u_a$ and $u_2 = u_b$. By integration, it results

$$\frac{1}{2} \|u_a(t) - u_b(t)\|^2 \leq \frac{1}{2} \|a - b\|^2 - \int_0^t \gamma(s) \|u_a(s) - u_b(s)\|^2 ds. \quad (4.3)$$

We may use similar arguments as in [20], we will see that the function $a \mapsto u_a(1)$ is a strict contraction on the convex compact set $D(A(0)) = D(A(1))$, that is,

$$\|u_a(1) - u_b(1)\| < \|a - b\|, \text{ if } \|a - b\| > 0.$$

In view of Lemma 5.4 [20] (if θ is a continuous real valued function such that

$$0 \leq \theta(t) \leq m - \int_0^t g(s)\theta(s)ds$$

with $m > 0$ and $g(\cdot) \in L^1_{\mathbb{R}^+}(I)$, then, $\theta(t) < m$ for all $t \in I$), it results from (4.3) that

$$\|u_a(1) - u_b(1)\| < \|a - b\|.$$

We conclude that the mapping $a \in D(A(0)) \mapsto u_a(1) \in D(A(1))$ has a unique fixed point which is the unique periodic solution for the problem (4.1). \square

By applying the tools given in Theorem 3.8, we obtain a new existence theorem of periodic solution for evolution problems governed by time-dependent subdifferential operators as follows.

Theorem 4.2. *Let $I := [0, 1]$. Assume that (H_1) - (H_2) hold, and that $\text{dom}\varphi(t, \cdot)$ is ball-compact for every $t \in I$. Let $F : I \times H \rightrightarrows H$ be a compact set-valued map satisfying (j) - (jj) - (jjj) . Suppose further that: there exists a function $\gamma(\cdot) \in L^2_{\mathbb{R}^+}(I)$ such that*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \gamma(t) \|x_1 - x_2\|^2,$$

for all $y_1 \in F(t, x_1)$, $y_2 \in F(t, x_2)$ and $t \in I$.

Suppose also that $\text{dom}\varphi(0, \cdot)$ is convex compact and $t \mapsto \text{dom}\varphi(t, \cdot)$ satisfies the periodic condition

$$\text{dom}\varphi(0, \cdot) = \text{dom}\varphi(1, \cdot).$$

Then, the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\varphi(t, u(t)) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \text{dom}\varphi(0, \cdot), \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ which is periodic, that is, $u(0) = u(1)$.

Proof. The proof is similar to the one of Theorem 4.1, so we omit it. \square

Conclusion. In this paper, we have established new existence results regarding first-order perturbed evolution problems involving time-dependent maximal monotone (resp. subdifferential) operators, in Hilbert spaces. The set-valued perturbation is assumed to be mixed semi-continuous. We have also investigated periodic solutions under suitable assumptions. Our results cover related perturbed convex sweeping processes. It will be interesting to extend these results to the more general setting of Banach spaces, and to deal with the differential inclusion of the form studied here governed by time and state dependent operator $A(t, x)$ instead of $A(t)$. Such studies are out of the scope of this manuscript and will be the subject of future works.

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