INSTITUT FOURIER

Probabilité

Excursion sets of correlated Random Fields

Application to continuum percolation theory

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Laboratoire 3SR
Université Grenoble Alpes (France)

3SR

UGA Université Grenoble Alpes

Motivations

Specimen

O. Stamati, E. Roubin, E. Andó and Y. Malecot, *Phase segmentation of concrete x-ray tomographic images at meso-scale: Validation with neutron tomography*, 2018.

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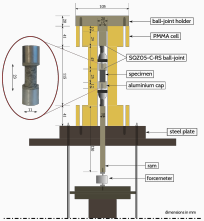

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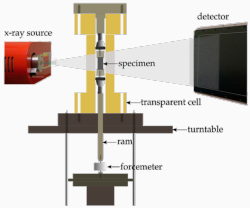

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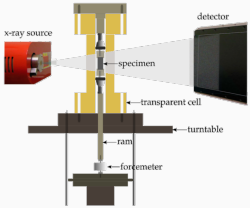

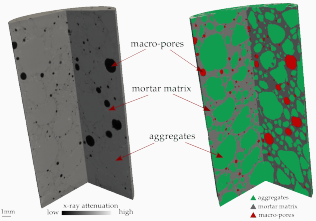

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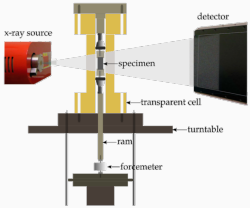

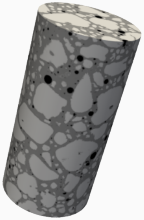

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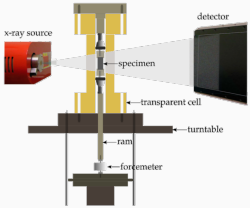

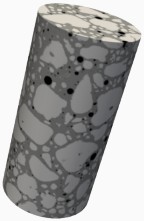

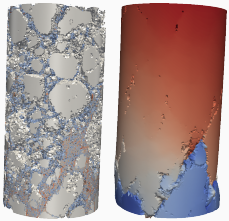

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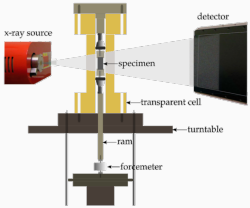
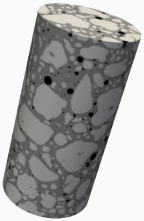
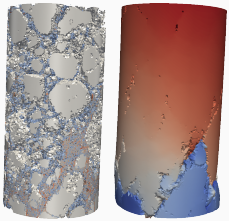


Morphological model





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Objectives being able to represent all kinds of morphologies

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... or at least dense granular morphologies

Outline

Morphological model based on correlated Random Fields

- Morphological models
- Excursions of correlated Random Fields

The Excursion Set Theory

- $N+1$ measures for $N-$dimensional spaces
- Expectations of the measures

Application to percolation

- Our positioning
- The Euler Characteristic: percolation criterion

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Morphological models

Goals:

- **Random** aspect of heterogeneities shape and positions
- **Discrete** aspect
- **Control** geometrical and topological quantities (volume, surface, size distribution of the heterogeneities...)

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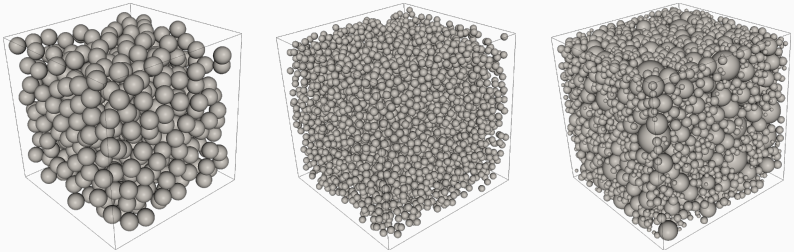
Hard spheres packing:



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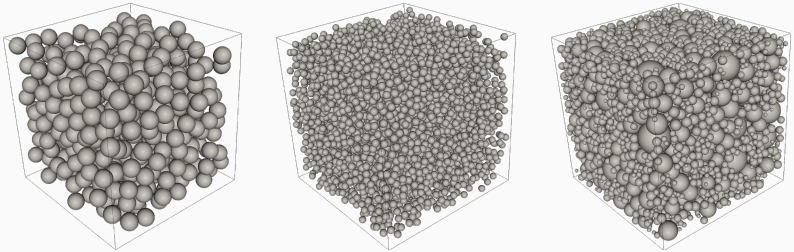

Simple, natural and efficient

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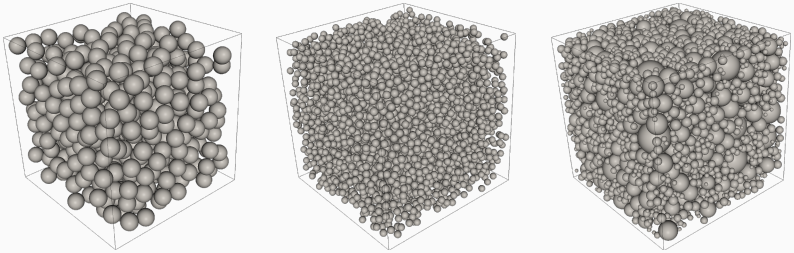

Simple, natural and efficient, one "kind" of morphology, ideal shapes, heavy

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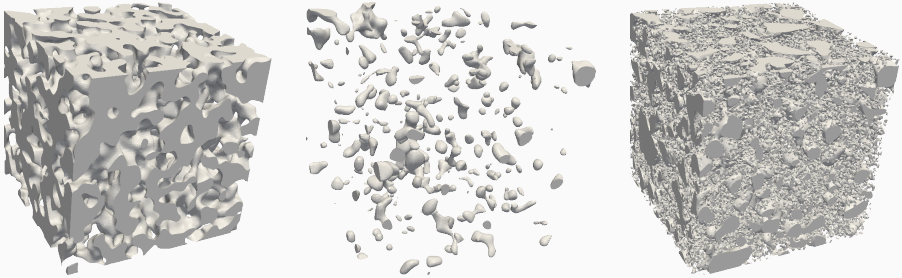
Morphological models

Excursion set of correlated Random Fields:



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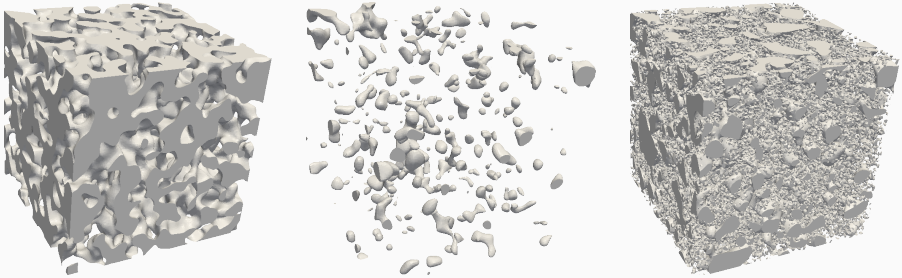

- Different “kinds” of morphologies, light, random shapes, evolutive

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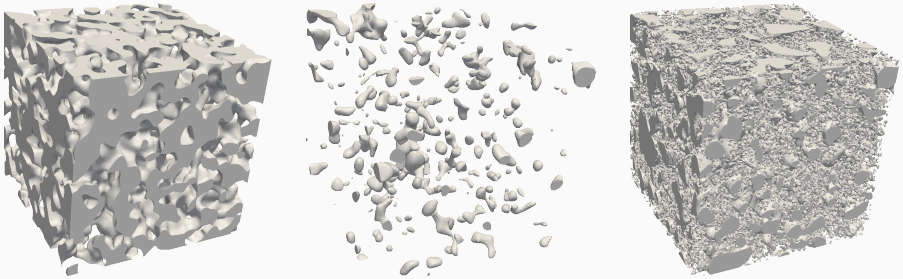

- Different "kinds" of morphologies, light, random shapes, evolutive
- Hard to control, distribution less natural, smooth surfaces (for now)

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Correlated Random Fields

Technically to define a **stricly stationary correlated Random Field** we have to define:

- A constant probability distribution over the spatial parameter $\boldsymbol{x}$. $g(\boldsymbol{x})$ can be seen as a RV X. A classical distribution is the **Gaussian distribution** $\mathcal{N}(\mu, \sigma)$ where μ is the mean value and σ the standard deviation:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- A covariance function wich depends on the distance between two points in space $d = \|\boldsymbol{x} - \boldsymbol{y}\|$. A classical choice is the **Gaussian covariance function**:

$$C(d) = \sigma^2 e^{-d^2/L_c^2}$$

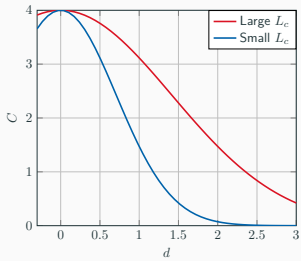
Correlated Random Fields

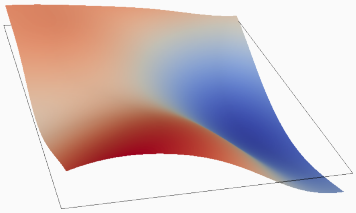
Correlation length L_c

The Gaussian correlation function

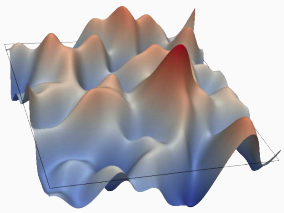
$$C(d) = \sigma^2 e^{-d^2/L_c^2}$$

has a single structural parameter L_c called the **correlation length**.





Large L_c



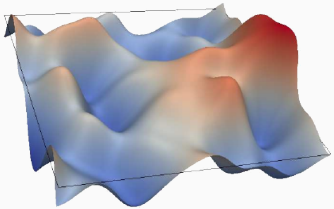
Small L_c

Correlated Random Fields

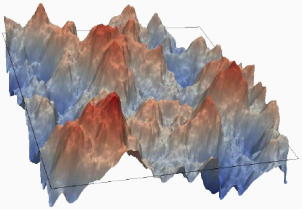
Out of topic... but other classes of covariance functions bring more flexibilty.

With the **Matérn class** we can play with the roughness (additional parameter ν):

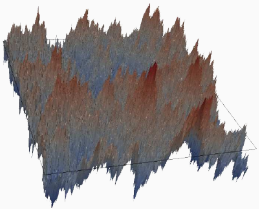
$$C(d) = \frac{\sigma^2}{\Gamma(\nu)2^{1-\nu}} \left(\frac{\sqrt{2\nu}d}{L_c} \right)^{\nu} K_\nu \left(\frac{\sqrt{2\nu}d}{L_c} \right)$$



$\nu \to \infty$
Gaussian



$\nu = 3/2$



$\nu = 1/2$
Exponential

Rasmussen and Williams, *Gaussian Processes for Machine Learning*, 2006.

Excursion sets

An **excursion set** $\mathcal{E}_s$ is the result of the **"threshold"** of a realisation of a RF:

$$\mathcal{E}_s = \{\boldsymbol{x} \in M \mid g(\boldsymbol{x}) \in \mathcal{H}_s\}$$

where M is the domain of definition of the RF and $\mathcal{H}_s$ the so called **Hitting Set**.

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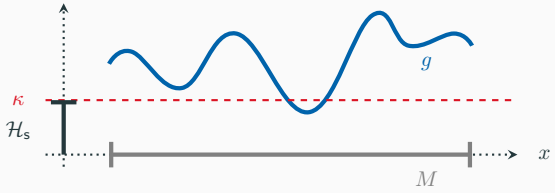

Excursion with "low" threshold

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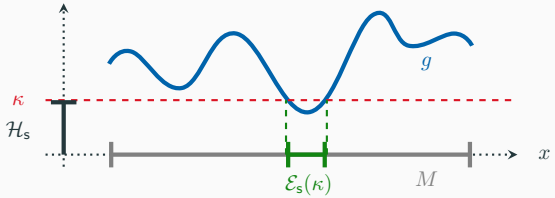

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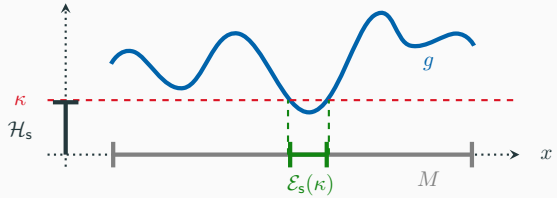

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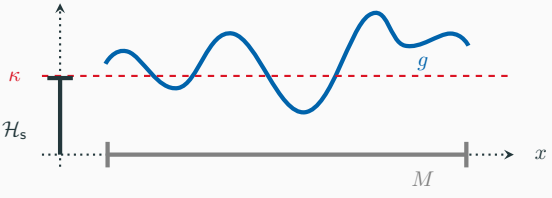

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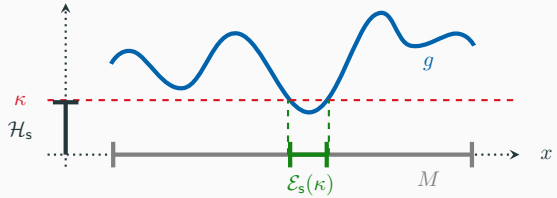

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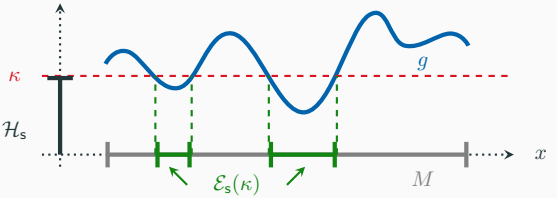

Excursion with "high" threshold

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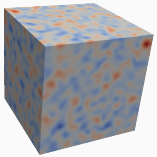
$g : \Omega \times \mathbb{R}^3 \mapsto \mathbb{R}$

Continuous aspect

parametric variability

Medium L_c



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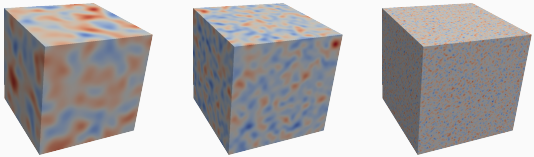

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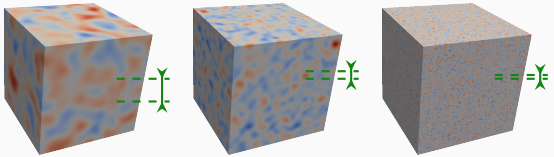

Heterogeneity sizes

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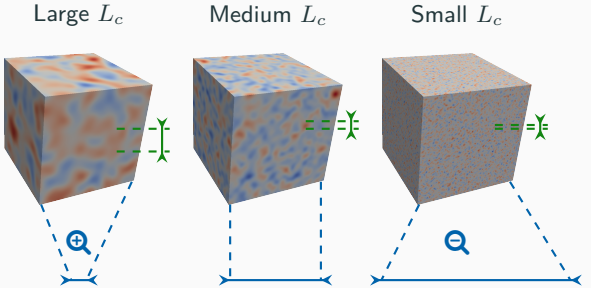

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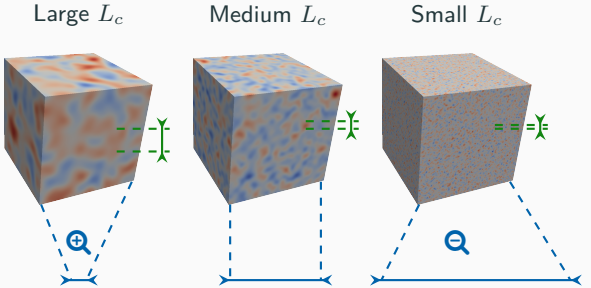

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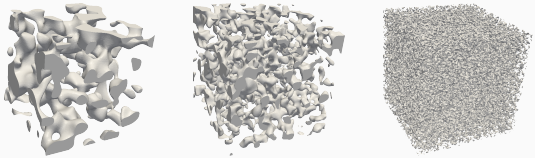
Discrete aspect

explicit morphology



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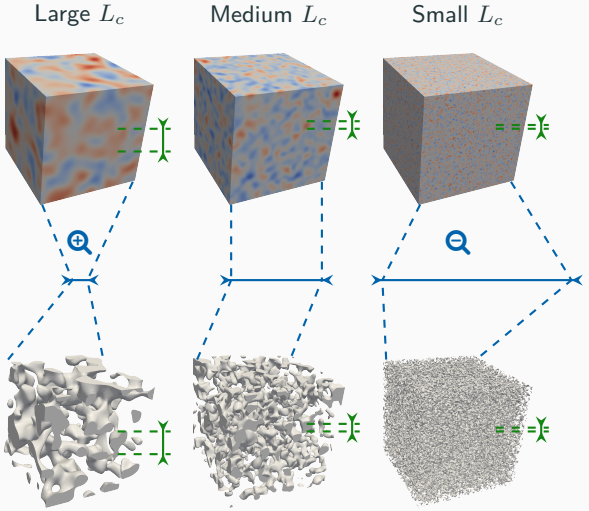
Continuous aspect

parametric variability



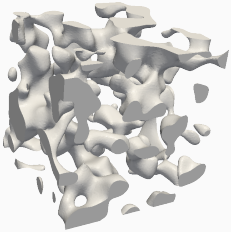
Excursion sets

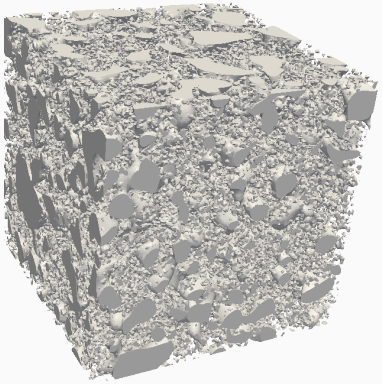
$\mathcal{E}_s = \{\boldsymbol{x} \in M \mid g(\boldsymbol{x}) \in \mathcal{H}_s\}$

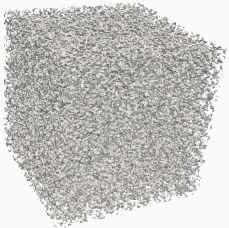
Discrete aspect

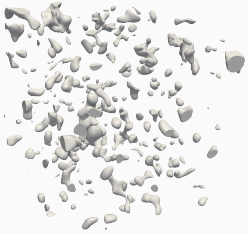
explicit morphology

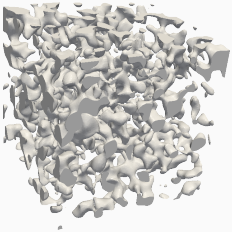
Excursion sets











The Excursion Set Theory

Outline

Morphological model based on correlated Random Fields

- Morphological models
- Excursions of correlated Random Fields

The Excursion Set Theory

- $N+1$ measures for $N-$dimensional spaces
- Expectations of the measures

Application to percolation

- Our positioning
- The Euler Characteristic: percolation criterion

Family of measures

It exists several **family of measures** (Minkowski functionals, Lipschitz-Killing curvatures...). In an $N-$dimensional space, the size of the family is $N+1$ where each element can be seen as a $n-$dimensional measure.
Each measure can be classified into two types:

- **geometrical measures** $(1 \leq n \leq N)$
- **topological measure** $(n = 0)$

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In 2D

$n = 2$**:** Surface area

$n = 1$**:** Total curvature

$n = 0$**:** Euler Characteristic

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$n = 2$: Surface area

$n = 1$: Total curvature

$n = 0$: Euler Characteristic

In 3D

$n = 3$: Volume

$n = 2$: Surface area

$n = 1$: Total curvature

$n = 0$: Euler Characteristic

The Euler Characteristic

The Euler Characteristic: a topological measure

The Euler Characteristic is a mathematical measure that gives information on the topology of the morphology.
It enumerates $n-$dimensional features.

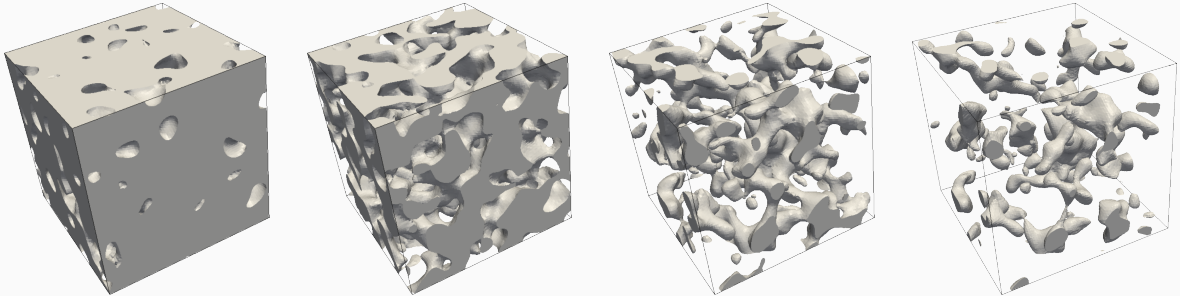
- In 2D

$$\chi = \#\{\text{connected components}\} - \#\{\text{holes}\}$$

- In 3D

$$\chi = \#\{\text{connected components}\} - \#\{\text{handles}\} + \#\{\text{holes}\}$$

Mean value of the measures over the threshold



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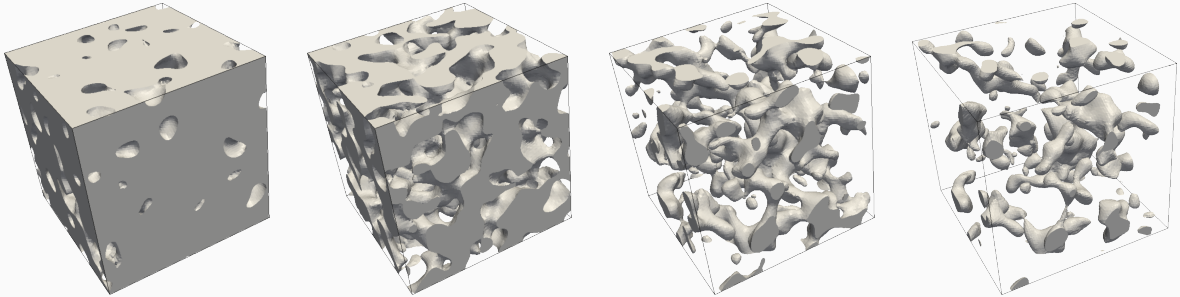
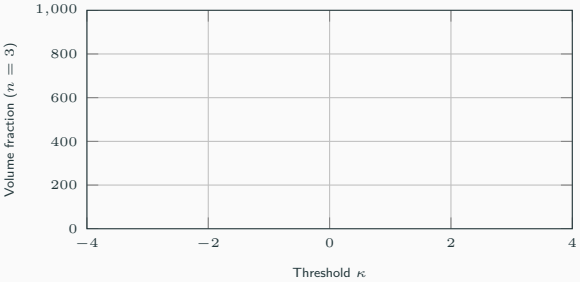
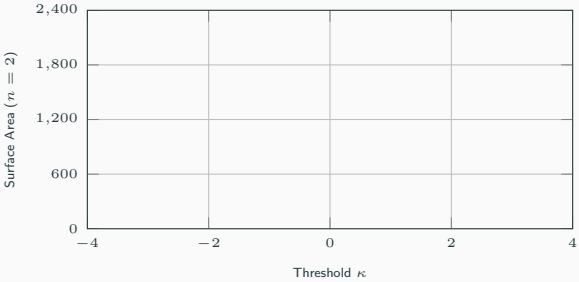
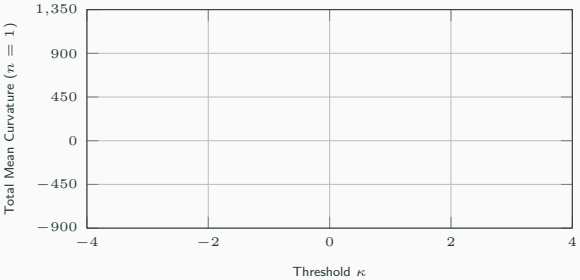


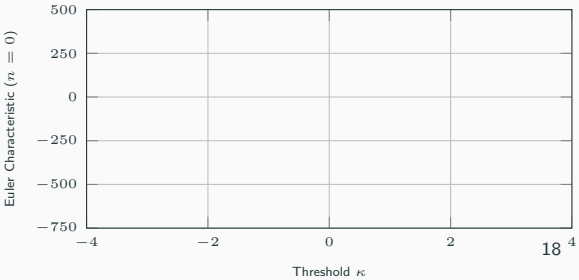
Evolution of the 4 measures?

Mean value of the measures over the threshold









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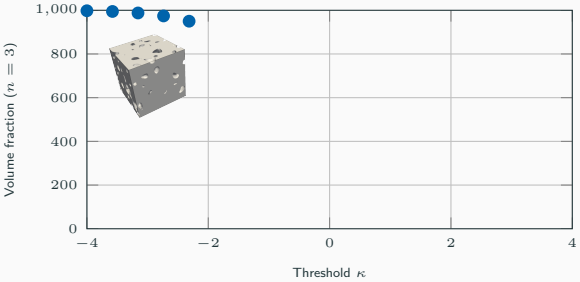

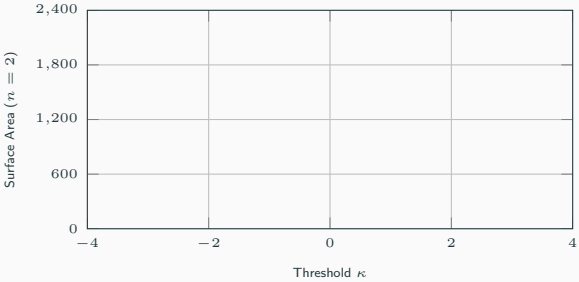

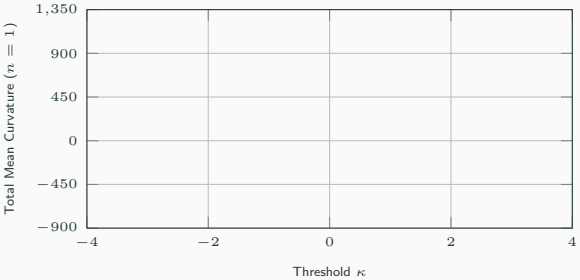

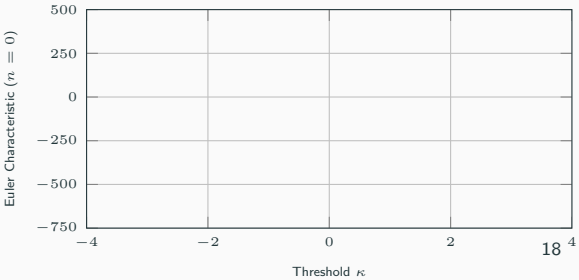

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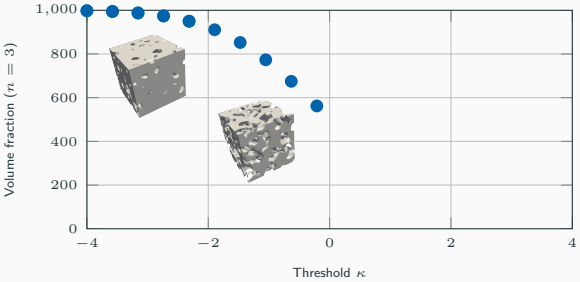

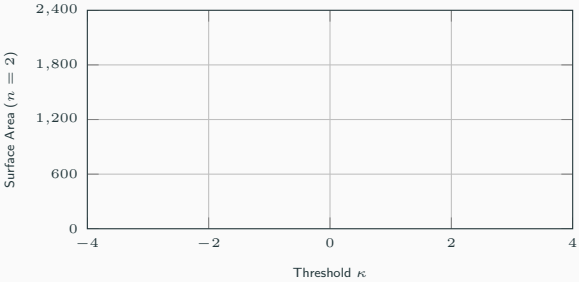

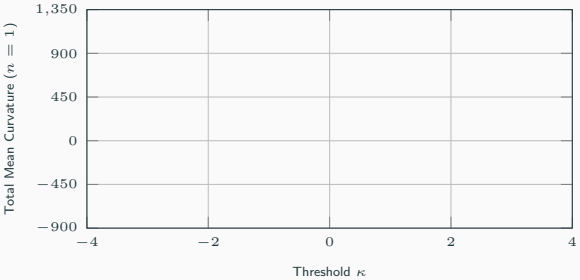

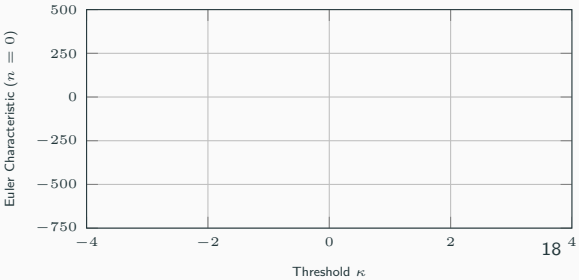

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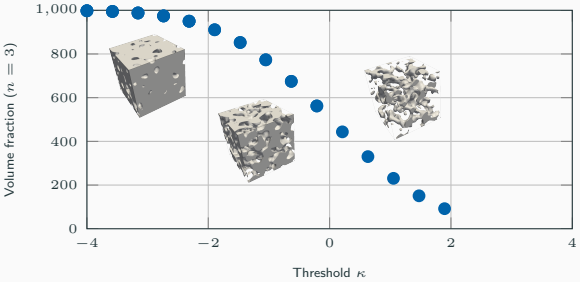

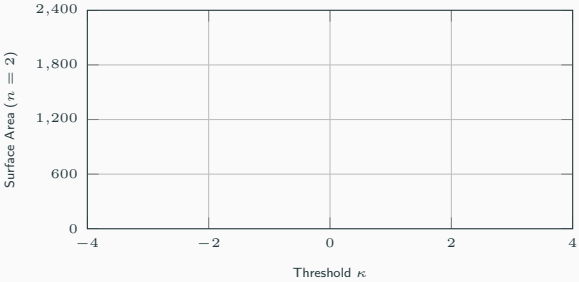

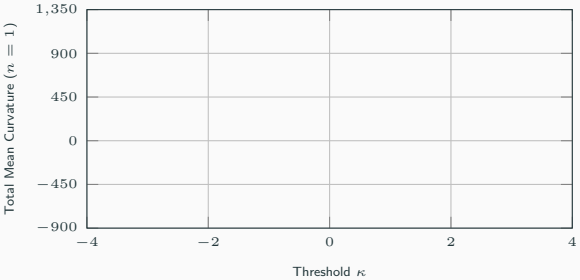

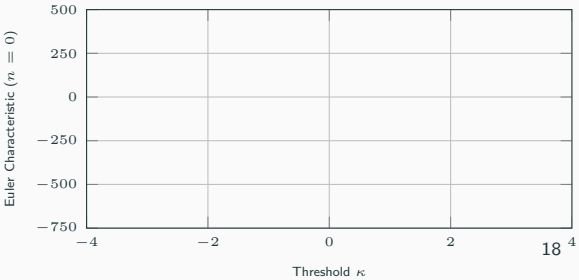

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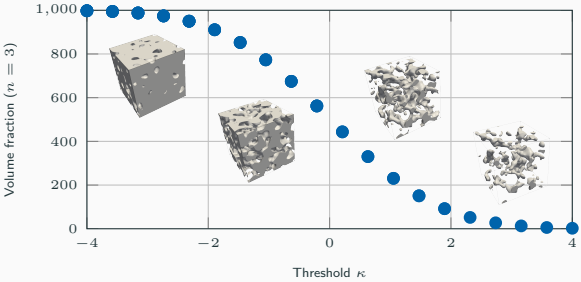

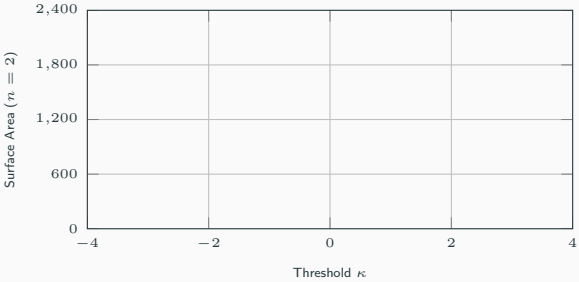

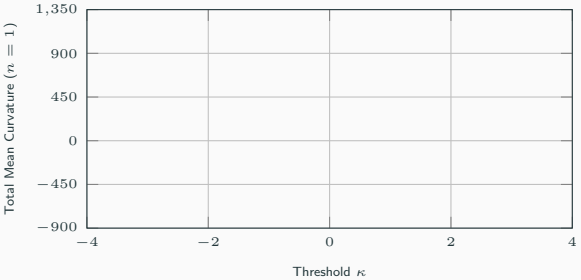

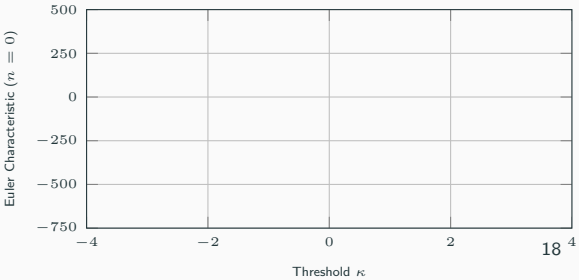

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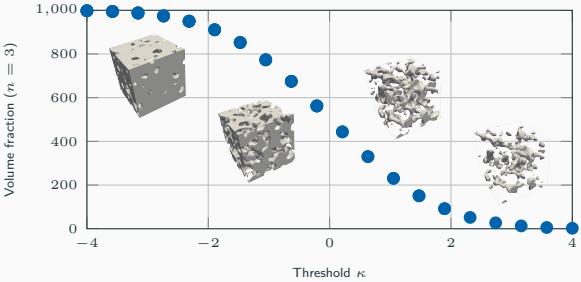

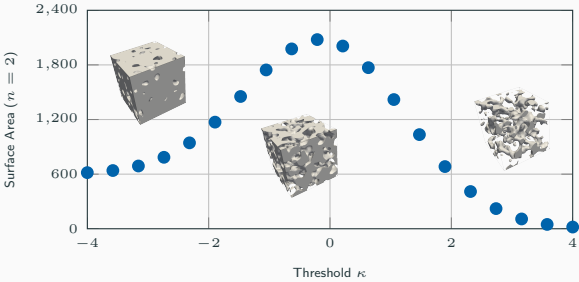

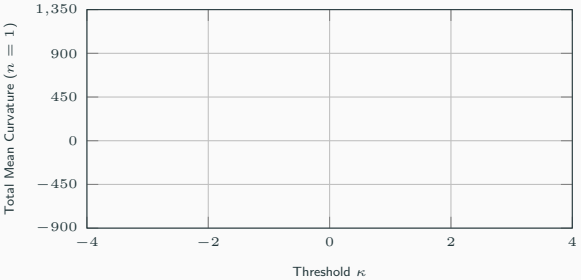

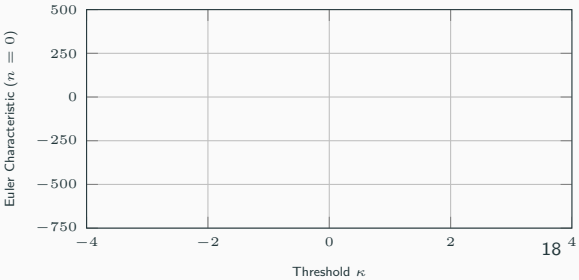

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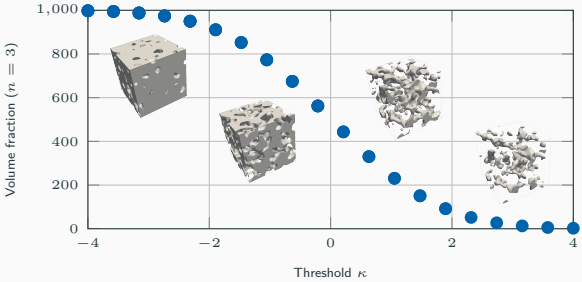

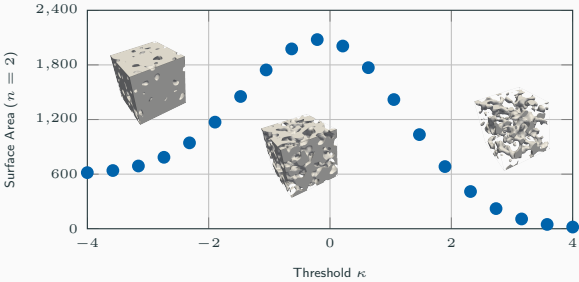

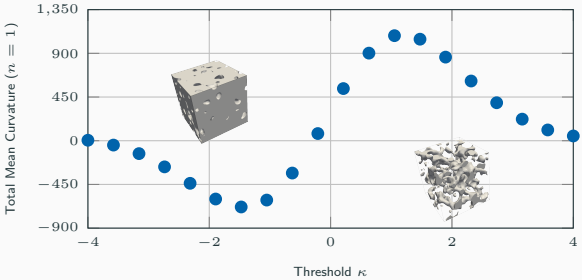

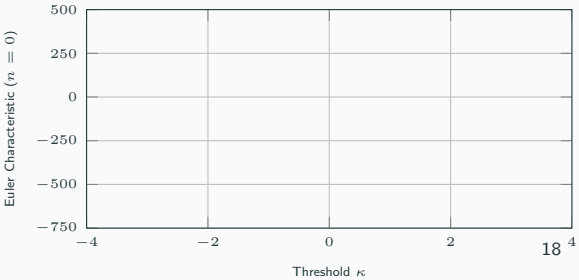

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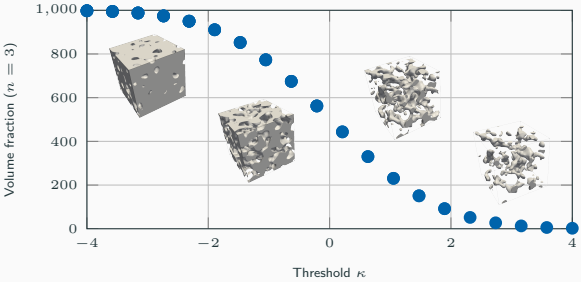

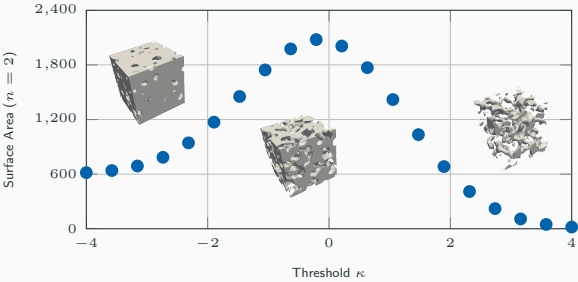

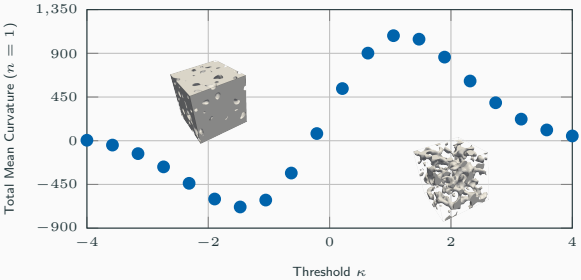

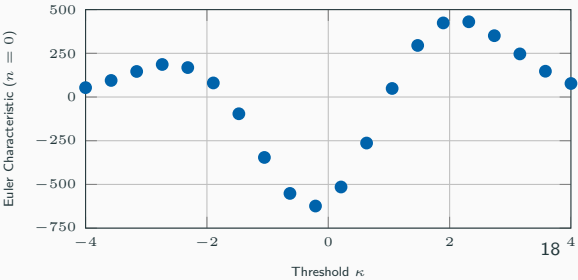

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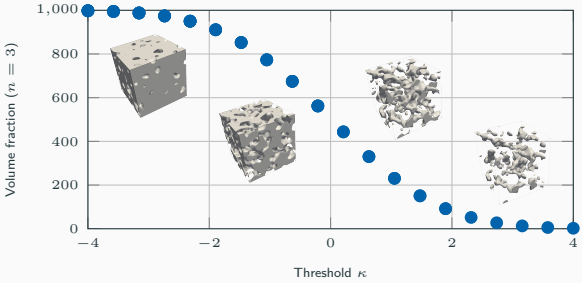

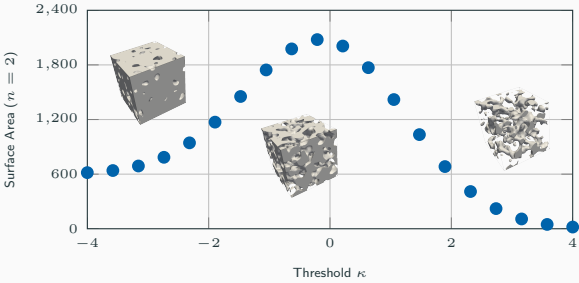

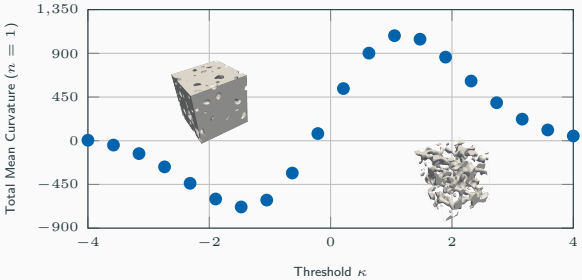

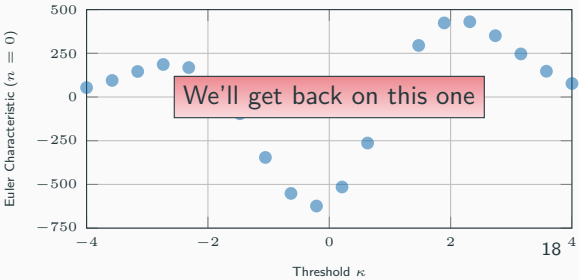

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The expectation formula

In the context of **excursion sets of correlated Random Fields** each measure $\mathcal{L}_j$ is a **Random Variable**.
They have a distribution that depends on:

- the parameters of the correlated Random Field ($C(\boldsymbol{x}, \boldsymbol{y}), f_X(x), M$)
- the hitting set (κ)

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We don't know the distribution but we know its **expected value**:

$$\mathbb{E}(\mathcal{L}_j(\mathcal{E}_\mathsf{s})) = f(j, L_c, \mu, \sigma, M, \kappa)$$

R. Adler, *Some new random field tools for spatial analysis*, 2008.

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$$\mathbb{E}\{\mathcal{L}_j(\mathcal{E}_\mathsf{s})\} = \sum_{i=0}^{N-j} \binom{i+j}{i} \frac{\omega_{i+j}}{\omega_i \omega_j} \left(\frac{\lambda_2}{2\pi}\right)^{i/2} \mathcal{L}_{i+j}(M)\, \mathcal{M}_i^\gamma(\kappa)$$

R. Adler, *Some new random field tools for spatial analysis*, 2008.

The expectation formula

Gaussian Minkowski functionals: $\mathcal{M}_i^{\gamma_k}(\mathcal{H}_s)$

- They measure the probability of the Random Field to be in the hitting set $\mathcal{H}_s \subset \mathbb{R}^k$.
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Kinematic formula

If $\boldsymbol{X} = X_i$ is a standard Gaussian vector of size k in which $X_i \sim \mathcal{N}(0, \sigma^2)$ are independant and $\mathcal{H}_\mathsf{s} \subset \mathbb{R}^k$:

$$\gamma_k(\mathcal{H}_\mathsf{s}) = P(\boldsymbol{X} \in \mathcal{H}_\mathsf{s}) = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}_\mathsf{s}} e^{-\|\boldsymbol{x}\|^2/2\sigma^2} d\boldsymbol{x}$$

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If $\mathcal{K}(A, \rho)$ is the tube of A or ray ρ we have the following Taylor expansion:

$$\gamma_k(\mathcal{K}(\mathcal{H}_s, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_i^{\gamma_k}(\mathcal{H}_s)$$

J. Taylor, *A Gaussian kinematic formula*, 2006.

The expectation formula

Application to scalar valued Gaussian Random Fields: $\mathcal{M}_i^{\gamma}(\mathcal{H}_{\mathsf{s}})$

Hitting set, Tube and expansions

$\mathcal{H}_{\mathsf{s}}$ and Tube $\quad \mathcal{H}_{\mathsf{s}} = [\kappa, \infty[\quad$ and $\quad \mathcal{K}(\mathcal{H}_{\mathsf{s}}) = [\kappa - \rho, \infty[$

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Identification of the Gaussian Minkowski Functionals

$$\mathcal{M}_i^\gamma(\mathcal{H}_s) = (-1)^j \bar{F}^{(i)}(\kappa)$$

The expectation formula

Volume Fraction

$$\mathbb{E}\{\Phi\} = \frac{1}{\sqrt{\pi}} \int_{\kappa/\sigma}^{\infty} e^{-t^2} dt$$

Euler Characteristic

With the scale ratio $\beta = \text{size}(M)/L_c$

$$\mathbb{E}\{\chi\} = \left[\frac{\beta^3}{\sqrt{2}\pi^2} \left(\frac{\kappa^2}{\sigma^2} - 1 \right) + \frac{3\beta^2}{\sqrt{2}\pi^{3/2}} \frac{\kappa}{\sigma} + \frac{3\beta}{\sqrt{2}\pi} \right] e^{-\kappa^2/2\sigma^2} + \frac{1}{\sqrt{\pi}} \int_{\kappa/\sigma}^{\infty} e^{-t^2} dt$$

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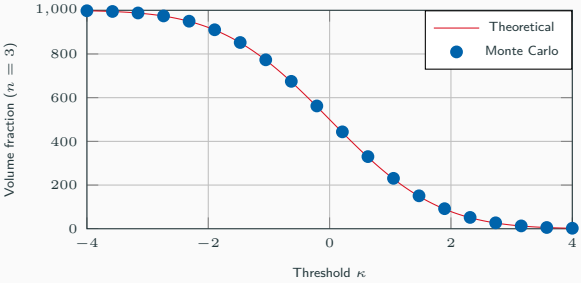
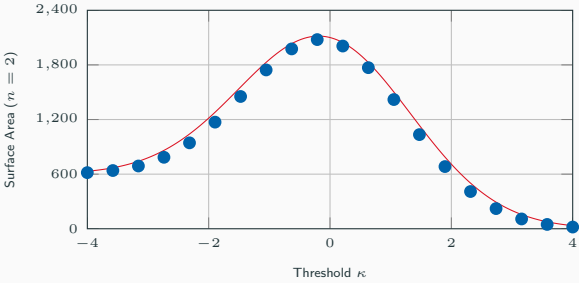
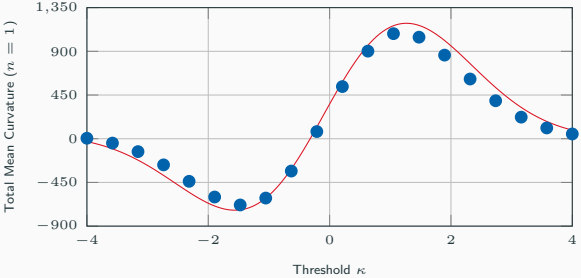
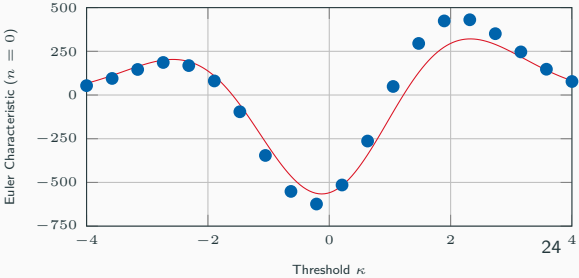
K.J. Worsley, *The geometry of random images*, 1996.

Mean value of the measures over the threshold









Application to percolation

Outline

Morphological model based on correlated Random Fields

- Morphological models
- Excursions of correlated Random Fields

The Excursion Set Theory

- $N+1$ measures for $N-$dimensional spaces
- Expectations of the measures

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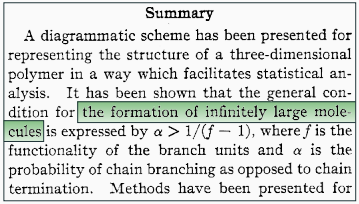
The pioners



Summary

A diagrammatic scheme has been presented for representing the structure of a three-dimensional polymer in a way which facilitates statistical analysis. It has been shown that the general condition for the formation of infinitely large molecules is expressed by $\alpha > 1/(f-1)$, where f is the functionality of the branch units and α is the probability of chain branching as opposed to chain termination. Methods have been presented for

Paul J. Flory, *Molecular size distribution in three dimensional polymers: Gelation*, 1941.

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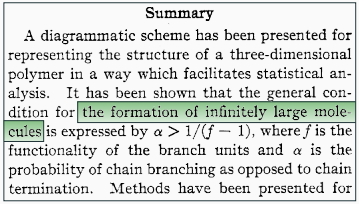

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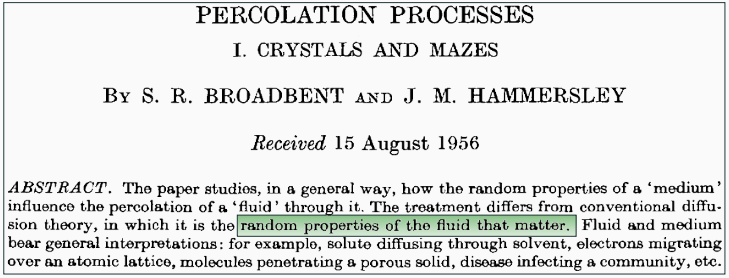

PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

By S. R. BROADBENT and J. M. HAMMERSLEY

Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

S. R. Broadbent and J. M. Hammersley, *Percolation process I and II*, 1957.

The Critical Percolation Probabilities p_c

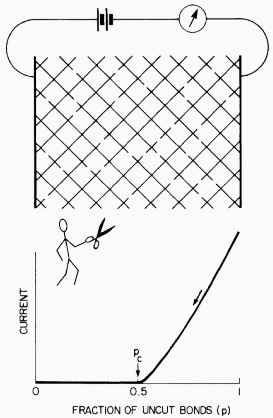

Figure 4.1 The randomly cut network as an example of percolation.

R. Zallen, *The Physics of Amorphous Solids: Chapter 4 The Percolation Model*, 1983.

From (7.1), if K is singular at p_c then it is also singular at $1 - p_c$, and if there is only one singularity these must be identical points, or

$$p_c = \tfrac{1}{2}. \qquad (7.2)$$

This establishes two important percolation probabilities as $\frac{1}{2}$—that for the site problem on the triangular lattice and that for the bond problem on the simple quadratic lattice. The result (7.2) holds for any fully triangulated lattice.

M. F. Sykes and J. W. Essam, *Exact Critical Percolation Probabilities for Site and Bond Problems in Two Dimensions*, 1964.

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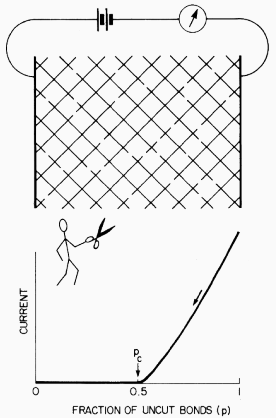

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- Lattices (graphs)
- Volumetric approach to regularise
  - R. Zallen, *Critical density in percolation processes*, 1970.
- Boolean models
  - K. Mecke, *Morphology of spatial patterns*, 1997.
  - R. Schneider and W. Weil, *Stochastic and Integral Geometry*, 2008.

Links between percolation theory and topology

Percolation and topological quantification

They are **two different concepts**.

Percolation: find the existence of clusters of the size of the system

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- Often with analytical solutions
- Often limited to boolean problems in infinite spaces

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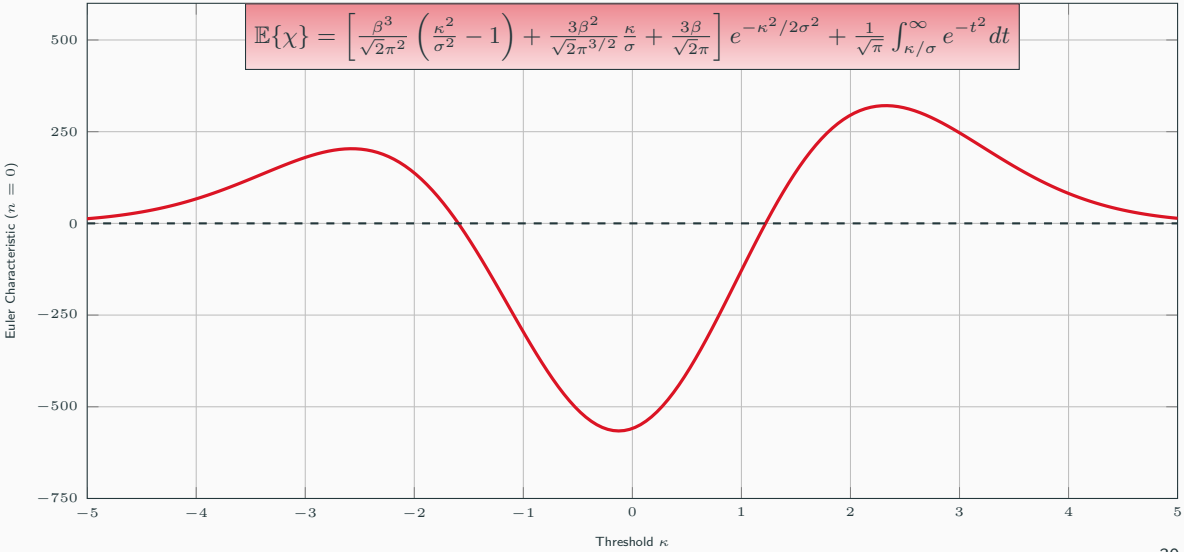

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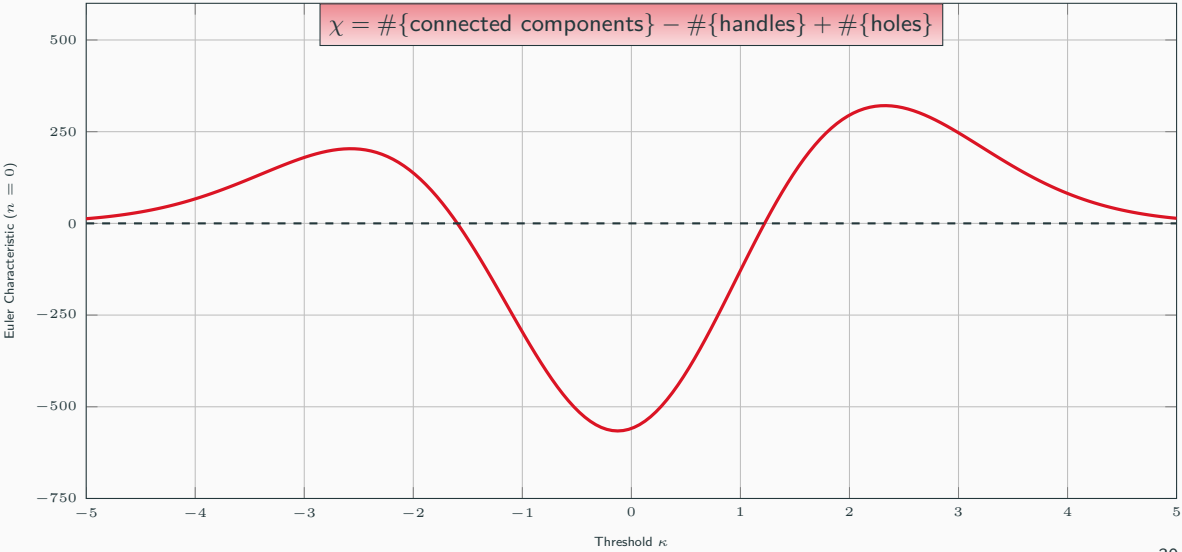

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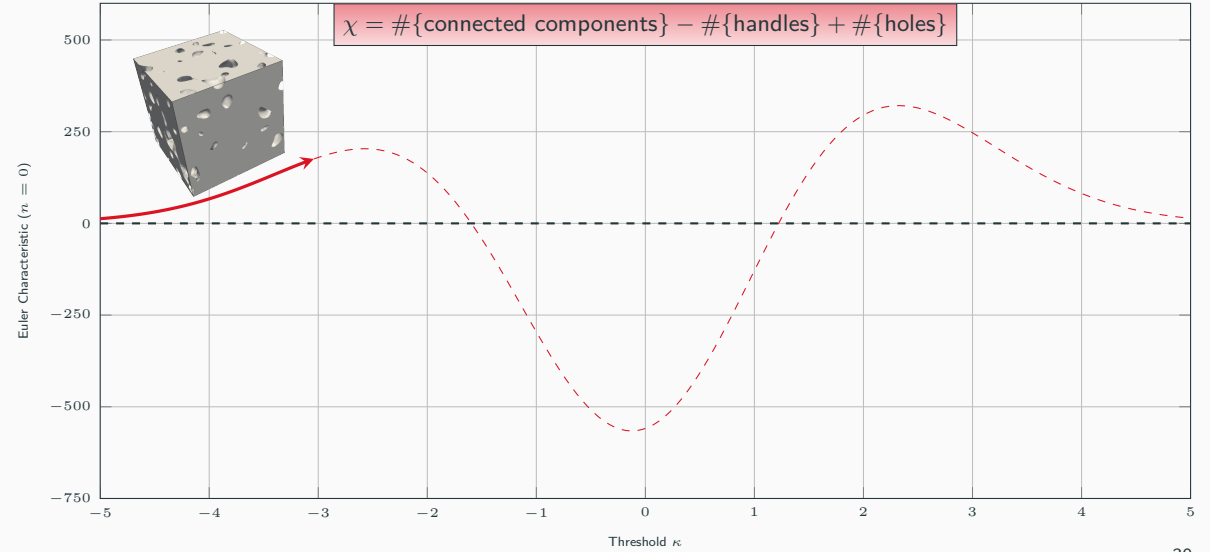

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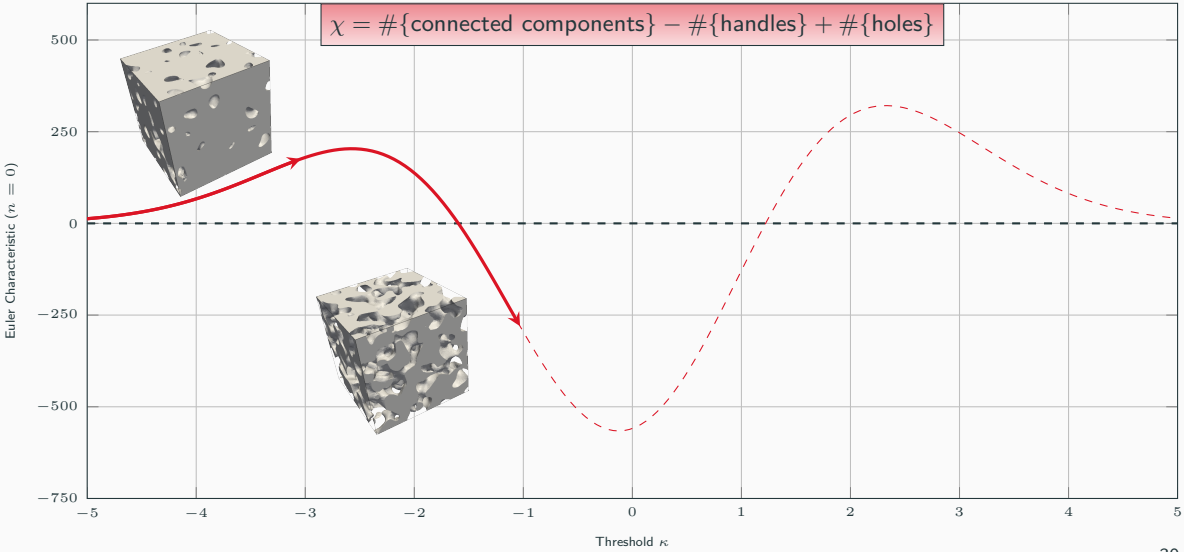

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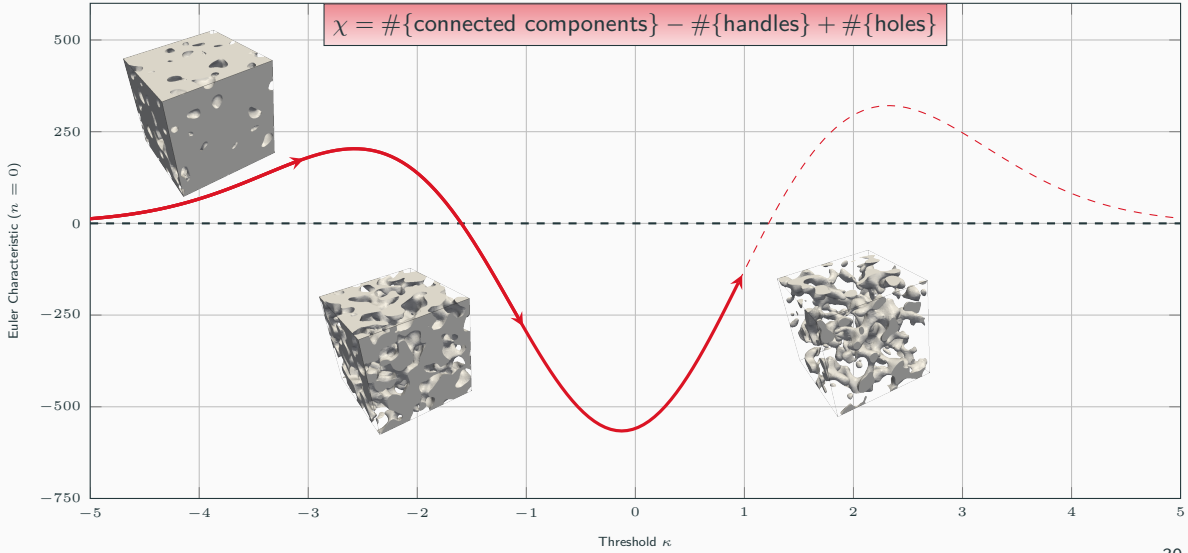

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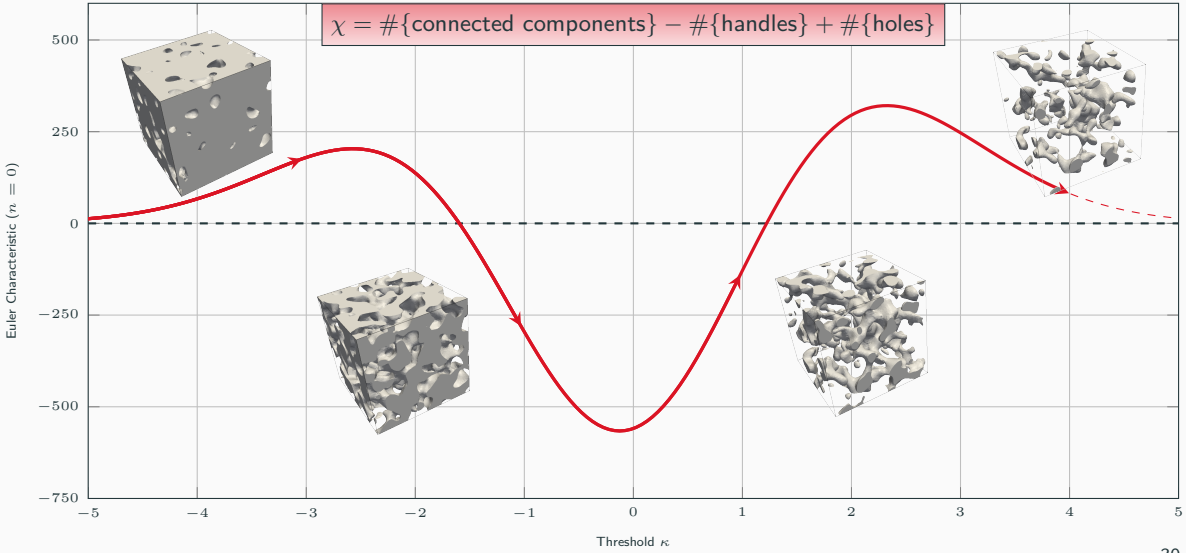

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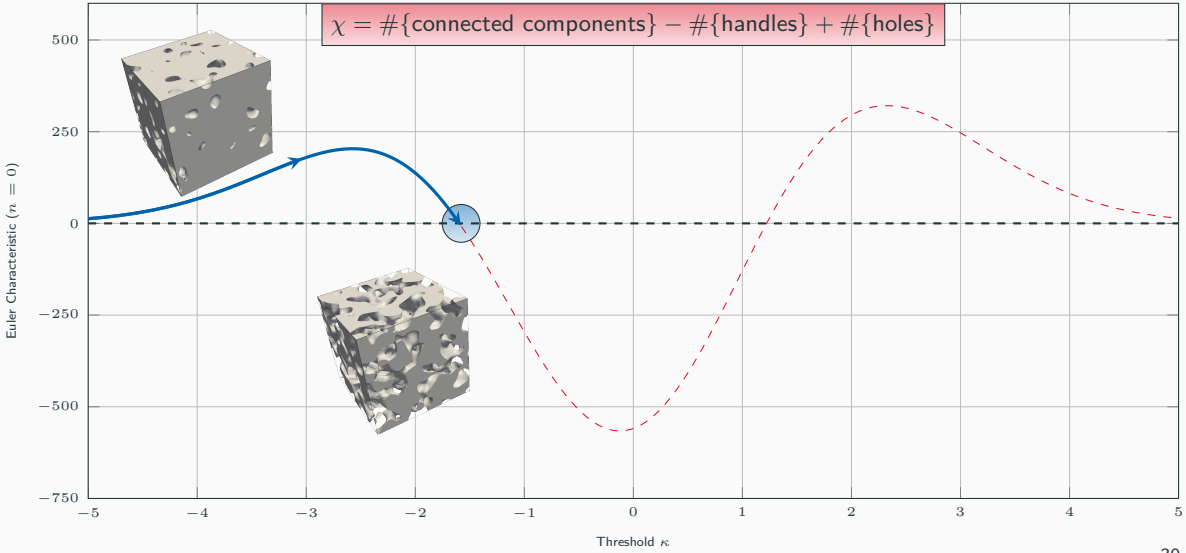

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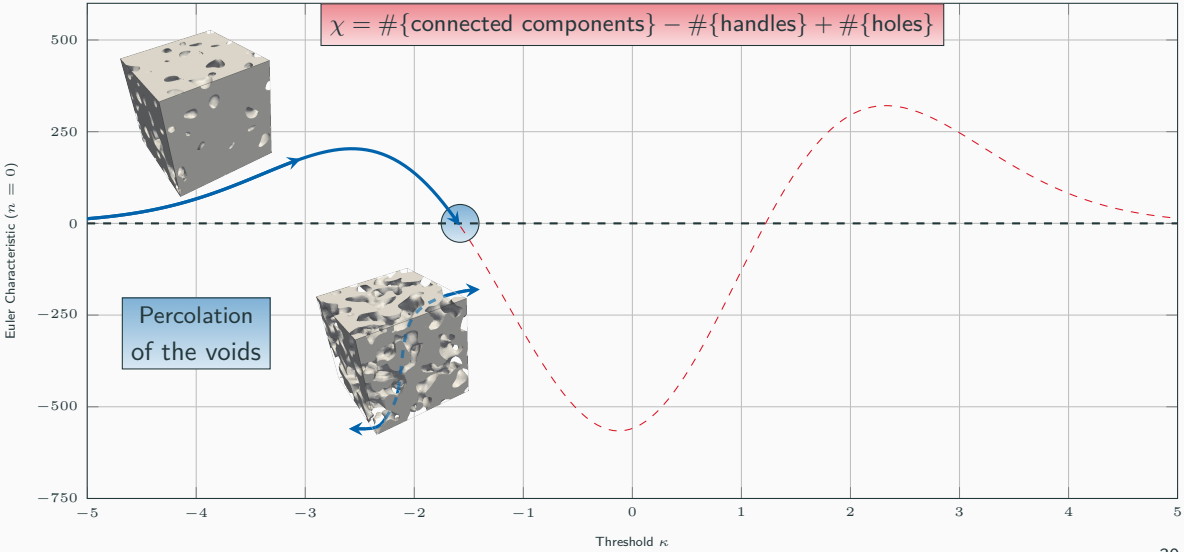

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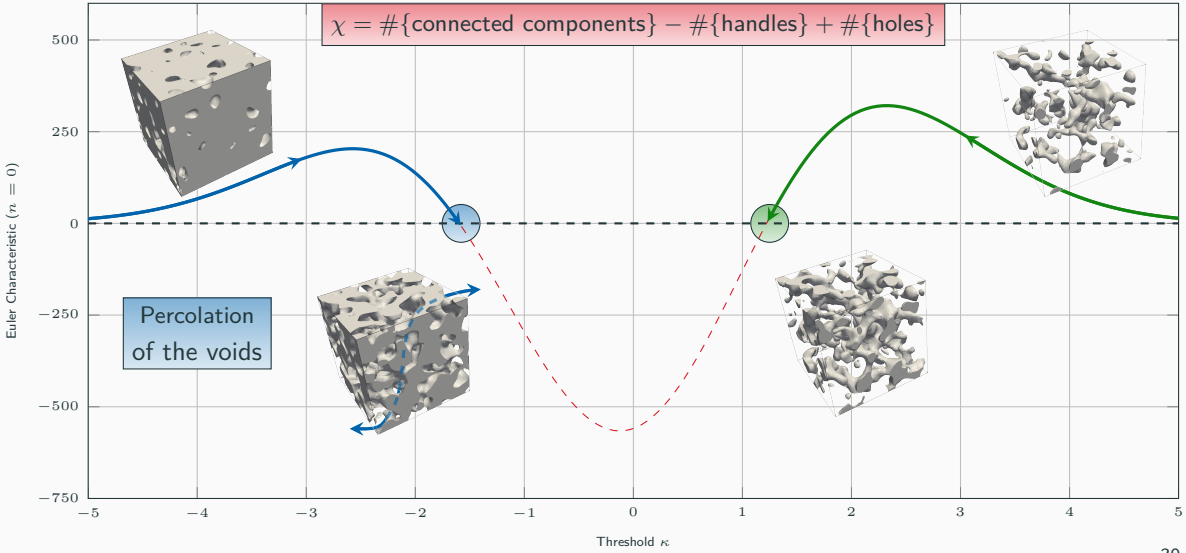

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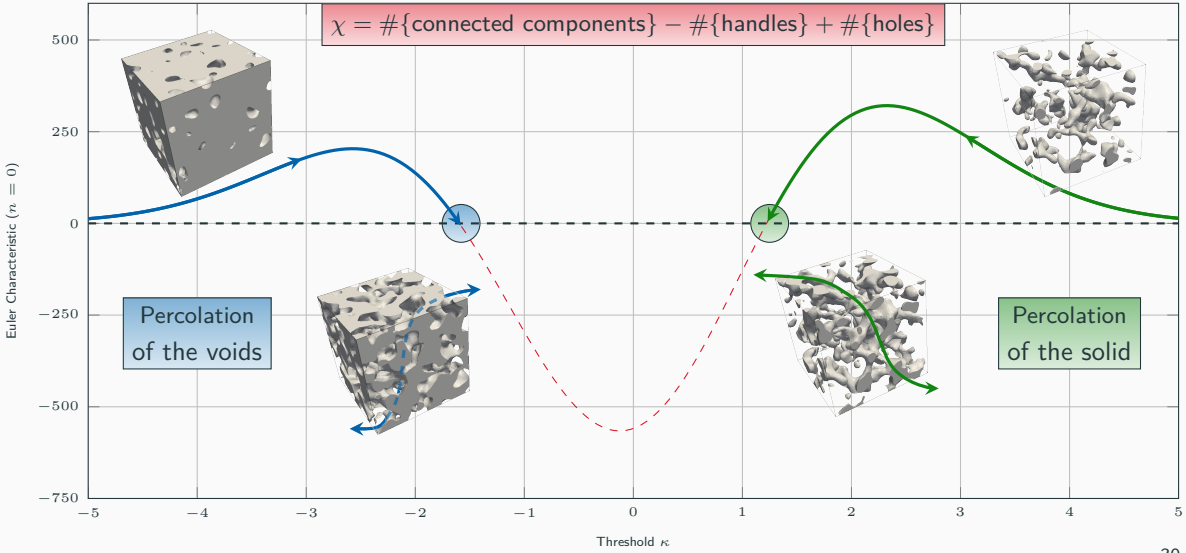

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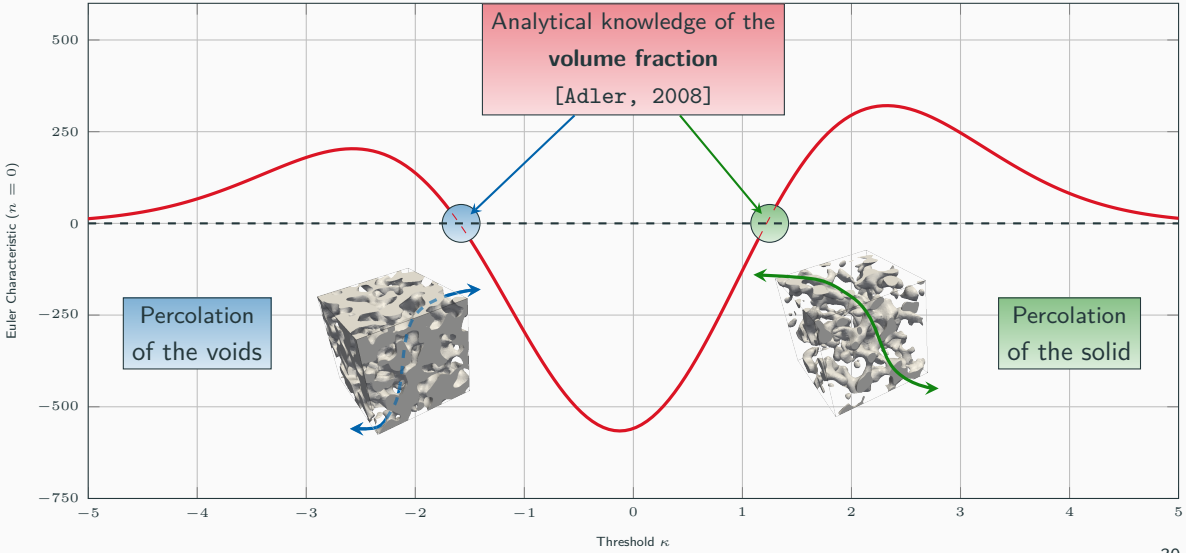
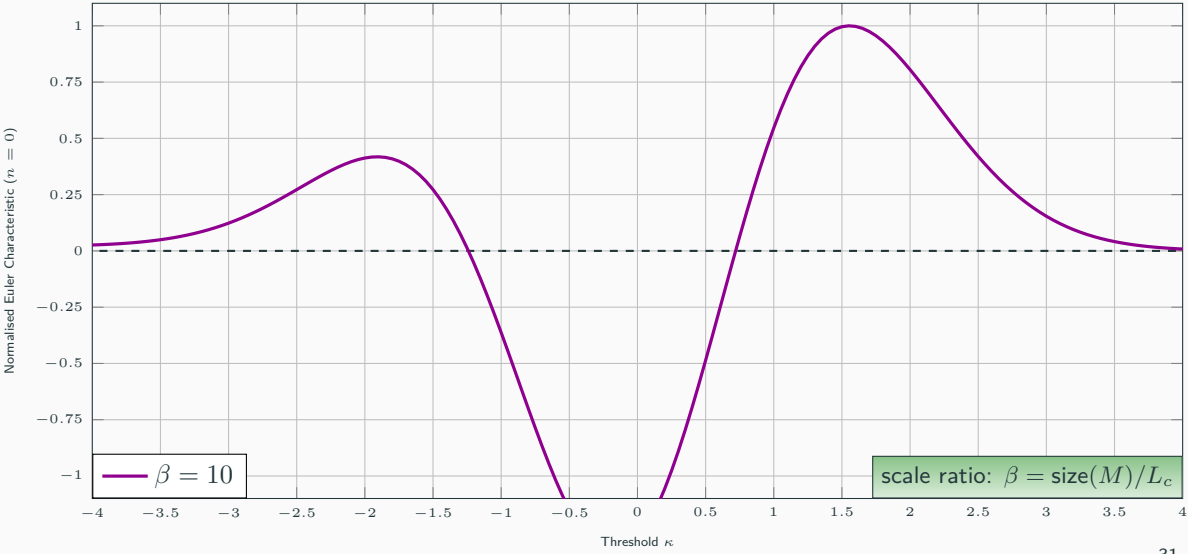


The Euler Characteristic: scale ratio



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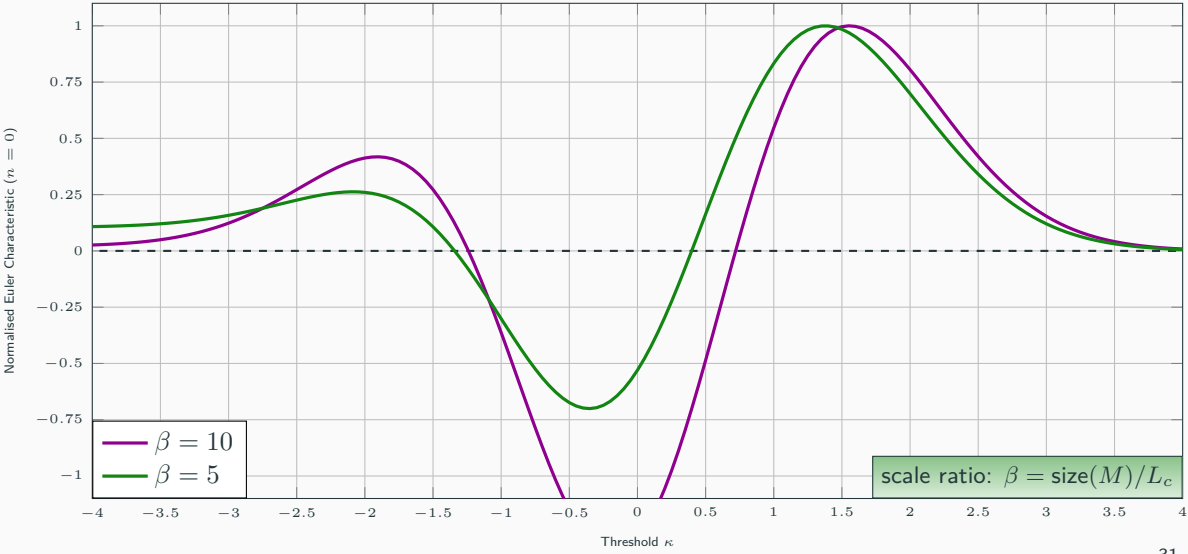

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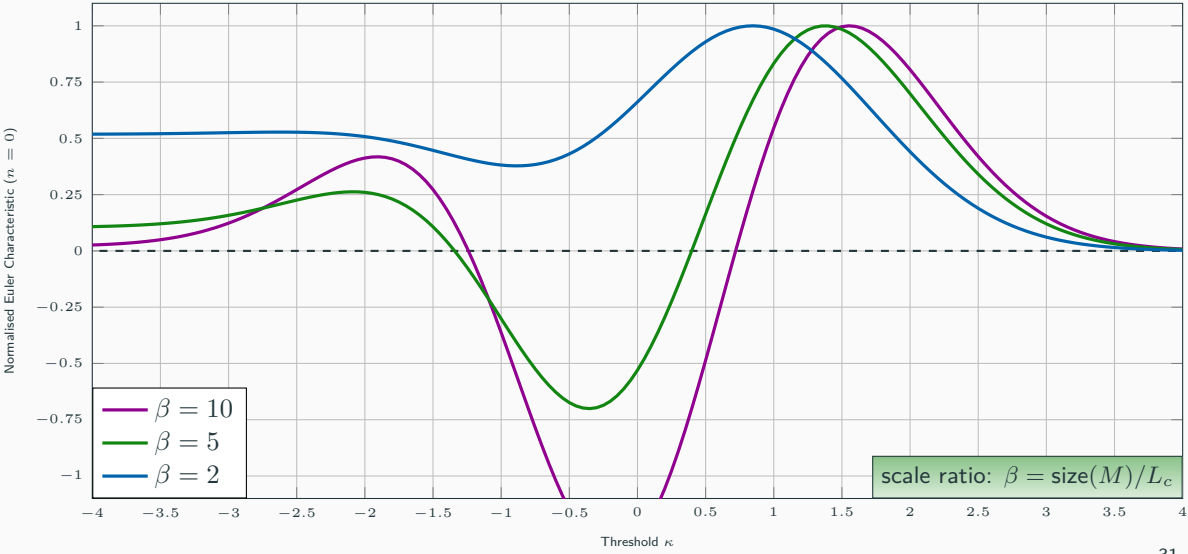

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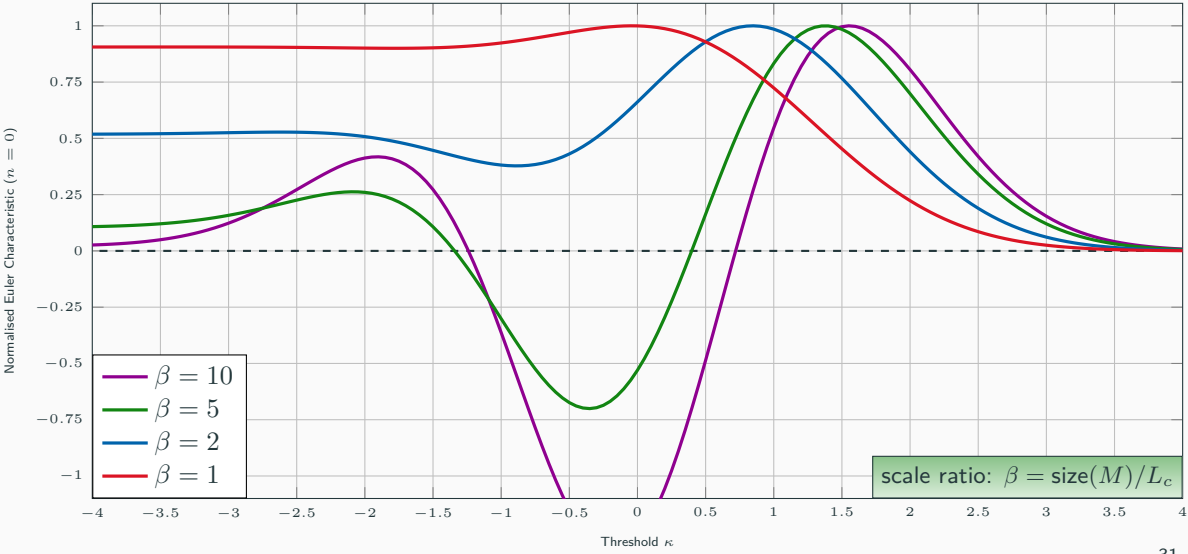
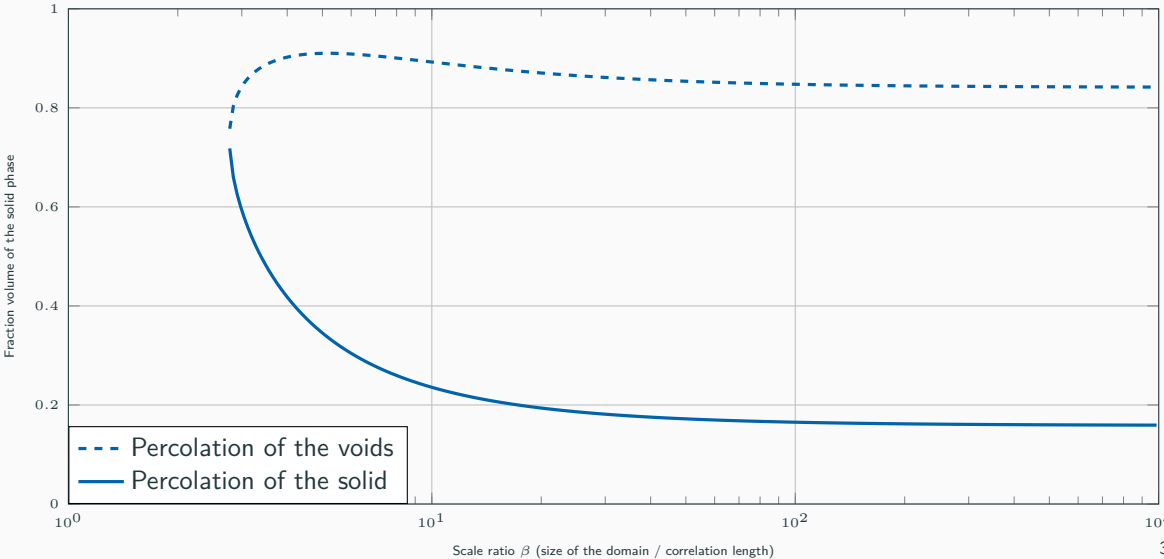


Phase diagram



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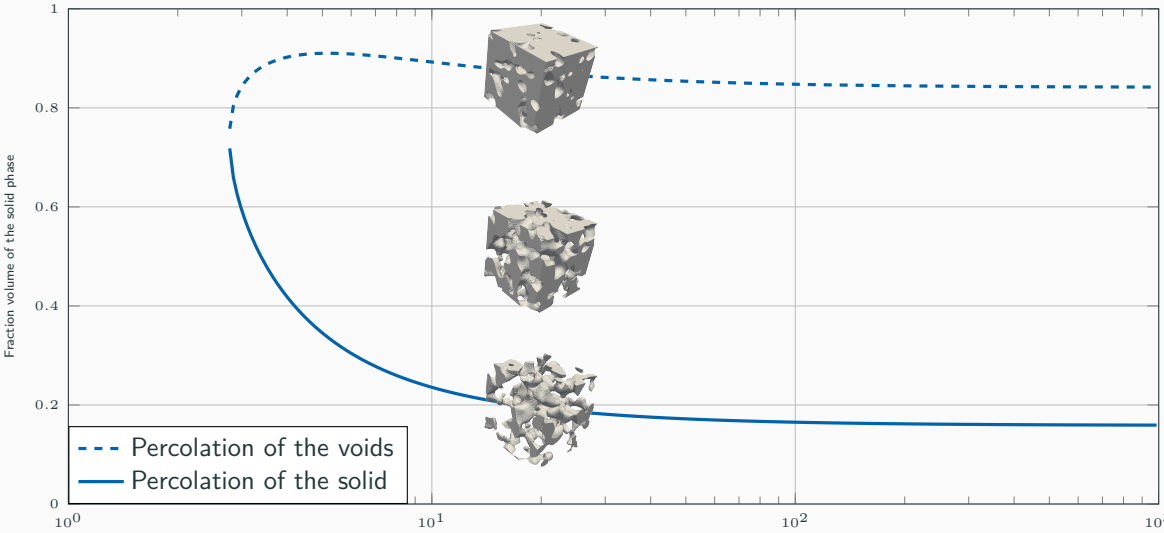

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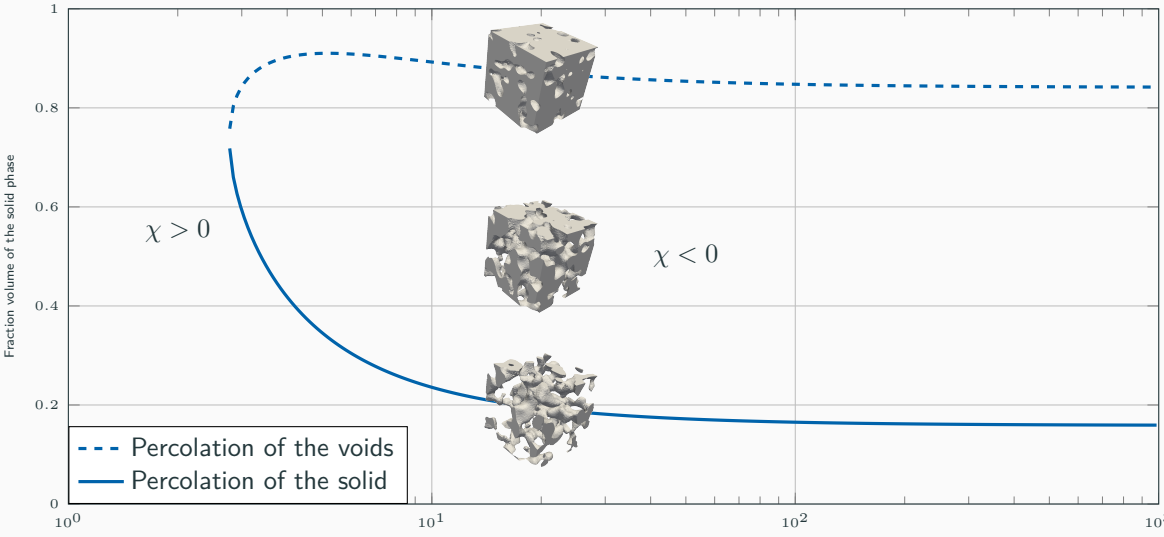

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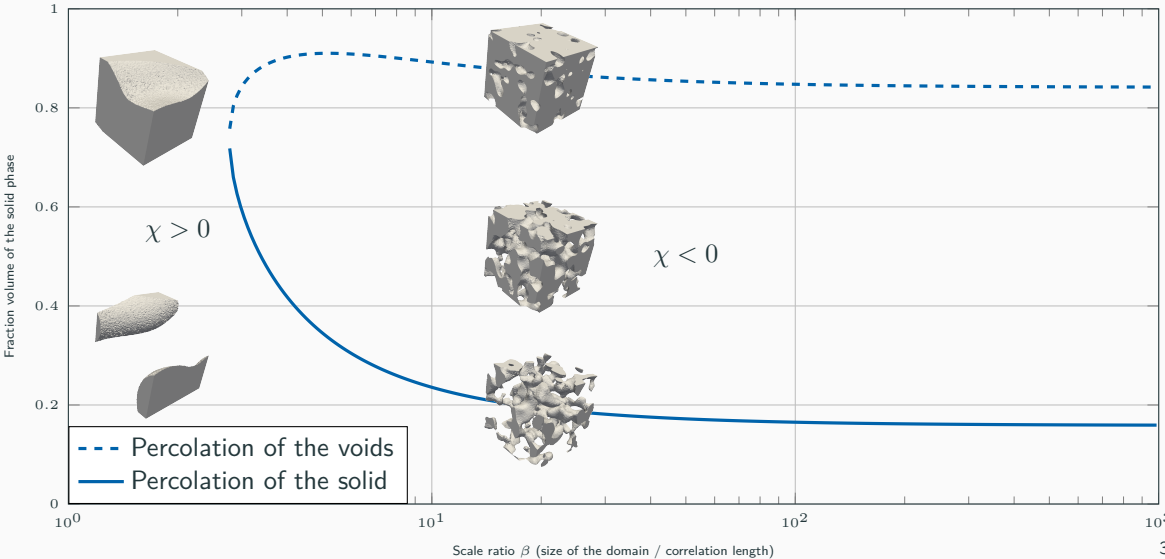

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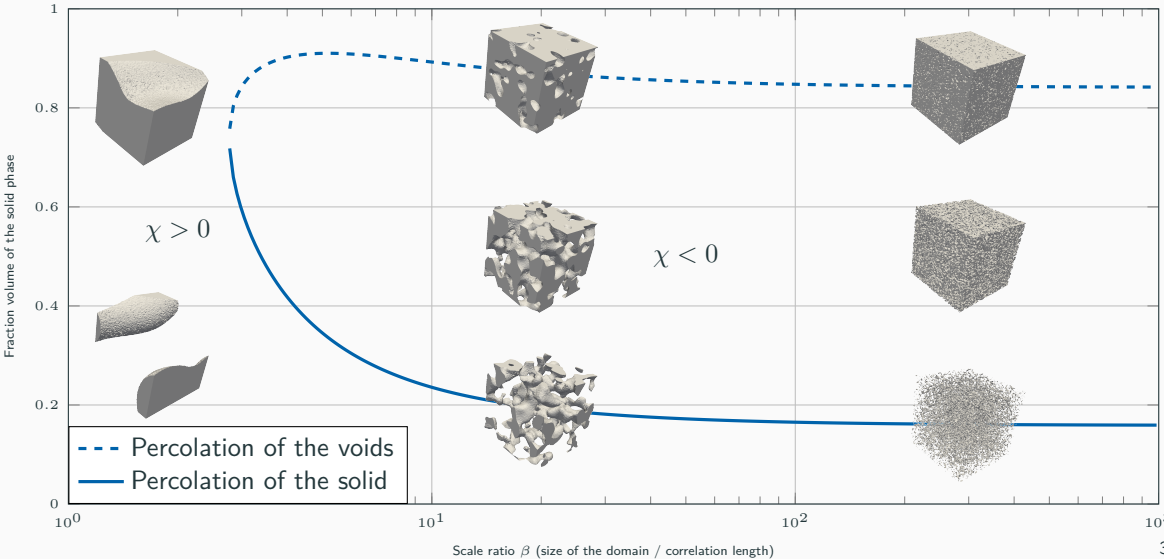

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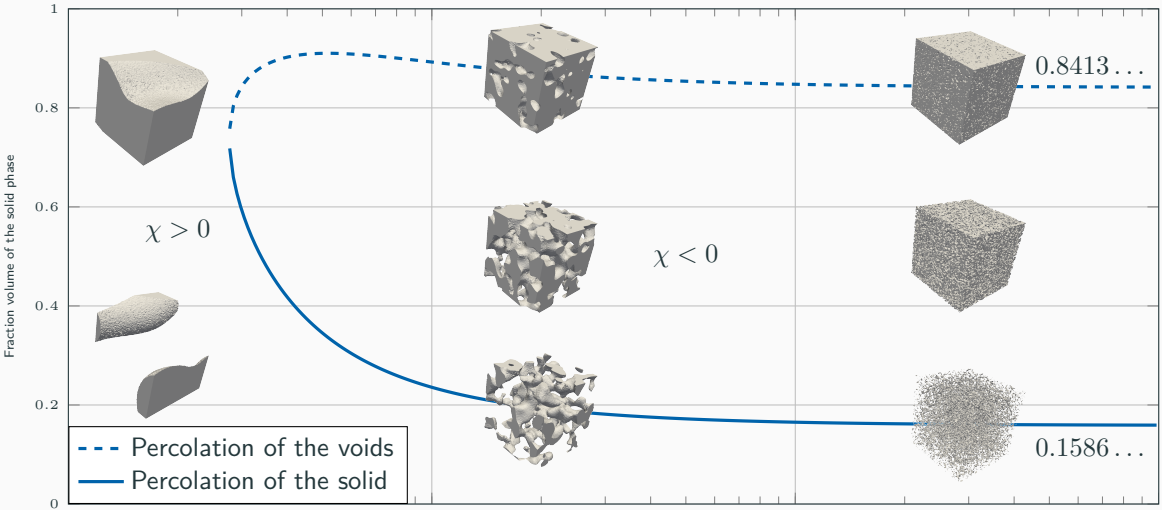

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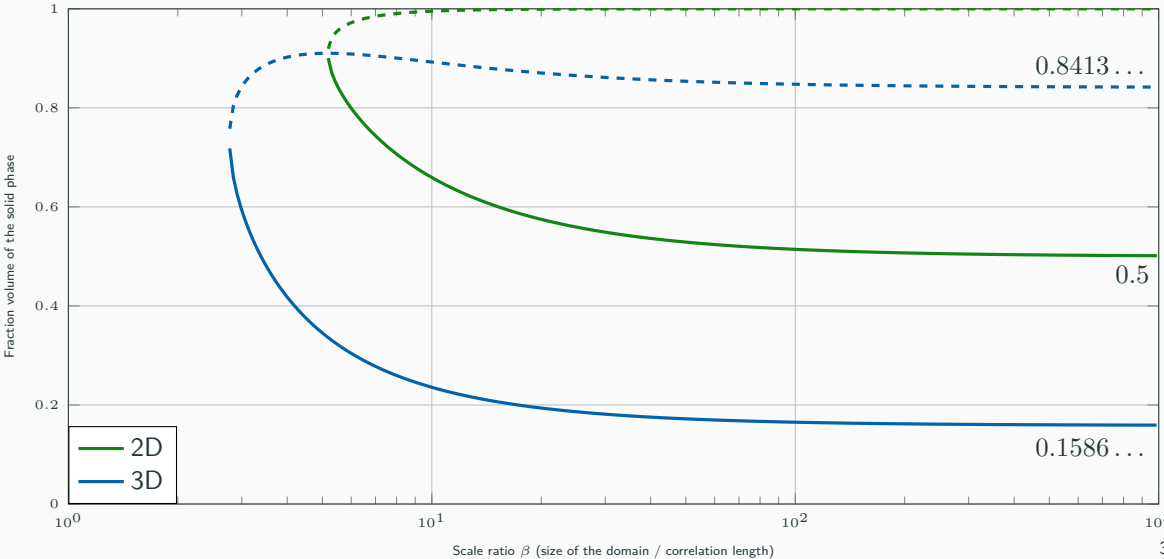

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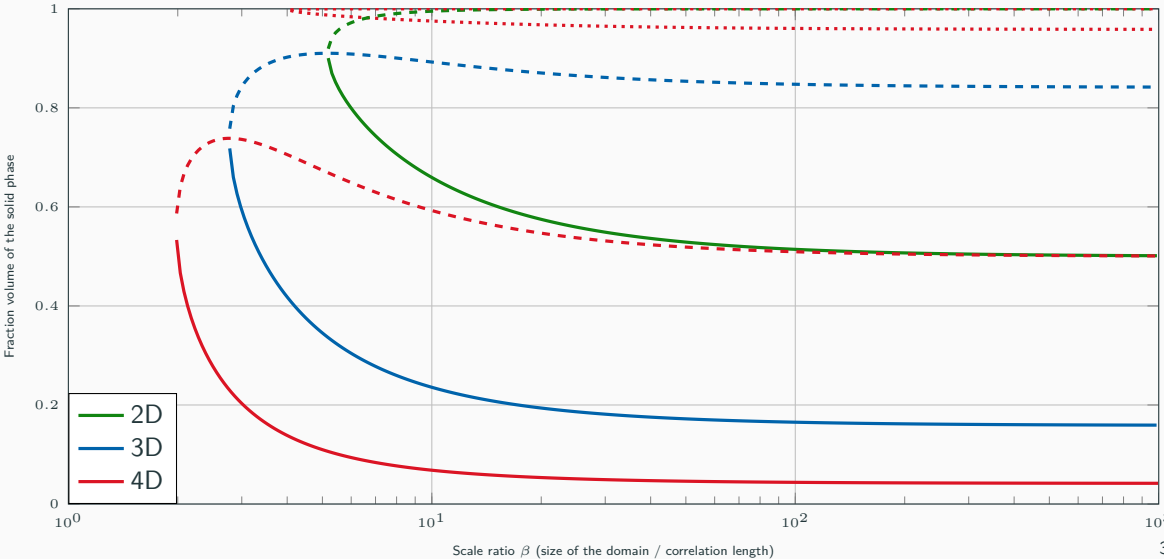
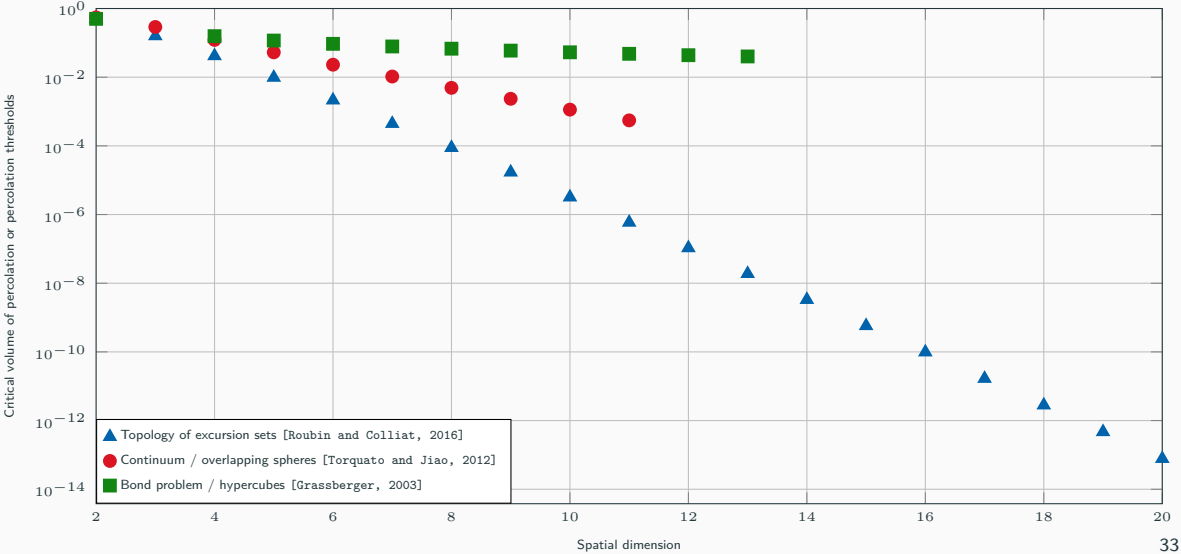


Percolation of the solid phase in N dimensions



Objectives being able to represent all kinds of morphologies

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- Seek help :)