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Adaptation to a heterogeneous patchy environment with nonlocal dispersion

Alexis Léculier* Sepideh Mirrihimi†

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Abstract

In this work, we provide an asymptotic analysis of the solutions to an elliptic integro-differential equation. This equation describes the evolutionary equilibria of a phenotypically structured population, subject to selection, mutation, and both local and non-local dispersion in a spatially heterogeneous, possibly patchy, environment. Considering small effects of mutations, we provide an asymptotic description of the equilibria of the phenotypic density. This asymptotic description involves a Hamilton-Jacobi equation with constraint coupled with an eigenvalue problem. Based on such analysis, we characterize some qualitative properties of the phenotypic density at equilibrium depending on the heterogeneity of the environment. In particular, we show that when the heterogeneity of the environment is low, the population concentrates around a single phenotypic trait leading to a unimodal phenotypic distribution. On the contrary, a strong fragmentation of the environment leads to multi-modal phenotypic distributions.

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1 Introduction

1.1 The model and motivations

In this work, we provide an asymptotic analysis of the equilibria of a non-local parabolic Lotka-Volterra type equation. Such equation arises in the study of adaptive evolution of phenotypically structured populations. The equation under study is the following

$$\begin{cases}
 -\partial_{xx}n_\varepsilon - \varepsilon^2\partial_{\theta\theta}n_\varepsilon + Ln_\varepsilon = n_\varepsilon(R(x, \theta) - \rho_\varepsilon(x)) & \text{in } \Omega \times]-A, A[, \\
 \rho_\varepsilon(x) = \int_{]-A, A[} n_\varepsilon(x, \theta)d\theta & \text{in }]-A, A[, \\
 Ln_\varepsilon(x, \theta) = \int_{\Omega} [n_\varepsilon(x, \theta) - n_\varepsilon(y, \theta)]K(x - y)dy & \text{in } \Omega \times]-A, A[, \\
 \partial_{\nu_x}n_\varepsilon(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta}n_\varepsilon(x, \pm A) = 0
 \end{cases} \tag{E}$$

with Ω a bounded subset of \mathbb{R} . Here, $n_\varepsilon(x, \theta)$ stands for the density of a population at equilibrium at position x with a phenotypical trait θ . The term $R(x, \theta)$ corresponds to the intrinsic growth rate of individuals of phenotype θ at position x . The term $\rho(x)$ corresponds to the total size of the population at position x . Via this term in the right hand side of (E), we take into account a mortality rate due to the competition between the individuals at the same position, independently of their traits. The trait of the parent is transmitted to the offspring. However the trait can be modified due to the mutations that we model by a Laplace term. We also consider that the species is subject to a local and a non-local dispersion in the space variable x . Indeed, in addition to a classical local dispersion term modeled by a Laplace term, we assume that the individuals can jump from position x to position y with a rate $K(x - y)$. Finally, we have denoted by ∂_{ν_x} , ∂_{ν_θ} the exterior derivatives with respect to the variables x and θ . The Neumann boundary condition with respect to x models the fact that the species cannot leave the domain. The Neumann boundary condition with respect to θ means that the mutants cannot be born with a trait in $] - A, A[^c$.

Several questions motivate our analysis. Can we determine extinction and survival criteria for such model? How would the population be spatially distributed at equilibrium? Would all the spatial domain be exploited close to its local carrying capacity or would we observe formation of clusters in certain zones of the environment? How the population would be distributed phenotypically? Would the population be adapted locally everywhere, or would we observe emergence of dominant traits? More specifically, would we observe emergence of generalist traits being adapted to an average environment, or specialist traits being adapted to certain zones of the environment? What will be the impact of the the fragmentation of the habitat on the phenotypical distribution of the population? Would it lead to the emergence of specialist traits?

An originality of our work lies in the non-local spatial dispersion operator L which allows to consider non-connected domains. While the role of the nonlocal dispersion and the fragmentation of the environment is significant in many situations, as in the adaptation of forest trees to the climate change because of the effect of the wind on the seeds or the pollens, very few theoretical works take it into account [21]. The non-local dispersion may have antagonistic effects on the population dynamics. On the one hand, it may allow the population to reach new favorable geographic regions which are not accessible by a local diffusion. On the other hand, it may also impede local adaptation by bringing individuals with locally maladapted traits from other regions.

1.2 The state of the art

Related models and the questions of spatial distribution and ecological niches were studied from a numerical point of view in the biological literature, considering local dispersion and a connected domain (see for instance [15, 29]). In particular, in [29] formation of clusters was observed in a closely related model. While the authors suggested that the formation of such clusters would be a result of the bounded domain or small mutational effects, they did not provide any analytical support for such hypothesis. Such type of models, again in the context of local dispersion, was later derived from stochastic individual based models by Champagnat and Méléard in [13] where further numerical studies were provided (see also [3] for a preliminary analysis of this model by Arnold, Desvillettes and Prévost). More recently, Alfaro, Coville and Raoul [2] studied a closely related model, considering again local dispersion but in unbounded domains. They proved propagation phenomena and existence of traveling front solutions for parabolic equations close to (E) (see also the related works [10, 1]). In the context of their study with unbounded domain, one expects indeed that the population would propagate and get adapted locally in every position in space attaining its local carrying capacity, which is in contrast with what we observe in bounded domains, in particular with what we obtain in the present work. However, in such model to our knowledge, yet there is no result characterizing rigorously the population distribution at the back of the front (see however the work of Berestycki, Jin and Silvestre in [6] in a particular case with spatially homogeneous growth rate).

A wide number of articles have also studied a closely related equation known as the "cane-toads" model where the growth rate is independent of the trait, but the trait influences the ability of dispersal leading to a θ coefficient in front of the diffusion term in space (see for instance [31, 7, 9, 8]). This equation is motivated by the propagation of the cane toads in Australia by taking into account the role of a phenotypical trait: the size of the legs of the toads. More closely to our work, the steady states of a "cane-toads" type model, in the regime of small mutations, were studied by Perthame and Souganidis in [28] and by Lam and Lou in [22]. In another related project, a model where similarly to (E) the growth rate, and not the dispersion rate, depends on the phenotype, but considering a discrete spatial structure, was studied by Mirrahimi and Gandon [26, 27]. In these works an asymptotic analysis of the steady states in the regime of small mutations was provided. In particular, it was shown that the presence of spatial heterogeneity can lead to polymorphic situations, that is the emergence of several dominant traits in the population.

In this work, we will use an approach based on Hamilton-Jacobi equations, which is adapted to study the small mutation regime (ε small). A closely related approach was first introduced in [19, 18], by Friedlin using probabilistic techniques and by Evans and Souganidis using deterministic tools, to study the propagation phenomena in reaction-diffusion equations. In the context of models from evolutionary biology and in the regime of small mutations, this method was suggested by Dieckmann, Jabin, Mishler and Perthame [14]. In [5], Barles and Perthame provide the first rigorous results within this approach and obtain a concentration phenomena considering homogeneous environments: as the mutational effects become small, the solution converges to a Dirac mass. In this case, the population at equilibrium is monomorphic (there is a single dominant trait in the population). We quote [4] which extends the main results of [5]. This approach was then widely extended to study more general models with heterogeneity.

In particular, in the context of the space heterogeneous environments, the works [10, 31, 28, 26] are within this framework. However, the analysis provided in these previous works do not allow to study problem (E). The closest work is the one in [10] which studies the propagation phenomenon in an unbounded domain, considering a different rescaling. Note also that none of the previous works considered a non-local dispersion operator, which adds significant difficulties to the analysis.

In an ecological context, fragmented environments and non-local spatial dispersion phenomena were studied by Léculier, Mirrahimi and Roquejoffre [23] and by Léculier and Roquejoffre [24]. Both mentioned works do not take into account any phenotypical structure. In [23], the authors study invasion phenomena in a Fisher-KPP equation involving a fractional Laplacian arising in a fragmented periodic environment with Dirichlet exterior conditions. In [24], the authors study the existence and uniqueness of bounded positive steady-states in a Fisher-KPP equation involving a fractional Laplacian in general fragmented environment with Dirichlet exterior conditions. One of the perspectives of the present work is to study models with other operators of dispersion, as the fractional Laplacian $(-\partial_{xx})^\alpha$, instead of $-\partial_{xx} + L$, and considering Dirichlet exterior conditions.

1.3 The assumptions and the notations

The domain $\Omega \subset \mathbb{R}$ is assumed to be bounded and composed of one or several connected components :

$$\Omega = \bigcup_{i=1}^n]a_i, b_i[\quad \text{with} \quad a_1 < b_1 < a_2 < \dots < a_n < b_n. \quad (\text{H1})$$

We assume that the growth rate verifies

$$R \in C^1(\overline{\Omega} \times [-A, A]) \quad \text{and} \quad \|R\|_{W^{1,\infty}(\Omega \times]-A, A])} < C_R. \quad (\text{H2})$$

Example 1. A typical example of growth rate is written

$$R(x, \theta) = r - g(bx - \theta)^2.$$

In this example, r is the maximal growth rate. The above quadratic term indicates that the optimal trait at position x is given by $\theta_o = bx$. The term b is the gradient of the environment: it indicates how fast the optimal trait varies as a function of a position in space. Moreover, g corresponds to the selection pressure. If g increases, the habitats becomes more hostile for unsuitable individuals.

We make the following assumptions on K

$$K \in C^1(\Omega), \quad K > 0, \quad K(x) = K(-x), \quad 0 < c_K < K < C_K \quad \text{and} \quad |\partial_x K| < C_K. \quad (\text{H3})$$

We introduce here two eigenvalues problems associated to the equation (E): let $\lambda(\theta, \rho)$ be the principal eigenvalue of the operator $-\partial_{xx} - L - [R(\cdot, \theta) - \rho]Id$ and μ_ε be the principal eigenvalue of the operator $-\partial_{xx} - \varepsilon^2 \partial_{\theta\theta} - L - R$ with Neumann boundary conditions:

$$\text{i.e.} \quad \begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - [R(\cdot, \theta) - \rho]\psi^\theta = \lambda(\theta, \rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi^\theta = 0 & \text{in } \partial\Omega \end{cases} \quad (1)$$

and

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2 \partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon \xi_\varepsilon & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega \times]-A, A[). \end{cases} \quad (2)$$

All along the article, we consider that the principal eigenfunctions (such as ψ or ξ_ε) are taken positive with L^2 norms equal to 1.

We make the following assumption:

$$\exists \theta_0 \in]-A, A[, \quad \text{such that} \quad \min_{\theta \in]-A, A[} \lambda(\theta, 0) = \lambda(\theta_0, 0) < 0. \quad (\text{H4})$$

Lemma 1. *Under the assumptions (H1)–(H4)*

$$\mu_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \lambda(\theta_0, 0).$$

It follows obviously that

$$\exists \varepsilon_0 > 0, \forall \varepsilon \in]0, \varepsilon_0[, \quad \mu_\varepsilon < \frac{\lambda(\theta_0, 0)}{2} < 0. \quad (3)$$

We postpone the proof of Lemma 1 to the Appendix and we make the hypothesis that (3) holds true.

1.4 The results and the strategy

First, we prove the following theorem which provides conditions for existence or non-existence of a solution of (E) for all small value ε .

Theorem 1. *Under the assumptions (H1)–(H4), for all $\varepsilon \in]0, \varepsilon_0[$, there exists a non-trivial positive bounded solution n_ε of (E). If Assumption (H4) does not hold and $\lambda(\theta_0, 0) > 0$, then there exists $\varepsilon_0 > 0$ small enough such that for all $\varepsilon < \varepsilon_0$, there does not exist a positive solution n_ε to (E).*

We expect indeed that in a dynamic version of (E), the solution would converge in long time to a nontrivial stationary solution, that is a solution to (E), when such non-trivial steady state exists and to 0 otherwise. Admitting such property, the theorem above provides us with conditions of survival and extinction of the population. The survival condition means indeed that there exists at least one trait θ such that $\lambda(\theta, 0) < 0$, so that such trait is viable in absence of competition.

The proof of Theorem 1 follows the one of Theorem 2.1 by Lam and Lou in [22] which treats the case of local diffusion. This proof relies on a topological degree argument. In section 2, we provide the additional arguments which allow to adapt the proof of [22] to the non-local operator L .

Next, we perform the Hopf-Cole transformation

$$n_\varepsilon(x, \theta) = e^{\frac{u_\varepsilon(x, \theta)}{\varepsilon}}. \quad (4)$$

This is the usual first step in the Hamilton-Jacobi approach (see [14], [5], [4]). The main idea in this approach is to first study the limit of u_ε as $\varepsilon \rightarrow 0$, and next obtain from this limit, information on the limit of the phenotypic density n_ε . The advantage of this transformation is that the limit of u_ε is usually a continuous function which solves a Hamilton-Jacobi equation, while the limit of n_ε is a measure. Performing such change of variable, we find that u_ε is solution to

$$\left\{ \begin{array}{l} -\frac{1}{\varepsilon} \partial_{xx} u_\varepsilon - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 \\ \quad + \int_{\Omega} \left[1 - e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} \right] K(x - y) dy = (R(x, \theta) - \rho_\varepsilon(x)) \quad \text{in } \Omega \times] - A, A[, \\ \rho_\varepsilon(x) = \int_{]-A, A[} n_\varepsilon(x, \theta) d\theta \quad \text{in }] - A, A[, \\ \partial_{\nu_x} u_\varepsilon(x, \theta) = 0 \text{ on } \partial\Omega \times] - A, A[, \quad \partial_{\nu_\theta} u_\varepsilon(x, \pm A) = 0. \end{array} \right. \quad (E_{HC})$$

We prove the following

Theorem 2. *Under the assumptions (H1)–(H4), as $\varepsilon \rightarrow 0$ along subsequences, there holds*

1. ρ_ε converges uniformly to ρ with

$$0 < c \leq \rho \leq C,$$

2. u_ε converges uniformly to u with u a viscosity solution of

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta, \rho), \\ \max u(\theta) = 0, \\ \partial_{\nu_\theta} u(\pm A) = 0, \end{cases} \quad (5)$$

where $\lambda(\theta, \rho)$ is the principal eigenvalue introduced in (1). Moreover, the limit u depends only on θ .

3. n_ε converges to n in the sense of measures. Moreover,

$$\text{supp } n \subset \Omega \times \{u(\theta) = 0\} \subset \Omega \times \{\lambda(\theta, \rho) = 0\}. \quad (6)$$

The theorem above allows us to characterize the phenotypic density n , at the limit as $\varepsilon \rightarrow 0$, via the Hamilton-Jacobi equation with constraint (5) coupled with the eigenvalue problem (1) and the inclusion properties (6). We expect indeed that n would be a sum of Dirac masses in θ as follows:

$$n(x, \theta) = \sum_{i=1}^d \rho_i(x) \delta(\theta - \theta_i).$$

In section 5, we will use the information obtained above to characterize the phenotypic density n in some particular situations. On the one hand, we will identify a situation where the phenotypic density at the limit will be a single Dirac mass in θ corresponding to a monomorphic population. On the other hand, we will show that a strong fragmentation of the environment will lead to polymorphic situations.

We present briefly heuristic arguments to understand how the second item of Theorem 2 can be recovered. First, we perform asymptotic developments of u_ε and ρ_ε with respect to the powers of ε

$$\text{i.e. } u_\varepsilon(x, \theta) = u_0(x, \theta) + \varepsilon u_1(x, \theta) + o(\varepsilon) \quad \text{and} \quad \rho_\varepsilon(x) = \rho_0(x) + o_\varepsilon(1).$$

Next, we implement such asymptotic developments into (E_{HC}) , it follows that

$$\begin{aligned} \frac{1}{\varepsilon} \left(-\partial_{xx} u_0 - 2|\partial_x u_0 \partial_x u_1| - \frac{|\partial_x u_0|^2}{\varepsilon} \right) + \int_{\Omega} [1 - e^{\frac{u_0(y, \theta) - u_0(x, \theta)}{\varepsilon} + u_1(y, \theta) - u_1(x, \theta) + o_\varepsilon(1)}] K(x - y) dy \\ - \partial_{xx} u_1 - |\partial_x u_1|^2 - |\partial_\theta u_0|^2 - [R - \rho_0] + o_\varepsilon(1) = 0. \end{aligned}$$

This equation suggests that

$$u_0(x, \theta) = u_0(\theta)$$

and passing to the limit $\varepsilon \rightarrow 0$, we deduce that

$$-|\partial_\theta u_0(\theta)|^2 = [R(x, \theta) - \rho_0(x)] + \partial_{xx} u_1(x, \theta) - |\partial_x u_1(x, \theta)|^2 - \int_{\Omega} [1 - e^{u_1(y, \theta) - u_1(x, \theta)}] K(x - y) dy. \quad (7)$$

Since the left hand side of (7) does not depend on x , it suggests that u_1 is such that $u_1 = \log(\psi^\theta)$ with ψ^θ the principal eigenfunction introduced in (1), that is

$$-|\partial_\theta u_0(\theta)|^2 = \left([R(x, \theta) - \rho_0(x)] \psi^\theta(x) + \partial_{xx} \psi^\theta(x) - L(\psi^\theta) \right) (\psi^\theta)^{-1}. \quad (8)$$

We deduce that u_0 is a formal solution of (5).

From a technical point of view, the convergence of $(u_\varepsilon)_{\varepsilon > 0}$ is proved using the Arzela-Ascoli Theorem and a perturbed test function argument (see [16]). To prove the uniform continuity, using the Bernstein's method, we prove that the first derivatives are bounded. These bounds rely on the establishment of Harnack type inequalities. Indeed, we prove the following regularity results on u_ε

Theorem 3. *Under assumptions (H1)–(H4), the following results hold true.*

1. [Harnack inequality] *There exists a constant $C > 0$ (independent of the choice of ε) such that for all intervals $I \subset] - A, A[$ with $|I| = \varepsilon$, there holds*

$$\sup_{(x,\theta) \in \Omega \times I} n_\varepsilon(x, \theta) \leq C \inf_{(x,\theta) \in \Omega \times I} n_\varepsilon(x, \theta). \quad (9)$$

2. [Lipschitz bounds] *There exists $C > 0$ such that for all ε small enough,*

$$|\partial_x u_\varepsilon| \leq C\varepsilon \quad \text{and} \quad |\partial_\theta u_\varepsilon| < C. \quad (10)$$

3. [Bounds on ρ_ε] *For all ε small enough, ρ_ε is uniformly bounded in $W^{2,p}(\Omega)$ for all $p \in [1, +\infty]$. Moreover, there exists $c, C > 0$ (independent of the choice of ε) such that*

$$c \leq \rho_\varepsilon \leq C. \quad (11)$$

4. [Bounds on u_ε] *The following holds true*

$$\lim_{\varepsilon \rightarrow 0} \sup_{(x,\theta) \in \Omega \times] - A, A[} u_\varepsilon \leq 0 \quad \text{and} \quad -a < \lim_{\varepsilon \rightarrow 0} \inf_{(x,\theta) \in \Omega \times] - A, A[} u_\varepsilon, \quad (12)$$

with $a > 0$.

Note that the combination of the local and the non-local diffusion terms makes the establishment of such regularity estimates non-standard (see for instance [4] and [25] where such types of estimates were obtained for related models with a local diffusion term).

Remark. *In 1. of Theorem 3, the interval I can be at the boundary of $] - A, A[$*

$$\text{i.e. } I =] - A, -A + \varepsilon[\quad \text{or} \quad I =]A - \varepsilon, A[.$$

1.5 Outline of the paper

In section 2, we provide some preliminary results and the existence of n_ε by proving Theorem 1. Next, in section 3, we prove the regularity results given by Theorem 3. Section 4 is devoted to the proof of Theorem 2. In section 5, we focus on the properties of the limit n . We illustrate the above properties by some numerical simulations in section 6.

The constants c, C are positive constants independent of the choice of ε and may change from line to line when there is no confusion possible.

2 Preliminary results

First, we establish a Hopf Lemma. It is obtained by a classical argument but for the sake of completeness and because of the presence of the less classical non-local operator L , we provide the proof. Next, we verify the existence of $\lambda(\theta, \rho)$ and then the existence of a non-trivial bounded solution n_ε .

2.1 A Hopf Lemma

In this section we prove the following Hopf Lemma

Lemma 2 (Hopf Lemma). *Let u be a smooth function defined on Ω such that*

$$-\partial_{xx}u + L(u) + c(x)u \geq 0, \quad (13)$$

with c a positive bounded smooth function. If there exists $x_0 \in \partial\Omega$ such that $\min_{x \in \Omega} u(x) = u(x_0) < 0$ then either u is constant or

$$\partial_{\nu_x} u(x) < 0. \quad (14)$$

The proof is in the spirit of the classical proof of the Hopf Lemma (see [17] p.250).

Proof. Up to a scaling, there is no loss of generality if we assume that $B(0, 1) \subset \Omega$ and $x_0 = 1$. Next, we define

$$v(x) = \left[e^{-\frac{3}{4}\lambda} - e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} \right] 1_{B(0,1)}(x),$$

for λ a positive constant. We underline that $v(x) = e^{-\frac{3}{4}\lambda} - 1$ in $B(0, \frac{1}{2})$. Next, we claim that by taking λ large enough, for all $x \in B(0, 1) \setminus B(0, \frac{3}{4})$ there holds

$$\begin{aligned} -\partial_{xx}v(x) &= 2\lambda e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} (2\lambda|x|^2 - 1) > 0 \\ \text{and } Lv(x) &\geq \int_{B(0,1)} e^{-\lambda \max(0, |y|^2 - \frac{1}{4})} K(x-y) dy - \int_{\Omega} e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} K(x-y) dy > 0. \end{aligned} \quad (15)$$

The first inequality of (15) follows from a straightforward computation. For the second inequality, according to the assumption (H3), we have

$$\liminf_{\lambda \rightarrow +\infty} \int_{B(0,1)} e^{-\lambda \max(0, |y|^2 - \frac{1}{4})} K(x-y) dy - \int_{\Omega} e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} K(x-y) dy \geq c_K |B(0, \frac{1}{2})| > 0.$$

Therefore, if λ is large enough, (15) holds true.

Next, we claim that if u is not constant, the minimum can not be reached in the interior of Ω . Otherwise, we deduce the existence of $x_1 \in \Omega$ such that $u(x_1) = \min_{x \in \Omega} u < 0$. Since c is positive, we have

$$-\partial_{xx}u(x_1) \leq 0, \quad Lu(x_1) < 0 \quad \text{and} \quad c(x_1)u(x_1) \leq 0.$$

Therefore, we deduce that

$$-\partial_{xx}u(x_1) + Lu(x_1) + c(x_1)u(x_1) < 0.$$

This is in contradiction with the assumption (13).

We deduce that $\min_{x \in \partial B(0, \frac{3}{4})} u(x_0) - u(x) < 0$. Next, taking ε small enough, there holds that

$$\forall x \in \partial B(0, \frac{3}{4}), \quad u(x_0) - u(x) - \varepsilon v(x) < 0.$$

Since $v = 0$ on $\partial B(0, 1)$ and by definition of x_0 , it follows

$$\forall x \in \partial B(0, 1), \quad u(x_0) - u(x) - \varepsilon v(x) \leq 0.$$

Moreover, according to (13) and (15) we have for all $x \in B(0, 1) \setminus B(0, \frac{3}{4})$ that

$$-\partial_{xx}(u(x_0) - u(x) - \varepsilon v(x)) + L(u(x_0) - u(x) - \varepsilon v(x)) + c(x)(u(x_0) - u(x) - \varepsilon v(x)) \leq 0.$$

We deduce thanks to the maximum principle that $u(x_0) - u(x) - \varepsilon v(x) \leq 0$, for all $x \in B(0, 1) \setminus B(0, \frac{3}{4})$.

We conclude that

$$\partial_{\nu_x} u(x_0) \leq -\partial_{\nu_x} \varepsilon v(x_0) = -\varepsilon 2\lambda e^{-\frac{3\lambda}{4}} < 0.$$

□

2.2 Existence of a principal eigenpair

Proposition 1. *For a fixed bounded smooth function ρ and a fixed value $\theta \in]-A, A[$ there exists a principal eigenvalue $\lambda(\theta, \rho)$ of the operator $-\partial_{xx}\psi + L(\psi) - (R(\cdot, \theta) - \rho)\psi$ with Neumann boundary conditions*

$$i.e. \quad \begin{cases} -\partial_{xx}\psi + L(\psi) - (R(\cdot, \theta) - \rho)\psi = \lambda(\theta, \rho)\psi & \text{in } \Omega, \\ \partial_{\nu_x}\psi = 0 & \text{on } \partial\Omega. \end{cases} \quad (16)$$

The associated eigenfunction ψ has a constant sign and is unique up to multiplication by a constant. Moreover, the function $\lambda(\theta, \rho)$ and ψ are C^1 with respect to θ and $\rho \in H^1(\Omega)$.

In the following, we will consider that ψ is positive and of L^2 norm equal to 1.

Proof. First, we prove the existence of the principal eigenpair by verifying that we can apply the Krein Rutman Theorem (see [30] p 122). Since it is classical, we do not provide all the details. The cone of functions where we apply the Krein-Rutmann Theorem is

$$K = \overline{\{u \in C^{1+\alpha}(\Omega) \mid u > 0 \text{ and } \partial_{\nu_x}u = 0\}}.$$

We define $\mathcal{L}(v)$ as the unique solution of

$$\begin{cases} -\partial_{xx}\mathcal{L}(v) + \int_{\Omega} [\mathcal{L}(v)(x) - \mathcal{L}(v)(y)]K(x-y)dy - (R(\cdot, \theta) - \rho - C)\mathcal{L}(v) = v & \text{in } \Omega, \\ \partial_{\nu_x}\mathcal{L}(v) = 0 & \text{on } \partial\Omega \end{cases}$$

where $C > \sup_{x \in \Omega} (R(x, \theta) - \rho(x))$ and $v \in K$. The operator \mathcal{L} is linear, compact thanks to the elliptic estimates. We have to prove that

$$\forall v \in K \setminus \{0\}, \quad \mathcal{L}(v) \in \text{int}(K).$$

Let v be in K with v not trivial. By elliptic regularity, it follows $\mathcal{L}(v) \in C^{1+\alpha}$ and $\partial_{\nu_x}\mathcal{L}(v) = 0$. It remains to prove that $\mathcal{L}(v) > 0$.

First we prove that if $\mathcal{L}(v)$ is constant then it is necessarily a positive constant. Next we prove that if $\mathcal{L}(v)$ varies then $\mathcal{L}(v) > 0$.

Assume that $\mathcal{L}(v) = c$. Let $\bar{x} \in \Omega$ be such that $v(\bar{x}) > 0$. Moreover, the choice of C gives $-(R(\bar{x}, \theta) - \rho(\bar{x}) - C) > 0$ and since $-\partial_{xx}\mathcal{L}(v) = L(\mathcal{L}(v)) = 0$, we deduce that

$$\mathcal{L}(v)(\bar{x}) = c = \frac{v(\bar{x})}{-(R(\bar{x}, \theta) - \rho(\bar{x}) - C)} > 0.$$

Next, we suppose that $\mathcal{L}(v)$ is not constant. Assume by contradiction that there exists x such that $\mathcal{L}(v)(x) \leq 0$. Let $\bar{x}' \in \bar{\Omega}$ be such that

$$\inf_{x \in \Omega} \mathcal{L}(v)(x) = \mathcal{L}(v)(\bar{x}').$$

Then either $\bar{x}' \in \Omega$ or $\bar{x}' \in \partial\Omega$. In the first case, we deduce that

$$-\partial_{xx}\mathcal{L}(v)(\bar{x}') \leq 0 \quad \text{and} \quad L(\mathcal{L}(v))(\bar{x}') < 0,$$

which leads to the following contradiction

$$0 \leq v(\bar{x}') = -\partial_{xx}\mathcal{L}(v)(\bar{x}') + L(\mathcal{L}(v))(\bar{x}') - (R(\bar{x}', \theta) - \rho(\bar{x}') - C)\mathcal{L}(v)(\bar{x}') < 0.$$

If $\bar{x}' \in \partial\Omega$, since $\mathcal{L}(v)$ is not constant, we deduce from Lemma 2 that $\partial_{\nu_x}\mathcal{L}(v)(\bar{x}') < 0$. It is in contradiction with the Neumann boundary condition.

We conclude that we can apply the Krein Rutman theorem and the conclusion follows.

Next, we focus on the regularity of λ and ψ with respect to θ and ρ . The result follows directly from the implicit function theorem applied to

$$G : (\phi, \lambda, \theta, \rho) \in H^1(\Omega) \times \mathbb{R} \times \mathbb{R} \times H^1(\Omega) \mapsto \left(-\partial_{xx}\phi + L\phi - [R(\cdot, \theta) - \rho + \lambda]\phi, \int_{\Omega} \phi(x)^2 dx - 1 \right).$$

The interested reader may refer to Theorem 2 of Chapter 11 of [17] for technical details in a finite dimensional setting. □

The existence of the solution of (2) is also due to the Krein-Rutman Theorem, therefore we do not provide the proof of existence.

2.3 Existence of a non-trivial solution of (E)

As mentioned in the introduction, we recall that the proof of existence of a non-trivial solution is an adaptation of the proof of Theorem 2.1 by Lam and Lou in [22]. The major difference is the presence of the integral operator L . Therefore, we only provide the main elements dealing with the integral operator L . We also skip the proof of non-existence of a non-trivial solution, when (H4) does not hold, which follows from classical arguments.

Proof of Theorem 1. We fix $\varepsilon \in]0, \varepsilon_0[$ (where ε_0 is given by (3)). Let $\tau \in [0, 1]$ and n_{τ} be a solution of

$$\begin{cases} -\partial_{xx}n_{\tau} - \varepsilon^2\partial_{\theta\theta}n_{\tau} + Ln_{\tau} = n_{\tau}(R - \tau\rho_{\tau} - (1 - \tau)n_{\tau}) & \text{in } \Omega \times]-A, A[, \\ \rho_{\tau}(x) = \int_{]-A, A[} n_{\tau}(x, \theta) d\theta & \text{in }]-A, A[, \\ \partial_{\nu_x}n_{\tau}(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_{\theta}}n_{\tau}(x, \pm A) = 0 \end{cases} \quad (E_{\tau})$$

It is well known that for $\tau = 0$, according to (3), there exists a non-trivial steady solution n_0 . As in [22], we prove that there exists a constant $C_{\varepsilon} > 1$ (which may depend on ε) such that we have for any $\tau \in [0, 1]$

$$C_{\varepsilon}^{-1} \leq \int_{-A}^A \int_{\Omega} n_{\tau} dx d\theta \leq C_{\varepsilon}.$$

Then one can conclude using a topological degree arguments.

The lower bound. Let v_{τ} be such that $n_{\tau} = \xi_{\varepsilon}v_{\tau}$ (where ξ_{ε} is provided by (2)). First, we remark that

$$\begin{aligned} L(v_{\tau}\xi_{\varepsilon}) &= v_{\tau}L(\xi_{\varepsilon}) + \xi_{\varepsilon}L(v_{\tau}) + \Lambda(v_{\tau}, \xi_{\varepsilon}) \\ \text{and } \Lambda(v_{\tau}, \xi_{\varepsilon})(x) &= \int_{\Omega} [(v_{\tau}(x) - v_{\tau}(y))(\xi_{\varepsilon}(y) - \xi_{\varepsilon}(x))]K(x - y)dy. \end{aligned}$$

Then v_{τ} is solution of

$$\begin{aligned} -\xi_{\varepsilon}\partial_{xx}v_{\tau} - 2\partial_x\xi_{\varepsilon}\partial_xv_{\tau} - \varepsilon^2\xi_{\varepsilon}\partial_{\theta\theta}v_{\tau} - 2\varepsilon^2\partial_{\theta}\xi_{\varepsilon}\partial_{\theta}v_{\tau} + \xi_{\varepsilon}L(v_{\tau}) + \Lambda(v_{\tau}, \xi_{\varepsilon}) + \mu_{\varepsilon}\xi_{\varepsilon}v_{\tau} \\ = -v_{\tau}\xi_{\varepsilon}[\tau\rho_{\tau} + (1 - \tau)n_{\tau}]. \end{aligned}$$

If we multiply it by $\frac{\xi_{\varepsilon}}{v_{\tau}}$, we obtain

$$\frac{-\partial_x(\xi_{\varepsilon}^2\partial_xv_{\tau}) - \varepsilon^2\partial_{\theta}(\xi_{\varepsilon}^2\partial_{\theta}v_{\tau}) + \xi_{\varepsilon}^2L(v_{\tau}) + \xi_{\varepsilon}\Lambda(v_{\tau}, \xi_{\varepsilon})}{v_{\tau}} = \xi_{\varepsilon}^2(-\mu_{\varepsilon} - \tau\rho_{\tau} - (1 - \tau)n_{\tau}).$$

Next, we integrate over all the domain

$$\begin{aligned}
& \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau) + \xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} dx d\theta \\
&= \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau} dx d\theta + \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} dx d\theta \\
&= I_1 + I_2.
\end{aligned}$$

We next prove that I_1 and I_2 are negative. For I_1 , by an integration by part, we have

$$I_1 = \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau} dx d\theta = - \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2}{v_\tau^2} (|\partial_x v_\tau|^2 + \varepsilon^2 |\partial_\theta v_\tau|^2) dx d\theta \leq 0.$$

For I_2 , using that K is even and the Fubini Theorem, we obtain

$$\begin{aligned}
I_2 &= \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2(x) L(v_\tau)(x) + \xi_\varepsilon(x) \Lambda(v_\tau, \xi_\varepsilon)(x)}{v_\tau(x)} dx d\theta \\
&= \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon(x)}{v_\tau(x)} \int_{\Omega} [(v_\tau(x) - v_\tau(y)) \xi_\varepsilon(y)] K(x - y) dy dx d\theta \\
&= - \int_{-A}^A \int_{\Omega} \int_{\Omega} \left[\frac{\xi_\varepsilon(x) \xi_\varepsilon(y)}{v_\tau(x) v_\tau(y)} (v_\tau(y) - v_\tau(x))^2 \right] K(x - y) dy dx d\theta - I_2
\end{aligned}$$

We deduce that

$$I_2 = - \frac{1}{2} \int_{-A}^A \int_{\Omega} \int_{\Omega} \left[\frac{\xi_\varepsilon(x) \xi_\varepsilon(y)}{v_\tau(x) v_\tau(y)} (v_\tau(y) - v_\tau(x))^2 \right] K(x - y) dy dx d\theta \leq 0.$$

Therefore, we have that

$$\int_{-A}^A \int_{\Omega} \xi_\varepsilon^2 [-\mu_\varepsilon - \tau \rho_\tau - (1 - \tau) n_\tau] dx d\theta \leq 0.$$

Thanks to (3), we conclude that for ε small enough

$$\frac{|\lambda(\theta_0, 0)|}{2} \leq -\mu_\varepsilon = -\mu_\varepsilon \int_{-A}^A \int_{\Omega} \xi_\varepsilon^2 dx d\theta \leq \sup(\xi_\varepsilon^2) [\tau + (1 - \tau)] \int_{-A}^A \int_{\Omega} n_\tau dx d\theta.$$

The upper bound. First, we remark that thanks to the Neumann boundary conditions and the parity of K , we have that

$$\int_{-A}^A \int_{\Omega} -\partial_{xx} n_\tau - \partial_{\theta\theta} n_\tau + L(n_\tau) dx d\theta = 0.$$

Therefore, if we integrate (E_τ) with respect to x and θ , we obtain

$$\begin{aligned}
\left(\frac{(1 - \tau)}{2A|\Omega|} + \frac{\tau}{|\Omega|} \right) \|n_\tau\|_{L^1}^2 &= \frac{\tau}{|\Omega|} \left(\int_{\Omega} \rho_\tau dx \right)^2 + \frac{(1 - \tau)}{2A|\Omega|} \left(\int_{-A}^A \int_{\Omega} n_\tau dx d\theta \right)^2 \\
&\leq \tau \int_{\Omega} \rho_\tau^2 dx + (1 - \tau) \int_{-A}^A \int_{\Omega} n_\tau^2 dx d\theta \\
&= \int_{-A}^A \int_{\Omega} R n_\tau dx d\theta \leq C_R \int_{-A}^A \int_{\Omega} n_\tau dx d\theta = C_R \|n_\tau\|_{L^1}.
\end{aligned}$$

Conclusion. It follows that there exists a bounded non trivial solution n_ε of (E). Moreover, we have indeed proved that there exists constants $c, C > 0$ such that

$$\frac{c}{\sup \xi_\varepsilon^2} \leq \int_{-A}^A \int_{\Omega} n_\varepsilon dx d\theta \leq C.$$

□

3 Regularity results

In this section we prove Theorem 3. The sub-sections correspond respectively to the proof of the item 1. 2. 3. and 4. of Theorem 3. But, we need an intermediate result : ρ_ε is uniformly bounded.

Lemma 3. *Under the assumptions (H1) – (H4), we have that for all $\varepsilon < \varepsilon_0$*

$$0 \leq \rho_\varepsilon \leq C_R,$$

(where C_R is introduced in (H2)). Moreover, there exists $C > 0$ such that for all ε small enough

$$\|\rho_\varepsilon\|_{W^{2,p}(\Omega)} \leq C.$$

Proof. **The L^∞ -bounds.** It is obvious that $\rho_\varepsilon > 0$. If we integrate (E) with respect to θ , we obtain

$$\begin{cases} -\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon = \int_{-A}^A R(\cdot, \theta)n_\varepsilon(\cdot, \theta)d\theta - \rho_\varepsilon^2 & \text{in } \Omega, \\ \partial_{\nu_x}\rho_\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \quad (E_\rho)$$

Recalling the L^∞ bounds on R (H2), it follows:

$$-\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon \leq C_R\rho_\varepsilon - \rho_\varepsilon^2.$$

We conclude thanks to the maximum principle that $\rho_\varepsilon \leq C_R$.

The $W^{2,p}(\Omega)$ bounds. Thanks to the L^∞ bounds on R, K, ρ (assumptions (H2), (H3) and the previous inequality), we may write (E_ρ) on the following form

$$-\partial_{xx}\rho_\varepsilon = f_\varepsilon$$

with $f_\varepsilon \in L^\infty(\Omega)$ uniformly bounded. The result follows from the standard elliptic estimates. \square

Corollary 1. *There exists a constant $C > 0$ such that for all ε small enough*

$$|\partial_x\rho_\varepsilon| \leq C. \quad (17)$$

3.1 A Harnack inequality

The first step to prove the first item of Theorem 3 is to prove the result in the interior of $\Omega \times]-A, A[$.

Theorem 4. *For all $(x_0, \theta_0) \in \Omega \times]-A, A[$, and $R_0 > 0$ such that*

$$B_{3R_0}(x_0) \times B_{3\varepsilon R_0}(\theta_0) \subset \Omega \times]-A, A[$$

there exists $C(R_0) > 0$ such that

$$\sup_{(x,\theta) \in B_{R_0}(x_0) \times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x, \theta) \leq C(R_0) \inf_{(x,\theta) \in B_{R_0}(x_0) \times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x, \theta). \quad (18)$$

Next, we prove that we can extend the solution thanks to a reflective argument (see Remark 9 p.275 in [11]).

We perform the following change of variable: $\tilde{n}(x, \theta) = n_\varepsilon(x, \varepsilon\theta)$. Therefore, we consider the following scaled equation

$$\begin{cases} -\partial_{xx}\tilde{n} - \partial_{\theta\theta}\tilde{n} + L\tilde{n} = \tilde{n}[\tilde{R} - \rho] & \text{in } \Omega \times]-\varepsilon^{-1}A, \varepsilon A[, \\ \partial_{\nu_x}\tilde{n} = \partial_{\nu_\theta}\tilde{n} = 0 & \text{in } \partial(\Omega \times]-\varepsilon^{-1}A, \varepsilon A[. \end{cases} \quad (E')$$

We have denoted by \tilde{R} the function $\tilde{R}(x, \theta) = R(x, \varepsilon\theta)$. Remark that \tilde{R} still verifies (H2).

Proof of Theorem 4. Let $(x_0, \theta_0) \in \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$ and a radius $R_0 > 0$ be such that $B_{3R_0}(x_0, \theta_0) \subset \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$. If we denote by $f(x, \theta) = \int_{\Omega} \tilde{n}(y, \theta) K(y - x) dy$, according to (H3) it follows that $f \in L^{\infty}(B_{2R_0}(x_0, \theta_0))$. From the classical Harnack inequality, the Theorem 9.20 and 9.22 pp. 244-246 in [20], and using (H2) we deduce the existence of $C_1 > 0$ (depending on R_0) such that

$$\begin{aligned} \sup_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) &\leq C_1 \inf_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) + C_1 \sup_{(x, \theta) \in B_{2R_0}(x_0, \theta_0)} |f(x, \theta)| \\ &\leq C_1 \inf_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) + C_1 C_K \sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx. \end{aligned} \quad (19)$$

The main element of the proof is to prove the following claim:

$$\exists C > 0 \quad \text{such that} \quad \sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C \inf_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta). \quad (20)$$

It is clear that if (20) holds true, the conclusion follows.

First, we integrate (E') with respect to x . It follows thanks to the Neumann boundary conditions that for all $\theta \in B_{3R_0}(\theta_0)$ we have

$$\begin{aligned} -\partial_{\theta\theta} \int_{\Omega} \tilde{n}(x, \theta) dx &= \frac{\int_{\Omega} \tilde{n}(x, \theta) \left(\tilde{R}(x, \theta) - \rho(x) - \int_{\Omega} K(x - y) dy \right) dx}{\int_{\Omega} \tilde{n}(x, \theta) dx} \int_{\Omega} \tilde{n}(x, \theta) dx \\ &\quad + \frac{\int_{\Omega} \int_{\Omega} \tilde{n}(y, \theta) K(x - y) dy dx}{\int_{\Omega} \tilde{n}(x, \theta) dx} \int_{\Omega} \tilde{n}(x, \theta) dx. \end{aligned}$$

Thanks to the L^{∞} -bounds on K, \tilde{R}, ρ (assumptions (H2), (H3) and Lemma 3) and the Fubini theorem, we have

$$\begin{aligned} -C &\leq \frac{\int_{\Omega} \tilde{n}(x, \theta) \left(\tilde{R}(x, \theta) - \rho(x) - \int_{\Omega} K(x - y) dy \right) dx}{\int_{\Omega} \tilde{n}(x, \theta) dx} < C \\ \text{and} \quad &\frac{\int_{\Omega} \int_{\Omega} \tilde{n}(y, \theta) K(x - y) dy dx}{\int_{\Omega} \tilde{n}(x, \theta) dx} \leq C_K |\Omega|. \end{aligned}$$

It follows

$$-C \int_{\Omega} \tilde{n}(x, \theta) dx \leq -\partial_{\theta\theta} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C \int_{\Omega} \tilde{n}(x, \theta) dx.$$

Hence, we apply the Harnack inequality to $\theta \in B_{3R_0}(\theta_0) \mapsto \int_{\Omega} \tilde{n}(x, \theta) dx$ into the ball $B_{2R_0}(\theta_0)$ and we deduce the existence of a constant $C_2 > 0$ such that

$$\sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx. \quad (21)$$

Next, thanks to the L^{∞} -bounds on K, \tilde{R}, ρ (assumptions (H2), (H3) and Lemma 3), it follows that in $\Omega \times B_{2R_0}(\theta_0)$

$$c_K \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(y, \theta) dy \leq c_K \int_{\Omega} \tilde{n}(y, \theta) dy \leq \int_{\Omega} \tilde{n}(y, \theta) K(x - y) dy \leq (-\partial_{xx} - \partial_{\theta\theta}) \tilde{n} + C \tilde{n}.$$

From an inequality developed by Krylov (we refer to Theorem 7.1 p33 in [12] and the reference therein), we deduce the existence of a constant $C_3 > 0$ such that

$$\inf_{B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_3 \inf_{B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta). \quad (22)$$

Combining the previous inequality with (21) and (22) yields to

$$\sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 C_3 \inf_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta).$$

This concludes the proof. \square

3.2 Lipschitz estimates

We prove 2. of Theorem 3 by the Bernstein method.

Proof of 2. of Theorem 3. We recall the main equation satisfied by u_ε :

$$-\frac{\partial_{xx} u_\varepsilon}{\varepsilon} - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 + \int_{\Omega} [1 - e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}}] K(x - y) dy = R(x, \theta) - \rho_\varepsilon \quad (23)$$

with Neumann boundary conditions. The first step is to differentiate (23) with respect to x and multiply it by $\frac{\partial_x u_\varepsilon}{\varepsilon^2}$:

$$\begin{aligned} & -\frac{\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon^3} - \frac{\partial_x \left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \right) \partial_x u_\varepsilon}{\varepsilon^2} + \int_{\Omega} e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} K(x - y) dy \frac{\partial_x u_\varepsilon^2}{\varepsilon^3} \\ & - \frac{\partial_x |\partial_\theta u_\varepsilon|^2 \partial_x u_\varepsilon}{\varepsilon^2} - \frac{\partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon} = \frac{\left(\int_{\Omega} [e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} - 1] \partial_x K(x - y) dy + \partial_x R - \partial_x \rho_\varepsilon \right) \partial_x u_\varepsilon}{\varepsilon^2}. \end{aligned}$$

Remarking that

$$\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{xx} (|\partial_x u_\varepsilon|^2)}{2} - (\partial_{xx} u_\varepsilon)^2 \quad \text{and} \quad \partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{\theta\theta} (|\partial_x u_\varepsilon|^2)}{2} - (\partial_{\theta x} u_\varepsilon)^2$$

yields to

$$\begin{aligned} & -\frac{\partial_{xx} \left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \right)}{2\varepsilon} + \frac{(\partial_{xx} u_\varepsilon)^2}{\varepsilon^3} - \frac{\partial_x \left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \right) \partial_x u_\varepsilon}{\varepsilon^2} \\ & + \int_{\Omega} e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} K(x - y) dy \frac{(\partial_x u_\varepsilon)^2}{\varepsilon^3} - \frac{\partial_x |\partial_\theta u_\varepsilon|^2 \partial_x u_\varepsilon}{\varepsilon^2} \\ & + \frac{(\partial_{\theta x} u_\varepsilon)^2}{\varepsilon} - \frac{\varepsilon \partial_{\theta\theta} \left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \right)}{2} = \frac{\left(\int_{\Omega} [e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} - 1] \partial_x K(x - y) dy + \partial_x R - \partial_x \rho_\varepsilon \right) \partial_x u_\varepsilon}{\varepsilon^2}. \end{aligned} \quad (24)$$

In the second step, we differentiate (23) with respect to θ and multiply by $\partial_\theta u_\varepsilon$. With computations similar to the ones presented above, we find

$$\begin{aligned} & -\frac{\partial_{xx} (|\partial_\theta u_\varepsilon|^2)}{2\varepsilon} + \frac{(\partial_{\theta x} u_\varepsilon)^2}{\varepsilon} - \partial_\theta \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \partial_\theta u_\varepsilon - \frac{\varepsilon}{2} \partial_{\theta\theta} (|\partial_\theta u_\varepsilon|^2) + \varepsilon (\partial_{\theta\theta} u_\varepsilon)^2 - \partial_\theta |\partial_\theta u_\varepsilon|^2 \partial_\theta u_\varepsilon \\ & + \int_{\Omega} \frac{(\partial_\theta u_\varepsilon(x)^2 - \partial_\theta u_\varepsilon(x) \partial_\theta u_\varepsilon(y))}{\varepsilon} e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} K(x - y) dy = \partial_\theta R \partial_\theta u_\varepsilon. \end{aligned} \quad (25)$$

Next, we introduce

$$p_\varepsilon(x, \theta) = \frac{|\partial_x u_\varepsilon(x, \theta)|^2}{\varepsilon^2} + |\partial_\theta u_\varepsilon(x, \theta)|^2. \quad (26)$$

If we combine (24) and (25) and we rewrite it in terms of p_ε , it follows

$$\begin{aligned}
& -\frac{\partial_{xx}p_\varepsilon}{2\varepsilon} - \frac{\varepsilon\partial_{\theta\theta}p_\varepsilon}{2} + \frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x, \theta) - \partial_\theta u_\varepsilon(x, \theta)\partial_\theta u_\varepsilon(y, \theta)] e^{\frac{u_\varepsilon(y, \theta) - u_\varepsilon(x, \theta)}{\varepsilon}} K(x - y) dy \\
& - \frac{\partial_x p_\varepsilon \partial_x u_\varepsilon}{\varepsilon^2} - \partial_\theta p_\varepsilon \partial_\theta u_\varepsilon + \frac{2(\partial_{x\theta} u_\varepsilon)^2}{\varepsilon} + \frac{(\partial_{xx} u_\varepsilon)^2}{\varepsilon^3} + \varepsilon(\partial_{\theta\theta} u_\varepsilon)^2 \\
& = \left(\int_{\Omega} [e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} - 1] \partial_x K(x - y) dy + \partial_x R - \partial_x \rho_\varepsilon \right) \frac{\partial_x u_\varepsilon}{\varepsilon^2} + \partial_\theta R \partial_\theta u_\varepsilon.
\end{aligned} \tag{27}$$

Let $(x_\varepsilon, \theta_\varepsilon)$ be such that

$$\sup_{(x, \theta) \in \Omega \times]-A, A[} p_\varepsilon(x, \theta) = p_\varepsilon(x_\varepsilon, \theta_\varepsilon).$$

Thanks to the Neumann boundaries conditions, we deduce that $(x_\varepsilon, \theta_\varepsilon) \notin \partial\Omega \times \partial(]-A, A[)$. Therefore, we distinguish three cases: either $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times]-A, A[$ or $(x_\varepsilon, \theta_\varepsilon) \in \partial\Omega \times]-A, A[$ or $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times \{\pm A\}$.

Case 1 : $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times]-A, A[$. First, we bound the right-hand-side of (27). Indeed, thanks to the Harnack inequality (first item of Theorem 3) and the L^∞ -bounds on the derivative of K , R and ρ_ε (assumptions (H2), (H3) and Corollary 1), it follows that

$$\left(\int_{\Omega} [e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} - 1] \partial_x K(x - y) dy + \partial_x R - \partial_x \rho_\varepsilon \right) \frac{\partial_x u_\varepsilon}{\varepsilon^2} + \partial_\theta R \partial_\theta u_\varepsilon \leq \frac{C\sqrt{p}}{\varepsilon}. \tag{28}$$

Next, we evaluate (27) at $(x_\varepsilon, \theta_\varepsilon)$. We claim that

$$\begin{aligned}
& -\partial_{xx}p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0, \quad -\partial_{\theta\theta}p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0, \quad \partial_x p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = \partial_\theta p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0 \\
& \text{and} \quad \frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_\theta u_\varepsilon(y, \theta_\varepsilon)] e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \geq 0.
\end{aligned} \tag{29}$$

Indeed, the first inequalities follow easily since $p(x_\varepsilon, \theta_\varepsilon) = \max p_\varepsilon$ and the last inequality holds true thanks to the following computations

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_\theta u_\varepsilon(y, \theta_\varepsilon)] e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \\
& \geq \frac{1}{\varepsilon} \left[\int_{\Omega} p_\varepsilon(x_\varepsilon, \theta_\varepsilon) e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \right. \\
& - \frac{1}{2} \int_{\Omega} \partial_\theta u_\varepsilon^2(x_\varepsilon, \theta_\varepsilon) e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \\
& \left. - \frac{1}{2} \int_{\Omega} \partial_\theta u_\varepsilon^2(y, \theta_\varepsilon) e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \right] \\
& \geq \frac{1}{2\varepsilon} \left[\int_{\Omega} p_\varepsilon(x_\varepsilon, \theta_\varepsilon) e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy - \int_{\Omega} p_\varepsilon(y, \theta_\varepsilon) e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y) dy \right] \\
& \geq 0.
\end{aligned}$$

We deduce thanks to (28) and (29) that

$$\begin{aligned}
\frac{1}{2\varepsilon} \left[\frac{\partial_{xx} u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon} + \varepsilon \partial_{\theta\theta} u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \right]^2 & \leq \frac{1}{\varepsilon} \left[\left(\frac{\partial_{xx} u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon} \right)^2 + (\varepsilon \partial_{\theta\theta} u_\varepsilon(x_\varepsilon, \theta_\varepsilon))^2 \right] \\
& \leq \frac{C\sqrt{p_\varepsilon(x_\varepsilon, \theta_\varepsilon)}}{\varepsilon}.
\end{aligned}$$

Hence, using the original equation (23), we deduce that

$$\left[-p_\varepsilon(x_\varepsilon, \theta_\varepsilon) + \int_{\Omega} (1 - e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}}) K(x_\varepsilon - y) dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \right]^2 \leq C\sqrt{p_\varepsilon(x_\varepsilon, \theta_\varepsilon)}. \tag{30}$$

Thanks to the L^∞ -bounds on K , R and ρ_ε (assumption (H2) (H3) and Lemma 3), it follows that $p_\varepsilon(x_\varepsilon, \theta_\varepsilon)$ is uniformly bounded with respect to ε . The conclusion follows.

Case 2 : $(x_\varepsilon, \theta_\varepsilon) \in \partial\Omega \times]-A, A[$. First remark that in this case, $p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = |\partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)|^2$. We claim that p_ε verifies also the Neumann boundary conditions at $(x_\varepsilon, \theta_\varepsilon)$. Indeed, according to the Neumann boundary conditions satisfied by u_ε , we can use a reflective argument and differentiate p_ε on the boundary. We obtain

$$\partial_x p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = \frac{2\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_{xx} u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon^2} + 2\partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_{x\theta} u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0$$

because

$$\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0 \quad \text{and} \quad \partial_{x\theta} u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0.$$

Since $p(x_\varepsilon, \theta_\varepsilon) = \max p_\varepsilon$, we deduce that

$$-\partial_{xx} p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0.$$

We conclude that (29) and (28) hold also true in this case and the conclusion follows from the same computations as in the previous case.

Case 3 : $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times \{\pm A\}$. This case is treated in the same manner as the previous case. \square

3.3 The bounds on ρ_ε

We recall the equation (E_ρ) satisfied by ρ_ε :

$$\begin{cases} -\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon = \int_{-A}^A R(x, \theta)n_\varepsilon(x, \theta)d\theta - \rho_\varepsilon^2 & \text{in } \Omega, \\ \partial_{\nu_x}\rho_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (E_\rho)$$

Proof of 3. of Theorem 3. The uniform bound from above and the $W^{2,p}$ bounds on ρ_ε are already provided in Lemma 3. Here we prove the uniform lower bound. We start by proving that $0 < c \leq \sup \rho_\varepsilon$. Next, we prove that $c < \rho_\varepsilon$ holds true in the whole domain Ω .

A lower bound on $\sup \rho_\varepsilon$. Assume by contradiction that there exists a sequence ε_k such that

$$\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{and} \quad \sup \rho_{\varepsilon_k} \xrightarrow[k \rightarrow +\infty]{} 0.$$

Next, if we multiply (E) by ξ_{ε_k} (introduced in (2)) and we integrate by part, we obtain

$$\mu_\varepsilon \int_{-A}^A \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta = - \int_{-A}^A \int_\Omega \rho_{\varepsilon_k} n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta.$$

We deduce thanks to (3) that for k large enough, it holds

$$\frac{|\lambda(\theta_0, 0)|}{2} \leq -\mu_{\varepsilon_k} \leq \sup \rho_{\varepsilon_k} \frac{\int_{-A}^A \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta}{\int_{-A}^A \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta}.$$

It is in contradiction with the hypothesis $\sup \rho_{\varepsilon_k} \xrightarrow[k \rightarrow +\infty]{} 0$. Therefore, there exists a constant $c > 0$ such that

$$\forall \varepsilon \in]0, \varepsilon_0[, \quad c \leq \sup \rho_\varepsilon. \quad (31)$$

The lower bound on ρ_ε in the whole domain Ω . Let $\varepsilon < \varepsilon_0$ and $x_0 \in \Omega$ be such that

$$\rho_\varepsilon(x_0) = \sup \rho_\varepsilon.$$

We conclude thanks to (31) and the Lipschitz estimates obtained in the second item of Theorem 3 that for all $x \in \Omega$

$$\rho_\varepsilon(x) = \int_{-A}^A e^{\frac{u_\varepsilon(x,\theta)}{\varepsilon}} d\theta = \int_{-A}^A e^{\frac{u_\varepsilon(x,\theta) - u_\varepsilon(x_0,\theta) + u_\varepsilon(x_0,\theta)}{\varepsilon}} d\theta \geq \rho_\varepsilon(x_0) e^{-C} \geq c e^{-C}.$$

□

3.4 The bounds on u_ε

Proof of 4. of Theorem 3. First, we prove that there exists $a > 0$ such that $-a < u_\varepsilon$. Thanks to the third item of Theorem 3, we know that there exists $c > 0$ such that for all ε small enough we have

$$c < \int_{-A}^A n_\varepsilon(x, \theta) d\theta.$$

We deduce the existence of $(x_0, \theta_0) \in \Omega \times]-A, A[$ such that

$$\frac{c}{2A} \leq n_\varepsilon(x_0, \theta_0).$$

Hence, it follows

$$\varepsilon \log\left(\frac{c}{2A}\right) \leq u_\varepsilon(x_0, \theta_0).$$

We conclude thanks to the Lipschitz estimates established in the second item of Theorem 3 that

$$\forall (x, \theta) \in \Omega \times]-A, A[, \quad -a \leq -2CA + \varepsilon[\log\left(\frac{c}{2A}\right) - C|\Omega|] \leq u_\varepsilon(x, \theta). \quad (32)$$

Next, we prove that $\lim_{\varepsilon \rightarrow 0} \sup_{(x,\theta) \in \Omega \times]-A, A[} u_\varepsilon(x, \theta) \leq 0$.

We prove it by contradiction. Assume that there exists $a > 0$ and sequences $\varepsilon_k, (x_k, \theta_k)$ such that

$$\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{and} \quad u_{\varepsilon_k}(x_k, \theta_k) > a.$$

Using the Lipschitz estimates provided by the second item of Theorem 3, it follows for all $\theta \in \left(B_{\frac{a}{4C}}(\theta_k) \cap]-A, A[\right)$

$$u_{\varepsilon_k}(x_k, \theta) = u_{\varepsilon_k}(x_k, \theta) - u_{\varepsilon_k}(x_k, \theta_k) + u_{\varepsilon_k}(x_k, \theta_k) \geq -C|\theta - \theta_k| + a \geq \frac{a}{2}$$

where C corresponds to the Lipschitz estimate given by (10). We deduce that

$$\rho_{\varepsilon_k}(x_k) \geq \min(2A, \frac{a}{4C}) e^{\frac{a}{2\varepsilon_k}}.$$

We conclude that $\liminf_{k \rightarrow +\infty} \rho_{\varepsilon_k}(x_k) = +\infty$. This is in contradiction with the L^∞ bounds on ρ_ε established in the third item of Theorem 3.

□

4 Convergence to the Hamilton Jacobi equation

Proof of Theorem 2. We prove here the 3 items of Theorem 2.

Proof of 1. Thanks to the third item of Theorem 3, it follows that for ε small enough $0 < c \leq \rho_\varepsilon \leq C$ and $\|\rho_\varepsilon\|_{W^{2,p}(\Omega)} \leq C$. We deduce from the classical Sobolev injection (see [11]) that ρ_ε converges, along subsequences, strongly in $W^{1,p}(\Omega)$ and in particular uniformly to ρ and ρ verifies

$$0 < c \leq \rho \leq C.$$

Proof of 2. From the Lipschitz estimates and the bounds established in the second and the fourth items of Theorem 3, we deduce thanks to the Arzela-Ascoli Theorem that up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ converges locally uniformly to some continuous function u . Moreover, the limit function u does not depend on x .

We prove that u is a viscosity solution of

$$\begin{cases} -|\partial_\theta u|^2 = -\lambda(\theta, \rho), \\ \partial_{\nu_\theta} u(\pm A) = 0, \end{cases}$$

with $\lambda(\theta, \rho)$ the principal eigenvalue of (1). First, we focus on the equation in the interior of the domain and then we treat the boundary conditions.

The interior equation. We recall that for a fixed value θ , Proposition 1 provides the existence of a sequence of principal eigenvalues $\lambda(\theta, \rho_\varepsilon)$ associated with a sequence of positive eigenfunctions $(\psi_\varepsilon^\theta)_{\varepsilon>0}$ of the operator $-\partial_{xx} + L - (R(x, \theta) - \rho_\varepsilon)$ with Neumann boundary conditions:

$$\text{i.e. } \begin{cases} -\partial_{xx}\psi_\varepsilon^\theta + L(\psi_\varepsilon^\theta) - (R(x, \theta) - \rho_\varepsilon)\psi_\varepsilon^\theta = \lambda(\theta, \rho_\varepsilon)\psi_\varepsilon^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi_\varepsilon^\theta = 0 & \text{on } \partial\Omega. \end{cases} \quad (33)$$

Since $\psi_\varepsilon^\theta > 0$, we introduce

$$\Psi_\varepsilon^\theta = \ln(\psi_\varepsilon^\theta).$$

Let ϕ be test function such that $u - \phi$ has a strict maximum at $\theta \in]-A, A[$. Then, there exists $(x_\varepsilon, \theta_\varepsilon) \in \overline{\Omega} \times]-A, A[$ such that

$$\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \theta \quad \text{and} \quad \max_{(x, \theta) \in \overline{\Omega} \times]-A, A[} u_\varepsilon(x, \theta) - \phi(\theta) - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x) = u_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon).$$

We distinguish two cases: either $x_\varepsilon \in \Omega$ or $x_\varepsilon \in \partial\Omega$.

Case 1: $x_\varepsilon \in \Omega$. Since u_ε is a classical solution of (E_{HC}) , we deduce that it is also a viscosity solution, therefore

$$\begin{aligned} & -\frac{\partial_{xx}(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))}{\varepsilon} - \frac{[\partial_x(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2}{\varepsilon^2} + \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy \\ & - \varepsilon \partial_{\theta\theta}(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)) - [\partial_\theta(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2 - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \leq 0. \end{aligned}$$

Remarking that ϕ does not depend on x and the θ value is fixed in $\Psi_\varepsilon^{\theta_\varepsilon}$, we deduce that

$$\begin{aligned} & -\partial_{xx}\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2 + \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \\ & - \varepsilon \partial_{\theta\theta}\phi(\theta_\varepsilon) - [\partial_\theta\phi(\theta_\varepsilon)]^2 \leq 0. \end{aligned} \quad (34)$$

Next, we observe that (33) implies

$$-\partial_{xx}\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2 + \int_\Omega [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon - y)dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) = \lambda(\theta_\varepsilon, \rho_\varepsilon).$$

Therefore, passing to the limit $\varepsilon \rightarrow 0$, thanks to the continuity of $\lambda(\theta, \rho)$ with respect to θ and ρ (Proposition 1), it follows that

$$-[\partial_\theta\phi(\theta)]^2 \leq -\lambda(\theta, \rho).$$

Case 2 : $x_\varepsilon \in \partial\Omega$. First, we remark that in this case,

$$-\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = -\partial_x \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) = 0.$$

Therefore, we deduce that

$$-\partial_x [u_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)] = 0.$$

Moreover, since $(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) = \max(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon})$, we have firstly by a reflective argument that

$$-\partial_{xx}(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) \geq 0, \quad (35)$$

and secondly, we have

$$u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \leq \varepsilon[\Psi_\varepsilon(y) - \Psi_\varepsilon(x_\varepsilon)]. \quad (36)$$

The inequalities (35) and (36) lead to

$$\begin{aligned} & -\partial_{xx}\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) \leq -\partial_{xx}u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \\ \text{and} \quad & \int_\Omega [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon - y)dy \leq \int_\Omega [1 - e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}}]K(x_\varepsilon - y)dy. \end{aligned}$$

Therefore, the conclusion follows from similar computation as above.

The boundary conditions. Let ϕ be a test function such that $u - \phi$ has a strict maximum at A (the proof works the same for $-A$). Then, there exists $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times [-A, A]$ such that

$$\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} A \quad \text{and} \quad \max_{(x, \theta) \in \bar{\Omega} \times [-A, A]} u_\varepsilon - \phi = (u_\varepsilon - \phi)(x_\varepsilon, \theta_\varepsilon).$$

We distinguish two cases: $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times]-A, A[$, $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times \{A\}$.

Case 1 : $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times]-A, A[$. In this case, by a similar analysis as above, we deduce that

$$-[\partial_\theta\phi(\theta_\varepsilon)]^2 \leq -\lambda(\theta_\varepsilon, \rho) + o_\varepsilon(1).$$

Case 2 : $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times \{A\}$. In this case, since the maximum is reached on the boundary, we deduce thanks to the boundary conditions of u_ε

$$-\partial_{\nu_\varepsilon}\phi(\theta_\varepsilon) = \partial_{\nu_\theta}(u_\varepsilon - \phi)(x_\varepsilon, \theta_\varepsilon) \geq 0.$$

Taking the inferior limit, we conclude that

$$\min(-[\partial_\theta\phi(A)]^2 + \lambda(A, \rho), \partial_{\nu_\theta}\phi(\pm A)) \leq 0,$$

which corresponds to the boundary condition in the viscosity sense.

Finally, u is a sub-solution of (5) in a viscosity sense. With similar arguments, u is also a super-solution. We conclude that u is a viscosity solution of (5).

Next, we prove that $\sup_{\theta \in [-A, A]} u(\theta) = 0$.

If it does not hold true, it follows that $\sup_{\theta \in [-A, A]} u(\theta) < -a < 0$. Hence for ε small enough, we deduce that $\max_{(x, \theta) \in \Omega \times [-A, A]} u_\varepsilon(x, \theta) < -\frac{a}{2}$, which implies that $\rho_\varepsilon < c$ for ε sufficiently small. This is in contradiction with the third item of Theorem 3. We conclude that $\max_{\theta \in [-A, A]} u(\theta) = 0$.

Proof of 3. Thanks to the L^∞ bounds on ρ_ε (third point of Theorem 3), we deduce that

$$c \leq \|n_\varepsilon\|_{L^1(\Omega \times]-A, A])} \leq C.$$

It follows that n_ε converges in the sense of the measure to n and the measure n is non-negative and not trivial. We next prove that

$$\text{supp } n_\varepsilon \subset \Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}.$$

Indeed, let $\phi \in C_c^\infty(\Omega \times]-A, A[)$ be any positive test function such that

$$\text{supp } \phi \subset \Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}^c. \quad (37)$$

We prove that $\int_\Omega \int_{-A}^A \phi(x, \theta) n(x, \theta) dx d\theta = 0$.

To this end, we introduce $-a = \sup_{\text{supp } \phi} u$. According to (37), it follows that $a > 0$. We deduce that for all ε small enough and all $(x, \theta) \in \Omega \times \text{supp } \phi$, we have

$$u_\varepsilon(x, \theta) \leq -\frac{a}{2}. \quad (38)$$

We conclude that

$$\begin{aligned} \int_\Omega \int_{-A}^A \phi(x, \theta) n(x, \theta) d\theta dx &= \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) n(x, \theta) d\theta dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) n_\varepsilon(x, \theta) d\theta dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) e^{\frac{u_\varepsilon(x, \theta)}{\varepsilon}} d\theta dx \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) e^{\frac{-a}{2\varepsilon}} d\theta dx \\ &= 0. \end{aligned}$$

We finally prove that

$$\{u(\theta) = 0\} \subset \{\lambda(\theta, \rho) = 0\}. \quad (39)$$

To this end, note first that since u is a Lipschitz continuous function, it is a.e. differentiable. Therefore, (5) implies that

$$\lambda(\theta, \rho) \geq 0, \quad \text{for a.e. } \theta.$$

Moreover, since λ is continuous with respect to θ the above inequality holds indeed for all θ . To prove (39), it is therefore enough to prove that for any θ_0 such that $u(\theta_0) = 0$, we have

$$\lambda(\theta_0, \rho) \leq 0.$$

This property can be derived by testing the equation in (5) against the test function $\varphi(\theta) \equiv 0$ at the point θ_0 for a viscosity subsolution criterion.

This concludes the proof of 3. □

5 Qualitative properties of the phenotypic density n

The objective of this section is to show how the asymptotic results provided in Theorem 2 allows to obtain qualitative results on the phenotypic density n , at the limit as $\varepsilon \rightarrow 0$. Recall from Theorem 2 that as $\varepsilon \rightarrow 0$, n_ε tends weakly to a measure n and ρ_ε converges uniformly to a function ρ such that

$$\text{supp } n \subset \Omega \times \{\theta \mid u(\theta) = 0\} \subset \Omega \times \{\theta \mid \lambda(\theta, \rho) = \min_{\theta'} \lambda(\theta', \rho) = 0\},$$

with $u : [-A, A] \rightarrow \mathbb{R}$ the viscosity to

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta, \rho), & \theta \in (-A, A) \\ \partial_\theta u(\pm A) = 0, \\ \max u(\theta) = 0, \end{cases}$$

and $\lambda(\theta, \rho)$ the principal eigenvalue corresponding to the following problem:

$$\begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - [R(\cdot, \theta) - \rho]\psi^\theta = \lambda(\theta, \rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi^\theta = 0 & \text{in } \partial\Omega \end{cases}$$

We expect indeed that u would take its maximum at some distinct traits such that the phenotypic density n would have the following form:

$$n(x, \theta) = \sum_{i=1}^d \rho_i(x)\delta(\theta - \theta_i), \quad \rho(x) = \sum_{i=1}^d \rho_i(x).$$

Note that Theorem 2 guarantees convergence of $(n_\varepsilon, \rho_\varepsilon)$ to (n, ρ) , only along subsequences. It does not exclude possibility of multiple limits as $\varepsilon \rightarrow 0$. In subsection 5.1 we will show that, for a particular choice of R , there is at most one possible monomorphic limit (n, ρ) such that n is a single Dirac mass, i.e. $n(x, \theta) = \rho(x)\delta(\theta - \bar{\theta})$. In particular under symmetry conditions the only possible monomorphic outcome at the limit would be $n(x, \theta) = \rho(x)\delta(\theta)$. We next identify a situation in subsection 5.2 where the phenotypic density n is indeed monomorphic. Finally we show in subsection 5.3 that a strong fragmentation of the environment may lead to polymorphic situations.

Before providing our qualitative results, we prove the following technical result on the principal eigenvalue λ that will help us in the next subsections to obtain our qualitative results.

Proposition 2. *The following identity holds true:*

$$\partial_\theta \lambda(\theta, \rho) = - \int_{\Omega} \partial_\theta R(x, \theta) \psi^\theta(x)^2 dx. \quad (40)$$

Proof. First recall from Proposition 1 that λ and ψ^θ are C^1 functions with respect to θ . Differentiating (1) with respect to θ we deduce that

$$\partial_\theta \lambda(\theta, \rho) \psi^\theta + \lambda(\theta, \rho) \partial_\theta \psi^\theta = -\partial_{xx} \partial_\theta \psi^\theta + L(\partial_\theta \psi^\theta) - [R(\cdot, \theta) - \rho] \partial_\theta \psi^\theta - \partial_\theta R(\cdot, \theta) \psi^\theta. \quad (41)$$

We multiply (41) by ψ^θ and integrate by parts, recalling $\int_{\Omega} \psi^\theta(x)^2 dx = 1$, to obtain that

$$\partial_\theta \lambda(\theta, \rho) = - \int_{\Omega} \partial_\theta R(x, \theta) \psi^\theta(x)^2 dx + \int_{\Omega} \partial_x \partial_\theta \psi^\theta \partial_x \psi^\theta + L(\psi^\theta) \partial_\theta \psi^\theta - [R(x, \theta) - \rho - \lambda(\theta, \rho)] \partial_\theta \psi^\theta \psi^\theta dx. \quad (42)$$

We remark that multiplying (1) by $\partial_\theta \psi^\theta$ and integrating by part leads to

$$\int_{\Omega} \partial_x \psi^\theta \partial_x \partial_\theta \psi^\theta + L(\psi^\theta) \partial_\theta \psi^\theta - [R(x, \theta) - \rho - \lambda(\theta, \rho)] \psi^\theta \partial_\theta \psi^\theta dx = 0.$$

The conclusion follows. \square

Corollary 2. *Let $\bar{\theta} \in]-A, A[$ be one of the emergent traits of n , it follows that*

$$\int_{\Omega} \partial_{\theta} R(x, \bar{\theta}) \psi^{\bar{\theta}}(x)^2 dx = 0.$$

Next, we consider some examples with explicit expressions of R . We illustrate how Proposition 2 can be useful to characterize the emergent dominant traits.

Example A. We fix $\theta_0 \in]-A, A[$, and we define

$$R(x, \theta) = r - g(\theta - \theta_0)^2.$$

We assume that r is large enough such that (H4) holds true for $\theta = \theta_0$. From Corollary 2, we deduce that at any emergent trait $\bar{\theta}$, we have

$$2g(\bar{\theta} - \theta_0) \int_{\Omega} \psi^{\bar{\theta}}(x)^2 dx = 0.$$

We deduce that the unique emergent trait is $\bar{\theta} = \theta_0$. Therefore, the limit population is monomorphic. Of course, this example is a toy-model and does not take into account any spatial structure.

Example B. We define

$$R(x, \theta) = r - g(\theta - bx)^2. \quad (43)$$

From Corollary 2, we deduce that for any emergent trait $\bar{\theta}$, we have

$$\bar{\theta} = b \int_{\Omega} x \psi^{\bar{\theta}}(x)^2 dx.$$

It is not enough to conclude that the number of emergent traits is finite. However, we can still remark that the optimal traits are fixed point of the application

$$\chi : \theta \mapsto b \int_{\Omega} x \psi^{\theta}(x)^2 dx. \quad (44)$$

5.1 At most one possible monomorphic outcome

In this subsection, we restrict our study to monomorphic limits, that is when $n = \rho \delta(\theta - \bar{\theta})$. We will prove that, when R is given by (43), the problem admits at most one monomorphic limit.

Proposition 3. *Let $n_1(x, \theta) = \rho_1(x) \delta(\theta - \bar{\theta}_1)$ and $n_2(x, \theta) = \rho_2(x) \delta(\theta - \bar{\theta}_2)$ be two monomorphic limits of the problem. Then, $\rho_1 = \rho_2$. Additionally, if R is given by (43), then $\theta_1 = \theta_2$. If the domain Ω is symmetric with respect to $x = 0$, then the only possible emergent trait corresponding to a monomorphic population is $\bar{\theta} = 0$.*

To prove Proposition 3, we enlarge the definition of the Rayleigh quotient defined in Proposition 1 by including ρ in the definition:

$$\mathcal{R}(\theta, \rho, \phi) = \frac{\int_{\Omega} |\partial_x \phi|^2 dx + \frac{\int_{\Omega \times \Omega} [\phi(x) - \phi(y)]^2 K(x-y) dx dy}{2} - \int_{\Omega} [R(x, \theta_1) - \rho(x)] \phi^2 dx}{\int_{\Omega} \phi(x)^2 dx}, \quad (45)$$

and we will use the following lemma.

Lemma 4. *Assume that (n_{ε}) converges in $L^{\infty}(w^*(0, \infty); \mathcal{M}^1(\mathbb{R}^d))$ to $\rho(x) \delta(\theta - \theta_0)$. Then $\rho(\cdot)$ is the principal eigenfunction corresponding to the operator $-\Delta_x + L - [R(\cdot, \theta_i) - \rho]$.*

Proof. By passing weakly to the limit in the equation (E), we obtain that

$$-\partial_{xx}\rho + L\rho = \rho(R(x, \theta_0) - \rho),$$

which implies that ρ is the principal eigenfunction corresponding to the operator $-\Delta_x + L - [R(\cdot, \theta_0) - \rho]$. \square

Proof of Proposition 3. Let $n_1(x, \theta) = \rho_1(x)\delta(\theta - \theta_1)$ and $n_2(x, \theta) = \rho_2(x)\delta(\theta - \theta_2)$ be two possible limits. Our objective is to prove that

$$\theta_1 = \theta_2, \quad \rho_1 = \rho_2.$$

From Theorem 2 we obtain that

$$\min_{\theta} \lambda(\theta, \rho_i) = \lambda(\theta_i, \rho_i) = 0.$$

Let ψ_i be the eigenfunction associated to the operator $-\Delta_x + L - [R(\cdot, \theta_i) - \rho_i]$ ensured by Proposition 1 with $\|\psi_i\|_{L^2} = 1$. Notice that since the limit is monomorphic and thanks to Lemma 4, we have for each case $\rho_i = c_i\psi_i$ with $c_i > 0$. Using the Rayleigh quotient introduced in (45), we deduce that

$$\begin{aligned} 0 &= \int_{\Omega} c_1^2 \psi_1(x)^2 dx \times \mathcal{R}(\theta_1, c_1\psi_1, c_1\psi_1) \\ &= c_1^2 \int_{\Omega} |\partial_x \psi_1|^2 dx + c_1^2 \frac{\int_{\Omega \times \Omega} [\psi_1(x) - \psi_1(y)]^2 K(x-y) dx dy}{2} - c_1^2 \int_{\Omega} [R(x, \theta_1) - c_1\psi_1] \psi_1^2 dx, \end{aligned} \quad (46)$$

and

$$\begin{aligned} 0 &\leq \int_{\Omega} c_1^2 \psi_1(x)^2 dx \times \mathcal{R}(\theta_1, c_2\psi_2, c_1\psi_1) \\ &= c_1^2 \int_{\Omega} |\partial_x \psi_1|^2 dx + c_1^2 \frac{\int_{\Omega \times \Omega} [\psi_1(x) - \psi_1(y)]^2 K(x-y) dx dy}{2} - c_1^2 \int_{\Omega} [R(x, \theta_1) - c_2\psi_2] \psi_1^2 dx. \end{aligned} \quad (47)$$

The last inequality holds since thanks to Theorem 2 we have

$$0 \leq \lambda(\theta_1, c_2\psi_2) \leq \mathcal{R}(\theta_1, c_2\psi_2, c_1\psi_1).$$

By subtracting (46) to (47), we deduce that

$$\int_{\Omega} (c_1\psi_1(x))^2 dx \leq \int_{\Omega} (c_1\psi_1(x))^2 c_2\psi_2(x) dx. \quad (48)$$

Following similar computations, we obtain that

$$\int_{\Omega} (c_2\psi_2(x))^3 dx \leq \int_{\Omega} (c_2\psi_2(x))^2 c_1\psi_1(x) dx. \quad (49)$$

By combining (48) and (49), we deduce that

$$\int_{\Omega} [c_1\psi_1(x) - c_2\psi_2(x)]^2 (c_1\psi_1(x) + c_2\psi_2(x)) dx \leq 0.$$

Since $c_{1,2}\psi_{1,2} > 0$, we deduce that $c_1\psi_1 = c_2\psi_2$, and hence $\rho_1 = \rho_2$.

We next assume that R is given by (43) and conclude thanks to Corollary 2. Indeed, since $\partial_{\theta} R(x, \theta_i) = -2g(\theta_i - bx)$, we deduce that

$$\theta_1 = \frac{-b}{2g} \int_{\Omega} x c_1 \psi_1^2(x) dx = \frac{-b}{2g} \int_{\Omega} x c_2 \psi_2^2(x) dx = \theta_2.$$

Finally, note that under symmetry conditions, if $n(x, \theta) = \rho(x)\delta(\theta)$ is a possible monomorphic limit, then $\tilde{n}(x, \theta) = n(-x, -\theta)$ is also a possible monomorphic outcome. We hence deduce in this case that $\theta = 0$. \square

5.2 A small selection pressure leads to a monomorphic population

The next proposition gives a condition which ensures the existence of a set of parameters such that the limit is monomorphic.

Proposition 4. *Assume that R is given by (43). Under the hypothesis (H1)–(H4), there exists $g_0 > 0$ such that if $g \in]0, g_0[$ then there exists a unique optimal trait.*

Proof. According to the implicit function theorem, the functions $(\theta \mapsto \psi^\theta)$ and $(\theta \mapsto \partial_\theta \psi^\theta)$ depend continuously on the parameter g . Moreover, the following claims hold true:

$$\lambda(\theta, \rho) \text{ is non-decreasing with respect to } g, \quad (50)$$

and

$$\int_{\Omega} |\partial_\theta \psi^\theta(x)|^2 dx \xrightarrow{g \rightarrow 0} 0. \quad (51)$$

Claim (50) follows straightforward computations, therefore we do not provide the proof. The second one is proved in Appendix B. Next, we differentiate χ (defined by (44)) with respect to θ , to obtain that

$$\chi'(\theta) = 2b \int_{\Omega} x \psi^\theta(x) \partial_\theta \psi^\theta(x) dx.$$

Thanks to the Cauchy-Schwarz inequality, we deduce that

$$|\chi'(\theta)| \leq 2b \sup_{x \in \Omega} x \int_{\Omega} \psi^\theta(x)^2 dx \int_{\Omega} \partial_\theta \psi^\theta(x)^2 dx = 2b \sup_{x \in \Omega} x \int_{\Omega} \partial_\theta \psi^\theta(x)^2 dx \xrightarrow{g \rightarrow 0} 0.$$

Since the last inequality does not depend on the choice of θ , we deduce the existence of a uniform $g_0 > 0$ such that for all $g \in]0, g_0[$ we have

$$|\chi'(\theta)| < 1.$$

Thanks to Theorem 2 there exists a fixed point to χ . Moreover since χ is a contraction mapping, we recover that the fixed point $\bar{\theta}$ is unique. \square

5.3 A strong fragmentation of the environment leads to polymorphism

In this subsection we consider the growth rate given by (43) and spatial domains of the following type:

$$\Omega_d = (-d - a, -d) \cup (d, d + a).$$

We also assume that the function $K : \mathbb{R} \rightarrow \mathbb{R}^+$ is bounded.

Proposition 5. *For $d \geq d_0$, with d_0 a large enough constant, the trait 0 is not included in the support of the phenotypic density n . As a consequence, the population density may not be monomorphic.*

Proof. We first note that, for fixed d , there always exists $r(d)$ such that for all $r \geq r(d)$, (H4) is satisfied so that thanks to Theorem 1 the population persists. We thus can assume that, up to adjusting the constant r , we are in a situation where the population persists.

Let's suppose that 0 is included in the support of the phenotypic density n . Then this implies that

$$\lambda(0, \rho) = \inf_{\phi \in H^1(\Omega)} \mathcal{R}(0, \rho, \phi) = 0 \leq \lambda(\theta, \rho), \quad \text{for all } \theta \in [-A, A].$$

Let $\phi_0 \in H^1(\Omega)$ be such that

$$\inf_{\phi \in H^1(\Omega)} \mathcal{R}(0, \rho, \phi) = \mathcal{R}(0, \rho, \phi_0) = 0, \quad \|\phi_0\|_{L^2(\Omega)} = 1.$$

We choose

$$\theta_1 = -d - a/2, \quad \theta_2 = d + a/2.$$

We also define

$$\Omega_1 = [-d - a, -d], \quad \Omega_2 = [d, d + a], \quad \phi_0^{(1)} = \phi_0 \mathbf{1}_{\Omega_1}, \quad \phi_0^{(2)} = \phi_0 \mathbf{1}_{\Omega_2}.$$

We will prove that when d is large enough,

$$\|\phi_0^{(1)}\|_{L^2} \lambda(\theta_1, \rho) + \|\phi_0^{(2)}\|_{L^2} \lambda(\theta_2, \rho) < \lambda(0, \rho) = 0. \quad (52)$$

Since $\|\phi_0^{(1)}\|_{L^2} + \|\phi_0^{(2)}\|_{L^2} = 1$, the inequality (52) would imply

$$\min(\lambda(\theta_1, \rho), \lambda(\theta_2, \rho)) < 0$$

which is in contradiction with the positiveness of the eigenvalues $\lambda(\theta, \rho)$ established in 2. of Theorem 2.

We consider the function ϕ_i , for $i = 1, 2$, which minimizes $\mathcal{R}(0, \rho, \cdot)$ restricted to the set Ω_i , that is the following operator

$$\mathcal{R}(0, \rho, \Omega_i, \phi) = \frac{\int_{\Omega_i} |\partial_x \phi|^2 dx + \frac{\int_{\Omega_i \times \Omega_i} [\phi(x) - \phi(y)]^2 K(x-y) dx dy}{2} - \int_{\Omega_i} [R(x, 0) - \rho(x)] \phi^2 dx}{\|\phi\|_{L^2(\Omega_i)}^2},$$

and such that

$$\|\phi_i\|_{L^2(\Omega_i)} = 1.$$

Note that here ϕ_i has its support in Ω_i , while ϕ_0 has its support in Ω_d . To prove (52), it is enough to prove that

$$\|\phi_0^{(1)}\|_{L^2}^2 \mathcal{R}(\theta_1, \rho, \phi_1) + \|\phi_0^{(2)}\|_{L^2}^2 \mathcal{R}(\theta_2, \rho, \phi_2) < \mathcal{R}(0, \rho, \phi_0).$$

We compute

$$\begin{aligned} \|\phi_0^{(i)}\|_{L^2}^2 \mathcal{R}(\theta_i, \rho, \phi_i) &= \|\phi_0^{(i)}\|_{L^2}^2 [\mathcal{R}(\theta_i, \rho, \Omega_i, \phi_i) + \frac{1}{2} \int_{\Omega_i \times \Omega_j} \phi_i^2(x) K(x-y) dx dy] \\ &\leq \|\phi_0^{(i)}\|_{L^2}^2 [\mathcal{R}(\theta_i, \rho, \Omega_i, \phi_0^{(i)}) + \frac{1}{2} \int_{\Omega_i \times \Omega_j} \phi_i^2(x) K(x-y) dx dy] \\ &= \int_{\Omega_i} |\partial_x \phi_0^{(i)}|^2 dx + \frac{1}{2} \int_{\Omega_i \times \Omega_i} [\phi_0^{(i)}(x) - \phi_0^{(i)}(y)]^2 K(x-y) dx dy \\ &\quad - \int_{\Omega_i} [R(x, \theta_i) - \rho(x)] \phi_0^{(i)2}(x) dx + \frac{1}{2} \|\phi_0^{(i)}\|_{L^2}^2 \int_{\Omega_i \times \Omega_j} \phi_i^2(x) K(x-y) dx dy. \end{aligned}$$

Combining the inequality above, for $i = 1$ and $i = 2$, we obtain that

$$\begin{aligned} &\|\phi_0^{(1)}\|_{L^2}^2 \mathcal{R}(\theta_1, \rho, \phi_1) + \|\phi_0^{(2)}\|_{L^2}^2 \mathcal{R}(\theta_2, \rho, \phi_2) \leq \mathcal{R}(0, \rho, \phi_0) \\ &\quad + \int_{\Omega_1} [R(x, 0) - R(x, \theta_1)] \phi_0^2(x) dx + \int_{\Omega_2} [R(x, 0) - R(x, \theta_2)] \phi_0^2(x) dx \\ &\quad + \frac{1}{2} \|\phi_0^{(1)}\|_{L^2}^2 \int_{\Omega_1 \times \Omega_2} \phi_1^2(x) K(x-y) dx dy + \frac{1}{2} \|\phi_0^{(2)}\|_{L^2}^2 \int_{\Omega_2 \times \Omega_1} \phi_2^2(x) K(x-y) dx dy. \end{aligned}$$

We next note that

$$\int_{\Omega_i} [R(x, 0) - R(x, \theta_i)] \phi_0^2(x) dx = -g\theta_i \int_{\Omega_i} (2x - \theta_i) \phi_0^2(x) dx \leq -g(d^2 - a^2/4) \int_{\Omega_i} \phi_0^2(x) dx.$$

We also have

$$\int_{\Omega_i \times \Omega_j} \phi_i^2(x) K(x-y) dx dy \leq a \|K\|_{L^\infty(\mathbb{R})} \int_{\Omega_i} \phi_i^2(x) dx = a \|K\|_{L^\infty(\mathbb{R})}.$$

We conclude that

$$\|\phi_0^{(1)}\|_{L^2}^2 \mathcal{R}(\theta_1, \rho, \phi_1) + \|\phi_0^{(2)}\|_{L^2}^2 \mathcal{R}(\theta_2, \rho, \phi_2) \leq \mathcal{R}(0, \rho, \phi_0) - g(d^2 - a^2/4) + \frac{a}{2} \|K\|_{L^\infty(\mathbb{R})}.$$

Therefore, for d large enough, we obtain that

$$\|\phi_0^{(1)}\|_{L^2}^2 \mathcal{R}(\theta_1, \rho, \phi_1) + \|\phi_0^{(2)}\|_{L^2}^2 \mathcal{R}(\theta_2, \rho, \phi_2) < \mathcal{R}(0, \rho, \phi_0).$$

□

6 Some numerical illustrations

In this section, we illustrate the numerical solutions of (E), for some particular examples, considering the following type of growth rate

$$R(x, \theta) = r - g(bx - \theta)^2.$$

We recall from Example 1 that in this case, r is a maximal growth rate and $-g(bx - \theta)^2$ models the selection. The parameter g is the selection pressure whereas b is the gradient of the environment. We provide numerical examples where we vary this set of parameters.

To find a numerical solution of (E), we solve numerically the following parabolic equation:

$$\begin{cases} \partial_t n_\varepsilon - \partial_{xx} n_\varepsilon - \varepsilon^2 \partial_{\theta\theta} n_\varepsilon + L(n_\varepsilon) = [R - \rho_\varepsilon] n_\varepsilon & \text{in } \mathbb{R}^+ \times \Omega \times]-A, A[, \\ \rho_\varepsilon(t, x) = \int_{-A}^A n_\varepsilon(t, x, \theta) d\theta & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_{\nu_x} n_\varepsilon = \partial_{\nu_\theta} n_\varepsilon = 0, \\ n(t = 0, x, \theta) = n_0(x, \theta). \end{cases} \quad (E_t)$$

We implement equation (E_t) by a semi-implicit finite difference method. We stop the algorithm when we find a numerical steady state of (E_t) : a numerical solution of (E).

First, we underline that in all the numerical resolutions, the density of the population concentrates around one or several distinct trait(s). Moreover, these optimal traits are present everywhere in space thanks to the local and the non-local migration. However, the density of the population at the position x with an optimal trait θ_m depends on whether this trait θ_m is adapted or not to the position x .

Figure 1 illustrates the convergence of n_ε to a Dirac mass as ε goes to 0. The only variation is with respect to the parameter $\varepsilon = 0.1, 0.01$ and 0.001 .

Next, we focus on the qualitative properties established in Section 5, Figures 2 and 3 are numerical illustrations of Proposition 3. We fix Ω as a single connected component and we investigate the dependence on the parameter g . We recover numerically that as $g \rightarrow 0$ the limit density is monomorphic with a dominant trait centred at $\theta = 0$. For larger values of g , the phenotypic density concentrates around several distinct traits. For each simulations, we also provide the numerical distributions of ρ_ε (Figure 3), this density seems to be centred around the point which maximizes $R(\cdot, \bar{\theta})$ (where $\bar{\theta}$ is any emergent trait). Therefore, when the emergent trait is unique, ρ_ε is increasing on $(-2, 0)$ and then decreasing on $(0, 2)$ whereas the spatial distribution can be more involved whenever there exist several distinct emergent traits.

To conclude, we present in Figure 4 a numerical illustration of Proposition 5. Here, the free parameter is the distance between the two connected components of Ω . We recover that increasing the distance between the two connected components may lead to the emergence of new dominant traits.

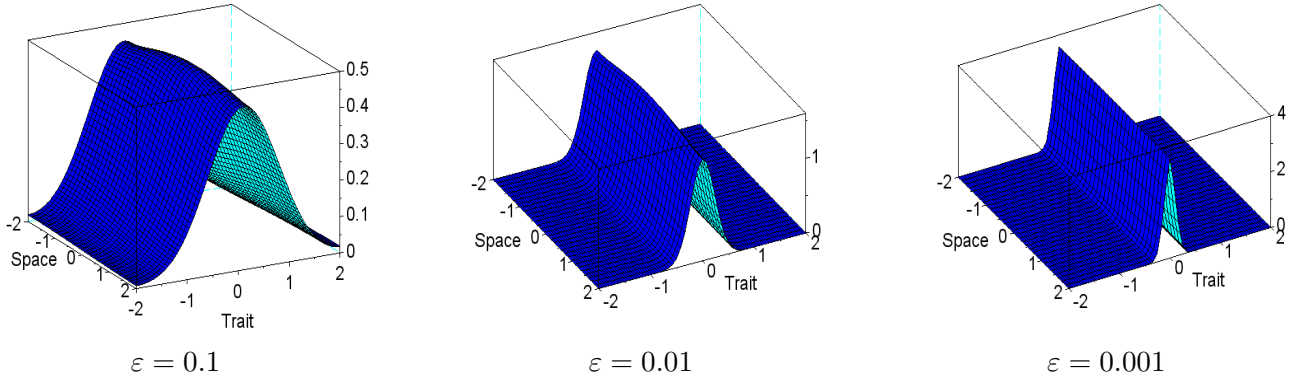


Figure 1: Variation of the numerical solutions of (E) with respect to ε . The others parameters are fixed as follow : $r = 1, b = 1, g = 0.1, \Omega = (-2, 2)$ and $A = 2$. We observe that the distribution of the population concentrates around the optimal trait $\theta = 0$.

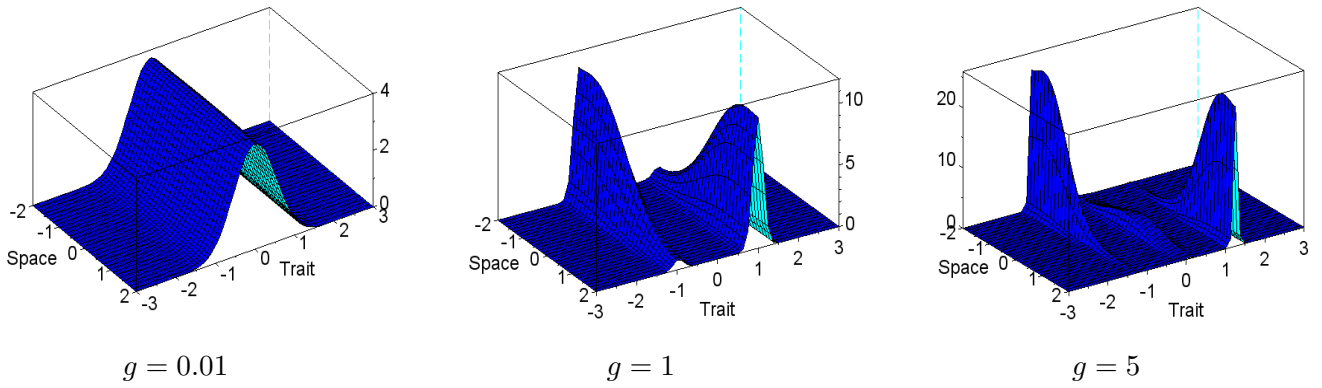


Figure 2: Variation of the numerical solutions n_ε of (E) with respect to g . The other parameters are fixed as follow: $r = 5, b = 1, \varepsilon = 0.01, \Omega = (-2, 2)$ and $A = 3$. We recover that if g is small then the population is monomorphic. For large values of g , there exist several distinct emergent traits.

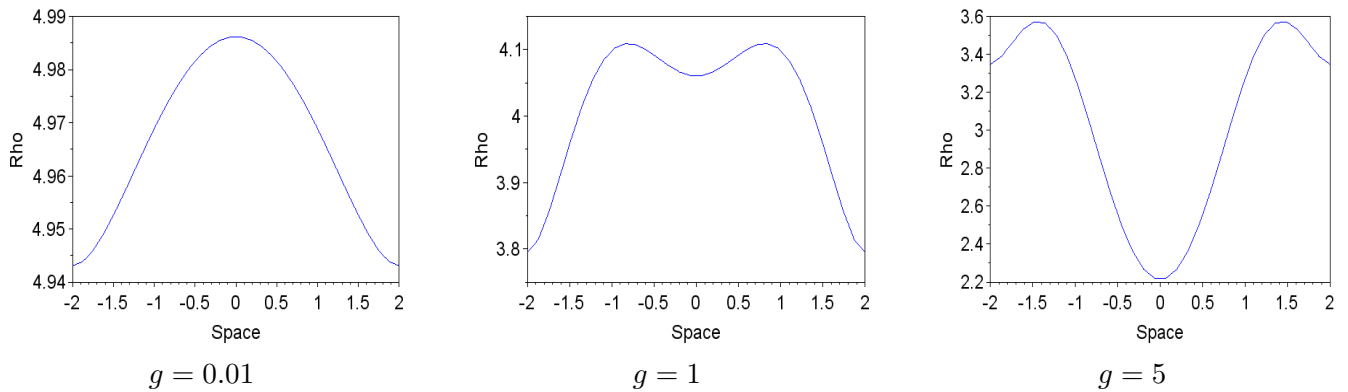


Figure 3: Variation of the numerical density ρ_ε of (E) with respect to g . The other parameters are fixed as follow: $r = 5, b = 1, \varepsilon = 0.01, \Omega = (-2, 2)$ and $A = 3$.

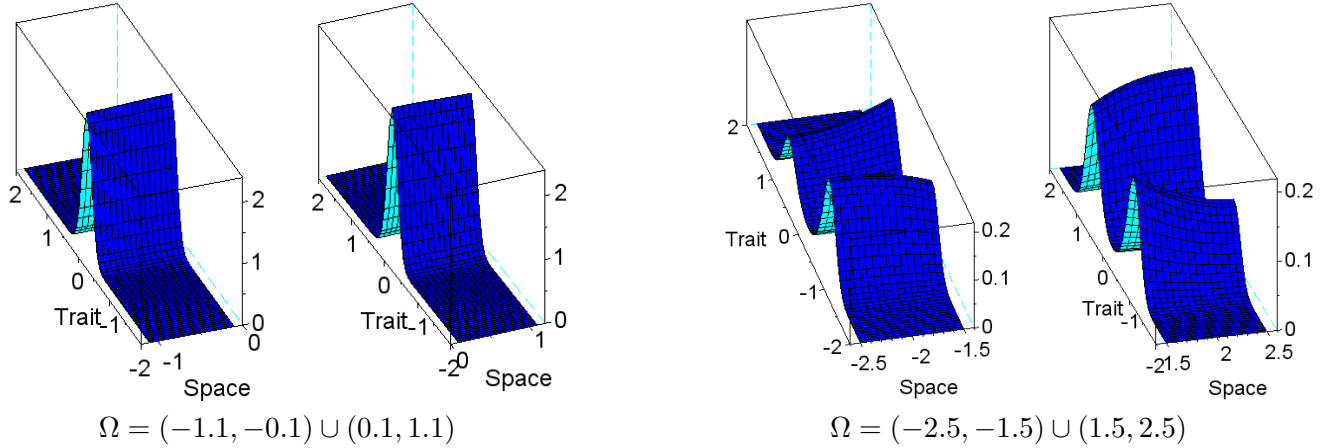


Figure 4: Variation of the numerical solutions n_ε of (E) with respect to the distance between the two connected components of Ω . The other parameters are fixed as follow: $r = 1$, $b = 1$, $g = 1$, $\varepsilon = 0.01$ and $A = 2$. We observe that for this set of parameters, increasing the distance induces a polymorphic density population.

Appendix A- Proof of Lemma 1

The proof of Lemma 1 follows essentially the steps of the proof of the convergence of u_ε (i.e. second item of Theorem 2). Therefore, we will only emphasize the differences between the two proofs. We made the choice to provide the proof of the convergence of u_ε rather than the convergence of μ_ε because it is the result that motivated the current study.

Proof of Lemma 1. We recall the equation satisfied by μ_ε and ξ_ε

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon\xi_\varepsilon & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega \times]-A, A[). \end{cases} \quad (2)$$

The existence of ξ_ε is ensured by the Krein-Rutman Theorem. Moreover, according to the Krein-Rutman Theorem, the sign of ξ_ε is constant. Therefore, we consider that $\xi_\varepsilon > 0$, $\|\xi_\varepsilon\|_{L^2} = 1$ and we define

$$v_\varepsilon = \varepsilon \ln(\xi_\varepsilon).$$

Next, we prove that μ_ε is bounded from below and above respectively by $-\sup R$ and $-\inf R$.

First, we focus on the upper bound. Let $(\bar{x}, \bar{\theta}) \in \bar{\Omega} \times [-A, A]$ be such that $\sup_{(x,\theta) \in \bar{\Omega} \times [-A,A]} \xi_\varepsilon(x, \theta) = \xi_\varepsilon(\bar{x}, \bar{\theta})$.

If $(\bar{x}, \bar{\theta}) \in \Omega \times]-A, A[$, it follows

$$(-\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L(\xi_\varepsilon))(\bar{x}, \bar{\theta}) \leq 0.$$

From (2), we deduce that

$$\mu_\varepsilon \leq -R(\bar{x}, \bar{\theta}) \leq -\inf R.$$

If $(\bar{x}, \bar{\theta})$ belongs to $\partial(\Omega \times]-A, A[)$, we conclude with a reflective argument and the same computations as in the previous case. In any case, for all $\varepsilon > 0$ we have

$$\mu_\varepsilon < -\inf R.$$

Next, we focus on the lower bound. Let $(\underline{x}, \underline{\theta}) \in \overline{\Omega} \times [-A, A]$ be such that $\inf_{(x, \theta) \in \overline{\Omega} \times [-A, A]} \xi_\varepsilon = \xi_\varepsilon(\underline{x}, \underline{\theta})$. With similar arguments as for the upper bound, we deduce that

$$-\sup R \leq -R(\underline{x}, \underline{\theta}) \leq \mu_\varepsilon.$$

Therefore, μ_ε is uniformly bounded from below and above thus μ_ε converges along subsequences to μ . Next, as we have established Lipschitz and uniform bounds on u_ε , we can prove that there exists a constant $C > 0$ such that

$$|\partial_x v_\varepsilon| < C\varepsilon, \quad |\partial_\theta v_\varepsilon| < C, \quad -C < \lim_{\varepsilon \rightarrow 0} \inf_{\Omega \times]-A, A[} v_\varepsilon, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sup_{\Omega \times]-A, A[} v_\varepsilon \leq 0.$$

Therefore, we deduce that v_ε converges along subsequences to v . Moreover, with similar computations as in the proof of the second item of Theorem 2, we deduce that v is a viscosity solution of

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, -\mu), \\ \max_{\theta \in [-A, A]} v(\theta) = 0. \end{cases} \quad (53)$$

Next, we claim that

$$\lambda(\theta, -\mu) = \lambda(\theta, 0) - \mu. \quad (54)$$

We postpone the proof of this claim to the end of this paragraph. Thanks to (53) and (54) we deduce that

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, 0) + \mu, \\ \max_{\theta \in [-A, A]} v(\theta) = 0. \end{cases}$$

Remark that $-\lambda(\theta, 0) + \mu \leq 0$ for all $\theta \in [-A, A]$. Next, we introduce $\theta_m \in [-A, A]$ such that

$$v(\theta_m) = \max_{\theta \in [-A, A]} v(\theta).$$

It follows that $\partial_\theta v(\theta_m) = 0$ and $-\lambda(\theta_m, 0) + \mu = 0 = \max(-\lambda(\theta, 0) + \mu)$. We deduce thanks to (H4) that

$$0 = \max(-\lambda(\theta, 0) + \mu) = -\min(\lambda(\theta, 0)) + \mu = -\lambda(\theta_0, 0) + \mu.$$

We conclude that

$$\lambda(\theta_0, 0) = \mu.$$

We finish the proof by remarking that the previous convergence result holds for any subsequence of μ_ε . Therefore, we conclude that

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon = \lambda(\theta_0, 0).$$

It remains to prove (54). Let ψ_μ^θ be the principal eigenfunction associated to the principal eigenvalue of $\lambda(\theta, -\mu)$ with μ a constant

$$\begin{cases} -\partial_{xx} \psi_\mu^\theta + L\psi_\mu^\theta - [R(\cdot, \theta) + \mu]\psi_\mu^\theta = \lambda(\theta, -\mu)\psi_\mu^\theta & \text{in } \Omega, \\ \partial_{\nu_x} \psi_\mu^\theta = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that

$$-\partial_{xx} \psi_\mu^\theta + L\psi_\mu^\theta - R(\cdot, \theta)\psi_\mu^\theta = (\lambda(\theta, -\mu) + \mu)\psi_\mu^\theta.$$

Since μ is constant, $\psi_\mu^\theta > 0$ and by the uniqueness of the positive eigenfunction of $-\partial_{xx} + L - R(\cdot, \theta)$ (up to a multiplication by a scalar), we deduce that $\lambda(\theta, -\mu) + \mu = \lambda(\theta, 0)$. \square

Appendix B- Proof of (51)

Proof. According to the implicit function theorem, the eigenfunction ψ is regular with respect to its parameters g and θ . In this appendix, we will take into account in the notations this dependence with respect to the parameters: we will denote ψ by $\psi^{\theta,g}$, $\lambda(\theta)$ by $\lambda(\theta, g)$ and R by R_g .

Since $\|\psi^{\theta,g}\|_{L^2(\Omega)}^2 = 1$, we deduce that $\int_{\Omega} \psi^{\theta,g}(x) \partial_{\theta} \psi^{\theta,g}(x) dx = 0$. If we differentiate (16) with respect to θ , we have that $\partial_{\theta} \psi^{\theta,g}$ is solution in Ω of

$$-\partial_{xx} \partial_{\theta} \psi^{\theta,g} + L(\partial_{\theta} \psi^{\theta,g}) - [R_g(\cdot, \theta) - \rho] \partial_{\theta} \psi^{\theta,g} - \partial_{\theta} R_g(\cdot, \theta) \psi^{\theta,g} = \lambda(\theta, g) \partial_{\theta} \psi^{\theta,g} + \partial_{\theta} \lambda(\theta, g) \psi^{\theta,g}.$$

We then multiply the above equation by $\psi^{\theta,g}$ and integrate over Ω to obtain that

$$\begin{aligned} & \int_{\Omega} \left(\partial_x (\partial_{\theta} \psi^{\theta,g}) \right)^2 dx - \frac{\int_{\Omega \times \Omega} [\partial_{\theta} \psi^{\theta,g}(x) - \partial_{\theta} \psi^{\theta,g}(y)]^2 K(x-y) dy dx}{2} - \int_{\Omega} [R_g(x, \theta) - \rho(x)] \partial_{\theta} \psi^{\theta,g}(x)^2 dx \\ &= \lambda(\theta, g) \int_{\Omega} \left(\partial_{\theta} \psi^{\theta,g} \right)^2 dx + \int_{\Omega} \partial_{\theta} R_g(x, \theta) \psi^{\theta,g}(x) \partial_{\theta} \psi^{\theta,g}(x) dx + \partial_{\theta} \lambda(\theta, g) \int_{\Omega} \psi^{\theta,g}(x) \partial_{\theta} \psi^{\theta,g}(x) dx. \end{aligned}$$

Remarking that $\partial_{\theta} R_0 = 0$ and recalling that $\int_{\Omega} \psi^{\theta,0}(x) \partial_{\theta} \psi^{\theta,0}(x) dx = 0$, we deduce that $\partial_{\theta} \psi^{\theta,0}$ belongs to the eigenspace associated to the principal eigenvalue $\lambda(\theta, 0)$ of the operator $-\partial_{xx} + L - [R_0 - \rho]$. Since this space is one dimensional, engendered by $\psi^{\theta,0}$ and using again that $\psi^{\theta,0}$ is orthogonal to $\partial_{\theta} \psi^{\theta,0}$, we conclude that

$$\int_{\Omega} \left(\partial_{\theta} \psi^{\theta,g}(x) \right)^2 dx \xrightarrow{g \rightarrow 0} 0.$$

□

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