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FUNCTIONS RELATED TO JACOBI THETA FUNCTIONS AND APPLICATIONS I

ABD RAOUF CHOUIKHA

Abstract. In this paper we highlight trigonometric holomorphic functions of two variables $f_j(v, \tau), j = 1, 2, 3, 4$ where $Im \tau > 0$, and $|Imv| < \frac{1}{2}Im\tau$. These functions are related to the four theta functions:

$\theta_j(v, \tau) = \theta_j(0, \tau)f_j(v, \tau)f_j(v + 1, \tau), j = 3, 4$;

$\theta_1(v, \tau) = (\pi \sin(\pi v))^{\theta_1(0, \tau)}f_1(v, \tau)f_1(v + 1, \tau)$;

$\theta_2(v, \tau) = \cos(\pi v)\theta_2(0, \tau)f_2(v, \tau)f_2(v + 1, \tau)$.

We propose to describe some properties of these functions. In particular the quotient $\frac{f_j(v, \tau)}{f_j(v + 1, \tau)}$ which makes it possible to deduce remarkable identities.

Various aspects will be considered, expressions in the form of infinite products and their convergence, their development in Fourier series, the convergence of their derivatives, and so on. We of course find again the classical expansions of theta functions in trigonometric forms or infinite products.

1. Introduction

Throughout this paper we assume $Im \tau > 0$ and take $q = e^{i\pi \tau}$. Jacobi theta functions for $j = 1, 2, 3, 4$ are defined as ([5] or [8])

$\theta_1(v, \tau) = 2 \sum_{n \geq 0} (-1)^n q^{(n + \frac{1}{2})^2} \sin((2n + 1)\pi v)$

$\theta_2(v, \tau) = 2 \sum_{n \geq 0} q^{(n + \frac{1}{2})^2} \cos((2n + 1)\pi v)$

$\theta_3(v, \tau) = 1 + 2 \sum_{n \geq 1} q^n \cos(2n\pi v)$

$\theta_4(v, \tau) = 1 + 2 \sum_{n \geq 1} (-1)^n q^n \cos(2n\pi v)$

These four theta functions can be extended to complex values for $v$ and $q$ such that $|q| < 1$. All four theta functions are entire and periodic functions of $v$, [1], [2].

Using the Jacobi triple product identity, we can find the infinite product representation ([5], [9]) for $\theta_j(v, \tau)$, namely:

$\theta_1(v, \tau) = 2q^\frac{1}{4} \sin v \prod_{n \geq 1} (1 - q^{2n})(1 - q^{2n}e^{2i\pi v})(1 - q^{2n}e^{-2i\pi v})$

$= 2q^\frac{1}{4} \sin v(q^2; q^2)^\infty(q^2e^{2i\pi v}; q^2)^\infty(q^2e^{-2i\pi v}; q^2)^\infty$

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\[ \theta_2(v, \tau) = \prod_{n \geq 1} (1 - q^{2n})(1 + q^{2n}e^{2\pi iv})(1 + q^{2n}e^{-2\pi iv}) \]
\[ = 2q^{\frac{1}{2}} \cos(vq^2; \tau^2) \infty (-q^2e^{2\pi iv}; \tau^2) \infty (-q^2e^{-2\pi iv}; \tau^2) \infty \]
\[ \theta_3(v, \tau) = \prod_{n \geq 1} (1 - q^{2n})(1 + q^{2n-1}e^{2\pi iv})(1 + q^{2n-1}e^{-2\pi iv}) \]
\[ = (q^2; \tau^2) \infty (-qe^{2\pi iv}; \tau^2) \infty (-qe^{-2\pi iv}; \tau^2) \infty \]
\[ \theta_4(v, \tau) = \prod_{n \geq 1} (1 - q^{2n})(1 - q^{2n-1}e^{2\pi iv})(1 - q^{2n-1}e^{-2\pi iv}) \]
\[ = (q^2; \tau^2) \infty (qe^{2\pi iv}; \tau^2) \infty (qe^{-2\pi iv}; \tau^2) \infty \]

where \( q = e^{i\pi v}, v \in C, \tau \in H_+, \) and \( (a; b)_\infty = \prod_{n \geq 0} (1 - ab^n). \)

Trigonometric expansions of theta functions can be proved by the methods of residue calculus as described by [5, p.358] and [8].

It is known that from these products we may deduce the Fourier expansions of \( \log(\theta_j(v, \tau)) \) and \( \theta_j' \).

The four Jacobi theta functions are naturally related. Starting from one of them we may obtain the other three by simple calculation.

Differentiating \( \theta_1(v, \tau) \) with respect to \( v \) and putting \( v \to 0 \) yields

\[ \theta_1'(0, \tau) = 2q^{\frac{1}{2}}(q^2; \tau^2)^3. \]

Setting \( z = 0 \) in the other three expressions we obtain

\[ \theta_2(0, \tau) = 2q^{\frac{1}{2}}(q^2; \tau^2) \infty (-q^2; \tau^2) \infty, \]  \[ \theta_3(0, \tau) = (q^2; \tau^2) \infty (-q; q^2) \infty, \]
\[ \theta_4(0, \tau) = (q^2; \tau^2) \infty (q; q^2) \infty. \]

The aim of this paper is to represent \( \theta_j(v, \tau) \) as product

\[ f_j(v, \tau) f_j(v + 1, \tau) \]

and to give a complete description of these functions \( f_j(v, \tau) \). We will see in the sequel that not only the products but also the quotients \( \frac{f_j(v, \tau)}{f_j(v + 1, \tau)} \) seem play a particular role.

In particular, we derive their representation as infinite products, their Fourier series expansions, as well as for \( \log(f_j(v, \tau)) \) and for \( \frac{f_j}{f_j'}(v, \tau) \). Some other significant properties and equalities will be derived, as the following identity valid for any integer \( n \)

\[ \left( \frac{1 - \cos \pi u}{1 + \cos \pi u} \right)^n \prod_{k \geq 0} \frac{e^{\pi ku} - q^k}{e^{\pi ku} + q^k} \frac{e^{-\pi ku} + q^k}{e^{-\pi ku} - q^k} = \prod_{k \geq 0} \frac{e^{\pi ku} - q^{k-n}}{e^{\pi ku} + q^{k-n}} \frac{e^{-\pi ku} + q^{k+n}}{e^{-\pi ku} - q^{k+n}}. \]

Moreover, using these infinite products we prove

\[ \frac{\theta_4(2^{n+1}v, 2^{n+1}\tau)}{\theta_4(0, 2^{n+1}\tau)} = \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \prod_{p=0}^{n} \frac{\theta_4(2^pv, 2^p\tau)}{\theta_4(0, 2^p\tau)}, \]
\[ \frac{\theta_3(2^{n+1}v, 2^{n+1}\tau)}{\theta_3'(0, 2^{n+1}\tau)} = \frac{\theta_3(v, \tau)}{\theta_3'(0, \tau)} \prod_{p=0}^{n} \frac{\theta_3(2^pv, 2^p\tau)}{\theta_3'(0, 2^p\tau)}, \]
\[ \frac{\theta_2(2^{n+1}v, 2^{n+1}\tau)}{\theta_2'(0, 2^{n+1}\tau)} = \frac{\theta_2(v, \tau)}{\theta_2'(0, \tau)} \prod_{p=0}^{n} \frac{\theta_2(2^pv, 2^p\tau)}{\theta_2'(0, 2^p\tau)}. \]
generalizing the known cases \( n = 0, n = 1 \) see [7, p. 23].

2. Functions defined by an infinite product

Consider the complex functions of two variables \( f_j, j = 1, 2, 3, 4 \) defined by the infinite products

\[
\begin{align*}
f_4(v, \tau) &= \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right) \right], \\
f_3(v, \tau) &= \prod_{k \geq 0} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos (k + \frac{1}{2}) \pi \tau} \right) \right], \\
f_1(v, \tau) &= \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\sin k \pi \tau} \right) \right], \\
f_2(v, \tau) &= \prod_{k \geq 1} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos k \pi \tau} \right) \right]
\end{align*}
\]

where \( v \in C \) and \( \tau \) belongs to the half plane \( (3 \tau : \text{Im} \tau > 0) \).

Remarks: These products may also be rewritten under different forms. For example

\[
\begin{align*}
f_4(v, \tau) &= 2 \prod_{k \geq 0} \left( \frac{\cos \left( \frac{k}{2} \pi \tau + \frac{\pi}{4} \right)}{\sin (k + \frac{1}{2}) \pi \tau} \right) \\
&= \prod_{k \geq 0} \left( \frac{\cos \left( \frac{k}{2} \pi \tau + \frac{\pi}{4} \right)}{\cos \left( \frac{k}{2} \pi \tau \right)} \right) \left( \frac{\sin \left( \frac{k}{2} \pi \tau - \frac{\pi}{4} \right)}{\sin \left( \frac{k}{2} \pi \tau \right)} \right), \\
f_1(v, \tau) &= \prod_{k \geq 0} \left( \frac{\cos \left( \frac{k}{2} \pi \tau + \frac{\pi}{4} \right)}{\cos \left( \frac{k}{2} \pi \tau \right)} \right) \left( \frac{\sin \left( \frac{k}{2} \pi \tau - \frac{\pi}{4} \right)}{\sin \left( \frac{k}{2} \pi \tau \right)} \right).
\end{align*}
\]

The following ensures the convergence of these products.

**Proposition 2-1** Consider the following sequences of functions

\[
\begin{align*}
(U_{k,4})(v, \tau) &= \left( \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right), \\
(U_{k,3})(v, \tau) &= \left( \frac{\sin \pi v}{\cos (k + \frac{1}{2}) \pi \tau} \right), \\
(U_{k,1})(v, \tau) &= \left( \frac{\sin \pi v}{\sin k \pi \tau} \right), \\
(U_{k,2})(v, \tau) &= \left( \frac{\sin \pi v}{\cos k \pi \tau} \right).
\end{align*}
\]

For \( v \) such that \( 0 < \text{Im} v < \frac{1}{4} \text{Im} \tau \), the sums \( \sum_{k \geq 0} |U_{k,j}(v, \tau)|, \quad j = 1, 2, 3, 4 \) converges uniformly. Equivalently the infinite products \( \prod_{k \geq 0} (1 - U_{k,j}(v, \tau)) \) are uniformly convergent.

**Proof** We demonstrate that the infinite products are absolutely and uniformly convergent by application of the M-test (see, e.g., [4] E.T. Copson, Theory of Functions of a Complex Variable, pp. 104-6) which states that an infinite product \( \prod (1 + g_k(v)) \) converges uniformly and absolutely in a bounded closed region if:

\( g_k(v) \) are such that \( |g_k(v)| \leq M_k \) and \( \sum_k M_k \) is convergent. Indeed, write \( \tau = \tau_1 + i\tau_2 \) where \( \tau_1 \in \mathbb{R} \), \( \tau_2 > 0 \). Then

\[
\left| \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right|^2 \leq \left| \frac{\sin \frac{\pi \tau_2}{2}}{\sin (k + \frac{1}{2}) \pi (\tau_1 + i\tau_2)} \right|^2.
\]
Moreover, the series \( \sum M_{k,4} = \sinh \frac{\pi \tau}{2} \sum_k \left( \frac{1}{\sinh(k + \frac{1}{2}) \pi \tau} \right) \)
converges.

By the same way, applying again the M-test for the other \( U_{k,j}(v, \tau) \) we get
\[
| \frac{\sin \pi v}{\cos(k + \frac{1}{2}) \pi \tau} | \leq M_{k,3}, \quad | \frac{\sin \pi v}{\sin k \pi \tau} | \leq M_{k,1}, \quad | \frac{\sin \pi v}{\cos k \pi \tau} | \leq M_{k,2}
\]
Moreover, the series \( \sum_k M_{k,j}, j = 1, 2, 3 \) converge.

Replacing \( v \) by \( v + 1, v + 2 \) and \( \tau \) by \( \tau + 1, \ldots \) we then obtain the following

**Proposition 2-2** The above function \( f_j(v, \tau) \) verifies the following properties
for \( \Im \tau > 0, \Im v < \frac{1}{2} \Im \tau \)

1. For \( j = 3, 4 \), one has
   \( f_j(v + 2, \tau) = f_j(v, \tau) = f_j(v, \tau + 4), \quad f_j(v + 1, \tau) = f_j(-v, \tau) = f_j(v, \tau + 2), \quad f_j(v, \tau + 3) = f_j(v + 1, \tau + 1). \)

2. For \( j = 1, 2 \) one has
   \( f_j(v + 2, \tau) = f_j(v, \tau) = f_j(v, \tau + 2) \quad f_j(v + 1, \tau + 2) = f_j(-v, \tau + 1). \)

Moreover,
\[
f_4(v, \tau + 1) = f_3(v, \tau), \quad f_1(v, \tau + 1) = f_2(v, \tau).
\]
\[
f_1(v, \tau + \frac{1}{2}) = f_4(v, \tau), \quad f_2(v, \tau + \frac{1}{2}) = f_3(v, \tau).
\]
\[
f_4(v + \frac{\tau}{2}, \tau) = f_1(v, \tau) \left( e^{i \pi v} - 1 \right) \prod_{k \geq 1} \frac{1 - q^{2k}}{1 - q^{2k+1}} = f_1(v, \tau) \left( e^{i \pi v} - 1 \right) \frac{f_4(0, \tau)}{f_1(0, \tau)}
\]

**Proof** These identities are direct consequence of definitions of \( f_j(v, \tau) \) and from the following relations
\[
f_4(v, \tau + 1) = \prod_{k \geq 0} \left[ 1 - \left( (-1)^k \frac{\sin \pi v}{\cos(k + \frac{1}{2}) \pi \tau} \right) \right] = f_3(v, \tau),
\]
\[
f_1(v, \tau + 1) = \prod_{k \geq 1} \left[ 1 - \left( (-1)^k \frac{\sin \pi v}{\cos k \pi \tau} \right) \right] = f_2(v, \tau).
\]

The following gives a triple product identity for \( f_j(v, \tau) \)

**Proposition 2-3** The functions \( f_j(v, \tau) \) may also be expressed
\[
f_4(v, \tau) = \prod_{k \geq 0} \left( \frac{e^{-i \pi v} + q^{k + \frac{1}{2}}(e^{i \pi v} - q^{k + \frac{1}{2}})}{1 - q^{2k+1}} \right) = \prod_{k \geq 0} (e^{-i \pi v} + q^{k + \frac{1}{2}})(e^{i \pi v} - q^{k + \frac{1}{2}})(1 + q^{k})
\]
\[ f_1(v, \tau) = \prod_{k \geq 1} \frac{\left( e^{-i\pi v} + q^k \right) \left( e^{i\pi v} - q^k \right)}{1 - q^{2k}}. \]

\[ f_4(v + \frac{\tau}{2}, \tau) = \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+\frac{1}{2}}}{e^{-i\pi v} - q^{k+\frac{1}{2}}}, \]

\[ f_4(v + 1, \tau) = \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+\frac{1}{2}}}{e^{-i\pi v} - q^{k+\frac{1}{2}}}. \]

Theorem 2-4  For an integer \( n \) one has

\[ f_4(v + n\tau, \tau) = \prod_{k \geq 0} \frac{\left( e^{-i\pi v} + q^{k+n+\frac{1}{2}} \right) \left( e^{i\pi v} - q^{k-n+\frac{1}{2}} \right)}{1 - q^{2k+1}} \]

\[ = \prod_{k \geq 0} \frac{e^{-i\pi v} + q^{k+n+\frac{1}{2}}}{e^{i\pi v} + q^{k+n+\frac{1}{2}}} \cdot \frac{e^{i\pi v} - q^{k-n+\frac{1}{2}}}{e^{-i\pi v} - q^{k-n+\frac{1}{2}}}(1 + q^k). \]

\[ f_1(v + n\tau, \tau) = \prod_{k \geq 1} \frac{\left( e^{-i\pi v} + q^{k+n} \right) \left( e^{i\pi v} - q^{k-n} \right)}{1 - q^{2k}}. \]

In particular, the following identities hold

\[ \left( \frac{1 - \cos \pi v}{1 + \cos \pi v} \right)^n \prod_{k \geq 0} e^{i\pi v} - q^{k} \cdot \frac{e^{i\pi v} + q^{k}}{e^{-i\pi v} - q^{k}} = \prod_{k \geq 0} e^{i\pi v} - q^{k-n} \cdot \frac{e^{-i\pi v} + q^{k+n}}{e^{-i\pi v} - q^{k+n}}. \]

Proof  Incrementing \( v \) by \( v + \tau \) one has

\[ \frac{f_4(v + \tau, \tau)}{f_4(v + 1 + \tau, \tau)} = \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+\frac{1}{2}}}{e^{-i\pi v} - q^{k+\frac{1}{2}}} \]

\[ = \frac{e^{i\pi v} - q^{-\frac{1}{2}}}{e^{i\pi v} + q^{-\frac{1}{2}}} \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} - q^{k+\frac{1}{2}}}{e^{-i\pi v} + q^{k+\frac{1}{2}}} \]

\[ = \frac{e^{i\pi v} - q^{-\frac{1}{2}}}{e^{i\pi v} + q^{-\frac{1}{2}}} \cdot \frac{e^{-i\pi v} - q^{\frac{1}{2}}}{e^{-i\pi v} + q^{\frac{1}{2}}} \cdot f_4(v, \tau). \]

Then we deduce for any integer \( n \)

\[ \frac{f_4(v + n\tau, \tau)}{f_4(v + 1 + n\tau, \tau)} = \left( \frac{e^{i\pi v} - q^{-\frac{1}{2}}}{e^{i\pi v} + q^{-\frac{1}{2}}} \right)^n \left( \frac{e^{-i\pi v} - q^{\frac{1}{2}}}{e^{-i\pi v} + q^{\frac{1}{2}}} \right)^n \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+\frac{1}{2}}}{e^{-i\pi v} - q^{k+\frac{1}{2}}} \]

\[ = \left( \frac{1 - \cos \pi (v + \frac{\tau}{2})}{1 + \cos \pi (v + \frac{\tau}{2})} \right)^n \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k+\frac{1}{2}}}{e^{i\pi v} + q^{k+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+\frac{1}{2}}}{e^{-i\pi v} - q^{k+\frac{1}{2}}} \]

\[ = \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k-n+\frac{1}{2}}}{e^{i\pi v} + q^{k-n+\frac{1}{2}}} \cdot \frac{e^{-i\pi v} + q^{k+n+\frac{1}{2}}}{e^{-i\pi v} - q^{k+n+\frac{1}{2}}} \]

Equivalently, replacing \( v \) by \( v - \frac{\tau}{2} \) it yields
\[(1 - \cos \pi v)^n \prod_{k \geq 0}^n \frac{e^{i\pi v} - q^k}{e^{i\pi v} + q^k} = \prod_{k \geq 0} \frac{e^{i\pi v} - q^{k-n}}{e^{i\pi v} + q^{k-n}} \]

By the same way one obtains
\[
\frac{f_1(v + n\tau, \tau)}{f_1(v + 1 + n\tau, \tau)} = \left(\frac{e^{i\pi v} - q^{-1}}{1 + \cos \pi v}\right)^n \prod_{k \geq 1}^n \frac{e^{i\pi v} - q^k}{e^{i\pi v} + q^k} \]
\[
= \prod_{k \geq 1} \frac{e^{i\pi v} - q^{k-n}}{e^{i\pi v} + q^{k-n}} \frac{e^{-i\pi v} + q^k}{e^{-i\pi v} - q^k}.
\]

The following yields trigonometric expansions of \(\log (f_j(v, \tau)), \ j = 1, 2, 3, 4\)

**Proposition 2-5** The logarithmic expression \(\log (f_j(v, \tau))\) may be expressed as
\[
\log (f_4(v, \tau)) = \sum_{k \geq 0}^n \frac{1 - e^{-i\pi v}q^{k+\frac{1}{2}}}{1 - q^{2k+1}} = \sum_{p \geq 0} \sum_{k \geq 0}^n \frac{1}{p} \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau}\right)^p, \]
\[
\log (f_3(v, \tau)) = - \sum_{p \geq 1} \sum_{k \geq 0}^n \frac{1}{p} \left(\frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau}\right)^p.
\]
\[
\log (f_1(v, \tau)) = - \sum_{p \geq 1} \sum_{k \geq 0}^n \frac{1}{p} \left(\frac{\sin \pi v}{\sin(k\pi \tau)}\right)^p, \ \log (f_2(v, \tau)) = - \sum_{p \geq 1} \sum_{k \geq 0}^n \frac{1}{p} \left(\frac{\sin \pi v}{\cos(k\pi \tau)}\right)^p.
\]

**Proof** Indeed, since we have
\[
\prod_{k \geq 0} (1 - U_k) = \prod_{k \geq 0} \exp(\log(1 - U_k)) = \exp \left(\sum_{k \geq 0} \log(1 - U_k)\right)
\]
then
\[
f_4(v, \tau) = \exp \left(\sum_{k \geq 0} \log \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau}\right)^p\right]\right).
\]
\[
\log (f_4(-v, \tau)) = \left[- \sum_{p \geq 1} \sum_{k \geq 0}^n \frac{(-1)^p}{p} \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau}\right)^p\right].
\]

On the other hand, since as we have seen by Proposition 1 :
\[
\left(\frac{\sin \frac{\pi v}{2}}{\sin(k + \frac{1}{2})\pi \tau}\right)^2 = M_{k, \frac{1}{2}}^2 < 1 \text{ then the following series is well defined}
\]
\[
\log (f_4(v, \tau)) = - \sum_{n \geq 0} \sum_{k \geq 0}^n \frac{1}{2n} \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau}\right)^{2n} - \sum_{n \geq 0} \sum_{k \geq 0}^n \frac{1}{2n+1} \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau}\right)^{2n+1}.
\]

By the same way one obtains the other series expansions for \(\log (f_j(v, \tau)), \ j = 1, 2, 3,\)
In the next result we derive Fourier expansions of $\log (f_j(v, \tau))$, $j = 1, 2, 3, 4$ as well as for its derivative.

**Theorem 2-6** A trigonometric expansion of the function $\log (f_4(v, \tau))$ is

$$\log (f_4(v, \tau)) = \sum_{n \geq 1} a_n(\tau) \cos(2n\pi v) + \sum_{m \geq 1} b_m(\tau) \sin((2m + 1)\pi v)$$

where

$$a_n(\tau) = \frac{q^n}{n(1 - q^{2n})}; \quad b_m(\tau) = (-1)^m \frac{q^{2m+1}}{(2m + 1)(1 - q^{4m+2})}$$

**Proof** Indeed, it suffices to express $\cos(2n\alpha)$ and $\sin((2m + 1)\pi v)$ in terms of $\sin \alpha$ and to identify the coefficients of the series

$$\cos(2n\alpha) = \frac{(-1)^n}{2}(2 \sin \alpha)^{2n} + \sum_{k=0}^{n-1} (-1)^{n+k+1} \frac{2n}{k+1} \left( \begin{array}{c} 2n - k + 1 \\ k \end{array} \right) (2 \sin \alpha)^{2n-2k-2}$$

$$= n \sum_{0 \leq p \leq n} \frac{(-1)^p (n + p - 1)!}{(2p)![(n - p)!]} (2 \sin \alpha)^{2p} = \sum_{p=0}^{n} b_{2p,n}(\sin \alpha)^{2p},$$

where

$$b_{2n,n} = (-1)^n 2^{2n-1}; \quad b_{0,n} = 1; \quad \text{and} \quad b_{2p,n} = (-1)^p 2^{2p} \frac{n + p - 1}{n - p} \frac{(n - p - 1)!}{(n - p)!} \text{ for } n \neq p.$$

By the same way we rewrite

$$\sin((2m + 1)\pi v) = \frac{(-1)^n}{2}(2 \sin \alpha)^{2n-1} + \sum_{k=0}^{n-1} (-1)^{n+k+1} \frac{2n}{k} \left( \begin{array}{c} 2n - k - 1 \\ k \end{array} \right) (2 \sin \alpha)^{2n-2k-1}$$

$$= n \sum_{0 \leq p \leq n} \frac{(-1)^p (n + p)!}{(2p)![(n - p)!]} (2 \sin \alpha)^{2p-1}.$$

After developing and identifying we find the coefficients of the Fourier expansions.

**Proposition 2-7** A trigonometric expansion of the functions $\frac{\partial}{\partial v} \log (f_j(v, \tau))$, $j = 1, 2, 3, 4$ is

$$\frac{1}{f_4} \frac{\partial f_4}{\partial v}(v, \tau) = \sum_{k \geq 0} \frac{-\pi \cos \pi v}{\sin(k + \frac{1}{2}) \pi \tau - \sin \pi v}, \quad \frac{1}{f_3} \frac{\partial f_3}{\partial v}(v, \tau) = \sum_{k \geq 0} \frac{-\pi \cos \pi v}{\cos(k + \frac{1}{2}) \pi \tau - \sin \pi v}$$

$$\frac{1}{f_1} \frac{\partial f_1}{\partial u}(u, \tau) = \sum_{k \geq 0} \frac{-\pi \cos \pi v}{\sin(k \pi \tau) - \sin \pi v}, \quad \frac{1}{f_2} \frac{\partial f_2}{\partial v}(v, \tau) = \sum_{k \geq 0} \frac{-\pi \cos \pi v}{\cos(k \pi \tau) - \sin \pi v}$$

$f_3$ and $f_4$ is defined in the "strip" $|Imv| < \frac{1}{2} Im\tau$, while $f_1$ and $f_2$ is defined in the "strip" $|Imv| < Im\tau$. 

7
3. Link with Jacobi theta functions

Consider the Ramanujan theta function

\[ \phi(a, b) = \sum_{k \in \mathbb{Z}} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}} \]

\[ \phi(a, b) = (-a, ab)(-b, ab) \infty(ab, ab) \infty \]

with \(|ab| < 1\). We denote here \((\alpha, \beta) \infty = \prod_{i \geq 1} (1 - \alpha \beta^i)\). When \(a = -qe^{2i\pi v}, b = -qe^{-2i\pi v}\) this function is related for example to the fourth theta function

\[ \theta_4(v, q) = \phi(-qe^{2i\pi v}, -qe^{-2i\pi v}). \]

From the knowledge of the zeros of \(\theta_4(v, q)\) it is possible to obtain infinite products representing this function.

3.1. Link with \(\theta_4(v, \tau)\). There is a natural relation with \(\theta_4(v, \tau)\)

**Theorem 3-1** Let \(\theta_4(v, \tau)\) the fourth Jacobi theta function. Then we have

\[ \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = f_4(v, \tau) f_4(-v, \tau) = f_4(v, \tau) f_4(v + 1, \tau) \]

where \(f_4\) and its logarithmic derivative satisfy the infinite product

\[ f_4(v, \tau) = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right) \right] \]

\[ \frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = \frac{1}{f_4} \frac{\partial f_4}{\partial v}(v, \tau) + \frac{1}{f_4} \frac{\partial f_4}{\partial v}(-v, \tau). \]

\(\theta_4\) is defined in the "strip" \(|{\text{Im}}v| < \frac{1}{2} \text{Im}\tau|\).

**Proof of Theorem 3-1** Write

\[ \log (f_4(v, \tau)) = \frac{1}{2} \log \left( \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right) = \sum_{n \geq 1} \frac{1}{2n + 1} \left( \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right)^{2n+1}. \]

Recall that ([5 p.358])

\[ \frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = 4\pi \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \sin 2n \pi v \]

\[ = -2\pi \sum_{n \geq 1} \frac{\sin 2n \pi v}{\sin n \pi \tau} \]

\[ = -\pi \sum_{k \geq 0} \left( \sin (k + \frac{1}{2}) \pi \tau \right)^2 - (\sin \pi v)^2 \]

On the other hand since \(\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = f_4(v, \tau) f_4(-v, \tau)\) then

\[ \frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = -\pi \sum_{k \geq 0} \frac{\cos \pi v}{\sin (k + \frac{1}{2}) \pi \tau} - \sin \pi v + \pi \sum_{k \geq 0} \frac{\cos \pi v}{\sin (k + \frac{1}{2}) \pi \tau} + \sin \pi v \]

\[ = \frac{1}{f_4} \frac{\partial f_4}{\partial v}(v, \tau) + \frac{1}{f_4} \frac{\partial f_4}{\partial v}(-v, \tau). \]
To state the convergence of $\frac{1}{f_4} \frac{\partial f_4}{\partial v}$ we will use the following ([4])

**Lemma 3-2** Let $(g_n)_n$ a sequence of holomorphic functions defined on $\Omega \in \mathbb{C}$ such that the series $\sum_n g_n$ converges on any compact set of $\Omega$. Notice $G$ the product of $(1 + g_n)$ on $\Omega$. Then the series of meromorphic functions $\sum_n \frac{\partial G}{\partial v}$ is normally convergent to $\frac{\partial G}{\partial v}$.

### 3.2. Link with other theta functions.

Recall the others functions $f_i(v, \tau)$, $i = 1, 2, 3$.

$$f_3(v, \tau) = \prod_{k \geq 0} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right],$$

$$f_1(v, \tau) = \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\sin k\pi \tau} \right)^2 \right],$$

$$f_2(v, \tau) = \prod_{k \geq 1} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos k\pi \tau} \right)^2 \right].$$

Notice also the following relations between them

$$f_3(v, \tau) = f_4(v, \tau + 1) = \prod_{k \geq 0} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right],$$

$$f_2(v, \tau) = f_1(v, \tau + 1) = \prod_{k \geq 1} \left[ 1 - \left( \frac{(-1)^k \sin \pi v}{\cos k\pi \tau} \right)^2 \right].$$

**Theorem 3-4** The other theta functions may be expressed

$$\theta_3(v, \tau) = \theta_3(0, \tau) f_3(v, \tau) f_3(v + 1, \tau),$$

$$\theta_1(v, \tau) = (\pi \sin(\pi v)) \theta'_1(0, \tau) f_1(v, \tau) f_1(v + 1, \tau),$$

$$\theta_2(v, \tau) = \cos(2\pi v) \theta_2(0, \tau) f_2(v, \tau) f_2(v + 1, \tau).$$

Moreover,

$$\frac{1}{f_j} \frac{\partial f_j}{\partial v}(v, \tau) = \frac{1}{f_j} \frac{\partial f_j}{\partial v}(v, \tau) + \frac{1}{f_j} \frac{\partial f_j}{\partial v}(-v, \tau), \quad j = 1, 2, 3.$$

Here $\theta_3$ is defined in the "strip" $|\text{Im} v| < \frac{1}{2} |\text{Im} \tau|$, while $\theta_1$ and $\theta_2$ is defined in the "strip" $|\text{Im} v| < |\text{Im} \tau|$.

We then deduce expression of Jacobi theta functions as infinite products

**Corollary 3-5** Let $q = e^{i\pi \tau}$, $|q| < 1$. The functions $\theta_j$, $j = 1, 2, 3, 4$ may also be expressed as infinite products

$$\theta_4(v, \tau) = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\cos 2\pi v - \cos(2k + 1)\pi \tau}{1 - \cos(2k + 1)\pi \tau},$$

$$\theta_3(v, \tau) = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\cos 2\pi v + \cos(2k + 1)\pi \tau}{1 + \cos(2k + 1)\pi \tau}.$$
\[
\frac{\theta_1(v, \tau)}{(\pi \sin \pi v)} \theta_1'(0, \tau) = \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\sin k\pi} \right)^2 \right] = \prod_{k \geq 1} \frac{\cos 2\pi v - \cos 2k\pi\tau}{1 - \cos 2k\pi\tau} \tag{3}
\]
\[
\frac{\theta_2(v, \tau)}{(\cos \pi v)} \theta_2'(0, \tau) = \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\cos k\pi} \right)^2 \right] = \prod_{k \geq 1} \frac{\cos 2\pi v + \cos 2k\pi\tau}{1 + \cos 2k\pi\tau} \tag{4}
\]

\(\theta_3\) and \(\theta_4\) is defined in the "strip" \(|Imv| < \frac{1}{2}Im\tau\), while \(\theta_1\) and \(\theta_2\) are defined in the "strip" \(|Imv| < Im\tau\).

**Proof** Indeed, we will only prove for \(\theta_4(v, \tau)\) the others will be deduced by the same way. Starting from (7) and notice that

\[
\log \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = -\sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{2p} \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 = -2\sum_{k \geq 0} \sum_{p \geq 1} \frac{1}{2p} (X_k)^{2p}
\]

where

\[X_k = \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)\]

On the other hand we may calculate

\[-2\sum_{p \geq 1} \frac{1}{2p} (X_k)^{2p} = \log(1 - (X_k)^2).\]

it follows that

\[
\log \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \sum_{k \geq 0} \log(1 - (X_k)^2) = \sum_{k \geq 0} \log \left( 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right)
\]

Therefore we obtain the infinite product of \(\theta_4\)

\[
\theta_4(v, \tau) = \theta_4(0, \tau) \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right].
\]

Similarly, the other expressions of theta functions \(\theta_j(v, \tau), j = 1, 2, 3\) are obtained by the same manner (details omitted).

**Remarks 3-5** Observe that we may derive

\[
\frac{\theta_3(1, \tau)}{\theta_3(0, \tau)} = \frac{\theta_3(0, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{1}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right],
\]

\[
\frac{\theta_3(\frac{1}{2}, \tau)}{\theta_3(0, \tau)} = \frac{\theta_3(0, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{1}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right].
\]

Thus,

\[
\prod_{k \geq 0} \left[ 1 - \left( \frac{1}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right] \left[ 1 - \left( \frac{1}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right] = 1.
\]

By the same way we have also

\[
\prod_{k \geq 1} \left[ 1 - \left( \frac{1}{\sin k\pi\tau} \right)^2 \right] \left[ 1 - \left( \frac{1}{\cos k\pi\tau} \right)^2 \right] = 1.
\]
On the other hands, since $\theta_2(v + \frac{1}{2}, \tau) = \theta_3(v, \tau)$, and $\theta_1(v + \frac{1}{2}, \tau) = \theta_2(v, \tau)$, we may derive

**Corollary 3-6** Let $q = e^{i\pi \tau}$ and $v \leq \frac{1}{2} \text{Im} \tau$ the infinite products satisfy the following relations

$$
\prod_{k \geq 0} \left[ 1 - \left( \frac{\cos \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right)^2 \right] = \frac{\theta_3(0, \tau)}{\theta_4(0, \tau)} \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos (k + \frac{1}{2}) \pi \tau} \right)^2 \right],
$$

$$
\prod_{k \geq 1} \left[ 1 - \left( \frac{\cos \pi v}{\sin k \pi \tau} \right)^2 \right] = \frac{\theta_2(0, \tau)}{\pi \theta_4'(0, \tau)} \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\cos k \pi \tau} \right)^2 \right].
$$

In particular for $v = 0$ we have

$$
\prod_{k \geq 0} \left[ 1 - \left( \frac{1}{\sin (k + \frac{1}{2}) \pi \tau} \right)^2 \right] = \frac{\theta_3(0, \tau)}{\theta_4(0, \tau)},
$$

$$
\prod_{k \geq 1} \left[ 1 - \left( \frac{1}{\sin k \pi \tau} \right)^2 \right] = \frac{\theta_2(0, \tau)}{\pi \theta_4'(0, \tau)}.
$$

With respect the quasi periodicity we have

$$
\theta_4(v, \tau) = -e^{i(2v+\pi)} \theta_4(v + \tau, \tau), \quad \theta_3(v, \tau) = e^{i(2v+\pi)} \theta_3(v + \tau, \tau)
$$

$$
\theta_1(v, \tau) = -e^{i(2v+\pi)} \theta_1(v + \tau, \tau), \quad \theta_2(v, \tau) = e^{i(2v+\pi)} \theta_2(v + \tau, \tau),
$$

which permits to derive

**Corollary 3-7** Let $q = e^{i\pi \tau}$ and $v \leq \frac{1}{2} \text{Im} \tau$ we have the relations

$$
\prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin (k + \frac{1}{2}) \pi \tau} \right)^2 \right] = -e^{i\pi (2v+\tau)} \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi (v + \tau)}{\sin (k + \frac{1}{2}) \pi \tau} \right)^2 \right],
$$

$$
\prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos (k + \frac{1}{2}) \pi \tau} \right)^2 \right] = e^{i\pi (2v+\tau)} \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi (v + \tau)}{\cos (k + \frac{1}{2}) \pi \tau} \right)^2 \right],
$$

$$
(\sin \pi v) \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin k \pi \tau} \right)^2 \right] = - (\sin \pi (v+\tau)) e^{i\pi (2v+\tau)} \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi (v + \tau)}{\sin k \pi \tau} \right)^2 \right],
$$

$$
(\cos \pi v) \prod_{k \geq 0} \left[ 1 - \left( \frac{\cos \pi v}{\cos k \pi \tau} \right)^2 \right] = (\cos \pi (v+\tau)) e^{i\pi (2v+\tau)} \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi (v + \tau)}{\cos k \pi \tau} \right)^2 \right].
$$

Another expression of theta functions as infinite products
Corollary 3-8 Let $q = e^{i\pi \tau}$, $|q| < 1$. The functions $\theta_j$, $j = 1, 2, 3, 4$ may also be expressed as

$$\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\sin[(k + \frac{1}{2})\pi \tau + \pi v] \sin[(k + \frac{1}{2})\pi \tau - \pi v]}{(\sin[(k + \frac{1}{2})\pi \tau])^2}$$

$$\frac{\theta_3(v, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \frac{\cos[(k + \frac{1}{2})\pi \tau + \pi v] \cos[(k + \frac{1}{2})\pi \tau - \pi v]}{(\cos[(k + \frac{1}{2})\pi \tau])^2}$$

$$\frac{\theta_1(v, \tau)}{(\pi \sin \pi v) \theta_1(0, \tau)} = \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\sin k\pi \tau} \right)^2 \right] = \prod_{k \geq 1} \frac{\sin[k\pi \tau + \pi v] \sin[k\pi \tau - \pi v]}{(\sin[k\pi \tau])^2}$$

$$\frac{\theta_2(v, \tau)}{(\cos \pi v) \theta_2(0, \tau)} = \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\cos k\pi \tau} \right)^2 \right] = \prod_{k \geq 1} \frac{\cos[k\pi \tau + \pi v] \cos[k\pi \tau - \pi v]}{(\cos[k\pi \tau])^2}$$

$\theta_3$ and $\theta_4$ is defined in the band $|Imv| < \frac{1}{2} Im\tau$, while $\theta_1$ and $\theta_2$ are defined in the band $|Imv| < Im\tau$.

From Corollary 3-5 we also deduce the following

Corollary 3-9 Let $q = e^{i\pi \tau}$, $|q| < 1$. The logarithmic derivative of functions $\theta_j$, $j = 1, 2, 3, 4$ with respect to $v$ may be expressed

$$\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = -\pi \sum_{k \geq 0} \frac{\sin 2\pi v}{(\sin(k + \frac{1}{2})\pi \tau)^2 - (\sin \pi v)^2}$$

$$\frac{1}{\theta_3} \frac{\partial \theta_3}{\partial v}(v, \tau) = \pi \sum_{k \geq 0} \frac{\sin 2\pi v}{(\cos(k + \frac{1}{2})\pi \tau)^2 - (\sin \pi v)^2}$$

$$\frac{1}{\theta_2} \frac{\partial \theta_2}{\partial v}(v, \tau) = -(\tan \pi v) + \pi \sum_{k \geq 1} \frac{\sin 2\pi v}{(\cos k\pi \tau)^2 - (\sin \pi v)^2}$$

$$\frac{1}{\theta_1} \frac{\partial \theta_1}{\partial v}(v, \tau) = (\cot \pi v) - \pi \sum_{k \geq 1} \frac{\sin 2\pi v}{(\sin k\pi \tau)^2 - (\sin \pi v)^2}$$

$\theta_3$ and $\theta_4$ is defined in the "strip" $|Imv| < \frac{1}{2} Im\tau$, while $\theta_1$ and $\theta_2$ is defined in the "strip" $|Imv| < Im\tau$.

Proof Indeed, by logarithmic differentiation (11) and (13) with respect to $v$ we obtain $\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}$ and $\frac{1}{\theta_1} \frac{\partial \theta_1}{\partial v}$. The others are deduced in incrementing $v$ by $\frac{1}{2}$

$$\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v + \frac{1}{2}, \tau) = \frac{1}{\theta_3} \frac{\partial \theta_3}{\partial v}(v, \tau), \quad \frac{1}{\theta_2} \frac{\partial \theta_2}{\partial v}(v + \frac{1}{2}, \tau) = \frac{1}{\theta_2} \frac{\partial \theta_2}{\partial v}(v, \tau).$$

For the convergence of $\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}$ we use again the above Lemma 3-2.

Thanks to this expression in infinite products of the Jacobi theta functions we are able to find again several known theta identities. In particular, the identities derived from the Landen transformation, [7, p. 18]
Corollary 3-10  

Let \( q = e^{i\pi \tau}, \quad |q| < 1. \) Then the followings identities hold

\[
\frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_3(v, \tau) \theta_4(v, \tau)}{\theta_3(0, \tau) \theta_4(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin 2\pi v}{\sin(2k + 1)\pi \tau} \right)^2 \right],
\]

\[
\frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_3(v, \tau) \theta_2(v, \tau)}{\theta_3(0, \tau) \theta_4(0, \tau)} = (\pi \sin \pi v) \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin 2\pi v}{\sin(2k + 1)\pi \tau} \right)^2 \right],
\]

\[
\frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_3(v - \frac{1}{4}, \tau) \theta_3(v + \frac{1}{4}, \tau)}{\theta_3(0, \tau) \theta_4(0, \tau)},
\]

\[
\frac{\theta_2(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_2(v - \frac{1}{4}, \tau) \theta_2(v + \frac{1}{4}, \tau)}{\theta_3(0, \tau) \theta_4(0, \tau)}.
\]

**Proof**

\[
\theta_4(v, \tau) = \theta_4(0, \tau) \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \right],
\]

\[
\theta_3(v, \tau) = \theta_3(0, \tau) \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right].
\]

We then deduce

\[
\frac{\theta_4(v, \tau) \theta_4(v, \tau)}{\theta_4(0, \tau) \theta_3(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \right] \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right].
\]

However,

\[
1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right] = 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 + 4 \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2
\]

\[
= 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 + 4 \left( \frac{\sin \pi v}{\sin(2k + 1)\pi \tau} \right)^2 \left( \frac{\sin \pi v}{\sin(2k + 1)\pi \tau} \right)^2
\]

\[
= 1 - \left( \frac{2(\sin \pi v)}{\sin(2k + 1)\pi \tau} \right)^2 + \left( \frac{2(\sin \pi v)}{\sin(2k + 1)\pi \tau} \right)^2
\]

\[
= 1 - \left( \frac{\sin \pi v}{\sin(2k + 1)\pi \tau} \right)^2.
\]

By the same way we may prove

\[
1 - \left( \frac{\sin \pi v}{\sin k \pi \tau} \right)^2 \left[ 1 - \left( \frac{\sin \pi v}{\cos k \pi \tau} \right)^2 \right] = 1 - \left( \frac{\sin \pi v}{\sin(2k \pi \tau)} \right)^2.
\]

Since

\[
\theta_4(0, 2\tau) = \sqrt{\theta_3(0, \tau) \theta_4(0, \tau)} = \frac{\theta_4'(0, \tau) \theta_2(0, \tau)}{2 \theta_4'(0, 2\tau)}
\]

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we then deduce
\[
\frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \theta_1(v, \tau) \theta_2(v, \tau) = (\pi \sin \pi v) \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin 2\pi v}{\sin 2k\pi \tau} \right)^2 \right].
\]

The following extends and generalizes the preceding Corollary 3-10

**Theorem 3-11** Let \( q = e^{i\pi \tau}, \quad |q| < 1. \) Then the following identities hold
\[
\frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_4(v, \tau) \theta_3(v, \tau)}{\theta_1(v, \tau) \theta_2(v, \tau)}, \quad \frac{\theta_4(2v, 2\tau)}{\theta_4(0, 2\tau)} = \frac{\theta_1(v, \tau) \theta_2(v, \tau)}{\theta_1(v, \tau) \theta_2(v, \tau)}
\]
\[
\frac{\theta_4(4v, 4\tau)}{\theta_4(0, 4\tau)} = \frac{\theta_4(v, \tau) \theta_4(v, \tau) \theta_4(\frac{\tau}{2} + v, \tau) \theta_4(\frac{\tau}{2} - v, \tau)}{\theta_1(v, \tau) \theta_3(v, \tau) \theta_3(\frac{\tau}{2} + v, \tau) \theta_3(\frac{\tau}{2} - v, \tau)},
\]
\[
\frac{\theta_4(4v, 4\tau)}{\theta_1(0, 4\tau)} = \frac{\theta_4(\frac{\tau}{2} + v, \tau) \theta_4(\frac{\tau}{2} + v, \tau) \theta_4(\frac{\tau}{2} - v, \tau) \theta_4(\frac{\tau}{2} - v, \tau)}{\theta_1(\frac{\tau}{4}, \tau)^4}.
\]

More generally, for any integer \( n \) we have
\[
\frac{\theta_4(2^{n+1}v, 2^{n+1}\tau)}{\theta_4(0, 2^{n+1}\tau)} = \frac{\theta_4(v, \tau) \prod_{p=0}^{n} \theta_3(2^{p}v, 2^{p}\tau)}{\theta_3(0, 2^{p}\tau)},
\]
\[
\frac{\theta_4(2^{n+1}v, 2^{n+1}\tau)}{\theta_1'(0, 2^{n+1}\tau)} = \frac{\theta_1(v, \tau) \prod_{p=0}^{n} \theta_3(2^{p}v, 2^{p}\tau)}{\theta_2(0, 2^{p}\tau)}.
\]

**Proof of Theorem 3-11** Consider the following trigonometric equalities
\[
\sin(2t + 2x) \sin(2t - 2x) = 4 \sin(t + x) \cos(t + x) \sin(t - x) \cos(t - x),
\]
\[
\sin(4t + 4x) \sin(4t - 4x) = 4 \sin(2t + 2x) \cos(2t + 2x) \sin(2t - 2x) \cos(2t - 2x)
\]
\[
= 16 \sin(t + x) \cos(t + x) \cos(t - x) \cos(2t + 2x) \cos(2t - 2x).
\]

Recall the expansions (Corollary 3-8)
\[
\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \frac{\sin((k + \frac{1}{2})\pi \tau + \pi v)}{(\sin((k + \frac{1}{2})\pi \tau))^2},
\]
\[
\frac{\theta_1(v, \tau)}{\pi \sin \pi v \theta'(0, \tau)} = \prod_{k \geq 0} \frac{\sin[k \pi \tau + \pi v]}{(\sin[k \pi \tau]^2}.
\]

Then we deduce
\[
\frac{\theta_4(4v, 4\tau)}{\theta_4(0, 4\tau)} (\sin 2(2k + 1)\pi \tau)^2 = 16 \frac{\theta_4(v, \tau) \theta_3(v, \tau) \theta_3(2v, 2\tau)}{\theta_4(0, \tau) \theta_3(0, \tau) \theta_3(0, 2\tau)}
\]
\[
\times (\sin(k + \frac{1}{2})\pi \tau)^2 (\cos(k + \frac{1}{2})\pi \tau)^2 (\sin(2k + 1)\pi \tau)^2,
\]
\[ \frac{\theta_4(4v, 4\tau)}{\theta_4'(0, 4\tau)} (\sin 4k\pi/\tau)^2 = 16 \frac{\theta_4(v, \tau) \theta_3(v, \tau) \theta_2(2v, 2\tau)}{\theta_4'(0, \tau) \theta_2(0, \tau) \theta_1(2, 2\tau)} (\sin k\pi/\tau)^2 (\cos 2k\pi/\tau)^2. \]

That means after simplifications

\[ \frac{\theta_4(4v, 4\tau)}{\theta_4'(0, 4\tau)} = \frac{\theta_4(v, \tau) \theta_3(v, \tau) \theta_2(2v, 2\tau)}{\theta_4'(0, \tau) \theta_3(0, \tau) \theta_1(0, 2\tau)} = \frac{\theta_4(v, \tau) \theta_3(v, \tau) \theta_2(4v, 2\tau)(\sin (\pi/2 - \theta_1(v, \tau))(\sin (\pi/2 + \theta_1(v, \tau))))}{\theta_4'(0, \tau) \theta_3(0, \tau) \theta_1(0, 2\tau)} \]

\[ \frac{\theta_4(8v, 8\tau)}{\theta_4'(0, 8\tau)} (\sin(8(k + 1/2)\pi/\tau))^2 = 4 \frac{\theta_4(4v, 4\tau)}{\theta_4'(0, 4\tau)} (\sin(4(k + 1/2)\pi/\tau))^2 \]

Thus

\[ \frac{\theta_4(8v, 8\tau)}{\theta_4'(0, 8\tau)} = \frac{\theta_4(4v, 4\tau) \theta_3(4v, 4\tau)}{\theta_4'(0, 4\tau) \theta_3(0, 4\tau)}. \]

On the other hand

\[ \frac{\theta_3(4v, 4\tau)}{\theta_3'(0, 4\tau)} = \frac{\theta_4(4v, 2\tau)(\sin (\pi/2 - \theta_1(4v, 2\tau))(\sin (\pi/2 + \theta_1(4v, 2\tau))))}{\theta_3'(0, \tau) \theta_3(0, \tau) \theta_3(2\tau, 2\tau)} \]

\[ \frac{\theta_3(8v, 8\tau)}{\theta_3'(0, 8\tau)} = \frac{\theta_4(8v, 2\tau)(\sin (\pi/2 - \theta_1(8v, 2\tau))(\sin (\pi/2 + \theta_1(8v, 2\tau))))}{\theta_3'(0, \tau) \theta_3(0, \tau) \theta_3(4\tau, 4\tau)} \]

By the same way we have

\[ \frac{\theta_2(8v, 8\tau)}{\theta_2'(0, 8\tau)} = \frac{\theta_4(8v, 2\tau)(\sin (\pi/2 - \theta_1(8v, 2\tau))(\sin (\pi/2 + \theta_1(8v, 2\tau))))}{\theta_2'(0, \tau) \theta_2(0, \tau) \theta_1(4\tau, 4\tau)}. \]

The last part of Theorem 3-11 may be proved by induction on \( n \).

3.3. Link with elliptic functions. Consider now the zeta function of Jacobi. It is defined by

\[ Z(z, k) = \frac{1}{2K} \log \theta_4(v, \tau), \]

where \( v = \frac{\tau}{2K} \) and \( K = 2 \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \) is the complete elliptic integral of the first kind and the modulus is such that \( 0 < k < 1 \).

We have

Corollary 3-12 The zeta function of Jacobi has the following form

\[ Z(z, k) = Z_1(z, k) - Z_2(z, k) = \frac{\pi}{2K} \sum_{k \geq 0} \frac{\sin(\pi 2v)}{\sin^2(\pi v) - \sin^2(\pi 2v)}.\]

\[ Z_1(z, k) = \frac{1}{f_4(v, \tau)} \frac{\partial f_4(-v, \tau)}{\partial u}(u, \tau), \quad Z_2(z, k) = \frac{1}{f_4} \frac{\partial f_4}{\partial u}(-u, \tau). \]
where \( v = \frac{1}{2K} \) satisfies \(| \sin \pi v \, | < | (\sin(\frac{1}{2}) \pi \tau) \| . \)
In particular, the logarithmic derivatives of theta functions can be written under the forms

\[
\frac{\theta_4'(v, \tau)}{\theta_4(v, \tau)} = 4\pi \sin(\pi 2v) \sum_{k \geq 0} q^{2k+1} \frac{1}{1 - 2q^{2k+1} \cos 2\pi v + q^{4k+2}}
\]

\[
\frac{\theta_3'(v, \tau)}{\theta_3(v, \tau)} = -4\pi \sin(\pi 2v) \sum_{k \geq 0} q^{2k+1} \frac{1}{1 + 2q^{2k+1} \cos 2\pi v + q^{4k+2}}
\]

\[
\frac{\theta_2'(v, \tau)}{\theta_2(v, \tau)} = -\tan(\pi v) - 4\pi \sin(\pi 2v) \sum_{k \geq 0} q^{2k+2} \frac{1}{1 + 2q^{2k+2} \cos 2\pi v + q^{4k+4}}
\]

\[
\frac{\theta_1'(v, \tau)}{\theta_1(v, \tau)} = \cot(\pi v) + 4\pi \sin(\pi 2v) \sum_{k \geq 0} q^{2k+2} \frac{q^{2k+2}}{1 + 2q^{2k+2} \cos 2\pi v + q^{4k+4}}.
\]

Moreover, the equations for \( \theta_1 \) and \( \theta_2 \) are valid in the strip \(| \text{Im}v | < \text{Im} \tau \), those for \( \theta_3 \) and \( \theta_4 \) are valid in the strip \(| \text{Im}v | < \frac{1}{2} \text{Im} \tau \).

Indeed,

\[
Z(z, k) = \frac{1}{2K} \frac{d}{dz} \log \theta_4(v, \tau) = \frac{\pi}{2K} \sin(\pi 2v) \sum_{k \geq 0} \sum_{p \geq 1} \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2}) \pi \tau} \right)^{2p}.
\]

Suppose the variable \( v \) satisfies

\[
| \frac{\sin \pi v}{(\sin(\frac{1}{2}) \pi \tau)} | < 1.
\]

We then obtain

\[
\left( \frac{\sin \pi v}{(\sin(k + \frac{1}{2}) \pi \tau)} \right)^2 \sum_{p \geq 0} \left( \frac{\sin \pi v}{(\sin(k + \frac{1}{2}) \pi \tau)} \right)^{2p} = \left( \frac{\sin \pi v}{(\sin(k + \frac{1}{2}) \pi \tau)} \right)^2 = \frac{(\sin \pi v)^2}{(\sin(k + \frac{1}{2}) \pi \tau)^2 - (\sin \pi v)^2}.
\]

Therefore, the result follows. The domain of convergence for these series may be extended to the strip \(| \text{Im}v | < \frac{1}{2} \text{Im} \tau \) (see for example [7 p.489]).

Notice that the zeta function of Jacobi also has a Fourier expansion

\[
Z(z, k) = \frac{2\pi}{K} \sum_{n \geq 1} q^n \frac{\sin n\pi z}{1 - q^{2n}}.
\]
3.4. Link with classical infinite products. From Corollaries 3-4 and 3-5 we find again by another way the classical infinite products for $\theta_j(v, \tau)$.

We then deduce the following classical expressions of theta functions as infinite products (see [9, p.485])

**Corollary 3-13**

\[
\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \left( \frac{1 - 2q^{2k+1} \cos 2\pi v + q^{4k+2}}{(1 - q^{2k+1})^2} \right)
\]

\[
\frac{\theta_4(v, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left( \frac{1 - 2q^{2k+1} \cos 2\pi v + q^{4k+2}}{1 + q^{2k+1}} \right)
\]

\[
\frac{\theta_4(v, \tau)}{\theta_1'(0, \tau)} = \sin \pi v \prod_{k \geq 1} \left( \frac{1 - 2q^{2k} \cos 2\pi v + q^{4k}}{(1 - q^{2k})^2} \right)
\]

\[
\frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} = \cos \pi v \prod_{k \geq 1} \left( \frac{1 + 2q^{2k} \cos 2\pi v + q^{4k}}{1 + q^{2k}} \right)
\]

**Proof** Starting with the classical expansions of the theta functions as infinite product [9]:

\[
\theta_4(v, \tau) = \prod_{k \geq 0} (1 - q^{2k})(1 - 2q^{2k+1} \cos 2\pi v + q^{4k+2})
\]

\[
\theta_3(v, \tau) = \prod_{k \geq 0} (1 - q^{2k})(1 + 2q^{2k+1} \cos 2\pi v + q^{4k+2})
\]

\[
\theta_1(v, \tau) = 2q^{1/4} \sin \pi v \prod_{k \geq 1} (1 - q^{2k})(1 - 2q^{2k} \cos 2\pi v + q^{4k})
\]

\[
\theta_2(v, \tau) = 2q^{1/4} \cos \pi v \prod_{k \geq 1} (1 - q^{2k})(1 + 2q^{2k} \cos 2\pi v + q^{4k}).
\]

In particular

\[
\theta_4(0, \tau) = \prod_{k \geq 0} (1 - q^{2k})(1 - q^{2k+1})^2, \quad \theta_3(0, \tau) = \prod_{k \geq 0} (1 - q^{2k})(1 + q^{2k+1})^2
\]

\[
\theta_2(0, \tau) = 2q^{1/4} \prod_{k \geq 1} (1 - q^{2k})(1 + q^{2k})^2, \quad \theta_1'(0, \tau) = 2\pi q^{1/4} \prod_{k \geq 1} (1 - q^{2k})^3.
\]

Thus, since $q = e^{i\pi \tau}$ Corollary 4 yields

\[
\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \left( \frac{1 - 2q^{2k+1} \cos 2\pi v + q^{4k+2}}{(1 - q^{2k+1})^2} \right)
\]

\[
\frac{\theta_3(v, \tau)}{\theta_3(0, \tau)} = \prod_{k \geq 0} \left[ 1 - \left( \frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi \tau} \right)^2 \right] = \prod_{k \geq 0} \left( \frac{1 + 2q^{2k+1} \cos 2\pi v + q^{4k+2}}{(1 + q^{2k+1})^2} \right)
\]

\[
\frac{\theta_1(v, \tau)}{\theta_1'(0, \tau)} = (\pi \sin \pi v) \theta_1'(0, \tau) \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\sin k \pi \tau} \right)^2 \right] = \prod_{k \geq 1} \left( \frac{1 - 2q^{2k} \cos 2\pi v + q^{4k}}{(1 - q^{2k})^2} \right)
\]
\[
\frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} = (\cos \pi v) \theta_2(0, \tau) \prod_{k \geq 1} \left[ 1 - \left( \frac{\sin \pi v}{\cos k\pi \tau} \right)^2 \right] = \prod_{k \geq 1} \left( \frac{(1 + 2q^{2k} \cos 2\pi v + q^{4k})}{(1 + q^{2k})^2} \right).
\]

\[
\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = \prod_{k \geq 0} \frac{(q^{k + \frac{1}{2}})^2 (q^{-2k-1} - 2 \cos 2\pi v + q^{2k+1})}{(1 - q^{2k+1})^2} = 2 \prod_{k \geq 0} \frac{q^{-2k-1} - 2 \cos 2\pi v + q^{2k+1}}{(\sin (k + \frac{1}{2})\pi \tau)^2} = 4 \prod_{k \geq 0} \frac{- \cos (2k + 1)\pi \tau + \cos 2\pi v}{(\sin (k + \frac{1}{2})\pi \tau)^2} = \prod_{k \geq 0} \frac{(\sin (k + \frac{1}{2})\pi \tau)^2 - (\sin \pi v)^2}{(\sin (k + \frac{1}{2})\pi \tau)^2}.
\]

Repetition of this calculation (details omitted) for \(\theta_3(v, \tau), \theta_2(v, \tau), \theta_1(v, \tau)\) yields the other expressions.

References


