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Abstract

We consider spin-flip dynamics of configurations in $\{-1,1\}^{\mathbb{Z}^d}$, and study the time evolution of concentration inequalities. For "weakly interacting" dynamics we show that the Gaussian concentration bound is conserved in the course of time and it is satisfied by the unique stationary Gibbs measure. Next we show that, for a general class of translation-invariant spin-flip dynamics, it is impossible to evolve in finite time from a low-temperature Gibbs state towards a measure satisfying the Gaussian concentration bound. Finally, we consider the time evolution of the weaker uniform variance bound, and show that this bound is conserved under a general class of spin-flip dynamics.

Keywords and phrases: concentration inequalities, spin-flip dynamics, relative entropy, space-time cluster expansion, analytic vectors, Gaussian concentration bound, uniform variance bound.

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1 Introduction

Concentration inequalities are important tools to understand the fluctuation properties of general observables $f(\sigma_1, \ldots, \sigma_n)$ of n random variables $(\sigma_1, \ldots, \sigma_n)$, where n is large but finite. For bounded random variables which are independent (or weakly dependent) typically one can obtain so-called Gaussian concentration bounds for the fluctuations of $f(\sigma_1, \ldots, \sigma_n)$ about its expectation. In the context of lattice spin systems, one has, e.g., $\sigma_i \in \{-1, +1\}$, with $i \in [-n, n]^d \cap \mathbb{Z}^d$, and these random variables are distributed according to a Gibbs measure. The "weak dependence" between them means for instance that we are in the Dobrushin uniqueness regime, which is for instance the case at "high enough" temperature for every finite-range potential, or for low temperature with a "high enough" external magnetic field. In this case a Gaussian concentration bound holds [9]. In contrast, regimes of non-uniqueness are known in which weaker concentration bounds, such as moment bounds, hold [5]. In [6] it is shown that the Gaussian concentration bound implies uniqueness of equilibrium states (translation-invariant Gibbs measures). In [2], many applications of these concentration bounds are given (speed of convergence of the empirical measure in the sense of Kantorovich distance, fluctuation bounds in

the Shannon-McMillan-Breiman theorem, fluctuation bounds for the first occurrence of a pattern, etc).

In this paper we are interested in the time evolution of the Gaussian concentration bound under a stochastic evolution. More precisely we study the following questions in the context of spin-flip dynamics of lattice spin systems:

1. When started from a probability measure satisfying the Gaussian concentration bound, do we have this bound at later times?

2. When started from a probability measure which does not satisfy the Gaussian concentration bound, can this bound be obtained at finite times?

This is motivated on one hand by the phenomenon of Gibbs-non-Gibbs transitions [7], and on the other by previous study of similar questions in the context of finite-dimensional diffusion processes [4].

In the context of time evolution of Gibbs measures, one has generically two scenarios. In the high-temperature regime, i.e., high-temperature initial Gibbs measure, and high-temperature dynamics, the time evolved measure is generically high-temperature Gibbs, and results of this type are proved via some form of high-temperature (cluster, polymer) expansion, see [7], [12]. In the regime where the dynamics is high-temperature and the initial measure is low-temperature, one typically has Gibbs-non-Gibbs transitions, i.e., after a finite time the time-evolved measure is no longer a Gibbs measure, and sometimes (e.g. for independent spin-flip dynamics starting from a low-temperature Ising state with positive small magnetic field) the measure can become Gibbs again. In the context of time-evolution of concentration inequalities, in [4] results are restricted to dynamics of diffusive type, in finite dimensional context. Here we are interested in the setting of translation-invariant spin-flip dynamics in infinite volume, which is precisely the context of Gibbs-non-Gibbs transitions in [7]. Guided by the intuition coming from this context, one expects that a high-temperature dynamics should conserve the Gaussian concentration bound. We prove this result in the present paper, using the expansion in [12], i.e., under the condition that the flip rates are sufficiently close to the rates of an independent spin-flip dynamics. Next we show that whenever one starts from a low-temperature initial state, i.e., in the non-uniqueness regime, then for any local spin-flip dynamics, at any later time the distribution cannot satisfy the Gaussian concentration bound. This can be thought of as a result showing that in finite time one cannot obtain "high-temperature properties" when initially started from a "low-temperature state". This result is shown via an analyticity argument, which shows that two different initial measures can never coincide in finite time, together with the fact that if a measure satisfies the Gaussian concentration bound, then its lower relative entropy density with respect to any

other translation-invariant measure is strictly positive. I.e., the existence of two time evolved measures with zero relative entropy density excludes the possibility that one of them satisfies the Gaussian concentration bound.

The rest of our paper is organized as follows. We start with introducing the basic context on Gibbs measures and spin-flip dynamics in Section 2. In Sections 3 and 4 we treat the weakly interacting dynamics. In Section 5 we prove that the Gaussian concentration bound cannot be obtained in finite time if one starts from an initial Gibbs measure in a non-uniqueness ("low-temperature") regime. In this section we also prove a non-degeneracy result, based on analyticity, which is of independent interest.

2 Setting: lattice spin systems, Gibbs measures, Markovian dynamics

We consider the state space of Ising spins on the lattice $\mathbb{Z}^d$, i.e., $\Omega = \{-1,1\}^{\mathbb{Z}^d}$. For elements $\sigma \in \Omega$, called "spin-configurations", we denote $\sigma_i \in \{-1,1\}$ the value of the spin at lattice site $i \in \mathbb{Z}^d$. When we say "a probability measure μ on Ω", we mean a probability measure on the Borel-σ-field of Ω, equipped with the standard product of discrete topologies, which makes Ω into a compact metric space. For $\eta \in \Omega$ we denote $\tau_i\eta$ the shifted or translated configuration is defined via $(\tau_i\eta)_j = \eta_{i+j}$. A function $f : \Omega \to \mathbb{R}$ is called local if it depends only on a finite number of coordinates. By the Stone-Weierstrass theorem, the set of local functions is dense in the Banach space of continuous functions $\mathscr{C}(\Omega)$, equipped with the supremum norm. For $f : \Omega \to \mathbb{R}$ we denote $\tau_i f$ the function defined via $\tau_i f(\eta) = f(\tau_i\eta)$.

For a function $f : \Omega \to \mathbb{R}$ we denote the discrete gradient

$$\nabla_i f(\sigma) = f(\sigma^i) - f(\sigma)$$

where σ^i denotes the configuration obtained from σ by flipping the symbol at lattice site $i \in \mathbb{Z}^d$. We further denote

$$\delta_i f = \sup_{\sigma\in\Omega} \nabla_i f(\sigma).$$

We think of $\delta_i f$ as "the Lipschitz constant in the coordinate σ_i". The symbol δf means the collection of $\delta_i f, i \in \mathbb{Z}^d$, i.e., the "vector" of Lipschitz constants. For $p \geq 1$ we define

$$\|\delta f\|_p = \left(\sum_{i\in\mathbb{Z}^d} (\delta_i f)^p\right)^{\frac{1}{p}}.$$

For a continuous function $f : \Omega \to \mathbb{R}$ and a probability measure μ on Ω, we will write either $\mathbb{E}_\mu(f)$ or $\int f \,\mathrm{d}\mu$ for the integral of f with respect to μ. We can now define what we mean by a Gaussian concentration for a given probability measure on Ω.

Definition 2.1 (Gaussian Concentration Bound)**.**
A probability measure μ on Ω is said to satisfy the Gaussian concentration bound with constant $C > 0$, abbreviated GCB(C)*, if for all continuous $f : \Omega \to \mathbb{R}$ we have*

$$\mathbb{E}_\mu\left(\mathrm{e}^{f-\mathbb{E}_\mu(f)}\right) \leq \mathrm{e}^{C\|\delta f\|_2^2}. \tag{1}$$

Observe that $\|\delta f\|_2$ is always finite for local functions. Note that a function $f : \Omega \to \mathbb{R}$ is local if and only if there exists a finite subset of $\mathbb{Z}^d$ (depending of course on f) such that $\delta_i f = 0$ for all i outside of that subset. For non-local continuous functions, inequality (1) is meaningful only when $\|\delta f\|_2 < +\infty$. By a standard argument (exponential Chebyshev inequality applied to λf, $\lambda > 0$, and then optimization over λ), the bound (1) implies the "sub-gaussian" concentration inequality

$$\mu\big(f - \mathbb{E}_\mu(f) \geq u\big) \leq \mathrm{e}^{-\frac{u^2}{4C\|\delta f\|_2^2}}$$

for all $u > 0$.

2.1 Gibbs measures

In the context of Gibbs measures, the Gaussian concentration bound is satisfied in the so-called high-temperature regime, and more generally in regimes where the unique Gibbs measure is sufficiently close to a product measure such as the Dobrushin uniqueness regime. In this subsection we provide some basic background material on Gibbs measures which we need in the sequel. We refer to [8] for more details and further background. Let $\mathscr{S}$ denote the set of finite subsets of $\mathbb{Z}^d$. For $\Lambda \subset \mathbb{Z}^d$, we denote by $\mathscr{F}_\Lambda$ the σ-field generated by $\{\sigma_i, i \in \Lambda\}$.

Definition 2.2. *A uniformly absolutely summable potential is a map $U : \mathscr{S} \times \Omega \to \mathbb{R}$ with the following properties:*

1. *$U(A, \cdot)$ only depends on $\sigma_i, i \in A$.*
2. *Uniform absolute summability:*

$$\sup_{i \in \mathbb{Z}^d} \sum_{\substack{A \in \mathscr{S} \\ A \ni i}} \sup_{\sigma \in \Omega} |U(A, \sigma)| < +\infty.$$

A potential is called translation invariant if $U(A + i, \sigma) = U(A, \tau_i \sigma)$ for all $A \in \mathscr{S}, \sigma \in \Omega$, $i \in \mathbb{Z}^d$.

Given a uniformly absolutely summable potential U, and $\Lambda \in \mathscr{S}$, we denote the finite-volume Hamiltonian with boundary condition $\eta \in \Omega$:

$$H_\Lambda^\eta(\sigma_\Lambda) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \sigma_\Lambda \eta_{\Lambda^c})$$

and the corresponding finite-volume Gibbs measure with boundary condition η

$$\mu_\Lambda^\eta(\sigma_\Lambda) = \frac{\mathrm{e}^{-H_\Lambda^\eta(\sigma_\Lambda)}}{Z_\Lambda^\eta}$$

where $Z_\Lambda^\eta = \sum_{\sigma_\Lambda \in \Omega_\Lambda} \mathrm{e}^{-H_\Lambda^\eta(\sigma_\Lambda)}$, the partition function with boundary condition η, is the normalizing constant (and where Ω_Λ is the restriction of Ω to Λ).

Definition 2.3. *Let U be a uniformly absolutely summable potential. A measure μ is called a Gibbs measure with potential U if its conditional probabilities satisfy*

$$\mu\big(\sigma_\Lambda | \mathscr{F}_{\Lambda^c}\big)(\eta) = \mu_\Lambda^\eta(\sigma_\Lambda)$$

for all $\Lambda \in \mathscr{S}$, for all σ, and for μ-almost every η. We will write $\mu \in \mathscr{G}(U)$ to mean that μ is a Gibbs measure for U.

We say that U satisfies the strong uniqueness condition if

$$c(U) := \sup_{i \in \mathbb{Z}^d} \frac{1}{2} \sum_{A \ni i} (|A| - 1) \sup_{\sigma, \eta \in \Omega} |U(A, \sigma) - U(A, \eta)| < 1. \tag{2}$$

If U satisfies (2) then the set of Gibbs measures $\mathscr{G}(U)$ is a singleton (unique Gibbs measure, no phase transition). The condition (2) implies the well-known Dobrushin uniqueness condition (cf. [8] chapter 8).

If U is translation invariant then $\mathscr{G}(U)$ contains at least one translation-invariant Gibbs measure.

The following result is a particular case of the main theorem in [9] which states that, under the Dobrushin uniqueness condition, one has the Gaussian concentration bound (1).

Theorem 2.1 ([9])**.** *If U satisfies (2) then $\mu \in \mathscr{G}(U)$ satisfies* GCB(C) *with $C = \frac{1}{2(1-c(U))^2}$.*

From the proof, one easily infers that also all the finite-volume Gibbs measures μ_Λ^η satisfy GCB(C) whenever U satisfies (2), with a constant C that neither depend on the boundary condition η nor on the volume Λ.

2.2 Relative entropy density and large deviations

Translation-invariant Gibbs measures with a translation-invariant uniformly absolutely summable potential satisfy a level-3 large deviation principle with the relative entropy density as rate function [8, Chapter 15]. Let U be a translation-invariant uniformly absolutely summable potential, and $\mu \in \mathscr{G}(U)$ be a translation-invariant Gibbs measure. Let ν be a translation-invariant probability measure on Ω. The relative entropy density is defined to be the limit

$$h(\nu|\mu) = \lim_{n \to \infty} \frac{h_{\Lambda_n}(\nu|\mu)}{|\Lambda_n|} \tag{3}$$

with $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$, $|\Lambda_n| = (2n+1)^d$, and

$$h_{\Lambda_n}(\nu|\mu) = \sum_{\sigma_{\Lambda_n} \in \Omega_{\Lambda_n}} \nu(\sigma_{\Lambda_n}) \log \frac{\nu(\sigma_{\Lambda_n})}{\mu(\sigma_{\Lambda_n})}.$$

The relative entropy density exists for any $\mu \in \mathscr{G}(U)$ translation-invariant Gibbs measure, and ν any translation-invariant probability measure. Moreover, the relative entropy density is the rate function of the so-called level 3 large deviation principle, i.e., in the sense of the large deviation principle, it holds that

$$\mu\left(\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n} \delta_{\tau_i\sigma} \approx \nu\right) \asymp \mathrm{e}^{-|\Lambda_n|\, h(\nu|\mu)}\,.$$

(This is of course an informal statement where "$\approx \nu$" means a neighborhood of ν in weak topology, and "$\asymp$" means asymptotic equivalence after taking the logarithm and dividing out by $|\Lambda_n|$.) In general, i.e., if μ is not a Gibbs measure, the limit defining (3) might not exist, in that case we define the lower relative entropy density as

$$h_*(\nu|\mu) = \liminf_{n\to\infty} \frac{h_{\Lambda_n}(\nu|\mu)}{|\Lambda_n|}.$$

The following elementary lemma, which we formulate in the context of a finite set, with a Markov transition matrix, shows that the relative entropy is decreasing under the action of a Markov kernel.

Lemma 2.1. *Let $P(x,y)$ be a Markov transition function on a finite set S, $x, y \in S$, i.e., $P(x,y) \geq 0, \sum_{y\in S} P(x,y) = 1$ for all $x \in S$. Let μ, ν be two probability measures on S and let*

$$H(\mu|\nu) = \sum_{x\in S} \mu(x) \log \frac{\mu(x)}{\nu(x)}$$

denote their relative entropy. Define $\mu P(y) = \sum_{x\in S} \mu(x) P(x,y)$ and similarly νP. Then we have

$$H(\mu P|\nu P) \leq H(\mu|\nu).$$

Proof. Define $\mu_{12}(x,y) = \mu(x)P(x,y)$ and similarly $\nu_{12}(x,y) = \nu(x)P(x,y)$. These define two joint distributions of a random variable (X,Y) on $S \times S$. Then the first marginals of μ_{12}, ν_{12} are μ, resp. ν, and the second marginals are μP, resp. νP. Moreover, because $\sum_{y\in S} P(x,y) = 1$, we get

$$\begin{aligned} H(\mu_{12}|\nu_{12}) &= \sum_{(x,y)\in S\times S} \mu(x)P(x,y) \log \frac{\mu(x)P(x,y)}{\nu(x)P(x,y)} = \sum_{x\in S} \mu(x) \log \frac{\mu(x)}{\nu(x)} \\ &= H(\mu|\nu). \end{aligned}$$

Therefore, by the chain rule for relative entropy (see e.g. Lemma 4.18 in [10]) we obtain

$$H(\mu|\nu) = H(\mu P|\nu P) + D$$

where D is the conditional divergence of X "knowing" Y, i.e.,

$$D = \sum_{y\in S} \mu P(y) \sum_{x\in S} \mu_{12}(x|y) \log \frac{\mu_{12}(x|y)}{\nu_{12}(x|y)}.$$

Because D is non-negative, we obtain the desired inequality. □

2.3 Dynamics: definitions and basic inequalities

2.3.1 Dynamics and generator

The basic question we are interested in is how the inequality GCB(C) is affected by applying a Markovian dynamics to the probability measure μ. For this dynamics, we consider spin-flip dynamics with flip rates $c(i,\sigma)$ at site $i \in \mathbb{Z}^d$ satisfying the following assumptions.

Condition A:

1. Strict positivity: $\inf_{i\in\mathbb{Z}^d,\,\sigma\in\Omega} c(i,\sigma) > 0$.
2. Locality:
$$\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \sup_{\sigma\in\Omega} \big(c(i,\sigma^j) - c(i,\sigma)\big) < +\infty.$$

This condition ensures existence of the dynamics with generator L defined below in (4).

In section 3 we will consider weakly interacting dynamics and need more stringent conditions:

Condition C:

1. Strict positivity: $\inf_{i\in\mathbb{Z}^d,\,\sigma\in\Omega} c(i,\sigma) > 0$.
2. Finite-range property: There exists $R > 0$ such that $c(i,\sigma)$ depends only on σ_j, for j such that $|j-i| \le R$.

If $c(i,\sigma) = c(0,\tau_i\sigma)$, $\sigma \in \Omega$, $i \in \mathbb{Z}^d$, then we say that the flip rates are translation invariant where we remind the notation $(\tau_i\sigma)_j = \sigma_{i+j}$.

The dynamics is defined via the Markov pre-generator L acting on local functions via

$$Lf(\sigma) = \sum_{i\in\mathbb{Z}^d} c(i,\sigma)\big(f(\sigma^i) - f(\sigma)\big). \tag{4}$$

As proved in [11, Chapter 1], under Condition A, the closure of L (in $\mathscr{C}(\Omega)$ equipped with the supremum norm) generates a unique Feller process. This process generated by L is denoted $\{\sigma(t), t \geq 0\}$, and $\sigma_i(t)$ denotes the spin at time t at lattice site i. We denote $\mathbb{E}_\sigma$ expectation in the process $\{\sigma(t), t \geq 0\}$ starting from σ, and $\mathbb{P}_\sigma$ the corresponding path-space measure. We denote the semigroup $S(t)f(\sigma) = \mathbb{E}_\sigma[f(\sigma(t))]$, which acts as a Markov semigroup of contractions on $\mathscr{C}(\Omega)$. Via duality, $S(t)$ acts on probability measures, and for μ a probability measure on Ω, we denote by $\mu S(t)$ the time-evolved measure, determined by the equation

$$\int f \, \mathrm{d}\mu S(t) = \int S(t) f \, \mathrm{d}\mu.$$

We also introduce the non-linear semigroup $V(t)f = \log S(t)\, \mathrm{e}^f$, which is a family of non-linear operators satisfying the semigroup property, i.e., $V(t+s) = V(t)V(s)$, $s, t \geq 0$. This non-linear semigroup appears naturally in the context of time-evolution of the Gaussian concentration bound.

Finally, notice that

$$\left(\mathrm{e}^{V(t)f}\right)(\sigma) = \int \mathrm{e}^{f(\xi)} \, \delta_\sigma S(t)(\mathrm{d}\xi) \tag{5}$$

whereas

$$S(t)f(\sigma) = \int f(\xi) \, \delta_\sigma S(t)(\mathrm{d}\xi). \tag{6}$$

2.3.2 Some basic facts for spin-flip dynamics

In the study of existence and ergodicity properties of the Markovian dynamics $\{\sigma(t) : t \geq 0\}$ an important role is played by the matrix indexed by sites $i, j \in \mathbb{Z}^d$ and defined by

$$\Gamma_{ij} = \sup_{\sigma \in \Omega} \big(c(i, \sigma^j) - c(i, \sigma)\big).$$

We have the pointwise estimate (see [11, Chapter 1])

$$\delta_i S(t) f \leq (\mathrm{e}^{t\Gamma} \, \delta f)_i, \; i \in \mathbb{Z}^d, \; t \geq 0$$

where $\mathrm{e}^{t\Gamma} \, \delta f$ denotes the bounded operator (in $\ell^1(\mathbb{Z}^d)$) $\mathrm{e}^{t\Gamma}$ working on the "column vector" δf. If the rates are translation invariant, i.e., then we have $\Gamma_{ij} = \gamma(j-i)$, i.e., Γ acts as a convolution operator:

$$(\Gamma \delta f)_i = (\gamma * \delta f)_i = \sum_{j \in \mathbb{Z}^d} \gamma(i-j) \, \delta_j f$$

and as a consequence

$$(\mathrm{e}^{t\Gamma} \, \delta f)_i = \sum_{j \in \mathbb{Z}^d} \gamma_t(i-j) \, \delta_j f.$$

The so-called uniform ergodic regime, or "$M < \varepsilon$ regime" (see [11]), is the regime where the dynamics admits a unique invariant measure to which every initial measure converges exponentially fast in the course of time. In that case there exists $\alpha > 0$ such that

$$\|\delta S(t) f\|_2^2 \leq \| \mathrm{e}^{t\Gamma}\, \delta f\|_2^2 \leq \mathrm{e}^{-\alpha t}\, \|\delta f\|_2^2 \tag{7}$$

see [3, Theorem 3.3]. In general, for a spin-flip dynamics generated by (4), we have that Γ is a bounded operator in $\ell^2(\mathbb{Z}^d)$, i.e.,

$$\|\delta S(t) f\|_2^2 \leq K(t) \|\delta f\|_2^2 \tag{8}$$

for some time dependent constant $K(t) > 0$. Finally, we mention a useful fact about the relative entropy density. Using the elementary Lemma 2.1, and finite-volume approximations, one obtains the following implication for a translation invariant spin-flip dynamics with rates satisfying condition A

$$h(\nu|\mu) = 0 \quad \Rightarrow \quad h\big(\nu S(t) \big|\, \mu S(t)\big) = 0,\ \forall t > 0.$$

This will be used later on, in Section 4.

3 Time evolution of the Gaussian concentration bound

In this section we show conservation of the Gaussian concentration bound under weakly interacting spin-flip dynamics, i.e., dynamics sufficiently close to independent spin-flip dynamics.

More precisely if we start the process $\{\sigma(t) : t \geq 0\}$ from a probability measure μ satisfying GCB(C), then we are interested in the following questions:

1. Is it the case that under the time evolution $\{\sigma(t), t \geq 0\}$, the time-evolved measure $\mu S(t)$ still satisfies GCB(C_t), and if yes, how does the constant C_t evolve?

2. If the dynamics admits a unique stationary measure ν, does this measure satisfy GCB(C)?

3.1 A general result and conservation of GCB for independent dynamics

We start with the following general result.

Theorem 3.1. *Let $\{\sigma(t), t \geq 0\}$ be such that for all $\sigma \in \Omega$ the probability measure $\delta_\sigma S(t)$ satisfies* GCB(D_t) *where the constant D_t does not depend*

on σ. Let μ be a probability measure satisfying $\mathrm{GCB}(C_\mu)$*. Then, for all local functions f we have*

$$\log \int \mathrm{e}^{f-\int f \,\mathrm{d}\mu S(t)} \,\mathrm{d}\mu S(t) \le D_t \|\delta f\|_2^2 + C_\mu \|\delta(S(t)f)\|_2^2. \tag{9}$$

As a consequence, we obtain the following results:

1. *$\mu S(t)$ satisfies* $\mathrm{GCB}(C(\mu,t))$ *with $C(\mu,t) \le D_t + K(t)C_\mu$, where $K(t)$ is defined in (8).*

2. *In the uniformly ergodic case ($M < \varepsilon$ regime, cf. (7)), there exists $\alpha > 0$ such that $\mu S(t)$ satisfies* $\mathrm{GCB}(C(\mu,t))$ *with*

$$C(\mu,t) \le D_t + C_\mu \,\mathrm{e}^{-\alpha t}\,.$$

If furthermore, $\sup_t D_t < \infty$, then also the unique stationary measure ν satisfies $\mathrm{GCB}(C_\nu)$ *with $C_\nu \le \sup_t D_t < +\infty$.*

Proof. Start from the left-hand side of (9). Use that (5), (6) to rewrite

$$\begin{aligned}
&\int \mathrm{e}^{f-\int f \,\mathrm{d}\mu S(t)} \,\mathrm{d}\mu S(t) \\
&= \left(\int \left(S(t)\,\mathrm{e}^{f} \right)(\sigma) \,\mathrm{d}\mu(\sigma) \right) \mathrm{e}^{-\int f \,\mathrm{d}\mu S(t)} \\
&= \int \left[\left(\int \mathrm{e}^{f(\xi)-\int f(\zeta)\,\delta_\sigma S(t)(d\zeta)} \,\delta_\sigma S(t)(\mathrm{d}\xi) \right) \mathrm{e}^{S(t)f(\sigma)-\int S(t)f(\zeta)\,\mathrm{d}\mu(\zeta)} \right] \mathrm{d}\mu(\sigma) \\
&\le \mathrm{e}^{D_t\|\delta f\|_2^2} \int \mathrm{e}^{S(t)f(\sigma)-\int S(t)f(\zeta)\,\mathrm{d}\mu(\zeta)} \,\mathrm{d}\mu(\sigma) \\
&\le \mathrm{e}^{D_t\|\delta f\|_2^2}\, \mathrm{e}^{C_\mu\|\delta S(t)f\|_2^2}\,.
\end{aligned}$$

In the two last steps we first used that $\delta_\sigma S(t)$ satisfies $\mathrm{GCB}(D_t)$, i.e., we have the inequality

$$\int \mathrm{e}^{f(\xi)-\int f(\xi)\,\delta_\sigma S(t)(\mathrm{d}\xi)} \,\delta_\sigma S(t)(\mathrm{d}\xi) \le \mathrm{e}^{D_t\|\delta f\|_2^2}$$

for all σ. Second, we used the fact that μ satisfies $\mathrm{GCB}(C_\mu)$. The consequences (1) and (2) now follow immediately. □

The following corollary shows that for independent spin-flip dynamics, Gaussian concentration is conserved.

Corollary 3.1. *Assume that in the process $\{\sigma(t), t \ge 0\}$ the coordinates $\{\sigma_i(t) : t \ge 0\}$ evolve independently. If μ satisfies* $\mathrm{GCB}(C_\mu)$*, then there exists $\alpha > 0$ such that at any later time, $\mu S(t)$ satisfies* $\mathrm{GCB}(C(\mu,t))$*, with*

$$C(\mu,t) = \mathrm{e}^{-\alpha t}\, C_\mu + D_t \tag{10}$$

with $\sup_t D_t < +\infty$.

Proof. First notice that if $\mathbb{P}$ is a product measure on $\{-1,1\}^{\mathbb{Z}^d}$ then $\mathbb{P}$ satisfies GCB(C) with a constant C that is not depending on the marginal distributions, see [1]. For independent spin-flip dynamics, $\delta_\sigma S(t)$ is a product measure. Therefore, for that case, the assumption of Theorem 3.1 is satisfied, with D_t uniformly bounded as a function of t. Furthermore, because the flip rates are assumed to be bounded from below, the process $\{\sigma(t), t \geq 0\}$ is uniformly ergodic, and as a consequence we obtain (10). □

3.2 Weakly interacting spin-flip dynamics

The result for independent spin-flip dynamics (i.e., Corollary 3.1) can be generalized to a setting of weakly interacting dynamics, which was studied before in [12] in the context of time-evolution of Gibbs measures. The setting is such that the rates are sufficiently close to the rates of independent rate 1 spin-flip dynamics, such that a space-time cluster expansion can be set up. In particular, these conditions imply that there exists a unique invariant measure which is a Gibbs measure in the Dobrushin uniqueness regime.

More precisely, the assumptions on the rates are those of condition C, with one extra assumption forcing the rates to be close to a constant:

$$c(i,\sigma) = 1 + \varepsilon(i,\sigma), \text{ with } \sup_{\sigma\in\Omega} |\varepsilon(i,\sigma)| < \varepsilon_0 \tag{11}$$

where $\varepsilon_0 \in (0,1)$ is a constant depending on the dimension, specified in [12].

The important implication of the space-time cluster expansion developed in [12] which we need in our context is the following. The measure $\delta_\sigma S(t)$ is a Gibbs measure which is in the Dobrushin uniqueness regime, uniformly in $t > 0$ and σ. More precisely, $\delta_\sigma S(t)$ is a Gibbs measure with uniformly absolutely summable potential U^t_σ satisfying

$$\sup_{i\in\mathbb{Z}^d} \sum_{\substack{A\in\mathscr{S}\\ A\ni i}} |A| \sup_{\sigma,\eta\,\in\,\Omega, t\geq 0} \left|U^t_\sigma(A,\eta)\right| < 1. \tag{12}$$

More precisely, in [12] an exponential norm

$$\sup_{i\in\mathbb{Z}^d} \sum_{\substack{A\in\mathscr{S}\\ A\ni i}} \mathrm{e}^{a|A|} \sup_{\sigma,\eta\,\in\,\Omega, t\geq 0} \left|U^t_\sigma(A,\eta)\right|$$

where $a > 0$ is small enough, is shown to be finite, and going to zero when $\varepsilon_0 \to 0$, which is stronger than (12).

Using Theorem 3.1, combined with Theorem 2.1, we obtain the following result.

Theorem 3.2. *Let $\{\sigma(t), t \geq 0\}$ be a spin-flip dynamics satisfying the conditions C, and the extra weak interaction condition (11). Then we have*

1. *If μ satisfies* GCB(C_μ)*, then there exists $C(\mu, t) < \infty$ such that $\mu S(t)$ satisfies* GCB($C(\mu, t)$)*.*

2. *The unique stationary measure ν satisfies* GCB(C_ν) *for some $C_\nu < \infty$.*

4 No-go from low-temperature Gibbs measures to Gaussian concentration bound

In this section we consider the other regime, i.e., starting from an initial distribution where GCB is not satisfied, such as a translation-invariant Gibbs measure in the non-uniqueness regime. we prove some results showing that it is impossible to go from such a Gibbs measure in the non-uniqueness regime towards a probability measure which satisfies GCB(C) in finite time. One can interpret this result as the fact that one cannot acquire in finite time strong "high-temperature" properties from a low-temperature initial state.

We start with an abstract "non-degeneracy" condition on the Markov semigroup.

Definition 4.1 (Non-degenerate Markov semigroup)**.**
We say that the Markov semigroup $(S_t)_{t\geq 0}$ of a spin-flip dynamics is non-degenerate if for every pair of probability measures $\mu \neq \nu$, we have $\mu S(t) \neq \nu S(t)$ for all $t > 0$.

Then we have the following general result which shows that under the evolution of a non-degenerate semigroup one cannot go from "low temperature" to "high temperature" in finite time.

Theorem 4.1. *Let $\mu^+ \neq \mu^-$ denote two translation-invariant Gibbs measures for the same translation-invariant potential. Assume that the Markov semigroup is non-degenerate. Then for all $t > 0$, $\mu^+ S(t)$ cannot satisfy* GCB(C)*.*

Proof. Because $\mu^+ \neq \mu^-$ are two translation-invariant Gibbs measures for the same translation-invariant potential, we conclude that $h(\mu^-|\mu^+) = 0$ and, as a consequence, $h(\mu^- S(t)|\mu^+ S(t)) = 0$, for all $t > 0$. By non-degeneracy, $\mu^- S(t) \neq \mu^+ S(t)$. By [6], we have that if $\mu^+ S(t)$ satisfies GCB(C), then for all ν translation invariant $h_*(\nu|\mu S(t)) > 0$, which contradicts $h(\mu^- S(t)|\mu^+ S(t)) = 0$. □

The following lemma shows that independent spin-flip is non-degenerate.

Lemma 4.1. *Let μ, ν be two different probability measures on Ω. If $S(t)$ denotes the semigroup of independent rate one spin-flip dynamics, then at any later time $t > 0$, $\mu S(t) \neq \nu S(t)$.*

Proof. Define, for $A \in \mathscr{S}$, $\sigma_A = \prod_{i\in A} \sigma_i$. Then we have $L\sigma_A = -2|A|\sigma_A$ and as a consequence,

$$S(t)\,\sigma_A = \mathrm{e}^{-2|A|t}\,\sigma_A. \tag{13}$$

If $\mu S(t) = \nu S(t)$ for some $t > 0$ then it follows from (13) that

$$\mathrm{e}^{-2|A|t} \int \sigma_A \,\mathrm{d}\mu = \mathrm{e}^{-2|A|t} \int \sigma_A \,\mathrm{d}\nu$$

and therefore $\int \sigma_A \,\mathrm{d}\mu = \int \sigma_A \,\mathrm{d}\nu$. Because linear combinations of the functions σ_A are uniformly dense in $\mathscr{C}(\Omega)$, we conclude that $\mu = \nu$, which leads to a contradiction. □

In the next subsection, we use analyticity arguments to show non-degeneracy for general translation-invariant local spin-flip dynamics.

4.1 Analyticity and non-degeneracy of local spin-flip dynamics

In this section we show that for general local translation-invariant spin-flip dynamics, for μ a probability measure on Ω, and for a (uniformly) dense set of continuous functions f the map $t \mapsto \int S(t) f \,\mathrm{d}\mu$ can be analytically extended to a strip in the complex plane of which the width does not depend on μ. This implies non-degeneracy in the sense of Definition 4.1. We start with setting up the necessary notation.

We remind the notation $\sigma_B = \prod_{i\in B} \sigma_i$ for B a finite subset of $\mathbb{Z}^d$. For a finite set $B \subset \mathbb{Z}^d$ we define the associated translation-invariant operator

$$L_B = \sum_{i\in\mathbb{Z}^d} \sigma_{B+i} \nabla_i.$$

In case $B = \emptyset$ we make the convention $\sigma_B = 1$, i.e., $L_\emptyset = \sum_{i\in\mathbb{Z}^d} \nabla_i$ is the generator of rate 1 independent spin flips.

A general local translation-invariant spin-flip generator can then be written in terms of these "building block" operators as follows

$$\mathscr{L}_{\mathscr{B}} := \sum_{B\in\mathscr{B}} \lambda(B) L_B \tag{14}$$

where $\mathscr{B}$ is a finite collection of finite subsets of $\mathbb{Z}^d$, and where $\lambda : \mathscr{B} \to \mathbb{R}$. For notational simplicity, we suppressed the dependence on the coefficient $\lambda(\cdot)$ in (14). In the following lemma we produce a uniform estimate for $L_{B_n} L_{B_{n-1}} \cdots L_{B_1} \sigma_A$.

Lemma 4.2. *We have the uniform estimate*

$$\begin{aligned} &\|L_{B_n} L_{B_{n-1}} \cdots L_{B_1} \sigma_A\|_\infty \\ &\le 2^n |A|(|A| + |B_1|)(|A| + |B_1| + |B_2|) \cdots (|A| + |B_1| + \cdots + |B_{n-1}|). \end{aligned} \tag{15}$$

Proof. First notice that the bound holds when $A = \emptyset$ because in that case $L_{B_n} L_{B_{n-1}} \cdots L_{B_1} \sigma_A = 0$. So we consider $A \neq \emptyset$. Let us first deal with $n = 1$. Notice that

$$\nabla_i \, \sigma_A = -2 \, \sigma_A \mathbb{1}(i \in A)$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. Next notice that $\sigma_G \sigma_F = \sigma_{G \Delta F}$ for G, F finite subsets of $\mathbb{Z}^d$ and $G \Delta F = (G \cap F^c) \cup (F \cap G^c)$ the symmetric difference. Then we compute

$$L_{B_1} \sigma_A = -2 \sum_{i \in A} \sigma_{(B_1 + i) \Delta A}.$$

As a consequence

$$\| L_{B_1} \sigma_A \|_\infty \leq 2 \, |A|.$$

Let us denote for n sets $C_1, \ldots, C_n$

$$\Delta_{i=1}^n C_i = C_1 \Delta C_2 \Delta \cdots \Delta C_n.$$

Then, by iteration, using $\| \sigma_A \|_\infty = 1$, we obtain

$$\begin{aligned} & L_{B_n} L_{B_{n-1}} \cdots L_{B_1} \sigma_A \\ & \quad = (-2)^n \sum_{i_1 \in A} \sum_{i_2 \in (B_1 + i_1) \Delta A} \cdots \sum_{i_n \in A \Delta (\Delta_{k=1}^{n-1} (B_k + i_k))} \sigma_{A \Delta (\Delta_{k=1}^n (B_k + i_k))}. \end{aligned}$$

Now use that $|C \Delta D| \leq |C| + |D|$, and $\| \sigma_{A \Delta (\Delta_{k=1}^n (B_k + i_k))} \|_\infty = 1$, to further estimate

$$\begin{aligned} & \left\| \sum_{i_1 \in A} \sum_{i_2 \in (B_1 + i_1) \Delta A} \cdots \sum_{i_n \in A \Delta (\Delta_{k=1}^{n-1} (B_k + i_k))} \sigma_{A \Delta (\Delta_{k=1}^n (B_k + i_k))} \right\|_\infty \\ & \leq |A| (|A| + |B_1|) \cdots (|A| + |B_1| + \cdots + |B_{n-1}|). \end{aligned}$$

The lemma is proved. □

We can then estimate $\mathscr{L}_{\mathscr{B}}^n \, \sigma_A$.

Lemma 4.3. *Let $\mathscr{B} \in 2^{\mathbb{Z}^d}$ denote a finite set consisting of finite subsets of $\mathbb{Z}^d$, and let $\mathscr{L}_{\mathscr{B}}$ as in (14). Denote $K := \max_{B \in \mathscr{B}} |B|$ and $M := \max_{B \in \mathscr{B}} |\lambda(B)|$. Then we have*

$$\| (\mathscr{L}_{\mathscr{B}}^n \, \sigma_A) \|_\infty \leq 2^n M^n |\mathscr{B}|^n (|A| + K)^n n!. \tag{16}$$

As a consequence

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, (\mathscr{L}_{\mathscr{B}}^n \, \sigma_A)$$

is a uniformly convergent series for $t < t_0$ with $t_0 = \frac{1}{2M|\mathscr{B}|(|A|+K)}$.

Proof. We have

$$\mathscr{L}_{\mathscr{B}}^n \sigma_A = \sum_{B_n \in \mathscr{B}} \dots \sum_{B_1 \in \mathscr{B}} \left(\prod_{i=1}^n \lambda(B_i) \right) L_{B_n} \cdots L_{B_1}(\sigma_A).$$

The result then follows via (15) using that $|B| \leq K$ for $B \in \mathscr{B}$ via the inequality

$$\begin{aligned} &|A|(|A| + |B_1|) \cdots (|A| + |B_1| + \cdots + |B_{n-1}|) \\ &\leq |A|(|A| + K) \cdots (|A| + (n-1)K)) \leq (|A| + K)^n n!. \end{aligned}$$

The consequence is immediate from (16). □

Proposition 4.1. *Let $\mathscr{L}_{\mathscr{B}}$ denote a local translation-invariant spin-flip generator as in (14). The set of analytic vectors is uniformly dense in the set of continuous functions.*

Proof. The set of analytic vectors is by definition the set of functions such that there exists $t > 0$ such that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\mathscr{L}_{\mathscr{B}}^n f\|_\infty$$

is a convergent series. Let us denote by $\mathscr{A}$ the set of analytic vectors. Notice that $\mathscr{A}$ is a vector space.

By Lemma 4.3 it follows that $\sigma_A \in \mathscr{A}$ for all finite $A \subset \mathbb{Z}^d$. As a consequence, $\mathscr{A}$ contains all local functions and as we saw before, the set of local functions is uniformly dense in $\mathscr{C}(\Omega)$. □

Proposition 4.2. *Let μ and ν denote two probability measures on the configuration space Ω. Let $\mathscr{L}_{\mathscr{B}}$ denote the generator of a translation-invariant local spin-flip dynamics as in (14). Let $S(t)$ denote the corresponding semigroup. Let $\mathscr{A}$ denote the set of analytic vectors. Then for every $f \in \mathscr{A}$, the map*

$$\psi_f(t) : t \mapsto \int S(t) f \, \mathrm{d}\mu - \int S(t) f \, \mathrm{d}\nu$$

extends analytically to the set

$$\Sigma_r := \{z \in \mathbb{C} : \mathrm{dist}(z, \mathbb{R}^+) \leq r\}$$

for some $r > 0$ which depends on f (but not on μ, ν).

Proof. By assumption, there exists $r > 0$ such that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\mathscr{L}_{\mathscr{B}}^n f\|_\infty$$

converges for $t \leq r$, which implies that $\psi_f(z)$ can be extended analytically in

$$B(0,r) = \{z \in \mathbb{C} : |z| \leq r\} \subset \mathbb{C}.$$

Now notice that the same holds when we replace f by $S(s)f$, by the contraction property:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\mathscr{L}_{\mathscr{B}}^n (S(s)f)\|_\infty = \sum_{n=0}^{\infty} \frac{t^n}{n!} \|S(s)(\mathscr{L}_{\mathscr{B}}^n f)\|_\infty \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|\mathscr{L}_{\mathscr{B}}^n f\|_\infty.$$

More precisely, for all s, $\psi_{S(s)f}(\cdot)$ can be extended analytically in

$$B(0,r) = \{z \in \mathbb{C} : |z| \leq r\} \subset \mathbb{C}$$

where r does not depend on s. This implies the statement of the proposition, because, via the semigroup property

$$\psi_{S(s)f}(t) = \psi_f(s+t).$$

The proof is finished. □

Corollary 4.1. *Let μ and ν denote two probability measures on the configuration space Ω. Let $\mathscr{L}_{\mathscr{B}}$ denote the generator of a translation-invariant local spin-flip dynamics as in (14). Let $S(t)$ denote the corresponding semigroup. If $\mu \neq \nu$ then $\mu S(t) \neq \nu S(t)$ for all $t > 0$.*

Proof. Assume on the contrary that $\mu S(t) = \nu S(t)$ for some $t > 0$, then by the semigroup property $\mu S(s) = \nu S(s)$ for all $s \geq t$. Let $f \in \mathscr{A}$ be an analytic vector such that $\int f \, \mathrm{d}\mu \neq \int f \, \mathrm{d}\nu$. Then it follows that the function $\psi_f(s) = \int S(s)f \, \mathrm{d}\mu - \int S(s)f \, \mathrm{d}\nu$ satisfies $\psi_f(0) \neq 0$. On the other hand, because $\mu S(s) = \nu S(s)$ for all $s \geq t$, it follows $\psi_f(s) = 0$ for all $s \geq t$. This contradicts the analyticity of ψ_f. □

4.2 Generalization to a class of infinite-range dynamics

The assumption of finite range for the translation-invariant flip rates can be replaced by an appropriate decay condition on the rates. This is specified below. We assume now that the generator is of the form

$$\mathscr{L}_{\mathscr{B}} = \sum_{B \in \mathscr{B}} \lambda(B) \, \sigma_B L_B$$

where as before $L_B = \sum_{i \in \mathbb{Z}^d} \sigma_{B+i} \nabla_i$. We assume now that $\mathscr{B}$ is an infinite set of finite subsets of $\mathbb{Z}^d$ and that we have the bound

$$\sum_{B \in \mathscr{B} : |B| = k} |\lambda(B)| \leq c \, \psi(k) \tag{17}$$

where $c \in (0, +\infty)$ is a constant and where $\psi(k)$ is a positive measure on the natural numbers such that for some $u > 0$

$$\sum_{k=0}^{\infty} \mathrm{e}^{uk}\, \psi(k) = F(u) < +\infty. \tag{18}$$

In the following lemma we obtain a bound which allows us to estimate $\|\mathscr{L}_{\mathscr{B}}^n\, \sigma_A\|_\infty$.

Lemma 4.4. *Let ψ be a positive measure on $\mathbb{N}$ such that (18) holds for some $u > 0$. Then for any positive integer n we have*

$$\sum_{0 \le k_1, \dots, 0 \le k_n} \prod_{j=1}^{n} \left(1 + \sum_{\ell=1}^{j} k_\ell\right) \prod_{m=1}^{n} \psi\big(k_m\big) \le \mathrm{e}^u\, n!\, u^{-n} F(u)^n.$$

Proof. We have

$$\begin{aligned}
&\sum_{0 \le k_1, \dots, 0 \le k_n} \prod_{j=1}^{n} \left(1 + \sum_{\ell=1}^{j} k_\ell\right) \prod_{m=1}^{n} \psi\big(k_m\big) \\
&\le \sum_{0 \le k_1, \dots, 0 \le k_n} \left(1 + \sum_{\ell=1}^{n} k_\ell\right)^n \prod_{m=1}^{n} \psi\big(k_m\big) \\
&= \mathrm{e}^u \sum_{0 \le k_1, \dots, 0 \le k_n} \mathrm{e}^{-u\left(1+\sum_{j=1}^{n} k_j\right)} \left(1 + \sum_{\ell=1}^{n} k_\ell\right)^n \prod_{m=1}^{n} \Big(\mathrm{e}^{u\, k_m}\, \psi\big(k_m\big)\Big) \\
&\le \mathrm{e}^u\, n!\, u^{-n} F(u)^n
\end{aligned}$$

where we used that $v^n\, \mathrm{e}^{-v}/n! < 1$, for all $v > 0$ and n. □

We can then show that the bound of Lemma (15) still holds.

Proposition 4.3. *Under (17) and (18), we have the bound*

$$\|\mathscr{L}_{\mathscr{B}}^n\, \sigma_A\|_\infty \le n!\, \kappa^n,\ n \ge 1,$$

for some $\kappa > 0$. As a consequence, local functions are analytic vectors, and the Markovian dynamics generated by $\mathscr{L}_{\mathscr{B}}$ is non-degenerate.

Proof. We estimate as in the proof of Lemma 4.3, using (18). Let $u > 0$

be as in (18), and $A \neq \emptyset$. Then

$$
\begin{aligned}
&\|\mathscr{L}_{\mathscr{B}}^{n}\,\sigma_A\|_\infty \\
&\le 2^n \sum_{B_1\in\mathscr{B}} \cdots \sum_{B_n\in\mathscr{B}_n} \\
&\quad\times \sum_{i_1\in A} \sum_{i_2\in(B_1+i_1)\Delta A} \cdots \sum_{i_n\in A\Delta(\Delta_{k=1}^{n-1}(B_k+i_k))} \|\sigma_{A\Delta(\Delta_{k=1}^{n}(B_k+i_k))}\|_\infty \prod_{i=1}^{n} |\lambda(B_i)| \\
&\le 2^n \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \sum_{B_1\in\mathscr{B},|B_1|=k_1} \cdots \sum_{B_n\in\mathscr{B},|B_n|=k_n} |A|(|A|+k_1)\cdots \\
&\qquad\times\ (|A|+k_1+k_2+\cdots+k_n)\prod_{i=1}^{n}|\lambda(B_i)| \\
&\le 2^n c^n \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} |A|(|A|+k_1)\cdots(|A|+k_1+k_2+\cdots+k_n)\prod_{i=1}^{n}\psi(k_i) \\
&\le |A|^n\, 2^n c^n\, \mathrm{e}^u\, n!\, u^{-n} F(u)^n \\
&\le n!\,\kappa^n
\end{aligned}
$$

for some $0 < \kappa < +\infty$. With this bound, we can proceed as in the proof of the local case (Lemma 4.3, Propositions 4.1, 4.2). □

5 Uniform variance bound

In this section we consider the time dependent behavior of a weaker concentration inequality, which we call the "uniform variance bound". In the context of Gibbs measures, contrarily to GCB, this inequality can still hold in the non-uniqueness regime (for the ergodic equilibrium states), see [5] for a proof of this inequality for the low-temperature pure phases of the Ising model.

Definition 5.1 (Uniform Variance Bound)**.**
We say that μ satisfies the uniform variance bound with constant C (abbreviation UVB(C)*) if for all $f : \Omega \to \mathbb{R}$ continuous*

$$\mathbb{E}_\mu\big[(f - \mathbb{E}_\mu(f))^2\big] \le C\|\delta f\|_2^2. \tag{19}$$

Notice that, in contrast with the Gaussian concentration bound, the inequality (19) is homogeneous, i.e., if (19) holds for f then for all $\lambda \in \mathbb{R}$, it also holds for λf. Furthermore, if (19) holds for a subset of continuous functions which is uniformly dense in $\mathscr{C}(\Omega)$ (such as the set of local functions), then it holds for all $f \in \mathscr{C}(\Omega)$ by standard approximation arguments.

This implies that if we can show the validity of (19) for a set of functions $\mathcal{D} \subset \mathscr{C}(\Omega)$ such that $\cup_{\lambda\in[0,+\infty)}\lambda\mathcal{D}$ contains all local functions, we obtain the validity of (19) for all $f \in \mathscr{C}(\Omega)$.

The following proposition shows that a weak form of Gaussian concentration is equivalent with the uniform variance bound.

Definition 5.2 (Weak Gaussian Concentration Bound)**.**
We say that a probability measure μ satisfies the weak Gaussian concentration bound with constant C if for every $f : \Omega \to \mathbb{R}$ continuous there exists $\lambda_0 = \lambda_0(f) > 0$ such that for all $\lambda \leq \lambda_0$

$$\mathbb{E}_\mu\left(\mathrm{e}^{\lambda(f-\mathbb{E}_\mu(f))}\right) \leq \mathrm{e}^{C\lambda^2\|\delta f\|_2^2}. \tag{20}$$

Proposition 5.1. *A probability measure μ satisfies the weak Gaussian concentration bound with constant C if and only if it satisfies the uniform variance bound.*

Proof. Assume that μ satisfies the weak Gaussian concentration bound with constant C. From (20) we derive, for $f : \Omega \to \mathbb{R}$ continuous,

$$\mathrm{Var}_\mu(f) = \lim_{\lambda\to 0}\frac{\mathbb{E}_\mu\left(\mathrm{e}^{\lambda(f-\mathbb{E}_\mu(f))}\right)-1}{\lambda^2} \leq \lim_{\lambda\to 0}\frac{\mathrm{e}^{C\lambda^2\|\delta f\|_2^2}-1}{\lambda^2} = C\|\delta f\|_2^2.$$

which is the uniform variance bound. Conversely, assume that the uniform variance bound holds, and let $f : \Omega \to \mathbb{R}$ be a continuous function. Then use the elementary inequality $\mathrm{e}^{\lambda x} - 1 - \lambda x \leq \frac{\lambda^2 \mathrm{e}\, x^2}{2}$, valid for for $0 \leq \lambda x \leq 1$, together with $\mathrm{e}^x \geq 1 + x$, to conclude that for $\lambda \leq \frac{1}{2\|f\|_\infty+1}$, we have

$$\mathbb{E}_\mu\left(\mathrm{e}^{\lambda(f-\mathbb{E}_\mu(f))}\right) \leq 1 + \frac{\lambda^2 \mathrm{e}\mathrm{Var}_\mu(f)}{2} \leq 1 + \frac{\lambda^2\,\mathrm{e}}{2}\,C\|\delta f\|_2^2 \leq \mathrm{e}^{\frac{\lambda^2\mathrm{e}}{2}C\|\delta f\|_2^2}.$$

□

The following theorem is the analogue of Theorem 3.1 for the uniform variance bound.

Theorem 5.1. *Assume that for all σ, the probability measure $\delta_\sigma S(t)$ satisfies* $\mathrm{UVB}(C(\sigma,t))$*. If μ satisfies* $\mathrm{UVB}(C)$ *and is such that $\int C(\sigma,t)\,\mathrm{d}\mu(\sigma) < +\infty$, then also $\mu S(t)$ satisfies* $\mathrm{UVB}(C(\mu,t))$ *with*

$$C(\mu,t) \leq CK(t) + \int C(\sigma,t)\,\mathrm{d}\mu(\sigma)$$

where $K(t)$ is as in (8).

Proof. Let $f : \Omega \to \mathbb{R}$ be a continuous function. Then we compute, using (8):

$$\begin{aligned}
\mathrm{Var}_{\mu S(t)}(f) &= \int f^2 \, \mathrm{d}\mu S(t) - \left(\int f \, \mathrm{d}\mu S(t) \right)^2 \\
&= \int \left(S(t)(f^2) - (S(t)f)^2 \right) \mathrm{d}\mu + \mathrm{Var}_{\mu}(S(t)f) \\
&= \int \mathrm{Var}_{\delta_\sigma S(t)}(f) \, \mathrm{d}\mu(\sigma) + C \| \delta S(t) f \|_2^2 \\
&\leq \left(\int C(\sigma, t) \, \mathrm{d}\mu(\sigma) \right) \| \delta f \|_2^2 + C K(t)^2 \| \delta f \|_2^2 .
\end{aligned}$$

The theorem is proved. □

Corollary 5.1. *Assume that the spin-flip rates satisfy the weak interaction condition of Section 3.2, then the dynamics conserves the uniform variance bound.*

Proof. Under the weak interaction condition, $\delta_\sigma S(t)$ satisfies GCB(C) with a constant that does not depend on σ. By Proposition 5.1 $\delta_\sigma S(t)$ satisfies UVB(C) with a constant that does not depend on σ. The conclusion follows from Theorem 5.1. □

The following theorem shows that the high-temperature condition of corollary 5.1 is not necessary, and in fact, the uniform variance inequality is robust under any local spin-flip dynamics, i.e., under the condition C of Section 2.3.

Theorem 5.2. *Assume that μ satisfies the uniform variance inequality (19). Let $S(t)$ denote the semigroup of a spin-flip dynamics condition A of Section 2.3. Then $\mu S(t)$ satisfies the uniform variance inequality for all $t > 0$.*

Proof. Let us denote the time-dependent quadratic form

$$\psi(t; f, g) = S(t)(fg) - (S(t)f)(S(t)g)$$

as well as the usual carré du champ quadratic form

$$\Gamma(f, g) = L(fg) - gLf - fLg.$$

Notice that

$$\mathrm{Var}_{\delta_\sigma S(t)}(f) = \psi(t; f, f)(\sigma). \tag{21}$$

An simple explicit computation shows that

$$\Gamma(f, f) = \sum_{i \in \mathbb{Z}^d} c(i, \sigma)(f(\sigma^i) - f(\sigma))^2$$

which by the boundedness of the rates implies the estimate

$$\|\Gamma(f,f)\|_\infty \le \hat{c}\,\|\delta f\|_2^2 \tag{22}$$

with $\hat{c} = \sup_{\sigma\in\Omega, i\in\mathbb{Z}^d} c(i,\sigma)$. We then compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\psi(t;f,f)) &= L(S(t)f^2) - 2S(t)fLS(t)f \\ &= L\left[S(t)f^2 - (S(t)f)(S(t)f)\right] + 2\Gamma(S(t)f, S(t)f).\end{aligned}$$

As a consequence, using $\psi(0;f,f)=0$, by the variation of constants method we obtain

$$\psi(t;f,f) = 2\int_0^t S(t-s)\,\Gamma(S(s)f, S(s)f)\,\mathrm{d}s.$$

Therefore, using (22) combined with the contraction property of the semigroup, we obtain, via (8)

$$\|\psi(t;f,f)\|_\infty \le 2\,\hat{c}\int_0^t \|\delta S(s)f\|_2^2\,\mathrm{d}s \le 2\,\hat{c}\left(\int_0^t K(s)^2\,\mathrm{d}s\right)\|\delta f\|_2^2.$$

Now use (21) to conclude

$$\mathrm{Var}_{\delta_\sigma S(t)}(f) \le C\|\delta f\|_2^2$$

with $C = 2\,\hat{c}\left(\int_0^t K(s)^2\,\mathrm{d}s\right)$ not depending on σ.

Via Theorem 5.1, we obtain the statement of the theorem. □

Remark 5.1. *Remark that we did not use the finite range character of the spin-flip rates, neither the translation invariance. I.e., as soon as the flip rates are uniformly bounded, and are such that the Markovian dynamics with these rates can be defined, we obtain that the uniform variance bound is conserved in the course of time.*

Finally, we show the analogue of Theorem 3.1 for more general inequalities including moment inequalities.

Definition 5.3. *Let $\mathcal{F}:\mathbb{R}\to\mathbb{R}$ be a convex function, $\mathcal{J}:[0,\infty)\to\mathbb{R}$ a continuous increasing function, and $C>0$ a constant. Then we say that μ satisfies the $(\mathcal{F},\mathcal{J},C)$ inequality if for all continuous $f:\Omega\to\mathbb{R}$ with $\|\delta f\|_2 < +\infty$ we have*

$$\int \mathcal{F}(f - \mathbb{E}_\mu(f))\,\mathrm{d}\mu \le \mathcal{J}(C\|\delta f\|_2).$$

To fit the examples we saw so far: we have UVB(a) corresponds to $\mathcal{F}(x) = x^2, \mathcal{J}(x) = x^2, C = \sqrt{a}$, whereas GCB($a$) corresponds to $\mathcal{F}(x) = \mathrm{e}^x, \mathcal{J}(x) = \mathrm{e}^{x^2}, C = \sqrt{a}$. More general moment inequalities correspond to $\mathcal{F}(x) = |x|^p, \mathcal{J}(x) = |x|^p$.

The following theorem is then the analogue of Theorem 3.1 for the $(\mathcal{F}, \mathcal{J}, C)$ inequality.

Theorem 5.3. *Assume that $\delta_\sigma S(t)$ satisfies the $(\mathcal{F}, \mathcal{J}, C)$ inequality with constant C that does not depend on σ. Then if μ satisfies the $(\mathcal{F}, \mathcal{J}, C_\mu)$ inequality, so does $\mu S(t)$ for all $t > 0$.*

Proof. We write, using $p_t(\sigma, \mathrm{d}\eta)$ for the transition probability measure starting from σ, and abbreviating $\int f \,\mathrm{d}\mu S(t) =: \mu(t, f)$

$$
\begin{aligned}
&\int \mathcal{F}(f - \mu(t,f)) \,\mathrm{d}\mu S(t) \\
&= \int p_t(\sigma, \mathrm{d}\eta)\, \mathcal{F}\left(f(\eta) - \int f(\xi)\, p_t(\sigma, \mathrm{d}\xi) + \int f(\xi)\, p_t(\sigma, \mathrm{d}\xi) - \mu(t,f) \right) \mathrm{d}\mu(\sigma) \\
&\le \frac{1}{2} \int p_t(\sigma, \mathrm{d}\eta)\, \mathcal{F}\left(2\left(f(\eta) - \int f(\xi)\, p_t(\sigma, \mathrm{d}\xi) \right) \right) \mathrm{d}\mu(\sigma) \\
&\qquad + \frac{1}{2} \int \mathcal{F}\left(2\left(S(t)f - \int S(t)f \,\mathrm{d}\mu \right) \right) \mathrm{d}\mu(\sigma) \\
&\le \frac{1}{2} \mathcal{J}(2C\|\delta f\|_2) + \frac{1}{2} \mathcal{J}(2C_\mu \|\delta S(t) f\|_2) \\
&\le \frac{1}{2} \mathcal{J}(2C\|\delta f\|_2) + \frac{1}{2} \mathcal{J}\big(2C_\mu \sqrt{K(t)}\, \|\delta f\|_2\big) \\
&\le \mathcal{J}\big(\big(2C + 2C_\mu \sqrt{K(t)}\,\big) \|\delta f\|_2 \big).
\end{aligned}
$$

Here in the last two steps we used (8), combined with the fact that $\mathcal{J}$ is increasing. □

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