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# Optimal Harvesting of Resources with Regime-Switching Stochastic Gilpin-Ayala Biomass Growth

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## Abstract

In this article, we provide a general framework for analyzing the optimal harvest of a renewable resource (i.e. fish, shrimp) assuming that the price and biomass evolve stochastically and harvesters have a constant relative risk aversion (CRRA). In order to take into account the impact of a sudden change in the environment linked to the ecosystem, we assume that the biomass are governed by a stochastic differential equation of the 'Gilpin-Ayala' type, with regime change in the parameters of the drift and variance. Under the above assumptions, we find the optimal effort to be deployed by the collector (fishery for example) in order to maximize the expected utility of its profit function. To do this, we give the proof of the existence and uniqueness of the value function, which is derived from the Hamilton-Jacobi-Bellman equations associated with this problem, by resorting to a definition of the viscosity solution.

## Keywords

Stochastic Gilpin-Ayala, CRRA utility, Viscosity solutions, Renewable Resources, Optimal Effort.

## 1 Introduction

Throughout the world, renewable resources have catapulted to the forefront of the environmental issues and economic developments. Many researchers have taken a particular interest in fishery fields. They used some stochastic biological production functions for resource growth for purposes of sustainable development. Some stochastic models that account for sudden, unforeseeable events are quite similar in most resource problems such as forestry, fishery, and so on. In general, these problems involve finding an optimal sustainable economic policy in continuous time with stochastic price or biomass growth.

Evolutionary ecology can be viewed as a branch of biology. She is particularly interested in describing and understanding the variability of ecological systems (e.g. fisheries or forestry), through biodemographic processes. It also allows the study or identification of the distribution of abundance of different types of organisms or species in an ecological system. As a corollary, it involves understanding the evolutionary properties of species and their interactions, depending on abiotic or biotic factors of their environment. The environment of a fish species, for example, includes both physical properties dependent on insolation, climate, geology, human activities exploiting ecosystem services, as well as other organisms that share its habitat. In

particular, population ecology allows the study of how the size of populations of species living together in groups changes over time or space ([1]).

Sustainable development comes down to considering how best to manage, on the basis of very long-term objectives, interactions between natural and social sources of variability. From an economic point of view, the behavior of an entrepreneur will be different depending on whether its investment cycle is shorter or longer than that of the ecosystem it operates. Economic or environmental decisions should be taken under constraint of maintaining the viability of ecosystems, in order to sustain the ecosystem services (forestry, fisheries, etc.) by economic agents.

In the early 1970s, economists were interested in renewable resources and the environment, over-exploitation of open access fisheries, pollution etc., long after pioneering work such as that of [2], [3]. In economics, the literature considers resources as stocks to be exploited rationally. The problem is therefore the optimal rate of extraction. *Since [4] considered more difficult by the existence of a renewal ([5]).* A fortiori, this living resource depends on others with which it interacts within an ecosystem. The evolution over time of their exploitation is a logic of both flow and variability. Authors such as Volterra in the 1930's, [6], among others, extended Hotelling's work in the case of resources renewable. Against all evidence, the assumptions of the model of [7], lead to hold the fishermen solely responsible for fluctuations in abundance. Such a model does not take into account the possible impact on the stock, of marine pollution, a tsunami, drastic climate change or an epidemic. Indeed, as indicated above, a living organism modifies the environment in which it lives, and adapts to exogenous modifications (i.e. natural disasters, extreme pollution, climate, etc.) impacting this environment. In the living world, the viability of a complex organism, a tree, a fish, relies on the combination of many and different rhythms (microorganisms, cells and certain functions). Organisms are alive when they have the faculties to reproduce, to modify the environment in which they evolve and to adapt to exogenous modifications of this environment. These organisms interact, exchange matter, energy, information. They confront each other, cooperate, coexist. They fluctuate in complex ways, at multiple spatial and temporal scales, as their interactions also fluctuate.

Based on the work of [7], scientific research has proposed several alternative models capable of taking into account exogenous environmental factors. Among these models, the stochastic version of the Gilpin-Ayala differential equation ([8]) is increasingly used. However, this equation considers constant drift and variance. Like other authors, to characterize the stochastic dynamics of the stock, we consider in our article, a stochastic differential equation of the 'Gilpin-Ayala' type, with a jump component, and with a drift and a variance liable to change according to the regime. The model considered makes it possible to take into account the impact of possible extreme events (exogenous or not), liable to significantly reduce the stock of resources.

In general, the authors are concerned with the question of how to maximize profit, which is just the difference between revenues and costs. More recently, it is recognized that fishers are risk averse. We introduced in the commonly used model the economic notion of utility function to take into account the fisher's risk attitude. This has made economic model more complex, but richer.

Even though, economics has enhanced our understanding of the problems of resources management, the biological dynamics of the fishery resource that underline the optimization problem make the fisheries more difficult than forestry problems or others. There is a wide variety of population growth models in the literature. In this work, we focus on Gilpin-Ayala growth model for fishery resources, our results may be easily adapted to many other fields.

The model that has been traditionally considered by many authors is the logistic model described by the ordinary differential equation:

$$dX_t = X_t [r - aX_t] dt, \quad (1)$$

where  $X_t$  stands for the population size at time  $t$ ,  $r > 0$  represents the growth rate of the species while  $a > 0$  is the environmental carrying capacity. Many authors ([9], [10], [11]) claimed that a little more complicated model was needed and proposed their following model:

$$dX_t = X_t [r - aX_t^\lambda] dt, \quad (2)$$

where  $\lambda > 0$  denotes the parameter to modify the classical deterministic logistic model, which is often called GA parameter.

The stochastic nature of renewable resource should be considered in the modeling process. To this end, stochastic versions Gilpin-Ayala model have been studied by many authors ([12], [8], [13], [14]):

$$dX_t = X_t [r - aX_t^\lambda] dt + \sigma X_t dW(t), \quad (3)$$

where  $\sigma$  is volatility and  $W(t)$  is the standard Brownian motion.

The rest of the paper is organized as follows. Section 2 briefly describes the stochastic dynamics of the fishing population and prices, specifies the decision rule and the model is formulated. Having derived the stochastic dynamic programming related to our problem, we show in section 3 that the value function is viscosity solution, while section 4 concludes the paper.

## 2 The model setup

Throughout this paper we let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e. it is increasing and right continuous while  $\mathcal{F}_0$  contains all P-null sets). Let  $W(t)$  and  $W_Y(t)$ ,  $t \geq 0$ , be two standard Brownian motions defined on this probability space which are supposed to be independent.

Let  $\alpha(t)$  be a right-continuous-time Markov chain,  $\mathcal{F}_t$ -adapted with finite state space  $\mathcal{S} = \{1, 2\}$  and generator  $\mathcal{Q} = (q_{ij}) \in \mathbb{R}^2 \times \mathbb{R}^2$ . We consider in this paper that the Markov chain  $\alpha(t)$  is irreducible, which is equivalent to the condition that  $q_{ij} > 0$  for  $i \neq j$ . We assume that the Markov chain  $\alpha(\cdot)$  is independent of the Brownian motions  $W_Y(\cdot)$  and  $W(\cdot)$ ,  $t$  is time,  $t \in [0, T]$  and  $T$  is finite-horizon of time.

### 2.1 Switching jump-diffusion Gilpin-Ayala population model

We set an SDE under regime switching of the form:

$$dX_t = X_t [r_{\alpha(t)} - a_{\alpha(t)} X_t^\lambda - q E_{\alpha(t)}(t)] dt + \sigma_{\alpha(t)} X_t dW(t); \quad \alpha(t) = 1, 2 \quad (4)$$

where  $r_{\alpha(t)}$  intrinsic rate of growth in regime  $\alpha(t)$ ,  $E_{\alpha(t)}$  is the fishing effort exerted on the population at time which depends on the current regime  $\alpha(t)$ ,  $q > 0$  is the catchability coefficient,  $\lambda > 0$  denotes the parameter to modify the classical deterministic logistic model, which is often called GA parameter,  $a_{\alpha(t)} = r_{\alpha(t)} / K_{\alpha(t)} > 0$ ,  $K_{\alpha(t)}$  is the environmental carrying capacity and  $\sigma_{\alpha(t)}$  is volatility in regime  $\alpha(t)$ . With initial value  $X_0 = x_0 \in (0, K)$ ,  $\alpha(t) = i$ .

As said above, population equations may suffer abrupt environmental shocks. Introducing Levy jump into equations, we have extended Eq. (4) to consider the effect of bio-disaster in optimal exploitation of fishing resources. Therefore, the corresponding equations may be expressed as follows

$$dX_t = X_{t-} [r_i - a_i X_{t-}^\lambda - q E_i(t)] dt + \sigma_i X_{t-} dW(t) + \int_{\mathbb{R} \setminus \{0\}} \eta(t, X_{t-}, i, z) \tilde{N}_i(dt, dz); \quad i = 1, 2 \quad (5)$$

where  $X_{t-}$  is the left limits of  $X_t$ ,  $\tilde{N}_i(dt, dz) = N_i(dt, dz) - dt \nu_i(dz)$  denotes the compensated Poisson measure which is independent of  $W(t)$ .  $N_i(dt, dz)$  is a stationary Poisson random measure on  $[0, T] \times \mathbb{R} \setminus \{0\}$  with intensity measure  $dt \nu_i(dz)$ .  $\nu_i(dz)$  is the Lévy Measure for the jump size when the Markov chain  $\alpha(t)$  is in state  $i$ , i.e. a  $\sigma$ -finite Borel measure on  $\mathbb{R} \setminus \{0\}$  with the property  $\int_{\mathbb{R} \setminus \{0\}} \min(1, z^2) \nu_i(dz) < +\infty$ .

And  $\eta : [0, T] \times \mathbb{R} \times \mathcal{S} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function with respect to  $\nu$  and is measurable satisfying the following assumption: there exist  $\rho : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $\int_{\mathbb{R} \setminus \{0\}} \rho^2(z) \nu_i(dz) < +\infty$ , such that for all  $t, s \in [0, T]$ ,  $x, y \in \mathbb{R}$  and  $i \in \mathcal{S}$ ,

$$|\eta(t, x, i, z) - \eta(t, y, i, z)| < \rho(z) |x - y| \quad (6)$$

$$|\eta(t, x, i, z)| < \rho(z) (1 + |x|). \quad (7)$$

Readers are referred to [15] for more precise properties of random measures.

When consider  $\eta(t, i, z) = 0$  for all  $(t, i, z) \in [0, T] \times \mathcal{S} \times \mathbb{R}$ , i.e there is no jump, many authors (see [16], [17], [12], [8]) showed that Eq. (5) has a unique global positive solution for any given positive initial value and represented by

$$X_{t,i}^\lambda = \frac{\exp \left\{ \lambda \left( \int_0^t \left[ r_i - qE_i(s) - \frac{1}{2}\sigma_i^2 \right] ds + \int_0^t \sigma_i dW(s) \right) \right\}}{1/X_{0,i}^\lambda + \lambda a_i \int_0^t \exp \left\{ \lambda \left( \int_0^s \left[ r_i - qE_i(\tau) - \frac{1}{2}\sigma_i^2 \right] d\tau + \int_0^s \sigma_i dW(\tau) \right) \right\} ds}, \quad t \geq 0.$$

## 2.2 Mean - reverting price

The unit price of fish,  $Y_t$ , will depend on the mean (or long-term) price  $\bar{Y}_i$ . Further, let the unit price evolve as the geometric mean-reverting stochastic process

$$dY_t = \theta_i(\bar{Y}_0 - Y_t)dt + \sigma_Y Y_t dW_Y(t) \quad (8)$$

with initial condition  $Y(0) = y_0$ , where the parameters are positive constants,  $\theta$  is the reversion speed,  $\bar{p}_0$  is a maximum price,  $\sigma_Y$  is the volatility of the price.  $W_Y(t)$  is standardized Brownian motion as set before. Model (8) is a generalization of the Orstein-Uhlenbeck process, where we use the level-dependent volatility instead of the constant volatility. The mean price,  $\bar{Y}_0$ , attracts the prices in its direction. In others words, when  $Y_t > \bar{Y}_0$  the trend term  $\theta(\bar{Y}_0 - Y_t) < 0$  and therefore  $Y_t$  decreases and when  $Y_t < \bar{Y}_0$  a similar argument establishes that  $Y_t$  grows.

## 2.3 Specifying the Decision Problem

The mostly used cost function in fishery management is the quadratic cost function (see [18], [19], [20], [21]). It can be written as

$$c(E_t) = (c_1 + c_2 E_t)E_t$$

where  $c_1, c_2 > 0$  are constants.

The profit is the difference between revenue and cost. Here  $\pi(t)$  is the instantaneous profit from the harvest of the stock biomass and is given as:

$$\pi(t) = qE_t X_t Y_t - (c_1 + c_2 E_t)E_t \quad (9)$$

where,  $qE_t X_t$  denotes the volume of harvest and  $Y_t$  the actual price of the harvest at the time of decision making.

We consider, as [21], that the fisher is risk-averse and we define the power utility function as

$$U(x) = (x^{1-\gamma})/(1-\gamma), \quad (10)$$

where  $\gamma$  is the constant relative risk aversion (CRRA) coefficient,  $\gamma = 0$  denotes risk neutrality,  $\gamma < 0$  and  $\gamma > 0$  implies risk loving and risk aversion, respectively. When  $\gamma = 1$ ,  $U(x) = \ln(x)$ .

For a time  $t$  in the horizon  $[0, T]$ , our problem is to maximize the present value  $V_i(t, x, y)$  for each  $i \in \mathcal{S}$  defined as:

$$V_i(t, x, y) = \mathbb{E}_{x,y,i} \left[ \int_t^T e^{-\beta(s-t)} \frac{\pi(X_s, Y_s, E_s, i)^{1-\gamma}}{1-\gamma} ds + e^{-\beta(T-t)} V_i(X_T) \right]. \quad (11)$$

Here  $\mathbb{E}_{x,y,i}$  is the conditional expectation given  $X(t) = x$ ,  $Y(t) = y$  and  $\alpha(t) = i$  under  $\mathbb{P}$ , where  $T$  is the finite time horizon  $\beta > 0$  is a discount factor.

We say that the control process  $E(t)$  is admissible if the following three conditions are satisfied:

1. the SDE (5) for the state process  $X(t)$  has a unique strong solution;
2. the SDE (8) for the state process  $Y(t)$  has a unique strong solution;
3.  $\mathbb{E}_{x,y,i} \left[ \int_t^T \left| e^{-\beta(s-t)} \frac{\pi(X_t^x, Y_t^y, E_t, i)^{1-\gamma}}{1-\gamma} \right| dt + \left| e^{-\beta(T-t)} V(X_T) \right| \right] < \infty$ .

The effort is bounded then  $\mathcal{A}$ , the set of admissible control, is bounded. The fisher strives to maximize total expected discounted utility of both profit and terminal biomass. Therefore, the value function can be written as:

$$v_i(t, x, y) = \sup_{E \in \mathcal{A}_i} V_i(t, x, y). \quad (12)$$

### 3 Dynamic Programming and Viscosity Solutions

In this section, we need to prove existence of the viscosity solution starting to prove some properties of the value function, which will be the viscosity solution.

We shall make the following assumptions: there exists  $\rho > 0$  such that for all  $s, t \in [0, T]$ ,  $x, x' \in \mathbb{R}_+$ ,  $y, y' \in \mathbb{R}_+$  and  $E \in \mathcal{A}$

$$|l(t, x, y, E) - l(s, x', y', E)| + |m(x, y) - m(x', y')| \leq \rho [|t - s| + |x - x'| + |y - y'|], \quad (13)$$

and the global linear growth conditions:

$$|l(t, x, y, E)| + |m(x, y)| \leq \rho [1 + |x| + |y|]. \quad (14)$$

Here  $l(t, x, y, E) = \frac{\pi(X_t^x, Y_t^y, E_t, i)^{1-\gamma}}{1-\gamma}$ , and  $m(x, y) = V(X_T^x)$ .

Let's consider the following two functions  $f$  and  $g$  defined by:

$$f(t, X_{t-}, E_i(t)) = X_{t-} (r_i - a_i X_{t-}^\lambda - q E_i(t)) \quad \text{and} \quad g(t, X_{t-}, E_i(t)) = \sigma_i X_{t-}.$$

Let us define the operator  $I_1$ ,  $I_2$  and  $I_3$  of the value function

$$\begin{aligned} I_1(t, x, y, D_{(x,y)} v_i, D_{(x,y)}^2 v_i) &= \theta(\bar{Y}_0 - y) \frac{\partial v_i}{\partial y} + f(t, x, E_i(t)) \frac{\partial v_i}{\partial x} + \frac{1}{2} \sigma_y^2 \frac{\partial^2 v_i}{\partial y^2} + \frac{1}{2} g^2(t, x, E_i(t)) \frac{\partial^2 v_i}{\partial x^2} \\ &= (f(t, x, E_i(t)), \theta(\bar{Y}_0 - y)) \cdot D_{(x,y)} v_i + \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_i^2 x^2 \end{pmatrix} \cdot D_{(x,y)}^2 v_i \right] \end{aligned}$$

$$I_2(t, x, y, v_i) = q_{ij} (v_j(t, x, y) - v_i(t, x, y)).$$

$$\begin{aligned} I_3(t, x, y, D_{(x,y)} v_i) &= \int_{\mathbb{R} \setminus \{0\}} \left[ v_i(t, x + \eta(t, x, i, z), y) - v_i(t, x, y) - \eta(t, x, i, z) \frac{\partial v_i(t, x, y)}{\partial x} \right] \nu_i(dz) \\ &= \int_{\mathbb{R} \setminus \{0\}} [v_i(t, x + \eta(t, x, i, z), y) - v_i(t, x, y) - (\eta(t, x, i, z), 0) \cdot D_{(x,y)} v_i] \nu_i(dz). \end{aligned}$$

The Hamilton-Jacobi-Bellman equations associated with the problem (12) is:

$$\frac{\partial v_i}{\partial t} + \sup_{E \in \mathcal{A}_i} \left\{ -\beta v_i + \frac{\pi(x, y, E_i(t))^{1-\gamma}}{1-\gamma} + I_1(t, x, y, D_{(x,y)} v_i, D_{(x,y)}^2 v_i) + I_2(t, x, y, v_i) + I_3(t, x, y, D_{(x,y)} v_i) \right\} = 0, \quad (15)$$

$$v_i(T, x, y) = \kappa(x, y) \frac{x^{1-\gamma}}{1-\gamma} \quad \text{for } i, j \in \{0, 1\} \quad \kappa > 0. \quad (16)$$

To the best of our knowledge, there is not in general a smooth solution of Eq. (15). Consequently, these should be interpreted in a weaker sense, notably in the framework of viscosity solutions introduced by [22].

### 3.1 Regularity of value functions

In this section, we study the growth and continuity properties of the value functions. First, we need some estimates on the moments of the price and population processes.

**Lemma 3.1.** *For any  $k \in [0, 2]$  there exists  $C = C(k, K, T) > 0$  such that for all  $h, t \in [0, T]$ ,  $x, y, x', y' \in \mathbb{R}_+$ :*

$$\begin{aligned} \mathbb{E}|X_h^{t,x}|^k &\leq C(1 + |x|^k); & \mathbb{E}|Y_h^{t,y}|^k &\leq C(1 + |y|^k). \\ \mathbb{E}|X_h^{t,x} - x|^k &\leq C(1 + |x|^k)h^{\frac{k}{2}}; & \mathbb{E}|Y_h^{t,y} - y|^k &\leq C(1 + |y|^k)h^{\frac{k}{2}}. \\ \mathbb{E}|X_h^{t,x} - X_h^{t,x'}|^k &\leq C|x - x'|^2; & \mathbb{E}|Y_h^{t,y} - Y_h^{t,y'}|^k &\leq C|y - y'|^2. \\ \mathbb{E}\left[\sup_{0 \leq s \leq h} |X_h^{t,x}|\right]^k &\leq C(1 + |x|^k)h^{\frac{k}{2}}; & \mathbb{E}\left[\sup_{0 \leq s \leq h} |Y_h^{t,y}|\right]^k &\leq C(1 + |y|^k)h^{\frac{k}{2}}. \end{aligned}$$

Proof of Lemma 3.1 see Appendix A.

**Proposition 3.1.** *For any  $i \in \mathcal{S}$ , the value function denoted by  $v_i(t, x, y)$  satisfies a linear growth condition and is also Lipschitz in couple  $(x, y)$  uniformly in  $t$ . There exists a constant  $C > 0$ , such that*

$$0 \leq v_i(s, x_s, y_s) \leq C(1 + |x_s| + |y_s|), \quad \forall (s, x_s, y_s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+.$$

$$|v_i(s, x_s, y_s) - v_i(s, x'_s, y'_s)| \leq C(|x_s - x'_s| + |y_s - y'_s|), \quad \forall s \in [0, T], \quad x_s, x'_s \in \mathbb{R}_+, \quad y_s, y'_s \in \mathbb{R}_+.$$

**Proposition 3.2.** *Under assumptions (13) and (14) the value function  $v \in \mathcal{C}^0([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ . More precisely, there exists a constant  $C > 0$  such that for all  $t, s \in [0, T]$ ,  $x, x_s \in \mathbb{R}_+$ ,  $y, y_s \in \mathbb{R}_+$ ,*

$$|v_i(t, x, y) - v_i(s, x_s, y_s)| \leq C \left[ (1 + |x| + |y|)|s - t|^{\frac{1}{2}} + |x - x_s| + |y - y_s| \right].$$

Proofs of proposition 3.1 and proposition 3.2 are rejected in appendices B and C.

### 3.2 The Value Function, Viscosity Solution of Hamilton-Jacobi-Bellman equation

In this section, we will first define what we mean by viscosity solutions. Then we will prove that the value function is a viscosity solution.

We consider the Hamilton-Jacobi-Bellman equations as follows:

$$\begin{aligned} \frac{\partial v_i}{\partial t}(t, x, y) + \sup_{E \in \mathcal{A}_i} \left\{ -\beta v_i(t, x, y) + \frac{\pi^{1-\gamma}}{1-\gamma} + \theta(\bar{y}_0 - y) \frac{\partial v_i}{\partial y}(t, x, y) + x \left[ r_i - a_i x^\lambda - q E_i(t) \right] \frac{\partial v_i}{\partial x}(t, x, y) \right. \\ \left. + \frac{1}{2} \sigma_Y^2 \frac{\partial^2 v_i}{\partial y^2}(t, x, y) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v_i}{\partial x^2}(t, x, y) + q_{ij} [v_j(t, x, y) - v_i(t, x, y)] \right. \\ \left. + \int_{\mathbb{R} \setminus \{0\}} \left[ v_i(t, x + \eta(t, x, i, z), y) - v_i(t, x, y) - \eta(t, x, i, z) \frac{\partial v_i(t, x, y)}{\partial x} \right] \nu_i(dz) \right\} = 0. \quad (17) \end{aligned}$$

And the related systems:

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sup_{E \in \mathcal{A}_i} \left\{ -\beta u_i + l + I_1(s, x_s, y_s, D_{(x,y)} u_i, D_{(x,y)}^2 u_i) + I_2(s, x_s, y_s, u_i) + I_3(s, x_s, y_s, D_{(x,y)} u_i) \right\} = 0, \\ u_i(T, x_s, y_s) = \kappa^\gamma \frac{x_s^{1-\gamma}}{1-\gamma} \quad \text{for } i, j \in \{0, 1\} \quad \kappa > 0 \quad (i, s, x_s, y_s) \in \mathcal{S} \times [0, T_i] \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \quad (18)$$

Using a notion of viscosity solution introduced by [22], we prove below the existence and uniqueness of a solution of (18). To do this, we define the set of measurable functions on  $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$  with polynomial growth of degree  $q \geq 0$  as,

$$\mathcal{C}_q([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) = \{ \phi : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+, \text{ measurable} \mid \exists C > 0, |\phi(t, x, y)| \leq C(1 + |x|^q + |y|^q) \}.$$

**Definition 3.1.** We say that  $u_i \in \mathcal{C}^0([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  is called

- i. a viscosity subsolution of (18) if for any  $i \in \mathcal{S}$ ,  $u_i(T, x, y) \leq \kappa^\gamma \frac{x^{1-\gamma}}{1-\gamma}$ , for all  $x \in \mathbb{R}_+$ ,  $y \in \mathbb{R}_+$  and for all functions  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  and  $(\bar{t}, \bar{x}, \bar{y})$  such that  $u_i - \phi$  attains its local maximum at  $(\bar{t}, \bar{x}, \bar{y})$ ,

$$\frac{\partial \phi}{\partial s} + \sup_{E \in \mathcal{A}_i} \left\{ -\beta u_i + l + I_1(\bar{t}, \bar{x}, \bar{y}, D_{(x,y)} u_i, D_{(x,y)}^2 u_i) + I_2(\bar{t}, \bar{x}, \bar{y}, u_i) + I_3(\bar{t}, \bar{x}, \bar{y}, D_{(x,y)} u_i) \right\} \geq 0, \quad (19)$$

- ii. a viscosity supersolution of (18) if for any  $i \in \mathcal{S}$ ,  $u_i(T, x, y) \geq \kappa^\gamma \frac{x^{1-\gamma}}{1-\gamma}$ , for all  $x \in \mathbb{R}_+$ ,  $y \in \mathbb{R}_+$  and if for all functions  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  and  $(\underline{t}, \underline{x}, \underline{y})$  such that  $u_i - \phi$  attains its local minimum at  $(\underline{t}, \underline{x}, \underline{y})$ ,

$$\frac{\partial \phi}{\partial s} + \sup_{E \in \mathcal{A}_i} \left\{ -\beta u_i + l + I_1(\underline{t}, \underline{x}, \underline{y}, D_{(x,y)} u_i, D_{(x,y)}^2 u_i) + I_2(\underline{t}, \underline{x}, \underline{y}, u_i) + I_3(\underline{t}, \underline{x}, \underline{y}, D_{(x,y)} u_i) \right\} \leq 0, \quad (20)$$

- iii. a viscosity solution of (18) if it is both a viscosity sub- and a supersolution of equation (18).

**Theorem 3.1.** Under assumption (13), the value function  $v$  is a viscosity solution of Eq. (17).

**Proof of Theorem 3.1.** See Appendix D.

### 3.3 Uniqueness, Comparison Principle

In this section, we will use the notion of parabolic superjet and subjet defined by Crandall, Ishii and Lions [22]. We state comparison principles, from which we obtain the uniqueness of the solution of the coupled system of partial differential equations.

**Definition 3.2.** Given  $v \in \mathcal{C}^o([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S})$  and  $(t, x, y, i) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S}$ , we define the parabolic superjet:

$$\mathcal{P}^{2,+}v(t, x, y, i) = \left\{ (c, q, M) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}^2 : v(s, x', y', i) \leq v(t, x, y, i) + c(s-t) + q \cdot ((x' - x), (y' - y)) \right. \\ \left. + \frac{1}{2}((x' - x), (y' - y)) \cdot M((x' - x), (y' - y)) + o(|((x' - x), (y' - y))|^2) \text{ as } (s, x', y') \rightarrow (t, x, y) \right\}$$

and its closure:

$$\bar{\mathcal{P}}^{2,+}v(t, x, y, i) = \left\{ (c, q, M) = \lim_{n \rightarrow \infty} (c_n, q_n, M_n) \text{ with } (c_n, q_n, M_n) \in \mathcal{P}^{2,+}v(t_n, x_n, y_n, i) \text{ and} \right. \\ \left. \lim_{n \rightarrow \infty} (t_n, x_n, y_n, v(t_n, x_n, y_n, i)) = (t, x, y, v(t, x, y, i)) \right\}.$$

Similarly, we define the parabolic subjet  $\bar{\mathcal{P}}^{2,-}v(t, x, y, i) = -\bar{\mathcal{P}}^{2,+}(-v)(t, x, y, i)$  and its closure  $\bar{\mathcal{P}}^{2,-}v(t, x, y, i) = -\bar{\mathcal{P}}^{2,+}(-v)(t, x, y, i)$ .

It is proved in [23] that

$$\mathcal{P}^{2,+(-)}v(t, x, y, i) = \left\{ \left( \frac{\phi}{\partial t}(t, x, y, i), D_{(x,y)}\phi(t, x, y, i), D_{(x,y)}^2\phi(t, x, y, i) \right) \right. \\ \left. \text{and } v - \phi \text{ has a global maximum (minimum) at } (t, x, y, i) \right\}.$$

The previous notions lead to new definition of viscosity solutions.

**Definition 3.3.**  $u_i \in \mathcal{C}^0([0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  satisfying the polynomial growth condition is a viscosity solution of (18) if

- (1) for any test-function  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$ , if  $(t, x, y)$  is a local maximum point of  $u_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  and if  $(c, q, L_1) \in \bar{\mathcal{P}}^{2,+}u(t, x, y, i)$  with  $c = \partial\phi(t, x, y)/\partial t$ ,  $q = D_{(x,y)}\phi(t, x, y)$  and  $L_1 \leq D_{(x,y)}^2\phi(t, x, y)$ , then

$$c + \sup_{E \in \mathcal{A}_i} \{-\beta v_i + l(x, y, E_i) + I_1(t, x, y, q, L_1) + I_2(t, x, y, \phi) + I_3(t, x, y, q)\} \leq 0,$$

in this case  $u$  is a viscosity subsolution;

- (2) for any test-function  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$ , if  $(t, x, y)$  is a local minimum point of  $u_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  and if  $(c, q, L_2) \in \bar{\mathcal{P}}^{2,-}u(t, x, y, i)$  with  $c = \partial\phi(t, x, y)/\partial t$ ,  $q = D_{(x,y)}\phi(t, x, y)$  and  $L_2 \geq D_{(x,y)}^2\phi(t, x, y)$ , then

$$c + \sup_{E \in \mathcal{A}_i} \{-\beta v_i + l(x, y, E_i) + I_1(t, x, y, q, L_2) + I_2(t, x, y, \phi) + I_3(t, x, y, q)\} \geq 0,$$

in this case  $u$  is a viscosity supersolution.

The authors [24] proved that definitions 3.2 and 3.3 are equivalent. The second definition is particular suitable for the discussion of a maximum principle which is the backbone of the uniqueness problem for the viscosity solutions theory.

Before state next lemma, we first introduce the inf and sup-convolution operations we are going to use.

**Definition 3.4.** For any usc (upper semi-continuous) function  $U : \mathbb{R}^m \rightarrow \mathbb{R}$  and any lsc (lower semi-continuous) function  $V : \mathbb{R}^m \rightarrow \mathbb{R}$ , we set

$$R^\alpha[U](z, r) = \sup_{|Z-z| \leq 1} \left\{ U(Z) - r \cdot (Z - z) - \frac{|Z - z|}{2\alpha} \right\}$$

$$R_\alpha[V](z, r) = \inf_{|Z-z| \leq 1} \left\{ V(Z) + r \cdot (Z - z) + \frac{|Z - z|}{2\alpha} \right\}$$

$R^\alpha[U](z, r)$  is called the modified sup-convolution and  $R_\alpha[V](z, r)$  the modified inf-convolution. Notice that  $R_\alpha[V](z, r) = -R^\alpha[-U](z, r)$ .

**Lemma 3.2.** (nonlocal Jensen-Ishii's lemma [24])

For any  $i \in \mathcal{S}$ , let  $u_i(\cdot, \cdot, \cdot)$  and  $v_i(\cdot, \cdot, \cdot)$  be, respectively, a usc and lsc function defined on  $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$  and  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2)$  if  $(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$  is a zero global maximum point of  $u_i(t, x, y) - v_i(t, x', y') - \phi(t, (x, y), (x', y'))$  and if  $c - d := D_t\phi(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2))$ ,  $q := D_{(x,y)}\phi(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2))$ ,  $r := -D_{(x',y')}\phi(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2))$ , then for any  $K > 0$ , there exists  $\alpha(K) > 0$  such that, for any  $0 < \alpha < \alpha(K)$ , we have: there exist sequences  $t_k \rightarrow \hat{t}$ ,  $(x_k, y_k) \rightarrow (\hat{x}_1, \hat{y}_1)$ ,  $(x'_k, y'_k) \rightarrow (\hat{x}_2, \hat{y}_2)$ ,  $q_k \rightarrow q$ ,  $r_k \rightarrow r$ , matrices  $M_k, N_k$  and a sequence of functions  $\phi_k$ , converging to the function  $\phi_\alpha := R^\alpha[\phi]((x, y), (x', y'), (q, r))$  uniformly in  $\mathbb{R}_+^2 \times \mathbb{R}_+^2$  and in  $C^2(B((\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)), K))$ , such that

$$u_i(t_k, (x_k, y_k)) \rightarrow u_i(\hat{t}, (\hat{x}_1, \hat{y}_1)), \quad v_i(t_k, (x'_k, y'_k)) \rightarrow v_i(\hat{t}, (\hat{x}_2, \hat{y}_2))$$

$(t_k, (x_k, y_k), (x'_k, y'_k))$  is a global maximum of  $u_i(\cdot, (\cdot, \cdot)) - v_i(\cdot, (\cdot, \cdot)) - \phi(\cdot, (\cdot, \cdot), (\cdot, \cdot))$

$$(c_k, q_k, M_k) \in \bar{\mathcal{P}}^{2,+}u_i(t_k, (x_k, y_k)); \quad (-d_k, r_k, N_k) \in \bar{\mathcal{P}}^{2,-}v_i(t_k, (x'_k, y'_k))$$

$$-\frac{1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_k & 0 \\ 0 & -N_k \end{pmatrix} \leq D_{(x,y),(x',y')}^2\phi(t_k, (x_k, y_k), (x'_k, y'_k)).$$

Here  $c_k - d_k = \nabla_t\phi(t_k, (x_k, y_k), (x'_k, y'_k))$ ,  $q_k = \nabla_{(x,y)}\phi(t_k, (x_k, y_k), (x'_k, y'_k))$ ,  $r_k = \nabla_{(x',y')}\phi(t_k, (x_k, y_k), (x'_k, y'_k))$  and  $\phi_\alpha(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)) = \phi(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2))$ ,  $\nabla\phi_\alpha(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)) = \nabla\phi(\hat{t}, (\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2))$ .

We refer the reader to the mentioned paper for a proof. Now we can state our comparison result.

**Theorem 3.2.** (*comparison principle*): *If  $u_i(t, x, y)$  and  $v_i(t, x, y)$  are continuous in  $(t, x, y)$  and are, respectively, viscosity subsolution and supersolution of the HJB system (17) with at most linear growth, then*

$$u_i(t, x, y) \leq v_i(t, x, y) \text{ for all } (t, x, y, i) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S}.$$

**Proof of Theorem 3.2.** *See E.*

The following corollary follows from Theorems 3.1 and 3.2.

**Corollary 3.1.** *The value function  $v$  is a unique viscosity solution of Eq. (17) that has at most a linear growth.*

## 4 Conclusion

In this paper we have from a theoretical point of view, analyzed the expected utility profit maximizing. The main contribution of this paper is related to the risk aversion of the fisher with the assumption of a regime switching environment where the dynamic of stock resource is dependent and therefore affected by season. Given that these assumptions also are valid in other types of renewable resources, the model here may have wider applications. Possible examples include forestry resources. The numerical simulations will be examined in further research.

## A Proof of Lemma 3.1

1. Let  $k \in [0, 2]$  and  $h = s - t$ .

According to Hölder inequality  $\mathbb{E}\eta^k \leq [\mathbb{E}\eta^2]^{k/2}$  for  $\forall k \geq 0$ ,

$$\mathbb{E}|X_h^{t,x}|^k \leq [\mathbb{E}|X_h^{t,x}|^2]^{k/2}. \quad (21)$$

Given that

$$dX_t = f(t, X_{t-}, E_i(t), X_t)dt + g(t, X_{t-}, E_i(t))dw(t) + \int_{\mathbb{R} \setminus \{0\}} \eta(t, X_{t-}, i, z)\tilde{N}_i(dt, dz); \quad i = 1, 2.$$

According to the elementary inequality  $|\sum_{i=1}^M a_i|^2 \leq M \sum_{i=1}^M |a_i|^2$ ,

$$\begin{aligned} |X_h^{t,x}|^2 \leq & 4 \left[ |x|^2 + \left| \int_0^h f(u+t, X_u^{t,x}, i) du \right|^2 + \left| \int_0^h g(u+t, X_u^{t,x}, i) dw_u \right|^2 \right. \\ & \left. + \left| \int_0^h \int_{\mathbb{R} \setminus \{0\}} \eta(t, X_{t-}, i, z)\tilde{N}_i(dt, dz) \right|^2 \right]. \end{aligned}$$

Using Itô-isometry and Fubini's theorem

$$\begin{aligned} \mathbb{E}|X_h^{t,x}|^2 \leq & 4 \left[ |x|^2 + \int_0^h \mathbb{E} \left| f(u+t, X_u^{t,x}, i) \right|^2 du + \int_0^h \mathbb{E} \left| g(u+t, X_u^{t,x}, i) \right|^2 du \right. \\ & \left. + \int_0^h \int_{\mathbb{R} \setminus \{0\}} \left| \eta(t, X_{t-}, i, z) \right|^2 \nu_i(dz) du \right], \end{aligned}$$

Using the growth condition on  $f, g$  and  $\eta$ , there exists  $C_1 \in \mathbb{R}$  such that

$$\mathbb{E}|X_h^{t,x}|^2 \leq C_1 \left\{ 1 + |x|^2 + \int_0^h \mathbb{E}|X_u^{t,x}|^2 du \right\}.$$

Applying Gronwall's inequality we obtain

$$\mathbb{E}|X_h^{t,x}|^2 \leq C_1 e^{C_1 h} [1 + |x|^2]$$

i.e

$$\mathbb{E}|X_h^{t,x}|^2 \leq C [1 + |x|^2]. \quad (22)$$

Using (21) and elementary inequalities  $(a_1 + a_2)^k \leq 2^{k-1}(|a_1|^k + |a_2|^k)$  and  $(\sqrt{|a_1 + a_2|} \leq \sqrt{|a_1|} + \sqrt{|a_2|})$  we deduce

$$\mathbb{E}|X_h^{t,x}|^k \leq C [1 + |x|^k].$$

The same reasoning gives us

$$\mathbb{E}|Y_h^{t,y}|^k \leq C [1 + |y|^k].$$

2. We have

$$\begin{aligned} |X_h^{t,x} - x|^2 \leq & 3 \left[ \left| \int_0^h f(u+t, P_u^{t,x}, i) du \right|^2 + \left| \int_0^h g(u+t, X_u^{t,x}, i) dw_u \right|^2 \right. \\ & \left. + \left| \int_0^h \int_{\mathbb{R} \setminus \{0\}} \eta(t, X_{t-}, i, z) \tilde{N}_i(dt, dz) \right|^2 \right]. \end{aligned}$$

Similar arguments as above we obtain

$$\mathbb{E}|X_h^{t,x} - x|^2 \leq C_1 \int_0^h [1 + \mathbb{E}|X_u^{t,x}|^2] du,$$

using (22) we deduce

$$\mathbb{E}|X_h^{t,x} - x|^2 \leq C(1 + |x|^2)h.$$

Hence

$$\mathbb{E}|X_h^{t,x} - x|^k \leq C(1 + |x|^k)h^{k/2}.$$

3. Let us define the process  $X_s^{t,x} - X_s^{t,x'}$ . Put  $\bar{f}(u+t, X_u^{t,x}, X_u^{t,x'}, i) = f(u+t, X_u^{t,x}, i) - f(u+t, X_u^{t,x'}, i)$ ,  $\bar{g}(u+t, X_u^{t,x}, X_u^{t,x'}, i) = g(u+t, X_u^{t,x}, i) - g(u+t, X_u^{t,x'}, i)$  and  $\bar{\eta}(t, X_u^{t,x}, X_u^{t,x'}, i, z) = \eta(t, X_u^{t,x}, i, z) - \eta(t, X_u^{t,x'}, i, z)$ . Then Applying Itô's formula, we have

$$\begin{aligned} \mathbb{E}|X_h^{t,x} - X_h^{t,x'}|^2 \leq & 4 \left( |x - x'|^2 + \mathbb{E} \left| \int_0^h \bar{f}(u+t, X_u^{t,x}, X_u^{t,x'}, i) du \right|^2 + \mathbb{E} \left| \int_0^h \bar{g}(u+t, X_u^{t,x}, X_u^{t,x'}, i) dw_u \right|^2 \right. \\ & \left. + \left| \int_0^h \int_{\mathbb{R} \setminus \{0\}} \bar{\eta}(t, X_u^{t,x}, X_u^{t,x'}, i, z) \tilde{N}_i(dt, dz) \right|^2 \right). \end{aligned}$$

$$\mathbb{E}|X_h^{t,x} - X_h^{t,x'}|^2 \leq C \left( |x - x'|^2 + \int_0^h \mathbb{E}|X_u^{t,x} - X_u^{t,x'}|^2 du \right).$$

Hence

$$\mathbb{E}|X_h^{t,x} - X_h^{t,x'}|^2 \leq C|x - x'|^2.$$

Similar arguments as above we deduce

$$\mathbb{E}|X_h^{t,x} - X_h^{t,x'}|^2 \leq C|x - x'|^2; \quad \mathbb{E}|Y_h^{t,y} - Y_h^{t,y'}|^k \leq C|y - y'|^2.$$

4. Using Doob's inequality for martingale. We get

$$\mathbb{E} \left[ \sup_{0 \leq s \leq h} |X_h^{t,x} h|^k \right] \leq C(1 + |x|^k)h^{\frac{k}{2}}; \quad \mathbb{E} \left[ \sup_{0 \leq s \leq h} |Y_h^{t,y}|^k \right] \leq C(1 + |y|^k)h^{\frac{k}{2}}.$$

## B Proof of Proposition 3.1

We first show that  $v$  is Lipschitz in  $(x, y)$ , uniformly in  $t$  and its linear growth condition.

$$v_i(s, x_s, y_s) = \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T e^{-\beta(u-s)} l(i, u, X_u^{s,x_s}, Y_u^{s,y_s}, E_u) du + e^{-\beta(T-s)} m(X_T^{s,x_s}, Y_T^{s,y_s}) \right].$$

1. Using elementary inequality  $|\sup A - \sup B| \leq \sup |A - B|$ , from Lipschitz condition (13) of the article on  $l, m$ , and from estimates (Lemma 3.1), with  $k=1$ , we have

$$\begin{aligned} & |v_i(s, x_s, y_s) - v_i(s, x'_s, y'_s)| \\ & \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T e^{-\beta(u-s)} \left( l(i, u, X_u^{s,x_s}, Y_u^{s,y_s}, E_u) - l(i, u, X_u^{s,x'_s}, Y_u^{s,y'_s}, E_u) \right) du \right. \\ & \quad \left. + e^{-\beta(T-s)} \left( m(X_T^{s,x_s}, Y_T^{s,y_s}) - m(X_T^{s,x'_s}, Y_T^{s,y'_s}) \right) \right] \\ & \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left| l(i, u, X_u^{s,x_s}, Y_u^{s,y_s}, E_u) - l(i, u, X_u^{s,x'_s}, Y_u^{s,y'_s}, E_u) \right| du \right. \\ & \quad \left. + \left| m(X_T^{s,x_s}, Y_T^{s,y_s}) - m(X_T^{s,x'_s}, Y_T^{s,y'_s}) \right| \right] \\ & \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left( |X_u^{s,x_s} - X_u^{s,x'_s}| + |Y_u^{s,y_s} - Y_u^{s,y'_s}| \right) du + \left( |X_T^{s,x_s} - X_T^{s,x'_s}| + |Y_T^{s,y_s} - Y_T^{s,y'_s}| \right) \right] \\ & \leq \sup_{E \in \mathcal{A}_i} \left[ \int_s^T \mathbb{E} \left( |X_u^{s,x_s} - X_u^{s,x'_s}| + \mathbb{E} |Y_u^{s,y_s} - Y_u^{s,y'_s}| \right) du + \left( \mathbb{E} |X_T^{s,x_s} - X_T^{s,x'_s}| + \mathbb{E} |Y_T^{s,y_s} - Y_T^{s,y'_s}| \right) \right] \\ & |v_i(s, x_s, y_s) - v_i(s, x'_s, y'_s)| \leq C \left( |x_s - x'_s| + |y_s - y'_s| \right). \end{aligned}$$

2. From linear growth condition (14) on  $l, m$ , and from estimates (Lemma 3.1), with  $k=1$ , we obtain

$$\begin{aligned} |v_i(s, x_s, y_s)| & \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left| l(i, u, X_u^{s,x_s}, Y_u^{s,y_s}, E_u) \right| du + \left| m(X_T^{s,x_s}, Y_T^{s,y_s}) \right| \right] \\ |v_i(s, x_s, y_s)| & \leq \rho \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left( 1 + |X_u^{s,x_s}| + |Y_u^{s,y_s}| \right) du + \left( 1 + |X_T^{s,x_s}| + |Y_T| \right) \right] \\ |v_i(s, x_s, y_s)| & \leq \rho \sup_{E \in \mathcal{A}_i} \left[ \int_s^T \left( 1 + \mathbb{E} |X_u^{s,x_s}| + \mathbb{E} |Y_u^{s,y_s}| \right) du + \left( 1 + \mathbb{E} |X_T^{s,x_s}| + \mathbb{E} |Y_T^{s,y_s}| \right) \right] \\ |v_i(s, x_s, y_s)| & \leq C (1 + |x_s| + |y_s|). \end{aligned}$$

## C Proof of Proposition 3.2

Let  $0 \leq t < s \leq T$ . To prove continuity property in time  $t$ , we use the dynamic programming principle.

$$\begin{aligned} v_i(t, x, y) & = \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_t^T e^{-\beta(u-t)} l(i, u, X_u^{t,x}, Y_u^{t,y}, E_u) du + e^{-\beta(T-t)} m(X_T^{t,x}, Y_T^{t,y}) \right] \\ & = \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_t^s e^{-\beta(u-t)} l(i, u, X_u^{t,x}, Y_u^{t,y}, E_u) du + e^{-\beta(s-t)} v(s, X_s^{t,x}, Y_s^{t,y}, i) \right] \\ & = \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_0^{s-t} e^{-\beta u} l(i, t+u, X_{t+u}^{t,x}, Y_{t+u}^{t,y}, E_{t+u}) du + e^{-\beta(s-t)} v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) \right]. \end{aligned}$$

$$\begin{aligned}
0 \leq v_i(t, x, y) - v_i(s, x_s, y_s) &= \sup_{E \in \mathcal{A}_i} \mathbf{E} \left[ \int_0^{s-t} e^{-\beta(u)} l(i, u, X_u^{t,x}, Y_u^{t,y}, E_u) du \right. \\
&\quad \left. + e^{-\beta(s-t)} (v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) - v(s, x_s, y_s, i)) \right. \\
&\quad \left. + (e^{-\beta(s-t)} - 1) v(s, x_s, y_s, i) \right].
\end{aligned}$$

Applying linear growth condition (13) of the article on  $l$ , noting that  $0 \leq 1 - e^{-\beta(s-t)} \leq \beta(s-t)$  and  $v$  satisfies proposition (3.1) of the article, we deduce that:

$$\begin{aligned}
&|v_i(t, x, y) - v_i(s, x_s, y_s)| \\
&\leq \sup_{E \in \mathcal{A}_i} \mathbf{E} \left[ \int_0^{s-t} |l(i, u, X_u^{t,x}, Y_u^{t,y}, E_u)| du + \left| e^{-\beta(s-t)} (v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) - v(s, x_s, y_s, i)) \right| \right. \\
&\quad \left. + \left| (e^{-\beta(s-t)} - 1) v(s, x_s, y_s, i) \right| \right] \\
&\leq (s-t)^{\frac{1}{2}} \left( \int_0^{s-t} \sup_{E \in \mathcal{A}_i} \mathbf{E} |l(i, u, X_u^{t,x}, Y_u^{t,y}, E_u)|^2 du \right)^{\frac{1}{2}} + \sup_{E \in \mathcal{A}_i} \mathbf{E} \left| e^{-\beta(s-t)} (v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) - v(s, x_s, y_s, i)) \right| \\
&\quad + \sup_{E \in \mathcal{A}_i} \mathbf{E} \left| (e^{-\beta(s-t)} - 1) v(s, x_s, y_s, i) \right| \\
&\leq (s-t)^{\frac{1}{2}} \left( \int_0^{s-t} \rho^2 \sup_{E \in \mathcal{A}_i} (1 + \mathbf{E}|X_u^{t,x}| + \mathbf{E}|p_u^{t,y}|)^2 du \right)^{\frac{1}{2}} + \sup_{E \in \mathcal{A}_i} \mathbf{E} \left| (v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) - v(s, x_s, y_s, i)) \right| \\
&\quad + \beta |s-t| \sup_{E \in \mathcal{A}_i} \mathbf{E} |v(s, x_s, y_s, i)| \\
&\leq |s-t|^{\frac{1}{2}} \left( \int_0^{s-t} \rho \sup_{E \in \mathcal{A}_i} (1 + \mathbf{E}|X_u^{t,x}| + \mathbf{E}|Y_u^{t,y}|) du \right) + \sup_{E \in \mathcal{A}_i} \mathbf{E} \left| (v(s, X_{s-t}^{t,x}, Y_{s-t}^{t,y}, i) - v(s, x_s, y_s, i)) \right| \\
&\quad + \beta |s-t| \sup_{E \in \mathcal{A}_i} \mathbf{E} |v(s, x_s, y_s, i)| \\
&\leq C' \left( |s-t|^{\frac{1}{2}} \int_0^{s-t} (1 + \mathbf{E}|B_u^{t,x}| + \mathbf{E}|P_u^{t,y}|) du + (|x-x_s| + |y-y_s|) + \beta(1 + |x_s| + |y_s|)|s-t| \right) \\
&\leq C \left[ (1 + |x| + |y|)|s-t|^{\frac{1}{2}} + |x-x_s| + |y-y_s| \right].
\end{aligned}$$

## D Proof of Theorem 3.1

We establish the viscosity super- and sub-solution properties, respectively in the following two steps.

Step 1.  $v_i(t, x, y)$ ,  $i = 1, 2$  is a viscosity super-solution of Eq. (17).

We already know that  $v \in \mathcal{C}^0([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ . We first note that  $v_i(T, x, y) = \kappa^\gamma \frac{x^{1-\gamma}}{1-\gamma}$  so, the boundary condition at time  $t = T$  is clearly satisfied. Let  $(s, x_s, y_s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$ ,  $i \in \mathcal{S}$  and  $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  such that  $v_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  has a local minimum at  $(s, x_s, y_s)$ . Let  $\mathbf{N}(x_s, y_s)$  to be a neighborhood of  $(s, x_s, y_s)$  where  $v_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  take its minimum, we introduce a new test-function  $\psi$  as follows:

$$\psi(\cdot, \cdot, \cdot, j) = \begin{cases} \phi(\cdot, \cdot, \cdot) + [v_i(s, x_s, y_s) - \phi(s, x_s, y_s)], & \text{if } j = i, \\ v_i(\cdot, \cdot, \cdot), & \text{if } j \neq i. \end{cases} \quad (23)$$

This helps us to suppose without any loss of generality that this minimum is equal to 0.

Let  $\tau_\alpha$  be the first jump time of  $\alpha(t)$  ( $= \alpha(t)^{x_s, y_s, i}$ ), i.e.  $\tau_\alpha = \min\{t \geq s : \alpha(t) \neq i\}$ . Then  $\tau_\alpha > s$ , a.s. Let  $\theta_s \in (s, \tau_\alpha)$  be such that the state  $(X_t^{x_s, i}, Y_t^{y_s, i})$  starts at  $(x_s, y_s)$  and stays in  $\mathbb{N}(x_s, y_s)$  for  $s \leq t \leq \theta_s$ . Applying the generalized Itô's formula to the switching process  $e^{-\beta t} \psi(t, X_t, Y_t, \alpha(t))$ , taking integral from  $t = s$  to  $t = \theta_s \wedge h$ , where  $h > 0$  is a positive constant, and then taking expectation we have

$$\begin{aligned}
& \mathbb{E}_{x_s, y_s, i} \left[ e^{-\beta \theta_s \wedge h} \psi(\theta_s \wedge h, X_{\theta_s \wedge h}, Y_{\theta_s \wedge h}, \alpha(\theta_s \wedge h)) \right] \\
&= e^{-\beta s} \psi(s, X_s, Y_s, i) + \mathbb{E}_{x_s, y_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ -\beta \psi(t, X_t, Y_t, \alpha(t)) + \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial t} \right. \right. \\
&\quad + X_t \left[ r_i - a_i X_t^\lambda - q E_i(t) \right] \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} + \theta(\bar{Y}_0 - Y_t) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial y} \\
&\quad + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial x^2} + \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial y^2} + q_{\alpha(t)j} (\psi(t, X_t, Y_t, j) - \psi(t, X_t, Y_t, \alpha(t))) \\
&\quad \left. \left. + \int_{\mathbb{R} \setminus \{0\}} \left( \psi(t, X_t + \eta(t, X_t, Y_t, \alpha(t), z), Y_t) - \psi(t, X_t, Y_t, \alpha(t)) \right. \right. \right. \\
&\quad \left. \left. \left. - \eta(t, X_t, Y_t, \alpha(t), z) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} \right) \nu_i(dz) \right\} dt \right], \quad \alpha(t) \neq j. \quad (24)
\end{aligned}$$

From hypothesis, for any  $t \in [s, \theta_s \wedge h]$

$$v_i(t, X_t^{x_s}, Y_t^{y_s}) \geq \phi(t, X_t^{x_s}, Y_t^{y_s}) + v_i(s, x_s, y_s) - \phi(s, x_s, y_s) \geq \psi(t, X_t^{x_s}, Y_t^{y_s}, i). \quad (25)$$

Moreover, we have  $q_{ij}(v_j(t, X_t, Y_t) - v_i(t, X_t, Y_t)) \leq q_{ij}(\psi(t, X_t, Y_t, j) - \psi(t, X_t, Y_t, i))$ , by definition of  $\psi$ . Recalling that  $(X_s^{y_s}, Y_s^{y_s}) = (x_s, y_s)$  and using Eq. (23) and Eq. (25) we have

$$\begin{aligned}
& \mathbb{E}_{x_s, y_s, i} \left[ e^{-\beta \theta_s \wedge h} \psi(\theta_s \wedge h, X_{\theta_s \wedge h}, Y_{\theta_s \wedge h}, \alpha(\theta_s \wedge h)) \right] \geq \\
&+ e^{-\beta s} v_i(s, x_s, y_s) + \mathbb{E}_{x_s, y_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ -\beta v_i(t, X_t, Y_t) + \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial t} \right. \right. \\
&\quad + X_t \left[ r_i - a_i X_t^\lambda - q E_i(t) \right] \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} + \theta(\bar{Y}_0 - Y_t) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial y} \\
&\quad + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial x^2} + \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial y^2} \\
&\quad \left. \left. + q_{ij}(v_j(t, X_t, Y_t) - v_i(t, X_t, Y_t)) \right. \right. \\
&\quad \left. \left. + \int_{\mathbb{R} \setminus \{0\}} \left( \psi(t, X_t + \eta(t, X_t, Y_t, \alpha(t), z), Y_t) - \psi(t, X_t, Y_t, \alpha(t)) \right. \right. \right. \\
&\quad \left. \left. \left. - \eta(t, X_t, Y_t, \alpha(t), z) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} \right) \nu_i(dz) \right\} dt \right]. \quad (26)
\end{aligned}$$

By Bellman's principle

$$\begin{aligned}
e^{-\beta s} \psi(s, x_s, y_s, i) &= e^{-\beta s} v_i(s, x_s, y_s) = \sup_{E \in \mathcal{A}_i} \mathbb{E}_{x_s, y_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} l(i, t, X_t^{s, x_s}, Y_t^{s, y_s}, E_t) dt \right. \\
&\quad \left. + e^{-\beta(\theta_s \wedge h)} v_i(\theta_s \wedge h, X_{\theta_s \wedge h}^{s, x_s}, Y_{\theta_s \wedge h}^{s, y_s}) \right] \\
&\geq \sup_{E \in \mathcal{A}_i} \mathbb{E}_{x_s, y_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} l(i, t, X_t^{s, x_s}, Y_t^{s, y_s}, E_t) dt \right. \\
&\quad \left. + e^{-\beta(\theta_s \wedge h)} \psi(\theta_s \wedge h, X_{\theta_s \wedge h}^{s, x_s}, Y_{\theta_s \wedge h}^{s, y_s}, i) \right]. \quad (27)
\end{aligned}$$

Setting  $\tau = \mathbb{E}(\theta_s \wedge h)$  combining (26) and (27) and multiplying both sides by  $1/(\tau - s) > 0$ , we obtain

$$\begin{aligned}
\sup_{E \in \mathcal{A}_i} \mathbb{E}_{x_s, y_s, i} &\left[ \frac{1}{\tau - s} \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ \beta v_i(t, X_t, Y_t) - \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial t} \right. \right. \\
&- X_t \left[ r_i - a_i X_t^\lambda - q E_i(t) \right] \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} - \theta(\bar{Y}_0 - Y_t) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial y} \\
&- \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial y^2} \\
&\quad \left. - q_{ij} [v_j(t, X_t, Y_t) - v_i(t, X_t, Y_t)] \right. \\
&- \int_{\mathbb{R} \setminus \{0\}} \left( \psi(t, X_t + \eta(t, X_t, Y_t, \alpha(t), z), Y_t) - \psi(t, X_t, Y_t, \alpha(t)) \right. \\
&\quad \left. \left. - \eta(t, X_t, Y_t, \alpha(t), z) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} \right) \nu_i(dz) - l(i, t, X_t, Y_t, E_t) \right\} dt \Big] \geq 0. \quad (28)
\end{aligned}$$

Letting  $\tau \downarrow s$  and using the dominated convergence theorem, it turns out that

$$\begin{aligned}
e^{-\beta s} &\left[ - \frac{\partial \psi(s, x_s, y_s, i)}{\partial t} + \inf_{E \in \mathcal{A}_i} \left\{ \beta v_i(s, x_s, y_s) - \right. \right. \\
&\quad \left. \left. X_s \left[ r_i - a_i x_s^\lambda - q E_i(s) \right] \frac{\partial \psi(s, x_s, y_s, i)}{\partial x} - \theta(\bar{Y}_0 - y_s) \frac{\partial \psi(s, x_s, y_s, i)}{\partial y} \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \sigma^2 x_s^2 \frac{\partial^2 \psi(s, x_s, y_s, i)}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \psi(s, x_s, y_s, i)}{\partial y^2} \right. \right. \\
&\quad \left. \left. - q_{ij} [v_j(s, x_s, y_s) - v_i(s, x_s, y_s)] \right. \right. \\
&\quad \left. \left. - \int_{\mathbb{R} \setminus \{0\}} \left( \psi(s, x_s + \eta(s, x_s, y_s, i, z), y_s) - \psi(s, x_s, y_s, i) \right. \right. \right. \\
&\quad \left. \left. \left. - \eta(s, x_s, y_s, i, z) \frac{\partial \psi(s, x_s, y_s, i)}{\partial x} \right) \nu_i(dz) - l(i, s, x_s, y_s, E_s) \right\} \right] \geq 0. \quad (29)
\end{aligned}$$

This shows that the value function  $v_i(t, x, y)$ ,  $i = 1, 2$ , satisfies the viscosity super-solution property (20).

Step 2.  $v_i(t, x, y)$ ,  $i = 1, 2$ , is a viscosity sub-solution of (17).

We argue by contradiction. Assume that there exist an  $i_0 \in \mathcal{S}$ , a point  $(s, x_s, y_s) \in [0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$  and a testing function  $\phi_{i_0} \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  such that  $v_{i_0}(\cdot, \cdot, \cdot) - \phi_{i_0}(\cdot, \cdot, \cdot)$

has a local maximum at  $(s, x_s, y_s)$  in a bounded neighborhood  $\mathbb{N}(x_s, y_s)$ ,  $v_{i_0}(s, x_s, y_s) = \phi_{i_0}(s, x_s, y_s)$ , and

$$\begin{aligned} \min \left[ -\frac{\partial \phi_{i_0}(s, x_s, y_s)}{\partial t} + \inf_{E \in \mathcal{A}_{i_0}} \left\{ \beta v_{i_0}(s, x_s, y_s) - \right. \right. \\ \left. \left. x_s \left[ r_{i_0} - a_i x_s^\lambda - q E_{i_0}(s) \right] \frac{\partial \phi_{i_0}(s, x_s, y_s)}{\partial x} - \theta(\bar{Y}_0 - y_s) \frac{\partial \phi_{i_0}(s, x_s, y_s)}{\partial y} \right. \right. \\ \left. \left. - \frac{1}{2} \sigma^2 x_s^2 \frac{\partial^2 \phi_{i_0}(s, x_s, y_s)}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \phi_{i_0}(s, x_s, y_s)}{\partial y^2} - q_{i_0 j} [v_j(s, x_s, y_s) - v_{i_0}(s, x_s, y_s)] \right. \right. \\ \left. \left. - \int_{\mathbb{R} \setminus \{0\}} \left( \phi_{i_0}(s, x_s + \eta(s, x_s, y_s, i_0, z), y_s) - \phi_{i_0}(s, x_s, y_s) \right) \right. \right. \\ \left. \left. - \eta(s, x_s, y_s, i_0, z) \frac{\partial \phi_{i_0}(s, x_s, y_s)}{\partial x} \right) \nu_{i_0}(dz) - l(i_0, s, x_s, y_s, E_s) \right], \quad v_{i_0}(T, x_s, y_s) - \kappa^\gamma \frac{x_s^{1-\gamma}}{1-\gamma} \right] > 0, \quad i_0 \neq j. \end{aligned} \quad (30)$$

By the continuity of all functions involved in (30) ( $v_{i_0}, \frac{\partial \phi_{i_0}}{\partial x}, \frac{\partial^2 \phi_{i_0}}{\partial x^2}, q_{ij}, l, \dots$ ), there exists a  $\delta > 0$  and an open ball  $\mathbb{B}_\delta(x_s, y_s) \subset \mathbb{N}(x_s, y_s)$  such that,

$$\begin{aligned} -\frac{\partial \phi_{i_0}(t, x, y)}{\partial t} + \inf_{E \in \mathcal{A}_{i_0}} \left\{ \beta v_{i_0}(t, x, y) - \right. \\ \left. x \left[ r_{i_0} - a_i x^\lambda - q E_{i_0}(t) \right] \frac{\partial \phi_{i_0}(t, x, y)}{\partial x} - \theta(\bar{Y}_0 - y) \frac{\partial \phi_{i_0}(t, x, y)}{\partial y} \right. \\ \left. - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi_{i_0}(t, x, y)}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \phi_{i_0}(t, x, y)}{\partial y^2} - q_{i_0 j} [v_j(t, x, y) - v_{i_0}(t, x, y)] \right. \\ \left. - \int_{\mathbb{R} \setminus \{0\}} \left( \phi_{i_0}(t, x + \eta(t, x, y, i_0, z), y_s) - \phi_{i_0}(t, x, y) \right) \right. \\ \left. - \eta(t, x, y, i_0, z) \frac{\partial \phi_{i_0}(t, x, y)}{\partial x} \right) \nu_{i_0}(dz) - l(i_0, t, x, y, E_t) \left. \right\} > \delta, \quad i_0 \neq j, \quad (t, x, y) \in \mathbb{B}_\delta(x_s, y_s) \end{aligned} \quad (31)$$

and

$$v_{i_0}(T, x, y) - \kappa^\gamma \frac{x^{1-\gamma}}{1-\gamma} > \delta \quad (t, x, y) \in \mathbb{B}_\delta(x_s, y_s).$$

Let  $\theta_\delta = \min\{t \geq s : (t, X_t, Y_t) \notin \mathbb{B}_\delta(x_s, y_s)\}$  be the first exit time of  $(t, X_t, Y_t)$  ( $= (t, X_t^{s, x_s}, Y_t^{s, y_s})$ ) from  $\mathbb{B}_\delta(x_s, y_s)$ . Let  $\theta_s = \theta_\delta \wedge \tau_\alpha$  where  $\tau_\alpha$  is the first stopping time of  $\alpha(t)^{x_s, y_s, i_0}$ . Then  $\theta_s > 0$ , a.s.. For  $0 \leq t \leq \theta_s$ , we have

$$\begin{aligned} \beta v_{i_0}(t, X_t, Y_t) - \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial t} \\ - X_t \left[ r_{i_0} - a_i X_t^\lambda - q E_{i_0}(t) \right] \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial x} - \theta(\bar{Y}_0 - Y_t) \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial y} \\ - \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \phi_{i_0}(t, X_t, Y_t)}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \phi_{i_0}(t, X_t, Y_t)}{\partial y^2} - q_{i_0 j} [v_j(t, X_t, Y_t) - v_{i_0}(t, X_t, Y_t)] \\ - \int_{\mathbb{R} \setminus \{0\}} \left( \phi_{i_0}(t, X_t + \eta(t, X_t, Y_t, i_0, z), y_s) - \phi_{i_0}(t, X_t, Y_t) \right) \\ - \eta(t, X_t, Y_t, i_0, z) \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial x} \Big) \nu_{i_0}(dz) - l(i_0, t, X_t, Y_t, E_t) > \delta, \quad i_0 \neq j, \quad (t, X_t, Y_t) \in \mathbb{B}_\delta(x_s, y_s) \end{aligned} \quad (32)$$

and

$$v_{i_0}(T, x, y) - \kappa^\gamma \frac{x^{1-\gamma}}{1-\gamma} > \delta \quad (t, x, y) \in \mathbf{B}_\delta(x_s, y_s). \quad (33)$$

As previously, we can replace  $\phi_{i_0}$  by a new test-function  $\psi$  defined as follows:

$$\psi(\cdot, \cdot, \cdot, j) = \begin{cases} \phi_{i_0}(\cdot, \cdot, \cdot), & \text{if } j = i_0, \\ v_{i_0}(\cdot, \cdot, \cdot), & \text{if } j \neq i_0. \end{cases} \quad (34)$$

For any first exit time  $\tau \in [s, T]$ . Applying Itô's formula to the switching process  $e^{-\beta t} \psi(t, X_t, Y_t, \alpha(t))$ , taking integral from  $t = s$  to  $t = (\theta_s \wedge \tau)^-$  and then taking expectation yield

$$\begin{aligned} & \mathbb{E}_{x_s, y_s, i} \left[ e^{-\beta \theta \wedge \tau} \psi(\theta_s \wedge \tau, X_{\theta_s \wedge \tau}, Y_{\theta_s \wedge \tau}, \alpha(\theta_s \wedge \tau)) \right] \\ &= e^{-\beta s} v_i(s, x_s, y_s) + \mathbb{E}_{x_s, y_s, i} \left[ \int_s^{(\theta_s \wedge \tau)^-} e^{-\beta t} \left\{ -\beta \psi(t, X_t, Y_t, \alpha(t)) + \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial t} \right. \right. \\ & \quad + X_t \left[ r_i - a_i X_t^\lambda - q E_i(t) \right] \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} + \theta(\bar{Y}_0 - Y_t) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial y} \\ & \quad + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial x^2} + \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \psi(t, X_t, Y_t, \alpha(t))}{\partial y^2} \\ & \quad \left. + q_{\alpha(t)j} (\psi(t, X_t, Y_t, j) - \psi(t, X_t, Y_t, \alpha(t))) \right. \\ & \quad \left. + \int_{\mathbb{R} \setminus \{0\}} \left( \psi(t, X_t + \eta(t, X_t, Y_t, \alpha(t), z), Y_t) - \psi(t, X_t, Y_t, \alpha(t)) \right. \right. \\ & \quad \left. \left. - \eta(t, X_t, Y_t, \alpha(t), z) \frac{\partial \psi(t, X_t, Y_t, \alpha(t))}{\partial x} \right) \nu_i(dz) \right\} dt \Bigg], \quad \alpha(t) \neq j. \quad (35) \end{aligned}$$

in which we used  $\mathbb{E}_{x_s, y_s, i} \left[ e^{-\beta \theta_s \wedge \tau} \psi(\theta_s \wedge \tau, X_{\theta_s \wedge \tau}, Y_{\theta_s \wedge \tau}, \alpha(\theta_s \wedge \tau)) \right] = \mathbb{E}_{x_s, y_s, i} \left[ e^{-\beta \theta_s \wedge \tau} \psi(\theta_s \wedge \tau, X_{\theta_s \wedge \tau}, Y_{\theta_s \wedge \tau}, \alpha(\theta_s \wedge \tau)) \right]$  due to continuity. Noting that the integrand in the RHS of (35) is continuous in t. Using (32), (33) and that  $v_{i_0}(t, X_t, Y_t) \leq \phi_{i_0}(t, X_t, Y_t)$  in (35). Also noting that  $\alpha(t) = i_0$  for  $0 \leq t \leq \theta_s$ , it follows

$$\begin{aligned} & e^{-\beta s} v_{i_0}(s, x_s, y_s) \\ & \geq \mathbb{E}_{x_s, y_s, i_0} \left[ e^{-\beta \theta_s \wedge \tau} \phi_{i_0}(\theta_s \wedge \tau, X_{\theta_s \wedge \tau}, Y_{\theta_s \wedge \tau}, \alpha(\theta_s \wedge \tau)) \right. \\ & \quad + \int_s^{(\theta_s \wedge \tau)^-} e^{-\beta t} \left\{ \beta v_{i_0}(t, X_t, Y_t) - \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial t} \right. \\ & \quad - X_t \left[ r_{i_0} - a_i X_t^\lambda - q E_{i_0}(t) \right] \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial x} - \theta(\bar{Y}_0 - Y_t) \frac{\partial \phi_{i_0}(t, X_t, Y_t)}{\partial y} \\ & \quad - \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 \phi_{i_0}(t, X_t, Y_t)}{\partial x^2} - \frac{1}{2} \sigma_Y^2 \frac{\partial^2 \phi_{i_0}(t, X_t, Y_t)}{\partial y^2} \\ & \quad \left. \left. - q_{i_0 j} [v_j(t, X_t, Y_t) - v_{i_0}(t, X_t, Y_t)] \right. \right. \\ & \quad \left. \left. - \int_{\mathbb{R} \setminus \{0\}} \left( \psi(t, X_t + \eta(t, X_t, Y_t, i_0, z), Y_t) - \psi(t, X_t, Y_t, i_0) \right. \right. \right. \\ & \quad \left. \left. \left. - \eta(t, X_t, Y_t, i_0, z) \frac{\partial \psi(t, X_t, Y_t, i_0)}{\partial x} \right) \nu_i(dz) \right\} dt \right], \quad i_0 \neq j \quad (36) \end{aligned}$$

i.e

$$\begin{aligned}
& e^{-\beta s} v_{i_0}(s, x_s, y_s) \\
& \geq \mathbb{E}_{x_s, y_s, i_0} \left[ e^{-\beta \tau} v_{i_0}(\tau, X_\tau, Y_\tau, \alpha(\tau)) \mathbf{1}_{\{\tau < \theta_s\}} + e^{-\beta \theta_s} v_{i_0}(\theta, X_{\theta_s}, Y_{\theta_s}, \alpha(\theta_s)) \mathbf{1}_{\{\tau \geq \theta_s\}} \right. \\
& \quad \left. + \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} \{l(i_0, t, X_t, Y_t, E_t) + \delta\} dt \right] \\
& \geq \mathbb{E}_{x_s, y_s, i_0} \left[ e^{-\beta \tau} \left[ \kappa^\gamma \frac{X_\tau^{1-\gamma}}{1-\gamma} + \delta \right] \mathbf{1}_{\{\tau < \theta_s\}} + e^{-\beta \theta_s} v_{i_0}(\theta_s, X_{\theta_s}, Y_{\theta_s}, \alpha(\theta_s)) \mathbf{1}_{\{\tau \geq \theta_s\}} \right. \\
& \quad \left. + \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} \{l(i_0, t, X_t, Y_t, E_t) + \delta\} dt \right] \\
& \geq \mathbb{E}_{x_s, y_s, i_0} \left[ + \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} \{l(i_0, t, X_t, Y_t, E_t)\} dt + e^{-\beta \theta} v_{i_0}(\theta_s, X_{\theta_s}, Y_{\theta_s}, \alpha(\theta_s)) \mathbf{1}_{\{\tau \geq \theta_s\}} \right. \\
& \quad \left. + e^{-\beta \tau} \left[ \kappa^\gamma \frac{X_\tau^{1-\gamma}}{1-\gamma} \right] \mathbf{1}_{\{\tau < \theta_s\}} \right] + \delta \mathbb{E}_{x_s, y_s, i_0} \left[ \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} dt + e^{-\beta \tau} \mathbf{1}_{\{\tau < \theta_s\}} \right]. \quad (37)
\end{aligned}$$

Now considering the estimate of the term  $\mathbb{E}_{x_s, y_s, i_0} \left[ \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} dt + e^{-\beta \tau} \mathbf{1}_{\{\tau < \theta_s\}} \right]$ , there exists a positive constant  $C_0$  such that,

$$\mathbb{E}_{x_s, y_s, i_0} \left[ \int_s^{(\theta_s \wedge \tau)} e^{-\beta t} dt + e^{-\beta \tau} \mathbf{1}_{\{\tau < \theta_s\}} \right] \geq C_0 (1 - \mathbb{E}_{x_s, y_s, i_0} [e^{-\beta \tau \alpha}]).$$

It follows that

$$\begin{aligned}
& v_{i_0}(s, x_s, y_s) \\
& \geq \sup_{\tau \in [s, T], E \in \mathcal{A}} \mathbb{E}_{x_s, y_s, i_0} \left[ + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \{l(i_0, t, X_t, Y_t, E_t)\} dt \right. \\
& \quad \left. + e^{-\beta \theta} v_{i_0}(\theta, X_\theta, Y_\theta, \alpha(\theta)) \mathbf{1}_{\{\tau \geq \theta\}} + e^{-\beta \tau} \left[ \kappa^\gamma \frac{X_\tau^{1-\gamma}}{1-\gamma} \right] \mathbf{1}_{\{\tau < \theta_s\}} \right] \\
& \quad + C_0 \delta (1 - \mathbb{E}_{x_s, y_s, i_0} [e^{-\beta \tau \alpha}]) \quad (38)
\end{aligned}$$

which is a contradiction to the DP principle since  $\mathbb{E}_{x_s, y_s, i_0} [e^{-\beta \tau \alpha}] < 1$ . Therefore the value function  $v_i(t, x, y)$ ,  $i = 1, 2$ , is a viscosity sub-solution of the system (18).

This completes the proof of Theorem 3.1.

## E Proof of Theorem 3.2

For  $\varrho, \epsilon, \delta, \lambda > 0$ , we define the auxiliary functions  $\phi : (0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$  and  $\Xi : [0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{S}$  by

$$\phi(t, (x, y), (x', y')) = \frac{\varrho}{t} + \frac{1}{2\epsilon} |(x, y) - (x', y')|^2 + \delta e^{\lambda(T-t)} (|(x, y)|^2 + |(x', y')|^2)$$

and

$$\Xi(t, (x, y), (x', y'), i) = v_i(t, x, y) - u_i(t, x', y') - \phi(t, (x, y), (x', y')).$$

By using the linear growth of  $v_i$  and  $u_i$ , we have for each  $i \in \mathcal{S}$

$$\lim_{|(x, y)| + |(x', y')| \rightarrow \infty} \Xi(t, (x, y), (x', y'), i) = -\infty.$$

Then, since  $v_i$  and  $u_i$  are uniformly continuous with respect to  $(t, x, y)$  on each compact subset of  $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S}$  and that  $\mathcal{S}$  is a finite set,  $\Xi$  attains its global maximum at some finite point belonging to a compact  $K \subset [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{S}$  say,  $(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon})$ . Observing that  $2\Xi(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \geq \Xi(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon}) + \Xi(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon})$  and using the uniform continuity of  $v_i$  and  $u_i$  on  $K$  we have

$$\begin{aligned} & \frac{1}{\epsilon} |(x_{1\delta\epsilon}, y_{1\delta\epsilon}) - (x_{2\delta\epsilon}, y_{2\delta\epsilon})|^2 \\ & \leq v_i(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - v_i(t_{\delta\epsilon}, (x_{2\delta\epsilon}, y_{2\delta\epsilon})) + u_i(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - u_i(t_{\delta\epsilon}, (x_{2\delta\epsilon}, y_{2\delta\epsilon})) \\ & \leq 2C|(x_{1\delta\epsilon}, y_{1\delta\epsilon}) - (x_{2\delta\epsilon}, y_{2\delta\epsilon})|. \end{aligned}$$

Thus,

$$|(x_{1\delta\epsilon}, y_{1\delta\epsilon}) - (x_{2\delta\epsilon}, y_{2\delta\epsilon})| \leq 2C\epsilon \quad (39)$$

where  $C$  is a positive constant independent of  $\varrho, \epsilon, \delta, \lambda$ . From the inequality,

$$2\Xi(T, (0, 0), (0, 0), \alpha_{\delta\epsilon}) \leq 2\Xi(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon})$$

and the linear growth for  $v_i$  and  $u_i$ , we have:

$$\begin{aligned} \delta(|(x_{1\delta\epsilon}, y_{1\delta\epsilon})|^2 + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})|^2) & \leq e^{-\lambda(T-t_{\delta\epsilon})} \left[ v_i(t_{\delta\epsilon}, x_{1\delta\epsilon}, y_{1\delta\epsilon}) - v_i(T, 0, 0) \right. \\ & \quad \left. + u_i(T, 0, 0) - u_i(t_{\delta\epsilon}, x_{2\delta\epsilon}, y_{2\delta\epsilon}) \right] \\ & \leq e^{-\lambda(T-t_{\delta\epsilon})} C_2 (1 + |(x_{1\delta\epsilon}, y_{1\delta\epsilon})| + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})|). \end{aligned} \quad (40)$$

It follows that

$$\frac{\delta(|(x_{1\delta\epsilon}, y_{1\delta\epsilon})|^2 + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})|^2)}{(1 + |(x_{1\delta\epsilon}, y_{1\delta\epsilon})| + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})|)} \leq C_2.$$

Consequently, there exists  $C_\delta > 0$  such that,

$$|(x_{1\delta\epsilon}, y_{1\delta\epsilon})| + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})| \leq C_\delta. \quad (41)$$

This inequality implies that for any fixed  $\delta \in (0, 1)$ , the sets  $\{(x_{1\delta\epsilon}, y_{1\delta\epsilon}), \epsilon > 0\}$  and  $\{(x_{2\delta\epsilon}, y_{2\delta\epsilon}), \epsilon > 0\}$  are bounded by  $C_\delta$  independent of  $\epsilon$ . It follows from inequalities (40) and (41) that, possibly if necessary along a subsequence, named again  $(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon})$  that there exists  $(x_{1\delta_0}, y_{1\delta_0}) \in \mathbb{R}_+^2$ ,  $t_{\delta_0} \in (0, T]$  and  $\alpha_{\delta_0} \in \mathcal{S}$  such that:  $\lim_{\epsilon \downarrow 0} (x_{1\delta\epsilon}, y_{1\delta\epsilon}) = (x_{1\delta_0}, y_{1\delta_0}) = \lim_{\epsilon \downarrow 0} (x_{2\delta\epsilon}, y_{2\delta\epsilon})$ ,  $\lim_{\epsilon \downarrow 0} t_{\delta\epsilon} = t_{\delta_0}$ ,  $\lim_{\epsilon \downarrow 0} \alpha_{\delta\epsilon} = \alpha_{\delta_0}$ .

If  $t_{\delta\epsilon} = T$  then writing that  $\Xi(t, (x, y), (x, y), \alpha_{\delta\epsilon}) \leq \Xi(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon})$ , we have

$$\begin{aligned} u_i(t, x, y) - v_i(t, x, y) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}(|(x, y)|^2) \\ & \leq u_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - v_i(T, (x_{2\delta\epsilon}, y_{2\delta\epsilon})) - \frac{\varrho}{T} \\ & \quad - \frac{1}{2\epsilon} |(x_{1\delta\epsilon}, y_{1\delta\epsilon}) - (x_{2\delta\epsilon}, y_{2\delta\epsilon})|^2 - \delta(|(x_{1\delta\epsilon}, y_{1\delta\epsilon})|^2 + |(x_{2\delta\epsilon}, y_{2\delta\epsilon})|^2) \\ & \leq u_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - v_i(T, (x_{2\delta\epsilon}, y_{2\delta\epsilon})) \\ & = [u_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - v_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon}))] \\ & \quad + [v_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) - v_i(T, (x_{2\delta\epsilon}, y_{2\delta\epsilon}))] \\ & \leq C_1 |(x_{1\delta\epsilon}, y_{1\delta\epsilon}) - (x_{2\delta\epsilon}, y_{2\delta\epsilon})| \end{aligned}$$

where the last inequality follows from the uniform continuity of  $v_i$  and by assumption that  $u_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon})) = \kappa\gamma \frac{x_{1\delta\epsilon}^{1-\gamma}}{1-\gamma} = v_i(T, (x_{1\delta\epsilon}, y_{1\delta\epsilon}))$ . Sending  $\varrho, \epsilon, \delta \downarrow 0$  and using estimate (39), we have:  $u_i(t, x, y) \leq v_i(t, x, y)$ .

Assume now that  $t_{\delta\epsilon} < T$ .

Applying Lemma 3.2 with  $u_i$ ,  $v_i$  and  $\phi(t, (x, y), (x', y'))$  at the point  $(t_{\delta\epsilon}, (x_{1\delta\epsilon}, y_{1\delta\epsilon}), (x_{2\delta\epsilon}, y_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \in (0, T) \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{S}$ , for any  $\zeta \in (0, 1)$  there are  $d \in \mathbb{R}, M_{\delta\epsilon}, N_{\delta\epsilon} \in \mathbb{S}^2$  such that:

$$\begin{aligned} & \left( d - \frac{\varrho}{t_{\delta\epsilon}^2} - \lambda \delta e^{\lambda(T-t_{\delta\epsilon})} (|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2), \frac{1}{\epsilon} ((x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})) \right. \\ & \quad \left. + 2\delta e^{\lambda(T-t_{\delta\epsilon})} (x_{\delta\epsilon}, y_{\delta\epsilon}), M_{\delta\epsilon} + 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \in \bar{\mathcal{P}}^{2,+} u(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}, i) \\ & \left( d, \frac{1}{\epsilon} ((x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})) - 2\delta e^{\lambda(T-t_{\delta\epsilon})} (x'_{\delta\epsilon}, y'_{\delta\epsilon}), N_{\delta\epsilon} - 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \in \bar{\mathcal{P}}^{2,-} v(t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon}, i) \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{\zeta} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} & \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \leq D_{(x,y),(x',y')}^2 \phi(t_{\delta\epsilon}, (x_{\delta\epsilon}, y_{\delta\epsilon}), (x'_{\delta\epsilon}, y'_{\delta\epsilon})) \\ & \quad + \zeta \left( D_{(x,y),(x',y')}^2 \phi(t_{\delta\epsilon}, (x_{\delta\epsilon}, y_{\delta\epsilon}), (x'_{\delta\epsilon}, y'_{\delta\epsilon})) \right)^2 \\ & \leq \frac{\epsilon + \zeta(2 + 4\delta\epsilon e^{\lambda(T-t)})}{\epsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (2\delta + 4\zeta\delta^2\epsilon e^{\lambda(T-t)}) e^{\lambda(T-t)} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

Letting  $\delta \downarrow 0$  and taking  $\zeta = \frac{\epsilon}{2}$ , we obtain

$$-\frac{1}{\epsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \leq \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

It follows that

$$\begin{aligned} & (x_{\delta\epsilon}, y_{\delta\epsilon}) M_{\delta\epsilon} \begin{pmatrix} x_{\delta\epsilon} \\ y_{\delta\epsilon} \end{pmatrix} - (x'_{\delta\epsilon}, y'_{\delta\epsilon}) N_{\delta\epsilon} \begin{pmatrix} x'_{\delta\epsilon} \\ y'_{\delta\epsilon} \end{pmatrix} \\ & = ((x_{\delta\epsilon}, y_{\delta\epsilon}), (x'_{\delta\epsilon}, y'_{\delta\epsilon})) \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} x_{\delta\epsilon} \\ y_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} x'_{\delta\epsilon} \\ y'_{\delta\epsilon} \end{pmatrix} \end{pmatrix} \\ & \leq ((x_{\delta\epsilon}, y_{\delta\epsilon}), (x'_{\delta\epsilon}, y'_{\delta\epsilon})) \left[ \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right] \begin{pmatrix} \begin{pmatrix} x_{\delta\epsilon} \\ y_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} x'_{\delta\epsilon} \\ y'_{\delta\epsilon} \end{pmatrix} \end{pmatrix} \\ & \leq \frac{2}{\epsilon} |(x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2. \end{aligned}$$

Furthermore, the definition of the viscosity subsolution  $u_i$  and supersolution  $v_i$  implies that

$$\begin{aligned} \min & \left[ \beta u_{i_0}(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - \left( d - \frac{\varrho}{t_{\delta\epsilon}^2} - \lambda \delta e^{\lambda(T-t_{\delta\epsilon})} (|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2) \right) \right. \\ & \quad + \inf_{E \in \mathcal{A}_{i_0}} \left\{ -x_{\delta\epsilon} \left[ r_{i_0} - a_i x_{\delta\epsilon}^\lambda - q E_i(s) \right] \left( \frac{1}{\epsilon} (x_{\delta\epsilon} - x'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} x_{\delta\epsilon} \right) \right. \\ & \quad - \theta(\bar{Y}_0 - y_{\delta\epsilon}) \left( \frac{1}{\epsilon} (x_{\delta\epsilon} - y'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} y_{\delta\epsilon} \right) - \frac{1}{2} (\sigma x_{\delta\epsilon}, \sigma y) (M_{\delta\epsilon} + 2\delta e^{\lambda(T-t)} I) \begin{pmatrix} \sigma x_{\delta\epsilon} \\ \sigma y \end{pmatrix} \\ & \quad - q_{i_0 j} [u_j(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - u_{i_0}(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon})] \\ & \quad \left. \left. - l(i_0, t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}, E_{t_{\delta\epsilon}}) \right\}, \quad u_{i_0}(T, x_{\delta\epsilon}, y_{\delta\epsilon}) - \kappa^\gamma \frac{x_{\delta\epsilon}^{1-\gamma}}{1-\gamma} \right] \leq 0, \quad i_0 \neq j \end{aligned}$$

and

$$\begin{aligned} \min & \left[ \beta v_{i_0}(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - d + \inf_{E \in \mathcal{A}_{i_0}} \left\{ -x_{\delta\epsilon} \left[ r_{i_0} - a_i x_{\delta\epsilon}^\lambda - q E_i(s) \right] \left( \frac{1}{\epsilon} (x_{\delta\epsilon} - x'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} x'_{\delta\epsilon} \right) \right. \\ & \quad - \theta(\bar{Y}_0 - y_{\delta\epsilon}) \left( \frac{1}{\epsilon} (y_{\delta\epsilon} - y'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} y'_{\delta\epsilon} \right) - \frac{1}{2} (\sigma x'_{\delta\epsilon}, \sigma P) (N_{\delta\epsilon} - 2\delta e^{\lambda(T-t)} I) \begin{pmatrix} \sigma x'_{\delta\epsilon} \\ \sigma Y \end{pmatrix} \\ & \quad - q_{i_0 j} [v_j(t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon})] \\ & \quad \left. \left. - l(i_0, t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon}, E_{t_{\delta\epsilon}}) \right\}, \quad v_{i_0}(T, x_{\delta\epsilon}, y_{\delta\epsilon}) - \kappa^\gamma \frac{x_{\delta\epsilon}^{1-\gamma}}{1-\gamma} \right] \geq 0, \quad i_0 \neq j. \end{aligned}$$

Let us define operators  $A^E(x, v, \phi, X, Z)$  and  $B^E(x, v)$ .

$$A^E(t, x, y, w, X, YZ) = x \left[ r_{i_0} - a_i x^\lambda - q E_i(t) \right] X + \theta(\bar{Y}_0 - y) Y + \frac{1}{2} w Z w'$$

$$B^E(t, x, y, v) = q_{i_0 j} [v_j(t, x, y) - v_{i_0}(t, x, y)].$$

By subtracting these last two inequalities and remarking that  $\min(x; y) - \min(z; t) \leq 0$  implies either  $x - z \leq 0$  or  $y - t \leq 0$ , we divide our consideration into two cases:

Case 1

$$\begin{aligned} & \beta [u_{i_0}(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon})] + \frac{\varrho}{t_{\delta\epsilon}^2} + \lambda \delta e^{\lambda(T-t_{\delta\epsilon})} (|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2) \\ & \leq \sup_{E \in \mathcal{A}_{i_0}} \left\{ l(i_0, t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}, E_{t_{\delta\epsilon}}) - l(i_0, t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon}, E_{t_{\delta\epsilon}}) \right\} \\ & \quad + \sup_{E \in \mathcal{A}_{i_0}} \left\{ A^E \left( t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}, (\sigma x_{\delta\epsilon}; \sigma P), \frac{1}{\epsilon} (x_{\delta\epsilon} - x'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} x_{\delta\epsilon}, \right. \right. \\ & \quad \left. \left. \frac{1}{\epsilon} (y_{\delta\epsilon} - y'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} y_{\delta\epsilon}, M_{\delta\epsilon} + 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \right. \\ & \quad - A^E \left( t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon}, (\sigma b'_{\delta\epsilon}; \sigma P), \frac{1}{\epsilon} (x_{\delta\epsilon} - x'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} x'_{\delta\epsilon}, \frac{1}{\epsilon} (y_{\delta\epsilon} - y'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} y'_{\delta\epsilon}, \right. \\ & \quad \left. \left. N_{\delta\epsilon} - 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \right\} \\ & \quad + \sup_{E \in \mathcal{A}_{i_0}} \left\{ B^E(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon} u) - B^E(t_{\delta\epsilon}, x'_{\delta\epsilon}, y'_{\delta\epsilon} v) \right\} \equiv \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \quad (42) \end{aligned}$$

In view of condition (13) on  $l$  and from estimate (??), we have the classical estimates of  $\mathcal{J}_1$  and  $\mathcal{J}_2$ :

$$\begin{aligned}\mathcal{J}_1 &\leq C|(x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})| \\ \mathcal{J}_2 &\leq C\left(\frac{1}{\epsilon}|(x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2 + 2\delta e^{\lambda(T-t_{\delta\epsilon})}(1 + |(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2)\right).\end{aligned}$$

Using the Lipschitz condition for  $u$  and  $v$ , we have

$$\mathcal{J}_3 \leq 2C|(x_{\delta\epsilon}, y_{\delta\epsilon}) - (x'_{\delta\epsilon}, y'_{\delta\epsilon})|.$$

Writing that  $\Xi(t, (x, y), (x, y), i) \leq \Xi(t_{\delta\epsilon}, (x_{\delta\epsilon}, y_{\delta\epsilon}), (x_{\delta\epsilon}, y_{\delta\epsilon}), i)$  for  $i \in \mathcal{S}$  and using the inequality (42),

$$\begin{aligned}u_i(t, x, y) - v_i(t, x, y) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(x, y)|^2 &\leq \\ v_i(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - u_i(t_{\delta\epsilon}, x_{\delta\epsilon}, y_{\delta\epsilon}) - \frac{\varrho}{t_{\delta\epsilon}} - 2\delta e^{\lambda(T-t)}|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 &\leq \\ \frac{1}{\beta}[\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3] - \frac{\varrho}{\beta t_{\delta\epsilon}^2} - \frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2) &\end{aligned}$$

this implies

$$\begin{aligned}u_i(t, x, y) - v_i(t, x, y) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(x, y)|^2 &\leq \\ \frac{1}{\beta}[\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3] - \frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(x_{\delta\epsilon}, y_{\delta\epsilon})|^2 + |(x'_{\delta\epsilon}, y'_{\delta\epsilon})|^2). &\end{aligned}$$

Sending  $\epsilon \downarrow 0$ , with the above estimates of  $(\mathcal{J}_1) - (\mathcal{J}_2) - (\mathcal{J}_3)$ , we obtain:

$$u_i(t, x, y) - v_i(t, x, y) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(x, y)|^2 \leq \frac{2\delta}{\beta}e^{\lambda(T-t_0)}\left[C(1 + 2|(x_0, y_0)|^2) - \lambda|(x_0, y_0)|^2\right].$$

Choose  $\lambda$  sufficiently large positive ( $\lambda \geq 2C$ ) and send  $\varrho, \delta \rightarrow 0^+$  to conclude that  $u_i(t, x, y) \leq v_i(t, x, y)$ .

Case 2 The second case occurs if

$$u_{i_0}(T, x_{\delta\epsilon}, y_{\delta\epsilon}) - v_{i_0}(T, x_{\delta\epsilon}, y_{\delta\epsilon}) \leq 0$$

and finally that  $u_i(t, x, y) \leq v_i(t, x, y)$ .

This completes the proof.

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