

# An Attack of the Conjecture of Lehmer by the Dynamical Zeta function of the $\beta$ -shift

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- 1 Introduction
- 2 Lehmer's Conjecture
- 3 Dynamical zeta function  $\zeta_\beta(z)$ , Generalized Fredholm Determinant, Parry Upper function  $f_\beta(z)$ , of the  $\beta$ -shift

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Minoration of the Mahler measure  $M(\alpha)$ , of the Weil height  $h(\alpha)$ , for  $\alpha$  a reciprocal algebraic integer



arithmetical dynamical systems associated with the house  $\beta = |\bar{\alpha}|$   
(dynamical zeta functions and its variants)

- toral automorphisms with the companion matrices of the minimal polynomials (Boyle, D. Lind, K. Schmidt, Einsiedler, Everest, Ward,...),
- $\beta$ -shift, for  $\beta$  real  $> 1$  (Rényi '57, Parry '60).

In Limit problems (Conjecture of Lehmer, etc) disadvantage of toral automorphisms : fixed degrees. Advantage  $\beta$ -shift : varying continuously  $\beta$  .

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Definition : Mahler measure : for

$$P(X) = a_0(X - \alpha_1)(X - \alpha_2) \dots (X - \alpha_n) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n \in \mathbb{Z}[X], \quad a_0a_n \neq 0$$

then

$$M(P) := |a_0| \prod_{i, |\alpha_i| > 1} |\alpha_i|.$$

multiplicativity :  $P = P_1 \times P_2 \times \dots \times P_m, \Rightarrow M(P) = M(P_1) \dots M(P_m)$ .

smallest Mahler measure : for  $|a_0| = 1$  (over algebraic integers).

$\alpha$  alg. number,  $\deg \alpha = d$ ,  $P_\alpha$  his minimal polynomial,  $M(\alpha) := M(P_\alpha)$ .

Absolute logarithmic, Weil height of  $\alpha$  :

$$h(\alpha) := \frac{\text{Log}M(\alpha)}{d}$$

fact :  $M(\alpha) = M(\alpha^{-1})$ ;  $M(\alpha) = \alpha$  if  $\alpha \in S$  (set of Pisot numbers) or  $T =$  set of Salem numbers).  $M(\alpha) = 1$  if  $\alpha$  is a root of unity.

Kronecker's Theorem(1857) : Let  $\alpha$  be an algebraic integer. Then  $M(\alpha) = 1$  iff  $\alpha$  is a root of unity.

Adler Marcus (1979) (topological entropy and equivalence of dynamical systems), Perron-Frobenius theory) :

$$\{M(\alpha) \mid \alpha \text{ alg. number}\} \subset \mathbb{P}_{Perron},$$

$$\{M(P) \mid P \in \mathbb{Z}[X]\} \subset \mathbb{P}_{Perron}.$$

Two strict inclusions (Dubickas 2004, Boyd 1981).

Definition :  $\alpha \in \mathbb{P}_{Perron}$  if  $\alpha = 1$  or if  $\alpha > 1$  is a real algebraic integer, for which the conjugates  $\alpha^{(i)}$  satisfy  $|\alpha^{(i)}| < \alpha$  (i.e. dominant root  $> 1$ ).

Conjecture of Lehmer : there exists  $c > 0$  such that

$$M(\alpha) \geq 1 + c$$

for any algebraic number  $\alpha \neq 0$  which is not a root of unity,

i.e. the interval  $(1, 1 + c) \cap \mathbb{P}_{Perron}$  is deprived of any value of Mahler measure of any algebraic number.

– > values : discontinuity at 1.



## Lehmer's problem (1933)

in the exhaustive search for large prime numbers : *if  $\varepsilon$  is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \dots + a_r$$

*where the  $a_i$ s are integers, such that the absolute value of the product of those roots of  $f$  which lie outside the unit circle, lies between 1 and  $1 + \varepsilon$ ... Whether or not the problem has a solution for  $\varepsilon < 0.176$  we do not know.*

Dobrowolski (1979) : for any reciprocal algebraic integer  $\alpha$  of degree  $d$ ,

$$M(\alpha) > 1 + (1 - \varepsilon) \left( \frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d > d_1(\varepsilon).$$

For  $P \in \mathbb{Z}[X]$ , define  $P^*(X) = X^{\deg P} P(1/X)$ .  $P$  is said *reciprocal* if  $P^* = P$ .

Smyth (1971) : If  $P$  is nonreciprocal, then

$$M(P) \geq \Theta = 1.3247\dots = \text{smallest Pisot number.}$$

Pisot number :  $|\alpha^{(i)}| < 1$  and  $\alpha > 1$  algebraic integer. “smallest” due to Siegel '44 : with minimal polynomial  $X^3 - X - 1$ .

→ : the polynomials of small Mahler measure **are all reciprocal**.

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- Consider the dynamical system  $([0, 1], T_\beta)$ , where  $1 < \beta < (1 + \sqrt{5})/2$  is in general a real number, tends to  $1^+$ , and

$$T_\beta : x \mapsto \beta x \pmod{1} = \{\beta x\}.$$

is the  $\beta$ -transformation.

- Specialize  $\beta > 1$  to run over reciprocal algebraic integers, as houses of nonzero algebraic integers which are not roots of unity :  $\beta = |\overline{\alpha}|$ .
- Use generalized Fredholm Theory : the  $\beta$ -shift is compact ( $=: X$ , with  $\Sigma$  Borel  $\sigma$ -algebra). The  $T_\beta$ -invariant probability measure  $\mu_\beta$  of the  $\beta$ -shift is unique (Rényi '57), ergodic (Parry, '60), maximal (Hofbauer '78), abs. cont. w.r.t the Lebesgue measure  $dt$ . The Perron-Frobenius operator  $P_{T_\beta}$ , on  $L^1(X, \Sigma, \mu_\beta)$ , defined by

$$\int_A P_{T_\beta} f d\mu_\beta = \int_{T_\beta^{-1}(A)} f d\mu_\beta \quad \text{is not compact (Takahashi, Ito Takahashi).}$$

- Use the transfer operators associated with  $T_\beta$ , the dynamical zeta function (Artin - Mazur) :

$$\zeta_\beta(z) := \exp \left( \sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n} z^n \right), \quad \mathcal{P}_n = \#\{x \in [0, 1] \mid T_\beta^n(x) = x\}.$$

The Parry Upper function at  $\beta$  is, up to the sign, is the generalized Fredholm determinant associated with  $T_\beta$  (Baladi - Keller '90) :

$$f_\beta(z) = -\frac{1}{\zeta_\beta(z)}$$

Explicitly (by ergodic theory) :

$$f_\beta(z) = -(1 - \beta z) \left( \sum_{n=0}^{\infty} T_\beta^n(1) z^n \right) = -1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots$$

with the digits coming from the orbit of 1 under  $T_\beta$  :

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{ \beta \} \rfloor, t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor, \dots, t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor, i \geq 1.$$

## Properties :

- the digits  $t_i$  belong to the finite alphabet  $\{-1, 0, 1\}$  : the analytic function  $f_\beta(z)$  obeys the Carlson-Polya dichotomy,
- moderate lacunarity controlled by two topological entropies  $\text{Log} M(\beta)$  and  $\text{Log} \beta$  (Verger-Gaugry, '06),
- (complete lexicographical ordering - Parry '60)  $1 < \gamma < \beta$  iff

$$(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$$

where the Rényi  $\gamma$ -expansion of 1 is  $d_\gamma(1) = 0.t'_1 t'_2 t'_3 \dots$ , resp. the Rényi  $\beta$ -expansion of 1 is  $d_\beta(1) = 0.t_1 t_2 t_3 \dots$

allow

$$\beta \rightarrow 1^+.$$

let  $\theta_n$  be the unique root of the trinomial  $G_n(z) := -1 + z + z^n$  in  $(0, 1)$ .

$$\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$$

$$-1 + z + z^n$$

$$-1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

where  $m_1 - n \geq n-1, m_{q+1} - m_q \geq n-1$

$$\theta_{n-1}^{-1}$$

$$-1 + z + z^{n-1}$$

*Reciprocal algebraic integers  $\beta$  are such that  $f_\beta(z)$  is never a polynomial.*

$(\theta_n^{-1})$  tends to 1.  $\beta$  tends to 1 equivalently  $n$  tends to infinity.

Def. :  $n$  is called the **dynamical degree of  $\beta$** , denoted by  $\text{dyg}(\beta)$ .

## The Parry Upper function at $\beta$

$$\theta_n^{-1} < \beta < \theta_{n-1}^{-1} \quad -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

satisfies (Conditions of admissibility of Parry) :

$$m_1 - n \geq n - 1, m_{q+1} - m_q \geq n - 1 \text{ for } 1 \leq q$$

[lexicographical ordering implies moderate gappiness]

its zeroes := eigenvalues of the transfer operators, := poles of  $\zeta_\beta(z)$ .

Method : 1) The  $\beta$ -shift is coupled with the introduction of a new method of **divergent series** (asymptotic expansions à la Poincaré (1895) in Celestial Mechanics) to obtain asymptotic expressions of  $n$  of the roots of the trinomials

$$G_n(x) = -1 + x + x^n, \quad n \geq 6, n \rightarrow \infty.$$

Then  $\mathbf{M}(\theta_n^{-1})$  is obtained as an asymptotic expansion of  $n$  from the lenticuli of roots in  $|\arg(z)| < \pi/3$ , with  $n$  tending to  $\infty$ ,

2) extension to any  $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ , with reduced lenticuli of roots.



Define  $\Omega_n :=$  open subset of  $|z| < 1$  (to avoid a neighbourhood of  $|z| = 1$ ).  $f_\beta(z)$  has no zero in  $\Omega_n$  except the lenticular zeroes.

2 relations, with uniform convergence on every  $K$  in  $\Omega_n$ ,

- existence of lenticuli of roots in  $\Omega_n$  (factorization type theorem) of the polynomial sections

$$\lim_{N \rightarrow \infty} s_N(z) = f_\beta(z)$$

- rewriting of the coefficient vector of  $P_\beta(z)$  along  $(s_N)$  to that of  $f_\beta(z)$  ( $A_m(z)$  integer polynomials) :

$$\lim_{m \rightarrow \infty} A_m(z) P_\beta(z) = f_\beta(z)$$

Hence, **every zero of  $P_\beta(z)$  is a zero of  $f_\beta(z)$** . Conversely, since  $1/\beta$  is a simple zero of  $P_\beta(z)$  and of  $f_\beta(z)$ , that lenticuli are limits of the lenticuli of the unique irreducible nonreciprocal factors of the  $s_N(z)$ , then the **complete lenticulus of  $f_\beta(z)$  is a lenticulus of zeroes of  $P_\beta(z)$** .

$$M(\beta) := \prod_i \max\{1, |\beta^{(i)}|\} = M(\beta^{-1})$$

then consider all the zeroes of the minimal reciprocal polynomial  $P_\beta$  inside the unit disk, which (lie inside) constitute the lenticulus to obtain the minorant :

$$M_r(\beta) :=: M_{lenticulus}(\beta) := \prod_{i \text{ lenticular}} \min\{1, |\beta^{(i)}|^{-1}\}$$

expressed as an asymptotic expansion of  $\text{dyg}(\beta)$ .

lenticulus  $\mathcal{L}_{\theta_n^{-1}}$  of simple zeroes in  $\arg(z) \in (-\pi/3, +\pi/3)$ ,  $n = 71$  and  $= 12$ .

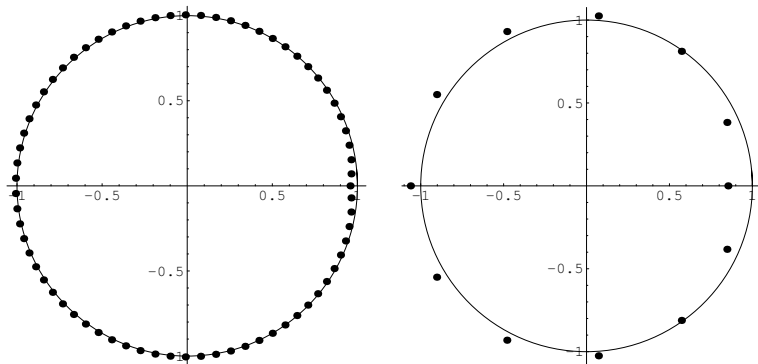
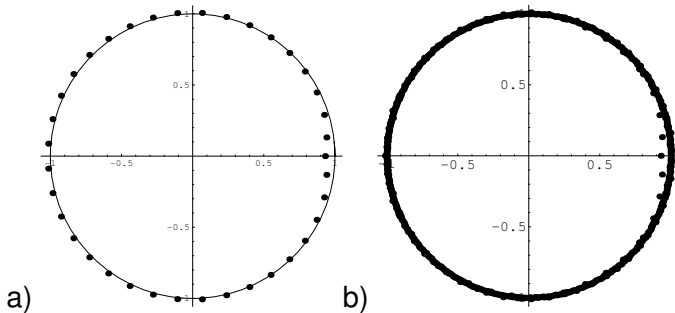
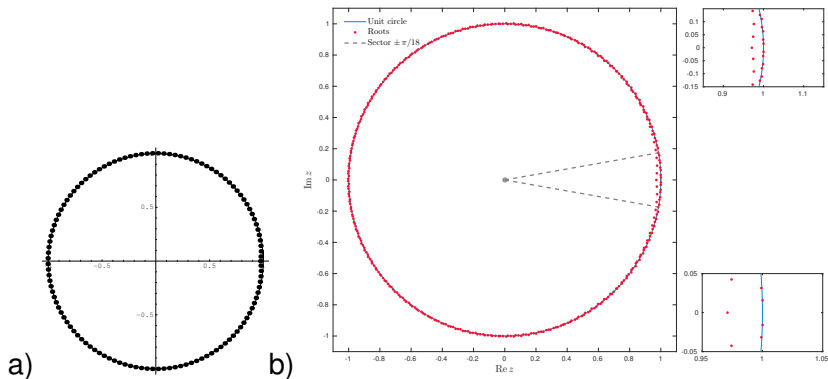


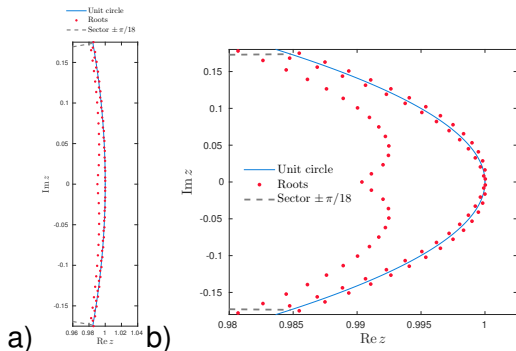
FIGURE: Roots of  $G_{71}(z)$ ,  $G_{12}(z)$ .



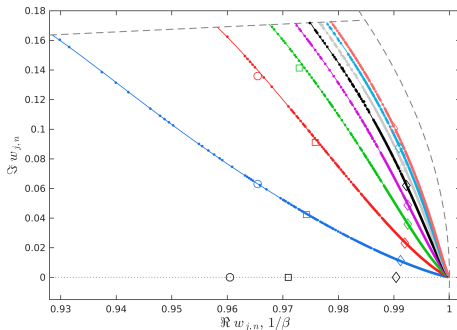
**FIGURE:** a) The 37 zeroes of  $G_{37}(x) = -1 + x + x^{37}$ , b) The 649 zeroes of  $f(x) = G_{37}(x) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649} = G_{37}(x) + x^{81} + \dots + x^{649}$ . The lenticulus of roots of  $f$  (having 3 simple zeroes) is obtained by a very slight deformation of the restriction of the lenticulus of roots of  $G_{37}$  to the angular sector  $|\arg z| < \pi/18$ , off the unit circle. The other roots (nonlenticular) of  $f$  can be found in a narrow annular neighbourhood of  $|z| = 1$ .



**FIGURE:** a) Zeroes of  $G_{121}$ , b) Zeroes of  $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$ . On the right the distribution of the roots of  $f$  is zoomed twice in the angular sector  $-\pi/18 < \arg(z) < \pi/18$ . The lenticulus of roots of  $f$  has 7 zeroes.



**FIGURE:** The representation of the 27 zeroes of the lenticulus of  $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$  in the angular sector  $-\pi/18 < \arg z < \pi/18$  in two different scalings in  $x$  and  $y$  (in a) and b)). In this angular sector the other zeroes of  $f$  can be found in a thin annular neighbourhood of the unit circle. The real root  $1/\beta > 0$  of  $f$  is such that  $\beta$  satisfies :  $1.00970357\dots = \theta_{481}^{-1} < \beta = 1.0097168\dots < \theta_{480}^{-1} = 1.0097202\dots$



**FIGURE:** Universal curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class  $\mathcal{B}$ . These curves are continuous, semi-fractal. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies in the boundary of Solomyak's fractal [?]. The lenticular roots of the previous polynomials  $f$  are represented by the respective symbols  $\circ$ ,  $\square$ ,  $\diamond$ . The dashed lines represent the unit circle and the top boundary of the angular sector  $|\arg z| < \pi/18$ .

Let  $\kappa := 0.171573\dots$  be the value of the maximum of the function

$a \rightarrow \kappa(1, a) := \frac{1 - \exp(-\frac{\pi}{a})}{2\exp(\frac{\pi}{a}) - 1}$  on  $(0, \infty)$ . Let  $S := 2 \arcsin(\kappa/2) = 0.171784\dots$

Denote

$$\Lambda_r \mu_r := \exp\left(\frac{-1}{\pi} \int_0^S \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right)$$

$= 1.15411\dots$ , a value slightly below Lehmer's number  $1.17628\dots$

## Theorem

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \Lambda_r \mu_r.$$

It is the limit lenticular contribution of the Parry Upper function.



Further : **identify** the lenticuli of zeroes of the Parry Upper function at  $\beta$  with lenticuli of conjugates of  $\beta$ , so that

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \lim_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta) = \Lambda_r \mu_r.$$

Further : the asymptotic expansions of the roots of  $f_\beta(z)$  lying in  $\beta$  gives an asymptotic expansion of the lenticular minorant of the Mahler measure :

### Theorem (Dobrowolski type minoration)

*Let  $\beta$  be a nonzero algebraic integer which is not a root of unity such that  $\text{dyg}(\beta) \geq 260$ . Then*

$$M(\beta) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left( \frac{1}{\text{Log}(\text{dyg}(\beta))} \right)$$

Lehmer's Conjecture :

Theorem (VG '17)

*For any nonzero algebraic integer  $\alpha$  which is not a root of unity,*

$$M(\alpha) \geq \theta_{259}^{-1} = 1.016126\dots$$

Schinzel Zassenhaus's Conjecture :

Theorem (VG '17)

*Schinzel-Zassenhaus's conjecture is true. Let  $\alpha$  be a nonzero algebraic integer which is not a root of unity. Then*

$$|\alpha| \geq 1 + \frac{C}{\deg(\alpha)} \quad \text{with } C = \theta_{259}^{-1} - 1 = 0.016126\dots$$

Pf. : Let  $\alpha \neq 0$  be an algebraic integer which is not a root of unity. Since  $M(\alpha) = M(\alpha^{-1})$  there are three cases to be considered :

- (i) the house of  $\alpha$  satisfies  $|\bar{\alpha}| \geq \theta_5^{-1}$ ,
- (ii) the dynamical degree of  $\alpha$  satisfies :  $6 \leq \text{dyg}(\alpha) < 260$ ,
- (iii) the dynamical degree of  $\alpha$  satisfies :  $\text{dyg}(\alpha) \geq 260$ .

In case (i),  $M(\alpha) \geq \theta_5^{-1} \geq \theta_{259}^{-1}$  and the claim holds true.

In the second case, since  $M(\alpha)$  is the product of  $|\bar{\alpha}|$  by the moduli of the conjugates of modulus  $> 1$ , we have  $M(\alpha) \geq |\bar{\alpha}|$ , therefore

$$M(\alpha) \geq \theta_{259}^{-1}.$$

In case (iii), the Dobrowolski type inequality gives the following lower bound of the Mahler measure

$$M(\alpha) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(\text{dyg}(\alpha))} \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(259)}, = 1.14843\dots$$

This lower bound is numerically greater than  $\theta_{259}^{-1} = 1.016126\dots$

Therefore, in any case, the lower bound  $\theta_{259}^{-1}$  of  $M(\alpha)$  holds true. We deduce the general minorant on  $M$ .

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