

# An attack of the Conjecture of Lehmer by the dynamical zeta function of the $\beta$ -shift

*Jean-Louis Verger-Gaugry*

*Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS,  
F - 73000 Chambéry, France.*

*email: Jean-Louis.Verger-Gaugry@univ-smb.fr*

**Abstract.** The present work proposes an attack of the Conjecture of Lehmer by the dynamical zeta function of the  $\beta$ -shift to prove that this Conjecture is true (arXiv.org > math > arXiv:1709.03771). In 1933 Lehmer asked the question about the existence of integer polynomials having a Mahler measure different of one, smaller than Lehmer's number (and arbitrarily close to one). The problem of Lehmer became a Conjecture, stating that there exists a universal lower bound  $> 1$  to the Mahler measures of the nonzero algebraic integers which are not roots of unity. The problem of the minoration of the Mahler measure of algebraic integers is a very deep one and has been extended in the theory of heights in arithmetic geometry.

The main ingredients arise from the Rényi-Parry dynamical system of real algebraic numbers and will be briefly presented. They involve: (i) the properties of the Parry Upper functions  $f_{|\alpha|}(z)$  associated with the dynamical zeta functions  $\zeta_{|\alpha|}(z)$  of the Rényi-Parry arithmetical dynamical systems, for  $\alpha$  an algebraic integer of house " $|\alpha|$ " greater than 1, tending to one, (ii) the discovery of lenticuli of poles of  $\zeta_{|\alpha|}(z)$  which uniformly equidistribute at the limit on a limit "lenticular" arc of the unit circle, for  $|\alpha|$  tending to  $1^+$ , giving rise to a continuous lenticular minorant  $M_r(|\alpha|)$  of the Mahler measure  $M(\alpha)$ , (iii) the Poincaré asymptotic expansions of these poles and of this minorant  $M_r(|\alpha|)$  as a function of the dynamical degree.

The same arguments apply to the Conjecture of Schinzel-Zassenhaus, also allow to obtain an inequality improving those of Dobrowolski and Voutier. Whether Lehmer's number is the smallest Mahler measure  $> 1$  of algebraic integers remains open.