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AN INTRINSIC GEOMETRIC FORMULATION OF HYPER-ELASTICITY, PRESSURE POTENTIAL AND NON-HOLONOMIC CONSTRAINTS

B. KOLEV AND R. DESMORAT

ABSTRACT. Isotropic hyper-elasticity, altogether with the equilibrium equation and the usual boundary conditions, are formulated directly on the body \mathcal{B} , a three-dimensional compact and orientable manifold with boundary equipped with a mass measure. Pearson–Sewell–Beatty pressure potential is formulated in an intrinsic geometric manner. It is shown that Poincaré’s formula extended to infinite dimension, provides, in a straightforward manner, the optimal (non-holonomic) constraints for such a pressure potential to exist.

1. INTRODUCTION

Finite strain theory for solids requires the comparison of the deformed configuration with a reference configuration (usually the initial unloaded structure). In modern continuum mechanics formulation, these configurations are described using mappings from some abstract manifold with boundary, the *body* \mathcal{B} , into the *ambient (usually Euclidean) space* \mathcal{E} [52, 51]. From the pure differential geometry point of view, the body does not have to be embedded in space [34, 41] and the formulation of continuum mechanics, more precisely of the boundary conditions, directly on the body boundary $\partial\mathcal{B}$ seems to remain an open question in the general case [22, 17]. Note that Noll [34, 35] did introduce the denomination “intrinsic” as synonymous of “defined on the body \mathcal{B} ”. Rougée [41, 42] did call the associated finite strain framework as an *intrinsic Lagrangian framework*, while the standard Lagrangian approach was meant as a formulation on the reference configuration.

Local *hyper-elasticity* with prescribed displacement and conservative loads can be formulated on the reference configuration as a variational problem, with an explicit Lagrangian (a potential energy) depending on the transformation φ — of the reference configuration into the deformed one — and on its first derivatives (see [4, 8, 30]). The further question of the existence and the determination of a Lagrangian (a potential) for the prescribed pressure boundary condition in finite strain theory has arisen. A partial answer has been given by Pearson [37] when studying elastic beams instabilities, and later improved by Sewell [47, 46] and Beatty [5] (see also [39]). A simplified, less general, formulation has been summarized by Ball [4]. Contrary to the case of small strains and displacements, in finite strain theory, *non-holonomic constraints* are required for a pressure potential to exist. Such conditions, have been formulated in [5], in a modern form, but with three drawbacks.

- (1) The Beatty conditions are not optimal: they are twofold where theoretically only one condition is expected [39];
- (2) The validity domain of the mathematical techniques involved to obtain the potential for *prescribed pressure* has not been mentioned by the authors;
- (3) Since these conditions are not expressed in an intrinsic way, it is difficult to recast them on the body (instead of on the reference configuration).

In the present work, we address these three issues by formulating the general problem of the existence and the obtention of a Lagrangian from virtual work as an extension of Poincaré’s lemma to the framework of differential geometry in infinite dimension [2, 23, 36, 28]. In order to

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exhibit a Lagrangian in hyper-elasticity, expressed either on the reference configuration or, more demanding, in an intrinsic way on the body, we focus on the common boundary conditions: of standard *dead load*, mainly to illustrate the proposed general methodology (no novelty is sought here, except for the formulation on the body), of *prescribed pressure*, for which we will obtain optimal *non-holonomic constraints*.

The outline of the paper is as follows. In [section 2](#), we pay close attention to the geometrical foundation of solids continuum mechanics, and we do not identify the body (an abstract manifold with boundary) with some reference configuration. Although this point of view has already been mentioned by Truesdell and Noll [[33](#), [34](#)] and more recently by several authors [[54](#), [25](#), [10](#), [6](#), [50](#), [22](#)], an intrinsic formulation using modern tools in differential geometry is expected. In this direction, we insist on the fundamental role played by the *manifold of all the Riemannian metrics* — introduced in solids mechanics by Rougée [[41](#), [43](#), [44](#)] and Fiala [[15](#), [16](#), [17](#)] — in the formulation of hyper-elasticity ([section 3](#)).

The ultimate goal is to formulate finite strain's theory and the principle of virtual power entirely on the body \mathcal{B} , not necessarily embedded in \mathbb{R}^3 . Meanwhile, we are led to consider virtual powers \mathcal{W} as one-forms on the configuration space (the space of embeddings [section 4](#)), therefore to formulate continuum mechanics in the framework of differential geometry in infinite dimension. To obtain a primitive of the virtual work of the applied surface forces, possibly configuration dependent, we use an infinite dimensional extension of Poincaré's lemma, which asserts that a closed form α on \mathbb{R}^n is always locally exact (*i.e.* there exists β such as $\alpha = d\beta$), and which is proven using the following explicit formula (see [Appendix B](#))

$$dK + Kd = \text{id},$$

where d is the exterior derivative and K , the *Poincaré integrator*. Indeed, writing

$$d(K\alpha) = \alpha - Kd\alpha,$$

where K is understood as an integrator operator, $\beta = K\alpha$ appears as the sought potential, and $Kd\alpha = 0$, as the constraints for β to be a primitive of α .

Finally, we explain how to formulate Lagrangians (potentials) for surface forces (boundary conditions) and illustrate the method through *dead loads* ([section 5](#)) and *prescribed pressure* ([section 6](#)), both on the body and on the reference configuration. In the case of dead loads, the proposed potential is close to the one derived in [[22](#)], but with the notable difference that our formulation of the displacement uses the affine structure of space. The proposed geometric formulation of the prescribed pressure boundary condition is new. The associated non-holonomic constraints (formally $Kd\mathcal{W} = 0$) for a pressure potential to exist is formulated both on the body and on the reference configuration.

2. KINEMATICS OF FINITE STRAINS

In continuum mechanics, the ambient space \mathcal{E} is represented by a three-dimensional Euclidean affine space. Denoting \mathbf{q} the Euclidean metric on \mathcal{E} , it is better to consider this space as a Riemannian manifold $(\mathcal{E}, \mathbf{q})$ and forget, as far as possible, this affine structure of space. The material medium is parameterized by a three-dimensional compact and orientable manifold with boundary, noted \mathcal{B} , the *body*. This manifold \mathcal{B} is equipped with a *volume form* μ , the *mass measure* [[52](#)].

Remark 2.1. It is common, when possible, to refer to *material coordinates* (*i.e.* defined on the body \mathcal{B}) as uppercase letters \mathbf{X} while *spatial coordinates* (*i.e.* defined on the space \mathcal{E}) are represented by lowercase letters \mathbf{x} .

A *configuration* of a material medium is represented by a smooth orientation-preserving *embedding* (particles cannot occupy the same point in space)

$$p : \mathcal{B} \rightarrow \mathcal{E},$$

sometimes referred to as a *placement* in mechanics and its image $\Omega = p(\mathcal{B})$ is usually called a *configuration system*. The *configuration space* in continuum mechanics is thus the set, noted

$\text{Emb}(\mathcal{B}, \mathcal{E})$, of smooth embeddings of \mathcal{B} in \mathcal{E} . This set can be endowed with a differential manifold structure of infinite dimension [27] (a *Fréchet* manifold). Since \mathcal{E} is an affine space, $\text{Emb}(\mathcal{B}, \mathcal{E})$ is just an open set of the *Fréchet topological vector space* $C^\infty(\mathcal{B}, \mathcal{E})$ (see Appendix D). The tangent space to $\text{Emb}(\mathcal{B}, \mathcal{E})$ at a point $p \in \text{Emb}(\mathcal{B}, \mathcal{E})$ is described as follows. Let $p(t)$ be a curve in $\text{Emb}(\mathcal{B}, \mathcal{E})$ such that $p(0) = p$ and $(\partial_t p)(0) = \mathbf{V}$, then the tangent space at $p \in \text{Emb}(\mathcal{B}, \mathcal{E})$ is the set

$$T_p \text{Emb}(\mathcal{B}, \mathcal{E}) = \{\mathbf{V} \in C^\infty(\mathcal{B}, T\mathcal{E}); \pi \circ \mathbf{V} = p\},$$

where \mathbf{V} is described by the following diagram:

$$\begin{array}{ccc} T\mathcal{B} & \xrightarrow{Tp} & T\mathcal{E} \\ \pi \downarrow & \mathbf{V} \nearrow & \downarrow \pi \\ \mathcal{B} & \xrightarrow{p} & \mathcal{E} \end{array}$$

We recognize \mathbf{V} as a *Lagrangian velocity*. The tangent bundle $T\text{Emb}(\mathcal{B}, \mathcal{E})$ of the configuration space $\text{Emb}(\mathcal{B}, \mathcal{E})$ is thus the set of *virtual Lagrangian velocities*.

A motion in continuum mechanics corresponds to a curve $p(t)$ in $\text{Emb}(\mathcal{B}, \mathcal{E})$ (a *path of embeddings*). To this motion is associated its Lagrangian velocity

$$\partial_t p(t, \mathbf{X}) = \mathbf{V}(t, \mathbf{X})$$

and its (right) Eulerian velocity

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{V}(t, p^{-1}(t, \mathbf{x})).$$

which is a vector field on $\Omega = p(\mathcal{B})$. But one can also introduces the *left Eulerian velocity*

$$\mathbf{U}(t, \mathbf{X}) := (Tp^{-1} \cdot \mathbf{V})(t, \mathbf{X}).$$

which is a vector field on \mathcal{B} (see for instance [3]). These two vector fields are better described by the following diagram:

$$\begin{array}{ccc} T\mathcal{B} & \xrightarrow{Tp} & T\Omega \\ \left(\begin{array}{c} \pi \downarrow \\ \mathbf{U} \left(\begin{array}{c} \mathbf{V} \nearrow \\ p \end{array} \right) \\ \downarrow \pi \\ \mathcal{B} \end{array} \right) & & \left(\begin{array}{c} \downarrow \pi \\ \mathbf{u} \end{array} \right) \\ \mathcal{B} & \xrightarrow{p} & \Omega \end{array}$$

Remark 2.2. Introducing the notions of *pull-back/push-forward* (see Appendix A), left and right Eulerian velocities are related to each other as follows,

$$\begin{aligned} \mathbf{U} &= p^* \mathbf{u} = Tp^{-1} \circ \mathbf{u} \circ p, & (\mathbf{U} \text{ is the pull-back of } \mathbf{u}), \\ \mathbf{u} &= p_* \mathbf{U} = Tp \circ \mathbf{U} \circ p^{-1}, & (\mathbf{u} \text{ is the push-forward of } \mathbf{U}). \end{aligned}$$

It is common to introduce a *reference configuration* $\Omega_0 := p_0(\mathcal{B})$ with $\mathbf{x}_0 = p_0(\mathbf{X})$. This allows for the definition of the mapping

$$\varphi := p \circ p_0^{-1}, \quad \Omega_0 \rightarrow \Omega,$$

usually called the *transformation* [14, 21, 52, 54]. This means that $\mathbf{x} = p(\mathbf{X}) = \varphi(\mathbf{x}_0)$ are the Eulerian coordinates (on the deformed configuration Ω), whereas \mathbf{x}_0 and \mathbf{X} are the Lagrangian coordinates, respectively on the reference configuration Ω_0 and the body \mathcal{B} .

The linear tangent mappings $Tp_0 : T\mathcal{B} \rightarrow T\Omega_0$, $Tp : T\mathcal{B} \rightarrow T\Omega$ and $T\varphi : T\Omega_0 \rightarrow T\Omega$, will be denoted respectively by

$$\mathbf{F}_0 := Tp_0 = \frac{\partial p_0}{\partial \mathbf{X}}, \quad \mathbf{F} := Tp = \frac{\partial p}{\partial \mathbf{X}} = \mathbf{F}_\varphi \mathbf{F}_0, \quad \mathbf{F}_\varphi := T\varphi = \frac{\partial \varphi}{\partial \mathbf{x}_0}.$$

The *displacement*, either defined on the reference configuration $\Omega_0 = p_0(\mathcal{B})$ as

$$(2.1) \quad \boldsymbol{\xi}(\varphi) = \varphi - \text{id},$$

or on the body \mathcal{B} , as

$$(2.2) \quad \boldsymbol{\xi}(p) = \boldsymbol{\xi}(\varphi) \circ p_0 = p - p_0,$$

has to be understood as a *vector valued function*, with values in space \mathcal{E} . Here, we strongly use the affine structure of space \mathcal{E} , contrary to [22] where the rigorous formulation of the displacement requires, then, the use of the *Riemannian exponential mapping* on \mathcal{E} (considered as a general Riemannian manifold with no affine structure).

2.1. Metric states. To each embedding p corresponds a Riemannian metric

$$\boldsymbol{\gamma} = p^* \mathbf{q}$$

on the body \mathcal{B} . Note that the Riemannian curvature of $\boldsymbol{\gamma}$ vanishes when \mathcal{B} is of dimension 3 (this is no longer true in shell theory where \mathcal{B} is a manifold of dimension 2). For both fluids and solids, the state of stress is determined by the “metric state” [52, 34]. It is however necessary to introduce a reference configuration p_0 to formalize geometrically hyper-elasticity of solids, which leads to a reference Riemannian metric

$$\boldsymbol{\gamma}_0 = p_0^* \mathbf{q}.$$

In classical theory of finite strains, two strain tensors play a fundamental role:

- (1) the (covariant) *right Cauchy–Green tensor*

$$\mathbf{C} := \varphi^* \mathbf{q} = \mathbf{q} \mathbf{F}_\varphi^t \mathbf{F}_\varphi,$$

defined as the *pull-back* to Ω_0 of the Euclidean metric \mathbf{q} on Ω ,

- (2) the (contravariant) *left Cauchy–Green tensor*

$$\mathbf{b} := \varphi_* \mathbf{q}^{-1} = \mathbf{F}_\varphi \mathbf{F}_\varphi^t \mathbf{q}^{-1},$$

defined as the *push-forward* to Ω of the inverse Euclidean metric \mathbf{q}^{-1} on Ω_0 ,

where $(\cdot)^t$ denotes the transpose relative to the metric \mathbf{q} .

The relations between the two Cauchy–Green tensors and the two metric tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_0$, defined on the body \mathcal{B} are the following

$$p_0^* \mathbf{C} = p^* \mathbf{q} := \boldsymbol{\gamma}, \quad \text{and} \quad p^* \mathbf{b} = p_0^* \mathbf{q}^{-1} := \boldsymbol{\gamma}_0^{-1}.$$

Remark 2.3. Using the reference metric tensor $\boldsymbol{\gamma}_0$, we can then build the mixed tensor $\boldsymbol{\gamma}_0^{-1} \boldsymbol{\gamma}$. Then, $\widehat{\mathbf{C}} = \mathbf{q}^{-1} \mathbf{C}$ and $\widehat{\mathbf{b}} = \mathbf{b} \mathbf{q}$ are related to each other by

$$(2.3) \quad \boldsymbol{\gamma}_0^{-1} \boldsymbol{\gamma} = p_0^* \widehat{\mathbf{C}} = p^* \widehat{\mathbf{b}}.$$

2.2. Strain rate. Traditionally, the *strain rate* (on Ω) is defined by the mixed tensor

$$\widehat{\mathbf{d}} := \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^t),$$

where $\nabla \mathbf{u}$ is the covariant derivative of the Eulerian velocity \mathbf{u} and $(\nabla \mathbf{u})^t$ means the transpose (relative to the metric \mathbf{q}) of the linear operator $\boldsymbol{w} \mapsto \nabla_{\boldsymbol{w}} \mathbf{u}$. It appears, however, that it is more interesting to introduce its covariant form

$$\mathbf{d} := \mathbf{q} \widehat{\mathbf{d}} = \frac{1}{2} \mathbf{L}_{\mathbf{u}} \mathbf{q},$$

where $\mathbf{L}_{\mathbf{u}}$ means the Lie derivative with respect to \mathbf{u} . We get then

$$(2.4) \quad \varphi^* \mathbf{d} = \frac{1}{2} \partial_t \mathbf{C}, \quad p^* \mathbf{d} = \frac{1}{2} \partial_t \boldsymbol{\gamma},$$

which are direct consequences of lemma A.3 (see also [34, 41, 44]).

2.3. Mass measure. Let μ be the mass measure on the body \mathcal{B} . To each embedding p corresponds a volume form $p_*\mu$ on the configuration $\Omega = p(\mathcal{B})$. Besides, the Riemannian metric \mathbf{q} induces on Ω a volume form, noted $\text{vol}_{\mathbf{q}}$ (see [Appendix C](#)), which writes

$$\text{vol}_{\mathbf{q}} = dx^1 \wedge dx^2 \wedge dx^3,$$

using the canonical coordinates on \mathbb{R}^3 . Since two volume forms are always proportional to each other, we get

$$(2.5) \quad p_*\mu = \rho \text{vol}_{\mathbf{q}},$$

which defines the *mass density* $\rho = p_*\mu/\text{vol}_{\mathbf{q}}$ as a scalar function on the space domain Ω . Similarly, a density $\rho_0 = p_{0*}\mu/\text{vol}_{\mathbf{q}}$ is defined on the reference configuration Ω_0 . The conservation of mass is obtained using the identity

$$\mu = p_0^*(\rho_0 \text{vol}_{\mathbf{q}}) = p^*(\rho \text{vol}_{\mathbf{q}}),$$

from which we deduce,

$$(2.6) \quad \rho_0 \text{vol}_{\mathbf{q}} = p_{0*}p^*(\rho \text{vol}_{\mathbf{q}}) = \varphi^*(\rho \text{vol}_{\mathbf{q}}) = (\varphi^*\rho)J_{\varphi} \text{vol}_{\mathbf{q}}, \quad \begin{cases} J_{\varphi} = \det \mathbf{F}_{\varphi}, \\ (\varphi^*\rho) = \rho \circ \varphi. \end{cases}$$

Its infinitesimal form

$$\partial_t \rho + \nabla_{\mathbf{u}} \rho + \rho \text{div } \mathbf{u} = 0$$

is obtained by deriving along a path of embeddings $p(t)$ and using lemma [A.3](#). Indeed, we get

$$\partial_t \mu = 0 = \partial_t (p^*(\rho \text{vol}_{\mathbf{q}})) = p^*(\partial_t(\rho \text{vol}_{\mathbf{q}}) + L_{\mathbf{u}}(\rho \text{vol}_{\mathbf{q}})),$$

where $L_{\mathbf{u}}$ is the Lie derivative relative to \mathbf{u} , and from which we deduce that

$$(\partial_t \rho) \text{vol}_{\mathbf{q}} + (L_{\mathbf{u}} \rho) \text{vol}_{\mathbf{q}} + \rho L_{\mathbf{u}} \text{vol}_{\mathbf{q}} = (\partial_t \rho + \nabla_{\mathbf{u}} \rho + \rho \text{div } \mathbf{u}) \text{vol}_{\mathbf{q}} = 0.$$

The Riemannian volume forms vol_{γ} and vol_{γ_0} and the mass measure μ are all proportional. They are related by

$$\mu = (p^*\rho) \text{vol}_{\gamma} = (p_0^*\rho_0) \text{vol}_{\gamma_0},$$

which leads us to define two mass densities on the body

$$\rho_{\gamma} := p^*\rho = \rho \circ p, \quad \rho_{\gamma_0} := p_0^*\rho_0 = \rho_0 \circ p_0.$$

3. ROUGÉE'S GEOMETRIC FORMULATION OF HYPER-ELASTICITY

Rougée suggested in [[43](#), [44](#)] to define an elasticity law as a vector field on $\text{Met}(\mathcal{B})$, the manifold of all the Riemannian metrics on the body \mathcal{B} , in other words as a section

$$F : \text{Met}(\mathcal{B}) \rightarrow T\text{Met}(\mathcal{B})$$

of the tangent vector bundle $T\text{Met}(\mathcal{B})$. It turns out that $\text{Met}(\mathcal{B})$ is an *open convex set* of the infinite dimensional vector space

$$\Gamma(S^2T^*\mathcal{B}),$$

of smooth covariant symmetric second-order tensors fields (see [Appendix D](#)). The tangent space $T_{\gamma}\text{Met}(\mathcal{B})$ is thus canonically identified with the vector space $\Gamma(S^2T^*\mathcal{B})$. This space can be interpreted as the space of virtual deformation tensor fields ϵ (linearized deformations around metric state γ). The tangent vector bundle $T\text{Met}(\mathcal{B})$ is trivial and writes

$$T\text{Met}(\mathcal{B}) = \text{Met}(\mathcal{B}) \times \Gamma(S^2T^*\mathcal{B}).$$

The cotangent vector space $T^*\text{Met}(\mathcal{B})$ is a space of *tensor-distributions*. It is interpreted as the space of *virtual powers of internal forces*.

Remark 3.1. Any pullback metric $\gamma = p^*\mathbf{q}$, as defined in [section 2](#), naturally belongs to this set $\text{Met}(\mathcal{B})$. Note however that each pullback metric $\gamma = p^*\mathbf{q}$ has vanishing curvature so that $\text{Met}(\mathcal{B})$ is in fact much bigger than just the set of all pullback metrics.

The manifold $\text{Met}(\mathcal{B})$ of Riemannian metrics on \mathcal{B} can be equipped with a (weak) Riemannian structure, by setting

$$(3.1) \quad G_\gamma^\mu(\epsilon^1, \epsilon^2) := \int_{\mathcal{B}} \text{tr}(\gamma^{-1} \epsilon^1 \gamma^{-1} \epsilon^2) \mu, \quad \epsilon^1, \epsilon^2 \in T_\gamma \text{Met}(\mathcal{B}),$$

where $\text{tr}(\gamma^{-1} \epsilon^1 \gamma^{-1} \epsilon^2) = \gamma^{ij} \epsilon_{ik}^1 \gamma^{kl} \epsilon_{jl}^2$, in a local coordinate system. Riemannian structures on the manifold of Riemannian metrics on a given manifold have been extensively studied, see for instance [12, 18, 20, 9]. The metric (3.1) was introduced by Rougée [43, 44] (see also [15]) and seems well adapted for the geometrical formulation of several concepts in solid mechanics. For this reason, we will call it the *Rougée metric*. This metric induces a linear injective (but not surjective) mapping

$$T_\gamma \text{Met}(\mathcal{B}) \rightarrow T_\gamma^* \text{Met}(\mathcal{B}), \quad \eta \mapsto G_\gamma^\mu(\eta, \cdot).$$

The range of this mapping in $T_\gamma^* \text{Met}(\mathcal{B})$ corresponds to *virtual powers with density*. In other words, an element \mathcal{P}_γ belongs to this range if it writes

$$\mathcal{P}_\gamma(\epsilon) = \int_{\mathcal{B}} (\theta : \epsilon) \mu, \quad \text{where } \theta = \gamma^{-1} \eta \gamma^{-1},$$

for some $\eta \in T_\gamma \text{Met}(\mathcal{B})$, defining on the body the *Rougée stress tensor* θ [41, 42] as the density of the power \mathcal{P}_γ . An elasticity law (in the Cauchy sense) writes thus as

$$(3.2) \quad \theta = \gamma^{-1} F(\gamma) \gamma^{-1},$$

where F is a vector field on $\text{Met}(\mathcal{B})$. This formula is better understood using the following diagram

$$\begin{array}{ccc} T\text{Met}(\mathcal{B}) & \xrightarrow{G^\mu} & T^*\text{Met}(\mathcal{B}) \\ F \left(\begin{array}{c} \downarrow \pi \\ \nearrow \theta = \gamma^{-1} F(\gamma) \gamma^{-1} \end{array} \right. & & \\ \text{Met}(\mathcal{B}) & & \end{array}$$

The *Noll intrinsic stress tensor* [34], also defined on the body \mathcal{B} , is

$$\mathfrak{S} := \rho_\gamma \theta, \quad \rho_\gamma = p^* \rho.$$

It is such that

$$\mathcal{P}_\gamma(\epsilon) = \int_{\mathcal{B}} (\mathfrak{S} : \epsilon) \text{vol}_\gamma, \quad \epsilon \in T_\gamma \text{Met}(\mathcal{B}).$$

Both \mathfrak{S} and θ are symmetric contravariant tensor fields on \mathcal{B} .

By *push-forward* on $\Omega = p(\mathcal{B})$, we recover then the Kirchhoff and Cauchy stress tensors

$$(3.3) \quad \tau = p_* \theta = \mathbf{q}^{-1} p_*(F(\gamma)) \mathbf{q}^{-1}, \quad \sigma = \rho \tau = p_* \mathfrak{S}.$$

Remark 3.2. Using the expression of the deformation rate (2.4), the evaluation

$$\mathcal{P}_\gamma\left(\frac{1}{2} \partial_t \gamma\right) = \int_{p(\mathcal{B})} p_* \theta : p_* \left(\frac{1}{2} \partial_t \gamma\right) p_* \mu = \int_{\Omega} \rho \tau : \mathbf{d} \text{vol}_\mathbf{q} = \int_{\Omega} \sigma : \mathbf{d} \text{vol}_\mathbf{q} = -\mathcal{P}^{int}$$

is recognized as the opposite of the power of internal forces for a Cauchy medium.

In this geometric framework, the law (3.2) is said to be *hyper-elastic* (or elastic in Green's sense) if F is the *gradient* (for the Rougée metric G^μ) of a functional \mathcal{H} defined on $\text{Met}(\mathcal{B})$. In that case

$$F = \text{grad}^{G^\mu} \mathcal{H},$$

where $\text{grad}^{G^\mu} \mathcal{H}$ is defined implicitly by

$$d_\gamma \mathcal{H} . \epsilon = G_\gamma^\mu(\text{grad}^{G^\mu} \mathcal{H}, \epsilon) = \int_{\mathcal{B}} \text{tr}(\gamma^{-1}(\text{grad}^{G^\mu} \mathcal{H}) \gamma^{-1} \epsilon) \mu, \quad \forall \epsilon \in T_\gamma \text{Met}(\mathcal{B}),$$

meaning that

$$\mathcal{P}_\gamma(\epsilon) = \int_{\mathcal{B}} (\theta : \epsilon) \mu = d_\gamma \mathcal{H} . \epsilon.$$

Remark 3.3. There is a strong similarity between this geometric framework and Einstein-Hilbert's formulation of general relativity [26, 13, 49]. In the second case, the metric γ on the body is replaced by a Lorentzian metric g on space-time \mathcal{U} , the Rougée stress tensor θ becomes the *stress-energy tensor* and the vector field $F(\gamma) = \text{grad}^{G^\mu} \mathcal{H}$ is replaced by the Einstein tensor

$$S(g) = \text{grad}^{G^E} \mathcal{H} = a\text{Ric}(g) - \frac{1}{2}(aR(g) + b)g,$$

which is the gradient of the Einstein-Hilbert functional

$$\mathcal{H}(g) = \int_{\mathcal{U}} (aR(g) + b)\text{vol}_g,$$

for the *Ebin metric* [12]

$$G_g^E(\epsilon^1, \epsilon^2) := \int_{\mathcal{U}} \text{tr}(g^{-1}\epsilon^1 g^{-1}\epsilon^2)\text{vol}_g.$$

The two constants a and b are related to Newton's gravitation constant and to Einstein's cosmological constant [49, p. 344, Eq. (35.25)].

3.1. Isotropic hyper-elasticity formulated on the body. One may refer to [48, 51] and [6, Chapter 7] for a review of usual hyper-elasticity laws. Here, we choose to formulate hyper-elasticity on the body \mathcal{B} rather than on a reference configuration Ω_0 , using the metric $\gamma = p^*\mathbf{q}$ and the Rougée stress $\theta = p^*\boldsymbol{\tau}$. In that case, a reference metric $\gamma_0 = p_0^*\mathbf{q}$ is required in order to define the mixed tensor (2.3) and its invariants. The formulation of isotropic hyper-elasticity (in Green sense) on the body \mathcal{B} , is provided by the following result.

Theorem 3.4. *Let p_0 be a reference configuration and $\gamma_0 = p_0^*\mathbf{q}$. The isotropic local hyper-elasticity can be formulated on the body \mathcal{B} by a functional*

$$\mathcal{H}_{\gamma_0}(\gamma) = \int_{\mathcal{B}} 2\psi(I_1, I_2, I_3)\mu$$

defined on $\text{Met}(\mathcal{B})$, where $I_k = \text{tr}(\gamma_0^{-1}\gamma)^k$. We get

$$(3.4) \quad \theta = 2 \frac{\partial \psi}{\partial \gamma} = \sum_{k=1}^3 2k \left(\frac{\partial \psi}{\partial I_k}(\gamma) \right) (\gamma_0^{-1}\gamma)^{k-1} \gamma_0^{-1}.$$

Remark 3.5. In the argument of ψ , I_3 can be replaced by $\mathcal{J} = \sqrt{\det(\gamma_0^{-1}\gamma)}$, and then

$$\frac{\partial \mathcal{J}}{\partial \gamma} = \frac{1}{2} \mathcal{J} \gamma^{-1}.$$

Proof. We have

$$\delta \mathcal{H}_{\gamma_0} = \int_{\mathcal{B}} 2 \sum_{k=1}^3 \frac{\partial \psi}{\partial I_k} \delta I_k \mu,$$

where

$$\delta I_k = k \text{tr} \left[(\gamma_0^{-1}\gamma)^{k-1} \gamma_0^{-1} \delta \gamma \right] = k \text{tr} \left[\gamma^{-1} (\gamma (\gamma_0^{-1}\gamma)^k) \gamma^{-1} \delta \gamma \right].$$

We get thus

$$\begin{aligned} \delta \mathcal{H}_{\gamma_0} &= \int_{\mathcal{B}} \left(\sum_{k=1}^3 2k \frac{\partial \psi}{\partial I_k}(\gamma) \text{tr} \left[\gamma^{-1} (\gamma (\gamma_0^{-1}\gamma)^k) \gamma^{-1} \delta \gamma \right] \right) \mu \\ &= \int_{\mathcal{B}} \text{tr} \left[\gamma^{-1} \left(\sum_{k=1}^3 2k \left(\frac{\partial \psi}{\partial I_k}(\gamma) \right) \gamma (\gamma_0^{-1}\gamma)^k \right) \gamma^{-1} \delta \gamma \right] \mu. \end{aligned}$$

Therefore

$$\text{grad}^{G^\mu} \mathcal{H}_{\gamma_0} = \sum_{k=1}^3 2k \left(\frac{\partial \psi}{\partial I_k}(\gamma) \right) \gamma (\gamma_0^{-1}\gamma)^k$$

and

$$\boldsymbol{\theta} = \boldsymbol{\gamma}^{-1} (\text{grad}^{G^\mu} \mathcal{H}_{\boldsymbol{\gamma}_0}) \boldsymbol{\gamma}^{-1} = \sum_{k=1}^3 2k \left(\frac{\partial \psi}{\partial I_k}(\boldsymbol{\gamma}) \right) (\boldsymbol{\gamma}_0^{-1} \boldsymbol{\gamma})^{k-1} \boldsymbol{\gamma}_0^{-1} = 2 \frac{\partial \psi}{\partial \boldsymbol{\gamma}}.$$

which completes the proof. \square

The isotropic invariants of the mixed tensor $\boldsymbol{\gamma}_0^{-1} \boldsymbol{\gamma}$ can be rewritten using either the left Cauchy–Green tensor $\mathbf{b} = p_* \boldsymbol{\gamma}_0^{-1}$ (on Ω) or the right Cauchy–Green tensor $\mathbf{C} = p_{0*} \boldsymbol{\gamma}$ (on Ω_0), using the identities in remark 2.3. Indeed, as push-forward commutes with all tensor contractions, we have

$$p_* I_k = I_k \circ p^{-1} = \text{tr}(\mathbf{b}\mathbf{q})^k = \text{tr} \widehat{\mathbf{b}}^k,$$

and

$$p_{0*} I_k = I_k \circ p_0^{-1} = \text{tr}(\mathbf{q}^{-1} \mathbf{C})^k = \text{tr} \widehat{\mathbf{C}}^k,$$

from which we deduce that

$$\frac{\partial p_* I_k}{\partial \mathbf{b}} = k \mathbf{q} (\mathbf{b}\mathbf{q})^{k-1}, \quad \frac{\partial p_{0*} I_k}{\partial \mathbf{C}} = k (\mathbf{q}^{-1} \mathbf{C})^{k-1} \mathbf{q}^{-1}.$$

Consider now the second Piola-Kirchhoff stress tensor

$$(3.5) \quad \mathbf{S} := \varphi^* \boldsymbol{\tau} = p_{0*} \boldsymbol{\theta},$$

defined as the pull-back on Ω_0 , by the transformation $\varphi = p \circ p_0^{-1}$, of Kirchhoff stress tensor $\boldsymbol{\tau}$. Then, we recover, on the one hand, the usual hyper-elasticity law

$$(3.6) \quad \mathbf{S} = 2 \frac{\partial \psi_0}{\partial \mathbf{C}}, \quad \psi_0(\widehat{\mathbf{C}}) = \psi(I_1 \circ p_0^{-1}, I_2 \circ p_0^{-1}, I_3 \circ p_0^{-1}),$$

expressed on the reference configuration Ω_0 by *push-forward* of (3.4) as

$$\mathbf{S} = p_{0*} \boldsymbol{\theta} = 2k \left(\frac{\partial \psi}{\partial I_k}(\boldsymbol{\gamma}) \circ p_0 \right) (\mathbf{q}^{-1} \mathbf{C})^{k-1} \mathbf{q}^{-1} = 2 \frac{\partial \psi_0}{\partial \mathbf{C}},$$

and, on the other hand, the usual isotropic hyper-elasticity law

$$(3.7) \quad \boldsymbol{\tau} = 2 \mathbf{q}^{-1} \frac{\partial \psi_p}{\partial \mathbf{b}}, \quad \psi_p(\widehat{\mathbf{b}}) = \psi(I_1 \circ p^{-1}, I_2 \circ p^{-1}, I_3 \circ p^{-1}),$$

expressed on the deformed configuration Ω by *push-forward* of (3.4) as

$$\boldsymbol{\tau} = p_* \boldsymbol{\theta} = 2k \left(\frac{\partial \psi}{\partial I_k}(\boldsymbol{\gamma}) \circ p \right) (\mathbf{b}\mathbf{q})^{k-1} \mathbf{b} = 2 \mathbf{q}^{-1} \frac{\partial \psi_p}{\partial \mathbf{b}}.$$

Remark 3.6. Note that identifying the body with the reference configuration $\mathcal{B} \equiv \Omega_0$, and thus $p_0 \equiv \text{id}$, $p \equiv \varphi$, leads to the identification

$$\boldsymbol{\theta} \equiv \mathbf{S},$$

of the Rougée tensor with the second Piola-Kirchhoff tensor.

3.2. Example: Harth-Smith hyper-elasticity. As an example, let us consider *quasi-incompressible Harth-Smith hyper-elasticity* [24], formulated on the reference configuration Ω_0 by (3.6) with the isotropic free energy density

$$\psi_0 = h_1 \int_3^{\bar{I}_1(\widehat{\mathbf{C}})} \exp(h_3(I-3)^2) dI + 3h_2 \ln \left(\frac{\bar{I}_2(\mathbf{C})}{3} \right) + U(J_\varphi),$$

where h_1 , h_2 and h_3 are material parameters, $\widehat{\mathbf{C}} = \mathbf{q}^{-1} \mathbf{C}$ and $J_\varphi = \sqrt{\det \widehat{\mathbf{C}}}$. In order to properly model rubber quasi-incompressibility, the Penn invariants [38] \bar{I}_1 , \bar{I}_2 are used for the shear part of ψ_0 , defined on Ω_0 as

$$\bar{I}_1(\widehat{\mathbf{C}}) := (\det \widehat{\mathbf{C}})^{-\frac{1}{3}} \text{tr} \widehat{\mathbf{C}}, \quad \bar{I}_2(\mathbf{C}) := (\det \widehat{\mathbf{C}})^{-\frac{2}{3}} \text{tr}(\widehat{\mathbf{C}})^2,$$

together with the compressibility function

$$U(J) = \kappa(J \ln J - J + 1),$$

defining $K = \rho_0 \kappa$ as the compressibility, a material parameter [11]. Since $p_0^* \widehat{\mathbf{C}} = \gamma_0^{-1} \gamma$, we get

$$p_0^* J_\varphi = \mathcal{J}, \quad p_0^* \bar{I}_1(\widehat{\mathbf{C}}) = \mathcal{J}^{-\frac{1}{3}} I_1, \quad p_0^* \bar{I}_2(\widehat{\mathbf{C}}) = \mathcal{J}^{-\frac{2}{3}} I_2.$$

where $\mathcal{J} = (\det(\gamma_0^{-1} \gamma))^{1/2}$, $I_1 = \text{tr}(\gamma_0^{-1} \gamma)$ and $I_2 = \text{tr}(\gamma_0^{-1} \gamma \gamma_0^{-1} \gamma)$ are functions on \mathcal{B} . We deduce then, by theorem 3.4 and remark 3.5, that Hart–Smith hyper-elasticity can be formulated on the body \mathcal{B} , as (3.4), with

$$\psi = p_0^* \psi_0 = h_1 \int_3^{\mathcal{J}^{-\frac{1}{3}} I_1} \exp(h_3(I-3)^2) dI + 3h_2 \ln\left(\frac{\mathcal{J}^{-\frac{2}{3}} I_2}{3}\right) + U(\mathcal{J}).$$

4. THE VIRTUAL POWER AS A ONE-FORM ON THE CONFIGURATION SPACE

In this section, we fix our notations in order to recast the virtual power of applied surface forces as a 1-form on the configuration space and formulate an elasticity problem in a geometric manner, directly on the body \mathcal{B} . We start by recalling that the principle of virtual power in quasi-statics states that

$$\mathcal{P}^{int}(\mathbf{w}) + \mathcal{P}^{ext}(\mathbf{w}) = 0,$$

for any kinematically admissible virtual displacement field \mathbf{w} on Ω , sufficiently regular and vanishing on the prescribed displacement (Dirichlet) boundary $\partial\Omega^\xi \subset \partial\Omega$. In the sequel, we assume that $\partial\Omega^\xi \neq \emptyset$, in order to avoid some indeterminacy.

The virtual power of the internal forces \mathcal{P}^{int} writes

$$- \int_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\epsilon}) \text{vol}_{\mathbf{q}},$$

where $\boldsymbol{\epsilon}$ is a virtual strain field (a second order symmetric covariant tensor field) on Ω . It is further assumed that the virtual deformation field $\boldsymbol{\epsilon}$ derives from a virtual displacement field \mathbf{w} on Ω , which means that

$$\boldsymbol{\epsilon} := \frac{1}{2} \text{Lie}_{\mathbf{w}} \mathbf{q} = D\mathbf{w}^b, \quad \mathbf{w}^b = \mathbf{q}\mathbf{w}.$$

Here, the operator

$$(4.1) \quad D\mathbf{w}^b(X, Y) = \frac{1}{2} \left((\nabla_X \mathbf{w}^b)(Y) + (\nabla_Y \mathbf{w}^b)(X) \right)$$

is the *formal adjoint* of the divergence operator. One, then, has

$$\mathcal{P}^{int}(\mathbf{w}) = - \int_{\Omega} (\boldsymbol{\sigma} : D\mathbf{w}^b) \text{vol}_{\mathbf{q}} = - \int_{\partial\Omega} (\widehat{\boldsymbol{\sigma}} \mathbf{n} \cdot \mathbf{w}) da + \int_{\Omega} (\text{div } \boldsymbol{\sigma} \cdot \mathbf{w}) \text{vol}_{\mathbf{q}},$$

where \mathbf{n} is the unit outer normal on the boundary $\partial\Omega$, $da = i_{\mathbf{n}} \text{vol}_{\mathbf{q}}$ is the area element, and $\widehat{\boldsymbol{\sigma}} := \boldsymbol{\sigma} \mathbf{q}$ is the mixed form of Cauchy stress tensor.

The virtual power of external forces has for general expression

$$(4.2) \quad \mathcal{P}^{ext} = \mathcal{P}^{ext,v} + \mathcal{P}^{ext,s},$$

where

$$\mathcal{P}^{ext,v}(\mathbf{w}) = \int_{\Omega} (\mathbf{f}_v \cdot \mathbf{w}) \text{vol}_{\mathbf{q}},$$

and \mathbf{f}_v is the field of external forces per unit of volume. The second term $\mathcal{P}^{ext,s}$ corresponds to surface forces (boundary conditions of Neumann type). In the sequel, we will assume that $\mathcal{P}^{ext,s}$ consists in a so-called *dead load* term (DL) and/or a prescribed pressure term (P). It has thus the following expression [8, 39]

$$(4.3) \quad \mathcal{P}^{ext,s}(\mathbf{w}) = \int_{\Sigma_0^{(DL)}} (\mathbf{t}_0 \cdot \delta\varphi) da_0 - \int_{\Sigma^{(P)}} P (\mathbf{n} \cdot \mathbf{w}) da, \quad (\delta\varphi = \mathbf{w} \circ \varphi)$$

where

- the vector valued function \mathbf{t}_0 , with values in \mathcal{E} , is defined on the surface $\Sigma_0^{(DL)} \subset \partial\Omega_0$,
- the scalar function P (the pressure) is defined on the surface $\Sigma^{(P)} \subset \partial\Omega$,

with the property that

$$\text{int}\left(\Sigma^{(DL)}\right) \cap \text{int}\left(\Sigma^{(P)}\right) = \emptyset, \quad \text{and} \quad \Sigma^{(DL)} \cup \Sigma^{(P)} = \partial\Omega^t.$$

The virtual power of the pressure writes

$$\int_{\Sigma^{(P)}} P(\mathbf{w} \cdot \mathbf{n}) \, da = \int_{\Sigma^{(P)}} P i_{\mathbf{w}} \text{vol}_{\mathbf{q}},$$

using remark C.1. Now, by the change of variables formula, we get

$$\begin{aligned} \int_{\Sigma^{(P)}} P i_{\mathbf{w}} \text{vol}_{\mathbf{q}} &= \int_{p^{-1}(\Sigma^{(P)})} p^*(P i_{\mathbf{w}} \text{vol}_{\mathbf{q}}) = \int_{p^{-1}(\Sigma^{(P)})} p^*(P) i_{p^*\mathbf{w}} p^* \text{vol}_{\mathbf{q}} \\ (4.4) \qquad \qquad \qquad &= \int_{\Sigma_{\mathcal{B}}^{(P)}} (P \circ p) i_{\mathbf{F}^{-1}\delta p} \text{vol}_{\mathbf{q}}(\mathbf{F}\cdot, \mathbf{F}\cdot, \mathbf{F}\cdot) \\ &= \int_{\Sigma_{\mathcal{B}}^{(P)}} (P \circ p) \text{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot), \end{aligned}$$

where $\mathbf{F} = Tp$, $\delta p = \mathbf{w} \circ p$ and $\Sigma_{\mathcal{B}}^{(P)} := p^{-1}(\Sigma^{(P)}) \subset \partial\mathcal{B}$.

Remark 4.1. If we identify the body \mathcal{B} with a reference configuration Ω_0 , we get $p_0 \equiv \text{id}$, $p \equiv \varphi$, $\Sigma_{\mathcal{B}}^{(P)} \equiv \Sigma_0^{(P)} \subset \partial\Omega_0$, $\mathbf{F} \equiv \mathbf{F}_\varphi$ and $\delta\xi \equiv \delta\varphi$, and

$$\begin{aligned} \int_{\Sigma_{\mathcal{B}}^{(P)}} (P \circ p) \text{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot) &= \int_{\Sigma_0^{(P)}} (P \circ \varphi) \text{vol}_{\mathbf{q}}(\delta\xi, \mathbf{F}_\varphi\cdot, \mathbf{F}_\varphi\cdot) \\ &= \int_{\Sigma_0^{(P)}} (P \circ \varphi) (\det \mathbf{F}_\varphi) i_{\mathbf{F}_\varphi^{-1}\delta\xi} \text{vol}_{\mathbf{q}} \\ &= \int_{\Sigma_0^{(P)}} (P \circ \varphi) J_\varphi \mathbf{F}_\varphi^{-1} \delta\xi \cdot \mathbf{n}_0 \, da_0. \end{aligned}$$

We recover the well-known expression of the pressure work

$$\int_{\Sigma^{(P)}} P(\mathbf{w} \cdot \mathbf{n}) \, da = \int_{\Sigma_0^{(P)}} (P \circ \varphi) J_\varphi \mathbf{F}_\varphi^{-1} \delta\xi \cdot \mathbf{n}_0 \, da_0, \quad J_\varphi = \det \mathbf{F}_\varphi,$$

formulated on the reference configuration Ω_0 (see [37, 47, 4, 8, 30]). This expression is usually obtained, using Nanson's formula (C.4). One might note that

$$J_\varphi \mathbf{F}_\varphi^{-1} \delta\xi \cdot \mathbf{n}_0 \, da_0 = \text{cof}(\mathbf{F}_\varphi)^t \delta\xi \cdot \mathbf{n}_0 \, da_0 = \text{vol}_{\mathbf{q}}(\delta\xi, \mathbf{F}_\varphi\cdot, \mathbf{F}_\varphi\cdot),$$

where $\text{cof}(\mathbf{F}_\varphi)$ is the cofactor matrix of \mathbf{F}_φ .

Finally, introducing

$$\mathscr{W}_p^{\text{ext},s}(\delta p) := -\mathscr{P}^{\text{ext},s}(\delta p \circ p^{-1}),$$

we recast (4.3) on the body as

$$(4.5) \quad \mathscr{W}_p^{\text{ext},s}(\delta p) = - \int_{\Sigma_{\mathcal{B}}^{(DL)}} (\mathbf{t}_0 \circ p_0 \cdot \delta p) \, da_{\gamma_0} + \int_{\Sigma_{\mathcal{B}}^{(P)}} (P \circ p) \text{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot),$$

where we have set $\Sigma_{\mathcal{B}}^{(DL)} = p_0^{-1}(\Sigma_0^{(DL)}) \subset \partial\mathcal{B}$ and $da_{\gamma_0} = p_0^* da_0$. Thus, $\mathscr{W}^{\text{ext},s}$ is naturally interpreted as a 1-form (see Appendix B) on the configuration space $\text{Emb}(\mathcal{B}, \mathcal{E})$.

Remark 4.2 (Equilibrium equations). The adequate choice of virtual displacement fields vanishing on the boundary $\partial\Omega$ classically leads to the equilibrium equation, but expressed here, either on the deformed configuration Ω , as

$$(4.6) \quad \text{div} \boldsymbol{\sigma} + \mathbf{f}_v = 0,$$

or, on the body \mathcal{B} , as

$$(4.7) \quad \text{div}^\gamma \boldsymbol{\mathfrak{S}} + \mathbf{f}_v = 0, \quad \mathbf{f}_v = p^* \mathbf{f}_v = \mathbf{F}^{-1} \mathbf{f}_v \circ p,$$

where \mathbf{f}_v stands for the external forces density on \mathcal{B} and where $\operatorname{div}^\gamma \mathfrak{S} = \operatorname{tr}_{13}(\nabla^\gamma \mathfrak{S}) = p^* \operatorname{div} \sigma$, ∇^γ being the Riemannian covariant derivative corresponding to the metric $\gamma = p^* \mathbf{q}$.

5. A GENERAL SCHEME FOR LAGRANGIAN FORMULATIONS OF HYPER-ELASTICITY PROBLEMS

We now introduce a systematic method to build a Lagrangian (a potential energy) for an hyper-elasticity problem, when certain compatibility conditions are satisfied, concerning the surface forces. When these conditions are not satisfied, we formulate, anyway, *non-holonomic* constraints (*i.e.* which involve not only the embedding p , but also its variation δp), to try to bypass these restrictions. This method is illustrated by applying them to two types of Neumann conditions: *dead load (DL)* in section 5.2 and *prescribed pressure (P)* in section 6. These examples are classical. However, the way we recast them seems to be original and illustrates the power of differential geometry (in infinite dimension) to tackle the problem in a more conceptual and systematic way (and avoid the numerous typos found in the literature). Moreover, the proposed method allows us to improve non-holonomic constraints formulated by Beatty [5] to ensure the existence of a pressure potential. It is rather general and could, *a priori*, be applied to many other situations. In a connected but different direction, the interested reader may refer to [40, 53, 55] for discussions on the geometrization of point-wise holonomic constraints in finite strain theory.

The principle of virtual work, once recast on the body, involves diverse differential one-forms \mathcal{W}^k defined on $\operatorname{Emb}(\mathcal{B}, \mathcal{E})$, like $\mathcal{W}^{ext,s}$ in (4.5). This principle stipulates that an embedding p (which may be subject to some holonomic constraints) is solution of the mechanical problem iff the sum of these one-forms vanishes for all variations δp (which may be subject to some non-holonomic constraints). The goal here is to try to recast this problem as a Lagrangian variational problem. In the best case, when each involved one-form \mathcal{W}^k is exact, meaning that $\mathcal{W}^k = d\mathcal{L}^k$, then, each solution of the mechanical problem is an extremal of the Lagrangian

$$\mathcal{L} := \sum_k \mathcal{L}^k.$$

The good news is that Classical Differential Geometry furnishes tools (like Poincaré's lemma B.2) to decide if the problem admits a Lagrangian, and in that case, to calculate a Lagrangian. Otherwise, it allows to formulate explicitly non-holonomic constraints under which a Lagrangian may still be defined. These tools can be extrapolated to Differential Geometry in infinite dimension and we will now describe them, in the mechanical situations we are interested in. The approach used here is the same as the one adopted by Arnold in [2]: use classical results from finite dimensional geometry, extrapolate them in this extended infinite dimensional framework and then check that it works well. This check is made, in our case, by an explicit variation calculus.

The tool required to achieve our goal is the theory of *differential forms* which is briefly recalled in Appendix B (see also [7, 1, 29]). For a rigorous extension of this formalism to infinite dimensional spaces, one may consider useful to look at [23, 36, 28].

For our concern, one needs only to know that a one-form on $\operatorname{Emb}(\mathcal{B}, \mathcal{E})$ is just a linear functional on $C^\infty(\mathcal{B}, \mathcal{E})$, but depending smoothly on p and a two-form on $\operatorname{Emb}(\mathcal{B}, \mathcal{E})$ is a skew-symmetric bilinear functional on $C^\infty(\mathcal{B}, \mathcal{E})$, depending smoothly on p . Extrapolating the theory of differential forms, we define

- the *exterior derivative* of a 0-form on $\operatorname{Emb}(\mathcal{B}, \mathcal{E})$ (*i.e.* a functional \mathcal{L}), just as the first variation of \mathcal{L}

$$(d\mathcal{L})_p(\delta p) = \delta \mathcal{L} := \left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(p(s)),$$

where $p(s)$ is a path of embeddings with $p(0) = p$ and $\dot{p}(0) = \delta p$;

- the *exterior derivative* of a 1-form \mathcal{W} on $\operatorname{Emb}(\mathcal{B}, \mathcal{E})$ as

$$(d\mathcal{W})_p(\delta_1 p, \delta_2 p) := \delta_1(\mathcal{W}_p(\delta_2 p)) - \delta_2(\mathcal{W}_p(\delta_1 p)),$$

where

$$\delta_2(\mathcal{W}_p(\delta_1 p)) := \left. \frac{d}{ds} \right|_{s=0} \mathcal{W}_{p(s)}(\delta_1 p),$$

where $p(s)$ is a path of embeddings with $p(0) = p$ and $\dot{p}(0) = \delta_2 p$.

Remark 5.1. To write these two formulas, we have used the hypothesis that \mathcal{E} is an affine space, and thus that $\text{Emb}(\mathcal{B}, \mathcal{E})$ is an open set of the Fréchet vector space $C^\infty(\mathcal{B}, \mathcal{E})$ (see [Appendix D](#)). An embedding $p : \mathcal{B} \rightarrow \mathcal{E}$ or a variation of this embedding $\delta p : T\mathcal{B} \rightarrow T\mathcal{E}$ are thus considered as vector valued functions.

5.1. Poincaré's integrator. Our methodology is the following. Starting from a single 1-form \mathcal{W} in the list (\mathcal{W}^k) , we check first if this form is closed. In other words, we check if

$$(d\mathcal{W})_p(\delta_1 p, \delta_2 p) = \delta_1(\mathcal{W}_p(\delta_2 p)) - \delta_2(\mathcal{W}_p(\delta_1 p)) = 0,$$

with then two cases.

- **The case \mathcal{W} is closed** ($d\mathcal{W} = 0$). Then, a Lagrangian \mathcal{L} is obtained locally, using Poincaré's lemma [B.2](#), by the following procedure. Let $p_0 \in \text{Emb}(\mathcal{B}, \mathcal{E})$ be a reference configuration. The displacement

$$\boldsymbol{\xi}(p) = p - p_0,$$

corresponds to the radial vector field issued from p_0 on $\text{Emb}(\mathcal{B}, \mathcal{E})$. Consider now the neighborhood of p_0 , defined by

$$\mathcal{U}_{p_0} := \{p \in C^\infty(\mathcal{B}, \mathcal{E}); \|\mathbf{F} - \mathbf{F}_0\|_{L^\infty} < 1\},$$

which is an *open convex set* of $C^\infty(\mathcal{B}, \mathcal{E})$, contained in $\text{Emb}(\mathcal{B}, \mathcal{E})$ (see [Appendix D](#)). Then, Poincaré's formula [\(B.3\)](#) provides us with the primitive

$$\mathcal{L}(p) = \int_{-\infty}^0 [(\phi^t)^* i_{\boldsymbol{\xi}} \mathcal{W}] (p) dt,$$

defined on \mathcal{U}_{p_0} , and where $\phi^t(p) = e^t p + (1 - e^t)p_0$ is the flow of the radial field $\boldsymbol{\xi}(p) = p - p_0$. But

$$[(\phi^t)^* i_{\boldsymbol{\xi}} \mathcal{W}] (p) = \mathcal{W}_{\phi^t(p)}(\boldsymbol{\xi}(\phi^t(p))) = e^t \mathcal{W}_{\phi^t(p)}(\boldsymbol{\xi}(p)),$$

because $\boldsymbol{\xi}(\phi^t(p)) = e^t \boldsymbol{\xi}(p)$ and we get finally the sought Lagrangian for \mathcal{W}

$$(5.1) \quad \mathcal{L}(p) = \int_{-\infty}^0 e^t \mathcal{W}_{\phi^t(p)}(\boldsymbol{\xi}(p)) dt.$$

- **The case \mathcal{W} is not closed** ($d\mathcal{W} \neq 0$). Then, by [\(B.4\)](#), we get

$$d\mathcal{L} = \mathcal{W} - \int_{-\infty}^0 (\phi^t)^* i_{\boldsymbol{\xi}} d\mathcal{W} dt,$$

where

$$\begin{aligned} [(\phi^t)^* i_{\boldsymbol{\xi}} d\mathcal{W}]_p(\delta p) &= (d\mathcal{W})_{\phi^t(p)}(T\phi^t \cdot \boldsymbol{\xi}(p), T\phi^t \cdot \delta p) \\ &= e^{2t} (d\mathcal{W})_{\phi^t(p)}(\boldsymbol{\xi}(p), \delta p), \end{aligned}$$

because the linear tangent mapping $T_p \phi^t : T_p \text{Emb}(\mathcal{B}, \mathcal{E}) \rightarrow T_{\phi^t(p)} \text{Emb}(\mathcal{B}, \mathcal{E})$ writes

$$T_p \phi^t \cdot \delta p = e^t \delta p.$$

In this case, the integral

$$(5.2) \quad \int_{-\infty}^0 e^{2t} (d\mathcal{W})_{\phi^t(p)}(\boldsymbol{\xi}(p), \delta p) dt,$$

is the obstruction for [\(5.1\)](#) to be a primitive of \mathcal{W} and its vanishing can be interpreted as a *non-holonomic constraint*, required for [\(5.1\)](#) to become a Lagrangian for \mathcal{W} .

5.2. Example: Dead loads. Before addressing the more involving case of a prescribed pressure, let us illustrate our methodology on the well-known case of a dead load.

Remark 5.2. The related questions of how to recast locally such a boundary condition on the body \mathcal{B} , in a so-called intrinsic manner [34, 41, 22, 17], and of how to redefine a first Piola-Kirchhoff tensor on \mathcal{B} is answered in [Appendix E](#).

In the dead load case (see (4.5)), the corresponding one-form defined on $\text{Emb}(\mathcal{B}, \mathcal{E})$ writes

$$\mathcal{W}_p^{DL}(\delta p) = - \int_{\Sigma_{\mathcal{B}}^{(DL)}} (\delta p \cdot \mathbf{t}_0 \circ p_0) da_{\gamma_0}.$$

This form is obviously closed, since it does not depend explicitly on p . Thus, a Lagrangian (*i.e.* a potential for the applied surface force [8]) $\mathcal{L}^{DL}(p)$ exists for dead loads and is given by (5.1). After integration, it writes

$$\mathcal{L}^{DL}(p) = - \int_{\Sigma_{\mathcal{B}}^{(DL)}} ((p - p_0) \cdot \mathbf{t}_0 \circ p_0) da_{\gamma_0},$$

since

$$\mathcal{W}_{\phi^t(p)}^{DL}(\xi(p)) = - \int_{\Sigma_{\mathcal{B}}^{(DL)}} ((p - p_0) \cdot \mathbf{t}_0 \circ p_0) da_{\gamma_0}.$$

6. PRESCRIBED PRESSURE

The question of the existence of a Lagrangian (a potential energy) for hyper-elasticity problems with boundary conditions of the prescribed pressure type

$$(6.1) \quad \widehat{\boldsymbol{\sigma}} \mathbf{n}|_{\Sigma} = -P \mathbf{n},$$

on some part $\Sigma = \Sigma^{(P)}$ of the boundary $\partial\Omega$ has been addressed in several works [37, 47, 46, 5, 4] but rarely (never ?) directly on the body' boundary. The prescribed pressure boundary condition (6.1) becomes, by pull-back on $\partial\mathcal{B}$

$$(6.2) \quad \widehat{\boldsymbol{\mathfrak{G}}} \mathbf{N}|_{\Sigma_{\mathcal{B}}} = -(P \circ p) \mathbf{N}, \quad \Sigma_{\mathcal{B}} = p^{-1}(\Sigma),$$

where $\widehat{\boldsymbol{\mathfrak{G}}} = p^* \widehat{\boldsymbol{\sigma}}$ is Noll's stress tensor in its mixed form, and where $p^* \mathbf{n} = \mathbf{N}$ is the unit normal on $\partial\mathcal{B}$ for the metric $\boldsymbol{\gamma} = p^* \mathbf{q}$. In this section, we assume that P is uniform, *i.e.* $P = P_k$, on each component Σ^k of the boundary $\Sigma^{(P)} \subset \partial\Omega$, where the pressure is applied. The contribution to the virtual power of exterior forces \mathcal{P}^{ext} of these prescribed pressures writes thus

$$(6.3) \quad - \sum_k P_k \int_{\Sigma^k} (\mathbf{w} \cdot \mathbf{n}) da,$$

where \mathbf{w} is the virtual Eulerian velocity.

6.1. The pressure virtual power is not closed. The integral

$$\int_{\Sigma} P(\mathbf{w} \cdot \mathbf{n}) da$$

was recast on the body's boundary in (4.4). Here, for simplicity, we will assume that $P = 1$ is a constant function and we have thus

$$(6.4) \quad \mathcal{W}_p^P(\delta p) := \int_{\Sigma} (\mathbf{w} \cdot \mathbf{n}) da = \int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot)$$

The pressure case illustrate the second case, \mathcal{W} is not closed, of our general scheme in [subsection 5.1](#).

Lemma 6.1. *The differential form*

$$\mathcal{W}_p^P(\delta p) = \int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot)$$

defined on $\text{Emb}(\mathcal{B}, \mathcal{E})$ is not closed. Its exterior derivative writes

$$(6.5) \quad (d\mathcal{W}^P)_p(\delta p_1, \delta p_2) = \int_{\partial\Sigma_{\mathcal{B}}} (\delta p_2 \times \delta p_1) \cdot \mathbf{F} d\ell_{\mathcal{B}},$$

where $d\ell_{\mathcal{B}} = p_0^* d\ell_0$ is the oriented length element on $\partial\Sigma_{\mathcal{B}}$.

The proof we propose here is based on the following observation. An embedding $p : \mathcal{B} \rightarrow \mathcal{E}$ or a variation of this embedding $\delta p : \mathcal{B} \rightarrow T\mathcal{E}$ are considered as *vector valued functions* $\mathbf{f} : \mathcal{B} \rightarrow \mathcal{E}$. Now, if A is a vector field on \mathcal{B} , we can define the Lie derivative $L_A \mathbf{f}$ of \mathbf{f} relative to A . But, since \mathbf{f} is considered as a function, it depends only on the pointwise value of A (and not of its first derivatives) and we get moreover

$$T\mathbf{f}.A = L_A \mathbf{f}.$$

Proof of lemma 6.1. We have

$$\text{vol}_{\mathbf{q}}(\delta_1 p, \mathbf{F}.A, \mathbf{F}.B) = \text{vol}_{\mathbf{q}}(\delta_1 p, L_A p, L_B p),$$

A and B being vector fields on $\partial\mathcal{B}$. Let $\delta_2 p$ be a second variation of p , we get thus

$$\delta_2(\text{vol}_{\mathbf{q}}(\delta_1 p, \mathbf{F}\cdot, \mathbf{F}\cdot))(A, B) = \text{vol}_{\mathbf{q}}(\delta_1 p, L_A \delta_2 p, L_B p) + \text{vol}_{\mathbf{q}}(\delta_1 p, L_A p, L_B \delta_2 p),$$

and hence

$$\begin{aligned} (d\mathcal{W}^P)_p(\delta p_1, \delta p_2) &= \int_{\Sigma_{\mathcal{B}}} \delta_1 \text{vol}_{\mathbf{q}}(\delta_2 p, Tp\cdot, Tp\cdot) - \delta_2 \text{vol}_{\mathbf{q}}(\delta_1 p, Tp\cdot, Tp\cdot) \\ &= \int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\delta_2 p, L_A \delta_1 p, L_B p) + \text{vol}_{\mathbf{q}}(\delta_2 p, L_A p, L_B \delta_1 p) \\ &\quad - \text{vol}_{\mathbf{q}}(\delta_1 p, L_A \delta_2 p, L_B p) - \text{vol}_{\mathbf{q}}(\delta_1 p, L_A p, L_B \delta_2 p). \end{aligned}$$

Let us now introduce the 1-form α on $\partial\mathcal{B}$ defined by

$$\alpha(A) := \text{vol}_{\mathbf{q}}(\delta_2 p, \delta_1 p, L_A p).$$

By lemma B.1, we have then

$$\begin{aligned} d\alpha(A, B) &= \text{vol}_{\mathbf{q}}(L_A \delta_2 p, \delta_1 p, L_B p) - \text{vol}_{\mathbf{q}}(L_B \delta_2 p, \delta_1 p, L_A p) \\ &\quad + \text{vol}_{\mathbf{q}}(\delta_2 p, L_A \delta_1 p, L_B p) - \text{vol}_{\mathbf{q}}(\delta_2 p, L_B \delta_1 p, L_A p), \end{aligned}$$

Hence, using Stokes theorem, we get

$$(d\mathcal{W}^P)_p(\delta p_1, \delta p_2) = \int_{\Sigma_{\mathcal{B}}} d\alpha = \int_{\partial\Sigma_{\mathcal{B}}} \alpha,$$

and thus

$$(d\mathcal{W}^P)_p(\delta p_1, \delta p_2) = \int_{\partial\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\delta_2 p, \delta_1 p, \mathbf{F}\cdot) = \int_{\partial\Sigma_{\mathcal{B}}} (\delta_2 p \times \delta_1 p) \cdot \mathbf{F} d\ell_{\mathcal{B}}.$$

□

6.2. A pressure potential under non-holonomic constraints. Since \mathcal{W}^P is not closed, a Lagrangian for the prescribed pressure virtual power does not exist *a priori* [37, 47, 46, 5, 4, 39]. However, if we define the functional \mathcal{L}^P by (5.1), then we have

$$d\mathcal{L}^P(\delta p) = \mathcal{W}_p^P(\delta p),$$

provided that condition (5.2) is satisfied. We will now achieve the calculation of \mathcal{L}^P and the corresponding non-holonomic constraints. First, we have

$$\mathcal{W}_{\phi^t(p)}^P(\boldsymbol{\xi}(p)) = \int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\delta p, T\phi^t \cdot, T\phi^t \cdot)$$

and (5.1) writes

$$\mathcal{L}^P(p) = \int_{-\infty}^0 e^t \left(\int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(\boldsymbol{\xi}(p), (e^t \mathbf{F} + (1 - e^t) \mathbf{F}_0) \cdot, (e^t \mathbf{F} + (1 - e^t) \mathbf{F}_0) \cdot) \right) dt,$$

since $T\phi^t(p) = e^t \mathbf{F} + (1 - e^t) \mathbf{F}_0$. The calculation is then straightforward, making use of the multi-linearity of $\text{vol}_{\mathbf{q}}$, and leads to the Lagrangian (the pressure potential)

$$\mathcal{L}^P(p) = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} 2\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F} \cdot, \mathbf{F} \cdot) + (\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F} \cdot, \mathbf{F}_0 \cdot) + \text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0 \cdot, \mathbf{F} \cdot)) + 2\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0 \cdot, \mathbf{F}_0 \cdot).$$

The non-holonomic constraints are obtained by specifying (5.2) for $\mathcal{W} = \mathcal{W}^P$ and thus by computing the integral

$$\int_{-\infty}^0 e^{2t} (d\mathcal{W}^P)_{\phi^t(p)}(\boldsymbol{\xi}(p), \delta p) dt$$

where

$$(d\mathcal{W}^P)_p(\delta p_1, \delta p_2) = \int_{\partial \Sigma_{\mathcal{B}}} (\delta p_2 \times \delta p_1) \cdot \mathbf{F} d\ell_{\mathcal{B}}.$$

We get

$$\oint_{\partial \Sigma_{\mathcal{B}}} (\delta p \times \boldsymbol{\xi}(p)) \cdot \left(\int_{-\infty}^0 (e^{3t} \mathbf{F} + e^{2t} (1 - e^t) \mathbf{F}_0) dt \right) d\ell_{\mathcal{B}},$$

which is equal, after integration to

$$-\frac{1}{6} \oint_{\partial \Sigma_{\mathcal{B}}} (\boldsymbol{\xi}(p) \times \delta p) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}}.$$

Thus, the non-holonomic constraints, which must be satisfied, in order for \mathcal{L}^P to be a Lagrangian for \mathcal{W}^P write

$$\oint_{\partial \Sigma_{\mathcal{B}}} (\boldsymbol{\xi} \times \delta \boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}} = 0.$$

We will summarize these results in the following theorem, checking, this time, by a direct variation calculus, that \mathcal{L}^P is a *potential* for \mathcal{W}^P which is globally defined. So far, this was established only locally, on the neighborhood

$$\mathcal{U}_{p_0} = \{p \in C^\infty(\mathcal{B}, \mathcal{E}); \|\mathbf{F} - \mathbf{F}_0\|_{L^\infty} < 1\}$$

of the reference configuration p_0 .

Theorem 6.2. *Let us consider the functional*

$$(6.6) \quad \mathcal{L}^P(p) = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} 2\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F} \cdot, \mathbf{F} \cdot) + (\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F} \cdot, \mathbf{F}_0 \cdot) + \text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0 \cdot, \mathbf{F} \cdot)) + 2\text{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0 \cdot, \mathbf{F}_0 \cdot),$$

defined on $\text{Emb}(\mathcal{B}, \mathcal{E})$, where $\boldsymbol{\xi}(p) := p - p_0$ is the displacement field. Then

$$(6.7) \quad d\mathcal{L}^P(\delta p) = \mathcal{W}^P(\delta p) + \frac{1}{6} \oint_{\partial \Sigma_{\mathcal{B}}} (\boldsymbol{\xi} \times \delta \boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}},$$

$d\ell_{\mathcal{B}}$ being the vector length element on $\partial\Sigma_{\mathcal{B}}$. In particular, the condition for the functional \mathcal{L}^P to be a pressure potential (i.e. a primitive of \mathcal{W}^P) is thus

$$(6.8) \quad \oint_{\partial\Sigma_{\mathcal{B}}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}} = 0.$$

Proof of theorem 6.2. Set

$$(6.9) \quad \begin{aligned} \omega^1 &:= 2 \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}\cdot) \\ \omega^2 &:= \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}_0\cdot) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}\cdot) \\ \omega^3 &:= 2 \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}_0\cdot), \end{aligned}$$

and $\omega := \omega^1 + \omega^2 + \omega^3$, so that

$$(6.10) \quad \mathcal{L}^P = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} \omega, \quad \delta\mathcal{L}^P = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} \delta\omega.$$

We have

$$\begin{aligned} \delta\omega^1(A, B) &= 2 (\operatorname{vol}_{\mathbf{q}}(\delta p, L_A p, L_B p) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A \delta p, L_B p) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A p, L_B \delta p)), \\ \delta\omega^2(A, B) &= \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p, L_B p_0) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A \delta p, L_B p_0) \\ &\quad + \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p_0, L_B p) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A p_0, L_B \delta p), \\ \delta\omega^3(A, B) &= 2 \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p_0, L_B p_0), \end{aligned}$$

and thus

$$\begin{aligned} \delta\omega(A, B) &= \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p, L_B(2p + p_0)) + \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p_0, L_B(2p_0 + p)) \\ &\quad + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A \delta p, L_B(2p + p_0)) - \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_B \delta p, L_A(2p + p_0)). \end{aligned}$$

Now we need to do an integration by part to get rid of the terms that contain Lie derivatives of δp . To do this, we introduce the 1-form α on $\partial\mathcal{B}$ defined by

$$\alpha(A) := \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \delta p, L_A(2p + p_0)).$$

such that, by lemma B.1,

$$\begin{aligned} d\alpha(A, B) &= \operatorname{vol}_{\mathbf{q}}(L_A \boldsymbol{\xi}, \delta p, L_B(2p + p_0)) - \operatorname{vol}_{\mathbf{q}}(L_B \boldsymbol{\xi}, \delta p, L_A(2p + p_0)) \\ &\quad + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_A \delta p, L_B(2p + p_0)) - \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, L_B \delta p, L_A(2p + p_0)). \end{aligned}$$

Therefore, after simplification, we have

$$\delta\omega(A, B) = 6 \operatorname{vol}_{\mathbf{q}}(\delta p, L_A p, L_B p) + d\alpha(A, B),$$

and thus

$$\delta\mathcal{L}^P = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} \delta\omega = \int_{\Sigma_{\mathcal{B}}} \operatorname{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot) + \frac{1}{6} \int_{\partial\Sigma_{\mathcal{B}}} \alpha,$$

where the restriction of α to $\partial\Sigma_{\mathcal{B}}$ writes

$$\alpha = \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \delta p, (\mathbf{F}_0 + 2\mathbf{F})\cdot) = (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (\mathbf{F}_0 + 2\mathbf{F}) d\ell_{\mathcal{B}},$$

$d\ell_{\mathcal{B}}$ being the vector length element relative to the metric $\gamma_0 = p_0^* \mathbf{q}$ on \mathcal{B} . Finally

$$\delta\mathcal{L}^P = \mathcal{W}^P(\delta p) + \frac{1}{6} \oint_{\partial\Sigma_{\mathcal{B}}} (\boldsymbol{\xi} \times \delta p) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}},$$

which ends the proof. \square

7. LINK WITH OTHER EXISTING FORMULATIONS

The goal of this section is to relate the present work with other existing studies concerning the formulation of a pressure potential and the required constraints. These results are all expressed on a reference configuration Ω_0 and not on the body \mathcal{B} .

7.1. Recovering Pearson–Sewell potential and Beatty conditions. The first formulation of a pressure potential seems to have been produced in 1956 by Pearson [37, Eq. (25) on p. 142] and then reobtained by Sewell ten years later (see [47, Eq. (32) on p. 407] and [46, Eq. (89) on p. 341]). In all these works, the potential is written in components. Its intrinsic expression seems to have been given for the first time by Beatty in [5, Eq. (4.3) on p. 373] but with some typos. Its (corrected) expression writes:

$$(7.1) \quad \mathcal{L}^P(\varphi) = \frac{P}{3} \int_{\Sigma_0} \left((\operatorname{cof} \mathbf{F}_\varphi)^t \boldsymbol{\xi} + \frac{1}{2} \left((\operatorname{tr} \mathbf{F}_\varphi) \boldsymbol{\xi} - \mathbf{F}_\varphi \boldsymbol{\xi} \right) + \boldsymbol{\xi} \right) \cdot \mathbf{n}_0 \, da_0,$$

where $\boldsymbol{\xi} = \varphi - \operatorname{id}$ and $\operatorname{cof} \mathbf{F}_\varphi$ is the cofactor matrix of \mathbf{F}_φ . This expression corresponds to the potential \mathcal{L}^P given by (6.6), if we identify the body \mathcal{B} with a reference configuration Ω_0 , embedded in Euclidean space \mathcal{E} (see below for a proof). Indeed, then, we make the identifications:

$$(7.2) \quad \Sigma_{\mathcal{B}} = \Sigma_0, \quad p \equiv \varphi, \quad \mathbf{F}_0 \equiv \mathbf{Id}, \quad \mathbf{F} \equiv \mathbf{F}_\varphi, \quad \boldsymbol{\xi} \equiv \varphi - \operatorname{id}.$$

Then, our non-holonomic constraints (6.8), formulated on the body, recast on Ω_0 as

$$(7.3) \quad \oint_{\partial\Sigma_0} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F}_\varphi + \mathbf{Id}) d\ell_0 = 0, \quad \boldsymbol{\xi} = \varphi - \operatorname{id}.$$

This is an improvement compared to Beatty conditions [5, Eq. (4.6) on p. 374],

$$(7.4) \quad \oint_{\partial\Sigma_0} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot d\ell_0 = 0 \quad \text{and} \quad \oint_{\partial\Sigma_0} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot \mathbf{F}_\varphi d\ell_0 = 0,$$

which are stronger since (7.4) implies (7.3), but the converse does not hold.

Remark 7.1. Both conditions (7.3) and (7.4) are satisfied, in particular, when the variations $\delta\boldsymbol{\xi}$ vanish on the closed contour $\partial\Sigma_0$. More generally, in order for them to be verified it is sufficient that the virtual displacement $\delta p = \delta\boldsymbol{\xi} \in T\operatorname{Emb}(\mathcal{B}, \mathcal{E})$ remains collinear to the displacement $\boldsymbol{\xi} = p - p_0$ (i.e. $\boldsymbol{\xi} \times \delta\boldsymbol{\xi} = 0$) all along the contour $\partial\Sigma_{\mathcal{B}}$. Note finally that the constraints (7.3) and (6.8) are trivially satisfied when $\partial\Sigma = p(\partial\Sigma_{\mathcal{B}}) = \emptyset$, as in the case of an uniform pressure applied on the entire external surface of a structure or on the entire surface of a fully embedded cavity.

We conclude by providing a detailed calculation of how (6.6) recast as (7.1) when we identify the body \mathcal{B} with Ω_0 embedded in Euclidean space \mathcal{E} . Thanks to the identifications (7.2), \mathcal{L}^P writes

$$\mathcal{L}^P(\varphi) = \frac{P}{6} \int_{\Sigma_0} \omega^1 + \omega^2 + \omega^3,$$

where, according to the definition of the 2-forms ω^i by (6.9),

$$\omega^1 = 2\operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_\varphi \cdot, \mathbf{F}_\varphi \cdot) = 2(\det \mathbf{F}_\varphi) i_{\mathbf{F}_\varphi^{-1}\boldsymbol{\xi}} \operatorname{vol}_{\mathbf{q}} = 2J_\varphi (\mathbf{F}_\varphi^{-1}\boldsymbol{\xi} \cdot \mathbf{n}_0) \, da_0,$$

$$\omega^2 = \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_\varphi \cdot, \cdot) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \cdot, \mathbf{F}_\varphi \cdot) = ((\operatorname{tr} \mathbf{F}_\varphi) \boldsymbol{\xi} - \mathbf{F}_\varphi \boldsymbol{\xi}) \cdot \mathbf{n}_0 \, da_0,$$

$$\omega^3 = 2\operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \cdot, \cdot) = i_{\boldsymbol{\xi}} \operatorname{vol}_{\mathbf{q}} = 2(\boldsymbol{\xi} \cdot \mathbf{n}_0) \, da_0.$$

The first and third equalities are straightforward using (C.2). The second equality results from the following observation. The 2-form ω^2 is proportional to the area element $da_0 = i_{\mathbf{n}_0} \operatorname{vol}_{\mathbf{q}}$ of $\partial\Omega_0$ (see Appendix B). Therefore, if $\mathbf{e}_1, \mathbf{e}_2$ is a direct orthonormal basis of $T_{\mathbf{x}_0} \partial\Omega_0$, the tangent space to the boundary of Ω_0 , we have $\omega^2 = \lambda da_0$, where $\lambda = \omega^2(\mathbf{e}_1, \mathbf{e}_2)$. Thus, writing $\boldsymbol{\xi} = \xi^1 \mathbf{e}_1 + \xi^2 \mathbf{e}_2 + (\boldsymbol{\xi} \cdot \mathbf{n}_0) \mathbf{n}_0$, we get

$$\begin{aligned} \omega^2(\mathbf{e}_1, \mathbf{e}_2) &= \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \operatorname{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) \\ &= (\boldsymbol{\xi} \cdot \mathbf{n}_0) \operatorname{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + (\boldsymbol{\xi} \cdot \mathbf{n}_0) \operatorname{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) \\ &\quad + \xi^1 \operatorname{vol}_{\mathbf{q}}(\mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \xi^2 \operatorname{vol}_{\mathbf{q}}(\mathbf{e}_2, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) \\ &= (\boldsymbol{\xi} \cdot \mathbf{n}_0) \{ \operatorname{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \operatorname{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) + \operatorname{vol}_{\mathbf{q}}(\mathbf{F}_\varphi \mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2) \} \\ &\quad + \xi^1 \operatorname{vol}_{\mathbf{q}}(\mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \xi^2 \operatorname{vol}_{\mathbf{q}}(\mathbf{e}_2, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) - (\boldsymbol{\xi} \cdot \mathbf{n}_0) \operatorname{vol}_{\mathbf{q}}(\mathbf{F}_\varphi \mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2). \end{aligned}$$

But

$$\text{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \text{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) + \text{vol}_{\mathbf{q}}(\mathbf{F}_\varphi \mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2) = \text{tr } \mathbf{F}_\varphi$$

and

$$\begin{aligned} \xi^1 \text{vol}_{\mathbf{q}}(\mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_1, \mathbf{e}_2) + \xi^2 \text{vol}_{\mathbf{q}}(\mathbf{e}_2, \mathbf{e}_1, \mathbf{F}_\varphi \mathbf{e}_2) - (\boldsymbol{\xi} \cdot \mathbf{n}_0) \text{vol}_{\mathbf{q}}(\mathbf{F}_\varphi \mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2) \\ = -\text{vol}_{\mathbf{q}}(\mathbf{F}_\varphi \boldsymbol{\xi}, \mathbf{e}_1, \mathbf{e}_2) = -\mathbf{F}_\varphi \boldsymbol{\xi} \cdot \mathbf{n}_0, \end{aligned}$$

because $\text{vol}_{\mathbf{q}}(\mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2) = 1$. Hence

$$\lambda = \omega^2(\mathbf{e}_1, \mathbf{e}_2) = (\boldsymbol{\xi} \cdot \mathbf{n}_0) \text{tr } \mathbf{F}_\varphi - \mathbf{F}_\varphi \boldsymbol{\xi} \cdot \mathbf{n}_0 = ((\text{tr } \mathbf{F}_\varphi) \boldsymbol{\xi} - \mathbf{F}_\varphi \boldsymbol{\xi}) \cdot \mathbf{n}_0.$$

We can therefore rewrite (6.6) as (7.1).

7.2. An alternative pressure potential. A Lagrangian formulation (together with its non-holonomic constraints) for the pressure boundary conditions is not unique. There exists in the literature alternative formulations [4, 8, 39], valid under the stronger condition $\delta \boldsymbol{\xi} = 0$ on $\partial \Sigma_0$ (which implies (7.3)). Such an alternative pressure potential has been suggested in [4, Eq. (1.36)] or [8, Theorem 2.7-1]). It writes

$$(7.5) \quad \tilde{\mathcal{L}}^P(\varphi) := \frac{P}{3} \int_{\Sigma_0} \text{cof}(\mathbf{F}_\varphi)^t \varphi \cdot \mathbf{n} \, da_0,$$

and differs from \mathcal{L}^P given by (7.1) (and deduced from (6.6)). On the body, this potential recast as

$$\tilde{\mathcal{L}}^P(p) := \frac{P}{3} \int_{\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(p, \mathbf{F}\cdot, \mathbf{F}\cdot),$$

where $p = \varphi \circ p_0$, $\mathbf{F} = \mathbf{F}_\varphi \mathbf{F}_0$ and $\Sigma_{\mathcal{B}} = p_0(\Sigma_0)$. Its variation can be derived the same way as in the proof of theorem 6.2, and we get

$$(7.6) \quad \delta \tilde{\mathcal{L}}^P(\delta p) = \mathcal{W}^P(\delta p) + \frac{P}{6} \oint_{\partial \Sigma_{\mathcal{B}}} (p \times \delta p) \cdot 2\mathbf{F} d\ell_{\mathcal{B}}.$$

Thus, the non-holonomic constraints for $\tilde{\mathcal{L}}^P$ to be a pressure potential writes

$$\oint_{\partial \Sigma_{\mathcal{B}}} (p \times \delta p) \cdot \mathbf{F} d\ell_{\mathcal{B}} = 0.$$

In an odd way, these constraints depend on the embedding p itself, whereas (6.8) depends only on the displacement $\boldsymbol{\xi} = p - p_0$.

Remark 7.2. The two Lagrangians \mathcal{L}^P and $\tilde{\mathcal{L}}^P$ are both valid for mechanical problems for which $\delta \boldsymbol{\xi} = \delta p = 0$ on the boundary $\partial \Sigma_{\mathcal{B}}$ or if $\partial \Sigma_{\mathcal{B}} = \emptyset$. But $\tilde{\mathcal{L}}^P$ is not a pressure potential anymore when the virtual displacement $\delta \boldsymbol{\xi}$ only remains collinear to the displacement $\boldsymbol{\xi} = p - p_0$ all along the contour $\partial \Sigma_{\mathcal{B}}$.

We can calculate explicitly the difference between the variations of \mathcal{L}^P and $\tilde{\mathcal{L}}^P$. If we set,

$$\delta \mathcal{L}^P = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} \delta \omega, \quad \text{and} \quad \delta \tilde{\mathcal{L}}^P = \frac{1}{6} \int_{\Sigma_{\mathcal{B}}} \delta \tilde{\omega},$$

then, we have

$$\delta \omega - \delta \tilde{\omega} = d\beta,$$

where β is the following one-form on $\partial \mathcal{B}$,

$$\beta(A) := \text{vol}_{\mathbf{q}}(p, \delta p, \mathbf{L}_A p_0) - \text{vol}_{\mathbf{q}}(p_0, \delta p, \mathbf{L}_A(2p + p_0)).$$

By Stokes–Ampère formula, the two variations $\delta\mathcal{L}^P(\delta p)$ and $\delta\tilde{\mathcal{L}}^P(\delta p)$ of pressure potentials differ then by the contour integral

$$\begin{aligned} \frac{1}{6} \oint_{\partial\Sigma_{\mathcal{B}}} \beta &= \frac{1}{6} \oint_{\partial\Sigma_{\mathcal{B}}} \text{vol}_{\mathbf{q}}(p, \delta p, \mathbf{F}_0 \cdot) - \text{vol}_{\mathbf{q}}(p_0, \delta p, (2\mathbf{F} + \mathbf{F}_0) \cdot) \\ &= \frac{1}{6} \oint_{\partial\Sigma_{\mathcal{B}}} (p \times \delta p) \cdot \mathbf{F}_0 d\ell_{\mathcal{B}} - \frac{1}{6} \oint_{\partial\Sigma_{\mathcal{B}}} (p_0 \times \delta p) \cdot (2\mathbf{F} + \mathbf{F}_0) d\ell_{\mathcal{B}}. \end{aligned}$$

Therefore, $\delta\mathcal{L}^P = \delta\tilde{\mathcal{L}}^P$ when $\delta p = \delta\boldsymbol{\xi} = 0$ on $\Sigma_{\mathcal{B}}$ or if $\partial\Sigma_{\mathcal{B}} = \emptyset$.

8. CONCLUSION

We have formulated hyper-elasticity as a variational problem directly on the body \mathcal{B} , a three-dimensional compact and orientable manifold with boundary (equipped with a mass measure), and, key-point of the present work, not necessarily embedded as a reference configuration in space. Accordingly, we have formulated the dead load and pressure types boundary conditions on $\partial\mathcal{B}$. Concerning prescribed pressure, Poincaré’s lemma (extended to infinite dimension) has allowed us to obtain in a straightforward manner both the pressure potential and optimal non-holonomic constraints for such a potential to exist. The proposed methodology is based on the interpretation of the virtual powers as 1-forms on the configuration space $\text{Emb}(\mathcal{B}, \mathcal{E})$. It is general and can be applied to many others situations.

APPENDIX A. PULL-BACK, PUSH-FORWARD AND LIE DERIVATIVE

The fundamental concept in differential geometry that allows to pass from spatial variables defined on the deformed/actual configuration Ω , to material variables, defined on the body \mathcal{B} or on the reference configuration Ω_0 (and vice versa) are the operations *pull-back* and *push-forward* (see [30, 4.7 p. 68] or [51] or [29, Chapter V], for instance). More precisely, given a diffeomorphism $f : M \rightarrow N$ between two differentiable manifolds, the pullback f^* transforms a tensor field \mathbf{t} defined on N into a tensor field $f^*\mathbf{t}$ defined on M , while the push-forward f_* transforms a tensor field \mathbf{T} defined on M into a tensor field $f_*\mathbf{T}$ defined on N . The notion of pull-back and push forward naturally extend to the case where f is an embedding.

Example A.1. Two usual examples of pull-back/push-forward by $p \in \text{Emb}(\mathcal{B}, \mathcal{E})$ are

$$\mathbf{T} = p^*\mathbf{t} = \mathbf{F}^*(\mathbf{t} \circ p)\mathbf{F}, \quad \mathbf{t} = p_*\mathbf{T} = \mathbf{F}^{-*}\mathbf{T}\mathbf{F}^{-1} \circ p^{-1},$$

for \mathbf{t}, \mathbf{T} covariant of order 2, and

$$\mathbf{T} = p^*\mathbf{t} = \mathbf{F}^{-1}(\mathbf{t} \circ p)\mathbf{F}^{-*}, \quad \mathbf{t} = p_*\mathbf{T} = \mathbf{F}\mathbf{T}\mathbf{F}^* \circ p^{-1},$$

for \mathbf{t}, \mathbf{T} contravariant of order 2, where $\mathbf{F} = Tp : T\mathcal{B} \rightarrow T\Omega$ is linear tangent mapping and $\mathbf{F}^* : T^*\Omega \rightarrow T^*\mathcal{B}$ is its metric free transpose.

Remark A.2. Pull-back and push-forward operations are inverse to each other, meaning that $f^* = (f_*)^{-1} = (f^{-1})_*$. They commute moreover with any contraction between covariant and contravariant indices.

The *Lie derivative* is the infinitesimal version of the pull-back. Indeed, let \mathbf{u} be a vector field on M and $\varphi(t)$ its flow, let \mathbf{t} be a tensor field on M , the *Lie derivative* of \mathbf{t} with respect to \mathbf{u} , noted $L_{\mathbf{u}}\mathbf{t}$ is defined as

$$L_{\mathbf{u}}\mathbf{t} := \left. \frac{\partial}{\partial t} \right|_{t=0} \varphi(t)^*\mathbf{t}.$$

When $\mathbf{t} := \mathbf{v}$ is a vector field, $L_{\mathbf{u}}\mathbf{v}$ is just the *Lie bracket* $[\mathbf{u}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}]$ of \mathbf{u} and \mathbf{v} , and we have moreover

$$(A.1) \quad L_{[\mathbf{u}, \mathbf{v}]} \mathbf{t} = L_{\mathbf{u}} L_{\mathbf{v}} \mathbf{t} - L_{\mathbf{v}} L_{\mathbf{u}} \mathbf{t}.$$

The Lie derivative extends without difficulty to *time-dependent vector fields* $\mathbf{u}(t)$ [30, Section 1.6]. In that case, the flow $\varphi(t, s)$ of $\mathbf{u}(t)$ depends on two parameters and one defines

$$L_{\mathbf{u}(s)} \mathbf{t} := \left. \frac{\partial}{\partial t} \right|_{t=s} \varphi(t, s)^* \mathbf{t}.$$

The following result extends the property $\partial_t \varphi(t)^* \mathbf{t} = \varphi(t)^* L_{\mathbf{u}} \mathbf{t}$ (see [30, Section 1.6]), when the path of diffeomorphisms $\varphi(t)$ is replaced by a path of embeddings $p(t) : \mathcal{B} \rightarrow \mathcal{E}$.

Lemma A.3. *Let $p(t)$ be a path of embeddings, $\mathbf{u} = (\partial_t p) \circ p^{-1}$ be its (right) Eulerian velocity and \mathbf{t} be a tensor field defined along $p(t)$ (i.e. on $\Omega_{p(t)} = p(t)(\mathcal{B})$ and possibly time-dependent). Then*

$$\partial_t(p^* \mathbf{t}) = p^* (\partial_t \mathbf{t} + L_{\mathbf{u}} \mathbf{t}).$$

APPENDIX B. DIFFERENTIAL FORMS

For basic materials on differential forms one may look at [7, 1, 29]. Let $\Omega^k(M)$ be the space of *differential forms* of degree k , i.e. covariant tensor fields ω on the manifold M of order k , which are alternate. The *exterior derivative* is a differential operator of order one

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

which extends the differential of a function to differential forms of any degree. For instance, in any local coordinate system (x^i) , we have

$$(d\alpha)_{ij} = \partial_i \alpha_j - \partial_j \alpha_i,$$

for a 1-form α and

$$(d\omega)_{ijk} = \partial_i \alpha_{jk} - \partial_j \alpha_{ik} + \partial_k \alpha_{ij},$$

for a 2-form α . Given a vector field A on M , the *inner product*

$$i_A : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$$

is defined by

$$(i_A \alpha)_m(B_1, \dots, B_{k-1}) := \alpha_m(A(m), B_1, \dots, B_{k-1}),$$

for any $B_1, \dots, B_{k-1} \in T_m M$, if $k \geq 1$ and $i_A \alpha = 0$ if $k = 0$.

These two linear operators are related to each other and to the Lie derivative L_A by *Cartan's magic Formula*

$$(B.1) \quad L_A = d \circ i_A + i_A \circ d.$$

The following lemma is useful for our computations, in which $\text{vol}_{\mathbf{q}}$ is the Riemannian volume for Euclidean metric \mathbf{q} (see [Appendix C](#)).

Lemma B.1. *Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ be three vector valued functions, defined on a manifold M and let*

$$\alpha(A) = \text{vol}_{\mathbf{q}}(\mathbf{f}_1, \mathbf{f}_2, L_A \mathbf{f}_3), \quad A \in T_m M.$$

Then

$$(B.2) \quad \begin{aligned} d\alpha(A, B) &= \text{vol}_{\mathbf{q}}(L_A \mathbf{f}_1, \mathbf{f}_2, L_B \mathbf{f}_3) - \text{vol}_{\mathbf{q}}(L_B \mathbf{f}_1, \mathbf{f}_2, L_A \mathbf{f}_3) \\ &\quad + \text{vol}_{\mathbf{q}}(\mathbf{f}_1, L_A \mathbf{f}_2, L_B \mathbf{f}_3) - \text{vol}_{\mathbf{q}}(\mathbf{f}_1, L_B \mathbf{f}_2, L_A \mathbf{f}_3). \end{aligned}$$

with $A, B \in T_m M$.

Proof. By Cartan's formula (B.1), we have

$$d\alpha(A, B) = L_A \alpha(B) - L_B \alpha(A) - \alpha([A, B]),$$

where $[A, B] := L_A B$ is the Lie Bracket. But

$$L_A \alpha(B) = \text{vol}_{\mathbf{q}}(L_A \mathbf{f}_1, \mathbf{f}_2, L_B \mathbf{f}_3) + \text{vol}_{\mathbf{q}}(\mathbf{f}_1, L_A \mathbf{f}_2, L_B \mathbf{f}_3) + \text{vol}_{\mathbf{q}}(\mathbf{f}_1, \mathbf{f}_2, L_A L_B \mathbf{f}_3),$$

$$L_B \alpha(A) = \text{vol}_{\mathbf{q}}(L_B \mathbf{f}_1, \mathbf{f}_2, L_A \mathbf{f}_3) + \text{vol}_{\mathbf{q}}(\mathbf{f}_1, L_B \mathbf{f}_2, L_A \mathbf{f}_3) + \text{vol}_{\mathbf{q}}(\mathbf{f}_1, \mathbf{f}_2, L_B L_A \mathbf{f}_3),$$

and

$$\alpha([A, B]) = \text{vol}_{\mathbf{q}}(\mathbf{f}_1, \mathbf{f}_2, L_{[A, B]} \mathbf{f}_3).$$

Now, by (A.1), we have

$$L_A L_B \mathbf{f}_3 - L_B L_A \mathbf{f}_3 = L_{[A,B]} \mathbf{f}_3$$

and we get thus (B.2). \square

A differential form $\alpha \in \Omega^k(M)$ is said to be closed if $d\alpha = 0$. It is said to be exact if it writes $d\beta$ with $\beta \in \Omega^{k-1}(M)$. Since $d \circ d = 0$, *any exact form is closed*. The following result, attributed to Poincaré, implies that *any closed form defined on an open convex set of \mathbb{R}^n is exact*.

Lemma B.2 (Poincaré lemma). *Let $U \subset \mathbb{R}^n$ be a convex open set and $\alpha \in \Omega^k(U)$ ($1 \leq k \leq n$). If $d\alpha = 0$, then there exists $\beta \in \Omega^{k-1}(U)$ such that $\alpha = d\beta$.*

The important point is that the proof of this lemma is constructive. A primitive β of a closed differential form α defined on U is explicitly constructed. Indeed, let $\mathbf{x}_0 \in U$ and let $\phi^t(\mathbf{x}) = e^t \mathbf{x} + (1 - e^t) \mathbf{x}_0$ the flow of the radial vector field $X(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ defined on U . Define the linear operator $K : \Omega^k(U) \rightarrow \Omega^{k-1}(U)$ by

$$(B.3) \quad K\alpha = \int_{-\infty}^0 (\phi^t)^* i_X \alpha dt.$$

Then, using Cartan's formula (B.1) and the fact that pullbacks and exterior derivative commute, we get

$$d(\phi^t)^* i_X \alpha = (\phi^t)^* d i_X \alpha = (\phi^t)^* (L_X \alpha - i_X d\alpha) = (\phi^t)^* L_X \alpha,$$

and thus

$$dK\alpha = \int_{-\infty}^0 d(\phi^t)^* i_X \alpha dt = \int_{-\infty}^0 (\phi^t)^* L_X \alpha dt = \int_{-\infty}^0 \frac{d}{dt} ((\phi^t)^* \alpha) dt = \alpha.$$

Therefore, $\beta := K\alpha$ is a primitive of α .

Remark B.3. When the form α is not closed we have

$$(B.4) \quad dK\alpha = \alpha - \int_{-\infty}^0 (\phi^t)^* i_X d\alpha dt.$$

This last integral can be used to formulate non-holonomic constraints for a primitive to exist, even when α is not closed.

APPENDIX C. VOLUME FORMS AND AREA ELEMENTS

A *volume form* on a manifold M of dimension n is a differential form of degree n that does not vanish at any point. A manifold M which has a volume form is necessarily *orientable* [19, Section 1.G]. On an orientable Riemannian manifold (M, g) , there is a unique volume form, noted vol_g , and called the *Riemannian volume form* such that

$$(\text{vol}_g)_m(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1,$$

for any direct orthonormal basis of $T_m M$ and any point $m \in M$. In a local coordinate system (x^i) , this volume form writes

$$(C.1) \quad \text{vol}_g = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n.$$

Let us now consider an orientable manifold with boundary M . Let ω be a volume form on M and $x \in \partial M$. In a local chart in the vicinity of x , we can consider the form $i_{\mathbf{n}} \omega$ where \mathbf{n} is a vector with a x_1 component *strictly positive*. The class of this volume form defines the induced orientation on ∂M , known as the convention of the *outer normal*. For example, consider the prototype manifold $] - \infty, 0] \times \mathbb{R}^{n-1}$ with the orientation defined by the volume form $dx^1 \wedge \dots \wedge dx^n$. Then the orientation induced on the boundary \mathbb{R}^{n-1} of $] - \infty, 0] \times \mathbb{R}^{n-1}$ is represented by the volume form $dx^2 \wedge \dots \wedge dx^n$.

Let (M, g) be an oriented 3-dimensional Riemannian manifold with boundary ∂M . Then, the Riemannian metric g on M induces by restriction a Riemannian metric on ∂M . Let vol_g be the

Riemannian volume form on M and \mathbf{n} be the outer unit normal on the boundary ∂M . Then, one can show that the Riemannian volume on the 2-dimensional manifold ∂M is the 2-form

$$da := i_{\mathbf{n}} \text{vol}_g,$$

which is called the *area element* of ∂M .

Remark C.1. Let X be a vector field defined on ∂M (not necessarily tangent to ∂M). Then, as a 2-form on ∂M , we have the following identities

$$(C.2) \quad i_X \text{vol}_g = \langle X, \mathbf{n} \rangle i_{\mathbf{n}} \text{vol}_g = \langle X, \mathbf{n} \rangle da.$$

The following theorem happens to be extremely useful when deriving boundary conditions on the body.

Theorem C.2. *Let p, p_0 be two orientation-preserving embeddings from \mathcal{B} to \mathcal{E} . Let \mathbf{w} be a vector field defined on $\partial\Omega$ (not necessarily tangent to $\partial\Omega$). Then*

$$(C.3) \quad p^*(\rho(\mathbf{w} \cdot \mathbf{n}) da) = \rho_{\gamma_0}(\mathbf{w} \circ p \cdot \mathbf{q}^{-1} \mathbf{F}^{-*} \gamma_0 \mathbf{N}_0) da_{\gamma_0},$$

where $\mathbf{F}^* : T^* \mathcal{E} \rightarrow T^* \mathcal{B}$ is the (metric free) transpose of $\mathbf{F} = Tp : T\mathcal{B} \rightarrow T\mathcal{E}$, $\rho_{\gamma_0} = p_0^* \rho_0 = \rho_0 \circ p_0$ and da_{γ_0} is the area density on $\partial\mathcal{B}$ relative to the metric $\gamma_0 = p_0^* \mathbf{q}$.

Proof. Using (C.2), and the fact that $p^* \rho = \rho_{\gamma}$, we have

$$p^*(\rho(\mathbf{w} \cdot \mathbf{n}) da) = \rho_{\gamma} p^*((\mathbf{w} \cdot \mathbf{n}) da) = \rho_{\gamma} p^*(i_{\mathbf{w}} \text{vol}_{\mathbf{q}}) = \rho_{\gamma} i_{p^* \mathbf{w}} \text{vol}_{\gamma},$$

because p is an orientation-preserving Riemannian isometry. Now, mass conservation leads (see section 2) to

$$\mu = \rho_{\gamma} \text{vol}_{\gamma} = \rho_{\gamma_0} \text{vol}_{\gamma_0}.$$

Therefore, we deduce that

$$p^*(\rho(\mathbf{w} \cdot \mathbf{n}) da) = \rho_{\gamma_0} i_{p^* \mathbf{w}} \text{vol}_{\gamma_0} = \rho_{\gamma_0} \langle p^* \mathbf{w}, \mathbf{N}_0 \rangle_{\gamma_0} da_{\gamma_0}.$$

But

$$\langle p^* \mathbf{w}, \mathbf{N}_0 \rangle_{\gamma_0} = (\gamma_0 \mathbf{N}_0, \mathbf{F}^{-1} \mathbf{w} \circ p) = (\mathbf{F}^{-*} \gamma_0 \mathbf{N}_0, \mathbf{w} \circ p) = (\mathbf{q}^{-1} \mathbf{F}^{-*} \gamma_0 \mathbf{N}_0) \cdot \mathbf{w} \circ p,$$

where (\cdot, \cdot) is the duality bracket. This achieves the proof. \square

Remark C.3. If we identify the body \mathcal{B} with a reference configuration Ω_0 , and set thus $p_0 \equiv \text{id}$, $\gamma_0 \equiv \mathbf{q}$, $p \equiv \varphi$ and $\mathbf{F} \equiv \mathbf{F}_{\varphi}$, then, (C.3) writes

$$\varphi^*(\rho(\mathbf{w} \cdot \mathbf{n}) da) = \rho_0(\mathbf{w} \circ \varphi \cdot \mathbf{q}^{-1} \mathbf{F}_{\varphi}^{-*} \mathbf{q} \mathbf{n}_0) da_0 = \rho_0(\mathbf{w} \circ \varphi \cdot \mathbf{F}_{\varphi}^{-t} \mathbf{n}_0) da_0,$$

which we can recast as

$$\varphi^*((\mathbf{w} \cdot \mathbf{n}) da) = \frac{\rho_0}{\varphi^* \rho} (\mathbf{w} \circ \varphi \cdot \mathbf{F}_{\varphi}^{-t} \mathbf{n}_0) da_0 = J_{\varphi} (\mathbf{w} \circ \varphi \cdot \mathbf{F}_{\varphi}^{-t} \mathbf{n}_0) da_0,$$

because $\rho_0 / \varphi^* \rho = J_{\varphi}$ by (2.6). This equality is known in continuum mechanics as *Nanson's formula* [32] and often written in condensed form as

$$(C.4) \quad \mathbf{n} da = J_{\varphi} \mathbf{F}_{\varphi}^{-t} \mathbf{n}_0 da_0.$$

APPENDIX D. FRÉCHET TOPOLOGY ON THE SPACE OF EMBEDDINGS

The vector space $C^{\infty}(\mathcal{B}, \mathcal{E})$ is not a Banach space; its topology is not defined by a norm but by a countable family of *semi-norms*. These semi-norms can be described either by choosing a Riemannian metric on \mathcal{B} or by choosing an embedding p_0 of \mathcal{B} into \mathcal{E} . In both cases, one can prove that the defined topology does not depend of the particular choice of the metric or of the embedding. Here, we chose to describe this topology using an embedding p_0 . This means that we describe first a topology on $C^{\infty}(\Omega_0, \mathcal{E})$, where $\Omega_0 = p_0(\mathcal{B})$, and then a topology on $C^{\infty}(\mathcal{B}, \mathcal{E})$, using the invertible linear mapping

$$L_{p_0} : C^{\infty}(\mathcal{B}, \mathcal{E}) \rightarrow C^{\infty}(\Omega_0, \mathcal{E}), \quad p \mapsto \varphi := p \circ p_0^{-1}.$$

Thus, a subset $U \subset C^\infty(\mathcal{B}, \mathcal{E})$ is open if and only if $L_{p_0}(U)$ is open and we define the semi-norm of $p \in C^\infty(\mathcal{B}, \mathcal{E})$ as the semi-norm of $p \circ p_0^{-1} \in C^\infty(\Omega_0, \mathcal{E})$. The topology on $C^\infty(\Omega_0, \mathcal{E})$ is defined by the family of C^k -semi-norms

$$\|f\|_k := \max_{k_1+k_2+k_3=k} \left\{ \sup_{x \in \Omega_0} \frac{\partial^k f}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}}(x) \right\}.$$

For this topology, the *semi-balls*

$$B_k(f_0, r) := \{f \in C^\infty(\Omega_0, \mathcal{E}); \|f - f_0\|_k < r\}$$

are always open sets. The space $C^\infty(\mathcal{B}, \mathcal{E})$ is an example of a so-called *Fréchet space* [23, 45]. These spaces are not nice from the point of view of Analysis since essential results such as *Inverse mapping theorem* or *local existence of solutions* for ordinary differential equations are no longer true without hard-to-check additional hypotheses [23]. The set $\text{Emb}(\mathcal{B}, \mathcal{E})$ of smooth embeddings from \mathcal{B} to \mathcal{E} can be shown to be an open set of the vector space $C^\infty(\mathcal{B}, \mathcal{E})$ as a corollary of the following lemma (see also [31]).

Lemma D.1. *Let $p_0 \in \text{Emb}(\mathcal{B}, \mathcal{E})$. The neighborhood of p_0 , defined by*

$$\mathcal{U}_{p_0} := \{p \in C^\infty(\mathcal{B}, \mathcal{E}); \|p - p_0\|_1 < 1\}$$

is an open convex set of $C^\infty(\mathcal{B}, \mathcal{E})$, contained in $\text{Emb}(\mathcal{B}, \mathcal{E})$.

Proof. Note first that

$$L_{p_0}(\mathcal{U}_{p_0}) := \{\varphi \in C^\infty(\Omega_0, \mathcal{E}); \|\varphi - \text{id}\|_1 < 1\}$$

which is a convex open subset of $C^\infty(\Omega_0, \mathcal{E})$. By the very definition of the topology on $C^\infty(\mathcal{B}, \mathcal{E})$ and the fact that L_{p_0} is linear, we deduce that \mathcal{U}_{p_0} is a convex open subset of $C^\infty(\mathcal{B}, \mathcal{E})$. It remains to show that each $p \in \mathcal{U}_{p_0}$ is an embedding. We will prove that each $\varphi \in L_{p_0}(\mathcal{U}_{p_0})$ is an embedding and the conclusion follows straightforwardly. We will prove first that each vector valued function $\varphi \in L_{p_0}(\mathcal{U}_{p_0})$ is injective. To do so, let $\varphi \in \mathcal{U}_{p_0}$ and suppose that $p(\mathbf{x}) = p(\mathbf{y})$, we get then, thanks to the mean value theorem

$$\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} - \varphi(\mathbf{x}) + \varphi(\mathbf{y}) - \mathbf{y}\| = \|(\text{id} - \varphi)(\mathbf{x}) - (\text{id} - \varphi)(\mathbf{y})\| \leq \|\varphi - \text{id}\|_1 \|\mathbf{x} - \mathbf{y}\|$$

and thus $\mathbf{x} = \mathbf{y}$, because $\|\varphi - \text{id}\|_1 < 1$. Next, we will show that φ is an immersion, i.e that $\mathbf{F}_\varphi = T\varphi$ is injective. Thus, assume that $\mathbf{F}_\varphi \cdot \delta \mathbf{x} = \mathbf{F}_\varphi \cdot \delta \mathbf{y}$, then

$$\|\delta \mathbf{x} - \delta \mathbf{y}\| = \|(\mathbf{Id} - \mathbf{F}_\varphi)(\delta \mathbf{x}) - (\mathbf{Id} - \mathbf{F}_\varphi)\delta \mathbf{y}\| \leq \|\mathbf{F}_\varphi - \mathbf{Id}\|_0 \|\delta \mathbf{x} - \delta \mathbf{y}\| = \|\varphi - \text{id}\|_1 \|\delta \mathbf{x} - \delta \mathbf{y}\|$$

and thus $\delta \mathbf{x} = \delta \mathbf{y}$, because $\|\varphi - \text{id}\|_1 < 1$. To finish, we observe that an injective immersion from a compact manifold into space is always an embedding. Since it is assumed in solid mechanics that \mathcal{B} (and thus Ω_0) is compact, this achieves the proof. \square

APPENDIX E. THE FIRST PIOLA-KIRCHHOFF TENSOR ON THE BODY

When so-called dead loads are involved, *i.e.* loads per unit area of direction and intensity independent of the deformation of the medium, the associated Neumann condition corresponds to the prescribed stress traction vector \mathbf{t}_0 on the boundary part $\Sigma_0^{(DL)} \subset \partial\Omega_0$ (with $\Omega_0 = p_0(\mathcal{B})$ the reference configuration) but with values in space \mathcal{E} ,

$$(E.1) \quad \widehat{\mathbf{P}}\mathbf{n}_0 \Big|_{\Sigma_0^{(DL)}} = \mathbf{t}_0,$$

where $\widehat{\mathbf{P}}$ is the (mixed, two point [30, 25]) first Piola-Kirchhoff tensor,

$$\widehat{\mathbf{P}} := \rho_0 \mathbf{F}_\varphi \mathbf{S} \mathbf{q} = \rho_0 (\boldsymbol{\tau} \circ \varphi) \mathbf{F}_\varphi^{-*} \mathbf{q},$$

such that

$$\int_{\partial\Omega} (\widehat{\boldsymbol{\sigma}} \mathbf{n} \cdot \mathbf{w}) da = \int_{\partial\Omega_0} (\widehat{\mathbf{P}} \mathbf{n}_0 \cdot \delta \varphi) da_0,$$

for any virtual velocity $\mathbf{w} = \delta\varphi \circ \varphi^{-1}$. Recall that $\mathbf{S} = \varphi^*\boldsymbol{\tau} = \varphi^*(\boldsymbol{\sigma}/\rho)$ is the second Piola-Kirchhoff stress tensor (defined on Ω_0). We have then the following result.

Lemma E.1. *We have*

$$\int_{\partial\Omega} (\widehat{\boldsymbol{\sigma}}\mathbf{n} \cdot \mathbf{w}) da = \int_{\partial\mathcal{B}} (\widehat{\boldsymbol{\pi}}\mathbf{N}_0 \cdot \delta p) da_{\gamma_0},$$

for any virtual velocity $\mathbf{w} = \delta p \circ p^{-1}$, where

$$\widehat{\boldsymbol{\pi}} := \rho_{\gamma_0}(\boldsymbol{\tau} \circ p)\mathbf{F}^{-*}\mathbf{q} = \rho_{\gamma_0}\mathbf{F}\boldsymbol{\theta}\gamma_0,$$

is the mixed stress tensor defined on \mathcal{B} but with values on Euclidean space \mathcal{E} , with $\gamma_0 = p_0^*\mathbf{q}$, $\rho_{\gamma_0} = p_0^*\rho_0 = \rho_0 \circ p_0$, $\boldsymbol{\tau} := \boldsymbol{\sigma}/\rho$ and $\boldsymbol{\theta} = p^*\boldsymbol{\tau}$.

Remark E.2. Thanks to this result, the dead load boundary condition (E.1) rewrites on the body

$$\widehat{\boldsymbol{\pi}}\mathbf{N}_0|_{\Sigma_{\mathcal{B}}^{(DL)}} = \mathbf{t}_0 \circ p_0, \quad \Sigma_{\mathcal{B}}^{(DL)} = p_0^{-1}(\Sigma_0^{(DL)}).$$

Proof of lemma E.1. By the symmetry of $\boldsymbol{\sigma}$ and the change of variable formula, we get

$$\int_{\partial\Omega} (\widehat{\boldsymbol{\sigma}}\mathbf{n} \cdot \mathbf{w}) da = \int_{\partial\Omega} (\mathbf{n} \cdot \widehat{\boldsymbol{\sigma}}\mathbf{w}) da = \int_{\partial\mathcal{B}} p^*((\mathbf{n} \cdot \widehat{\boldsymbol{\sigma}}\mathbf{w}) da) = \int_{\partial\mathcal{B}} p^*(\rho(\mathbf{n} \cdot \widehat{\boldsymbol{\tau}}\mathbf{w}) da).$$

Therefore, using theorem C.2, we get

$$p^*((\widehat{\boldsymbol{\sigma}}\mathbf{n} \cdot \mathbf{w}) da) = p^*(\rho(\mathbf{n} \cdot \widehat{\boldsymbol{\tau}}\mathbf{w}) da) = \rho_{\gamma_0}((\widehat{\boldsymbol{\tau}}\mathbf{w}) \circ p \cdot \mathbf{q}^{-1}\mathbf{F}^{-*}\gamma_0\mathbf{N}_0)da_{\gamma_0}.$$

But

$$(\widehat{\boldsymbol{\tau}}\mathbf{w}) \circ p \cdot \mathbf{q}^{-1}\mathbf{F}^{-*}\gamma_0\mathbf{N}_0 = \mathbf{w} \circ p \cdot (\widehat{\boldsymbol{\tau}} \circ p)\mathbf{q}^{-1}\mathbf{F}^{-*}\gamma_0\mathbf{N}_0 = \mathbf{w} \circ p \cdot (\boldsymbol{\tau} \circ p)\mathbf{F}^{-*}\gamma_0\mathbf{N}_0,$$

and $\boldsymbol{\tau} \circ p = \mathbf{F}\boldsymbol{\theta}\mathbf{F}^*$, by the very definition of the push forward $\boldsymbol{\tau} = p_*\boldsymbol{\theta}$ (see A.1). We get thus finally

$$\int_{\partial\Omega} (\widehat{\boldsymbol{\sigma}}\mathbf{n} \cdot \mathbf{w}) da = \int_{\partial\mathcal{B}} \delta p \cdot \widehat{\boldsymbol{\pi}}\mathbf{N}_0 da_{\gamma_0},$$

where

$$\widehat{\boldsymbol{\pi}} := \rho_{\gamma_0}(\boldsymbol{\tau} \circ p)\mathbf{F}^{-*}\gamma_0 = \rho_{\gamma_0}\mathbf{F}\boldsymbol{\theta}\gamma_0,$$

which concludes the proof. \square

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