

Recent Progress on the Dynamical Zeta Function of the β -shift, and Lehmer's Problem

Jean-Louis VERGER-GAUGRY
(joint work with Denys DUTYKH)

CNRS, LAMA,
Univ. Savoie Mont Blanc

Linz UDT 2021, Feb22-25

Obj. : To study the integer polynomials of non-trivial Mahler measure $<$ Lehmer's number = 1.176280..., if they exist. Let $\beta > 1$ be a reciprocal algebraic integer. Assume β lying close to the limit point 1. To show how the dynamical zeta function

$$\zeta_{\beta}(z)$$

of the β -shift, β variable, can be used to identify the subcollection of lenticular zeroes of modulus < 1 of the minimal polynomial

$$P_{\beta}(x)$$

of β using **eventually periodic representations of $\mathbb{Q}(\beta)$ in the algebraic basis β** , and **study their (universal) locus inside Solomyak's fractal.**

application :

- (i) to any algebraic integer α with $\beta = |\overline{\alpha}| > 1$.
- (ii) problem of Lehmer, minoration of the Mahler measure of β .

Hyp. : P_β cannot be written $P_\beta(X) = R(X^r)$ for some irred. int. pol. R with $r \geq 2$.

In continuation of :

Boris Solomyak '94 : Conjugates of beta-numbers and the zerofree domain for a class of analytic functions.

Flatto, Lagarias, Poonen '94 : The zeta function of the beta-transformation.

Hofbauer and Keller '84 : Zeta functions and transfer operators for piecewise linear transformations.

Context : numeration dynamical system introduced by Rényi - Parry, $\beta > 1$,

$$([0, 1], T_\beta : x \rightarrow \{\beta x\}).$$

Contents

- 1 **Periodic representations in algebraic basis**
 - Minimal and Maximal alphabets - Main Theorem
 - Examples, Pisot numbers, maximal alphabets
 - Rewriting trails - Intermediate alphabets - Proof of Main Theorem
- 2 The class \mathcal{B} of lacunary polynomials - Dynamical zeta function - Problem of Lehmer
- 3 Solomyak's fractal - Spirals of poles - Universal self-intersecting curves of conjugates

Minimal and Maximal alphabets - Main Theorem

For a general complex number $\beta \in \mathbb{C}$, $|\beta| > 1$, and a finite alphabet $\mathcal{A} \subset \mathbb{C}$, we define the (β, \mathcal{A}) -representations as expressions of the form

$$\sum_{k \geq -L} a_k \beta^{-k}, \quad a_k \in \mathcal{A},$$

for some integer $L \in \mathbb{Z}$. They are Laurent series of $1/\beta$. We define

$$\text{Per}_{\mathcal{A}}(\beta) := \{x \in \mathbb{C} \mid x \text{ has an eventually periodic } (\beta, \mathcal{A})\text{-representation}\}.$$

In this note attention is focused on the complex numbers β which are real algebraic integers > 1 , close to 1, assuming that β has no conjugate on the unit circle, and on the finite alphabets $\mathcal{A} \subset \mathbb{Z}$, depending upon β , involved in the identity :

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta).$$

Such an identity always holds by the following theorem.

Minimal and Maximal alphabets - Main Theorem

For a general complex number $\beta \in \mathbb{C}$, $|\beta| > 1$, and a finite alphabet $\mathcal{A} \subset \mathbb{C}$, we define the (β, \mathcal{A}) -representations as expressions of the form

$$\sum_{k \geq -L} a_k \beta^{-k}, \quad a_k \in \mathcal{A},$$

for some integer $L \in \mathbb{Z}$. They are Laurent series of $1/\beta$. We define

$$\text{Per}_{\mathcal{A}}(\beta) := \{x \in \mathbb{C} \mid x \text{ has an eventually periodic } (\beta, \mathcal{A})\text{-representation}\}.$$

In this note attention is focused on the complex numbers β which are real algebraic integers > 1 , close to 1, assuming that β **has no conjugate on the unit circle**, and on the **finite alphabets** $\mathcal{A} \subset \mathbb{Z}$, depending upon β , involved in the identity :

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta).$$

Such an identity always holds by the following theorem.

Theorem (Kala - Vávra, '19)

Let $\beta \in \mathbb{C}$ be an algebraic number of degree d , $|\beta| > 1$, and let $a_d x^d - a_{d-1} x^{d-1} - \dots - a_1 x - a_0 \in \mathbb{Z}[x]$ be its minimal polynomial. Suppose that $|\beta'| \neq 1$ for any conjugate β' of β . Then there exists a finite alphabet $\mathcal{A} \subset \mathbb{Z}$ such that

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta). \quad (1)$$

Kala, Vávra '19 : Periodic representations in algebraic bases.

Vávra '21 : iterated functions (Hutchinson), generalization of Thurston's approach.

Previously :

Frougny, Pelantova, Svobodova '11 : Parallel addition in non-standard numeration system.

Baker, Masáková, Pelantová and Vávra '17 : On periodic representations in non-Pisot bases, +hyp. $1/a_d$ in $\mathbb{Z}[\beta, \beta^{-1}]$.

non-real bases β , $|\beta| > 1$: existence of a finite alphabet \mathcal{A} in \mathbb{C} : Daróczy and Kátai '88, Thurston '89.

Periodic representations (radix systems) :

Kovács and Környei '92 : On the Periodicity of the Radix Expansion.

Pethoe '99 : On the Periodic Expansion of Algebraic Numbers.

Theorem (D VG, '20)

Let $\gamma > 1$ be an algebraic integer, root of $S_\gamma(X) = X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$, with $s \geq 1, t_i \in \mathbb{Z}, |t_i| \leq 1$, not necess. irred., s.t. $|\gamma'| \neq 1$ for any conjugate γ' of γ . Let $P(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + a_d X^d \in \mathbb{Z}[X]$, $d := \deg P \geq 1$, $H := \max_{i=0, \dots, d} |a_i|$ the height of P . Assume $0 < \eta < 1$ and $0 \neq |P(\gamma)| < \eta$. Then $P(\gamma) \in \mathbb{Q}(\gamma)$ admits one eventually periodic representation

$$P(\gamma) = R(\gamma^{-1}) + \frac{1}{\gamma^L} \sum_{j=0}^{\infty} \frac{1}{\gamma^{jr}} T(\gamma^{-1}) \in \text{Per}_{\mathcal{A}}(\gamma) \quad (2)$$

with (i) alphabet $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$, independent of s and γ , (ii) $H(R) \leq m$, $\deg R \leq s - 1$, $H(T) \leq m$, $\deg T \leq s - 1$, (iii) preperiod

$$R(\gamma^{-1}) = \frac{a_w}{\gamma^w} + \frac{a_{w+1}}{\gamma^{w+1}} + \dots + \frac{a_{w+s-1}}{\gamma^{w+s-1}}, \quad a_j \in \mathcal{A}, j = w, \dots, w + s - 1, a_w \neq 0,$$

with $w \geq 1$ satisfying $\frac{\kappa_{\gamma, \mathcal{A}}}{\eta} \leq \gamma^{w-1}$ for some positive constant $\kappa_{\gamma, \mathcal{A}}$ depending upon γ and \mathcal{A} .

Theorem (D VG, '20)

Let $\gamma > 1$ be an algebraic integer, root of $S_\gamma(X) = X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$, with $s \geq 1, t_i \in \mathbb{Z}, |t_i| \leq 1$, not necess. irred., s.t. $|\gamma'| \neq 1$ for any conjugate γ' of γ . Let $P(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + a_d X^d \in \mathbb{Z}[X]$, $d = \deg P \geq 1$, $H = \max_{i=0, \dots, d} |a_i|$ the height of P . Assume $0 < \eta < 1$ and $0 \neq |P(\gamma)| < \eta$. Then $P(\gamma) \in \mathbb{Q}(\gamma)$ admits one eventually periodic representation

$$P(\gamma) = R(\gamma^{-1}) + \frac{1}{\gamma^L} \sum_{j=0}^{\infty} \frac{1}{\gamma^{jr}} T(\gamma^{-1}) \in \text{Per}_{\mathcal{A}}(\gamma) \quad (3)$$

with (i) alphabet $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$, independent of s and γ , (ii) $H(R) \leq m$, $\deg R \leq s - 1$, $H(T) \leq m$, $\deg T \leq s - 1$, (iii) preperiod

$$R(\gamma^{-1}) = \frac{a_w}{\gamma^w} + \frac{a_{w+1}}{\gamma^{w+1}} + \dots + \frac{a_{w+s-1}}{\gamma^{w+s-1}}, \quad a_j \in \mathcal{A}, j = w, \dots, w + s - 1, a_w \neq 0,$$

with $w \geq 1$ satisfying $\frac{\kappa_{\gamma, \mathcal{A}}}{\eta} \leq \gamma^{w-1}$ for some positive constant $\kappa_{\gamma, \mathcal{A}}$ depending upon γ and \mathcal{A} .

Theorem (D VG, '20)

Let $\gamma > 1$ be an algebraic integer, root of $S_\gamma(X) = X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$, with $s \geq 1, t_i \in \mathbb{Z}, |t_i| \leq 1$, not necess. irred., s.t. $|\gamma'| \neq 1$ for any conjugate γ' of γ . Let $P(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + a_d X^d \in \mathbb{Z}[X]$, $d = \deg P \geq 1$, $H = \max_{i=0, \dots, d} |a_i|$ the height of P . Assume $0 < \eta < 1$ and $0 \neq |P(\gamma)| < \eta$. Then $P(\gamma) \in \mathbb{Q}(\gamma)$ admits one eventually periodic representation

$$P(\gamma) = R(\gamma^{-1}) + \frac{1}{\gamma^L} \sum_{j=0}^{\infty} \frac{1}{\gamma^{jr}} T(\gamma^{-1}) \quad \in \text{Per}_{\mathcal{A}}(\gamma) \quad (4)$$

with (i) alphabet $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$, independent of s and γ , (ii) $H(R) \leq m$, $\deg R \leq s - 1$, $H(T) \leq m$, $\deg T \leq s - 1$, (iii) preperiod

$$R(\gamma^{-1}) = \frac{a_w}{\gamma^w} + \frac{a_{w+1}}{\gamma^{w+1}} + \dots + \frac{a_{w+s-1}}{\gamma^{w+s-1}}, \quad a_j \in \mathcal{A}, j = w, \dots, w + s - 1, a_w \neq 0,$$

with $w \geq 1$ satisfying $\frac{\kappa_{\gamma, \mathcal{A}}}{\eta} \leq \gamma^{w-1}$ for some positive constant $\kappa_{\gamma, \mathcal{A}}$ depending upon γ and \mathcal{A} .

On the Alphabets :

The alphabet \mathcal{A} is symmetrical : $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, satisfies

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta).$$

It is said **minimal** if $m = 1$.

It is said **maximal** if

$$m := \left\lceil \frac{|a_j| - 1}{2} \right\rceil + \sum_{i=0, i \neq j}^N |a_i|, \quad (5)$$

where $Q(X) = \sum_{i=0}^N a_i X^i \in P_{\beta}(X)\mathbb{Z}[X]$ has a_j as *dominant coefficient*, produced by the *companion matrices of the "iterates" of P_{β}* , with N *minimal* (Proposition 5.1 in Frougny Pelantová Svobodová '11, and Theorem 25 in Baker Masáková Pelantová Vávra '17).

A polynomial $Q(X) = \sum_{i=0}^d a_i X^i \in \mathbb{Z}[X]$ is said to have a *dominant coefficient*, if $\exists j \in \{0, 1, \dots, N\}$ such that $|a_j| > \sum_{i=0, i \neq j}^N |a_i|$.

Proposition (Frougny, Pelantová, Svobodová '11)

Let β be an algebraic integer, of degree d , $|\beta| > 1$, of minimal polynomial $P_\beta(X) = \prod_{j=1}^d (X - \beta_{(j)})$, with $\beta = \beta_{(1)}$ and $|\beta_{(j)}| \neq 1$ for $j = 2, 3, \dots, d$. Denote by j_0 the number of conjugates $\beta_{(j)}$ of β which have a modulus > 1 . Then, for any $t \geq 1$, there exist an integer N and a polynomial

$$Q(X) = X^{dN} + a_1 X^{(d-1)N} + a_2 X^{(d-2)N} + \dots + a_{d-1} X^N + a_d \in \mathbb{Z}[X]$$

such that $Q(\alpha) = 0$, setting $a_0 = 1$, with

$$|a_{j_0}| > t \sum_{i \in \{0, 1, 2, \dots, d\} \setminus \{j_0\}} |a_i|. \quad (6)$$

Take $t = 1$. Constructive.

Frougny, Pelantová Svobodová '11 : Parallel addition in non-standard numeration systems

“Iterates” of $P_\beta(X)$:

With

$$P_\beta(X) = \prod_{j=1}^d (X - \beta_{(j)})$$

Consider the characteristic polynomial of the companion matrix of

$$P_{\beta,n}(X) = \prod_{j=1}^d (X - \beta_{(j)}^n), \quad n \geq 1$$

For n large enough, it has a dominant coefficient. Take N the smallest integer allowing this fact : obtain the maximal alphabet. N always exists.

$$|a_{j_0}| > \frac{1}{2} \Delta_N(\alpha), \quad (7)$$

where $|P_{\beta,N}(1)| = \Delta_N(\beta) = |\prod_{j=1}^d (1 - \beta_{(j)}^N)|$ is the N -th Pierce number of β , and the maximal alphabet $\mathcal{A}_\beta^{(max)} = \{-m, \dots, 0, \dots, m\}$ is such that

$$m \geq \lceil 2^{-1}(2^{-1} \Delta_N(\beta) - 1) \rceil.$$

Interest : asymptotics of $\Delta_N(\beta)$ studied by Lehmer (1933) and Einsiedler, Everest and Ward (2000).

$$\Delta_N(\alpha) = \frac{\Delta_N(\alpha)}{\Delta_{N-1}(\alpha)} \times \frac{\Delta_{N-1}(\alpha)}{\Delta_{N-2}(\alpha)} \times \dots \times \frac{\Delta_2(\alpha)}{\Delta_1(\alpha)} \Delta_1(\alpha),$$

with $\Delta_1(\alpha) = |P_\alpha(1)|$.

From Lehmer '33,

$$M(\alpha) = \lim_{q \rightarrow \infty} \frac{\Delta_{q+1}(\alpha)}{\Delta_q(\alpha)},$$

we deduce, without taking into account the type of convergence towards $M(\alpha)$, as a rough estimate for the lower bound,

$$|a_{j_0}| > \frac{1}{2} M(\beta)^{N-1} |P_\beta(1)|,$$

and the approximate lower bound $\lceil 2^{-1}(2^{-1}M(\beta)^{N-1}|P_\beta(1)| - 1) \rceil$ for m .

Huge numerical values. These maximal alphabets do exist all the time but are not convenient.

General Problem

For β any real algebraic integer > 1 such that β has no conjugate on the unit circle, what is the smallest (symmetrical) alphabet $\mathcal{A} \subset \mathbb{Z}$ realizing

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta)?$$

 If β is a Pisot number the problem is solved (consider the Rényi β -expansions) :

$$\mathcal{A} = \{-1, 0, +1\}$$

(independent of β). It is said **minimal**.

Theorem (K. Schmidt, '80)

Let $\beta > 1$ be a real number.

(1) If $\mathbb{Q} \cap [0, 1] \subset \text{Per}_{\{0,1\}}(\beta)$, then β is either a Pisot or a Salem number.

(2) If β is a Pisot number, then $\text{Per}_{\{0,1\}}(\beta) = \mathbb{Q}(\beta) \cap [0, 1]$.

Examples, Pisot numbers, maximal alphabets $\mathcal{A}_{\beta_k}^{(max)}$

Consider the sequence of Pisot polynomials :

$$P_{2k}(z) = (1 - z^{2k}(1 + z - z^2))/(1 - z), \quad k \geq 1.$$

- (i) The dominant root of $P_{2k}(z)$ is denoted $\beta_k > 1$, Pisot number.
- (ii) All the other roots have a modulus < 1 .
- (iii) For all $k \geq 1$, we have : $\beta_k < (1 + \sqrt{5})/2$.
- (iv) The sequence $(\beta_k)_{k \geq 1}$ is an increasing sequence of Pisot numbers, with limit : $\lim_{k \rightarrow \infty} \beta_k = \frac{1 + \sqrt{5}}{2}$.

$k = 1$: $P_2(z) = z^3 - z - 1$; β_1 is the smallest Pisot number.

$k = \infty$: $\beta_\infty = (1 + \sqrt{5})/2$; dominant root of the trinomial $z^2 - z - 1$.

Dominance index N_j of β_∞ is 3, the maximal alphabet $\mathcal{A}_\tau^{(max)}$ is $= \{-3, \dots, +3\}$.

Figure : Growth rate of the maximal alphabet $\mathcal{A}_{\beta_k}^{(max)} = \{-m_k, \dots, m_k\}$ as a function of k .

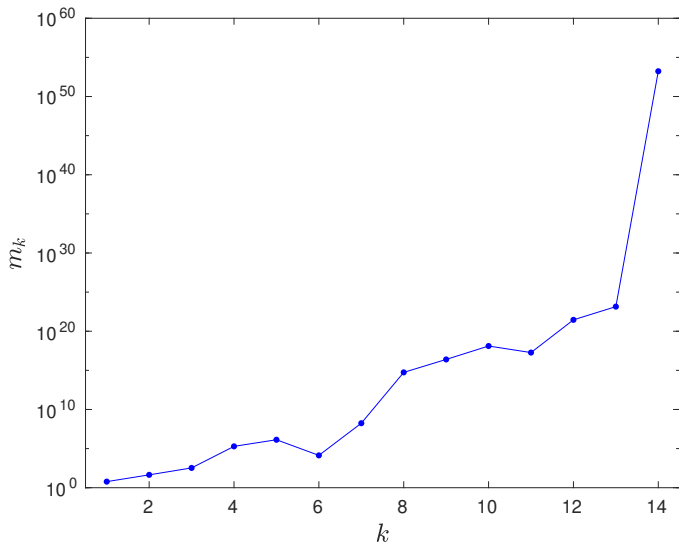


FIGURE: Maximal alphabets of the Pisot numbers β_k .

Theorem (D VG, '20)

Let $\gamma > 1$ be an algebraic integer, root of $S_\gamma(X) = X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$, with $s \geq 1, t_i \in \mathbb{Z}, |t_i| \leq 1$, not necess. irred., s.t. $|\gamma'| \neq 1$ for any conjugate γ' of γ . Let $P(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + a_d X^d \in \mathbb{Z}[X]$, $d = \deg P \geq 1$, $H = \max_{i=0, \dots, d} |a_i|$ the height of P . Assume $0 < \eta < 1$ and $0 \neq |P(\gamma)| < \eta$. Then $P(\gamma) \in \mathbb{Q}(\gamma)$ admits one eventually periodic representation

$$P(\gamma) = R(\gamma^{-1}) + \frac{1}{\gamma^L} \sum_{j=0}^{\infty} \frac{1}{\gamma^{jr}} T(\gamma^{-1}) \in \text{Per}_{\mathcal{A}}(\gamma) \quad (8)$$

with (i) alphabet $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$, independent of s and γ , (ii) $H(R) \leq m$, $\deg R \leq s - 1$, $H(T) \leq m$, $\deg T \leq s - 1$, (iii) preperiod

$$R(\gamma^{-1}) = \frac{a_w}{\gamma^w} + \frac{a_{w+1}}{\gamma^{w+1}} + \dots + \frac{a_{w+s-1}}{\gamma^{w+s-1}}, \quad a_j \in \mathcal{A}, j = w, \dots, w + s - 1, a_w \neq 0,$$

with $w \geq 1$ satisfying $\frac{\kappa_{\gamma, \mathcal{A}}}{\eta} \leq \gamma^{w-1}$ for some positive constant $\kappa_{\gamma, \mathcal{A}}$ depending upon γ and \mathcal{A} .

Rewriting trails - Intermediate alphabets - Proof of Main Theorem

Denote by

$$S_\gamma^*(X) = X^s S_\gamma(1/X) = 1 - t_1 X - t_2 X^2 - \dots - t_{s-1} X^{s-1} - t_s X^s$$

the reciprocal polynomial of $S_\gamma(X) = X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$. The coefficients t_i are in

$$\{-1, 0, +1\}.$$

The algebraic integer γ is the algebraic *base* and $S_\gamma^*(\gamma^{-1}) = 0$.

We want to express $P(\gamma)$ as a (γ, \mathcal{A}) -eventually periodic representation with a certain alphabet \mathcal{A} to be defined. This objective means that, first, we have to express $P(\gamma)$ as a Laurent series of $1/\gamma$.

We now introduce a construction, that we call “rewriting trail from “ S_γ^* ” to “ P ”, at γ^{-1} ”, to reach this objective, and which will allow us to show that a symmetrical alphabet $\mathcal{A} = \{-m, \dots, 0, \dots, m\}$ can be defined and is such that m depends upon H and $\deg(P)$, independently of s and γ .

The starting point is the identity

$$1 = 1$$

to which we add $0 = -S_\gamma^*(\gamma^{-1})$ in the right hand side. Then we define a rewriting trail from

$$1 = 1 - S_\gamma^*(\gamma^{-1}) = t_1\gamma^{-1} + t_2\gamma^{-2} + \dots + t_{s-1}\gamma^{-(s-1)} + t_s\gamma^{-s} \quad (9)$$

to

$$-a_1\gamma^{-1} - a_2\gamma^{-2} + \dots - a_{d-1}\gamma^{-(d-1)} - a_d\gamma^{-d} = 1 - P(\gamma^{-1}).$$

A rewriting trail will be a sequence of integer polynomials, whose role will consist in “restoring” the coefficients of $1 - P(\gamma^{-1})$ one after the other, from the left, by adding “0” conveniently at each step to both sides of (9).

At the first step we add $0 = (-a_1 - t_1)\gamma^{-1} S_\gamma^*(\gamma^{-1})$; and we obtain

$$1 = -a_1\gamma^{-1}$$

$$+(-(-a_1 - t_1)t_1 + t_2)\gamma^{-2} + (-(-a_1 - t_1)t_2 + t_3)\gamma^{-3} + \dots$$

so that the height of the polynomial

$$(-(-a_1 - t_1)t_1 + t_2)X^2 + (-(-a_1 - t_1)t_2 + t_3)X^3 + \dots$$

is $\leq H + 2$.

At the second step we add $0 = (-a_2 - (-(-a_1 - t_1)t_1 + t_2))\gamma^{-2} S_\gamma^*(\gamma^{-1})$.

Then we obtain

$$1 = -a_1\gamma^{-1} - a_2\gamma^{-2}$$

$$+ [(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]\gamma^{-3} + \dots$$

where the height of the polynomial

$$[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]X^3 + \dots$$

is $\leq H + (H + 2) + (H + 2) = 3H + 4$.

Iterating this process d times we obtain

$$1 = -a_1\gamma^{-1} - a_2\gamma^{-2} - \dots - a_d\gamma^{-d} \\ + \textit{polynomial remainder in } \gamma^{-1}.$$

Denote by $V(\gamma^{-1})$ this polynomial remainder in γ^{-1} , for some $V(X) \in \mathbb{Z}[X]$, and X specializing in γ^{-1} . If we denote the upper bound of the height of the polynomial remainder $V(X)$, at step q , by

$$\lambda_q H + v_q,$$

we readily deduce : $v_q = 2^q$, and $\lambda_{q+1} = 2\lambda_q + 1$, $q \geq 1$, with $\lambda_1 = 1$; then $\lambda_q = 2^q - 1$.

To summarize, we obtain a sequence $(A'_q(X))_{q \geq 1}$ of rewriting polynomials involved in this rewriting trail ; for $q \geq 1$, $A'_q \in \mathbb{Z}[X]$, $\deg(A'_q) \leq q$ and $A'_q(0) = -1$. The first polynomial $A'_1(X)$ is

$$-1 + (-a_1 - t_1)X.$$

The second polynomial $A'_2(X)$ is

$$-1 + (-a_1 - t_1)X + (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2,$$

etc.

For $q \geq \deg(P)$, all the coefficients of P are restored.

Denote by $(h'_{q,j})_{j=0,1,\dots,s-1}$ the s -tuple of integers produced by this rewriting trail, at step q . It is such that

$$A'_q(\gamma^{-1})S_\gamma^*(\gamma^{-1}) = -P(\gamma^{-1}) + \gamma^{-q-1} \left(\sum_{j=0}^{s-1} h_{q,j} \gamma^{-j} \right). \quad (10)$$

Then take $q = d$. The lhs of (10) is equal to 0. Thus

$$P(\gamma^{-1}) = \gamma^{-d-1} \left(\sum_{j=0}^{s-1} h_{d,j} \gamma^{-j} \right) \quad \implies \quad P(\gamma) = \sum_{j=0}^{s-1} h_{d,j} \gamma^{-j-1}.$$

The height of the polynomial $W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1}$ is

$$\leq (2^d - 1)H + 2^d.$$

We now assume $|P(\gamma)| < \eta$.

$|P(\gamma)| < \eta$: by Kala-Vavra's Theorem there exist an alphabet $\mathcal{A} \subset \mathbb{Z}$, a preperiod $R(X) \in \mathcal{A}[X]$, a period $T(X) \in \mathcal{A}[X]$ such that

$$W(\gamma^{-1}) = P(\gamma) = R(\gamma^{-1}) + \gamma^{-\deg R-1} \sum_{j=0}^{\infty} \frac{1}{\gamma^{j(\deg T+1)}} T(\gamma^{-1})$$

Since

$$S_{\gamma}^*(\gamma^{-1}) = 1 - t_1\gamma^{-1} - t_2\gamma^{-2} - \dots - t_{s-1}\gamma^{-s+1} - t_s\gamma^{-s} = 0$$

holds, we may assume $\deg R \leq s-1$, $\deg T \leq s-1$. Then, for X specialized at γ^{-1} , we have the identity

$$W(X) = R(X) + X^L \frac{T(X)}{1 - X^r} \quad (11)$$

for some positive integers L, r . The height of $(1 - X^r)W(X)$ is $\leq 2((2^d - 1)H + 2^d)$ and, with \mathcal{A} assumed $= \{-m, \dots, 0, \dots, +m\}$, the height of $(1 - X^r)R(X) + X^L T(X)$ is less than $3m$. Therefore

$$m \leq 2((2^d - 1)H + 2^d)/3.$$

We can take $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$.

Since the algebraic norm $N(\gamma)$ is equal to ± 1 we cannot expect the uniqueness of the representations $P(\gamma)$, by [Kovacs Környei '92]. However, for any (γ, \mathcal{A}) -eventually periodic representation of $P(\gamma)$

$$P(\gamma) = W(\gamma^{-1}) = \frac{a'_w}{\gamma^w} + \frac{a'_{w+1}}{\gamma^{w+1}} + \frac{a'_{w+2}}{\gamma^{w+2}} + \dots, \quad \text{with } |a'_j| \leq m, j = w, w+1, \dots$$

with $a'_w \neq 0$, the exponent w appearing in the first term tends to infinity if η tends to 0. Indeed, from Theorem 4, Remarks 5 to 7, in Frougny Pelantová Svobodová '11] (Thurston construction), there exists a positive real number $\kappa_{\gamma, \mathcal{A}} > 0$ such that w is the minimal integer such that

$$\gamma^{w-1} \geq \frac{\kappa_{\gamma, \mathcal{A}}}{|P(\gamma)|} \geq \frac{\kappa_{\gamma, \mathcal{A}}}{\eta}.$$

Contents

- 1 Periodic representations in algebraic basis
- 2 The class \mathcal{B} of lacunary polynomials - Dynamical zeta function - Problem of Lehmer**
- 3 Solomyak's fractal - Spirals of poles - Universal self-intersecting curves of conjugates

Definition

The collection of lacunary almost Newman polynomials of the type :

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

where $n \geq 2$, $s \geq 0$,

$$m_1 - n \geq n - 1, \quad m_{q+1} - m_q \geq n - 1 \quad \text{for } 1 \leq q < s,$$

is called the *class* \mathcal{B} . The case “ $s = 0$ ” corresponds to the trinomials $G_n(z) := -1 + z + z^n$.

The subclass $\mathcal{B}_n \subset \mathcal{B}$ is the set of polynomials $f(x) \in \mathcal{B}$ whose third monomial is exactly x^n , so that the union $\mathcal{B} = \cup_{n \geq 2} \mathcal{B}_n$ is disjoint.

Heuristics (“Asymptotic Reducibility Conjecture”) : 75% of the polynomials $f(x) \in \mathcal{B}$ are irreducible.

Theorem (Selmer '56)

Let $n \geq 2$. The trinomials $G_n(x) = -1 + x + x^n$ are irreducible if $n \not\equiv 5 \pmod{6}$, and, for $n \equiv 5 \pmod{6}$, are reducible as product of two irreducible factors whose one is the cyclotomic factor $x^2 - x + 1$, the other factor $(-1 + x + x^n)/(x^2 - x + 1)$ being nonreciprocal of degree $n - 2$.

By definition, for $n \geq 2$,

$\theta_n :=$ the unique root of the trinomial $-1 + x + x^n$ in the interval $(0, 1)$.

The algebraic integers $\theta_n^{-1} > 1$ are Perron numbers.

Scale of comparison : the sequence $(\theta_n^{-1})_{n \geq 2}$ is decreasing, tends to 1 if n tends to $+\infty$.

Theorem (Dutykh - Verger-Gaugry '18)

For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then

- (i) the nonreciprocal part C is nontrivial, irreducible and *never vanishes on the unit circle*,
- (ii) if $\gamma > 1$ denotes the real algebraic number uniquely determined by the sequence $(n, m_1, m_2, \dots, m_s)$ such that $1/\gamma$ is the unique real root of f in (θ_{n-1}, θ_n) , *$-C^*(X)$ is the minimal polynomial $P_\gamma(X)$ of γ , and γ is a nonreciprocal algebraic integer.*

Now let us assume the existence of a reciprocal algebraic integer β in the interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for some integer $n \geq 3$ (fixed) :

$$\theta_{n-1} < \beta^{-1} < \theta_n < 1.$$

β is canonically associated with

- (i) its minimal polynomial P_β , which is monic and reciprocal meaning $X^{\deg P_\beta} P_\beta(1/X) = P_\beta(X)$;

denote $d := \deg P_\beta$, $H :=$ the height of P_β ,

- (ii) the Parry Upper function $f_\beta(x)$ at β^{-1} , which is the generalized Fredholm determinant of the β -transformation $T_\beta : x \rightarrow \{\beta x\}$ which is a power series with coefficients in the alphabet $\{0, 1\}$ except the constant term equal to -1 , with distanciation between the exponents of the monomials :

$$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

where $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $q \geq 1$.

β^{-1} is the unique zero of $f_\beta(x)$ in the unit interval $(0, 1)$.

The analytic function $f_\beta(z)$ is related to the dynamical zeta function $\zeta_\beta(z)$ of the β -shift by :

$$f_\beta(z) = -1/\zeta_\beta(z).$$

Since β is reciprocal, with the two real roots β and $1/\beta$, the series $f_\beta(x)$ is never a polynomial, by Descartes's rule on sign changes on the coefficient vector. The algebraic integer β is associated with the infinite sequence of exponents (m_j) .

Takahashi '73
Ito and Takahashi '74
Lagarias '99

β fixed (n is fixed) :

All the polynomial sections

$$S_{\gamma_s}^*(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

of $f_\beta(x)$ are polynomials of the class \mathcal{B}_n .

For every $s \geq 1$, denote by

$$\gamma_s > 1$$

the (non-reciprocal) algebraic integer which is such that γ_s^{-1} is the unique zero in $(0, 1)$ of the polynomial section $S_{\gamma_s}^*(x)$ of $f_\beta(x)$.

We have : $\deg \gamma_s^{-1} = \deg S_{\gamma_s}^*$ if and only if $S_{\gamma_s}^*(x)$ is irreducible.

Moreover

$$f_\beta(\beta^{-1}) = 0$$

and

$$\lim_{s \rightarrow \infty} \gamma_s = \beta.$$

The integer $n \geq 3$ is fixed. For all s such that $\deg S_{\gamma_s}^* \geq \deg P_\beta$, the identity

$$\mathbb{Q}(\gamma_s) = \text{Per}_{\mathcal{A}}(\gamma_s),$$

holds with

$$\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}, \quad m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil.$$

By Theorem [Dutykh VG '18], for any $s \geq 0$,

γ_s^{-1} has no conjugate on the unit circle.

The polynomial value $P_\beta(\gamma_s) \in \mathbb{Q}(\gamma_s)$ is eventually periodic

$$P_\beta(\gamma_s) = R(\gamma_s^{-1}) + \frac{1}{\gamma_s^L} \sum_{j=0}^{\infty} \frac{1}{\gamma_s^{jr}} T(\gamma_s^{-1}) \in \text{Per}_{\mathcal{A}}(\gamma_s)$$

with L, r and $R(X), T(X)$, depending upon s .

This representation of $P_\beta(\gamma_s)$ starts as

$$= \frac{a_{w,(s)}}{\gamma_s^w} + \frac{a_{w+1,(s)}}{\gamma_s^{w+1}} + \dots + \frac{a_{w+m_s-1,(s)}}{\gamma_s^{w+m_s-1}} + \dots, \quad a_{j,(s)} \in \mathcal{A}, j = w, \dots, w + m_s - 1,$$

with $a_{w,(s)} \neq 0$ and $w = w_s \geq 1$, depending upon s , satisfies

$$\frac{\kappa_{\gamma_s, \mathcal{A}}}{\eta} \leq \gamma_s^{w_s-1}$$

for some positive constant $\kappa_{\gamma_s, \mathcal{A}}$ depending upon γ and \mathcal{A} . Since \mathcal{A} is independent of s , and that the sequence (γ_s) is convergent with limit $\beta > 1$, there exists a (true) constant $\kappa > 0$ such that

$$\frac{\kappa}{\eta} \leq \gamma_s^{w_s-1}$$

from Theorem 4, Rks 5 to 7, in Frougny Pelantová Svobodová '11.

Since $\lim_{s \rightarrow \infty} P_\beta(\gamma_s) = 0 = P_\beta(\beta)$, we take $\eta = \eta_s := |P_\beta(\gamma_s)|$. The sequence (η_s) tends to 0. We deduce $\lim_{s \rightarrow \infty} w_s = +\infty$.

Let $\Omega \neq \beta^{-1}$ be a zero of modulus < 1 of $f_\beta(x)$. We assume that it lies inside an open disk

$$D(z_{j,n}, r) \subset D(0, 1)$$

centered at a zero $z_{j,n}$ of $-1 + x + x^n$ of modulus < 1 , of radius $r > 0$ (small enough), which has the property that the only zero of $f_\beta(x)$ in $D(\Omega, r)$ is Ω and that it is a simple zero.

Under the above assumptions :

Proposition

$$f_\beta(\Omega) = 0 \quad \implies \quad P_\beta(\Omega) = 0.$$

Consequence : every lenticular pole of ζ_β is a Galois conjugate of $1/\beta$, i.e. of β . Such Galois conjugates constitute a lenticulus of conjugates.

Proof : the zero Ω is limit point of a sequence of zeroes of the polynomial sections of $f_\beta(x)$. As soon as $s \geq s_0$ for some s_0 , we assume that the disk $D(\Omega, r)$ contains only one zero of $S_{\gamma_s}^*(x)$.

Denote by r_s this zero.

r_s is a Galois conjugate of γ_s^{-1}

Denote by $\sigma_s : \gamma_s^{-1} \rightarrow r_s$ the \mathbb{Q} -automorphism of conjugation (Galois).

To summarize, for $s \geq s_0$:

$$f_\beta(\Omega) = f_\beta(\beta^{-1}) = 0, \quad S_{\gamma_s}^*(r_s) = S_{\gamma_s}^*(\gamma_s^{-1}) = 0, \quad r_s = \sigma_s(\gamma_s^{-1}),$$

$$|\Omega - \sigma_s(\gamma_s^{-1})| < r, \quad \lim_{s \rightarrow \infty} r_s = \Omega.$$

To show

$$P_\beta(\Omega) = 0$$

we consider

$$P_\beta(\gamma_s^{-1}),$$

apply the Theorem [D VG '20] above, conjugate by Galois σ_s , then take the limit $s \rightarrow \infty$.

Key idea : the eventual periodicity of the representations in base γ_s is conjugated term by term by σ_s .

The \mathbb{Q} -automorphisms of Galois theory (except z, \bar{z}) are not continuous but the eventual periodicity compensates this fact, once the image of γ_s^{-1} by the conjugation σ_s is such that $|\sigma_s(\gamma_s^{-1})| < 1$, to ensure convergence.

Then

$$\sigma_s(P_\beta(\gamma_s)) = W(r_s) = R(r_s) + r_s^L \frac{T(r_s)}{1 - r_s^r} = a_{w_s, (s)} r_s^{w_s} + a_{w_s+1, (s)} r_s^{w_s+1} + \dots \tag{12}$$

with

$$\frac{\kappa}{|P_\beta(\gamma_s)|} \leq \gamma_s^{w_s-1} \quad \text{and} \quad |r_s| < |\Omega| + r < 1, \quad s \geq s_0.$$

We have, with $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$,

$$|W(r_s)| \leq |r_s|^{w_s} \frac{m}{1 - |r_s|}$$

We deduce

$$P_\beta(\Omega) = \lim_{s \rightarrow \infty} W(r_s) = 0.$$

Contents

- 1 Periodic representations in algebraic basis
- 2 The class \mathcal{B} of lacunary polynomials - Dynamical zeta function - Problem of Lehmer
- 3 Solomyak's fractal - Spirals of poles - Universal self-intersecting curves of conjugates**

Let

$$\mathcal{W} := \{h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0, 1]\}$$

be the class of power series defined on $|z| < 1$ equipped with the topology of uniform convergence on compact sets of $|z| < 1$. The space \mathcal{W} is compact and convex. Let

$$\mathcal{G} := \{\lambda \mid |\lambda| < 1, \exists h(z) \in \mathcal{B} \text{ such that } h(\lambda) = 0\} \subset \{z \mid |z| < 1\}$$

be the set of zeroes of the power series belonging to \mathcal{W} . The domain $D(0, 1) \setminus \mathcal{G}$ is star-convex due to the fact that :

$$h(z) \in \mathcal{W} \implies h(z/r) \in \mathcal{W}, \quad \text{for any } r > 1.$$

Denote by $\partial\mathcal{G}_S$ the “spike” : $[-1, \frac{1}{2}(1 - \sqrt{5})]$ on the negative real axis.

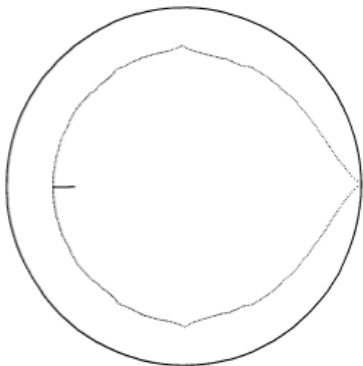


FIG. 1. The set \mathcal{S} is bounded by the dotted line and the unit circle; it also contains a spike along the real axis.

Theorem (Solomyak)

- (i) *The union $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ is closed, symmetrical with respect to the real axis, has a cusp at $z = 1$ with logarithmic tangency,*
- (ii) *the boundary $\partial\mathcal{G}$ is a continuous curve, given by $\phi \rightarrow |\lambda_\phi|$ on $[0, \pi)$, taking its values in $[\frac{\sqrt{5}-1}{2}, 1)$, with $|\lambda_\phi| = 1$ if and only if $\phi = 0$. It admits a left-limit at π^- , $1 > \lim_{\phi \rightarrow \pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1 + \sqrt{5})$, the left-discontinuity at π corresponding to the extremity of $\partial\mathcal{G}_S$.*
- (iii) *at all points $\rho_\phi e^{i\phi} \in \mathcal{G}$ such that ϕ/π is rational in an open dense subset of $(0, 2)$, $\partial\mathcal{G}$ is non-smooth,*
- (iv) *there exists a nonempty subset of transcendental numbers L_{tr} , of Hausdorff dimension zero, such that $\phi \in (0, \pi)$ and $\phi \notin \mathcal{K} \cup \pi\mathbb{Q} \cup \pi L_{tr}$ implies that the boundary curve $\partial\mathcal{G}$ has a tangent at $\rho_\phi e^{i\phi}$ (smooth point).*

obj. : to visualize the sets of lenticuli of poles of $\zeta_\beta(z)$ for all $\beta \in (1, (1 + \sqrt{5})/2)$, inside this fractal.

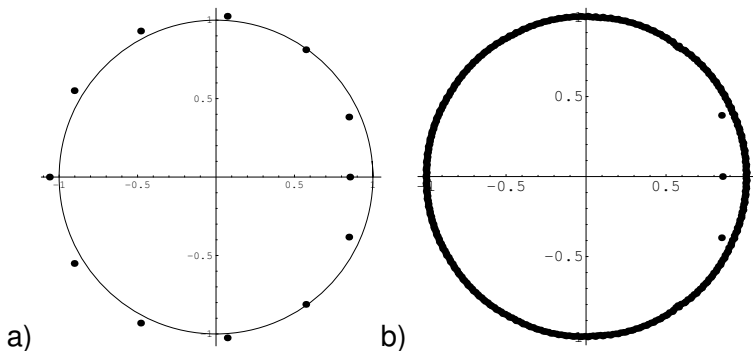


FIGURE: a) The 12 zeroes of G_{12} , b) The 385 zeroes of $f(x) = -1 + x + x^{12} + x^{250} + x^{385}$. The lenticulus of roots of the trinomial $-1 + x + x^{12}$ can be guessed, slightly deformed and almost “complete”. It is well separated from the other roots, and off the unit circle.

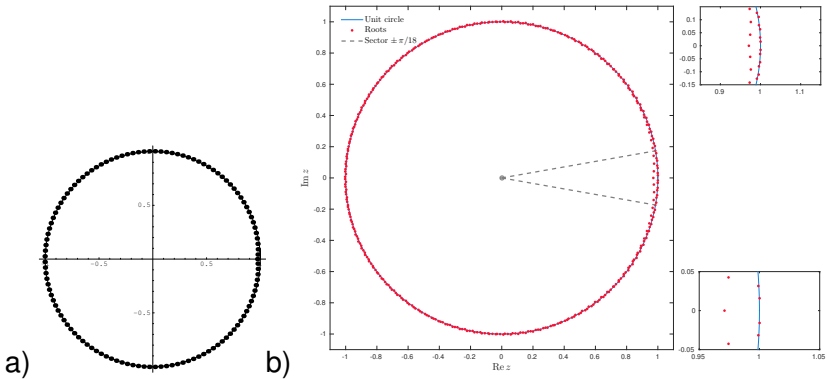


FIGURE: a) Zeroes of G_{121} , b) Zeroes of $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$. The lenticulus of roots of f has 7 zeroes.

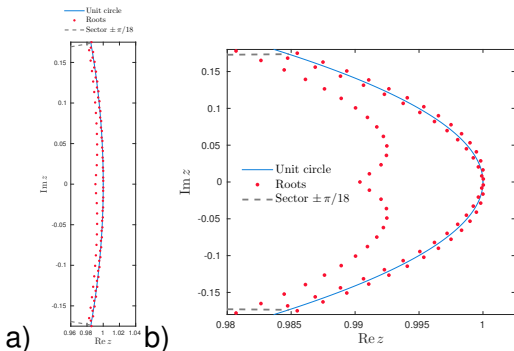


FIGURE: The representation of the 27 zeroes of the lenticulus of $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$ in the angular sector $-\pi/18 < \arg z < \pi/18$ in two different scalings in x and y (in a) and b)). In this angular sector the other zeroes of f can be found in a thin annular neighbourhood of the unit circle. The real root $1/\beta > 0$ of f is such that β satisfies : $1.00970357\dots = \theta_{481}^{-1} < \beta = 1.0097168\dots < \theta_{480}^{-1} = 1.0097202\dots$

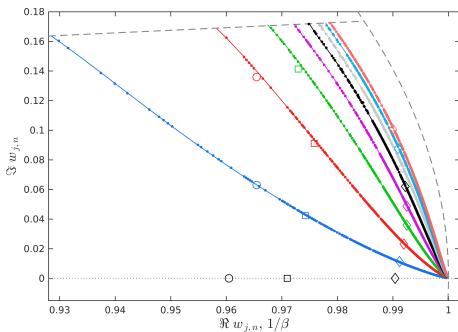


FIGURE: Curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class \mathcal{B} . These (universal) curves are continuous. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies inside the boundary of Solomyak's fractal. The lenticular roots of some polynomial sections are represented by the respective symbols \circ , \square , \diamond . The dashed lines represent the unit circle and the top boundary of the angular sector $|\arg z| < \pi/18$. The complete set of curves, i.e. the locus of lenticuli, is obtained by symmetrization with respect to the real axis.

blue curve = curve of the first conjugates above \mathbb{R} : boundary of Solomyak's fractal.

Galois conjugates : on the other curves, + slight perturbations observed.

Need : to zoom on the fine structure of these curves to answer the questions left by Boris Solomyak.

Numerical challenge : the unit circle is very often the natural boundary of $\zeta_\beta(z)$. The lenticuli of roots become closer and closer to the unit circle, when β tends to 1.

Collab. : Denys Dutykh (Monte-Carlo methods)

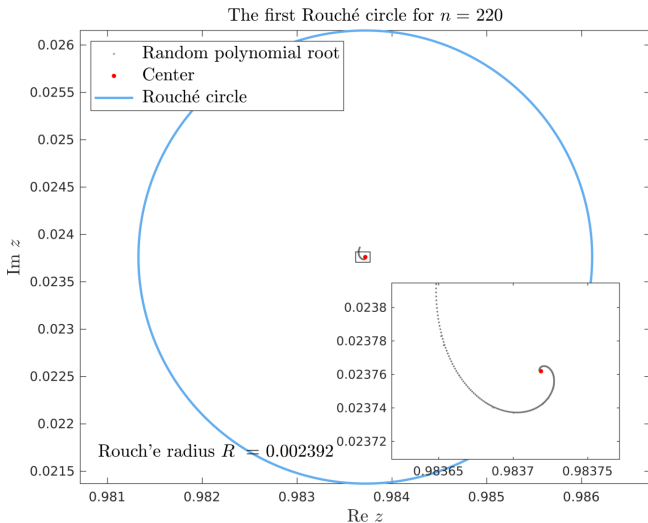


FIGURE: $n = 220$: The first spiral of conjugates. The blue circle is the Rouché disk allowing to detect an unique simple zero inside.

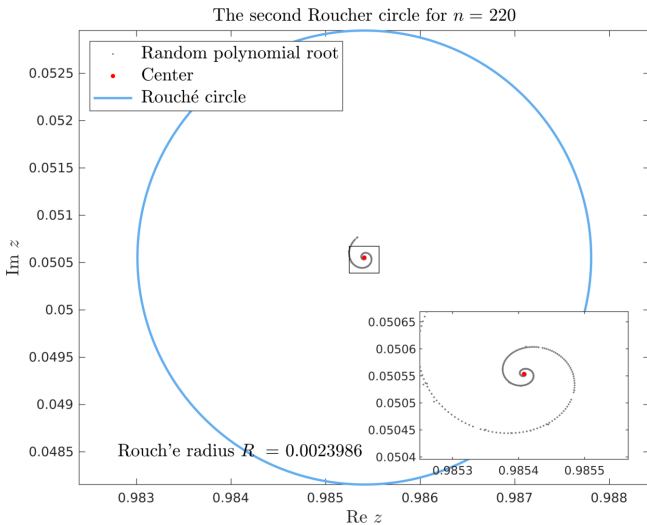


FIGURE: $n = 220$: The second spiral of conjugates.

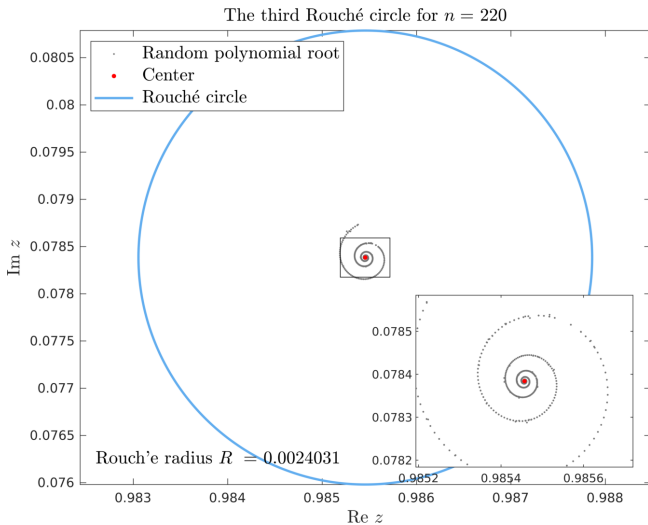


FIGURE: $n = 220$: The third spiral of conjugates.

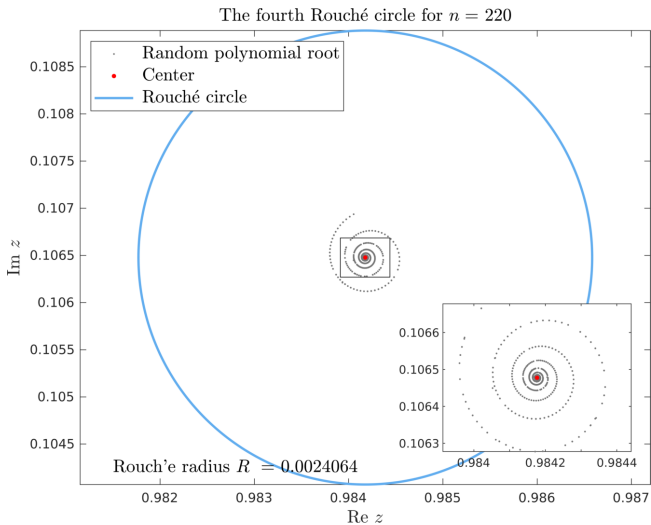


FIGURE: $n = 220$: The fourth spiral of conjugates.

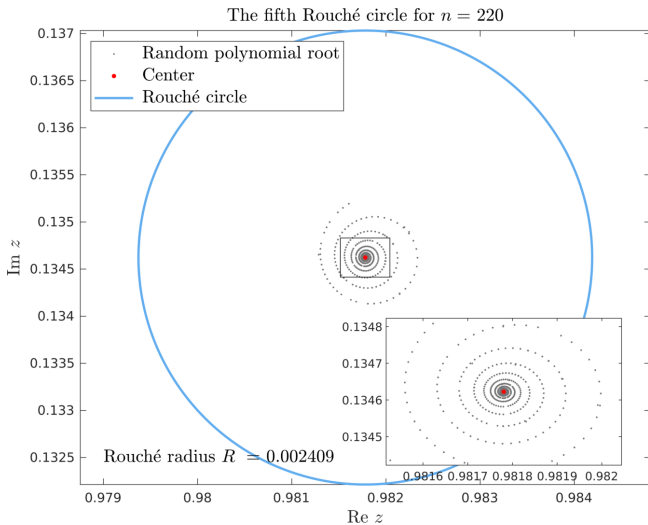


FIGURE: $n = 220$: The fifth spiral of conjugates.

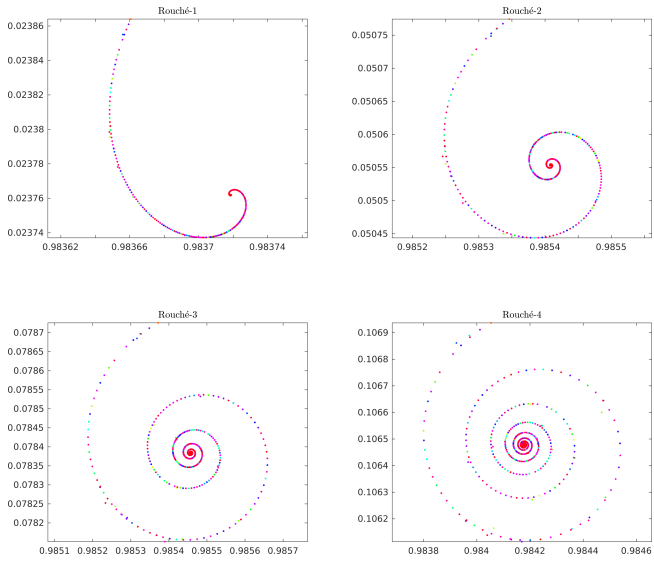


FIGURE: $n = 220$: The first four spirals.

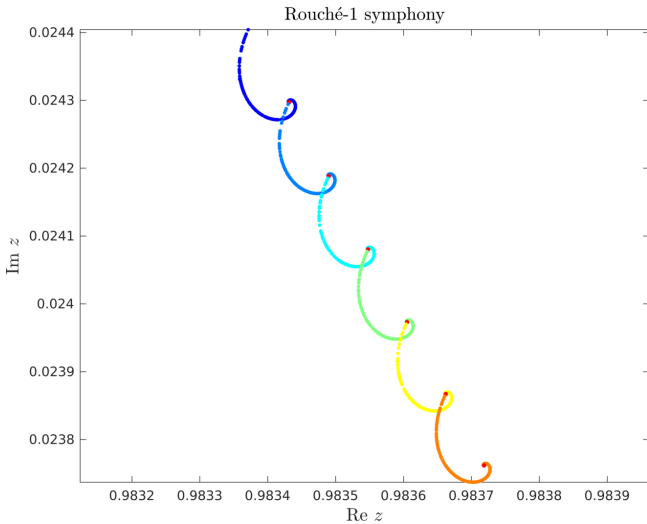


FIGURE: $n = 215$ to $n = 220$: The first curve.

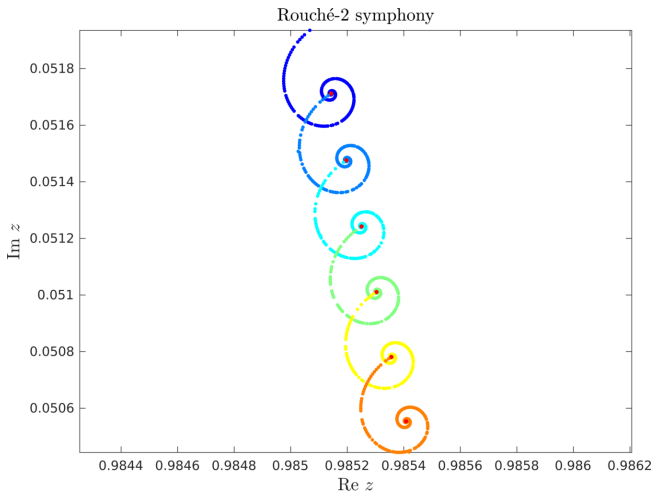


FIGURE: $n = 215$ to $n = 220$: The second curve.

