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Conjunctive grammars, cellular automata and logic

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Abstract: The expressive power of the class `Conj` of *conjunctive languages*, i.e. languages generated by the *conjunctive grammars* of Okhotin, is largely unknown, while its restriction `LinConj` to *linear conjunctive grammars* equals the class of languages recognized by *real-time one-way one-dimensional* cellular automata. We prove two weakened versions of the open question `Conj` $\subseteq$? `RealTimeCA`: 1) it is true for *unary* languages; 2) `Conj` $\subseteq$ `RealTime2OCA`, i.e. any conjunctive language is recognized by a *real-time one-way two-dimensional* cellular automaton. Interestingly, we express the rules of a conjunctive grammar in two *Horn logics*, which *exactly characterize* the complexity classes `RealTimeCA` and `RealTime2OCA`.

Keywords: Computational complexity, Real-time, One-way/two-way communication, Grid-circuit, Unary language, Descriptive complexity, Existential second-order logic, Horn formula.

1 Introduction

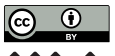
For decades, logic has maintained close relationships with, on the one hand, computational models [27] and computational complexity [3], in particular through descriptive complexity [6, 14, 19, 9, 12, 2], and on the other hand with formal language theory and grammars [7, 19].

Conjunctive grammars versus logic: Okhotin [23] wrote that "context-free grammars may be thought of as a *logic* for inductive description of syntax in which the propositional connectives available... are restricted to *disjunction only*". Thus, twenty years ago, the same author introduced *conjunctive grammars* [20] as an extension of context-free grammars by adding an explicit *conjunction* operation within the grammar rules.

As shown by Okhotin [20], conjunctive grammars – and more generally, Boolean grammars [21, 23] – inherit the parsing algorithms of the ordinary context-free grammars, without increasing their computational complexity. However, the expressive power of these grammars is largely unknown. The fact that the class `Conj` of languages generated by conjunctive grammars has many closure properties – it is trivially closed under reverse, concatenation, Kleene closure, disjunction and conjunction – suggests that this class has equivalent definitions in computational complexity and/or logic.

Conjunctive grammars versus real-time cellular automata: Note that the `LinConj` subclass of languages generated by *linear* conjunctive grammars was found to be equal to the `Trellis` class of languages recognized by *trellis* automata [22], or equivalently, *one-way real-time* cellular automata. Faced with this result, it is tempting to ask the following question: is the larger class `Conj` equal to the class `RealTimeCA` of languages recognized by *two-way* real-time cellular automata? Either answer to this question has strong consequences:

- If `Conj` = `RealTimeCA` then each of the two classes will benefit from the closure properties of the other class; in particular, `RealTimeCA` would be closed under reverse, which was shown by [13] to imply `RealTimeCA` = $\texttt{LinearTime}_{\texttt{CA}}$, i.e. real-time is nothing but *linear time* for cellular automata, a surprising positive answer to a longstanding open question [5, 24, 26].
- If `Conj` $\neq$ `RealTimeCA` then `Conj` $\subsetneq$ `DSPACE`(n) or `RealTimeCA` $\subsetneq$ `DSPACE`(n): any of these strict inclusions would be a striking result.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Real-time is the minimal time of cellular automata (CA). Recall that RealTimeCA (resp. Trellis) is the class of languages recognized in real-time by *one-dimensional* CA with *two-way* (resp. *one-way*) communication and input word given in parallel. We know the strict inclusion Trellis $\subsetneq$ RealTimeCA. The robustness of these classes is attested by their characterization by two sub-logics of ESO – the *existential second-order* logic, which characterizes NP – with *Horn formulas* as their first-order parts[1], and called respectively pred-ESO-HORN and incl-ESO-HORN, see [10, 11]. For short, we write RealTimeCA = pred-ESO-HORN and Trellis = incl-ESO-HORN.

Results of this paper: This paper focuses on the relationships between the class of conjunctive languages and the real-time classes of cellular automata. Although we do not know the answer to the question Conj =? RealTimeCA or even to the question of the inclusion Conj $\subseteq$? RealTimeCA, we prove two weakened versions of this inclusion:

1. $\text{Conj}_1 \subseteq \text{RealTimeCA}_1$: The inclusion holds when restricted to *unary* languages[2].
2. Conj $\subseteq$ RealTime2OCA: The inclusion holds for real-time of *two-dimensional one-way* cellular automata (2-OCA). (We have RealTimeCA $\subseteq$ RealTime2OCA.)

To grasp the scope of inclusion (1), it is important to note that unlike the subclass CFL_1 of the unary languages of the class of context-free languages, which is reduced to regular languages, $\text{CFL}_1 = \text{Reg}_1$, the class Conj_1 was shown by Jez [15] to be much larger than Reg_1. Understanding its precise expressiveness seems as difficult a problem to us as for Conj.

Our inclusion (2) improves the inclusion CFL $\subseteq$ RealTime2SOCA, where RealTime2SOCA denotes the class of languages recognized by real-time *sequential* two-dimensional one-way cellular automata, proved by Terrier [25], who uses a result by King [16] and improves results by Kosaraju [18] and Chang et al. [4]. Terrier's result derives transitively from (2): CFL $\subseteq$ Conj $\subseteq$ RealTime2OCA $\subseteq$ RealTime2SOCA.

Logic as a bridge from problems and grammars to real-time CAs: Logic has been the basis of logic programming and database queries for decades, especially Horn logic through the Prolog and Datalog programming languages [1, 17, 9]. Likewise, the above-mentioned logical characterizations of real-time complexity classes of CAs, RealTimeCA = pred-ESO-HORN and Trellis = incl-ESO-HORN, have been used to easily show that several problems belong to the RealTimeCA or Trellis class by inductively expressing/programming the problems in the corresponding Horn logic, see [10, 11].

In this paper, the same logic programming method is adopted. We prove inclusion (1), $\text{Conj}_1 \subseteq \text{RealTimeCA}_1$, by expressing a unary language generated by a conjunctive grammar in the pred-ESO-HORN logic. Inclusion (1) follows, by the equality pred-ESO-HORN = RealTimeCA. Similarly, to prove inclusion (2), Conj $\subseteq$ RealTime2OCA, we first design a logic denoted incl-pred-ESO-HORN so that incl-pred-ESO-HORN = RealTime2OCA. Then, we express any conjunctive language in this logic, proving that it belongs to RealTime2OCA, as claimed.

Thus, the heart of each proof consists in presenting a formula of a certain *Horn logic*, which *inductively* expresses how a word is generated by a conjunctive grammar: the Horn clauses of the formula *naturally* imitate the rules of the grammar.

Our proof method and the paper structure: After Section 2 gives some definitions, Sections 3 and 4 present inclusions (1) and (2) and their proofs with a common plan: Subsection 3.1 (resp. 4.1) expresses the inductive generating process of a conjunctive grammar, assumed in binary (Chomsky) normal form in the logic pred-ESO-HORN (resp.

[1] The class ESO-HORN of languages defined by existential second-order formulas with Horn formulas as their first-order parts is exactly PTIME, see [8].

[2] The subclass of the unary languages of a class of languages $\mathcal{C}$ is denoted $\mathcal{C}_1$.

incl-pred-ESO-HORN). Subsection 3.2 (resp. 4.2) shows that any formula of this logic can be normalized into a formula which mimics the computation of a two-dimensional (resp. three-dimensional) *grid-circuit* called Grid (resp. Cube); Subsection 3.3 (resp. 4.3) translates the grid-circuit into a real-time one-dimensional CA (resp. two-dimensional OCA). Note that we prove the equivalence of our logics with grid-circuits and CA real-time[3]: pred-ESO-HORN = Grid = RealTimeCA and incl-pred-ESO-HORN = Cube = RealTime2OCA. Section 5 gives a conclusion with a diagram of the known relations between the Conj class and the CA complexity classes studied here, for the general case and for the unary case.

2 Preliminaries

2.1 Conjunctive grammars and their binary normal form

Conjunctive grammars extend context-free grammars with a conjunction operation.

▶ **Definition 1.** *A* conjunctive grammar *is a tuple* $G = (\Sigma, N, P, S)$ *where* Σ *is the finite set of* terminal *symbols,* N *is the finite set of* nonterminal *symbols,* $S \in N$ *is the* initial *symbol, and* P *is the finite set of* rules*, each of the form* $A \to \alpha_1 \& ... \& \alpha_m$*, for* $m \geq 1$ *and* $\alpha_i \in (\Sigma \cup N)^+$*. The set of words,* $L(A)$*,* generated *by any* $A \in N$ *is defined by induction: if the rules for* A *are* $A \to \alpha_1^1 \& ... \& \alpha_{m_1}^1 \mid \cdots \mid \alpha_1^k \& ... \& \alpha_{m_k}^k$*, then* $L(A) := \bigcup_{i=1}^{k} \bigcap_{j=1}^{m_i} L(\alpha_j^i)$*. The* language generated *by the grammar* G *is* $L(S)$*.*

We will mainly use the binary normal form of conjunctive grammars, which extends the Chomsky normal form of context-free grammars. Each conjunctive grammar can be rewritten in an equivalent binary normal form [20, 23].

▶ **Definition 2** (Binary normal form [20])**.** *A conjunctive grammar* $G = (\Sigma, N, P, S)$ *is in* binary normal form *if each rule in* P *has one of the two following forms:*

- *a long rule:* $A \to B_1C_1 \& ... \& B_mC_m$ $(m \geq 1, B_i, C_j \in N)$*;*
- *a short rule:* $A \to a$ $(a \in \Sigma)$*.*

2.2 Elements of logic

The underlying structure we will use to encode an input word $w = w_1 \dots w_n$ on its index interval $[1, n] = \{1, \dots, n\}$ uses the *successor* and *predecessor* functions and the monadic predicates min and max as its *only* arithmetic functions/predicates:

▶ **Definition 3** (structure encoding a word)**.** *Each nonempty word* $w = w_1 \dots w_n \in \Sigma^n$ *on a fixed finite alphabet* Σ *is represented by the first-order structure*
$\langle w \rangle := ([1, n]; (Q_s)_{s \in \Sigma}, \mathtt{min}, \mathtt{max}, \mathtt{suc}, \mathtt{pred})$
of domain $[1, n]$*, monadic predicates* Q_s*,* $s \in \Sigma$*,* min *and* max *such that* $Q_s(i) \iff w_i = s$*,* $\mathtt{min}(i) \iff i = 1$*, and* $\mathtt{max}(i) \iff i = n$*, and unary functions* suc *and* pred *such that* $\mathtt{suc}(i) = i + 1$ *for* $i < n$ *and* $\mathtt{suc}(n) = n$*,* $\mathtt{pred}(i) = i - 1$ *for* $i > 1$ *and* $\mathtt{pred}(1) = 1$*.*
Let $\mathcal{S}_\Sigma$ *denote the signature* $\{(Q_s)_{s \in \Sigma}, \mathtt{min}, \mathtt{max}, \mathtt{suc}, \mathtt{pred}\}$ *of the structure* $\langle w \rangle$*.*

▶ **Notation 1.** *Let* $x + k$ *and* $x - k$ *abbreviate the terms* $\mathtt{suc}^k(x)$ *and* $\mathtt{pred}^k(x)$*, for a fixed integer* $k \geq 0$*. We will also use the intuitive abbreviations* $x = 1$*,* $x = n$ *and* $x > k$*, for a fixed integer* $k \geq 1$*, in place of the formulas* $\mathtt{min}(x)$*,* $\mathtt{max}(x)$ *and* $\neg \mathtt{min}(x - (k - 1))$*, respectively.*

[3] We have chosen to give here a simplified proof of the logical characterization pred-ESO-HORN = Grid = RealTimeCA already proved in [10] so that this paper is self-content, but above all because our proof of the similar result incl-pred-ESO-HORN = Cube = RealTime2OCA is an extension of it.

2.3 Cellular automata and real-time

▶ **Definition 4** (1-CA and 2-0CA). *A* d-dimensional cellular automaton *(CA) is a triple* $(\mathtt{S}, \mathcal{N}, \mathtt{f})$ *where* $\mathtt{S}$ *is the finite set of* states, $\mathcal{N} \subset \mathbb{Z}^d$ *is the* neighborhood, *and* $\mathtt{f} : \mathtt{S}^{|\mathcal{N}|} \to \mathtt{S}$ *is the* transition function. *We are interested in the following two special cases:*

- 1-CA*: It is a* one-dimensional two-way *cellular automaton* $(\mathtt{S}, \{-1, 0, 1\}, \mathtt{f})$*, for which the state* $\langle c, t\rangle$ *of any cell* c *at a time* $t > 1$ *is updated in this way:* $\langle c, t\rangle = \mathtt{f}(\langle c-1, t-1\rangle, \langle c, t-1\rangle, \langle c+1, t-1\rangle)$.
- 2-OCA*: It is a* two-dimensional one-way *cellular automaton* $(\mathtt{S}, \{(0,0), (-1,0), (0,-1)\}, \mathtt{f})$ *for which the state* $\langle c_1, c_2, t\rangle$ *of any cell* (c_1, c_2) *at a time* $t > 1$ *is updated in this way:* $\langle c_1, c_2, t\rangle = \mathtt{f}(\langle c_1, c_2, t-1\rangle, \langle c_1-1, c_2, t-1\rangle, \langle c_1, c_2-1, t-1\rangle)$.

▶ **Definition 5** (permanent and quiescent states). *In a CA, a state* $\sharp$ *is* permanent *if a cell in state* $\sharp$ *remains in this state forever. A state* λ *of a CA is* quiescent *if a cell in state* λ *remains in this state as long as the states of its neighborhood cells are quiescent or permanent.*

▶ **Definition 6** (CA as a word acceptor). *A cellular automaton* $(\mathtt{S}, \mathcal{N}, \mathtt{f})$ *with an* input alphabet $\Sigma \subset \mathtt{S}$*, a* permanent state $\sharp$*, a* quiescent state λ*, and a set of* accepting states $\mathtt{S}_{\mathtt{acc}} \subset \mathtt{S}$ *acts as a* word acceptor *if it operates on an input word* $w \in \Sigma^+$ *in respecting the following conditions (see Figure 1).*

Input. *For a* 1*-CA, the* i*-th symbol of the input* $w = w_1 \dots w_n$ *is given to the cell* i *at the initial time* 1*:* $\langle i, 1\rangle = w_i$*. All other cells are in the permanent state* $\sharp$*. For a* 2*-OCA, the* i*-th symbol of the input is given to the cell* $(i, 1)$ *at time* 1*:* $\langle i, 1, 1\rangle = w_i$*. At time* 1*, the cells* $(c_1, c_2) \in [1, n] \times [2, n]$ *are in the quiescent state* λ*, all other cells are in the permanent state* $\sharp$*.*

Output. *One specific cell called* output cell *gives the output, "accept" or "reject", of the computation. For a* 1*-CA, the output cell is the cell* 1*. For a* 2*-OCA, the output cell is* (n, n)*.*

Acceptance. *An input word is* accepted *by a* 1*-CA (resp.* 2*-CA)* at time t *if the output cell enters an accepting state at time* t*.*

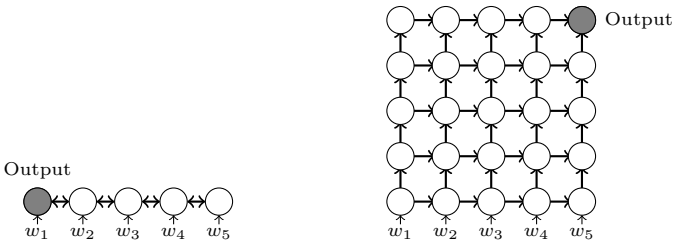

Figure 1 Input and output of a CA acting as a word acceptor

▶ **Definition 7** (RealTimeCA, RealTime2OCA). *A word is* accepted in real-time *by a* 1*-CA (resp.* 2*-OCA) if the word is accepted in minimal time for the output cell* 1 *(resp.* (n, n)*) to receive each of its letters. A language is* recognized in real-time *by a CA if it is the set of words that it accepts in real-time. The class* RealTimeCA *(resp.* RealTime2OCA*) is the class of languages recognized in real-time by a* 1*-CA (resp.* 2*-OCA).*

3 Real-time recognition of a unary conjunctive language

In this section, we prove our first main result:

▶ **Theorem 8.** $\mathtt{Conj}_1 \subseteq \mathtt{RealTimeCA}_1$.

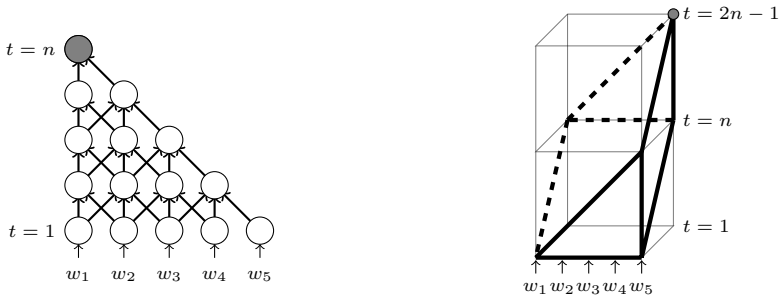

Figure 2 Space-time diagrams of `RealTimeCA` and `RealTime2OCA`

3.1 Expressing inductively a unary conjunctive language in logic

The generating process of a unary conjunctive language is naturally expressed in the logic `pred-ESO-HORN`, an inductive Horn logic whose only function is the predecessor function.

▶ **Definition 9** (`pred-ESO-HORN`). *A formula of* `pred-ESO-HORN` *is a formula* $\Phi := \exists \mathbf{R} \forall x \forall y \psi(x, y)$ *where* $\mathbf{R}$ *is a finite set of binary predicates and* ψ *is a conjunction of Horn clauses, of signature* $\mathcal{S}_\Sigma \cup \mathbf{R}$*, and of one the three following forms:*

- an input clause: $\texttt{min}(x) \wedge (\neg)\,\texttt{min}(y) \wedge Q_s(y) \to R(x, y)$ *with* $s \in \Sigma$ *and* $R \in \mathbf{R}$*;*
- a computation clause: $\delta_1 \wedge \ldots \wedge \delta_r \to R(x, y)$ *with* $R \in \mathbf{R}$ *and where each hypothesis* δ_i *is an atom* $S(x, y)$ *or a conjunction* $S(x - a, y - b) \wedge x > a \wedge y > b$*, with* $S \in \mathbf{R}$ *and* $a, b \geq 0$ *two integers such that* $a + b > 0$*;*
- a contradiction clause: $\texttt{max}(x) \wedge \texttt{max}(y) \wedge R(x, y) \to \bot$ *with* $R \in \mathbf{R}$*.*

We denote by `pred-ESO-HORN` *the class of languages defined by a formula of* `pred-ESO-HORN`*.*

▶ Remark 10. We will freely use equalities (resp. inequalities) $x = a$ and $y = b$ (resp. $x > a$, $y > b$), for constants a, b, in our formulas since they can be easily defined in `pred-ESO-HORN`. For example, the binary predicate $R^{x>2}$ of intuitive meaning $R^{x>2}(x, y) \iff x > 2$ is defined inductively by the following clauses where $R^{x=a}(x, y)$ means $x = a$:

- $\texttt{min}(x) \to R^{x=1}(x, y)$; $x > 1 \wedge R^{x=1}(x - 1, y) \to R^{x=2}(x, y)$;
- $x > 1 \wedge R^{x=2}(x - 1, y) \to R^{x>2}(x, y)$; $x > 1 \wedge R^{x>2}(x - 1, y) \to R^{x>2}(x, y)$.

Also, some other arithmetic predicates easily defined in `pred-ESO-HORN` will be used. For example, $y = 2x$ can be replaced by the atom $R^{y=2x}(x, y)$, where $R^{y=2x}$ is defined by the following two clauses using the predicates $R^{x=1}, R^{y=2}, R^{x>1}$ and $R^{y>2}$:

- $x = 1 \wedge y = 2 \to R^{y=2x}(x, y)$; $x > 1 \wedge y > 2 \wedge R^{y=2x}(x - 1, y - 2) \to R^{y=2x}(x, y)$.

▶ **Notation 2.** *More generally, let* $R^{\rho(x,y)}$ *denote a binary predicate whose meaning is* $R^{\rho(x,y)}(x, y) \iff \rho(x, y)$*, for a property or a formula* $\rho(x, y)$*. We will also use a set of binary arithmetic predicates denoted by* $\mathbf{R}_{\texttt{arith}}$*, which consists of* $R^{x=y}, R^{y=2x}$ *and* $R^{\rho(x,y)}$*, for* $\rho(x, y) := x \geq \lceil \frac{y}{2} \rceil$*, and the predicates used to define them in* `pred-ESO-HORN`*.*

Let us prove that for every unary conjunctive languages, their complements can be defined in $\texttt{pred-ESO-HORN}_1$.

▶ **Lemma 11.** *For each language* $L \subseteq a^+$*, if* $L \in \texttt{Conj}_1$ *then* $a^+ \setminus L \in$ `pred-ESO-HORN`*.*

Proof. Let $G = (\{a\}, N, P, S)$ be a conjunctive grammar in binary normal form which generates L. For each $A \in N$ and each unary word a^y, we have, according to the length y, the following equivalences which will be the basis of our induction:

- if $y = 1$, then $a^y \in L(A) \iff$ the short rule $A \to a$ belongs to P;
- if $y > 1$, then $a^y \in L(A) \iff$ there is a long rule $A \to B_1C_1\& \dots \& B_mC_m$ in P such that, for each $i \in \{1, \dots, m\}$, there exists $x \geq \lceil \frac{y}{2} \rceil$ such that either $a^x \in L(B_i)$ and $a^{y-x} \in L(C_i)$, or $a^{y-x} \in L(B_i)$ and $a^x \in L(C_i)$.

We want to construct a first-order formula $\forall x \forall y \psi_G(x, y)$ of signature $\mathcal{S}_\Sigma \cup \mathbf{R}$, for $\Sigma := \{a\}$ and the set of binary predicates $\mathbf{R} := \{\mathtt{Maj}_A, \mathtt{Min}_A \mid A \in N\} \cup \{\mathtt{Sum}_{BC} \mid B, C \in N\} \cup \mathbf{R}_{\mathtt{arith}}$ so that the formula $\Phi_G := \exists \mathbf{R} \forall x \forall y \psi_G$ belongs to `pred-ESO-HORN` and defines the language $a^+ \setminus L$. The intuitive meanings of the predicates $\mathtt{Maj}_A, \mathtt{Min}_A$ and $\mathtt{Sum}_{BC}$ are as follows:

- $\mathtt{Maj}_A(x, y) \iff \lceil \frac{y}{2} \rceil \leq x \leq y$ and $a^x \in L(A)$;
- $\mathtt{Min}_A(x, y) \iff \lceil \frac{y}{2} \rceil \leq x < y$ and $a^{y-x} \in L(A)$;
- $\mathtt{Sum}_{\mathtt{BC}}(x, y) \iff$ there is some x' with $\lceil \frac{y}{2} \rceil \leq x' \leq x$ such that either $a^{x'} \in L(B)$ and $a^{y-x'} \in L(C)$, or $a^{y-x'} \in L(B)$ and $a^{x'} \in L(C)$.

Note that for $x = y$, the above equivalence for $\mathtt{Maj}_A$ implies $\mathtt{Maj}_A(x, y) \iff a^y \in L(A)$.

Let us give and justify a list of Horn clauses whose conjunction ψ'_G defines the predicates $\mathtt{Maj}_A$, $\mathtt{Min}_A$ and $\mathtt{Sum}_{BC}$, using the arithmetic predicates of $\mathbf{R}_{\mathtt{arith}}$ (see Notation 2 and Remark 10), namely $R^{x=y}, R^{y=2x}$ and $R^{\rho(x,y)}$, for $\rho(x, y) := x \geq \lceil \frac{y}{2} \rceil$.

Short rules. Each rule $A \to a$ of P is expressed by the input clause:

- $\mathtt{min}(x) \wedge \mathtt{min}(y) \wedge Q_a(y) \to \mathtt{Maj}_A(x, y)$.

Induction on the length y. If we have for $y > 1$ the inequalities $\lceil \frac{y-1}{2} \rceil \leq x \leq y - 1$ and $x \geq \lceil \frac{y}{2} \rceil$ then $\lceil \frac{y}{2} \rceil \leq x \leq y$. This justifies the clause:

- $y > 1 \wedge \mathtt{Maj}_A(x, y - 1) \wedge x \geq \lceil \frac{y}{2} \rceil \to \mathtt{Maj}_A(x, y)$ for all $A \in N$.

For $y > 1$ and $y = 2x$, we have $a^x = a^{y-x}$ and $\lceil \frac{y}{2} \rceil \leq x < y$. This justifies the clause:

- $y > 1 \wedge \mathtt{Maj}_A(x, y - 1) \wedge y = 2x \to \mathtt{Min}_A(x, y)$ for all $A \in N$.

If for $x, y > 1$ we have the inequalities $\lceil \frac{y-1}{2} \rceil \leq x - 1 < y$, then $\lceil \frac{y}{2} \rceil \leq x < y$. Moreover, $a^{(y-1)-(x-1)} = a^{y-x}$. This justifies the clause:

- $x > 1 \wedge y > 1 \wedge \mathtt{Min}_A(x - 1, y - 1) \to \mathtt{Min}_A(x, y)$ for all $A \in N$.

Concatenation. For all $B, C \in N$, it is clear that the concatenation predicate $\mathtt{Sum}_{\mathtt{BC}}$ is defined inductively by the following three clauses:

- *initialization:* $\mathtt{Maj}_B(x, y) \wedge \mathtt{Min}_C(x, y) \to \mathtt{Sum}_{\mathtt{BC}}(x, y)$; $\mathtt{Min}_B(x, y) \wedge \mathtt{Maj}_C(x, y) \to \mathtt{Sum}_{\mathtt{BC}}(x, y)$;
- *induction:* $\neg \mathtt{min}(x) \wedge \mathtt{Sum}_{\mathtt{BC}}(x - 1, y) \to \mathtt{Sum}_{\mathtt{BC}}(x, y)$.

Long rules. Each rule $A \to B_1C_1\& \dots \& B_mC_m$ of P is expressed by the clause:

- $x = y \wedge \mathtt{Sum}_{\mathtt{B_1C_1}}(x, y) \wedge \dots \wedge \mathtt{Sum}_{\mathtt{B_mC_m}}(x, y) \to \mathtt{Maj}_A(x, y)$.

Thus, the formula $\forall x \forall y \psi'_G$ where ψ'_G is the conjunction of the above clauses defines the predicates $\mathtt{Maj}_A, \mathtt{Min}_A$, and $\mathtt{Sum}_{BC}$.

Definition of $a^+ \setminus L$. We have the equivalence $\mathtt{Maj}_S(n, n) \iff a^n \in L(S) \iff a^n \in L$. Therefore, the following contradiction clause expresses $a^n \notin L$:

- $\gamma_S := \mathtt{max}(x) \wedge \mathtt{max}(y) \wedge \mathtt{Maj}_S(x, y) \to \bot$.

Finally, observe that the formula $\Phi_G := \exists \mathbf{R} \forall x \forall y \psi_G$ where ψ_G is $\gamma_{\mathtt{arith}} \wedge \psi'_G \wedge \gamma_S$ and $\gamma_{\mathtt{arith}}$ is the conjunction of clauses that defines the arithmetic predicates of $\mathbf{R}_{\mathtt{arith}}$, belongs to `pred-ESO-HORN`. Since we have $\langle a^n \rangle \models \Phi_G \iff a^n \notin L$, as justified above, then the langage $a^+ \setminus L$ belongs to `pred-ESO-HORN`, as claimed. ◀

3.2 Equivalence of logic with grid-circuits

We introduce the *grid-circuit* as an intermediate object between our logic and the real-time cellular automaton: see Figure 3.

▶ **Definition 12.** *A* grid-circuit *is a tuple* $C := (\Sigma, (\texttt{Input}_n)_{n>0}, \mathbf{Q}, \mathbf{Q}_{\texttt{acc}}, \mathsf{g})$ *where*

- Σ *is the* input alphabet *and* $(\texttt{Input}_n)_{n>0}$ *is the family of* input functions $\texttt{Input}_n : \Sigma^n \times [1,n]^2 \to \Sigma \cup \{\$\}$ *such that, for* $w = w_1 \dots w_n \in \Sigma^n$, $\texttt{Input}_n(w,x,y) = w_y$ *if* $x = 1$ *and* $\texttt{Input}_n(w,x,y) = \$$ *otherwise,*
- $\mathbf{Q} \cup \{\sharp\}$ *is the finite set of* states *and* $\mathbf{Q}_{\texttt{acc}} \subseteq \mathbf{Q}$ *is the subset of* accepting states,
- $\mathsf{g} : (\mathbf{Q} \cup \{\sharp\})^2 \times (\Sigma \cup \{\$\}) \to \mathbf{Q}$ *is the* transition function.

▶ **Definition 13** (computation of a grid-circuit)**.** *The* computation C_w *of a grid-circuit* $C := (\Sigma, (\texttt{Input}_n)_{n>0}, \mathbf{Q}, \mathbf{Q}_{\texttt{acc}}, \mathsf{g})$ *on a* $w = w_1 \dots w_n \in \Sigma^n$ *is a regular grid of* $(n+1)^2$ *sites* $(x,y) \in [0,n]^2$*, each in a state* $\langle x,y \rangle \in \mathbf{Q} \cup \{\sharp\}$ *computed inductively:*

- *each site in* $\{0\} \times [0,n]$ *or* $[0,n] \times \{0\}$ *is in the particular state* $\sharp$*;*
- *the state of each site* $(x,y) \in [1,n]^2$ *is* $\langle x,y \rangle = \mathsf{g}(\langle x, y-1 \rangle, \langle x-1, y \rangle, \texttt{Input}_n(w,x,y))$.

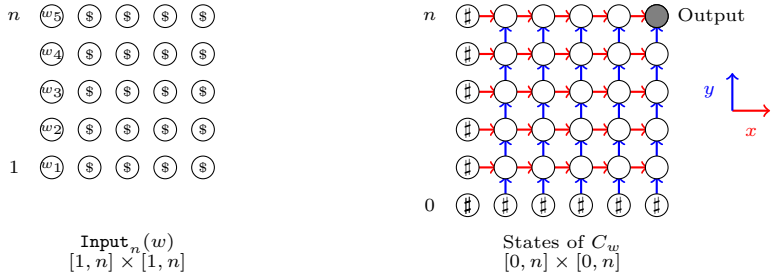

Figure 3 The grid-circuit

A word $w = w_1 \dots w_n \in \Sigma^n$ is *accepted* by the grid-circuit C if the output state $\langle n,n \rangle$ of C_w belongs to $\mathbf{Q}_{\texttt{acc}}$. The language *recognized* by C is the set of words it accepts. We denote by `Grid` the class of languages recognized by a grid-circuit.

Actually, our predecessor Horn logic is equivalent to grid-circuits.

▶ **Lemma 14.** *[10]* $\texttt{pred-ESO-HORN} = \texttt{Grid}$.

Proof. In some sense, a grid-circuit is the "normalized form" of a formula of `pred-ESO-HORN`. So, the inclusion $\texttt{Grid} \subseteq \texttt{pred-ESO-HORN}$ is proved in a straightforward way.

The first step of the proof of the converse inclusion $\texttt{pred-ESO-HORN} \subseteq \texttt{Grid}$ is to show that every formula $\Phi := \exists \mathbf{R} \forall x \forall y \psi(x,y)$ in `pred-ESO-HORN` is equivalent to a formula $\Phi' \in \texttt{pred-ESO-HORN}$ in which the only hypotheses of computation clauses are atoms $S(x,y)$ and conjunctions $S(x-1,y) \wedge x > 1$ and $S(x,y-1) \wedge y > 1$.

Elimination of atoms $R(x-a, y-b)$ for $a+b>1$: The idea is to introduce new "shift" predicates $R^{x-a',y-b'}$ for fixed integers $a', b' > 0$ with the intuitive meaning:
$R^{x-a',y-b'}(x,y) \iff R(x-a', y-b') \wedge x > a' \wedge y > b'$.
Let us explain the method by an example. Assume we have in ψ the Horn clause
(1) $x > 3 \wedge y > 2 \wedge S(x-3, y-2) \to T(x,y)$. This clause is replaced by the clause
(2) $S^{x-2,y-2}(x-1,y) \wedge x > 1 \to T(x,y)$
for which the predicates S^{x-1}, S^{x-2}, $S^{x-2,y-1}$ and $S^{x-2,y-2}$ are defined by the respective clauses: $x > 1 \wedge S(x-1,y) \to S^{x-1}(x,y)$, $x > 1 \wedge S^{x-1}(x-1,y) \to S^{x-2}(x,y)$, $y > 1 \wedge S^{x-2}(x,y-1) \to S^{x-2,y-1}(x,y)$, and $y > 1 \wedge S^{x-2,y-1}(x,y-1) \to S^{x-2,y-2}(x,y)$,

which imply together the clause $x > 2 \wedge y > 2 \wedge S(x-2, y-2) \to S^{x-2,y-2}(x,y)$ and then also $x > 3 \wedge y > 2 \wedge S(x-3, y-2) \to S^{x-2,y-2}(x-1,y)$.

It is clear that the formula $\Phi \coloneqq \exists \mathbf{R} \forall x \forall y \psi$ is equivalent to the formula $\Phi' \coloneqq \exists \mathbf{R}' \forall x \forall y \psi'$ where $\mathbf{R}' \coloneqq \mathbf{R} \cup \{S^{x-1}, S^{x-2}, S^{x-2,y-1}, S^{x-2,y-2}\}$ and ψ' is the conjunction $\psi_{\texttt{replace}} \wedge \psi_{\texttt{def}}$, where $\psi_{\texttt{replace}}$ is the formula ψ in which clause (1) is replaced by clause (2), and $\psi_{\texttt{def}}$ is the conjunction of the above clauses defining the new predicates of $\mathbf{R}'$.

Thus, any formula $\Phi \in$ `pred-ESO-HORN` is equivalent to a formula $\Phi' \in$ `pred-ESO-HORN` whose computation clauses only contain hypotheses of the following three forms: $R(x-1,y) \wedge x > 1$; $R(x,y-1) \wedge y > 1$; $R(x,y)$. The next step is to eliminate these $R(x,y)$.

Elimination of hypotheses $R(x,y)$ (sketch of proof): The first idea is to group together in each computation clause the hypothesis atoms of the form $R(x,y)$ and the conclusion of the clause. As a result, the formula can be rewritten in the form

$$\Phi \coloneqq \exists \mathbf{R} \forall x \forall y \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right]$$

where the C_i's are the input clauses and the contradiction clauses, and each computation clause is written in the form $\alpha_i(x,y) \to \theta_i(x,y)$, where $\alpha_i(x,y)$ is a conjunction of formulas of the only forms $R(x-1,y) \wedge x > 1$, $R(x,y-1) \wedge y > 1$, and $\theta_i(x,y)$ is a Horn clause in which *all* atoms are of the form $R(x,y)$.

The second idea is to "solve" the Horn clauses θ_i according to the input clauses and *all the possible* conjunctions of hypotheses α_i that may be true. Notice the two following facts: the hypotheses of the input clauses are input literals and the conjuncts of the α_i's are of the only forms $R(x-1,y) \wedge x > 1$, $R(x,y-1) \wedge y > 1$. So, we can prove by induction on the sum $x+y$ that the obtained formula Φ' in which no atom $R(x,y)$ appears as a clause hypothesis, is equivalent to the above formula Φ. The complete proof is given in Appendix A.

Transformation of the formula into a grid-circuit: Let $\mathbf{R} = \{R_1, \ldots, R_m\}$ denote the set of binary predicates of the formula. By a separation into cases of input clauses and computation clauses, it is easy to transform the formula into an equivalent formula $\Phi \coloneqq \exists \mathbf{R} \forall x \forall y \psi$ where ψ is a conjunction of clauses of the following forms (a-e), in which $s \in \Sigma$, $j \in [1,m]$, and A, B are (possibly empty) subsets of $[1,m]$:
(a) $x = 1 \wedge y = 1 \wedge Q_s(y) \to R_j(x,y)$;
(b) $x = 1 \wedge y > 1 \wedge Q_s(y) \wedge \bigwedge_{i \in A} R_i(x, y-1) \to R_j(x,y)$;
(c) $x > 1 \wedge y = 1 \wedge \bigwedge_{i \in A} R_i(x-1, y) \to R_j(x,y)$;
(d) $x > 1 \wedge y > 1 \wedge \bigwedge_{i \in A} R_i(x-1, y) \wedge \bigwedge_{i \in B} R_i(x, y-1) \to R_j(x,y)$;
(e) $x = n \wedge y = n \wedge R_j(x,y) \to \bot$.

Now, transform this formula into a grid-circuit $C \coloneqq (\Sigma, (\texttt{Input}_n)_{n>0}, \mathbf{Q}, \mathbf{Q}_{\texttt{acc}}, \mathtt{g})$. The idea is that the state of a site $(x,y) \in [1,n]^2$ is the set of predicates R_i such that $R_i(x,y)$ is true. Let $\mathbf{Q}$ be the power set of the set of $\mathbf{R}$ indices: $\mathbf{Q} \coloneqq \mathcal{P}([1,m])$. There are four types of transition (a-d) which mimic the clauses (a-d) above. These are, for $s \in \Sigma$ and $q, q' \in \mathbf{Q}$:
(a) $\mathtt{g}(\sharp, \sharp, s) = \{j \in [1,m] \mid$ there is a clause (a) with Q_s, and conclusion $R_j(x,y)\}$;
(b) $\mathtt{g}(q, \sharp, s) = \{j \in [1,m] \mid$ there is a clause (b) with Q_s, and $A \subseteq q$, and conclusion $R_j(x,y)\}$;
(c) $\mathtt{g}(\sharp, q, \$) = \{j \in [1,m] \mid$ there is a clause (c) with $A \subseteq q$, and conclusion $R_j(x,y)\}$;
(d) $\mathtt{g}(q, q', \$) = \{j \in [1,m] \mid \exists$ a clause (d) with $A \subseteq q$, $B \subseteq q'$, and conclusion $R_j(x,y)\}$.

Of course, the set of accepting states of C is determined by the contradiction clauses (e): $\mathbf{Q}_{\texttt{acc}} \coloneqq \{q \in \mathbf{Q} \mid q$ contains no j such that R_j occurs in a clause (e)$\}$.

We can easily check the equivalence, for each $w \in \Sigma^+$: $\langle w \rangle \models \Phi \iff C$ accepts w. Therefore, the inclusion `pred-ESO-HORN` $\subseteq$ `Grid` is proved. ◀

3.3 Grid-circuits are equivalent to real-time 1-CA

▶ **Lemma 15.** *[10]* `Grid` = `RealTimeCA`.

Proof. Figure 4 shows how `Grid` is simulated on `RealTimeCA` and Figure 5 shows how `RealTimeCA` is simulated on `Grid`. The proof is detailed in Appendix B. ◀

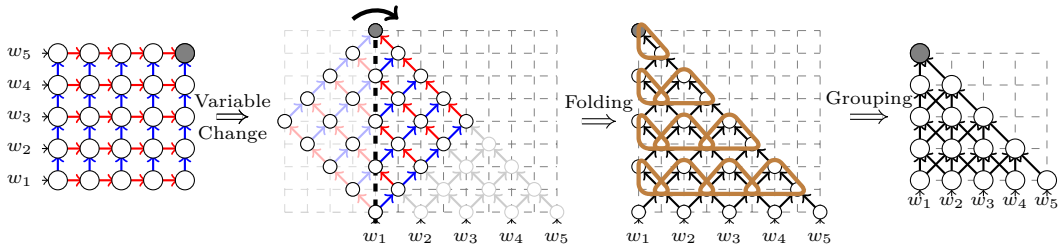

Figure 4 Simulation of `Grid` on `RealTimeCA`

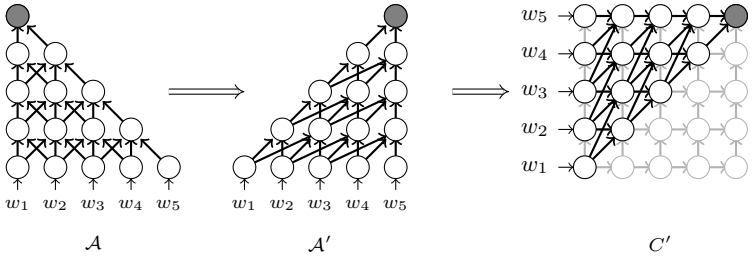

Figure 5 Simulation of `RealTimeCA` on the grid-circuit

Proof of Theorem 8. Lemmas 14 and 15 give us the following equalities of classes: `pred-ESO-HORN` = `Grid` = `RealTimeCA`. These equalities trivially hold when restricted to unary languages: $\texttt{pred-ESO-HORN}_1 = \texttt{Grid}_1 = \texttt{RealTimeCA}_1$.

From the fact that the class $\texttt{RealTimeCA}_1$ is closed under complement and from Lemma 11, we deduce $\texttt{Conj}_1 \subseteq \texttt{pred-ESO-HORN}_1 = \texttt{Grid}_1 = \texttt{RealTimeCA}_1$. ◀

4 Real-time recognition of a conjunctive language: the general case

Our second main result strengthens the inclusion `CFL` $\subseteq$ `RealTime2SOCA` of Terrier[4] [25]:

▶ **Theorem 16.** `Conj` $\subseteq$ `RealTime2OCA`.

4.1 Expressing a conjunctive language in logic: the general case

The generating process of a conjunctive language is naturally expressed in the Horn logic `incl-pred-ESO-HORN`. This is a hybrid logic with three first-order variables x, y, z, whose name means that it makes inductions on the variable interval $[x, y]$, by *inclusion*, and on the individual variable z, by *predecessor*.

[4] Recall that `RealTime2SOCA` is the class of languages recognized by *sequential* one-way two-dimensional cellular automata in real-time: this is the minimal time, $3n - 1$, for the output cell (n, n) to receive the n letters of the input word, communicated sequentially by the input cell $(1, 1)$.

▶ **Definition 17** (incl-pred-ESO-HORN). *A formula of* incl-pred-ESO-HORN *is a formula* $\Phi := \exists \mathbf{R} \forall x \forall y \forall z \psi(x,y,z)$ *where* $\mathbf{R}$ *is a finite set of ternary predicates, and* ψ *is a conjunction of Horn clauses, of signature* $\mathcal{S}_\Sigma \cup \mathbf{R} \cup \{=, \leq\}$*, and of the three following forms:*

- *an* input clause: $x = y \wedge \mathtt{min}(z) \wedge Q_s(x) \to R(x,y,z)$ *with* $s \in \Sigma$ *and* $R \in \mathbf{R}$*;*
- *a* computation clause: $\delta_1 \wedge \ldots \wedge \delta_r \to R(x,y,z)$ *with* $R \in \mathbf{R}$ *and where each hypothesis* δ_i *is an atom* $S(x,y,z)$ *or a conjunction* $S(x+a, y-b, z-c) \wedge x + a \leq y - b \wedge z > c$ *with* $S \in \mathbf{R}$ *and* $a, b, c \geq 0$ *three integers such that* $a + b + c > 0$*;*
- *a* contradiction clause: $\mathtt{min}(x) \wedge \mathtt{max}(y) \wedge \mathtt{max}(z) \wedge R(x,y,z) \to \bot$ *with* $R \in \mathbf{R}$.

We denote by incl-pred-ESO-HORN *the class of languages defined by a formula of* incl-pred-ESO-HORN.

▶ **Lemma 18.** *For each language* $L \subseteq \Sigma^+$*, if* $L \in$ Conj*, then* $\Sigma^+ \setminus L \in$ incl-pred-ESO-HORN .

Proof. The proof is a variation (an extension) of the proof of the same result, Lemma 11, in the unary case. This is why we insist on the differences. Let $G = (\Sigma, N, P, S)$ be a conjunctive grammar in binary normal form which generates L and let w be a word $w = w_1 \ldots w_n \in \Sigma^+$. For each $A \in N$ and each factor $w_{x,y} := w_x \ldots w_y$, we have, according to the length $y - x + 1$ of $w_{x,y}$, the following equivalences which will be the basis of our induction:

- if $x = y$, then $w_{x,y} \in L(A) \iff$ the short rule $A \to w_x$ belongs to P;
- if $x < y$, then $w_{x,y} \in L(A) \iff$ there is a long rule $A \to B_1C_1 \& \ldots \& B_mC_m$ in P such that, for each $i \in \{1, \ldots, m\}$, there exists $z \geq \lceil (y-x+1)/2 \rceil$ such that either $w_{x,x+z-1} \in L(B_i)$ and $w_{x+z,y} \in L(C_i)$, or $w_{x,y-z} \in L(B_i)$ and $w_{y-z+1,y} \in L(C_i)$.

Thus, a double induction is performed, on the index interval $[x,y]$ of a factor $w_{x,y}$ and on the maximal z among the lengths of the two sub-factors u, v of the m decompositions $w_{x,y} = uv$, $u \in L(B_i)$, $v \in L(C_i)$, for a long rule. This is naturally expressed in the logic incl-pred-ESO-HORN.

We want to construct a first-order formula $\forall x \forall y \forall z \psi_G$ of signature $\mathcal{S}_\Sigma \cup \mathbf{R} \cup \{=, \leq\}$, for the set of *ternary* predicates $\mathbf{R} := \{\mathtt{Pref}_A^{\mathtt{Maj}}, \mathtt{Pref}_A^{\mathtt{Min}}, \mathtt{Suff}_A^{\mathtt{Maj}}, \mathtt{Suff}_A^{\mathtt{Min}} \mid A \in N\} \cup \{\mathtt{Concat}_{BC} \mid B, C \in N\} \cup \mathbf{R}_{\mathtt{arith}}$, so that the formula $\Phi_G := \exists \mathbf{R} \forall x \forall y \forall z \psi_G$ belongs to incl-pred-ESO-HORN and defines the language $\Sigma^+ \setminus L$. The intuitive meanings of the predicates $\mathtt{Pref}_A^{\mathtt{Maj}}, \mathtt{Pref}_A^{\mathtt{Min}}, \mathtt{Suff}_A^{\mathtt{Maj}}, \mathtt{Suff}_A^{\mathtt{Min}}$ and $\mathtt{Concat}_{BC}$ are as follows:

- $\mathtt{Pref}_A^{\mathtt{Maj}}(x,y,z) \iff \left\lceil \frac{y-x+1}{2} \right\rceil \leq z \leq y - x + 1$ and $w_{x,x+z-1} \in L(A)$;
- $\mathtt{Pref}_A^{\mathtt{Min}}(x,y,z) \iff \left\lceil \frac{y-x+1}{2} \right\rceil \leq z \leq y - x$ and $w_{x,y-z} \in L(A)$;
- $\mathtt{Suff}_A^{\mathtt{Maj}}(x,y,z) \iff \left\lceil \frac{y-x+1}{2} \right\rceil \leq z \leq y - x + 1$ and $w_{y-z+1,y} \in L(A)$;
- $\mathtt{Suff}_A^{\mathtt{Min}}(x,y,z) \iff \left\lceil \frac{y-x+1}{2} \right\rceil \leq z \leq y - x$ and $w_{x+z,y} \in L(A)$;
- $\mathtt{Concat}_{BC}(x,y,z) \iff$ there is some z' with $\left\lceil \frac{y-x+1}{2} \right\rceil \leq z' \leq z$ such that either $w_{x,x+z'-1} \in L(B)$ and $w_{x+z',y} \in L(C)$, or $w_{x,y-z'} \in L(B)$ and $w_{y-z'+1,y} \in L(C)$.

Note that the above equivalences for $\mathtt{Pref}_A^{\mathtt{Maj}}$ and $\mathtt{Suff}_A^{\mathtt{Maj}}$ imply in the particular case $z = y - x + 1$ the equivalences $\mathtt{Pref}_A^{\mathtt{Maj}}(x,y,z) \iff \mathtt{Suff}_A^{\mathtt{Maj}}(x,y,z) \iff w_{x,y} \in L(A)$.

Let us give and justify a list of Horn clauses whose conjunction ψ'_G defines the predicates $\mathtt{Pref}_A^{\mathtt{Maj}}, \mathtt{Pref}_A^{\mathtt{Min}}, \mathtt{Suff}_A^{\mathtt{Maj}}, \mathtt{Suff}_A^{\mathtt{Min}}$ and $\mathtt{Concat}_{BC}$, using the arithmetic predicates $z = y - x + 1$, $y - x + 1 = 2z$, and $z \geq \left\lceil \frac{y-x+1}{2} \right\rceil$ easily defined in incl-pred-ESO-HORN.

Short rules. Each rule $A \to s$ of P is expressed by the two clauses:

- $x = y \wedge z = 1 \wedge Q_s(x) \to \mathtt{Pref}_A^{\mathtt{Maj}}(x,y,z)$; $x = y \wedge z = 1 \wedge Q_s(x) \to \mathtt{Suff}_A^{\mathtt{Maj}}(x,y,z)$.

Induction for prefixes. If we have for $x < y$ the inequalities $\left\lceil \frac{(y-1)-x+1}{2} \right\rceil \leq z \leq (y-1) - x + 1$ and $z \geq \left\lceil \frac{y-x+1}{2} \right\rceil$ then $\left\lceil \frac{y-x+1}{2} \right\rceil \leq z \leq y - x + 1$. This justifies the clause:

- $x \leq y - 1 \wedge \mathtt{Pref}_A^{\mathtt{Maj}}(x, y-1, z) \wedge z \geq \left\lceil \frac{y-x+1}{2} \right\rceil \to \mathtt{Pref}_A^{\mathtt{Maj}}(x,y,z)$, for all $A \in N$.

For $x < y$ and $y - x + 1 = 2z$, we have $w_{x,x+z-1} = w_{x,y-z}$ and $\lceil \frac{y-x+1}{2} \rceil \leq z \leq y - x$. This justifies the clause:

- $x \leq y - 1 \wedge \mathtt{Pref}_A^{\mathtt{Maj}}(x, y-1, z) \wedge y - x + 1 = 2z \to \mathtt{Pref}_A^{\mathtt{Min}}(x, y, z)$, for all $A \in N$.

For $x < y$ and $z > 1$ and $\left\lceil \frac{(y-1)-x+1}{2} \right\rceil \leq z-1 \leq (y-1)-x$, we have $\lceil \frac{y-x+1}{2} \rceil \leq z \leq y-x$. This justifies the clause:

- $x \leq y - 1 \wedge z > 1 \wedge \mathtt{Pref}_A^{\mathtt{Min}}(x, y-1, z-1) \to \mathtt{Pref}_A^{\mathtt{Min}}(x, y, z)$, for all $A \in N$.

Induction for suffixes. As this induction is symmetric to the one for prefixes, we do not justify the following list of induction clauses for the predicates $\mathtt{Suff}_A^{\mathtt{Maj}}$ and $\mathtt{Suff}_A^{\mathtt{Min}}$, $A \in N$:

- $x + 1 \leq y \wedge \mathtt{Suff}_A^{\mathtt{Maj}}(x+1, y, z) \wedge z \geq \lceil \frac{y-x+1}{2} \rceil \to \mathtt{Suff}_A^{\mathtt{Maj}}(x, y, z)$;
- $x + 1 \leq y \wedge \mathtt{Suff}_A^{\mathtt{Maj}}(x+1, y, z) \wedge y - x + 1 = 2z \to \mathtt{Suff}_A^{\mathtt{Min}}(x, y, z)$;
- $x + 1 \leq y \wedge z > 1 \wedge \mathtt{Suff}_A^{\mathtt{Min}}(x+1, y, z-1) \to \mathtt{Suff}_A^{\mathtt{Min}}(x, y, z)$.

Concatenation. For all $B, C \in N$, it is clear that the concatenation predicate $\mathtt{Concat}_{BC}$ is defined inductively by the following three clauses:

- *initialization:* $\mathtt{Pref}_B^{\mathtt{Maj}}(x, y, z) \wedge \mathtt{Suff}_C^{\mathtt{Min}}(x, y, z) \to \mathtt{Concat}_{BC}(x, y, z)$;
  $\mathtt{Pref}_B^{\mathtt{Min}}(x, y, z) \wedge \mathtt{Suff}_C^{\mathtt{Maj}}(x, y, z) \to \mathtt{Concat}_{BC}(x, y, z)$;
- *induction:* $z > 1 \wedge \mathtt{Concat}_{BC}(x, y, z-1) \to \mathtt{Concat}_{BC}(x, y, z)$.

Long rules. Each rule $A \to B_1C_1 \& \dots \& B_mC_m$ of P is expressed by the two clauses:

- $z = y - x + 1 \wedge \mathtt{Concat}_{B_1C_1}(x, y, z) \wedge \cdots \wedge \mathtt{Concat}_{B_mC_m}(x, y, z) \to \mathtt{Pref}_A^{\mathtt{Maj}}(x, y, z)$;
- $z = y - x + 1 \wedge \mathtt{Concat}_{B_1C_1}(x, y, z) \wedge \cdots \wedge \mathtt{Concat}_{B_mC_m}(x, y, z) \to \mathtt{Suff}_A^{\mathtt{Maj}}(x, y, z)$.

Thus, the formula $\forall x \forall y \forall z \psi'_G$ where ψ'_G is the conjunction of the above clauses defines the predicates $\mathtt{Pref}_A^{\mathtt{Maj}}, \mathtt{Pref}_A^{\mathtt{Min}}, \mathtt{Suff}_A^{\mathtt{Maj}}, \mathtt{Suff}_A^{\mathtt{Min}}$, and $\mathtt{Concat}_{BC}$.

Definition of $\Sigma^+ \setminus L$. We have the equivalence $\mathtt{Pref}_S^{\mathtt{Maj}}(1, n, n) \iff w \in L(S) \iff w \in L$. Therefore, the following contradiction clause expresses $w \notin L$:

- $\gamma_S \coloneqq \mathtt{min}(x) \wedge \mathtt{max}(y) \wedge \mathtt{max}(z) \wedge \mathtt{Pref}_S^{\mathtt{Maj}}(x, y, z) \to \bot$.

Finally, observe that the formula $\Phi_G \coloneqq \exists \mathbf{R} \forall x \forall y \forall z \psi_G$ where ψ_G is $\gamma_{\mathtt{arith}} \wedge \psi'_G \wedge \gamma_S$ and $\gamma_{\mathtt{arith}}$ is the conjunction of clauses that define the arithmetic predicates, belongs to `incl-pred-ESO-HORN`. Since we have $\langle w \rangle \models \Phi_G \iff w \notin L$, as justified above, then the langage $\Sigma^+ \setminus L$ belongs to `incl-pred-ESO-HORN`, as claimed. ◀

4.2 Equivalence of logic with cube-circuits

We now introduce the *cube-circuit*, an extension of the grid-circuit to three dimensions. It will make the link between our logic `incl-pred-ESO-HORN` and the class `RealTime2OCA`.

▶ **Definition 19.** *A* cube-circuit *is a tuple* $C \coloneqq (\Sigma, (\mathtt{Input}_n)_{n>0}, \mathbf{Q}, \mathbf{Q}_{\mathtt{acc}}, \mathtt{g})$ *where*

- Σ *is the* input alphabet *and* $(\mathtt{Input}_n)_{n>0}$ *is the family of* input functions $\mathtt{Input}_n : \Sigma^n \times [1, n]^3 \to \Sigma \cup \{\$\}$ *such that, for* $w = w_1 \dots w_n \in \Sigma^n$,
  $\mathtt{Input}_n(w, x, y, z) = w_y$ *if* $x = y$ *and* $z = 1$, *and* $\mathtt{Input}_n(w, x, y, z) = \$$ *otherwise,*
- $\mathbf{Q} \cup \{\sharp\}$ *is the finite set of* states *and* $\mathbf{Q}_{\mathtt{acc}} \subseteq \mathbf{Q}$ *is the subset of* accepting states,
- $\mathtt{g} : (\mathbf{Q} \cup \{\sharp\})^3 \times (\Sigma \cup \{\$\}) \to \mathbf{Q}$ *is the* transition function.

▶ **Definition 20** (computation of a cube-circuit)**.** *The* computation C_w *of a cube-circuit* $C \coloneqq (\Sigma, (\mathtt{Input}_n)_{n>0}, \mathbf{Q}, \mathbf{Q}_{\mathtt{acc}}, \mathtt{g})$ *on a word* $w = w_1 \dots w_n \in \Sigma^n$ *is a grid of* $(n+1)^3$ *sites* $(x, y, z) \in [1, n+1] \times [0, n]^2$, *each in a state* $\langle x, y, z \rangle \in \mathbf{Q} \cup \{\sharp\}$ *computed inductively:*

- *each site* (x, y, z) *such that* $x > y$ *or* $z = 0$ *is in the state* $\sharp$*;*

- *the state of each site* $(x, y, z) \in [1, n]^3$ *such that* $x \leq y$ *and* $z > 0$ *is* $\langle x, y\rangle = \mathtt{g}(\langle x+1, y, z\rangle, \langle x, y-1, z\rangle, \langle x, y, z-1\rangle, \mathtt{Input}_n(w, x, y, z))$.

A word $w = w_1 \dots w_n \in \Sigma^n$ is *accepted* by the cube-circuit C if the output state $\langle 1, n, n\rangle$ of C_w belongs to $\mathbf{Q}_{\mathtt{acc}}$. The language *recognized* by C is the set of words it accepts. We denote by `Cube` the class of languages recognized by a cube-circuit.

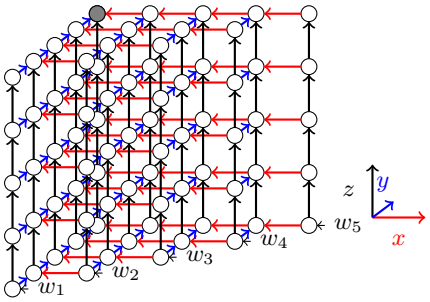

Figure 6 The cube-circuit

Actually, the logic `incl-pred-ESO-HORN` is equivalent to cube-circuits.

▶ **Lemma 21.** `incl-pred-ESO-HORN` = `Cube`.

Proof. The proof is similar to that of Lemma 14. The cube-circuit can be seen as the "normalized form" of a formula of `incl-pred-ESO-HORN`, proving the inclusion `Cube` ⊆ `incl-pred-ESO-HORN`. The proof of inverse inclusion is divided into the same three steps as for Lemma 14, which are easily adaptable to three variables: elimination of atoms $R(x+a, y-b, z-c)$ for $a+b+c > 1$, elimination of hypotheses $R(x, y, z)$ and transformation of the resulting formula into a cube-circuit. ◀

4.3 Cube-circuits are equivalent to real-time 2-OCA

One observes that by a one-to-one transformation, the computation C_w of a cube-circuit C on a word w is nothing else than the space-time diagram of a real-time 2-OCA on the input w. This yields:

▶ **Lemma 22.** `Cube` = `RealTime2OCA`.

Proof. The bijection between the sites (x, y, z) of the computation C_w of a cube-circuit C on a word w and the sites (c_1, c_2, t) of the space-time diagram of a real-time 2-OCA on the input w is depicted in Figure 7. We check that this bijection respects the communication scheme and the input/output sites of both computation models as shown in Figure 7. By this transformation, the transition function $\mathtt{g}$ of the cube-circuit, which is $\langle x, y, z\rangle = \mathtt{g}(\langle x+1, y, z\rangle, \langle x, y-1, z\rangle, \langle x, y, z-1\rangle, \mathtt{Input}_n(w, x, y, z))$ becomes the transition function $\mathtt{f}$ of the 2-OCA: $\langle c_1, c_2, t\rangle = \mathtt{f}(\langle c_1, c_2, t-1\rangle, \langle c_1-1, c_2, t-1\rangle, \langle c_1, c_2-1, t-1\rangle)$, and vice versa.

◀

Proof of Theorem 16. Lemmas 21 and 22 give us the following equalities of classes: `incl-pred-ESO-HORN` = `Cube` = `RealTime2OCA`.

From the fact that the class `RealTime2OCA` is closed under complement and from Lemma 18, we deduce `Conj` ⊆ `incl-pred-ESO-HORN` = `Cube` = `RealTime2OCA`. ◀

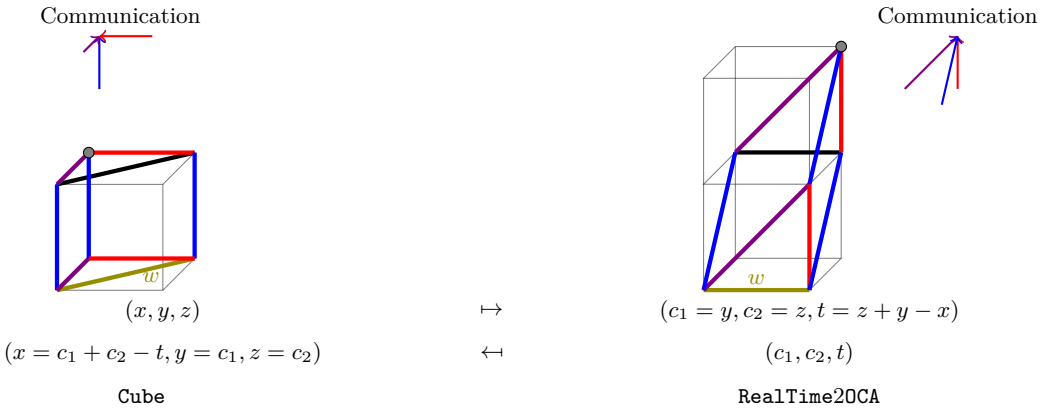

Figure 7 Bijection between the sites of C_w and the space-time sites of a 2-OCA on w

5 Conclusion

We have proved the inclusions $\texttt{Conj}_1 \subseteq \texttt{RealTimeCA}$ and $\texttt{Conj} \subseteq \texttt{RealTime2OCA}$ by expressing in two logics (proved equivalent to RealTimeCA and RealTime2OCA, respectively) the inductive process of a conjunctive grammar. Figure 8 recapitulates the known inclusions between the language classes that we have considered here. To grasp the expressive power of the Conj (resp. $\texttt{Conj}_1$) class, it would be important to obtain exact characterizations of this class in logic and/or computational complexity.

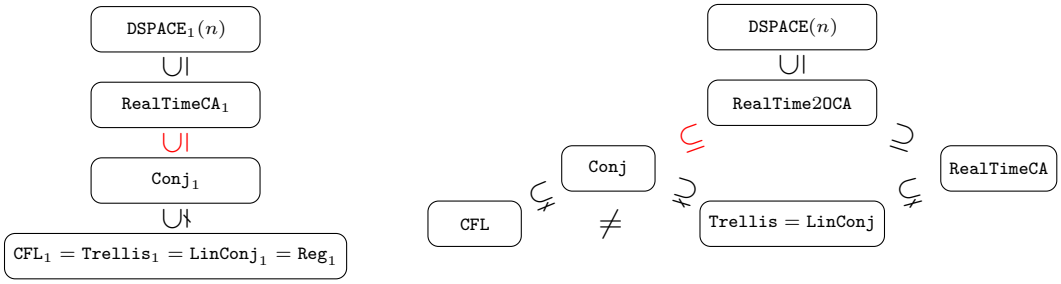

Figure 8 Relations between language classes over a unary or general alphabet

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Appendix A: Complement of proof for Lemma 14

Elimination of hypotheses $R(x,y)$: The first idea is to group together in each computation clause the hypothesis atoms of the form $R(x,y)$ and the conclusion of the clause. Accordingly, the formula obtained Φ can be rewritten in the form

$$\Phi \coloneqq \exists \mathbf{R} \forall x \forall y \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right]$$

where the C_i's are the input clauses and the contradiction clause and each computation clause is written in the form $\alpha_i(x,y) \to \theta_i(x,y)$ where $\alpha_i(x,y)$ is a conjunction of formulas of the only forms $R(x-1,y) \wedge \neg \texttt{min}(x)$, $R(x,y-1) \wedge \neg \texttt{min}(y)$ (*but not* $R(x,y)$), and $\theta_i(x,y)$ is a Horn clause whose *all* atoms are of the form $R(x,y)$.

We number $R_1, \ldots, R_m$ the computation predicates of $\mathbf{R}$. To each subset $J \subseteq [1,k]$ of the family of implications $(\alpha_i(x,y) \to \theta_i(x,y))_{i \in [1,k]}$ let us associate the set

$$K_J \coloneqq \{h \in [1,m] \mid \textstyle\bigwedge_{i \in J} \theta_i(x,y) \to R_h(x,y) \text{ is a tautology}\}.$$

Note that the notion of *tautology* used in the definition of K_J is "propositional" because all the atoms involved are of the form $R_i(x,y)$, i.e., refer to the same pair of variables (x,y). Also, note that the function $J \mapsto K_J$ is *monotonic*: for $J' \subseteq J$, we have $K_{J'} \subseteq K_J$ because $\bigwedge_{i \in J'} \theta_i(x,y) \to R_h(x,y)$ implies $\bigwedge_{i \in J} \theta_i(x,y) \to R_h(x,y)$.

Clearly, it is enough to prove the following claim:

▷ Claim 23. The formula Φ is equivalent to the following formula Φ', whose clauses have *no hypothesis* $R(x,y)$.

$$\Phi' \coloneqq \exists \mathbf{R} \forall x \forall y \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{J \subseteq [1,k]} \bigwedge_{h \in K_J} \left(\bigwedge_{i \in J} \alpha_i(x,y) \to R_h(x,y) \right) \right]$$

Proof of the implication $\Phi \Rightarrow \Phi'$: It is enough to prove the implication

$$\left[\bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right] \to \left[\bigwedge_{i \in J} \alpha_i(x,y) \to \bigwedge_{h \in K_J} R_h(x,y) \right]$$

for all set $J \subseteq [1,k]$. The implication to be proved can be equivalently written:

$$\left[\bigwedge_{i \in J} \alpha_i(x,y) \wedge \bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right] \to \bigwedge_{h \in K_J} R_h(x,y).$$

The sub-formula between brackets above implies the conjunction $\bigwedge_{i \in J} \theta_i(x,y)$. As the implication $\bigwedge_{i \in J} \theta_i(x,y) \to \bigwedge_{h \in K_J} R_h(x,y)$ is a tautology (by definition of K_J), the implication to be proved is a tautology too.

The converse implication $\Phi' \Rightarrow \Phi$ is more difficult to prove. It uses a folklore property of propositional Horn formulas easy to be proved:

▶ **Lemma 24** (Horn property: folklore). *Let F be a* strict Horn formula *of propositional calculus, that is a conjunction of clauses of the form $p_1 \wedge \ldots \wedge p_k \to p_0$ where $k \geq 0$ and the*

p_i's are propositional variables. Let F' be the conjunction of propositional variables q such that the implication $F \to q$ is a tautology. F has the same minimal model [5] as F'.

Proof of the implication $\Phi' \Rightarrow \Phi$: Let $\langle w \rangle$ be a model of Φ' and let $(\langle w \rangle, \mathbf{R})$ be the minimal model of the Horn formula

$$\varphi' \coloneqq \forall x \forall y \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{J \subseteq [1,k]} \bigwedge_{h \in K_J} \left(\bigwedge_{i \in J} \alpha_i(x,y) \to R_h(x,y) \right) \right].$$

It is enough to show that $(\langle w \rangle, \mathbf{R})$ also satisfies the formula

$$\varphi \coloneqq \forall x \forall y \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right].$$

As each α_i is a conjunction of formulas of the form $R(x-1,y) \wedge \neg\mathtt{min}(x)$, or $R(x,y-1) \wedge \neg\mathtt{min}(y)$, we make an induction on the domain $\{(a,b) \in [1,n]^2 \mid a+b \leq t\}$, for $t \in [1,2n]$. More precisely, we are going to prove, by recurrence on the integer $t \in [1,2n]$, that the minimal model $(\langle w \rangle, \mathbf{R})$ of φ' satisfies the "relativized" formula φ_t of the formula φ defined by

$$\varphi_t \coloneqq \forall x \forall y \left[x+y \leq t \to \left[\bigwedge_i C_i(x,y) \wedge \bigwedge_{i \in [1,k]} (\alpha_i(x,y) \to \theta_i(x,y)) \right] \right]$$

As the hypothesis $x+y \leq 2n$ holds for all x, y in the domain $[1,n]$, φ_{2n} is equivalent to φ on the structure $(\langle w \rangle, \mathbf{R})$.

Basis case: For $t=1$ the set $\{(a,b) \in [1,n]^2 \mid a+b \leq t\}$ is empty so that the "relativized" formula φ_1 is trivially true in the minimal model $(\langle w \rangle, \mathbf{R})$ of φ'.

Recurrence step: Suppose $(\langle w \rangle, \mathbf{R}) \models \varphi_{t-1}$, for an integer $t \in [2,2n]$. It is enough to show that, for each couple $(a,b) \in [1,n]^2$ such that $a+b=t$, we have $(\langle w \rangle, \mathbf{R}) \models \bigwedge_{i \in [1,k]} (\alpha_i(a,b) \to \theta_i(a,b))$. Let $J_{a,b}$ be the set of indices $i \in [1,k]$ such that the couple (a,b) satisfies α_i:

$$J_{a,b} \coloneqq \{i \in [1,k] \mid (\langle w \rangle, \mathbf{R}) \models \alpha_i(a,b)\}.$$

Recall that each $\alpha_i(a,b)$ is a (possibly empty) conjunction of atoms $R(a',b')$ with $(a',b') = (a-1,b)$ or $(a',b') = (a,b-1)$, therefore such that $a'+b' = t-1$. Let $J \subseteq [1,k]$ be any set. Let us examine the two possible cases:

1) $J \subseteq J_{a,b}$: then the conjunction $\bigwedge_{i \in J} \alpha_i(a,b)$ holds in $(\langle w \rangle, \mathbf{R})$; hence, in $(\langle w \rangle, \mathbf{R})$, the conjunction $\bigwedge_{h \in K_J} (\bigwedge_{i \in J} \alpha_i(a,b) \to R_h(a,b))$ is equivalent to $\bigwedge_{h \in K_J} R_h(a,b)$;

2) $J \setminus J_{a,b} \neq \emptyset$: then the conjunction $\bigwedge_{i \in J} \alpha_i(a,b)$ is false in $(\langle w \rangle, \mathbf{R})$; hence, the conjunction $\bigwedge_{h \in K_J} (\bigwedge_{i \in J} \alpha_i(a,b) \to R_h(a,b))$ holds in $(\langle w \rangle, \mathbf{R})$.

From (1) and (2), we deduce that in $(\langle w \rangle, \mathbf{R})$ the conjunction $\bigwedge_{J \subseteq [1,k]} \bigwedge_{h \in K_J} (\bigwedge_{i \in J} \alpha_i(a,b) \to R_h(a,b))$ is equivalent to the conjunction $\bigwedge_{J \subseteq J_{a,b}} \bigwedge_{h \in K_J} R_h(a,b)$, which can be simplified as $\bigwedge_{h \in K_{J_{a,b}}} R_h(a,b)$ because $J \subseteq J_{a,b}$ implies $K_J \subseteq K_{J_{a,b}}$. Consequently, for all $h \in [1,m]$, the minimal model $(\langle w \rangle, \mathbf{R})$ of the Horn formula φ' satisfies the atom $R_h(a,b)$ *iff* h belongs to $K_{J_{a,b}}$. By definition,

[5] For example, for $F \coloneqq p_1 \wedge p_3 \wedge (p_1 \wedge p_3 \to p_5) \wedge (p_1 \wedge p_2 \to p_4)$, we have $F' \coloneqq p_1 \wedge p_3 \wedge p_5$, which has the same minimal model I as F; this model is given by $I(p_1) = I(p_3) = I(p_5) = 1$ and $I(p_2) = I(p_4) = 0$.

$$K_{J_{a,b}} \coloneqq \{h \in [1,m] \mid \bigwedge_{i\in J_{a,b}} \theta_i(x,y) \to R_h(x,y) \text{ is a tautology}\}$$

or, equivalently,

$$K_{J_{a,b}} \coloneqq \{h \in [1,m] \mid \bigwedge_{i\in J_{a,b}} \theta_i(a,b) \to R_h(a,b) \text{ is a tautology}\}.$$

As a consequence of Lemma 24, the two conjunctions

$$\bigwedge_{i\in J_{a,b}} \theta_i(a,b) \text{ and } \bigwedge_{h\in K_{J_{a,b}}} R_h(a,b)$$

have the same minimal model, which is also the restriction of the minimal model $(\langle w\rangle, \mathbf{R})$ of φ' to the set of atoms $R_h(a,b)$, for $h \in [1,m]$. Therefore, if $i \in J_{a,b}$, then $(\langle w\rangle, \mathbf{R}) \models \theta_i(a,b)$. If $i \in [1,k] \setminus J_{a,b}$, then we have $(\langle w\rangle, \mathbf{R}) \models \neg\alpha_i(a,b)$, by definition of $J_{a,b}$. Therefore, for all $i \in [1,k]$, we get $(\langle w\rangle, \mathbf{R}) \models \neg\alpha_i(a,b) \vee \theta_i(a,b)$. In other words, for all (a,b) such that $a+b=t$, we have : $(\langle w\rangle, \mathbf{R}) \models \bigwedge_{i\in[1,k]}(\alpha_i(a,b) \to \theta_i(a,b))$ and then $(\langle w\rangle, \mathbf{R}) \models \varphi_t$.

This concludes the inductive proof that $(\langle w\rangle, \mathbf{R}) \models \varphi_t$, for all $t \in [1,2n]$, and then $\langle w\rangle \models \Phi$. This proves the converse implication $\Phi' \Rightarrow \Phi$. Claim 23 is demonstrated. □

Appendix B: Complement of proof for Lemma 15

`Grid` $\subseteq$ `RealTimeCA`. To prove this inclusion, we show how to simulate the computation of the grid-circuit on a real-time CA. The simulation is made by a geometric transformation that embeds the grid-circuit in the space-time diagram of a real-time CA. This transformation is divided into three steps:

1. a variable change: we apply to each site $(x,y) \in [1,n]^2$ of the grid-circuit the variable change $(x,y) \mapsto (c' = y - x + 1, t' = x + y - 1)$;
2. a folding: we fold the resulting diagram along the axis $c' = 1$: each site (c',t') with $c' < 1$ is send to its symmetric counterpart $(-c'+1, t')$;
3. a grouping: each site $(c,t) = (\lceil \frac{c'}{2} \rceil, \lceil \frac{t'}{2} \rceil)$ of the new diagram records the set of sites $\{(c'-1,t'-1),(c',t'),(c'+1,t'-1)\}$ with c' and t' odd and greater than 1.

The resulting diagram is the expected space-time diagram of a real-time CA, proving the inclusion.

`RealTimeCA` $\subseteq$ `Grid`. To simulate a real-time CA $\mathcal{A} = (\mathtt{S}, \mathtt{S}_{accept}, \{-1,0,1\}, \mathtt{f})$ on the grid, we first turn $\mathcal{A}$ into an equivalent CA $\mathcal{A}' = (\mathtt{S}, \mathtt{S}_{accept}, \{-2,-1,0\}, \mathtt{f})$. This transformation can be seen as the variable change $(c,t) \mapsto (c+t-1, t)$. The diagram of $\mathcal{A}'$ is then embedded on the grid-circuit C' by applying to its sites (c',t') the variable change $(c',t') \mapsto (t',c')$. The local and uniform communication of the embedded diagram can easily be carried out by the grid-circuit communication scheme.