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► **To cite this version:**

Yogesh J. Bagul, Ramkrishna M. Dhaigude, Christophe Chesneau, Marko Kostić. Tight exponential bounds for hyperbolic tangent. 2021. hal-03167430

**HAL Id: hal-03167430**

**<https://hal.science/hal-03167430>**

Preprint submitted on 12 Mar 2021

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# TIGHT EXPONENTIAL BOUNDS FOR HYPERBOLIC TANGENT

YOGESH J. BAGUL, RAMKRISHNA M. DHAIGUDE, CHRISTOPHE CHESNEAU,  
AND MARKO KOSTIĆ

ABSTRACT. We aim to obtain very tight exponential bounds for hyperbolic tangent by first establishing simple algebraic bounds for this function. Our inequalities refine a double inequality for  $\tanh x$  recently proved by Zhang and Chen. In addition, a graphical analysis is carried out, and a number of auxiliary lemmas may be of use on their own.

## 1. INTRODUCTION AND MAIN RESULTS

Hyperbolic tangent function usually denoted by  $\tanh$  occurs in many branches of mathematics, exact sciences and engineering and it is an important sigmoidal function. Sharp and tractable bounds of  $\tanh$  can be therefore useful in the fields of concern. Even so, very little can be found related to the bounds of this function in the literature. For instance, L. Zhu in [10] proved the following inequality:

$$\left(\frac{r^2 - x^2}{r^2 + x^2}\right)^\beta \leq \frac{\tanh x}{x} \leq \left(\frac{r^2 - x^2}{r^2 + x^2}\right)^\alpha, \quad 0 < x < r, \quad (1.1)$$

where  $\alpha \leq 0$  and  $\beta \geq r^2/6$ . Also, for all  $x \in (0, \infty)$ , the inequalities

$$\frac{1}{1 + 2 \log\left(\frac{\sinh x}{x}\right)} < \frac{\tanh x}{x} < \frac{1}{1 + \log\left(\frac{\sinh x}{x}\right)} \quad (1.2)$$

appeared in [4] and [6]. It can be shown that the inequalities in (1.1) are weaker than those in (1.2). On his side, Bhayo et al. [5] obtained a sharp algebraic bound for  $\tanh x$  as follows:

$$\tanh x < \frac{2x}{\sqrt{4x^2 + 9} - 1}, \quad x > 0. \quad (1.3)$$

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2010 *Mathematics Subject Classification.* 26A09, 26D07, 33B10.

*Key words and phrases.* Hyperbolic tangent, sigmoidal function, hyperbolic cosine, exponential bounds.

Very recently, Zhang and Chen [9] proposed an alternative by proving the following double inequality:

$$\sqrt{1 - \exp\left(-\frac{x^2}{\sqrt{x^2 + 1}}\right)} < \tanh x < \sqrt[3]{1 - \exp\left(-\frac{x^3}{\sqrt{x^3 + 1}}\right)}, \quad x > 0. \quad (1.4)$$

The bounds in (1.3) and (1.4) are tighter than the corresponding bounds in (1.1) and (1.2). Needless to say that the algebraic bounds are better over transcendental bounds due to their computational efficiency and hence an upper bound of  $\tanh x$  in (1.3) is better. However, in terms of sharpness the inequality (1.4) is strongest of all the inequalities listed above except in  $(0, \zeta)$ ,  $\zeta \approx 1.557$ , where an upper bound of  $\tanh x$  in (1.3) is the best. In this paper, we aim to refine the inequalities (1.3) and (1.4). Our main results are presented in the following theorems.

**Theorem 1.** *For  $x > 0$ , the inequalities*

$$\begin{aligned} \sqrt{1 - \exp\left\{\frac{(15)^{5/7}}{2} [(15)^{2/7} - (7x^2 + 15)^{2/7}]\right\}} &< \tanh x \\ &< \sqrt{1 - \exp\left\{\frac{2\sqrt{15}}{7} (\sqrt{15} - \sqrt{7x^2 + 15})\right\}} \end{aligned} \quad (1.5)$$

*hold.*

**Theorem 2.** *The inequalities*

$$\sqrt{1 - \exp\left[2\left(1 - \sqrt{1 + x^2}\right)\right]} < \tanh x < \sqrt{1 - \exp\left[3\left(1 - \sqrt{1 + \frac{2}{3}x^2}\right)\right]} \quad (1.6)$$

*are fulfilled for  $x > 0$ .*

We claim that the inequalities in (1.5) and (1.6) are very tight and they are clear refinements of the inequalities in (1.4). In order to support this claim,

- we point out that

$$1 - \sqrt{1 + x^2} = -\frac{x^2}{1 + \sqrt{1 + x^2}} \leq -\frac{x^2}{2\sqrt{1 + x^2}},$$

implying that the lower bound in (1.6) is uniformly better than the one in (1.4).

- a graphical analysis is performed for comparison of the involved lower and upper bounds in (1.4), (1.5) and (1.6) in Figures 1 and 2, respectively.

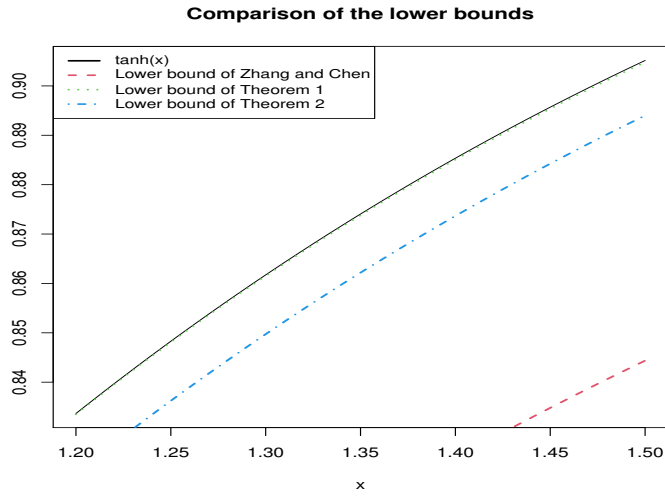


FIGURE 1. Selected plot of  $\tanh(x)$  and the presented lower bounds; the one by Zhang and Chen [9] and the two candidates in Theorems 1 and 2.

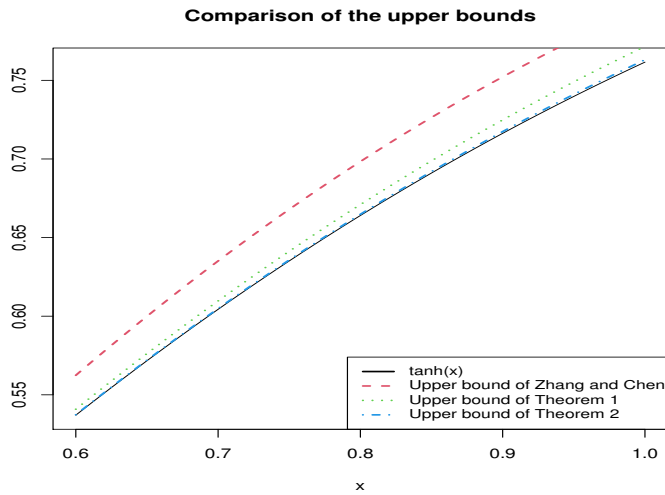


FIGURE 2. Selected plot of  $\tanh(x)$  and the presented upper bounds; the one by Zhang and Chen [9] and the two candidates in Theorems 1 and 2.

Note that, in Figure 1, the mentioned lower bounds are plotted for  $x \in (1.2, 1.5)$  for a zoom reason. In this setting, we can see that the black and green curves, which correspond to  $\tanh x$  and the lower bound of (1.5), are almost confounded. The related upper bounds are displayed in Figure 2 for  $x \in (0.6, 1)$ , still for a zoom reason. In this figure, the upper bound of (1.6) is the best.

In fact, with the help of any graphing calculator it can be found that the double inequality (1.6) is a refinement of (1.5) except in the interval  $(0, \gamma)$ , with  $\gamma \approx 4.279$ , where the lower bound of (1.5) dominates the corresponding lower bound of (1.6).

In the last section, while obtaining the proofs of Theorems 1 and 2 we also establish sharp exponential bounds for hyperbolic cosine function. The proofs of main results will be based on auxiliary results in Section 2. These auxiliary results give us simple algebraic bounds for  $\tanh x$ , which may be of independent interest. Section 3 contains the proofs of the main results. As a complementary study, a discussion on the sharpness of (1.3) near the point zero is finally given in Section 4.

## 2. LEMMAS

We begin by presenting some well-known results in Lemmas 1 and 2.

**Lemma 1.** *[1, p.10] Let  $f, g : [m, n] \rightarrow \mathbb{R}$  be two continuous functions which are differentiable on  $(m, n)$  and  $g'(x) \neq 0$  for  $x \in (m, n)$ . If  $f'(x)/g'(x)$  is increasing (or decreasing) on  $(m, n)$ , then the ratio functions  $[f(x) - f(m)]/[g(x) - g(m)]$  and  $[f(x) - f(n)]/[g(x) - g(n)]$  are also increasing (or decreasing) on  $(m, n)$ . If  $f'(x)/g'(x)$  is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.** *([8]) Let  $A(x) = \sum_{k=0}^{\infty} a_k x^k$  and  $B(x) = \sum_{k=0}^{\infty} b_k x^k$  be convergent for  $|x| < R$ , where  $a_k$  and  $b_k$  are real numbers for  $k = 0, 1, 2, \dots$  such that  $b_k > 0$ . If the sequence  $a_k/b_k$  is strictly increasing (or decreasing), then the function  $A(x)/B(x)$  is also strictly increasing (or decreasing) on  $(0, R)$ .*

Besides, we state and prove following important lemmas.

**Lemma 3.** *For  $a > 0$ , define the function*

$$f_a(x) = \frac{\log\left(\frac{\tanh x}{x}\right)}{\log\left(\frac{a}{a+x^2}\right)}, \quad x > 0.$$

*Then the function  $f_a(x)$  is strictly decreasing over  $(0, \infty)$  if  $a \geq 15/7$ .*

*Proof.* Let us decompose  $f_a(x)$  as

$$f_a(x) = \frac{\log\left(\frac{\tanh x}{x}\right)}{\log\left(\frac{a}{a+x^2}\right)} := \frac{g(x)}{g_a(x)}, \quad x > 0,$$

where  $g(x) = \log(\tanh x/x)$  and  $g_a(x) = \log[a/(a+x^2)]$  with  $g(0+) = 0 = g_a(0)$ . Upon differentiation, it comes

$$\frac{g'(x)}{g_a'(x)} = \frac{(a+x^2)\tanh x - x \operatorname{sech}^2 x}{2x^2 \tanh x} = \frac{(a+x^2)\sinh 2x - 2x}{2x^2 \sinh 2x} := \frac{1}{2} \frac{L(t)}{M(t)},$$

where  $L(t) = (4a+t^2)(\sinh t - t)$ ,  $M(t) = t^2 \sinh t$  and  $t = 2x$ . Using the well-established series expansion of  $\sinh t$ , we can write

$$\begin{aligned} L(t) &= (4a+t^2) \sum_{k=1}^{\infty} \frac{t^{2k+1}}{(2k+1)!} = \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} \frac{t^{2k+3}}{(2k+1)!} \\ &= -t^3 + \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{t^{2k+3}}{(2k+1)!} \end{aligned}$$

and

$$M(t) = \sum_{k=0}^{\infty} \frac{t^{2k+3}}{(2k+1)!}.$$

Therefore

$$\frac{L(t)}{M(t)} := 1 + \frac{A(t)}{B(t)},$$

where

$$\begin{aligned} A(t) &= -t^3 + \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+1}}{(2k+1)!} = -t^3 + \sum_{k=0}^{\infty} \frac{4a \cdot t^{2k+3}}{(2k+3)!} \\ &= \left(\frac{2a}{3} - 1\right) t^3 + \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+3}}{(2k+3)!} := \sum_{k=0}^{\infty} a_k t^{2k+3} \end{aligned}$$

and

$$B(t) = M(t) = \sum_{k=0}^{\infty} \frac{t^{2k+3}}{(2k+1)!} = t^3 + \sum_{k=1}^{\infty} \frac{t^{2k+3}}{(2k+1)!} := \sum_{k=0}^{\infty} b_k t^{2k+3}.$$

Here  $a_0 = 2a/3 - 1$ ,  $b_0 = 1$ ,  $a_k = 4a/(2k+3)!$ ,  $b_k = 1/(2k+1)!$ ,  $k \geq 1$  with  $k \in \mathbb{N}$ . Consider

$$\frac{a_k}{b_k} = \frac{2a}{(k+1)(2k+3)}, \quad k \geq 1.$$

Clearly  $\{a_k/b_k\}_{k=1}^{\infty}$  is a strictly decreasing sequence. By Lemma 2,  $A(t)/B(t)$  and hence  $L(t)/M(t)$  or  $g'(x)/g'_a(x)$  will be strictly decreasing if

$$\frac{a_0}{b_0} \geq \frac{a_1}{b_1},$$

which is equivalent to  $a \geq 15/7$ . Therefore, by Lemma 1,  $f_a(x)$  is strictly decreasing if  $a \geq 15/7$ . This completes the proof.  $\square$

**Corollary 1.** *Let  $x > 0$  and  $a \geq 15/7$ . Then the best possible constants such that*

$$\left(\frac{a}{a+x^2}\right)^{\alpha} < \frac{\tanh x}{x} < \left(\frac{a}{a+x^2}\right)^{\beta} \quad (2.1)$$

are  $\alpha = a/3$  and  $\beta = 1/2$ .

*Proof.* Since the function  $f_a(x)$  defined in Lemma 3 is strictly decreasing on  $(0, \infty)$  for  $a \geq 15/7$ , we have

$$f_a(0+) > f_a(x) > f_a(\infty), \quad x > 0.$$

The limits  $f_a(0+) = a/3$  and  $f_a(\infty) = 1/2$  give the required inequalities in (2.1).  $\square$

By putting  $a = 3$  in (2.1) we get the inequalities

$$\frac{3}{3+x^2} < \frac{\tanh x}{x} < \left(\frac{3}{3+x^2}\right)^{1/2}, \quad x > 0. \quad (2.2)$$

The left inequality of (2.2) already appeared in [2]. Similarly, by putting  $a = 15/7$  we obtain

$$\left(\frac{15}{15+7x^2}\right)^{5/7} < \frac{\tanh x}{x} < \left(\frac{15}{15+7x^2}\right)^{1/2}, \quad x > 0. \quad (2.3)$$

Inequalities (2.3) are the sharpest inequalities of kind (2.1). A graphical comparison says that the lower bound in (2.3) is finer than the corresponding lower bound in (1.4) for  $x \in (0, \varsigma)$  where  $\varsigma \approx 2.4126$ . The upper bound in (2.3) is not sharp enough, so we obtain better algebraic bounds for  $\tanh x$  in the next lemma.

**Lemma 4.** *The best possible constants  $\alpha$  and  $\beta$  such that*

$$\frac{1}{\sqrt{1+\alpha x^2}} < \frac{\tanh x}{x} < \frac{1}{\sqrt{1+\beta x^2}}, \quad x > 0 \quad (2.4)$$

are 1 and  $2/3$ , respectively.

*Proof.* Let

$$f(x) = \frac{x^2 - \tanh^2 x}{x^2 \tanh^2 x},$$

which can be written as

$$\begin{aligned} f(x) &= \frac{x^2 \cosh^2 x - \sinh^2 x}{x^2 \sinh^2 x} = \frac{x^2(1 + \cosh 2x) - (\cosh 2x - 1)}{x^2(\cosh 2x - 1)} \\ &= \frac{(x^2 + 1) + (x^2 - 1) \cosh 2x}{x^2(\cosh 2x - 1)}. \end{aligned}$$

Using a known series expansion of  $\cosh x$ , we get

$$\begin{aligned} f(x) &= \frac{(x^2 + 1) + (x^2 - 1) + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} (x^2 - 1)x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2}} \\ &= \frac{2x^2 + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2} - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2}} \\ &= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2} - \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!} x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2}} \\ &= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} \left[ 1 - \frac{2}{(k+1)(2k+1)} \right] x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2}} \\ &= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} \left[ \frac{2k^2 + 3k - 1}{2k^2 + 3k + 1} \right] x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2}} := \frac{\sum_{k=1}^{\infty} a_k x^{2k+2}}{\sum_{k=1}^{\infty} b_k x^{2k+2}}. \end{aligned}$$

Here  $a_k = (2^{2k}/(2k)!) [(2k^2 + 3k - 1)/(2k^2 + 3k + 1)]$  and  $b_k = 2^{2k}/(2k)!$ .  
Now

$$\frac{a_k}{b_k} = \frac{2k^2 + 3k - 1}{2k^2 + 3k + 1} := t_k.$$

Suppose that  $t_k < t_{k+1}$ , i.e.,

$$\frac{2k^2 + 3k - 1}{2k^2 + 3k + 1} < \frac{2k^2 + 7k + 4}{2k^2 + 7k + 6}.$$

Equivalently,

$$(2k^2 + 3k - 1)(2k^2 + 7k + 6) < (2k^2 + 3k + 1)(2k^2 + 7k + 4)$$

or after simplifying  $11k - 6 < 19k + 4$ , i.e.,  $8k + 10 > 0$ , which is obviously true for  $k \geq 1$ . Thus a sequence  $\{a_k/b_k\}$  is strictly increasing for  $k \geq 1$  and by Lemma 2, we conclude that the function  $f(x)$  is also strictly increasing for  $x > 0$ . Hence

$$f(0+) < f(x) < f(\infty).$$



Lastly, the limits  $f(0+) = 2/3$  and  $f(\infty) = 1$  prove the lemma.  $\square$

It should be noted that the upper bound in (2.4) is finer than the corresponding upper bound in (1.4) for  $x \in (0, \epsilon)$ , where  $\epsilon \approx 1.2952$ .

### 3. PROOFS OF MAIN RESULTS

In this section, we prove our main results.

**3.1. Proof of theorem 1.** Inequality (2.3) can be written as

$$t \left( \frac{15}{7t^2 + 15} \right)^{5/7} < \tanh t < t \left( \frac{15}{7t^2 + 15} \right)^{1/2}, \quad t > 0.$$

Let  $t \in (0, x)$  with  $x > 0$ . Then, an integration gives

$$\int_0^x t \left( \frac{15}{7t^2 + 15} \right)^{5/7} dt < \int_0^x \tanh t dt < \int_0^x t \left( \frac{15}{7t^2 + 15} \right)^{1/2} dt$$

i.e.,

$$\frac{(15)^{5/7}}{14} \int_0^x (7t^2 + 15)^{-5/7} \cdot 14t dt < \log(\cosh x) < \frac{\sqrt{15}}{14} \int_0^x \frac{1}{\sqrt{7t^2 + 15}} \cdot 14t dt$$

or

$$\frac{(15)^{5/7}}{4} \left[ (7x^2 + 15)^{2/7} - (15)^{2/7} \right] < \log(\cosh x) < \frac{\sqrt{15}}{7} \left( \sqrt{7x^2 + 15} - \sqrt{15} \right)$$

i.e.,

$$\begin{aligned} \exp \left\{ \frac{(15)^{5/7}}{4} \left[ (7x^2 + 15)^{2/7} - (15)^{2/7} \right] \right\} &< \cosh x \\ &< \exp \left\{ \frac{\sqrt{15}}{7} \left( \sqrt{7x^2 + 15} - \sqrt{15} \right) \right\}. \end{aligned}$$

By first squaring and then taking reciprocals, we have

$$\begin{aligned} \exp \left\{ \frac{(15)^{5/7}}{2} \left[ (7x^2 + 15)^{2/7} - (15)^{2/7} \right] \right\} &> \operatorname{sech}^2 x \\ &> \exp \left\{ \frac{2\sqrt{15}}{7} \left( \sqrt{7x^2 + 15} - \sqrt{15} \right) \right\}. \end{aligned}$$

Owing to the relation  $\operatorname{sech}^2 x = 1 - \tanh^2 x$ , we get

$$1 - \exp \left\{ \frac{(15)^{5/7}}{2} \left[ (15)^{2/7} - (7x^2 + 15)^{2/7} \right] \right\} < \tanh^2 x \\ 1 - \exp \left\{ \frac{2\sqrt{15}}{7} \left( \sqrt{15} - \sqrt{7x^2 + 15} \right) \right\}.$$

This gives inequality (1.5).  $\square$

**3.2. Proof of Theorem 2.** We write the inequality (2.4) as

$$\frac{t}{\sqrt{1+t^2}} < \tanh t < \frac{t}{\sqrt{1+\frac{2}{3}t^2}}, \quad t > 0.$$

Let  $t \in (0, x)$  with  $x > 0$ . Then, through an integration, it comes

$$\int_0^x \frac{1}{2\sqrt{1+t^2}} 2t \, dt < \int_0^x \tanh t \, dt < \frac{3}{4} \int_0^x \frac{\frac{4}{3}t}{\sqrt{1+\frac{2}{3}t^2}} \, dt$$

i.e.,

$$\sqrt{1+x^2} - 1 < \log(\cosh x) < \frac{3}{2} \left( \sqrt{1+\frac{2}{3}x^2} - 1 \right)$$

or

$$\exp \left( \sqrt{1+x^2} - 1 \right) < \cosh x < \exp \left[ \frac{3}{2} \left( \sqrt{1+\frac{2}{3}x^2} - 1 \right) \right].$$

By squaring and then taking reciprocals as in the proof of Theorem 1 we get the desired inequalities (1.6).  $\square$

If  $r > 0$  then, for  $x \in (0, r)$ , the inequalities

$$\exp(\lambda x^2) < \cosh x < \exp(x^2/2), \quad (3.1)$$

where  $\lambda = \log(\cosh r)/r^2$  and

$$\left( 1 + \frac{x^2}{3} \right)^{3/2} < \cosh x < \left( 1 + \frac{x^2}{3} \right)^\delta, \quad (3.2)$$

where  $\delta = \log(\cosh r)/\log(1+r^2/3)$  are proved in [3] and [7], respectively.

While giving proofs of Theorems 1 and 2, we in fact refined the inequalities (3.1) and (3.2) and obtained better exponential bounds for hyperbolic cosine. For  $x \in (0, r)$ , with  $r \rightarrow \infty$ , the refined inequalities for  $\cosh x$  are given as

$$\begin{aligned} \exp \left\{ \frac{(15)^{5/7}}{4} \left[ (7x^2 + 15)^{2/7} - (15)^{2/7} \right] \right\} &< \cosh x \\ &< \exp \left\{ \frac{\sqrt{15}}{7} \left( \sqrt{7x^2 + 15} - 15 \right) \right\} \end{aligned} \quad (3.3)$$

and

$$\exp \left( \sqrt{1 + x^2} - 1 \right) < \cosh x < \exp \left[ \frac{3}{2} \left( \sqrt{1 + \frac{2}{3}x^2} - 1 \right) \right]. \quad (3.4)$$

#### 4. ON THE INEQUALITY (1.3)

We now complete the previous study by discussing the sharpness of the inequality (1.3) near the point zero. First of all, observe that the inequality (1.3) can be written as

$$\frac{e^{2x} - 1}{e^{2x} + 1} \leq \frac{2x}{\sqrt{4x^2 + 9} - 1}, \quad x > 0,$$

i.e., as

$$\frac{e^x - 1}{e^x + 1} \leq \frac{x}{\sqrt{x^2 + 9} - 1}, \quad x > 0. \quad (4.1)$$

Putting  $t = e^x > 1$ , and observing that the mapping  $t \mapsto (t - 1)/(t + 1)$ ,  $t > 1$  is strictly increasing, it can be simply shown that the inequality (4.1) is equivalent on the interval  $(0, 4)$  with

$$e^x \leq \frac{1 + \frac{x}{\sqrt{x^2 + 9} - 1}}{1 - \frac{x}{\sqrt{x^2 + 9} - 1}} = \frac{g(x) + x}{g(x) - x}, \quad x \in [0, 4). \quad (4.2)$$

where  $g(x) := \sqrt{x^2 + 9} - 1$ ,  $x \in [0, 4)$ . Our idea is to find a differentiable function  $f : [0, 4) \rightarrow (0, \infty)$  such that  $f(0) = 2$ ,  $f(x) \geq g(x)$  for all  $x \in [0, 4)$  and

$$e^x \leq \frac{f(x) + x}{f(x) - x}, \quad x \in [0, 4).$$

Since

$$\frac{f(x) + x}{f(x) - x} \leq \frac{g(x) + x}{g(x) - x}, \quad x \in [0, 4),$$

we immediately get the following extension of (4.2):

$$e^x \leq \frac{f(x) + x}{f(x) - x} \leq \frac{g(x) + x}{g(x) - x}, \quad x \in [0, 4). \quad (4.3)$$

The first inequality in (4.3), which will be considered in what follows, is equivalent with

$$x \leq \ln \left( \frac{f(x) + x}{f(x) - x} \right), \quad x \in [0, 4).$$

Now let us set

$$F(x) := x - \ln \left( \frac{f(x) + x}{f(x) - x} \right), \quad x \in [0, 4).$$

Since  $f(0) = 2$ , we have  $F(0) = 0$ . Moreover

$$F'(x) = \frac{f^2(x) + 2xf'(x) - x^2 - 2f(x)}{(f(x) + x)(f(x) - x)}, \quad x \in [0, 4),$$

so that  $F'(x) \leq 0$ ,  $x \in [0, 4)$  if  $f^2(x) + 2xf'(x) - x^2 - 2f(x) \leq 0$ ,  $x \in [0, 4)$ . A detailed analysis of the class  $\mathcal{F}$  consisting of differentiable functions  $f : [0, 4) \rightarrow (0, \infty)$  such that  $f(0) = 2$ ,  $f(x) \geq g(x)$  for all  $x \in [0, 4)$  and

$$f^2(x) + 2xf'(x) - x^2 - 2f(x) \leq 0, \quad x \in [0, 4) \quad (4.4)$$

is far from being trivial and falls out from the scope of this paper. We only note that the solution  $y = f(x) = 2/(1 - Cx)$  of the associated Bernoulli differential equation

$$y' - \frac{y}{x} = -\frac{y^2}{2x}$$

presents a good candidate for an element of the class  $\mathcal{F}$ . In the particular case  $C = 1/4$ , the function  $y = 2/(1 - (x/4))$  belongs to  $\mathcal{F}$  since

$$\frac{2}{1 - (x/4)} \geq \sqrt{x^2 + 9} - 1, \quad x \in [0, 4);$$

see <https://www.desmos.com/calculator/2g7wpl8fri>. Unfortunately, if we disregard the term  $x^2$  in (4.4), then the solution of the associated Ricatti differential equation

$$y' - \frac{y}{x} + \frac{y^2}{2x} = 0,$$

given by  $y = 6Cx/(Cx^3 - 1)$  cannot belong to the class  $\mathcal{F}$  because  $y(0) = 0$  (the complete Ricatti differential equation associated to (4.4) is not solvable in quadratures). We finally note all the established results are checked and compared at <https://www.desmos.com/calculator>.

## REFERENCES

- [1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Maps*, John Wiley and Sons, New York, 1997.
- [2] Y. J. Bagul, *On simple Jordan type inequalities*, Turkish Journal of Inequalities, Vol. **3**, No. 1, pp. 1-6, 2019.
- [3] Y. J. Bagul *On exponential bounds of hyperbolic cosine*. Bull. Int. Math. Virtual Inst., Vol. **8**, No. 2, pp. 365-367, 2018.
- [4] Y. J. Bagul, *On a result of Bhayo and Sándor*, Anal. Math., 2021. Doi: 10.1007/s10476-020-0060-8
- [5] B. A. Bhayo, R. Klén and J. Sándor, *New trigonometric and hyperbolic inequalities*, Miskolc Math. Notes, Vol. **18**, No. 1, pp. 125-137, 2017. Doi: 10.18514/MMN.2017.1560
- [6] B. A. Bhayo and J. Sándor, *On certain old and new trigonometric and hyperbolic inequalities*, Anal. Math., Vol. **41**, pp. 3-15, 2015. Online: <https://doi.org/10.1007/s10476-015-0102-9>
- [7] R. M. Dhaigude, Y. J. Bagul, and V. M. Raut, *Generalized bounds for hyperbolic sine and hyperbolic cosine functions*, Tbilisi Math. J., Vol. **14**, No. 1, pp. 41-47, 2021. Doi: 10.32513/tmj/1932200813
- [8] V. Heikkala, M. K. Vamanamurthy and M. Vuorinen, *Generalized elliptic integrals*, Comput. Methods Funct. Theory, Vol. **9**, No. 1, pp. 75-109, 2009.
- [9] B. Zhang, and C.-P. Chen, *A double inequality for  $\tanh x$* , J. Inequal. and Appl., Vol. **2020**, 2020:19, 8 pp., 2020, Doi: 10.1186/s13660-020-2289-y
- [10] L. Zhu, and J. Sun, *Six new Redheffer-type inequalities for circular and hyperbolic functions*, Comput. Math. Appl., Vol. **56**, pp. 522-529, 2008. Doi: 10.1016/j.camwa.2008.01.012

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