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WHEN IS A REDUCTIVE GROUP SCHEME LINEAR?

PHILIPPE GILLE

ABSTRACT. We show that a reductive group scheme over a base scheme S admits a faithful linear representation if and only if its radical torus is isotrivial; that is, it splits after a finite étale cover.

to the 75-th anniversary of Gopal Prasad

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MSC 2000: 14L15, 20G35

1. INTRODUCTION

Let S be a scheme. Let \mathfrak{G} be an S -group scheme. It is natural to ask whether \mathfrak{G} is linear; that is, there exists a group monomorphism $\mathfrak{G} \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free \mathcal{O}_S -module of finite rank. In particular, \mathfrak{G} admits a faithful representation on \mathcal{E} . This holds for affine algebraic groups over a field [4, II, §2.3.3].

In the case S is locally noetherian and G is of multiplicative type of constant type and of finite type, Grothendieck has shown that G is linear if and if G is isotrivial, i.e. G is split by a finite étale extension of S [11, XI.4.6]. In particular there exist non linear tori of rank 2 over the local ring (at a node) of a nodal algebraic curve (*ibid*, X.1.6). Firstly we extend that criterion over an arbitrary base by using Azumaya and separable \mathcal{O}_S -algebras (Theorem 3.3).

Secondly we deal with the case \mathfrak{G} reductive; that is, \mathfrak{G} is smooth affine with reductive (connected) geometric fibers. In this case a faithful representation is necessarily a closed immersion [11, XVI.1.5]. Positive results on the linearity question are due to M. Raynaud [11, VI_B] and R. Thomason [13, 3.1] which is essentially the implication (i) \implies (ii) in the theorem below.

We can restrict our attention to the case when \mathfrak{G} is of constant type (recall that the type is a locally constant function on S); this implies that there exists a Chevalley \mathbb{Z} -group scheme G such that \mathfrak{G} is locally isomorphic to G_S for the étale topology [11, XXII.2.3, 2.5]. A short version of our main result is the following.

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Theorem 1.1. *The following are equivalent:*

- (i) *The radical torus $\text{rad}(\mathfrak{G})$ is isotrivial;*
- (ii) *\mathfrak{G} is linear.*

Furthermore if S is affine, the above are equivalent to

- (ii') *there exists a closed immersion $i : \mathfrak{G} \rightarrow \text{GL}_n$ with $n \geq 1$ which is a homomorphism.*

We recall that $\text{rad}(\mathfrak{G})$ is the maximal central subtorus of \mathfrak{G} [11, XXIV.4.3.6] and that (i) means that $\text{rad}(\mathfrak{G})$ splits after passing to a finite étale cover $S' \rightarrow S$. In the noetherian setting, a variant of the implication (i) \implies (ii) has been shown by Margaux who furthermore provided an $\text{Aut}(\mathfrak{G})$ -equivariant representation [9]. Note that condition (i) depends only on the quasi-split form of \mathfrak{G} and also that it is always satisfied if \mathfrak{G} is semisimple or if $\text{rad}(\mathfrak{G})$ is of rank one. Furthermore, if S is a semilocal scheme, Demazure's characterization of isotrivial group schemes [11, XXIV.3.5] permits us to deduce that \mathfrak{G} is isotrivial if and only if \mathfrak{G} is linear, see section 5.

Finally for $S = \text{Spec}(R)$ with R noetherian, we complete Thomason's approach by showing that linearity for \mathfrak{G} is equivalent to the resolution property (Th. 6.2).

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2. DEFINITIONS AND BASIC FACTS

2.1. Notation. We use mainly the terminology and notation of Grothendieck-Dieudonné [5, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [11, Exp. I.4].

(a) Let S be a scheme and let \mathcal{E} be a quasi-coherent sheaf over S . For each morphism $f : T \rightarrow S$, we denote by $\mathcal{E}_T = f^*(\mathcal{E})$ the inverse image of \mathcal{E} by the morphism f . We denote by $\mathbf{V}(\mathcal{E})$ the affine S -scheme defined by $\mathbf{V}(\mathcal{E}) = \text{Spec}(\text{Sym}^\bullet(\mathcal{E}))$; it represents the S -functor $Y \mapsto \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$ [5, 9.4.9].

(b) We assume now that \mathcal{E} is locally free of finite rank and denote by \mathcal{E}^\vee its dual. In this case the affine S -scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S -functor $Y \mapsto H^0(Y, \mathcal{E}_Y) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$ is representable by the affine S -scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [11, I.4.6].

The above applies to the locally free quasi-coherent sheaf $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$ over S so that we can consider the affine S -scheme $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is an S -functor in associative and unital algebras [5, 9.6.2]. Now we consider the S -functor $Y \mapsto \text{Aut}_{\mathcal{O}_Y}(\mathcal{E}_Y)$. It is representable by an open S -subscheme of $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is denoted by $\text{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

(c) If \mathcal{B} is a locally free \mathcal{O}_S -algebra (unital, associative) of finite rank, we recall that the functor of invertible elements of \mathcal{B} is representable by an affine S -group scheme which is denoted by $\mathrm{GL}_1(\mathcal{B})$ [3, 2.4.2.1].

For separable and Azumaya algebras, we refer to [8]. Note that in [3, §2.5.1], separable algebras are supposed furthermore to be locally free of finite rank.

If \mathcal{B} is a separable \mathcal{O}_S -algebra which is a locally free \mathcal{O}_S -algebra of finite rank, then $\mathrm{GL}_1(\mathcal{B})$ it is a reductive S -group scheme [3, 3.1.0.50].

(d) We use the theory and terminology of tori and multiplicative group schemes of [11]; see also Oesterlé's survey [10].

2.2. Finite étale covers. The next lemma is a consequence of the equivalence of categories describing finite étale \mathcal{O}_S -algebra of rank N [3, §2.5.2]; it admits a simple direct proof.

Lemma 2.1. *Let N be a positive integer and let \mathcal{C} be a finite étale \mathcal{O}_S -algebra of rank N . Then there exists a finite étale cover T of S of degree $N!$ such that $\mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{\sim} (\mathcal{O}_T)^N$.*

Proof. We proceed by induction on N , the case $N = 1$ being obvious. We put $S' = \mathrm{Spec}(\mathcal{O}_{\mathcal{C}})$, this is a finite étale cover of S of degree N . Since the diagonal map $S' \rightarrow S' \times_S S'$ is closed and open [6, 4.17.4.2] there exists a decomposition $\mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}'_S = \mathcal{O}_{S'} \times \mathcal{C}'$ where \mathcal{C}' is a finite étale $\mathcal{O}_{S'}$ -algebra of rank $N - 1$. Applying the induction process to \mathcal{C}' provides a finite étale cover T of S' of degree $(N - 1)!$ such that $\mathcal{C}' \otimes_{\mathcal{O}_{S'}} \mathcal{O}_T \cong (\mathcal{O}_T)^{N-1}$. Thus $\mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_T = \mathcal{O}_T \times (\mathcal{O}_T)^{N-1}$ and T is a finite étale cover of S of degree $N! = N \times (N - 1)!$. \square

2.3. Isotriviality. [11, XXIV.4] Let \mathcal{H} be a fppf S -sheaf in groups and let \mathcal{X} be a \mathcal{H} -torsor. We say that \mathcal{X} is *isotrivial* if there exists a finite étale cover S' of S which trivializes \mathcal{X} ; that is, satisfying $\mathcal{X}(S') \neq \emptyset$.

The notion of locally isotrivial (with respect to the Zariski topology) is then clear and there is also the following variant of *semilocally isotrivial*.

We say that \mathcal{X} is *semilocally isotrivial* if for each subset $\{s_1, \dots, s_n\}$ of points of S contained in an affine open subset of S , there exists an open subscheme U of S containing s_1, \dots, s_n such that $\mathcal{X} \times_S U$ is isotrivial over U .

A reductive S -group scheme \mathfrak{G} is *isotrivial* if it is split by a finite étale cover S' of S . An isotrivial reductive S -group scheme \mathfrak{G} is necessarily of constant type. If \mathfrak{G} is of constant type with underlying Chevalley group scheme G , \mathfrak{G} is isotrivial if and only if the $\mathrm{Aut}(G)$ -torsor $\mathrm{Isom}(G_S, \mathfrak{G})$ is isotrivial.

2.4. Rank one tori. The simplest case is that of $G = \mathbb{G}_{m,S}$, the split S -torus of rank 1. The S -functor $S' \mapsto \mathrm{Hom}_{S'-gp}(\mathbb{G}_{m,S'}, \mathbb{G}_{m,S'})$ is representable by the constant S -group scheme \mathbb{Z}_S [11, VIII.1.5]. It follows that the S -functor $S' \mapsto \mathrm{Isom}_{S'-gp}(\mathbb{G}_{m,S'}, \mathbb{G}_{m,S'})$ is representable by the constant S -group scheme $(\mathbb{Z}/2\mathbb{Z})_S = \mathrm{Aut}_{S-gp}(\mathbb{Z}_S)$. On the other hand, $(\mathbb{Z}/2\mathbb{Z})_S$ is the automorphism group of the split

étale cover $S \sqcup S \rightarrow S$ of degree 2. By definition an S -torus of rank one is a form of \mathbb{G}_m for the fpqc topology; in the other hand, a degree 2 étale cover is a form of $S \sqcup S$ for the finite étale topology (see for example Lemma 2.1) and a fortiori for the fpqc topology.

According to the faithfully flat descent technique (e.g. [11, XXIV.1.17]), there is then an equivalence of categories between the groupoid of rank one tori over S (resp. the groupoid of degree 2 étale covers of S) and the groupoid of $(\mathbb{Z}/2\mathbb{Z})_S$ -torsors.

More precisely one associates to an S -torus T of rank 1 (resp. to an étale cover E of degree 2) the $(\mathbb{Z}/2\mathbb{Z})_S$ -torsor $\text{Isom}_{gr}(\mathbb{G}_m, T)$ (resp. $\text{Isom}(S \sqcup S, E)$). The inverse map is given by twisting the split object by a given $(\mathbb{Z}/2\mathbb{Z})_S$ -torsor.

Let T be an S -torus of rank one and let $S' \rightarrow T$ be its associated étale cover of degree 2. Then T splits after base change to S' and so is isotrivial. Furthermore we claim that T is isomorphic to the quotient $Q = R_{S'/S}(\mathbb{G}_{m,S'})/\mathbb{G}_m$ where $R_{S'/S}(\mathbb{G}_{m,S'})$ stands for the Weil restriction. We consider the $(\mathbb{Z}/2\mathbb{Z})_S$ -equivariant exact sequences

$$1 \rightarrow \mathbb{G}_{m,S} \xrightarrow{\Delta} \mathbb{G}_{m,S} \times \mathbb{G}_{m,S} \xrightarrow{\Pi} \mathbb{G}_{m,S} \rightarrow 1$$

where $\Pi(x, y) = xy^{-1}$ and the $\mathbb{Z}/2\mathbb{Z}$ -action on $\mathbb{G}_{m,S} \times \mathbb{G}_{m,S}$ is by $(x, y) \mapsto (y, x)$. The $\mathbb{Z}/2\mathbb{Z}$ -action on the last factor is then by $x \mapsto x^{-1}$. Twisting this sequence by the $(\mathbb{Z}/2\mathbb{Z})_S$ -torsor $\text{Isom}(S \sqcup S, S')$ yields the desired exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow R_{S'/S}(\mathbb{G}_{m,S'}) \rightarrow Q \rightarrow 1$$

where the identification of the second term is left to the reader (it is similar to that of the proof of [11, XXIX.3.13]).

2.5. Linear representations. Let \mathfrak{G} be an S -group scheme. We say that \mathfrak{G} is *linear* if there exists a locally free \mathcal{O}_S -module \mathcal{E} of finite rank and a group homomorphism $\mathfrak{G} \rightarrow \text{GL}(\mathcal{E})$ which is a monomorphism.

The notion of locally linear S -group scheme is then clear and there is also the following variant of *semilocally linear*.

We say that \mathfrak{G} is *semilocally linear* if for each subset $\{s_1, \dots, s_n\}$ of points of S contained in an affine open subset of S , there exists an open subscheme U of S containing s_1, \dots, s_n such that $\mathfrak{G} \times_S U$ is linear over U .

Lemma 2.2. *Assume that $S = \text{Spec}(R)$ and let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank. Then $\text{GL}(\mathcal{E})$ embeds as a closed S -subgroup scheme in GL_n for some $n \geq 1$.*

Proof. Since the rank of \mathcal{E} is a locally constant function [5, Ch. 0, 5.4.1], we can assume that \mathcal{E} is locally free of constant rank r . Then $E = H^0(R, \mathcal{E})$ is a locally free R -module of rank r , so is finitely generated projective [12, Tag 00NX]. It follows that there exists an integer $n \geq 1$ and a decomposition $R^n = E \oplus E'$. The homomorphism $\text{GL}(\mathcal{E}) \rightarrow \text{GL}_n$ is a closed immersion. \square

Lemma 2.3. *Let \mathfrak{G} be an S -group scheme and let S' be a finite locally free cover of S . Then \mathfrak{G} is linear if and only if $\mathfrak{G} \times_S S'$ is linear.*

Proof. We denote by $p : S' \rightarrow S$ the structure map. If \mathfrak{G} is linear, then $\mathfrak{G} \times_S S'$ is linear. Conversely we assume that there exists a monomorphism $i : \mathfrak{G} \times_S S' \rightarrow \mathrm{GL}(\mathcal{E}')$ where \mathcal{E}' is a locally free $\mathcal{O}_{S'}$ -module of finite rank. We put $\mathcal{E} = p_*(\mathcal{E}')$, this is a locally free \mathcal{O}_S -module of finite rank. We consider the sequence of S -functors in S -groups

$$\mathfrak{G} \rightarrow R_{S'/S}(\mathfrak{G} \times_S S') \xrightarrow{R_{S'/S}(i)} R_{S'/S}(\mathrm{GL}(\mathcal{E}')) \rightarrow \mathrm{GL}(\mathcal{E})$$

where $R_{S'/S}$ stands for the Weil restriction and the first map is the diagonal map which is a monomorphism. Since the Weil restriction for S'/S transforms monomorphisms into monomorphisms, the map $R_{S'/S}(i)$ is also a monomorphism and so is the last map since $R_{S'/S}(\mathrm{GL}(\mathcal{E}'))(T) \subset \mathrm{GL}(\mathcal{E})(T)$ corresponds to automorphisms of $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ which are $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ -linear. Since all maps are monomorphisms, we conclude that \mathfrak{G} is linear. \square

3. TORI AND GROUP OF MULTIPLICATIVE TYPE

3.1. Maximal tori of linear groups. Let \mathcal{A} be an Azumaya \mathcal{O}_S -algebra. We consider the reductive S -group scheme $\mathrm{GL}_1(\mathcal{A})$. Let $\mathcal{B} \subset \mathcal{A}$ be a separable \mathcal{O}_S -subalgebra of \mathcal{A} which is a locally free \mathcal{O}_S -module of finite rank and which is locally a direct summand of \mathcal{A} as \mathcal{O}_S -module. We get a monomorphism of reductive S -group schemes $\mathrm{GL}_1(\mathcal{B}) \rightarrow \mathrm{GL}_1(\mathcal{A})$. In particular, if \mathcal{B} is commutative, \mathcal{B} is a finite étale \mathcal{O}_S -algebra and $\mathrm{GL}_1(\mathcal{B})$ is a torus. We come now to Grothendieck's definition of maximal étale subalgebras.

Definition 3.1. [7, Déf. 5.6]. *We say that a finite étale \mathcal{O}_S -subalgebra $\mathcal{C} \subset \mathcal{A}$ is maximal if \mathcal{C} is locally a direct summand of \mathcal{A} as \mathcal{O}_S -module and if the rank of $\mathcal{C} \otimes_{\mathcal{O}_S} \kappa(s)$ is the degree of $\mathcal{A}_s \otimes_{\mathcal{O}_S} \kappa(s)$ for each $s \in S$.*

If $\mathcal{C} \subset \mathcal{A}$ is maximal finite étale \mathcal{O}_S -subalgebra of \mathcal{A} , then the torus $\mathrm{GL}_1(\mathcal{C})$ is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$ since it is the case on geometric fibers. According to [7, §7.5], all maximal S -tori of $\mathrm{GL}_1(\mathcal{A})$ occur in that manner; this is part (3) of the following enlarged statement.

Proposition 3.2. *Let \mathfrak{S} be a subgroup scheme of multiplicative type of $\mathrm{GL}_1(\mathcal{A})$ and put $\mathcal{B} = \mathcal{A}^{\mathfrak{S}}$, the centralizer subalgebra of \mathfrak{S} .*

- (1) \mathcal{B} is a separable \mathcal{O}_S -algebra which is locally a direct summand of \mathcal{A} as \mathcal{O}_S -module.
- (2) Let \mathcal{C} be the center of \mathcal{B} ; this is a finite étale \mathcal{O}_S -algebra of positive rank which is locally a direct summand of \mathcal{B} (and \mathcal{A}) as \mathcal{O}_S -module. We have the closed immersions

$$\mathfrak{S} \subset \mathrm{GL}_1(\mathcal{C}) \subset \mathrm{GL}_1(\mathcal{A}).$$

- (3) If \mathfrak{S} is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$, then $\mathfrak{S} = \mathrm{GL}_1(\mathcal{C})$ and \mathcal{C} is a maximal finite étale \mathcal{O}_S -subalgebra of \mathcal{A} .

(4) If \mathfrak{S} is of constant type, then \mathfrak{S} is isotrivial.

Proof. According to [11, IX.2.5], the map $\mathfrak{S} \rightarrow \mathrm{GL}_1(\mathcal{A})$ is a closed immersion so that \mathfrak{S} is of finite type over S . As a preliminary observation we notice that (1), (2), (3) are local for the étale topology. We can assume that $S = \mathrm{Spec}(R)$, $\mathcal{A} = \mathrm{M}_n(R)$ and that $\mathfrak{S} = D(M)$ for M an abelian group. Note that M is finitely generated since \mathfrak{S} is of finite type (*ibid*, VII.2.1.b).

(1) We consider the M -grading $R^n = \bigoplus_{m \in M} R_m^n$. The R -modules $(R_m^n)_{m \in M}$ are finitely generated projective so locally free of finite rank. There is a finite subset $M' \subset M$ such that $R^n = \bigoplus_{m \in M'} R_m^n$. Then $\mathcal{B} = \prod_{m \in M'} \mathrm{End}_R(R_m^n)$ and each $\mathrm{End}_R(R_m^n)$ is a separable R -algebra which is locally free of finite rank [8, III, example 2.8]. Since a product of separable algebras is a separable algebra (*ibid*, III, proposition 1.7), it follows that \mathcal{B} is a separable R -algebra. Furthermore \mathcal{B} is locally a direct summand of $\mathrm{M}_n(R)$ as R -module.

(2) Let \mathcal{C} be the center of \mathcal{B} . We have R -monomorphisms of groups $\mathfrak{S} \subset \mathrm{GL}_1(\mathcal{C}) \subset \mathrm{GL}_1(\mathcal{A})$.

(3) We assume that \mathfrak{S} is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$ and want to establish that $\mathfrak{S} = \mathrm{GL}_1(\mathcal{C})$. So $\mathrm{GL}_1(\mathcal{C})$ is an R -torus containing \mathfrak{T} and since maximality holds also in the naive sense [11, XII.1.4], we conclude that $\mathfrak{T} = \mathrm{GL}_1(\mathcal{C})$.

(4) We come back to the initial setting (i.e. without localizing). We want to show that \mathfrak{S} is isotrivial. According to [5, ch.0, 5.4.1], for each integer $l \geq 0$, $S_l = \{s \in S \mid \mathrm{rank}(\mathcal{C}_{\kappa(s)}) = l\}$ is an open subset of S so that we have a decomposition in clopen subschemes $S = \coprod_{l \geq 0} S_l$. Without loss of generality we can assume

that \mathcal{C} is locally free of rank l . Lemma 2.1 provides a finite étale cover S' of S such that $\mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \cong (\mathcal{O}_{S'})^l$. Hence $\mathrm{GL}_1(\mathcal{C}) \times_S S' \cong (\mathbb{G}_m)^l \times_S S'$. It follows that $\mathfrak{S} \times_S S'$ is a subgroup S' -scheme of $(\mathbb{G}_m)^l \times_S S'$ of multiplicative type. According to [11, IX.2.11.(i)] there exists a partition in clopen subsets $S' = \sqcup_{i \in I} S'_i$ such that each $\mathfrak{S} \times_S S'_i$ is diagonalizable. Since $\mathfrak{S} \times_S S'$ is of constant type we conclude that $\mathfrak{S} \times_S S'$ is diagonalizable. \square

3.2. Characterization of isotrivial groups of multiplicative type.

Theorem 3.3. *Let \mathfrak{S} be an S -group scheme of multiplicative type of finite type and of constant type. Then the following are equivalent:*

- (i) \mathfrak{S} is isotrivial;
- (ii) \mathfrak{S} is linear;
- (iii) \mathfrak{S} is an S -subgroup scheme of an S -group scheme $\mathrm{GL}_1(\mathcal{A})$ where \mathcal{A} is an Azumaya \mathcal{O}_S -algebra.

Proof. (i) \implies (ii). This follows from Lemma 2.3.

(ii) \implies (iii). By definition we have a monomorphism $\mathfrak{S} \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free \mathcal{O}_S -module of finite rank. Since $\mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$ is an Azumaya \mathcal{O}_S -algebra, we get (iii).

(iii) \implies (i). This follows of Proposition 3.2.(4). \square

Examples 3.4. (a) Grothendieck constructed a scheme S and an S -torus \mathfrak{T} which is locally trivial of rank 2 but which is not isotrivial [11, §X.1.6] (e.g. S consists of two copies of the projective line over a field pinched at 0 and ∞). Theorem 3.3 shows that such an S -torus is not linear.

(b) Also there exists a local ring R and an R -torus \mathfrak{T} of rank 2 which is not isotrivial [11, §1.6]; the ring R can be taken as the local ring of an algebraic curve at a double point. Theorem 3.3 shows that such an R -torus is not linear.

4. REDUCTIVE CASE

For stating the complete version of our main result, we need more notation. As in the introduction, \mathfrak{G} is a reductive S -group scheme of constant type and G is the underlying Chevalley \mathbb{Z} -group scheme. We denote by $\mathrm{Aut}(G)$ the automorphism group scheme of G and we have an exact sequence of \mathbb{Z} -group schemes [11, th. XXIV.1.3]

$$1 \rightarrow G_{ad} \rightarrow \mathrm{Aut}(G) \rightarrow \mathrm{Out}(G) \rightarrow 1.$$

We remind the reader of the representability of the fppf sheaf $\underline{\mathrm{Isom}}(G_S, \mathfrak{G})$ by a $\mathrm{Aut}(G)_S$ -torsor $\mathrm{Isom}(G_S, \mathfrak{G})$ defined in [11, XXIV.1.8]. The contracted product

$$\mathrm{Isomext}(G_S, \mathfrak{G}) := \mathrm{Isom}(G_S, \mathfrak{G}) \wedge^{\mathrm{Aut}(G)_S} \mathrm{Out}(G)_S$$

is an $\mathrm{Out}(G)_S$ -torsor (*ibid*, 1.10) which encodes the isomorphism class of the quasi-split form of \mathfrak{G} .

Theorem 4.1. *The following are equivalent:*

- (i) *The torus $\mathrm{rad}(\mathfrak{G})$ is isotrivial;*
- (ii) *the $\mathrm{Out}(G)_S$ -torsor $\mathrm{Isomext}(G_S, \mathfrak{G})$ is isotrivial;*
- (iii) *\mathfrak{G} is linear;*
- (iv) *$\mathrm{rad}(\mathfrak{G})$ is linear.*

Furthermore if S is affine, we can take a faithful linear representation in some GL_n for (iii) and (iv).

Proof. (i) \implies (ii). We assume that $\mathrm{rad}(\mathfrak{G})$ is isotrivial and want to show that the $\mathrm{Out}(G)_S$ -torsor $\mathcal{F} = \mathrm{Isomext}(G_S, \mathfrak{G})$ is isotrivial. In other words, we want to show that there exists a finite étale cover S' of S such that $\mathfrak{G} \times_S S'$ is an inner form of G .

Without loss of generality we can assume that $\mathrm{rad}(\mathfrak{G})$ is a split torus. We quote now [11, XXIV.2.16] for the Chevalley group G over \mathbb{Z} which introduces the \mathbb{Z} -group scheme

$$H = \ker(\mathrm{Aut}(G) \rightarrow \mathrm{Aut}(\mathrm{rad}(G)));$$

furthermore there is an equivalence of categories between the category of H -torsors over S and the category of pairs (\mathfrak{M}, ϕ) where \mathfrak{M} is an S -form of G and $\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{M})$.

Since $\text{rad}(\mathfrak{G})$ is split, we choose an isomorphism $\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{G})$ and consider an H -torsor \mathfrak{P} mapping to an object isomorphic to (\mathfrak{G}, ϕ) . Furthermore the quoted reference provides an exact sequence of \mathbb{Z} -group schemes

$$1 \rightarrow G_{ad} \rightarrow H \xrightarrow{p} F \rightarrow 1$$

where F is finite étale over \mathbb{Z} , so is constant. We denote by $S' = \mathfrak{M} \wedge^{H_S} F_S$ the contracted product of \mathfrak{M} and F_S with respect to H_S ; this is an F -torsor over S , hence is a finite étale cover of S . It follows that $\mathfrak{P} \times_S S'$ admits a reduction to a $G_{ad, S'}$ -torsor \mathfrak{Q}' . Since the map $G_{ad} \rightarrow H \rightarrow \text{Aut}(G)$ is the canonical map, we conclude that $\mathfrak{G}_{S'} \cong {}^{\mathfrak{Q}'}G$ is an inner form of G .

(ii) \implies (iii). Our assumption is that there exists a finite étale cover S'/S which splits the $\text{Out}(G)$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$. Lemma 2.3 permits us to replace S by S' , so we can assume that the $\text{Out}(G)$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is trivial; that is, \mathfrak{G} is an inner form of G . There exists a G_{ad} -torsor \mathfrak{Q} over S such that $\mathfrak{G} \cong {}^{\mathfrak{Q}}G$. Since $G \rtimes G_{ad}$ is defined over \mathbb{Z} , it admits a faithful representation $\rho : G \rtimes G_{ad} \rightarrow \text{GL}_n$ over \mathbb{Z} [2, §1.4.5]. The map ρ is then G_{ad} -equivariant and can be twisted by the G_{ad} -torsor \mathfrak{Q} . We obtain a faithful representation ${}^{\mathfrak{Q}}G \rtimes {}^{\mathfrak{Q}}G_{ad} \rightarrow {}^{\mathfrak{Q}}\text{GL}_n = \text{GL}(\mathcal{E})$ where \mathcal{E} is the locally free \mathcal{O}_S -module of rank n which is the twist of $(\mathcal{O}_S)^n$ by the GL_n -torsor $\mathfrak{Q} \wedge^{G_{ad}} \text{GL}_n$. Thus ${}^{\mathfrak{Q}}G$ is linear.

(iii) \implies (iv). Obvious.

(iv) \implies (i). Since $\text{rad}(\mathfrak{G})$ is a form of $\text{rad}(G)$, it is of constant rank and Theorem 3.3 shows that $\text{rad}(G)$ is isotrivial.

Finally the refinement for S affine follows from Lemma 2.2. \square

Corollary 4.2. *Under the assumptions of \mathfrak{G} , let \mathfrak{G}^{qs} be the quasi-split form of \mathfrak{G} . Then \mathfrak{G} is linear if and only if \mathfrak{G}^{qs} is linear.*

The next corollary slightly generalizes a result by Thomason [13, cor. 3.2].

Corollary 4.3. *Assume that either*

- (i) S is locally noetherian and geometrically unibranch (e.g. normal);
- (ii) $\text{rad}(\mathfrak{G})$ is of rank ≤ 1 (in particular if G is semisimple).

Then \mathfrak{G} is linear.

Proof. In case (i), the torus $\text{rad}(\mathfrak{G})$ is isotrivial [11, X.5.16]. In case (ii), we have $\text{rad}(G) = 1$ or \mathbb{G}_m (since G is split), so that $\text{rad}(\mathfrak{G})$ is split by a quadratic étale cover of S (§2.4), hence is isotrivial. Hence Theorem 1.1 implies that \mathfrak{G} is linear. \square

The next corollary extends Demazure's characterization of locally isotrivial reductive group schemes [11, XXIV.4.1.5].

Corollary 4.4. *The following are equivalent:*

- (i) \mathfrak{G} is locally (resp. semilocally) isotrivial;
- (ii) The torus $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) isotrivial;
- (iii) the $\text{Out}(G)_S$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is locally (resp. semilocally) isotrivial;
- (iv) \mathfrak{G} is locally (resp. semilocally) linear;
- (v) $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) linear.

Proof. In view of Theorem 4.1, it remains to establish the equivalence (i) \iff (ii). Since this is precisely the quoted result [11, XXIV.3.5], the proof is complete. \square

Corollary 4.5. *Let \mathfrak{H} be a reductive S -subgroup scheme of \mathfrak{G} . If \mathfrak{G} is locally (resp. semilocally) isotrivial, then \mathfrak{H} is locally (resp. semilocally) isotrivial.*

Proof. Corollary 4.4 shows that \mathfrak{G} is locally (resp. semilocally) linear and so is \mathfrak{H} . Therefore \mathfrak{H} is locally (resp. semilocally) isotrivial. \square

5. THE SEMILOCAL CASE

We assume that $S = \text{Spec}(R)$ where R is a semilocal ring and continue to assume that the reductive S -group scheme \mathfrak{G} is of constant type. We remind the reader that \mathfrak{G} admits a maximal torus (Grothendieck, [11, XIV.3.20 and footnote]).

Corollary 5.1. *Let \mathfrak{T} be a maximal torus of \mathfrak{G} . The following are equivalent:*

- (i) \mathfrak{G} is isotrivial;
- (ii) The torus $\text{rad}(\mathfrak{G})$ is isotrivial;
- (iii) the $\text{Out}(G)_S$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is isotrivial;
- (iv) \mathfrak{G} is linear;
- (v) $\text{rad}(\mathfrak{G})$ is linear;
- (vi) \mathfrak{T} is linear;
- (vii) \mathfrak{T} is isotrivial.

Proof. From Corollary 4.4, we have the equivalences (i) \iff (ii) \iff (iii) \iff (iv) \iff (v). On the other hand, the equivalence (vi) \iff (vii) holds according to Theorem 3.3. Now we observe that the implications (iv) \implies (vi) and (vi) \implies (v) are obvious so the proof is complete. \square

6. EQUIVARIANT RESOLUTION PROPERTY

Definition 6.1. *Let \mathfrak{G} be a flat group scheme over S acting on a locally noetherian S -scheme \mathfrak{X} . One says that $(\mathfrak{G}, S, \mathfrak{X})$ has the resolution property (RE) for short) if for every coherent \mathfrak{G} -module \mathcal{F} on \mathfrak{X} , there is a locally free coherent \mathfrak{G} -module (i.e. a \mathfrak{G} -vector bundle \mathcal{E}) and a \mathfrak{G} -equivariant epimorphism $\mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$.*

We strengthen Thomason's results.

Theorem 6.2. *Let \mathfrak{G} be a reductive S -group scheme. We assume that S is separated noetherian and that $(1, S, S)$ satisfies the resolution property, e.g. S is affine or regular or admits an ample family of line bundles. Then the following are equivalent:*

- (i) \mathfrak{G} is linear;
- (ii) $\text{rad}(\mathfrak{G})$ is isotrivial;
- (iii) \mathfrak{G} satisfies (RE).

Proof. (i) \iff (ii). This is a special case of Theorem 4.1.

(ii) \implies (iii). This is Thomason's result [13, Theorem 2.18].

(iii) \implies (i). This is Thomason's result [13, Theorem 3.1], see also [11, VI_B.13.5]. \square

Remark 6.3. Example 3.4.(b) is an example of a local noetherian ring R and of a rank two non-isotrivial torus \mathfrak{T} . Theorem 6.2 shows that \mathfrak{T} does not satisfy (RE). This answers a question of Thomason [13, §2.3].

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