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WHEN IS A REDUCTIVE GROUP SCHEME LINEAR?

PHILIPPE GILLE

ABSTRACT. We show that a reductive group scheme over a base scheme S admits a faithful linear representation if and only if its radical torus is isotrivial, that is, it splits after a finite étale cover.

Keywords: Reductive group schemes, representations, tori, resolution property.

MSC 2000: 14L15, 20G35

1. INTRODUCTION

Let S be a scheme. Let \mathfrak{G} be an affine S -group scheme. It is natural to ask whether \mathfrak{G} is linear, that is there exists a group monomorphism $\mathfrak{G} \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free \mathcal{O}_S -module of finite rank. In particular, \mathfrak{G} admits a faithful representation on \mathcal{E} . This holds for affine algebraic groups over a field [5, II, §2.3.3].

We are mainly interested in the case where \mathfrak{G} is reductive, that is \mathfrak{G} is smooth affine with reductive (connected) geometric fibers. In this case a faithful representation is necessarily a closed immersion [10, XVI.1.5]. Positive results on the linearity question are due to M. Raynaud [10, VI_B] and R. Thomason [12, 3.1] which is essentially the implication (i) \implies (ii) in the theorem below.

We shall see that the existence of a faithful linear representation does not hold in general even for rank two tori. We can restrict our attention to the case when \mathfrak{G} is of constant type (recall that the type is a locally constant function on S); this implies that there exists a Chevalley \mathbb{Z} -group scheme G such that \mathfrak{G} is locally isomorphic to G_S for the étale topology [10, XXII.2.3, 2.5]. A short version of our main result is the following.

Theorem 1.1. *The following are equivalent:*

- (i) *The radical torus $\mathrm{rad}(\mathfrak{G})$ is isotrivial;*
- (ii) *\mathfrak{G} is linear.*

Furthermore if S is affine, the above are equivalent to

- (ii') *there exists a closed immersion $i : \mathfrak{G} \rightarrow \mathrm{GL}_n$ with $n \geq 1$.*

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We recall that $\text{rad}(\mathfrak{G})$ is the maximal central subtorus of \mathfrak{G} [10, XXIV.4.3.6] and that (i) means that $\text{rad}(\mathfrak{G})$ splits after passing to a finite étale cover $S' \rightarrow S$. In the noetherian setting, a variant of the implication (i) \implies (ii) has been shown by Margaux who furthermore provided an $\text{Aut}(\mathfrak{G})$ -equivariant representation [9]. Note that condition (i) depends only on the quasi-split form of \mathfrak{G} and also that it is always satisfied if \mathfrak{G} is semisimple or if $\text{rad}(\mathfrak{G})$ is of rank one. The result is new even in the case of tori, we deal first with this case in section 3. Furthermore, if S is a semilocal scheme, Demazure's characterization of isotrivial group schemes [10, XXIV.3.5] permits to deduce that \mathfrak{G} is isotrivial if and only if \mathfrak{G} is linear, see section 5.

Finally for $S = \text{Spec}(R)$ with R noetherian, we complete Thomason's approach by showing that linearity for \mathfrak{G} is equivalent to the resolution property (Th. 6.2).

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2. DEFINITIONS AND BASIC FACTS

2.1. Notation. We use mainly the terminology and notation of Grothendieck-Dieudonné [6, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [10, Exp. I.4].

(a) Let S be a scheme and let \mathcal{E} be a quasi-coherent sheaf over S . For each morphism $f : T \rightarrow S$, we denote by $\mathcal{E}_T = f^*(\mathcal{E})$ the inverse image of \mathcal{E} by the morphism f . We denote by $\mathbf{V}(\mathcal{E})$ the affine S -scheme defined by $\mathbf{V}(\mathcal{E}) = \text{Spec}(\text{Sym}^\bullet(\mathcal{E}))$; it is affine over S and represents the S -functor $Y \mapsto \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$ [6, 9.4.9].

(b) We assume now that \mathcal{E} is locally free and denote by \mathcal{E}^\vee its dual. In this case the affine S -scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S -functor $Y \mapsto H^0(Y, \mathcal{E}_Y) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$ is representable by the affine S -scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [10, I.4.6].

The above applies to the locally free coherent sheaf $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$ over S so that we can consider the affine S -scheme $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is an S -functor in associative commutative and unital algebras [6, 9.6.2]. Now we consider the S -functor $Y \mapsto \text{Aut}_{\mathcal{O}_Y}(\mathcal{E}_Y)$. It is representable by an open S -subscheme of $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is denoted by $\text{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

(c) If \mathcal{B} is a locally free \mathcal{O}_S -algebra of finite rank, we recall that the functor of invertible elements of \mathcal{B} is representable by an affine S -group scheme which is denoted by $\text{GL}_1(\mathcal{B})$ [4, 2.4.2.1].

For separable and Azumaya algebras, we refer to [8]. Note that in [4, §2.5.1], separable algebras are supposed furthermore to be locally free of finite rank.

If \mathcal{B} is a separable \mathcal{O}_S -algebra which is a locally free \mathcal{O}_S -algebra of finite rank, then $\text{GL}_1(\mathcal{B})$ it is a reductive S -group scheme [4, 3.1.0.50].

2.2. Isotriviality. [10, XXIV.4] Let \mathcal{H} be a fppf S -sheaf in groups and let \mathcal{X} be a \mathcal{H} -torsor. We say that \mathcal{X} is *isotrivial* if there exists a finite étale cover S' of S which trivializes \mathcal{X} , that is satisfying $\mathcal{X}(S') \neq \emptyset$.

The notion of locally isotrivial is then clear and there is also the following variant of *semilocally isotrivial*.

We say that \mathcal{X} is *semilocally isotrivial* if for each subset $\{s_1, \dots, s_n\}$ of points of S contained in an affine subset of S , there exists an open subscheme U of S containing s_1, \dots, s_n such that $\mathcal{X} \times_S U$ is isotrivial over U .

A reductive S -group scheme \mathfrak{G} is *isotrivial* if it is split by a finite étale cover S' of S . An isotrivial reductive S -group scheme \mathfrak{G} is necessarily of constant type. If \mathfrak{G} is of constant type with underlying Chevalley group scheme G , \mathfrak{G} is isotrivial if and only if the $\text{Aut}(G)$ -torsor $\text{Isom}(G_S, \mathfrak{G})$ is isotrivial.

2.3. Linear representations. Let \mathfrak{G} be an S -group scheme. We say that \mathfrak{G} is *linear* if there exists a locally free \mathcal{O}_S -module \mathcal{E} of finite rank and a group homomorphism $\mathfrak{G} \rightarrow \text{GL}(\mathcal{E})$ which is a monomorphism.

The notion of locally linear S -group scheme is then clear and there is also the following variant of *semilocally linear*.

We say that \mathfrak{G} is *semilocally linear* if for each subset $\{s_1, \dots, s_n\}$ of points of S contained in an affine subset of S , there exists an open subscheme U of S containing s_1, \dots, s_n such that $\mathfrak{G} \times_S U$ is linear over U .

Lemma 2.1. *Assume that $S = \text{Spec}(R)$ and let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank. Then $\text{GL}(\mathcal{E})$ embeds as closed S -subgroup scheme in GL_n for some $n \geq 1$.*

Proof. Since the rank of \mathcal{E} is a locally constant function [6, Ch. 0, 5.4.1], we can assume that \mathcal{E} is locally free of constant rank r . Then $E = H^0(R, \mathcal{E})$ is a locally free R -module of rank r , so is finitely generated projective [11, Tag 00NX]. It follows that there exists an integer $n \geq 1$ and a decomposition $R^n = E \oplus E'$. The homomorphism $\text{GL}(\mathcal{E}) \rightarrow \text{GL}_n$ is a closed immersion. \square

Lemma 2.2. *Let \mathfrak{G} be an affine S -group scheme and let S' be a finite locally free cover of S . Then \mathfrak{G} is linear if and only if $\mathfrak{G} \times_S S'$ is linear.*

Proof. We denote by $p : S' \rightarrow S$ the structure map. If \mathfrak{G} is linear, then $\mathfrak{G} \times_S S'$ is linear. Conversely we assume that there exists a closed embedding $i : \mathfrak{G} \times_S S' \rightarrow \text{GL}(\mathcal{E}')$ where \mathcal{E}' is a locally free $\mathcal{O}_{S'}$ -module of finite rank. We put $\mathcal{E} = p_*(\mathcal{E}')$, this is a locally free \mathcal{O}_S -module of finite rank. We consider the sequence of S -group homomorphisms

$$\mathfrak{G} \rightarrow R_{S'/S}(\mathfrak{G} \times_S S') \rightarrow R_{S'/S}(\text{GL}(\mathcal{E}')) \rightarrow \text{GL}(\mathcal{E})$$

where $R_{S'/S}$ stands for the Weil restriction [2, §7.6]. The second (resp. the first one) map is a closed embedding according to [2, §7.6, prop. 2.(b) (resp. end of page

197)]. The last one is a monomorphism; note that $R_{S'/S}(\mathrm{GL}(\mathcal{E}'))(T) \subset \mathrm{GL}(\mathcal{E})(T)$ corresponds to automorphisms of $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ which are $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ -linear.

Since all maps are monomorphisms, we conclude that \mathfrak{G} is linear thanks to [10, XVI.1.5]. \square

3. TORI

3.1. Maximal tori of linear groups. Let \mathcal{A} be an Azumaya \mathcal{O}_S -algebra. We consider the reductive S -group scheme $\mathrm{GL}_1(\mathcal{A})$. If $\mathcal{B} \subset \mathcal{A}$ is a separable \mathcal{O}_S -subalgebra of \mathcal{A} which is a locally free \mathcal{O}_S -module of finite rank, we get a monomorphism of reductive S -group schemes $\mathrm{GL}_1(\mathcal{B}) \rightarrow \mathrm{GL}_1(\mathcal{A})$. In particular, if \mathcal{B} is commutative, \mathcal{B} is a finite étale \mathcal{O}_S -algebra and $\mathrm{GL}_1(\mathcal{B})$ is a torus; furthermore if the rank of $\mathcal{B} \otimes_{\mathcal{O}_S} \kappa(s)$ is the degree of $\mathcal{A}_s \otimes_{\mathcal{O}_S} \kappa(s)$ for each $s \in S$ (we say then that \mathcal{B} is a maximal finite étale \mathcal{O}_S -algebra of \mathcal{A}) then $\mathrm{GL}_1(\mathcal{B})$ is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$.

According to [7, §7.5], all maximal S -tori of $\mathrm{GL}_1(\mathcal{A})$ occur in that manner; this is part (3) of the following enlarged statement.

Proposition 3.1. *Let \mathfrak{T} be a torus of $\mathrm{GL}_1(\mathcal{A})$ and put $\mathcal{B} = \mathcal{A}^{\mathfrak{T}}$, the centralizer subalgebra of \mathfrak{T} .*

- (1) \mathcal{B} is a separable \mathcal{O}_S -algebra which is locally free of finite rank.
- (2) Let \mathcal{C} be the center of \mathcal{B} ; this is a finite étale \mathcal{O}_S -algebra and we have the closed immersions

$$\mathfrak{T} \subset \mathrm{GL}_1(\mathcal{C}) \subset \mathrm{GL}_1(\mathcal{A}).$$

- (3) If \mathfrak{T} is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$, then $\mathfrak{T} = \mathrm{GL}_1(\mathcal{C})$ and \mathcal{C} is a maximal finite étale R -subalgebra of \mathcal{A} .
- (4) If \mathfrak{T} is of constant rank, then \mathfrak{T} is an isotrivial torus.

Proof. As a preliminary observation we notice that (1), (2), (3) are local for the étale topology. We can assume that $S = \mathrm{Spec}(R)$, $\mathcal{A} = M_n(R)$ and $\mathfrak{T} = (\mathbb{G}_{m,R})^r$.

(1) We denote by $M = \mathbb{Z}^r$ the character group of \mathfrak{T} and we consider the M -grading $R^n = \bigoplus_{m \in M} (R^n)_m$. The R -modules $(R^n)_m$ are finitely generated projective so locally free of finite rank. Then $\mathcal{B} = \prod_{m \in M} \mathrm{End}_R((R^n)_m)$ and each $\mathrm{End}_R((R^n)_m)$ is a separable R -algebra which is locally free of finite rank [8, III, example 2.8].

We are given a point $s \in \mathrm{Spec}(R)$ and want to find a localization $\mathrm{Spec}(R_f)$ containing s such that $\mathcal{B}_f = \mathcal{B} \otimes_R R_f$ is a separable R_f -algebra.

Let $M' \subset M$ be the finite subset of M consisting of the characters m satisfying $(R^n)_m \otimes_R \kappa(s) \neq 0$. Let R_f be a neighborhood of s such that $(R^n)_m$ is free for each $m \in M'$. The map $\bigoplus_{m \in M'} (R_f^n)_m \rightarrow (R_f^n)^n$ is then an isomorphism and we have $\mathcal{B}_f = \prod_{m \in M} \mathrm{End}_R((R^n)_m)_f = \prod_{m \in M'} \mathrm{End}_R((R^n)_m)_f$. Thus \mathcal{B}_f is separable over R_f and locally free of finite rank.

- (2) We have R -monomorphisms of groups $\mathfrak{T} \subset \mathrm{GL}_1(\mathcal{C}) \subset \mathrm{GL}_1(\mathcal{A})$.

(3) We assume that \mathfrak{T} is a maximal torus of $\mathrm{GL}_1(\mathcal{A})$ and want to establish that $\mathfrak{T} = \mathrm{GL}_1(\mathcal{C})$. We can localize as in (1) and deal then with the case the case $\mathcal{B}_f = \prod_{m \in M'} \mathrm{End}_R((R_n)_m)_f$ where $((R_n)_m)_f$ is free over R_f of rank n_m . We have $n = \sum_{m' \in M} n_m$ and $\mathcal{C}_f = \prod_{m \in M'} M_{n_m}(R_f) \subset M_n(R_f)$.

Claim 3.1.1. $n_m = 1$ for each $m \in M'$.

Over $\overline{\kappa(s)}$, the maximal split torus $\mathfrak{T} \times_R \overline{\kappa(s)}$ is $\mathrm{GL}_n(\overline{\kappa(s)})$ -conjugated to the diagonal $\overline{\kappa(s)}$ -torus of GL_n [1, cor. 11.3] so that $\mathcal{B} \otimes_R \overline{\kappa(s)} = \overline{\kappa(s)}^n$, $\sharp M' = n$ and $n_m = 1$ for each $m \in M'$. The Claim is established.

It follows that $\mathcal{C}_f = (R_f)^n \subset M_n(R_f)$. So $\mathrm{GL}_1(\mathcal{C})$ is an R -torus containing \mathfrak{T} and by maximality we conclude that $\mathfrak{T} = \mathrm{GL}_1(\mathcal{C})$. Since $\mathcal{C} \otimes_{\mathcal{O}_S} \kappa(s')$ is commutative of rank n for each $s' \in \mathrm{Spec}(R_f)$, we conclude that \mathcal{C} is a maximal finite étale R -subalgebra of \mathcal{A} .

(4) We come back to the initial setting (i.e. without localizing). We assume that \mathfrak{T} is of constant rank r . By (3), we have $\mathfrak{T} \subset \mathrm{GL}_1(\mathcal{C})$ where \mathcal{C} is a finite étale \mathcal{O}_S -algebra. According to [6, ch.0, 5.4.1], for each integer $l \geq 0$, $S_l = \{s \in S \mid \mathrm{rank}(\mathcal{C}_{\kappa(s)}) = l\}$ is an open subset of S so that we have a decomposition in clopen subschemes $S = \coprod_{l \geq 0} S_l$. Without loss of generality we can assume that \mathcal{C} is locally free of rank l .

We denote by Σ_l the constant S -group scheme associated to the permutation group of l letters and use now the correspondence of categories between that of finite étale \mathcal{O}_S -algebras of rank r and that of Σ_l -torsors [4, §2.5.2]. Then \mathcal{C} gives rise to a Σ_l -torsor \mathfrak{E} over S . Furthermore we have $\mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathfrak{E}} \cong (\mathcal{O}_{\mathfrak{E}})^l$. hence $\mathrm{GL}_1(\mathcal{C}) \times_S \mathfrak{E} \cong (\mathbb{G}_m)^l \times_S \mathfrak{E}$.

It follows that $\mathfrak{T} \times_S \mathfrak{E}$ is a subgroup \mathfrak{E} -scheme of $(\mathbb{G}_m)^l \times_S \mathfrak{E}$ of multiplicative type. According to [10, IX.2.11.(i)] there exists a partition in clopen subsets $\mathfrak{E} = \sqcup_{i \in I} \mathfrak{E}_i$ such that each $\mathfrak{T} \times_S \mathfrak{E}_i$ is diagonalizable. Since $\mathfrak{T} \times_S \mathfrak{E}$ is a torus of constant rank we conclude that $\mathfrak{T} \times_S \mathfrak{E}$ is diagonalizable so is isomorphic to $\mathbb{G}_{m, \mathfrak{E}}^r$. \square

3.2. Characterization of isotrivial tori.

Theorem 3.2. *Let \mathfrak{T} be an S -torus of constant rank r . Then the following are equivalent:*

- (i) \mathfrak{T} is isotrivial;
- (ii) \mathfrak{T} is linear;
- (iii) \mathfrak{T} is an subtorus of an S -group scheme $\mathrm{GL}_1(\mathcal{A})$ where \mathcal{A} is an Azumaya \mathcal{O}_S -algebra.

Proof. (i) \implies (ii). Let S' be a finite étale cover which splits \mathfrak{T} . We want to show that \mathfrak{T} is linear. According to [2, end of page 197], the map $\mathfrak{T} \rightarrow R_{S'/S}(\mathfrak{T} \times_S S')$ is a closed embedding. The isomorphism $R_{S'/S}(\mathfrak{T} \times_S S') \cong R_{S'/S}(\mathbb{G}_m^r)$ reduces the problem to the case of the induced torus $R_{S'/S}(\mathbb{G}_m)$. But $R_{S'/S}(\mathbb{G}_m) = \mathrm{GL}_1(\mathcal{O}_{S'})$ is an (maximal) torus of $\mathrm{GL}(\mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_{S'}))$ so we are done.

(ii) \implies (iii). By definition we have a monomorphism $\mathfrak{T} \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free \mathcal{O}_S -module of finite rank. Since $\mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$ is an Azumaya \mathcal{O}_S -algebra, we get (iii).

(iii) \implies (i). This follows of Proposition 3.1.(4). \square

Examples 3.3. (a) Grothendieck constructed a scheme S and an S -torus \mathfrak{T} which is locally trivial of rank 2 but which is not isotrivial [10, §X.1.6] (e.g. S consists of two copies of the projective line over a field pinched at 0 and ∞). Theorem 3.2 shows that such an S -torus is not linear.

(b) Also there exists a local ring R and an R -torus \mathfrak{T} of rank 2 which is not isotrivial [10, §1.6]; the ring R can be taken as the local ring of an algebraic curve at a double point. Theorem 3.2 shows that such an R -torus is not linear.

4. REDUCTIVE CASE

For stating the complete version of our main result, we need more notation. As in the introduction, \mathfrak{G} is a reductive S -group scheme of constant type and G is the underlying Chevalley \mathbb{Z} -group scheme. We denote by $\mathrm{Aut}(G)$ the automorphism group of G and we have an exact sequence of \mathbb{Z} -group schemes [10, th. XXIV.1.3]

$$1 \rightarrow G_{ad} \rightarrow \mathrm{Aut}(G) \rightarrow \mathrm{Out}(G) \rightarrow 1.$$

We remind the reader of the representability of the fppf sheaf $\underline{\mathrm{Isom}}(G_S, \mathfrak{G})$ by a $\mathrm{Aut}(G)_S$ -torsor $\mathrm{Isom}(G_S, \mathfrak{G})$ defined in [10, XXIV.1.8]. The contracted product

$$\mathrm{Isomext}(G_S, \mathfrak{G}) := \mathrm{Isom}(G_S, \mathfrak{G}) \wedge^{\mathrm{Aut}(G)_S} \mathrm{Out}(G)_S$$

is a $\mathrm{Out}(G)_S$ -torsor (*ibid*, 1.10) which encodes the isomorphism class of the quasi-split form of \mathfrak{G} .

Theorem 4.1. *The following are equivalent:*

- (i) *The torus $\mathrm{rad}(\mathfrak{G})$ is isotrivial;*
- (ii) *the $\mathrm{Out}(G)_S$ -torsor $\mathrm{Isomext}(G_S, \mathfrak{G})$ is isotrivial;*
- (iii) *\mathfrak{G} is linear;*
- (iv) *$\mathrm{rad}(\mathfrak{G})$ is linear.*

Furthermore if S is affine, we can take a faithful linear representation in some GL_n for (iii) and (iv).

Proof. (i) \implies (ii). We assume that $\mathrm{rad}(\mathfrak{G})$ is isotrivial and want to show that the $\mathrm{Out}(G)_S$ -torsor $\mathcal{F} = \mathrm{Isomext}(G_S, \mathfrak{G})$ is isotrivial. In other words, we want to show that there exists a finite étale cover S' of S such that $\mathfrak{G} \times_S S'$ is an inner form of G .

Without loss of generality we can assume that $\mathrm{rad}(\mathfrak{G})$ is a split torus. We quote now [10, XXIV.2.16] for the Chevalley group G over \mathbb{Z} which introduces the \mathbb{Z} -group scheme

$$H = \ker(\mathrm{Aut}(G) \rightarrow \mathrm{Aut}(\mathrm{rad}(G)));$$

furthermore there is an equivalence of categories between the category of H -torsors over S and the category of pairs (\mathfrak{M}, ϕ) where \mathfrak{M} is an S -form of G and $\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{M})$.

Since $\text{rad}(\mathfrak{G})$ is split, we choose an isomorphism $\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{G})$ and consider an H -torsor \mathfrak{P} mapping to an object isomorphic to (\mathfrak{G}, ϕ) . Furthermore the quoted reference provides an exact sequence of \mathbb{Z} -group schemes

$$1 \rightarrow G_{ad} \rightarrow H \xrightarrow{p} F \rightarrow 1$$

where F is finite étale over \mathbb{Z} , so is constant. We denote by $S' = \mathfrak{M} \wedge^{H_S} F_S$ the contracted product of \mathfrak{M} and F_S with respect to H_S , this is an F -torsor over S hence is a finite étale cover of S . It follows that $\mathfrak{P} \times_S S'$ admits a reduction to a $G_{ad, S'}$ -torsor \mathfrak{Q}' . Since the map $G_{ad} \rightarrow H \rightarrow \text{Aut}(G)$ is the canonical map, we conclude that $\mathfrak{G}_{S'} \cong {}^{\mathfrak{Q}'}G$ is an inner form of G .

(ii) \implies (iii). Our assumption is that there exists a finite étale cover S'/S which splits the $\text{Out}(G)$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$. Lemma 2.2 permits to replace S by S' , so we can assume that the $\text{Out}(G)$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is trivial, that is \mathfrak{G} is an inner form of G . There exists a G_{ad} -torsor \mathfrak{Q} over S such that $\mathfrak{G} \cong {}^{\mathfrak{Q}}G$. Since $G \rtimes G_{ad}$ is defined over \mathbb{Z} , it admits a faithful representation $\rho : G \rtimes G_{ad} \rightarrow \text{GL}_n$ over \mathbb{Z} [3, §1.4.5]. The map ρ is then G_{ad} -equivariant and can be twisted by the G_{ad} -torsor \mathfrak{Q} . We obtain a faithful representation ${}^{\mathfrak{Q}}G \rtimes {}^{\mathfrak{Q}}G_{ad} \rightarrow {}^{\mathfrak{Q}}\text{GL}_n = \text{GL}(\mathcal{E})$ where \mathcal{E} is the locally free \mathcal{O}_S -module of rank n which is the twist of $(\mathcal{O}_S)^n$ by the GL_n -torsor $\mathfrak{Q} \wedge^{G_{ad}} \text{GL}_n$. Thus ${}^{\mathfrak{Q}}G$ is linear.

(iii) \implies (iv). Obvious.

(iv) \implies (i). Since $\text{rad}(\mathfrak{G})$ is a form of $\text{rad}(G)$, it is of constant rank and Theorem 3.2 shows that $\text{rad}(G)$ is isotrivial.

Finally the refinement for S affine follows from Lemma 2.1. \square

Corollary 4.2. *Under the assumptions of \mathfrak{G} , let \mathfrak{G}^{qs} be the quasi-split form of \mathfrak{G} . Then \mathfrak{G} is linear if and only if \mathfrak{G}^{qs} is linear.*

The next corollary slightly generalizes a result by Thomason [12, cor. 3.2].

Corollary 4.3. *Assume that either*

- (i) S is normal and locally noetherian;
- (ii) $\text{rad}(G)$ is of rank ≤ 1 (in particular if G is semisimple).

Then \mathfrak{G} is linear.

Proof. In case (i), the torus $\text{rad}(\mathfrak{G})$ is isotrivial [10, X.5.16]. In case (ii), we have $\text{rad}(G) = 1$ or \mathbb{G}_m , so that $\text{rad}(\mathfrak{G})$ is split by a quadratic étale cover of S , hence is isotrivial. Hence Theorem 1.1 implies that \mathfrak{G} is linear. \square

The next corollary extends Demazure's characterization of locally isotrivial reductive group schemes [10, XXIV.4.1.5].

Corollary 4.4. *The following are equivalent:*

- (i) \mathfrak{G} is locally (resp. semilocally) isotrivial;
- (ii) The torus $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) isotrivial;
- (iii) the $\text{Out}(G)_S$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is locally (resp. semilocally) isotrivial;
- (iv) \mathfrak{G} is locally (resp. semilocally) linear;
- (v) $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) linear.

Proof. In view of Theorem 4.1, it remains to establish the equivalence (i) \iff (ii). Since this is precisely the quoted result [10, XXIV.3.5], the proof is complete. \square

Corollary 4.5. *Let \mathfrak{H} be a reductive S -subgroup scheme of \mathfrak{G} . If \mathfrak{G} is locally (resp. semilocally) isotrivial, then \mathfrak{H} is locally (resp. semilocally) isotrivial.*

Proof. Corollary 4.4 shows that \mathfrak{G} is locally (resp. semilocally) linear and so is \mathfrak{H} . Therefore \mathfrak{H} is locally (resp. semilocally) isotrivial. \square

5. THE SEMILOCAL CASE

We assume that $S = \text{Spec}(R)$ where R is a semilocal ring and continue to assume that the reductive S -group scheme \mathfrak{G} is of constant type. We remind the reader that \mathfrak{G} admits a maximal torus (Grothendieck, [10, XIV.3.20]).

Corollary 5.1. *Let \mathfrak{T} be a maximal torus of \mathfrak{G} . The following are equivalent:*

- (i) \mathfrak{G} is isotrivial;
- (ii) The torus $\text{rad}(\mathfrak{G})$ is isotrivial;
- (iii) the $\text{Out}(G)_S$ -torsor $\text{Isomext}(G_S, \mathfrak{G})$ is isotrivial;
- (iv) \mathfrak{G} is linear;
- (v) $\text{rad}(\mathfrak{G})$ is linear;
- (vi) \mathfrak{T} is linear;
- (vii) \mathfrak{T} is isotrivial.

Proof. From Corollary 4.4, we have the equivalences (i) \iff (ii) \iff (iii) \iff (iv) \iff (v). On the other hand, the equivalence (vi) \iff (vii) holds according to Theorem 3.2. Now we observe that the implications (iv) \implies (vi) and (vi) \implies (v) are obvious so the proof is complete. \square

6. EQUIVARIANT RESOLUTION PROPERTY

Definition 6.1. *Let \mathfrak{G} be a flat group scheme over S acting on an S -scheme \mathfrak{X} . One says that $(\mathfrak{G}, S, \mathfrak{X})$ has the resolution property (RE) for short if for every coherent \mathfrak{G} -module \mathcal{F} on \mathfrak{X} , there is a locally free coherent \mathfrak{G} -module (i.e. a \mathfrak{G} -vector bundle \mathcal{E}) and a \mathfrak{G} -equivariant epimorphism $\mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$.*

We strengthen Thomason's results.

Theorem 6.2. *Let \mathfrak{G} be a reductive S -group scheme. We assume that S is separated noetherian and that $(1, S, S)$ satisfies the resolution property, e.g. S is affine or regular or admits an ample family of line bundles. Then the following are equivalent:*

- (i) \mathfrak{G} is linear;
- (ii) $\text{rad}(\mathfrak{G})$ is isotrivial;
- (iii) \mathfrak{G} satisfies (RE).

Proof. (i) \iff (ii). This is a special case of Theorem 4.1.

(ii) \implies (iii). This is Thomason's result [12, Theorem 2.18].

(iii) \implies (i). This is Thomason's result [12, Theorem 3.1], see also [10, VI_B.13.5]. \square

Remark 6.3. Example 3.3.(b) is an example of a local noetherian ring R and of a rank two non-isotrivial torus \mathfrak{T} . Theorem 6.2 shows that \mathfrak{T} does not satisfy (RE). This answers a question of Thomason [12, §2.3].

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