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# NON-AMBIGUOUS TREES: NEW RESULTS AND GENERALISATION

JEAN-CHRISTOPHE AVAL, ADRIEN BOUSSICAULT, BÉRÉNICE DELCROIX-OGER, FLORENT HIVERT, AND PATXI LABORDE-ZUBIETA

ABSTRACT. We present a new definition of non-ambiguous trees (NATs) as labelled binary trees. We thus get a differential equation whose solution can be described combinatorially. This yields a new formula for the number of NATs. We also obtain  $q$ -versions of our formula. We finally generalise NATs to higher dimension.

## INTRODUCTION

Non-ambiguous trees (NATs for short) were introduced in a previous paper [ABBS14]. We propose in the present article a sequel to this work.

Tree-like tableaux [ABN13] are certain fillings of Ferrers diagram, in simple bijection with permutations or alternative tableaux [Pos07, Vie08]. They are the subject of an intense research activity in combinatorics, mainly because they appear as the key tools in the combinatorial interpretation of the well-studied model of statistical mechanics called PASEP: they naturally encode the states of the PASEP, together with the transition probabilities through simple statistics [CW07].

Among tree-like tableaux, NATs were defined as rectangular-shaped objects in [ABBS14]. In this way, they are in bijection with permutations  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$  such that the exceedances ( $\sigma_i > i$ ) are placed at the beginning of the word  $\sigma$ . Such permutations were studied by Ehrenborg and Steingrímsson [ES00], who obtained an explicit enumeration formula. Thanks to NATs, a bijective proof of this formula was described in [ABBS14].

In the present work, we define NATs as labelled binary trees (see Definition 1.1, which is equivalent to the original definition). This new presentation allows us to obtain many new results about these objects. The plan of the article is the following.

In Section 1, we (re-)define NATs as binary trees whose right and left

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children are respectively labelled with two sets of labels. We show how the generating series for these objects satisfies differential equations (Prop. 1.10), whose solution is quite simple and explicit (Prop. 1.11). A combinatorial interpretation of this expression involves the (new) notion of hooks in binary trees, linked to the notion of leaves in ordered trees. Moreover this expression yields a new formula for the number of NATs as a positive sum (see Proposition 1.12), where Ehrenborg-Steingrímsson's formula is alternating. It should be noted that Prop. 1.11 and Proposition 1.12 (in the case  $\alpha = \beta = 1$ ) were already proven by Clark and Ehrenborg [CE10]. To conclude with Section 1, we obtain  $q$ -analogues of our formula, which are similar to those obtained for binary trees in [BW89, HNT08] (see Theorem 1.28, the relevant statistics are either the number of inversions or the inverse major index).

Section 2 presents a generalisation of NATs in higher dimension. For any  $k \leq d$ , we consider NATs of dimension  $(d, k)$ , embedded in  $\mathbb{Z}^d$ , and with edges of dimension  $k$ <sup>1</sup>. The original case corresponds to dimension  $(2, 1)$ . Our main result on this question is a differential equation satisfied by the generating series of these new objects.

Finally, we study the (new) notion of hooks on binary trees in Section 3. We prove (through the use of generating series, and bijectively) that the number of hooks is distributed on binary trees as another statistics: the childleaf statistic, defined as the number of vertices who has at least one leaf as a child.

## 1. NON-AMBIGUOUS TREES

**1.1. Definitions.** We recall that a *binary tree* is a rooted tree whose vertices may have no child, one left child, one right child or both of them. The size of a binary tree is its number of vertices. Usually, it is considered that there is a unique binary tree with no vertex, it is called the *empty binary tree*. In this article, we consider that there are two binary trees of size 0: the *left empty binary tree* and the *right empty binary tree*, they are respectively notes  $\emptyset_L$  and  $\emptyset_R$ . Having no child in the left direction (resp. right direction) is the same as having the left (resp. right) empty subtree in this direction. We denote by  $\mathcal{BT}$  the set of binary trees. Given a binary tree  $B$ , we denote by  $\mathcal{V}_L(B)$  and  $\mathcal{V}_R(B)$  the set of left children (also called left vertices) and the set of right children (also called right vertices). By convention,  $\mathcal{V}_L(\emptyset_L) = \mathcal{V}_R(\emptyset_R) = -1$  and  $\mathcal{V}_L(\emptyset_R) = \mathcal{V}_R(\emptyset_L) = 0$ . We shall extend this notation to NATs. Let  $U$  and  $V$  be two vertices of a binary tree

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<sup>1</sup>A definition in terms of labelled trees is given in Subsection 2.1.

$B$ . If  $V$  is a vertex of the subtree of  $B$  whose root is  $U$ , then  $V$  is a *descendant* of  $U$  and  $U$  an *ancestor* of  $V$ .

We now define the notion of non-ambiguous trees:


**Definition 1.1.** A non-ambiguous tree (NAT)  $T$  is a labelling of a binary tree  $B$  such that:

- the left (resp. right) children are labelled from 1 to  $|\mathcal{V}_L(B)|$  (resp.  $|\mathcal{V}_R(B)|$ ), such that different left (resp. right) vertices have different labels. In other words, each left (right) label appears exactly once.
- if  $U$  and  $V$  are two left (resp. right) children in the tree, such that  $U$  is an ancestor of  $V$ , then the label of  $U$  in  $T$  is strictly greater than the label of  $V$ .

The underlying binary tree of a non-ambiguous tree is called its *shape*. By convention, there is a unique NAT whose shape is  $\emptyset_L$  (resp.  $\emptyset_R$ ) which is also denoted  $\emptyset_L$  (resp.  $\emptyset_R$ ). We denote by  $\mathcal{NAT}(B)$  the set of NATs of shape  $B$ .

It is sometimes useful to label the root as well. In this case, it is considered as both a left and right child so that it carries a pairs of labels, namely  $(|\mathcal{V}_L(T)| + 1, |\mathcal{V}_R(T)| + 1)$ . On pictures, to ease the reading, we color the labels of left and right vertices in red and blue respectively.

In [ABBS14], NATs were defined in a slightly different, more geometrical way. We recall it here, and show that the two definitions are equivalent. Formally, a (geometric) NAT of size  $n$  is a set  $A$  of  $n$  points  $(x, y) \in \mathbb{N} \times \mathbb{N}$  such that:

- (1)  $(0, 0) \in A$ ; we call this point the *root* of  $A$ ;
- (2) given a non-root point  $p = (x, y) \in A$ , there exists one point  $q = (x', y') \in A$  such that  $y' < y$  and  $x' = x$ , or one point  $r = (x', y') \in A$  such that  $x' < x$ ,  $y' = y$ , but not both (which means that the pattern  is avoided);
- (3) there is no empty line between two given points: if there exists a point  $p = (x, y) \in A$ , then for every  $x' < x$  (resp.  $y' < y$ ) there exists  $q = (x'', y'') \in A$  such that  $x'' = x'$  (resp.  $y'' = y'$ ).

To see that this presentation is equivalent to Definition 1.1, we refer to Figure 1 (top). Let us consider a NAT  $T$  presented as a labeled tree (Definition 1.1). Each vertex of  $T$  may be given a pair of coordinates  $(x, y)$  as follows. For a left (resp. right) child, its  $x$  (resp.  $y$ ) coordinate is its label in  $T$ . and its  $y$  (resp.  $x$ ) coordinate is the one of its closest ancestor which is a right child, or the root. We thus get a geometrical

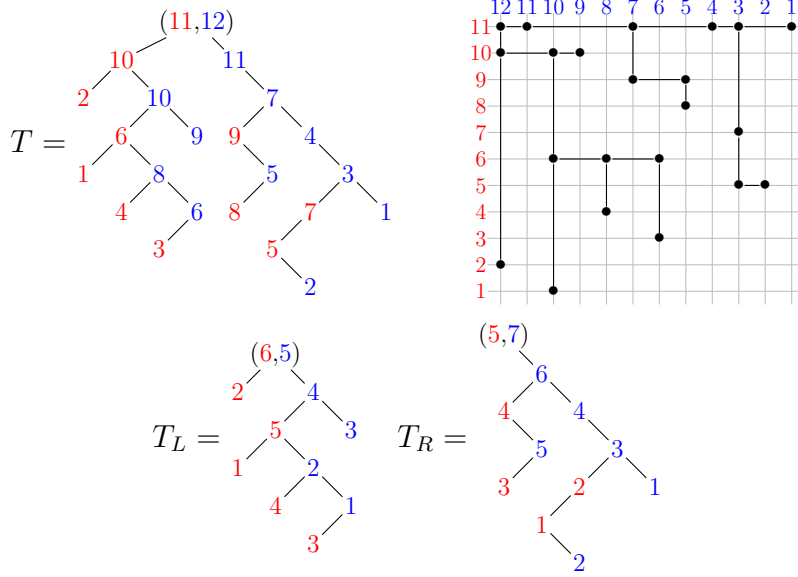


FIGURE 1. A non-ambiguous tree, its geometrical presentation, and its left and right subtrees

object, which can be drawn as shown in Figure 1 (top, right). With the minor change of coordinates  $(x', y') = (|\mathcal{V}_L(T)| + 1 - x, |\mathcal{V}_R(T)| + 1 - y)$ , we get a set of points which satisfies the geometrical definition. The only thing that needs to be checked is the avoidance of the pattern



. Let us suppose the faulty vertex  $p = (x, y)$  is a right child. Its  $y$  coordinate is its label in  $T$ , and we suppose that there is a vertex  $q = (x', y)$  with  $x' < x$ , thus  $q$  is not a descendant of  $p$ . The  $y$  coordinate of  $q$  is either its own label (if it is a right child), or inherited from an ancestor, a right child  $q' = (x'', y)$  with  $x'' < x' < x$ . It is absurd that  $p$  and  $q'$  have the same label in  $T$ .

Conversely, to go from the geometric version of a NAT to Definition 1.1, we just have to forget the redundant coordinate.

The top part of Figure 1 shows an example of a NAT, and illustrates the correspondence between the geometrical presentation of [ABBS14] and Definition 1.1. The dimension  $w_L(T) \times w_R(T)$  of the rectangle containing the geometrical presentation of  $T$ , is called the geometric size of  $T$  and satisfies

$$(w_L(T), w_R(T)) = (|\mathcal{V}_L(T)| + 1, |\mathcal{V}_R(T)| + 1).$$

By convention, the non-ambiguous trees  $\emptyset_L$  and  $\emptyset_R$  satisfies respectively  $(w_L(\emptyset_L), w_R(\emptyset_L)) = (0, 1)$  and  $(w_L(\emptyset_R), w_R(\emptyset_R)) = (1, 0)$ .

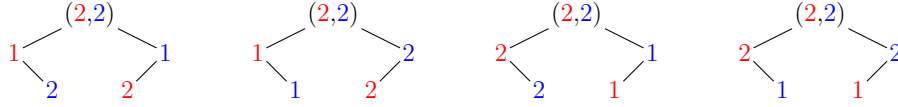

 FIGURE 2. An example of  $\mathcal{NAT}(B)$ 

Figure 2 gives an example of a class  $\mathcal{NAT}(B)$ , in the case where the binary tree is  $B = \cdot \swarrow \cdot \searrow \cdot$ .

**1.2. Differential equations on non-ambiguous trees.** The goal of this section is to get (new) formulas for the number of NATs with prescribed shape. The crucial argument is the following remark: let  $T$  be a NAT whose shape is a non-empty binary tree  $B = \begin{matrix} \bullet \\ \swarrow \searrow \\ L \quad R \end{matrix}$ . Restricting the labellings of the left and right children of  $T$  to  $L$  and  $R$  gives non-decreasing labelling of their respective left and right children. Note that the root of  $L$  (resp.  $R$ ) is a left (resp. right) child in  $T$ . By renumbering the labels so that they are consecutive numbers starting from 1, we get two non-ambiguous labellings for  $L$  and  $R$ , that is two non-ambiguous trees  $T_L$  and  $T_R$ . See Figure 1 for an example.

Conversely, knowing the labelling of  $L$  and  $R$ , to recover the labelling of  $T$ , one has to choose which labels among  $\{1, \dots, \mathcal{V}_L(T)\}$  will be used for  $L$  (including its root) and the same for right labels. As a consequence:

$$|\mathcal{NAT}(\begin{matrix} \bullet \\ \swarrow \searrow \\ L \quad R \end{matrix})| = \binom{|\mathcal{V}_L(T)|}{|\mathcal{V}_L(R)|} \binom{|\mathcal{V}_R(T)|}{|\mathcal{V}_R(L)|} |\mathcal{NAT}(L)| |\mathcal{NAT}(R)|. \quad (1)$$

Our first step is to recover the hook-length formula for the number of NATs of fixed shape ([ABBS14]). We use the method from [HNT08], namely, applying recursively a bilinear integro-differential operator called here a *pumping function* along a binary tree.

First of all, we consider the  $\mathbb{Q}$ -vector space  $\mathbb{Q}\mathcal{NAT}$  of formal sums of non-ambiguous trees and identifies  $\mathcal{NAT}(B)$  with the formal sum of its elements. We consider also the  $\mathbb{Q}$ -vector spaces  $\mathbb{Q}\mathcal{NAT}_L$  and  $\mathbb{Q}\mathcal{NAT}_R$  generated respectively by  $\mathcal{NAT} \setminus \{\emptyset_R\}$  and  $\mathcal{NAT} \setminus \{\emptyset_L\}$ . Let  $\mathbb{M}$  be the linear map

$$\mathbb{M} : \mathbb{Q}\mathcal{NAT}_L \times \mathbb{Q}\mathcal{NAT}_R \mapsto \mathbb{Q}\mathcal{NAT}$$

sending a pair of non-ambiguous trees  $(T_1, T_2)$  to the formal sum of NATs  $T$  such that  $T_L = T_1$  and  $T_R = T_2$ . The main remark is that  $\mathbb{Q}\mathcal{NAT}(B)$  can be computed by a simple recursion using  $\mathbb{M}$ .

**Lemma 1.2.** *The formal sum  $\mathbb{Q}\mathcal{NAT}(B)$  of non-ambiguous trees of shape  $B$  satisfies the following recursion: if  $B = \emptyset_L$  or  $B = \emptyset_R$  then  $\mathbb{Q}\mathcal{NAT}(B) = B$ , else*

$$\mathbb{Q}\mathcal{NAT}\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ L \quad R \end{array}\right) = \mathbb{M}(\mathbb{Q}\mathcal{NAT}(L), \mathbb{Q}\mathcal{NAT}(R)).$$

To count non-ambiguous trees, and as suggested by the binomial coefficients in (1), we shall use *doubly exponential generating functions* in two variables  $x$  and  $y$ , where  $x$  and  $y$  count the geometric size: the weight of a NAT  $T$  is  $\Phi(T) := \frac{x^{w_L(T)} y^{w_R(T)}}{w_L(T)! w_R(T)!}$ . We extend  $\Phi(T)$  by linearity to a map  $\mathbb{Q}\mathcal{NAT} \mapsto \mathbb{Q}[[x, y]]$ . Consequently,  $\Phi(\mathcal{NAT}(B))$  is the generating series of the non-ambiguous trees of shape  $B$ . Thanks to (1) the image in  $\mathbb{Q}[[x, y]]$  of the bilinear map  $\mathbb{M}$  under the map  $\Phi$  is a simple differential operator:

**Definition 1.3.** *The pumping function  $\mathbb{B}$  is the bilinear map  $\mathbb{Q}[[x, y]] \times \mathbb{Q}[[x, y]] \mapsto \mathbb{Q}[[x, y]]$  defined by*

$$\mathbb{B}(f, g) = \int_0^x \int_0^y \partial_y f(u, v) \cdot \partial_x g(u, v) du dv. \quad (2)$$

We further define recursively, for any binary tree  $B$  an element  $\mathbb{B}(B) \in \mathbb{Q}[[x, y]]$  by

$$\mathbb{B}(\emptyset_L) = y, \quad \mathbb{B}(\emptyset_R) = x \quad \text{and} \quad \mathbb{B}\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ L \quad R \end{array}\right) = \mathbb{B}(\mathbb{B}(L), \mathbb{B}(R)). \quad (3)$$

A simple computation gives us that for  $f = \frac{x^{a_1} y^{b_1}}{a_1! b_1!}$  and  $g = \frac{x^{a_2} y^{b_2}}{a_2! b_2!}$  one has

$$\mathbb{B}(f, g) = \binom{a_1 + a_2 - 1}{a_2 - 1} \binom{b_1 + b_2 - 1}{b_1 - 1} \frac{x^{a_1 + a_2}}{(a_1 + a_2)!} \frac{y^{b_1 + b_2}}{(b_1 + b_2)!}.$$

Whence for  $(T_1, T_2)$  a pair of NATs in  $\mathbb{Q}\mathcal{NAT}_L \times \mathbb{Q}\mathcal{NAT}_R$ , we get:

$$\begin{aligned} \mathbb{B}(\Phi(T_1), \Phi(T_2)) &= \binom{w_L(T_1) + w_L(T_2) - 1}{w_L(T_2) - 1} \binom{w_R(T_1) + w_R(T_2) - 1}{w_R(T_1) - 1} \\ &\quad \times \frac{x^{w_L(T_1) + w_L(T_2)}}{(w_L(T_1) + w_L(T_2))!} \frac{y^{w_R(T_1) + w_R(T_2)}}{(w_R(T_1) + w_R(T_2))!} \end{aligned}$$

Thus (1) may be rewritten as

**Lemma 1.4.** *For  $(T_1, T_2)$  a pair of NATs in  $\mathbb{Q}\mathcal{NAT}_L \times \mathbb{Q}\mathcal{NAT}_R$ , one has*

$$\Phi(\mathbb{M}(T_1, T_2)) = \mathbb{B}(\Phi(T_1), \Phi(T_2)).$$

We derive from these two lemmas the following proposition.

**Proposition 1.5.** *For any non-empty binary tree  $B$ ,*

$$\Phi(\mathcal{NAT}(B)) = \mathbb{B}(B).$$

*Proof.* We apply Lemmas 1.2 and 1.4 to write (with  $L$  and  $R$  the left and right sub-tree of  $B$ ):

$$\begin{aligned} \Phi(\mathcal{NAT}(B)) &= \Phi(\mathbb{M}(\mathbb{Q}\mathcal{NAT}(L), \mathbb{Q}\mathcal{NAT}(R))) \\ &= \Phi(\mathbb{M}(\sum_{T_1 \in \mathcal{NAT}(L)} T_1, \sum_{T_2 \in \mathcal{NAT}(R)} T_2)) \\ &= \sum_{T_1 \in \mathcal{NAT}(L)} \sum_{T_2 \in \mathcal{NAT}(R)} \Phi(\mathbb{M}(T_1, T_2)) \\ &= \sum_{T_1 \in \mathcal{NAT}(L)} \sum_{T_2 \in \mathcal{NAT}(R)} \mathbb{B}(\Phi(T_1), \Phi(T_2)) \\ &= \mathbb{B}\left(\sum_{T_1 \in \mathcal{NAT}(L)} \Phi(T_1), \sum_{T_2 \in \mathcal{NAT}(R)} \Phi(T_2)\right) = \mathbb{B}(\mathbb{B}(L), \mathbb{B}(R)). \end{aligned}$$

□

We are now able to recover the hook-length formula of [ABBS14] for non-ambiguous trees of a given shape.

**Proposition 1.6.** *Let  $B$  be a binary tree. For each non-root left (resp. right) vertex  $U$ , we denote  $\mathcal{E}_L(U)$  (resp.  $\mathcal{E}_R(U)$ ) the number of left (resp. right) vertices of the subtree with root  $U$  (itself included in the count). Then*

$$|\mathcal{NAT}(B)| = \frac{|\mathcal{V}_L(B)|! \cdot |\mathcal{V}_R(B)|!}{\prod_{U:\text{left child}} \mathcal{E}_L(U) \cdot \prod_{U:\text{right child}} \mathcal{E}_R(U)}. \quad (4)$$

*Proof.* Proposition 1.5 may be rewritten as:

$$\mathbb{B}(B) = |\mathcal{NAT}(B)| \frac{x^{w_L(B)} y^{w_R(B)}}{w_L(B)! w_R(B)!}. \quad (5)$$



So, we get by a simple computation :

$$\begin{aligned}
\mathbb{B}(B) &= \mathbb{B}\left(\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ L \quad R \end{array}\right) = \mathbb{B}(\mathbb{B}(L), \mathbb{B}(R)) \\
&= \mathbb{B}\left(|\mathcal{NAT}(L)| \frac{x^{w_L(L)} y^{w_R(L)}}{w_L(L)! w_R(L)!}, |\mathcal{NAT}(R)| \frac{x^{w_L(R)} y^{w_R(R)}}{w_L(R)! w_R(R)!}\right) \\
&= \frac{|\mathcal{NAT}(L)| \cdot |\mathcal{NAT}(R)|}{w_L(L)! w_R(L)! w_L(R)! w_R(R)!} \mathbb{B}(x^{w_L(L)} y^{w_R(L)}, x^{w_L(R)} y^{w_R(R)}) \\
&= \frac{|\mathcal{NAT}(L)| \cdot |\mathcal{NAT}(R)| w_R(L) w_L(R)}{w_L(L)! w_R(L)! w_L(R)! w_R(R)!} \frac{x^{w_L(L)+w_L(R)} y^{w_R(L)+w_R(R)}}{(w_L(L) + w_L(R)) \cdot (w_R(L) + w_R(R))} \\
&= \frac{w_R(L) w_L(R)}{(w_L(L) + w_L(R)) \cdot (w_R(L) + w_R(R))} \mathbb{B}(L) \cdot \mathbb{B}(R) \\
&= \frac{w_R(L) w_L(R)}{w_L(B) w_R(B)} \mathbb{B}(L) \cdot \mathbb{B}(R)
\end{aligned}$$

We thus have

$$\mathbb{B}(B) = |\mathcal{NAT}(B)| \frac{x^{w_L(B)} y^{w_R(B)}}{w_L(B)! w_R(B)!}$$

And,

$$\begin{aligned}
\mathbb{B}(B) &= \frac{w_R(L) w_L(R)}{w_L(B) w_R(B)} \mathbb{B}(L) \cdot \mathbb{B}(R) \\
&= \frac{w_R(L) w_L(R)}{w_L(B) w_R(B)} |\mathcal{NAT}(L)| \frac{x^{w_L(L)} y^{w_R(L)}}{w_L(L)! w_R(L)!} \cdot |\mathcal{NAT}(R)| \frac{x^{w_L(R)} y^{w_R(R)}}{w_L(R)! w_R(R)!}
\end{aligned}$$

which, together with (5) gives:

$$\begin{aligned}
|\mathcal{NAT}(B)| \frac{x^{w_L(B)} y^{w_R(B)}}{w_L(B)! w_R(B)!} &= \frac{w_R(L) w_L(R)}{w_L(B) w_R(B)} \cdot |\mathcal{NAT}(L)| \frac{x^{w_L(L)} y^{w_R(L)}}{w_L(L)! w_R(L)!} \\
&\quad \cdot |\mathcal{NAT}(R)| \frac{x^{w_L(R)} y^{w_R(R)}}{w_L(R)! w_R(R)!} .
\end{aligned}$$

We obtain

$$\begin{aligned}
\frac{|\mathcal{NAT}(B)|}{(w_L(B) - 1)! (w_R(B) - 1)!} &= w_R(L) w_L(R) \frac{|\mathcal{NAT}(L)| \cdot |\mathcal{NAT}(R)|}{w_L(L)! w_R(L)! w_L(R)! w_R(R)!} \\
&= \frac{1}{w_L(L) w_R(R)} \frac{|\mathcal{NAT}(L)|}{(w_L(L) - 1)! (w_R(L) - 1)!} \\
&\quad \cdot \frac{|\mathcal{NAT}(R)|}{(w_L(R) - 1)! (w_R(R) - 1)!} .
\end{aligned}$$

We deduce that

$$\frac{|\mathcal{NAT}(B)|}{\mathcal{V}_L(B)!\mathcal{V}_R(B)!} = \frac{1}{w_L(L)w_R(R)} \cdot \frac{|\mathcal{NAT}(L)|}{\mathcal{V}_L(L)!\mathcal{V}_R(L)!} \cdot \frac{|\mathcal{NAT}(R)|}{\mathcal{V}_L(R)!\mathcal{V}_R(R)!}.$$

The coefficient  $\frac{1}{w_L(L)w_R(R)}$  recursively gives the denominator of the hook-length formula, and we get (4).  $\square$

Let  $\mathfrak{G}$  be the *exponential generating function of non-ambiguous trees* with weight  $\Phi$ :

$$\mathfrak{G}(x, y) := \sum_{T \in \mathcal{NAT}} \Phi(T) = \sum_{T \in \mathcal{NAT}} \frac{x^{w_L(T)} x^{w_R(T)}}{w_L(T)! w_R(T)!}. \quad (6)$$

and  $\mathfrak{N}$  its derivative  $\mathfrak{N} = \partial_x \partial_y \mathfrak{G}$ . Naturally, they are linked by the relation

$$\mathfrak{G}(x, y) = y + x + \int_0^x \int_0^y \mathfrak{N}(u, v) du dv.$$

They both are solutions of a fixed point differential equation.

**Proposition 1.7.** *The generating function  $\mathfrak{N}$  and  $\mathfrak{G}$  can be computed by the following fixed point differential equations:*

$$\mathfrak{G} = y + x + \int_x \int_y \partial_x \mathfrak{G} \cdot \partial_y \mathfrak{G} \quad \text{and} \quad \mathfrak{N} = \left(1 + \int_x \mathfrak{N}\right) \cdot \left(1 + \int_y \mathfrak{N}\right) \quad (7)$$

*Proof.* The first equation is a consequence of the definition of the bilinear map  $\mathbb{B}$ :

$$\begin{aligned} \mathfrak{G} &= \sum_{B \in \mathcal{BT}} \mathbb{B}(B) \\ &= y + x + \sum_{(L, R) \in \mathcal{BT}_L \times \mathcal{BT}_R} \mathbb{B}\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ L \quad R \end{array}\right) \\ &= y + x + \sum_{(L, R) \in \mathcal{BT}_L \times \mathcal{BT}_R} \mathbb{B}(\mathbb{B}(L), \mathbb{B}(R)) \\ &= y + x + \mathbb{B}(\mathfrak{G} - x, \mathfrak{G} - y) \\ &= y + x + \mathbb{B}(\mathfrak{G}, \mathfrak{G}), \end{aligned}$$

with  $\mathcal{BT}_L = \mathcal{BT} \setminus \{\emptyset_R\}$  and  $\mathcal{BT}_R = \mathcal{BT} \setminus \{\emptyset_L\}$ . To prove the second equation, remark that the first equation implies the identity

$$\partial_x \partial_y \mathfrak{G} = \partial_x \mathfrak{G} \cdot \partial_y \mathfrak{G}$$

and moreover that we have

$$\partial_x \mathfrak{G} = 1 + \int_y \mathfrak{N} \quad \text{and} \quad \partial_y \mathfrak{G} = 1 + \int_x \mathfrak{N}.$$

□

From these identities, a closed formula can be computed for  $\mathfrak{N}$  and  $\mathfrak{G}$ . The expression of  $\mathfrak{N}$  was already proven in [CE10] using permutations.

**Proposition 1.8.** *The doubly exponential generating functions for non-ambiguous trees are given by*

$$\mathfrak{G} = x+y-\log(1-(e^x-1)(e^y-1)), \quad \text{and} \quad \mathfrak{N} = \frac{e^{x+y}}{(1-(e^x-1)(e^y-1))^2}.$$

*Proof.* We know that  $\mathfrak{G}$  is a solution of

$$\begin{cases} \partial_x \partial_y f = \partial_x f \times \partial_y f, \\ f(x, y) = f(y, x) \end{cases} \quad (8)$$

This system of equation satisfies the two following properties:

- if  $s$  is a solution of Equation 8 then for each power series  $\varphi$  with constant term equal to zero,  $f(\varphi(x), \varphi(y))$  is also a solution;
- if we fix the initial condition  $f(x, 0)$ , there exists a unique formal power series solution to Equation 8.

Let  $f$  be a particular solution. Let us consider the notation  $f_x := \partial_x f$  and  $f_y := \partial_y f$ , then

$$\partial_x f_y \cdot \partial_y f_x = f_x^2 f_y^2.$$

We suppose that  $\partial_x f_y = f_y^2$  and  $\partial_y f_x = f_x^2$ , hence

$$f_y = \frac{-1}{x + c_1(y)} \quad \text{and} \quad f_x = \frac{-1}{y + c_2(x)}.$$

Since  $f_x^2 = \partial_y f_x = f_x f_y$ , we get  $f_y = f_x$ , which implies

$$x + c_1(y) = y + c_2(x).$$

As a consequence,  $c_1(z) = c_2(z) = z + c$  with  $c$  a real number. Finally

$$f(x, y) = -\ln(x + y + c).$$

Conversely,  $-\ln(x + y + c)$  satisfies Equation (8). It remains to find a real number  $c$  and formal power series  $\varphi$  such that  $\mathfrak{G}(x, 0) = -\ln(\varphi(x) + c)$ . Since  $\mathfrak{G}(x, 0) = x$ , we get  $\varphi(x) = e^{-x} - c$ . Moreover, the condition  $\varphi(0) = 0$  implies  $c = 1$ . As a consequence

$$\mathfrak{G}(x, y) = -\ln(e^{-x} + e^{-y} - 1),$$

which can be rewritten as

$$\mathfrak{G}(x, y) = y + x - \ln(1 - (e^x - 1)(e^y - 1)).$$

Differentiating with respect to  $x$  and  $y$ , we find the expression of  $\mathfrak{N}$ . □

In the context of the PASEP, it is natural to consider the following statistics.

**Definition 1.9.** *The leftmost branch of a binary tree  $B$  is the set of vertices  $\{s_0, \dots, s_k\}$  such that  $s_0$  is the root of  $B$ ,  $s_k$  is a leaf and  $s_{i+1}$  is the left child of  $s_i$ , for each  $i < k$ . Similarly, we define the rightmost branch of a binary tree. We denote by  $\mathcal{L}_0(B)$  and  $\mathcal{R}_0(B)$  the number of non-root vertices respectively in the leftmost and rightmost branches.*

We extend these definitions to non-ambiguous trees. For example, in Figure 1, we have  $\mathcal{L}_0(T) = 2$  and  $\mathcal{R}_0(T) = 5$ . These statistics correspond to the parameters  $\alpha$  and  $\beta$  in the PASEP. Let us define the following  $(\alpha, \beta)$ -generating function for non-ambiguous trees:

$$\mathfrak{N}(x, y; \alpha, \beta) = \sum_{T \in \mathcal{NAT}} \frac{x^{|\mathcal{V}_L(T)|} \cdot y^{|\mathcal{V}_R(T)|} \cdot \alpha^{\mathcal{L}_0(T)} \cdot \beta^{\mathcal{R}_0(T)}}{|\mathcal{V}_L(T)|! \cdot |\mathcal{V}_R(T)|!}.$$

It satisfies an  $(\alpha, \beta)$ -analogue of the identity of Proposition 1.7.

**Proposition 1.10.** *A differential equation for  $\mathfrak{N}(x, y; \alpha, \beta)$  is*

$$\mathfrak{N}(x, y; \alpha, \beta) = \left(1 + \alpha \int_x \mathfrak{N}(u, y; \alpha, 1) du\right) \cdot \left(1 + \beta \int_y \mathfrak{N}(x, v; 1, \beta) dv\right).$$

*Proof.* We just need to define a new pumping function:

$$\mathbb{B}^{(\alpha, \beta)}(f, g) = \alpha\beta \mathbb{B}(f|_{\beta=1}, g|_{\alpha=1})$$

and deduce the expected differential equation.  $\square$

The solution of the new differential equation is given by Proposition 1.11, a bijective proof is given in Section 1.3

**Proposition 1.11.** *The  $(\alpha, \beta)$ -exponential generating function for non-ambiguous trees is equal to*

$$\mathfrak{N}(x, y; \alpha, \beta) = \frac{e^{\alpha x + \beta y}}{(1 - (e^x - 1)(e^y - 1))^{\alpha + \beta}}.$$

If we develop this expression we obtain an  $(\alpha, \beta)$ -analogue of the enumeration of non ambiguous-trees of fixed geometric size. In order to express it, we need the following definitions. We denote  $q^{(n)} := q(q+1) \cdots (q+n-1)$  the rising factorial and  $S_{2,q}(n, k)$  the  $q$ -analogue of the Stirling numbers of the second kind such that, if we consider a set partition,  $q$  counts the number of elements different from  $n$  in the subset containing  $n$ .

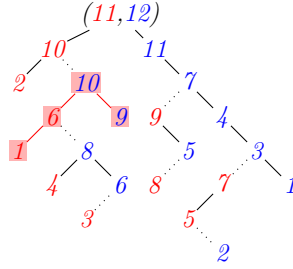


FIGURE 3. Hooks on a non-ambiguous tree

**Proposition 1.12.** *Let  $i$  and  $j$  be two positive integers. The  $(\alpha, \beta)$ -analogue of the number of NATs of geometric size  $i \times j$  is*

$$\left[ \frac{x^{i-1} y^{j-1}}{(i-1)! (j-1)!} \right] \mathfrak{N}(x, y; \alpha, \beta) = \sum_{p \geq 1} (p-1)! (\alpha + \beta)_{p-1} S_{2, \alpha}(i, p) S_{2, \beta}(j, p).$$

**1.3. Combinatorial interpretation with the zigzag bijection of Burstein.** The purpose of this subsection is to explain combinatorially the propositions 1.11 and 1.12. In order to do so, we use the “zigzag” bijection of Burstein, denoted  $\varphi$  [Bur07, Theorem 4.2] between non-ambiguous trees and permutations with all their exceedances at the beginning.

First, let us introduce the statistic that corresponds to the integer  $p$  in the enumeration formula of Proposition 1.12.

**Definition 1.13.** *Let  $B$  be a binary tree and  $v$  one of its node. The hook of a vertex  $v$  is the union of  $\{v\}$ , its leftmost branch and its rightmost branch. We say that  $v$  is the root of its hook. There is a unique way to partition the vertices in hooks. The number of hooks in such a partition is the hook number of the tree and it is denoted by  $\text{hook}(T)$ . We extend this definition to non-ambiguous trees.*

**Remark 1.14.** *We can obtain recursively the unique partition of a binary tree in hooks by extracting the root’s hook and iterating the process on each tree of the remaining forest.*

**Example 1.15.** *On the left part of Figure 3, we represented in red the hook of 10. The partition of vertices in hooks is obtained by removing the dotted edges. The hook number of the tree is 8.*

The correspondence between  $p$  and this new statistic is proven thereafter.

From now on, we juggle between geometric representation and the labelled binary tree representation of non-ambiguous trees. Let  $T$  be





**Corollary 1.18.** *Let  $w_L$  and  $w_R$  be two positive integers. The map  $\psi$  is a bijection between non-ambiguous trees of geometric size  $w_L \times w_R$  and cycles of  $\llbracket 0, w_L + w_R \rrbracket$  such that all their exceedances are at positions  $0, \dots, w_R - 1$ .*

In order to ease future explanations, we number independently rows and columns. We replace the integers  $\llbracket 0, w_R - 1 \rrbracket$  with  $\llbracket 1, w_R \rrbracket$  by using the map  $i \mapsto (w_R - i)$ , and the integers  $\llbracket w_R, w_R + w_L - 1 \rrbracket$  with  $\llbracket 1, w_L \rrbracket$  by using the map  $i \mapsto (i - w_R + 1)$ , as shown in Figure 5. If we denote  $T$  the non-ambiguous tree of this figure, then,  $\psi(T)$  is equal to

$$(9 \ 5 \ 8 \ 3 \ 12 \ 4 \ 4 \ 14 \ 2 \ 7 \ 15 \ 13 \ 10 \ 6 \ 2 \ 9 \ 7 \ 5 \ 11 \ 6 \ 1 \ 1 \ 8 \ 3).$$

The cycles appearing with this renumbering are the followings.

**Definition 1.19.** *Let  $i$  and  $j$  be two positive integers. We call 2-coloured block decreasing cycles of size  $i \times j$ , the cycles of the set  $\llbracket 1, i \rrbracket \cup \llbracket 1, j \rrbracket$  such that, if the image of an element  $a$  is an element  $b$  of the same colour then  $a > b$ .*

Using this definition, the map  $\psi$  is a bijection between non-ambiguous trees of geometric size  $i \times j$  and 2-coloured block decreasing cycles of size  $i \times j$ . Moreover, the number of blue blocks of the 2-coloured block decreasing cycles has a simple interpretation over non-ambiguous trees.

**Lemma 1.20.** *Let  $T$  be a non-ambiguous tree, then the hook number of  $T$  is equal to the number of blue block in  $\psi(T)$ .*

*Proof.* In order to prove this lemma, we show that the red integers at the right of the right branches of the hooks are the ones with a blue preimage. It is a consequence of the property that in the case of a zigzag originating from a blue (resp. red) integer, the end point of an horizontal step is a non-root point of the right branch (resp. the root) of a hook or a vertical step of the south-east border, and the end point of a vertical step is the root (resp. a non-root point of the left branch) of a hook or an horizontal step of the south-east border. In particular, the starting point of an horizontal step of a zigzag originating from a blue (resp. red) is a point (resp. non-root) of the right (resp. left) branch of a hook, which proves the result.  $\square$

For example, if  $T$  is the non-ambiguous tree of the Figure 5, its hook number is 7 which also the number of blue blocks in  $\psi(T)$ . From this lemma, we deduce the following proposition.

**Proposition 1.21.** *Let  $i, j$  and  $p$  be positive integers. The number of non-ambiguous trees of geometric size  $i \times j$  and hook number  $p$  is*

$$(p - 1)! p! S_2(i, p) S_2(j, p).$$



Moreover, the doubly exponential generating series of non-empty non-ambiguous trees  $T$  with weight

$$z^{\text{hook}(T)} \frac{x^{w_L(T)} y^{w_R(T)}}{w_L(T)! w(T)!},$$

is

$$-\ln(1 - z(e^x - 1)(e^y - 1)).$$

*Proof.* It is clear that the enumeration formula counts the number of 2-coloured block decreasing cycles of size  $i \times j$  with  $p$  blue blocks. Hence, using the Corollary 1.18 and the Lemma 1.20 we get the desired result.

The formula of the generating series is obtained as the composition of the generating series of pairs composed of a non-empty blue set and a non-empty red set in which we added  $z$  to count the number of pairs,  $z(e^x - 1)(e^y - 1)$ , with the generating series of cycles  $-\ln(1 - u)$ .  $\square$

We extend naturally this proposition with the parameters  $\alpha$  and  $\beta$ .

**Theorem 1.22.** *Let  $i, j$  and  $p$  be positive integers. The number of non-ambiguous trees of geometric size  $i \times j$  and hook number  $p$  is*

$$(p - 1)! (\alpha + \beta)_{p-1} S_{2,\alpha}(i, p) S_{2,\beta}(j, p).$$

Moreover, the doubly exponential generating series of non-empty non-ambiguous trees  $T$  with weight

$$\alpha^{\mathcal{L}_0(T)} \beta^{\mathcal{R}_0(T)} z^{\text{hook}(T)} \frac{x^{w_L(T)-1} y^{w_R(T)-1}}{(w_L(T) - 1)! (w_R(T) - 1)!},$$

is

$$\frac{ze^{\alpha x + \beta y}}{(1 - z(e^x - 1)(e^y - 1))^{\alpha + \beta}}.$$

*Proof.* A non-ambiguous tree can be decomposed in three pieces:

- the hook  $h$  of the root of which we removed all the vertices which have a child which is not part of  $h$ , we can represent it as a pair of the set of non-root red labels and the set of non-root blue labels,
- the set of non-ambiguous trees attached to the left branch of  $h$
- and the set of non-ambiguous trees attached to the right branch of  $h$ .

The exponent of  $\alpha$  (resp.  $\beta$ ) is equal to the number of non-root red (resp. blue) labels in the hook of the root plus the number of sub-non-ambiguous trees attached to the left (resp. right) branch of  $h$ . Hence, the generating series of  $h$  is

$$ze^{\alpha x} e^{\beta y},$$

the generating series of the sub-non-ambiguous trees attached to the left branch of  $h$  is

$$e^{-\alpha \ln(1-z(e^x-1)(e^y-1))},$$

the one of those attached to the right branch of  $h$  is

$$e^{-\beta \ln(1-z(e^x-1)(e^y-1))}.$$

Taking the product we obtain the desired formula.

For the enumeration formula, let us consider a non-ambiguous tree  $T$ . We use the same idea of decomposition. We delete the topmost row and the leftmost column, before using the zigzags paths. For example, the non-ambiguous tree of Figure 5 gives us

$$(3) (13) (6 \ 2 \ 9 \ 7 \ 5 \ 11 \ 10) (1 \ 8 \ 6 \ 1) (8 \ 5) (7 \ 3 \ 12 \ 4 \ 4 \ 14 \ 2).$$

Red (resp. blue) parentheses means that the corresponding sub-non-ambiguous tree is attached to the leftmost (resp. rightmost) branch. We regroup the red (resp. blue) fixed point in a same set, adding also the red (resp. blue) label of the root. This way, we obtain a partition of red (resp. blue) labels in  $p = \text{hook}(T)$  non-empty sets. The number of non-root points in the leftmost column (resp. topmost row), with no right (resp. left) child, is equal to the number of elements minus 1 in the subset containing the biggest element. This explains the  $\alpha$ -analogue (resp.  $\beta$ -analogue) of  $S_2$ . Let us order and number the  $p - 1$  other blue subsets with respect to their biggest element and pair each blue subset with the red block to its right (in the same cycle). Keeping the same example, we obtain

$$\begin{array}{ccccccc} & & 6 & 5 & 4 & 3 & 2 & 1 \\ & & \{9\} & \{8\} & \{7 \ 3\} & \{6 \ 2\} & \{5\} & \{4\} & \{1\} \\ \{15 \ 13 \ 3\} & \{5\} & \{12 \ 4\} & \{9 \ 7\} & \{11 \ 10\} & \{14 \ 2\} & \{8 \ 6 \ 1\} & . \end{array}$$

In a general setting, there are  $(p - 1)!$  pairing possibilities. Let us now replace each pair with its corresponding number. We get

$$(6) (5 \ 2) (4 \ 3) (1).$$

In the end, in addition to the two partitions, we have a permutation of size  $p - 1$  decomposed in cycles and whose cycles are coloured in red or in blue. Each red (resp. blue) cycle counts for an  $\alpha$  (resp.  $\beta$ ), hence, the generating polynomial of such permutations is  $(\alpha + \beta)_{p-1}$ . We finally get the desired formula.  $\square$

As stated in the introduction, Proposition 1.11 and Theorem 1.12 (in the case  $\alpha = \beta = 1$ ) were already proven by Clark and Ehrenborg [CE10]. In the proof they gave, the statistic  $p$  is interpreted on permutations as follows.

**Definition 1.23.** Let  $i, j$  and  $n$  be positive integers such that  $n = i + j - 1$ . Let  $p$  be a permutation of size  $n$  such that all its exceedances are at position  $\llbracket 1, j - 1 \rrbracket$ . The CE-statistic <sup>2</sup> of  $p$  is the positive integer

$$\text{CE}(p) = |\{u \in \llbracket 1, j - 1 \rrbracket, p(u) > i\}| + 1.$$

The Lemma 1.20 tells us that the hook statistic corresponds to the number of blue blocks in 2-coloured block decreasing cycles. The bijection between 2-coloured block decreasing cycles and permutations will all their exceedances at the beginning that we will study is  $\Theta = \varphi \circ \psi^{-1}$ . The following lemma describes the difference between the number of blue blocks and the CE-statistic.

**Lemma 1.24.** Let  $i$  and  $j$  be positive integers and  $c$  a 2-coloured block decreasing cycle of size  $i \times j$ . Then, the number of blue blocks of  $c$  is equal to the CE-statistic of  $\Theta(c)$

- minus 1, if  $j$  has a blue element to its right and
- plus 1, if  $1$  has a blue integer, different from  $j$  to its left.

*Proof.* Let  $p$  be a permutation of size  $n = i + j - 1$  such that all its exceedances are at position  $\llbracket 1, j - 1 \rrbracket$ . If we keep the interpretation with red and blue integer, then the CE-statistic becomes

$$|\{u \in \llbracket 1, j - 1 \rrbracket, p(u) \in \llbracket 2, i \rrbracket\}| + 1.$$

Hence, the CE-statistic is the number of blue blocks of  $p$  without  $1$  to their right plus 1. Hence, we should study how the number of blue blocks behave with respect to  $\Theta$ . Using Lemma 1.17 we obtain the conditions of Lemma 1.24.  $\square$

The previous lemma tells us that  $\Theta$  is not sufficient to prove the equidistribution between the hook statistic and the CE-statistic. We need one last involution. Let  $m$  be the representative of a 2-coloured block decreasing sequence  $c$  of size  $i \times j$  such that  $j$  is to the left of  $m$ . The word  $m$  can be factorised as  $m = j b_1 \cdots b_k 1 m'$  where the  $b_i$  are maximal blocks of same colours. Let  $\omega$  be the involution such that if  $k$  is even then a representative of  $\omega(c)$  is  $m = j b_2 b_1 \cdots b_k b_{k-1} 1 m'$ , and if  $k$  is odd then  $\omega(c) = c$ .

**Proposition 1.25.** Let  $c$  be a 2-coloured block decreasing sequence. The number of blue blocks of  $c$  is equal to the CE-statistic of  $\Theta(\omega(c))$ .

**Corollary 1.26.** The hook statistic on non-ambiguous trees and the CE-statistic on permutations with all their exceedances at the beginning are equidistributed.

<sup>2</sup>The "+1" doesn't appear in the definition of Clark and Ehrenborg. We introduced it because there is a shift between the hook statistic and the CE-statistic.

**1.4.  $q$ -analogues of the hook formula.** As for binary trees, there exists  $q$ -analogues of the hook formula for NATs of a given shape associated to either the number of inversions or the major index. There are two ingredients: first we need to associate two permutations to a non-ambiguous tree, and second we need to give a  $q$ -analogue of the bilinear map  $\mathbb{B}$ . It turns out that it is possible to use two different  $q$  namely  $q_R$  and  $q_L$  for the derivative and integral in  $x$  and  $y$ .

The first step to formulate a  $q$ -hook formula is to associate to any non-empty non-ambiguous tree  $T$  a pair of permutations

$$\sigma(T) = (\sigma_L(T), \sigma_R(T)) \in \mathfrak{S}_{\mathcal{V}_L(T)} \times \mathfrak{S}_{\mathcal{V}_R(T)}.$$

**Definition 1.27.** *Let  $T$  be a non-ambiguous tree. Then  $\sigma_L(T)$  is obtained by performing a left postfix reading of the left labels: precisely we recursively read trees  $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ L \quad R \end{array}$  by reading the left labels of  $L$ , then the left labels of  $R$  and finally the label of the root if it is a left child. The permutation  $\sigma_R(T)$  is defined similarly reading right labels, starting from the right subtree, then the left subtree and finally the root.*

If we take back the example of Figure 1 we get the two permutations  $\sigma_L(T) = (2, 1, 4, 3, 6, 10, 8, 9, 5, 7)$  and  $\sigma_R(T) = (1, 2, 3, 4, 5, 7, 11, 9, 6, 8, 10)$ .

Recall that the *number of inversions* of a permutation  $\sigma \in \mathfrak{S}_n$  is the number of  $i < j \leq n$  such that  $\sigma(i) > \sigma(j)$ . A *descent* of  $\sigma$  is a  $i < n$  such that  $\sigma(i) > \sigma(i+1)$  and the *inverse major index* of  $\sigma$  is the sum of the descents of  $\sigma^{-1}$ . Finally for a repetition free word  $w$  of length  $l$  we write  $\text{Std}(w)$  the permutations in  $\mathfrak{S}_l$  obtained by renumbering  $w$  keeping the order of the letters. For example  $\text{Std}(36482) = 24351$ . We define as usual the  $q$ -integer  $[n]_q := \frac{1-q^n}{1-q}$ , and the  $q$ -factorial  $[n]_q! := \prod_{i=1}^n [i]_q$ .

**Theorem 1.28.** *For a non-ambiguous tree  $T$  and a statistic  $S \in \{\text{Inv}, \text{iMaj}\}$ , define*

$$w_S(T) := q_L^{S(\sigma_L(T))} q_R^{S(\sigma_R(T))}. \quad (9)$$

*Then, for any non-empty binary tree  $B$*

$$\sum_{T \in \text{NAT}(B)} w_S(T) = \frac{|\mathcal{V}_L(B)|_{q_L!} \cdot |\mathcal{V}_R(B)|_{q_R!}}{\prod_{U: \text{left child}} [\mathcal{E}_L(U)]_{q_L} \cdot \prod_{U: \text{right child}} [\mathcal{E}_R(U)]_{q_R}}. \quad (10)$$

Going back to the non-ambiguous tree of Figure 1, the inversions numbers are  $\text{Inv}(\sigma_L(T)) = 11$  and,  $\text{Inv}(\sigma_R(T)) = 7$  so that  $w_{\text{Inv}}(T) = q_L^{11} q_R^7$ . For the inverse major index, we get the permutations  $\sigma_L(T)^{-1} = (2, 1, 4, 3, 9, 5, 10, 7, 8, 6)$  and  $\sigma_R(T)^{-1} = (1, 2, 3, 4, 5, 9, 6, 10, 8, 11, 7)$ . Consequently,  $\text{iMaj}(\sigma_L(T)) = 1 + 3 + 5 + 7 + 9 = 25$  and  $\text{iMaj}(\sigma_R(T)) =$

$6 + 8 + 10 = 24$  so that  $w_{\text{iMaj}}(T) = q_L^{25} q_R^{24}$ . Note that it is possible to read directly  $w_S(T)$  on  $T$ .

The argument of the proof follows the same path as for the hook formula, using pumping functions. Recall that the  $q$ -derivative and  $q$ -integral are defined as  $\partial_{x,q} x^n := [n]_q x^{n-1}$  and  $\int_{x,q} u^n du := \frac{x^{n+1}}{[n+1]_q}$ . Then the  $(q_L, q_R)$ -analogue of the pumping function is given by

$$\mathbb{B}_q(f, g) = \int_{x, q_L} \int_{y, q_R} \partial_{x, q_L} g(u, v) \cdot \partial_{y, q_R} f(u, v) du dv. \quad (11)$$

We also define recursively  $\mathbb{B}_q(B)$  by  $\mathbb{B}_q(\emptyset) := x + y$  and  $\mathbb{B}_q\left(\begin{array}{c} \bullet \\ L \quad R \end{array}\right) = \mathbb{B}_q(\mathbb{B}_q(L), \mathbb{B}_q(R))$ . Then the main idea is to go through a pumping function on pairs of permutations. We write  $\mathbb{Q}\mathfrak{S}$  the vector space of formal sums of permutations. For any permutation  $\sigma \in \mathfrak{S}_n$  we write  $\int \sigma = \sigma[n+1]$  the permutation in  $\mathfrak{S}_{n+1}$  obtained by adding  $n+1$  at the end. Again we extend  $\int$  by linearity.

**Definition 1.29.** *The pumping function on permutation is the bilinear map  $\mathbb{B}\mathfrak{S} : \mathbb{Q}\mathfrak{S} \times \mathbb{Q}\mathfrak{S} \mapsto \mathbb{Q}\mathfrak{S}$  defined for  $\sigma \in \mathfrak{S}_m$  and  $\mu \in \mathfrak{S}_n$  by*

$$\mathbb{B}\mathfrak{S}(\sigma, \mu) = \sum_{\substack{uv \in \mathfrak{S}_{m+n+1} \\ \text{Std}(u) = \int \sigma \\ \text{Std}(v) = \mu}} uv.$$

*We define also a pumping function on pairs of permutations*

$$\mathbb{B}\mathfrak{S}^2((\sigma_L, \sigma_R), (\mu_L, \mu_R)) := (\mathbb{B}\mathfrak{S}(\sigma_L, \mu_L), \mathbb{B}\mathfrak{S}(\mu_R, \sigma_R))$$

For example  $\mathbb{B}\mathfrak{S}(21, 12) = 21345 + 21435 + 21534 + 31425 + 31524 + 41523 + 32415 + 32514 + 42513 + 43512$ . Note that for two non-empty non-ambiguous tree  $C, D$

$$\sum_{T \in \mathbb{M}(C, D)} \sigma_L(T) = \mathbb{B}\mathfrak{S}(\sigma_L(C), \sigma_L(D))$$

$$\text{and} \quad \sum_{T \in \mathbb{M}(C, D)} \sigma_R(T) = \mathbb{B}\mathfrak{S}(\sigma_R(D), \sigma_R(C)).$$

The central argument is the following commutation property:

**Proposition 1.30.** *For a statistic  $S \in \{\text{Inv}, \text{iMaj}\}$ , and  $(\sigma_L, \sigma_R) \in \mathfrak{S}_m \times \mathfrak{S}_n$ , define*

$$\Psi_S((\sigma_L, \sigma_R)) := q_L^{S(\sigma_L)} \frac{x^{m+1}}{[m+1]_{q_L}!} q_R^{S(\sigma_R)} \frac{y^{n+1}}{[n+1]_{q_L}!}. \quad (12)$$

*Then for any pairs  $\sigma = (\sigma_L, \sigma_R)$  and  $\mu = (\mu_L, \mu_R)$ , one has*

$$\Psi_S(\mathbb{B}\mathfrak{S}^2(\sigma, \mu)) = \mathbb{B}_q(\Psi_S(\sigma), \Psi_S(\mu)).$$

*Proof.* We need to prove that for  $\tau \in \mathfrak{S}_m$  and  $\pi \in \mathfrak{S}_n$ ,

$$\sum_{\substack{\theta=uv \in \mathfrak{S}_{n+m+1} \\ \text{Std}(u)=\int \tau \\ \text{Std}(v)=\pi}} q^{S(\theta)} = q^{S(\tau)+S(\pi)} \binom{m+n+1}{m+1}_q.$$

The case  $S = \text{Inv}$  is easier to prove. The  $q$ -binomial consists in choosing a permutation  $\theta$  such that  $\theta(1) < \dots < \theta(m+1)$  and  $\theta(m+2) < \dots < \theta(m+n+1)$ . The term  $q^{S(\tau)+S(\pi)}$  comes from the reordering of the  $\theta(i)$  in order to have  $\text{Std}(\theta(1) \dots \theta(m+1)) = \int \tau$  and  $\text{Std}(\theta(m+2) \dots \theta(m+n+1)) = \pi$ .

In order to prove the case  $S = \text{iMaj}$ , we consider the two following equations

$$\sum_{\substack{\theta=uv \in \mathfrak{S}_{n+m+1} \\ n+m+1 \in u \\ \text{Std}(u)=\int \tau \\ \text{Std}(v)=\pi}} q^{\text{iMaj}(\theta)} = q^{\text{iMaj}(\tau)+\text{iMaj}(\pi)+n} \binom{m+n}{n}_q, \quad (13)$$

and


$$\sum_{\substack{\theta=uv \in \mathfrak{S}_{n+m+1} \\ n+m+1 \in v \\ \text{Std}(u)=\int \tau \\ \text{Std}(v)=\pi}} q^{\text{iMaj}(\theta)} = q^{\text{iMaj}(\tau)+\text{iMaj}(\pi)} \binom{m+n}{n-1}_q. \quad (14)$$

Equation 13 is a consequence of Equation 36 in [HNT08]. We prove Equation 14 by induction on  $n+m$  and distinguishing the two cases  $n+m \in u$  and  $n+m \in v$ . Finally, making the sum of Equations 13 and 14 finishes the proof.  $\square$

As a consequence, noting that  $w_S(T) = \Phi_S(\sigma(T))$ , one finds that for any non-empty non-ambiguous trees  $C$  and  $D$ ,

$$\sum_{T \in \mathbb{M}(C,D)} w_S(T) = \Phi_S(\mathbb{B}\mathfrak{S}^2(\sigma(C), \sigma(D))) = \mathbb{B}_q(w_S(C), w_S(D)).$$

Applying this recursively on the structure of a binary tree  $B$ , we have that  $\sum_{T \in \mathcal{NAT}(B)} w_S(T) = \mathbb{B}_q(B)$ . Unfolding the recursion for  $\mathbb{B}_q(B)$ , gives finally Theorem 1.28.

We conclude this section by an example. Let  $B =$  . Then one finds that the  $q$ -hook formula gives  $(q_R^3 + q_R^2 + q_R + 1)(q_L^2 + q_L + 1)(q_R + 1)$ . Expanding this expression, one finds that the coefficient of  $q_R^2 q_L$  is 2. For the iMaj statistic it corresponds to the two following



- (2) for any descendant  $U$  of  $V$ , if the  $i$ -th component of  $U$  and  $V$  are different from  $\bullet$ , then the  $i$ -th component of  $V$  is strictly greater than the  $i$ -th component of  $U$ ;
- (3) for each  $i \in \llbracket 1, d \rrbracket$ , all the  $i$ th components, different from  $\bullet$ , are pairwise distinct;
- (4) the set of  $i$ th components different from  $\bullet$  of every vertices in the tree is an interval whose minimum is 1.

The set of non-ambiguous trees of dimensions  $(d, k)$  is denoted by  $\mathcal{NAT}_{d,k}$ .

We write  $\mathcal{NAT}_{d,k}$  for a non-ambiguous tree (of dimensions  $(d, k)$ ). Figure 6 gives an example of a  $\mathcal{NAT}_{3,1}$  and a  $\mathcal{NAT}_{3,2}$ .

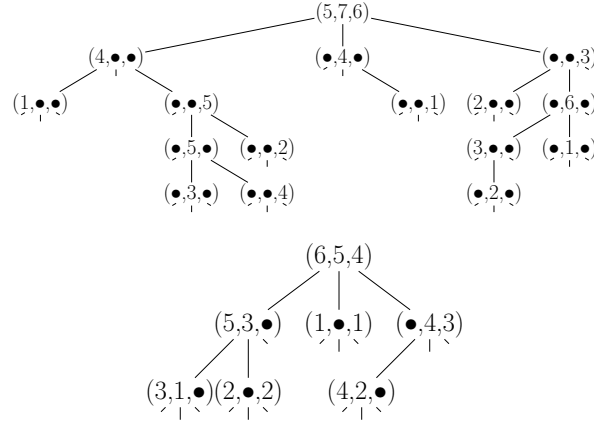


FIGURE 6. A  $\mathcal{NAT}$  of dimension  $(3, 1)$  (above) and a  $\mathcal{NAT}$  of dimension  $(3, 2)$  (below).

**Definition 2.3.** The geometric size of a  $\mathcal{NAT}_{d,k}$  is the  $d$ -tuple of integers  $(w_1, \dots, w_d)$  which labels the root of the  $\mathcal{NAT}_{d,k}$ , it is denoted by  $w_1 \times \dots \times w_d$ . The  $\pi$ -size of a  $\mathcal{NAT}_{d,k}$  is the number of vertices in the tree of direction  $\pi$ , the set of such vertices is denoted by  $\mathcal{V}_\pi$ .

Proposition 2.4 gives the relation between the geometric size and the  $\pi$ -size of a non-ambiguous trees.

**Proposition 2.4.** Let  $M$  be a  $\binom{d}{k}$ -ary tree, the root label is constant on elements of  $\mathcal{NAT}_{d,k}$  of shape  $M$  ( $\mathcal{NAT}_{d,k}(M)$ ):

$$w_i(M) := w_i = \sum_{\pi \in \Pi_{d,k} \mid i \in \pi} |\mathcal{V}_\pi(M)| + 1.$$



**2.2. Associated differential equations.** In this section, we denote respectively by  $x_{\{i_1, \dots, i_k\}}$  the product  $x_{i_1} \times \dots \times x_{i_k}$ , by  $\partial_{\{i_1, \dots, i_k\}}$  the operator  $\partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_k}}$  and by  $\int_{\{i_1, \dots, i_k\}}$  the operator  $\int_{x_{i_1}} \int_{x_{i_2}} \dots \int_{x_{i_k}}$ . As for non-ambiguous trees (Proposition 1.6), there is a hook formula for the number of non-ambiguous trees with fixed underlying tree. Let  $M$  be a  $\binom{d}{k}$ -ary tree, for each vertex  $U$  we denote by  $\mathcal{E}_i(U)$  the number of vertices, of the subtree whose root is  $U$  (itself included in the count), whose direction contains  $i$ . Then,

$$|\mathcal{NAT}_{d,k}(M)| = \frac{\prod_{i=1}^d (w_i(M) - 1)!}{\prod_{\substack{U: \text{child of direction} \\ \text{containing } i}} \mathcal{E}_i(U)}.$$

Let  $\mathfrak{N}_{d,k}(x_1, \dots, x_d)$  be the exponential generating function of generalised non-ambiguous trees

$$\mathfrak{N}_{d,k}(x_1, \dots, x_d) := \sum_{T \in \mathcal{NAT}_{d,k}^*} \prod_{i=1}^d \frac{x_i^{w_i(T)-1}}{(w_i(T) - 1)!}$$

There is a  $(d, k)$ -dimensional analogue of the fixed point differential Equation 7:

**Proposition 2.5.** *The exponential generating function  $\mathfrak{N}_{d,k}$  of generalised non-ambiguous trees satisfies the following differential equation*

$$\mathfrak{N}_{d,k}(x_1, \dots, x_d) = \prod_{\pi \in \Pi_{d,k}} \left( 1 + \int_{\pi} \mathfrak{N}_{d,k} \right). \quad (15)$$

*Proof.* The method is analogue to the method of Section 1.2, and goes through the use of a  $\binom{d}{k}$ -linear map and a pumping function for  $\binom{d}{k}$ -ary trees.  $\square$

The family of differential equations defined by Equation 15 can be rewritten using differential operators instead of primitives. We need to introduce the function  $\mathfrak{G}_{d,k} = \int_{\{1, \dots, d\}} \mathfrak{N}_{d,k} + \sum_{\pi \in \Pi_{d,d-k}} x_{\pi}$ . Then, we show that  $\mathfrak{G}_{d,k}$  satisfies the following differential equations:

**Proposition 2.6.** *The differential equation satisfied by  $\mathfrak{G}_{d,k}$  is*

$$\partial_1 \dots \partial_d \mathfrak{G}_{d,k} = \prod_{\pi \in \Pi_{d,d-k}} \partial_{\pi} \mathfrak{G}_{d,k}.$$

In the generic case, we are not able to solve those differential equations. We know that setting a variable  $x_d$  to 0 gives the generating function of NATs of lower dimension.

**Proposition 2.7.** *Let  $d > k \geq 1$ , then  $\mathfrak{N}_{d,k}|_{x_d=0} = \mathfrak{N}_{d-1,k}$ .*

For some specific values of  $d$  and  $k$  we have (at least partial) results.

**Proposition 2.8.** *Let  $k = d - 1$ , if we know a particular solution  $s(x_1, \dots, x_d)$  for*

$$\partial_1 \dots \partial_d \mathfrak{G}_{d,d-1} = \partial_1 \mathfrak{G}_{d,d-1} \times \dots \times \partial_d \mathfrak{G}_{d,d-1}$$

*then, for any function  $s_1(x_1), \dots, s_d(x_d)$ , the function  $s(s_1(x_1), \dots, s_d(x_d))$  is also a solution.*

**Proposition 2.9.** *Some non trivial rational functions are solutions of*

$$\partial_1 \dots \partial_d \mathfrak{G}_{d,1} = \prod_{\pi \in \Pi_{d,d-1}} \partial_\pi \mathfrak{G}_{d,1}.$$

*Proof (sketch).* We generalise the first part of the proof of Proposition 1.8. We define  $\mathfrak{G}_{(i)} = \partial_\pi \mathfrak{G}_{d,1}$  where  $i \in \llbracket 1, d \rrbracket$  and  $\pi = \llbracket 1, d \rrbracket \setminus \{i\}$ . We get the relation  $\partial_i \mathfrak{G}_{(i)} = \prod_{j=1}^d \mathfrak{G}_{(j)}$  and then  $\prod_{i=1}^d \partial_i \mathfrak{G}_{(i)} = \prod_{i=1}^d \mathfrak{G}_{(i)}^d$ . To obtain a particular solution, we just need to identify, in the previous equation, the term  $\partial_i \mathfrak{G}_{(i)}$  to the term  $\mathfrak{G}_{(i)}^d$ . We thus obtain some non trivial solutions for our equation, which are rational functions.  $\square$

Since dimension  $(2, 1)$  is the unique case where Proposition 2.8 and Proposition 2.9 can be applied at the same time, and the computation of  $\mathfrak{N}_{d,d}$  is straightforward, we have the following proposition.

**Proposition 2.10.** *We have the closed formulas:*

$$\mathfrak{N}_{2,1} = \mathfrak{N} \quad \text{and} \quad \mathfrak{N}_{d,d} = \sum_{n \geq 0} \frac{(x_1 \cdot \dots \cdot x_d)^n}{(n!)^d}.$$

We see  $\mathfrak{N}_{d,d}$  as is a kind of generalised Bessel function because

$$\mathfrak{N}_{2,2}(x/2, -x/2) = J_0(x)$$

where  $J_\alpha$  is the classical Bessel function. This supports our feeling that the general case leads to serious difficulties.

**2.3. Geometric interpretation.** As for non-ambiguous trees, we can give a geometric definition of non-ambiguous trees of dimensions  $(d, k)$  as follows. We denote by  $(e_1, \dots, e_d)$  the canonical basis of  $\mathbb{R}^d$  and  $(X_1, \dots, X_d)$  its dual basis, i.e.  $X_i$  is  $\mathbb{R}$ -linear  $X_i(e_i) = \delta_{i,j}$ . Let  $P \in \mathbb{R}^d$  and  $\pi = \{i_1, \dots, i_k\}$  a  $(d, k)$ -direction, we call *cone of origin  $P$  and direction  $\pi$*  the set of points  $C(P, \pi) := \{P + a_1 e_{i_1} + \dots + a_k e_{i_k} \mid (a_1, \dots, a_k) \in \mathbb{N}^k\}$ .

**Definition 2.11.** A geometric non-ambiguous tree of dimension  $(d, k)$  and box  $w_1 \times \dots \times w_d$  is a non-empty set  $\mathcal{V}$  of points of  $\mathbb{N}^d$  such that:

- (1)  $\mathcal{V}$  is contained in  $\llbracket 1, w_1 \rrbracket \times \dots \times \llbracket 1, w_d \rrbracket$ .
- (2)  $\mathcal{V}$  contains the point  $(w_1, \dots, w_d)$ , which is called the root,
- (3) For  $P \in \mathcal{V}$  different from the root, there exists a unique  $(d, k)$ -direction  $\pi = \{i_1, \dots, i_k\}$  such that the cone  $c(P, \pi)$  contains at least one point different from  $P$ . We say that  $P$  is of type  $\pi$ .
- (4) For each  $i \in \llbracket 1, d \rrbracket$ , for all  $l \in \llbracket 1, w_i - 1 \rrbracket$ , the affine hyperplane  $\{x_i = l\}$  contains exactly one point of type  $\pi$ . If we denote by  $\pi$  its type, then  $i \in \pi$ .
- (5) For  $P$  and  $P'$  two points of  $\mathcal{V}$  belonging to a same affine space of direction  $\text{Vect}(e_{i_1}, \dots, e_{i_k})$ , then, either  $\forall j \in \llbracket 1, k \rrbracket, X_{i_j}(P) > X_{i_j}(P')$ , or  $\forall j \in \llbracket 1, k \rrbracket, X_{i_j}(P') > X_{i_j}(P)$ .

**Proposition 2.12.** There is a simple bijection between the set of geometric non-ambiguous tree of box  $w_1 \times \dots \times w_d$  and the set of non-ambiguous tree of geometric size  $w_1 \times \dots \times w_d$ .

*Proof.* If  $k = d$ ,  $\mathcal{V}$  is of the form

$$\{(w, \dots, w), (w-1, \dots, w-1), \dots, (1, \dots, 1)\},$$

which corresponds exactly to non-ambiguous trees of dimension  $(d, d)$  defined in 2.2.

Let us now suppose that  $k < d$ .

**2.2 implies 2.11:**

Let  $T$  be a non-ambiguous tree of dimension  $(d, k)$ -defined with Definition 2.2 and let  $w_1 \times \dots \times w_d$  be its geometric size. The first step is to define the completed label of a vertex  $U$  by replacing the  $\bullet$  by integers in the vertices labels, we do it as follows. Let  $U$  be a vertex of  $T$  such that its  $i$ th component is a  $\bullet$  and let  $V$  be its parent. If the  $i$ th component of  $V$  is not a  $\bullet$ , then replace the  $i$ th component of  $U$  by the  $i$ th component of  $V$ . Else replace recursively the  $i$ th component of  $V$  and then do the replacement. It is equivalent to say that we replace the  $i$ th component of  $U$  with the  $i$ th component of the first ancestor of  $U$  with a  $i$ th component different from  $\bullet$ . Such an ancestor exists since

the root has no  $\bullet$  component. As a consequence, using 2.2.2 we deduce that for a vertex  $V$  of completed label  $(v_1, \dots, v_d)$ , if  $V$  has a child  $U$  indexed by a  $(d, k)$ -direction  $\pi$  and of completed label  $(u_1, \dots, u_d)$ , then for  $i \in \pi$ ,  $v_i > u_i$  and for  $i \notin \pi$ ,  $v_i = u_i$ . Moreover, 2.2.3 and the definition of completed labels implies that, for  $U$  and  $V$  two vertices, if there exists  $i$  such that  $u_i = v_i$ , then they have a common ancestor  $W$  such that  $w_i \neq \bullet$  and  $w_i = u_i = v_i$ . We denote  $\mathbf{a}_i(U)$ , or equivalently  $\mathbf{a}_i(V)$ , the vertex  $W$ .

Let  $V$  be a vertex of  $T$  of completed label  $(u_1, \dots, u_d)$ . We denote by  $P_V \in \mathbb{N}^d$  the point  $(u_1, \dots, u_d)$ . Let  $\mathcal{V}$  be the set of points  $\{P_V \mid V \text{ vertex of } T\}$ . We can prove that each vertex has a different completed label. Let us prove that  $\mathcal{V}$  satisfies the conditions of 2.11.

- (1) It is a consequence of 2.2.4, and Definition 2.3.
- (2)  $\mathcal{V}$  contains  $(w_1, \dots, w_d)$ , since  $(w_1, \dots, w_d)$  is the label of the root of  $T$ .
- (3) Let  $P_U$  be a point of  $\mathcal{V}$  different from  $(w_1, \dots, w_d)$ . Since  $U$  is not the root, it is a child indexed by a  $(d, k)$ -direction  $\pi$ , of a vertex  $V$ . Hence for  $i \notin \pi$ ,  $X_i(P_V) = X_i(P_U)$  and for  $i \in \pi$ ,  $X_i(P_V) > X_i(P_U)$ . So  $P_V$  is in the cone of origin  $P_U$  and direction  $\pi$ , in particular,  $P_U$  is of type  $\pi$ . Let us prove the uniqueness by contradiction. Suppose there is another  $(d, k)$ -direction  $\pi'$  such that the cone of origin  $P_U$  and direction  $\pi'$  contains a point  $P_{V'}$  different from  $P_U$ . Let  $i \in \pi \setminus \pi'$ . Then, by definition of the cone we have  $X_i(P_U) = X_i(P_{V'})$ . Since  $\mathbf{a}_i(U) = V$ , then  $\mathbf{a}_i(V') = V$ . Let  $i' \in \pi' \setminus \pi$  then  $X_{i'}(P_{V'}) > X_{i'}(P_U) = X_{i'}(P_V)$ , which is not possible since  $V$  is an ancestor of  $V'$ . In particular the type of  $P_U$  corresponds to the type of  $U$ .
- (4) Let  $i \in \llbracket 1, d \rrbracket$  and  $l \in \llbracket 1, w - 1 \rrbracket$ , by 2.2.4, there exists  $U$  such that  $u_i \neq \bullet$  and  $u_i = l$ . Let  $\pi$  be the index of  $u$  then  $i \in \pi$ , hence,  $P_U$  satisfies 2.11.4. Suppose there exists another point  $P_V$  satisfying 2.11.4, let  $\pi'$  be its type. Then  $V$  is indexed by  $\pi'$  and  $i \in \pi'$ . Thus,  $V$  is another vertex such that  $v_i \neq \bullet$  and  $v_i = l$  which is in contradiction with 2.2.3.
- (5) Let  $\mathcal{F}$  be an affine space of direction  $\text{Vect}(e_{i_1}, \dots, e_{i_k})$  containing a point  $P_U$ . We denote by  $\pi$  the set  $\{e_{i_1}, \dots, e_{i_k}\}$ . If  $U$  is the root, then 2.11.5 is satisfied. Else, let  $(V_0, V_1, \dots, V_m)$  be the sequence of ancestors of  $U$ , i.e  $V_0 = U$  for all  $j$ ,  $V_j$  is a child of  $V_{j+1}$  and  $V_m$  is the root of  $T$ . Let  $l$  be the index such that  $V_l$  is a child of  $V_{l+1}$  not indexed by  $\pi$  and  $\forall j \in \llbracket 0, l - 1 \rrbracket$ ,  $V_j$  is the child indexed by  $\pi$  of  $V_{j+1}$ ,  $V_l$  will be denoted  $V$ . Let  $\pi'$  be the

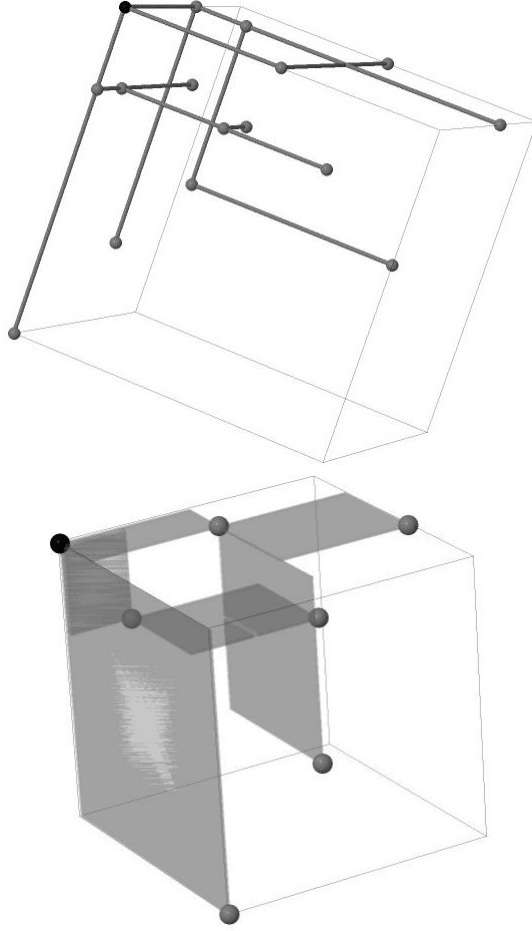


FIGURE 7. Geometric representation of the NATs of Figure 6.

$(d, k)$ -direction indexing  $V$  and let  $i \in \pi' \setminus \pi$ . If  $V$  is the root then  $\pi' = \llbracket 1, d \rrbracket$ . Let  $P_{U'}$  be another point of  $\mathcal{F}$ , since  $i \notin \pi$ , then  $u'_i = u_i$ . Hence, we have  $\mathbf{a}_i(U') = \mathbf{a}_i(U) = V$ . Since for all  $j \notin \pi$ ,  $u'_j = v_j$ , the path from  $U'$  to  $V$  contains only vertices indexed by  $\pi$ . Hence, by definition of  $V$ ,  $U$  is an ancestor of  $U'$  or the converse, which proves 2.11.5.

### 2.11 implies 2.2:

Let  $\mathcal{V}$  be a non-ambiguous tree defined with Definition 2.11. We start by constructing the underlying  $\binom{d}{k}$ -ary tree  $M$  of  $\mathcal{V}$ . The vertices of  $M$  correspond to the points of  $\mathcal{V}$ , in particular, the root of  $M$  corresponds to the root of  $\mathcal{V}$ . Let  $P$  be a point of  $\mathcal{V}$ , we denote by  $V_P$

the corresponding vertex of  $M$ . Let  $P$  be a point of  $\mathcal{V}$  different from the root, let  $\pi = \{e_{i_1}, \dots, e_{i_k}\}$  be the  $(d, k)$ -direction defined by 2.11.3 and let  $\mathcal{F}$  be the affine space  $P + \text{Vect}(e_{i_1}, \dots, e_{i_k})$ . Using 2.11.5, we can define without ambiguity the parent of  $V_P$  as the vertex  $V_{P'}$  such that  $P'$  is the closest point to  $P$  belonging to  $c(P, \pi)$ .  $V_P$  is the child of  $V_{P'}$  indexed by  $\pi$ , moreover, for all  $i \in \pi$ ,  $X_i(P) < X_i(P')$  and for all  $i \notin \pi$ ,  $X_i(P) = X_i(P')$ . The labelling is done as follows. We start by labelling the vertices  $V_P$  by the coordinates of  $P$ . Then for each vertex  $V$ , let  $\pi$  be its index, for all  $i \notin \pi$ , we replace the  $i$ th component of its label by  $\bullet$ . Thus, if the  $i$ th component of a vertex  $V_{P'}$  is equal to  $l$  then for a descendant  $V_P$  of  $V_{P'}$ ,  $X_i(P) \leq l$ , and if  $X_i(P) = l$  then the  $i$ th component of  $V_{P'}$  is  $\bullet$ . Let us prove that the condition of Definition 2.2 are satisfied.

- (1) By construction of the labels.
- (2) Let  $P$  and  $P''$  be two points such that  $V_P$  is a descendant of  $V_{P''}$  indexed by  $\pi$  such that  $i$  belongs to  $\pi$ . Let  $P'$  be the father of  $P$ , then  $X_i(P) < X_i(P') \leq X_i(P'')$ .
- (3) Let  $V_P$  be a vertex of  $M$  such that its  $i$ th component is different from  $\bullet$ . If  $V_P$  is the root, then all the vertices of  $M$  are its descendants, hence its  $i$ th label appears only once. Else,  $V_P$  is a child indexed by  $\pi$  of a vertex  $V_{P'}$ . In particular,  $\pi$  contains  $i$  since the  $i$ th component of  $V_P$  is not  $\bullet$ . Hence, by 2.11.4, the  $i$ th component of  $V_{P'}$  is unique.
- (4) 2.11.4 implies that for each  $i \in \llbracket 1, d \rrbracket$ , for all  $l \in \llbracket 1, w_i - 1 \rrbracket$  there is a point  $P$  of type  $\pi$  such that  $X_i(P) = l$  and  $i \in \pi$ , so that the  $i$ th component of  $V_P$  is equal to  $l$ . Moreover, the coordinates of the root are  $(w_1, \dots, w_d)$  and  $\mathcal{V}$  is contained in the box  $\llbracket 1, w_1 \rrbracket \times \dots \times \llbracket 1, w_d \rrbracket$ . Therefore, the set of  $i$ th components, different from  $\bullet$ , is the interval  $\llbracket 1, w_i \rrbracket$ .

□

### 3. A NEW STATISTIC ON BINARY TREES: THE HOOK STATISTIC

We present in this section a bijection between binary trees and ordered trees, sending the vertices to edges and the hook statistic defined in Definition 1.13 to the number of vertices having at least a child which is a leaf, what we will call the *child-leaf statistic*. The corresponding integer series appears as [Slo, A127157] in OEIS.

We denote by  $\mathcal{B}_p$  and  $\mathcal{O}_p$  respectively the exponential generating series of these trees, with these statistics, the variable  $x$  indexing the number of vertices in  $\mathcal{B}_p$  and the number of edges in  $\mathcal{O}_p$ , and  $t$  the statistic.

Then, these generating series satisfy:

**Proposition 3.1.** *The generating series of binary trees with hook statistic and ordered trees with the child-leaf statistic are given by the following functional equations:*

$$\mathcal{B}_p = 1 + xt \times \left( \frac{1}{1 - x\mathcal{B}_p} \right)^2$$

$$\mathcal{O}_p = \frac{1}{1 - x(\mathcal{O}_p - 1)} \times \left( 1 + xt \times \frac{1}{1 - x\mathcal{O}_p} \right)$$

*These generating series are equal.*

*Proof.* The first functional equation is obtained by considering the vertices in the hook of the root: there can be none or there is a root, a list of left descendant (whose right child is a binary tree) and a list of right descendant (whose left child is a binary tree).

The second functional equation is obtained by considering, if the ordered tree is not reduced to a vertex, the first leaf of the root from left to right, if it exists. Then, on the left side of this leaf, there is a list of ordered trees not reduced to a vertex and on the right side a list of ordered trees, if there is a leaf.

Then, by multiplying the preceding equations by  $1 - x\mathcal{B}_p$  and  $1 - x(\mathcal{O}_p - 1)$  respectively, they are equivalent to:

$$\mathcal{B}_p - x\mathcal{B}_p^2 = 1 - x\mathcal{B}_p + xt \frac{1}{1 - x\mathcal{B}_p}$$

$$\mathcal{O}_p - x\mathcal{O}_p^2 + x\mathcal{O}_p = 1 + xt \frac{1}{1 - x\mathcal{O}_p}.$$

□

Let us now exhibit a bijection between these two objects. This bijection comes from the following equation:

$$(\mathcal{B}_p - 1) - x(\mathcal{B}_p - 1) - x(\mathcal{B}_p - 1)^2 = xt \frac{1}{1 - x\mathcal{B}_p}.$$

This equation can be viewed as considering only binary trees whose root has no left descendants or ordered trees such that the leftmost child of the root is a leaf.

We obtain the following bijection:

**Proposition 3.2.** *The map  $\zeta$  sends a binary tree  $B$  to an ordered tree  $O$  by mapping:*

- *the leftmost descendant of the root, if it is a leaf, to an edge between the root and its only child*

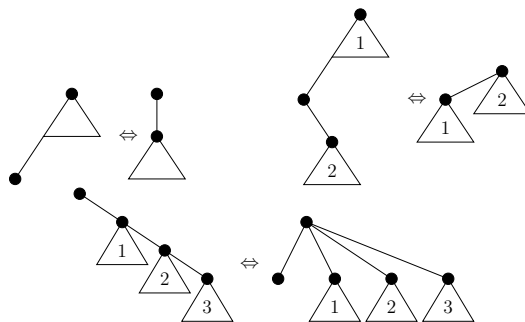


FIGURE 8. Bijection  $\zeta$

binary trees	
ordered trees	

TABLE 1. First terms of the bijection  $\zeta$

- the leftmost descendant of the root  $v$  to an edge between the root of the tree associated with the descendants of  $v$  and the root of the tree obtained from what is left
- the set of right descendants of the root to the set of children of the root.

It is a bijection between binary trees and ordered trees, sending the vertices to edges and the hook statistic to the child-leaf statistic.

We sum up this bijection on Figure 8.

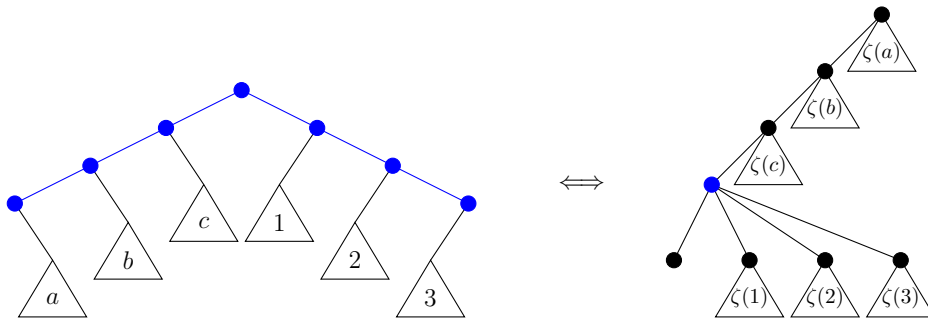
We present in Table 1 the first terms in the bijection.

Another way of describing recursively the bijection  $\zeta$  is given in Figure 9, the empty binary tree is still send to the ordered tree reduced to one vertex.

### PERSPECTIVES

In this work we gave new results about NATs, and we generalised the definition of NATs to higher dimensions with a choice on the dimension of the edges. It gives rise to several questions.



FIGURE 9. Bijection  $\zeta$ , alternative description.

We found nice formulas for the generating function and the generating polynomial of non-ambiguous trees that takes into account the number of points in the first column and in the first row. Those two parameters correspond to the parameters  $\alpha$  and  $\beta$  of the PASEP. We obtained those formulas with two different technics: by solving a differential equation and by decomposing non-ambiguous trees in hooks. In the context of the PASEP, it raises natural questions. Is it possible to introduce the parameter  $q$  of the PASEP in either one of them? What can we deduce from the hook decomposition of tree-like tableaux?

We found a polynomial analogue of the hook formula enumerating the NATs with a fixed binary tree, by adapting the methods of [HNT08]. Since we gave a hook formula also for NATs of higher dimension, we could extend the work to higher dimensions to get polynomial analogue.

The generating functions of generalised NATs satisfy differential equations similar to the case of NATs. While we were able to solve the case of NATs and get a nice closed form, we have not been able to tackle the general case. It would be interesting to find a generic way of solving these type of differential equations.

By generalising the NATs to higher dimension, we answered a question raised in the perspectives of [ABBS14]. In this paper, in addition to studying NATs, they also study NATs with a complete underlying binary tree, and they obtain nice combinatorial identities. It would be interesting to see if the same happens in higher dimension.

Finally, as mentioned in the introduction, NATs correspond to the tree-like tableaux of rectangular shape, so, we could generalise tree-like tableaux to higher dimension and hope to obtain again an insertion algorithm, a key property of tree-like tableaux.

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## REFERENCES

- [ABBS14] J.C. Aval, A. Boussicault, M. Bouvel, and M. Silimbani. Combinatorics of non-ambiguous trees. *Advances in Applied Mathematics*, 56:78–108, May 2014.
- [ABN13] J.-C. Aval, A. Boussicault, and P. Nadeau. Tree-like tableaux. *Electron. J. Combin.*, 20(4):Paper 34, 24, 2013.
- [Bur07] A. Burstein. On some properties of permutation tableaux. *Ann. Comb.*, 11(3-4):355–368, 2007.
- [BW89] A. Björner and M. L. Wachs. q-hook length formulas for forests. *J. Comb. Theory Ser. A*, 52(2):165–187, November 1989.
- [CE10] E. Clark and R. Ehrenborg. Explicit expressions for the extremal exceedance set statistics. *European J. Combin.*, 31(1):270–279, 2010.
- [CW07] S. Corteel and L. K. Williams. Tableaux combinatorics for the asymmetric exclusion process. *Adv. in Appl. Math.*, 39(3):293–310, 2007.
- [ES00] R. Ehrenborg and E. Steingrímsson. The exceedance set of a permutation. *Advances in Applied Mathematics*, 24(3):284 – 299, 2000.
- [HNT08] F. Hivert, J.C. Novelli, and J.Y. Thibon. Trees, functional equations, and combinatorial Hopf algebras. *European Journal of Combinatorics*, 29(7):1682–1695, 2008.
- [Pos07] A. Postnikov. Total positivity, grassmannians, and networks, 2007.
- [S<sup>+</sup>15] W. A. Stein et al. *Sage Mathematics Software (Version 6.10.beta1)*. The Sage Development Team, 2015. <http://www.sagemath.org>.
- [SCc08] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. <http://combinat.sagemath.org>.
- [Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. <http://oeis.org>.
- [SW07] E. Steingrímsson and L. K. Williams. Permutation tableaux and permutation patterns. *J. Combin. Theory Ser. A*, 114(2):211–234, 2007.
- [Vie08] X. Viennot. Alternative tableaux, permutations and partially asymmetric exclusion process. Slides of a talk at the Isaac Newton Institute in Cambridge, 2008.

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