



HAL
open science

Vortex pinning in the superfluid core of relativistic neutron stars

Aurélien Sourie, Nicolas Chamel

► **To cite this version:**

Aurélien Sourie, Nicolas Chamel. Vortex pinning in the superfluid core of relativistic neutron stars. Monthly Notices of the Royal Astronomical Society, 2021, 503 (1), pp.1407-1417. 10.1093/mnras/stab606 . hal-03164443

HAL Id: hal-03164443

<https://hal.science/hal-03164443>

Submitted on 6 Jul 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Vortex pinning in the superfluid core of relativistic neutron stars

Aurélien Sourie^{1,2}★ and Nicolas Chamel¹★

¹*Institut d'Astronomie et d'Astrophysique, Université Libre de Bruxelles, CP-226, B-1050 Brussels, Belgium*

²*Laboratoire Univers et Théories, Observatoire de Paris, Université PSL, CNRS, Université de Paris, F-92190 Meudon, France*

Accepted 2021 February 20. Received 2021 January 28; in original form 2020 July 14

ABSTRACT

Our recent Newtonian treatment of the smooth-averaged mutual-friction force acting on the neutron superfluid and locally induced by the pinning of quantized neutron vortices to proton fluxoids in the outer core of superfluid neutron stars is here adapted to the general-relativistic framework. We show how the local non-relativistic motion of individual vortices can be matched to the global dynamics of the star using the fully 4D covariant Newtonian formalism of Carter & Chamel. We derive all the necessary dynamical equations for carrying out realistic simulations of superfluid rotating neutron stars in full general relativity, as required for the interpretation of pulsar frequency glitches. The role of vortex pinning on the global dynamics appears to be non-trivial.

Key words: stars: interiors – stars: neutron – pulsars: general.

1 INTRODUCTION

Pulsar frequency glitches (Manchester 2017) are peculiar astrophysical phenomena that are thought to reveal the existence of superfluidity (Chamel 2017a) in the interior of neutron stars (NSs), the cold and dense remnants of gravitational (core-collapse) supernova explosions. The sudden spins up and the subsequent long relaxations, as observed in the emblematic Vela pulsar, were originally explained by the unpinning and creep of neutron quantized vortices in the NS crust (Anderson & Itoh 1975; Alpar et al. 1984a; Alpar, Langer & Sauls 1984b). However, the details of the vortex dynamics and the stellar regions involved during glitches still remain uncertain (see e.g. Haskell & Melatos 2015; Graber, Andersson & Hogg 2017; Haskell & Sedrakian 2018 for recent reviews). Indeed, it has been found that the presence of inhomogeneities in the crust tends to suppress superfluidity (Watanabe & Pethick 2017; Chamel 2017b; Sauls, Chamel & Alpar 2020), which may thus play a less important role than initially thought (Andersson et al. 2012; Chamel 2013; Delsate et al. 2016). On the other hand, angular momentum can also be stored in the superfluid core and different alternative astrophysical scenarios have been proposed (Sedrakian & Cordes 1999; Jahan-Miri 2002; Peralta et al. 2006; Pizzochero 2011; Ho et al. 2015; Pizzochero, Montoli & Antonelli 2020).

In particular, neutron vortices may pin to proton fluxoids in the core of NSs (Muslimov & Tsygan 1985; Sauls 1989; see also Alpar 2017, for a recent review), considering protons form a type-II superconductor, as first argued by Baym, Pethick & Pines (1969). Because a toroidal magnetic field is expected to be present in the outer core of an NS, in the region beneath the crust (see e.g. Sur, Haskell & Kuhn 2020 and references therein), vortex pinning is unavoidable. Therefore, that region of the core also contributes to glitches and their relaxation (Gügercinoğlu & Alpar 2014, 2020;

Gügercinoğlu 2017). Although vortices may cut through fluxoids depending on their velocity and on the pinning strength, Ruderman, Zhu & Chen (1998) estimated that this does not occur in Vela-like pulsars. The pinning of vortices to fluxoids may also drive crustal plate tectonics and play a key role in the evolution of the magnetic field (Srinivasan et al. 1990; Ruderman et al. 1998). Alternatively, Sedrakian & Sedrakian (1995) argued that fluxoids could actually be naturally nucleated in the vicinity of each vortex, thus forming ‘vortex clusters’. In either case, we have shown (Sourie & Chamel 2020b) that the rigid motion of vortices and fluxoids could explain specific timing features that have been recently observed in the Crab and Vela pulsars (Palfreyman et al. 2018; Shaw et al. 2018; Ashton et al. 2019).

Our analysis was carried out in the framework of Newtonian theory. Although the motions of individual vortices are locally non-relativistic, their typical velocities in the core are of order of 1 cm s^{-1} (Gügercinoğlu & Alpar 2016), the smooth-averaged hydrodynamics of the superfluid at the global scale of the star is prone to general-relativistic effects, especially in the most massive NSs (see e.g. Sourie et al. 2017; Gavassino et al. 2020 in the context of glitches). In particular, frame dragging in rotating NSs induces additional couplings between the superfluid and the rest of star, as first discussed by Carter (1975). We have shown that the ensuing coupling coefficients may be of comparable magnitude (although of opposite sign) as those due to the mutual entrainment induced by nuclear interactions (Sourie et al. 2017).

In this paper, we adapt our recent model of superfluid NSs (Sourie & Chamel 2020a,b) to the general-relativistic framework. To this end, we extend the analysis of Langlois, Sedrakian & Carter (1998) to allow for the pinning of vortices to fluxoids or the formation of vortex clusters in the outer core of NSs, based on the general formalism of dissipative superfluid mixtures developed by Carter & Chamel (2005b) in the Newtonian context. Using the fully 4D covariance of this formalism, we also show how the local non-relativistic vortex dynamics can be matched to the global hydrodynamic description of

* E-mail: aurelien.sourie@gmail.com (AS); nchamel@ulb.ac.be (NC)

the star. Our model of superfluid NS is presented in Section 2. Applications to quasi-stationary rotating NSs, including the calculation of the mutual-friction force, are discussed in Section 3. Unless stated otherwise, we shall set the speed of light $c = 1$.

2 COVARIANT TWO-CONSTITUENT SUPERFLUID HYDRODYNAMICS

2.1 Carter–Langlois–Sedrakian two-fluid model

In this section, we shall briefly review the mean features of the two-fluid model of Langlois et al. (1998; for a general description of superfluid NS, see e.g. Glampedakis, Andersson & Samuelsson 2011; Gusakov & Dommes 2016, and references therein).

Since electrically charged particles inside NSs are strongly coupled and essentially corotate with the crust and the magnetosphere, the outer core of a cold mature NS can be reasonably well described in terms of just two dynamically distinct fluids, namely (i) an inviscid neutron superfluid with 4-current $n_n^\mu = n_n u_n^\mu$ and (ii) a fluid made of protons and electrons with 4-current $n_p^\mu = n_p u_p^\mu$, u_n^μ and u_p^μ denoting the corresponding 4-velocities. This latter component will be referred to as the ‘normal’ fluid throughout the paper. In what follows, not only will the total baryon 4-current

$$n_b^\mu = n_n^\mu + n_p^\mu \quad (1)$$

be conserved, i.e.

$$\nabla_\mu n_b^\mu = 0, \quad (2)$$

but we will also neglect any kind of transfusive processes whereby one constituent is converted into the other. Each 4-current is therefore assumed to be separately conserved:¹

$$\nabla_\mu n_n^\mu = 0 \quad \text{and} \quad \nabla_\mu n_p^\mu = 0. \quad (3)$$

Following Langlois et al. (1998), the entropy current s^μ is not treated as an independent fluid, but is assumed to be expressible as

$$s^\mu = s u_p^\mu, \quad (4)$$

where s is the entropy density in the rest frame of the charged particles.

The local thermodynamic state of the system under consideration can be described by a Lagrangian density Λ , commonly referred to as the master function, which depends on both particle 4-current vectors n_n^μ and n_p^μ and on the entropy density s ,

$$\Lambda(n_n^\mu, n_p^\mu, s). \quad (5)$$

Variations of this quantity (keeping fixed the space–time metric²) lead to

$$\delta\Lambda = -\Theta \delta s + p_n^\mu \delta n_n^\mu + p_p^\mu \delta n_p^\mu, \quad (6)$$

where Θ is interpretable as the thermodynamic temperature of the system as measured in the rest frame of the normal fluid, and p_n^μ (resp. p_p^μ) denotes the canonical 4-momentum per baryon of the neutron superfluid (resp. the normal fluid). Let us remark that, due to non-dissipative entrainment effects arising from the nuclear interactions

¹Although such processes could be readily incorporated to the present model (Langlois et al. 1998), their impact on the superfluid dynamics of NSs is expected to be very weak (see e.g. Sourie et al. 2017).

²Variations of the total Lagrangian density (including the Einstein–Hilbert contribution) with respect to the space–time metric yield Einstein’s equations.

between neutrons and protons, the 4-momentum of a given fluid is not simply aligned with its corresponding 4-velocity but also depends on the 4-velocity of the second fluid (see e.g. Gusakov, Haensel & Kantor 2014; Sourie, Oertel & Novak 2016; Leinson 2018; Chamel & Allard 2019 for recent calculations of the coupling coefficients).

Carter and collaborators have developed an elegant action principle to derive the fluid equations of motion from the Lagrangian density Λ by considering variations of the fluid particle trajectories (see e.g. Carter 1989; Langlois et al. 1998; Carter & Langlois 1998 for details; see also Andersson & Comer 2020 for a review). Applied to the two-fluid model under consideration here, this procedure yields the following expression for the energy–momentum tensor of the system:

$$T_\nu^\mu = \Psi \delta_\nu^\mu + s^\mu \Theta_\nu + n_p^\mu p_\nu^p + n_n^\mu p_\nu^n, \quad (7)$$

where Ψ denotes the generalized pressure of the fluids, and Θ_ν is the thermal 4-momentum per ‘entropion’ as referred to by Carter, i.e. the 4-momentum per one unit of entropy dynamically conjugate to the entropy current, see e.g. equation (18) of Langlois et al. (1998). Note that the temperature can be alternatively interpreted as the chemical potential of entropions in the thermal rest frame (Carter 1989):

$$\Theta = -u_p^\mu \Theta_\mu. \quad (8)$$

This approach also leads to the following force laws [see equations (20) and (21) of Langlois et al. 1998]

$$f_\mu^p = n_p^\nu \varpi_{\nu\mu}^p \quad (9)$$

and

$$f_\mu^n = n_n^\nu \varpi_{\nu\mu}^n, \quad (10)$$

where (square brackets denoting antisymmetrization)

$$\varpi_{\nu\mu}^x = 2 \nabla_{[\nu} p_{\mu]}^x = \nabla_\nu p_\mu^x - \nabla_\mu p_\nu^x \quad (11)$$

stands for the vorticity 2-form averaged over scales larger than the intervortex separation (see below). The 4-covectors f_μ^p and f_μ^n involved in equations (9) and (10) are to be interpreted as the mean force densities acting on the normal fluid and the neutron superfluid, respectively.

At sufficiently small scale (but large enough for the hydrodynamic description to remain valid), the 4-momentum p_μ^n of the neutron superfluid is given by the gradient of the phase ϕ of the quantum condensate (see e.g. Carter & Langlois 1998):

$$p_\mu^n = \frac{m_n \kappa_n}{2\pi} \nabla_\mu \phi, \quad (12)$$

where $\kappa_n = h/(2m_n)$, h being the Planck constant and m_n the neutron rest mass. This relation implies that the superflow is irrotational, as characterized by the vanishing of the corresponding vorticity 2-form (11). Nevertheless, it is well known from laboratory experiments (see e.g. Yarmchuk, Gordon & Packard 1979) that the condition (12) can be locally violated through the formation of quantized vortices, each carrying a quantum of circulation κ_n .

The existence of such vortex filaments leads to the non-vanishing of the macroscopically averaged (i.e. averaged on scales much larger than the mean intervortex separation) vorticity 2-form $\varpi_{\mu\nu}^n$. However, the underlying presence of quantized vortices implies that $\varpi_{\mu\nu}^n$ must be of rank 2 instead of 4 (see e.g. Carter 1989; Langlois et al. 1998; Carter 2001; Andersson, Wells & Vickers 2016; Gavassino et al. 2020 for further discussions).

2.2 Interactions between vortices and the surrounding fluids

Having recalled the main features of the Carter–Langlois–Sedrakian two-fluid model, as well as the conditions imposed by superfluidity, the second law of thermodynamics can now be invoked to constrain the expression for the force density f_μ^n (10) exerted on the neutron superfluid by the vortex lines. A seminal work towards this direction was initiated by Langlois et al. (1998; see also Andersson et al. 2016; Gavassino et al. 2020 for similar studies). We shall extend the treatment of Langlois et al. (1998) by following the more general approach of dissipation in superfluid mixtures developed by Carter & Chamel (2005b) in a fully 4D covariant Newtonian framework.

Let us first introduce the thermal 4-force density defined by

$$f_\mu^\theta = 2s^v \nabla_{[v} \Theta_{\mu]} + \Theta_\mu \nabla_v s^v. \quad (13)$$

In the strictly conservative case, as characterized by the absence of any force other than those already included in the Lagrangian description, the separate force densities $f_{p\mu}$, $f_{n\mu}$, and f_μ^θ must all vanish. However, in the more general (dissipative) context under interest here, the force densities will be subject to the following relation (Carter & Chamel 2005b):

$$f_\mu^n + f_\mu^p + f_\mu^\theta = f_{\text{ext}\mu}, \quad (14)$$

where $f_{\text{ext}\mu} = \nabla_v T_\mu^v$ denotes the total external force density acting on the system, which arises from the loss of rotational energy through the emission of electromagnetic radiation and from internal heating. It can thus be decomposed into two forces acting separately on the charged and on the entropy components,

$$f_{\text{ext}\mu} = f_{\text{ext}\mu}^p + f_{\text{ext}\mu}^\theta, \quad (15)$$

considering that the neutron superfluid is not directly subject to any external force.

Without any loss of generality, the neutron force density f_μ^n can be decomposed as

$$f_\mu^n = f_{d\mu}^n + f_{c\mu}^n, \quad (16)$$

where $f_{d\mu}^n$ (resp. $f_{c\mu}^n$) is a dissipative (resp. conservative) force term, usually referred to as ‘drag’ (resp. ‘transverse’) force in the literature. Using this decomposition, equation (10) now reads

$$n_n^v \varpi_{v\mu}^n = f_{d\mu}^n + f_{c\mu}^n. \quad (17)$$

The conservative force term $f_{c\mu}^n$ was ignored by Langlois et al. (1998), as can be seen from their equation (41). The simplest prescription to ensure that equation (17) is compatible with the existence of two null eigenvectors, say w_1^μ and w_2^μ , for the neutron vorticity 2-form $\varpi_{\mu\nu}^n$ (as implied by the presence of quantized vortices), is to require that each force density covector is orthogonal to both w_1^μ and w_2^μ :

$$w_1^\mu f_{d\mu}^n = 0 = w_2^\mu f_{d\mu}^n \quad \text{and} \quad w_1^\mu f_{c\mu}^n = 0 = w_2^\mu f_{c\mu}^n, \quad (18)$$

where by definition

$$w_1^\mu \varpi_{\mu\nu}^n = 0 = w_2^\mu \varpi_{\mu\nu}^n. \quad (19)$$

In other words, $f_{d\mu}^n$ and $f_{c\mu}^n$ must be orthogonal to the 2D vortex worldsheet swept by the vectors w_1^μ and w_2^μ . Introducing the corresponding (orthogonal) projector \perp_μ^v , we must have

$$\perp_\mu^v f_{dv}^n = f_{d\mu}^n \quad \text{and} \quad \perp_\mu^v f_{cv}^n = f_{c\mu}^n. \quad (20)$$

By definition,

$$\perp_\mu^v \varpi_{\nu\rho}^n = \varpi_{\mu\rho}^n. \quad (21)$$

Contracting equation (21) again by the projector leads to

$$\perp_\nu^\mu \perp_\rho^v = \perp_\rho^\mu. \quad (22)$$

In view of equation (19), we also have

$$\perp_\nu^\mu w_1^v = 0 = \perp_\nu^\mu w_2^v. \quad (23)$$

The explicit form of the projector will be given in the next subsection, see equation (35). Since the vorticity is carried by vortex lines, one of the null eigenvectors w_1^μ or w_2^μ must be timelike, see Appendix C for further discussions. The orthogonality conditions (18) thus imply that $f_{d\mu}^n$ and $f_{c\mu}^n$ are spacelike.

The actual forms of $f_{d\mu}^n$ and $f_{c\mu}^n$ can be deduced from the second law of thermodynamics, which can be locally expressed as (see e.g. equation 23 of Carter & Chamel 2005b)

$$\Theta \nabla_\mu s^\mu + \mathcal{Q} \geq 0, \quad (24)$$

where $\mathcal{Q} = u_p^\mu f_{\text{ext}\mu}^\theta$ denotes the heat loss rate per unit volume in the thermal rest frame. In the following, we shall adopt the weak closure condition according to which the external force density acting on the normal fluid does not vanish but is restricted by the following relation (Carter & Chamel 2005b):

$$u_p^\mu f_{\text{ext}\mu}^p = 0. \quad (25)$$

In view of equations (9) and (10), we find the similar relations:

$$u_n^\mu f_\mu^n = 0 \quad \text{and} \quad u_p^\mu f_\mu^p = 0. \quad (26)$$

Contracting equation (15) with u_p^μ using equations (8), (13), (14), (25), and (26) yields

$$\Theta \nabla_\mu s^\mu + \mathcal{Q} = u_p^\mu f_\mu^n. \quad (27)$$

Since the force density $f_{c\mu}^n$ does not lead to any dissipation, the following relation should hold:

$$u_p^\mu f_{c\mu}^n = 0, \quad (28)$$

so that equation (27) reduces to

$$\Theta \nabla_\mu s^\mu + \mathcal{Q} = u_p^\mu f_{d\mu}^n. \quad (29)$$

An obvious way to make sure that $f_{c\mu}^n$ satisfies both equations (18) and (28) is to write

$$f_{c\mu}^n = \mathcal{A} \varpi_{\mu\nu}^n n_p^v, \quad (30)$$

with some unknown coefficient \mathcal{A} . On the other hand, the second law of thermodynamics (24) requires that $u_p^\mu f_{d\mu}^n$ be positive definite, or equivalently $u_{p\perp}^\mu f_{d\mu}^n > 0$ with $u_{p\perp}^\mu \equiv \perp_\nu^\mu u_p^v$ in view of equation (20). This condition can be fulfilled by expressing the dissipative force as

$$f_{d\mu}^n = C_r u_{p\perp\mu}, \quad (31)$$

(recalling that $f_{d\mu}^n$ hence also $u_{p\perp}^\mu$ are spacelike), where C_r is a positive coefficient. For later convenience, we introduce the (positive) resistivity coefficient $\mathcal{R} = C_r / \mathcal{N}_n$, where \mathcal{N}_n is the average surface density of vortex lines. The force balance equation (17) can finally be expressed as

$$n_n^v \varpi_{v\mu}^n = \mathcal{N}_n \mathcal{R} u_{p\perp\mu} + \mathcal{A} \varpi_{\mu\nu}^n n_p^v, \quad (32)$$

where, up to this point, \mathcal{R} and \mathcal{A} are left unspecified. Equation (32) represents the mutual-friction force, i.e. the effective force density exerted by the normal fluid on the neutron superfluid due to the averaged forces acting on individual vortices. This expression is very general since we have made no assumption on the spatial arrangement of vortex lines. Langlois et al. (1998) implicitly assumed that $\mathcal{A} = 0$. However, as we shall show in the next section, this coefficient is

non-zero whenever proton fluxoids are pinned to vortices or vortex clusters of the kind proposed by Sedrakian & Sedrakian (1995) are formed.

2.3 Matching between the local and global dynamics

The coefficients \mathcal{R} and \mathcal{A} appearing in equation (32) should be provided by an analysis of the local perturbations of the fluid flows induced by the motion of individual vortices. At such small scales, the space–time curvature is negligible (see e.g. the discussion in section 3.4 of Glendenning 1997) and the fluid dynamics is essentially non-relativistic. We have recently derived the Newtonian expression for the force per unit length exerted on a single vortex line in a mixture of superfluid neutrons, superconducting protons, and degenerate electrons, as can be typically found in the outer core of NSs (Sourie & Chamel 2020a). Allowing for the possible presence of vortex pinning on to proton fluxoids or vortex clusters of the kind proposed by Sedrakian & Sedrakian (1995), we considered the force acting on a vortex line to which a given number N_p of fluxoids are anchored.³ This number could be huge, of order $\sim 10^{13}$ (Sourie & Chamel 2020b). The resulting expression for the force acting in a vortex involves an unspecified dimensionless coefficient ξ , the so-called drag-to-lift ratio, which measures the amplitude of the microscopic drag force acting on each vortex line (which is thought to arise from the scattering of electrons off the magnetic field carried by the quantized lines). Although the drag-to-lift ratio associated with a vortex line pinned to N_p fluxoids is essentially unknown (see Sourie & Chamel 2020b), this parameter is expected to depend on N_p . The smooth-averaged force per unit volume exerted on the neutron superfluid at the macroscopic scale, as induced by the drag forces acting on individual vortices, was derived in Sourie & Chamel (2020b) within the Newtonian framework considering straight and infinitely rigid vortices, assumptions that remain applicable in general relativity (Gavassino et al. 2020). One can now use these results to determine the expressions for \mathcal{R} and \mathcal{A} .

To find the Newtonian limit of the force balance equation (32), we rely on the fully 4D-covariant Newtonian formulation developed by Carter & Chamel (2004, 2005a,b), based on the Milne–Cartan structure of the space–time. This approach allows for a more direct comparison with the general relativistic results (see e.g. Carter, Chachoua & Chamel 2006; Chamel 2008) than the Newtonian expressions from classical mechanics and hydrodynamics within the traditional ‘3+1’ space–time decomposition, as developed, e.g. by Prix (2004). As shown in Appendix A, the comparison between the Newtonian limit of equation (32) and results from Sourie & Chamel (2020b) leads to the following identification:

$$\mathcal{R} = m_n n_n \kappa_n \xi \quad \text{and} \quad \mathcal{A} = N_p, \quad (33)$$

from which we deduce that \mathcal{R} corresponds to the microscopic drag coefficient usually introduced in the literature (see e.g. Andersson, Sidery & Comer 2006). Using equation (33), the mutual-friction force (32) finally reads

$$f_\mu^{\text{mf}} = n_n^v \varpi_{\nu n}^n = n_n w^n \xi \perp_{\mu\nu} u_\nu^v - N_p n_p^v \varpi_{\nu\mu}^n, \quad (34)$$

which generalizes equation (76) from Langlois et al. (1998) to the case in which each neutron vortex line is pinned to N_p proton fluxoids. Introducing the space–time metric $g^{\mu\nu}$ with signature $(-, +, +, +)$,

³The N_p fluxoids are not superimposed but lie in the vicinity of the vortex, within distances much smaller than the intervortex spacing.

the projector to the surface orthogonal to the vortex worldsheet is explicitly given by (Langlois et al. 1998)

$$\perp_\nu^\mu = \frac{1}{(w^n)^2} g^{\mu\sigma} g^{\rho\tau} \varpi_{\sigma\tau}^n \varpi_{\nu\rho}^n = \frac{1}{(w^n)^2} \varpi^{n\mu\rho} \varpi_{\nu\rho}^n, \quad (35)$$

where the scalar

$$w^n = \sqrt{g^{\rho\mu} g^{\sigma\nu} \varpi_{\mu\nu}^n \varpi_{\rho\sigma}^n} / 2 \quad (36)$$

is related to the average surface density of vortex lines by

$$\mathcal{N}_n = \frac{w^n}{m_n \kappa_n}. \quad (37)$$

Let us remark that the microscopic expression of the drag-to-lift ratio ξ is likely to depend on various physical parameters, for which suitable relativistic definition must be used. For instance, the temperature in the covariant formulation must be understood as the scalar given by equation (8).

3 MUTUAL FRICTION IN A (QUASI-)STATIONARY AND AXISYMMETRIC SPACE–TIME

3.1 Space-time symmetries and fluid 4-velocities

We now restrict our study to (quasi-)stationary and axisymmetric space-times, as would be relevant for the modelling of rotating NSs (see e.g. Paschalidis & Stergioulas 2017 for a recent review). The validity of these assumptions is discussed at the beginning of section 3 of Langlois et al. (1998). In what follows, we thus assume that there exist two Killing vectors: k^μ for stationarity and h^μ for axisymmetry.⁴ The axisymmetry (resp. quasi-stationarity) of the fluid flows translates into the exact (resp. approximate) vanishing of the Lie derivative along h^μ (resp. k^μ) of any tensor field q associated with matter, i.e.

$$\mathcal{L}_{\bar{h}} q = 0 \quad \text{and} \quad \mathcal{L}_{\bar{k}} q \simeq 0, \quad (39)$$

where $\mathcal{L}_{\bar{h}} q$ denotes the Lie derivative of q along the vector u^μ .

Furthermore, the normal fluid is assumed to be rigidly rotating, so that its corresponding 4-velocity reads

$$u_p^\mu = \gamma_p (k^\mu + \Omega_p h^\mu), \quad (40)$$

where γ_p is a Lorentz-type factor and Ω_p is the (uniform) angular velocity of the normal fluid. Because charged particles are essentially locked to the pulsar’s magnetosphere, Ω_p coincides with the observed angular velocity of the pulsar. The 4-velocity of the neutron superfluid is taken as⁵

$$u_n^\mu = \gamma_n (k^\mu + \Omega_n h^\mu + \tilde{v}_n^\mu), \quad (41)$$

where γ_n is the relevant Lorentz-type factor and Ω_n is the (a priori non-uniform) angular velocity of the neutron superfluid. The last term allows for the possibility of a non-circular motion (expected to

⁴While h^μ is an exact Killing vector, k^μ is only an *approximate* Killing vector because of the small non-circular motion considered in the following. This means that k^μ satisfies the following condition:

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = \mathcal{O}\left(\frac{1}{L}\right), \quad (38)$$

where L is a length-scale very large with respect to the stellar radius (see equation 49 of Langlois et al. 1998).

⁵Note that equations (40) and (41) are not orthogonal decompositions.

be very small, see Langlois et al. 1998). The factors γ_p and γ_n are fixed by the normalization conditions

$$g_{\mu\nu}u_n^\mu u_n^\nu = -1 \quad \text{and} \quad g_{\mu\nu}u_p^\mu u_p^\nu = -1. \quad (42)$$

Further details on the decompositions (40) and (41) are given in Appendix B.

3.2 Evolution equations

Let us now express the mutual-friction force (34) in terms of Ω_p , Ω_n , and \tilde{v}_n^μ . We follow here the same approach as Langlois et al. (1998). Let us start by contracting equation (34) with $\varpi^{n\mu\nu}$. This leads to

$$w^n \perp_\mu^\nu \left(n_n^\mu + N_p n_p^\mu \right) = n_n \xi \varpi^{n\nu\mu} u_{p\mu}, \quad (43)$$

where we have used equation (35) and the orthogonality property $\perp_\mu^\rho \varpi^{n\mu\nu} = \varpi^{n\rho\nu}$. Introducing

$$\alpha_n = h^\nu p_\nu^n, \quad (44)$$

which can be interpreted as the angular momentum per neutron (see Section 3.4), a first dynamical equation is obtained by contracting equation (43) with h_ν , i.e.

$$w^n h_\perp^\mu \left(n_{n\mu} + N_p n_{p\mu} \right) = -n_n \xi u_p^\mu \nabla_\mu \alpha_n, \quad (45)$$

where $\nabla_\mu \alpha_n = w_{\mu\nu}^\nu h^\nu$, as deduced from $\mathcal{L}_{\tilde{h}} p_\mu^n = 0$, and we have introduced the short-hand notation $h_\perp^\mu = \perp_\nu^\mu h^\nu$. A second dynamical equation is derived by contracting equation (43) with $w_{\nu\rho}^\nu h^\rho$, which yields

$$\left(n_n^\mu + N_p n_p^\mu \right) \nabla_\mu \alpha_n = n_n w^n \xi h_\perp^\mu u_{p\mu}. \quad (46)$$

Using equations (40) and (41), the dynamical equations (45) and (46) lead to

$$(1+X)k_\mu h_\perp^\mu + \frac{\tilde{\xi}}{w^n} \dot{\alpha}_n = -h_\perp^\mu \tilde{v}_{n\mu} - (\Omega_n + X\Omega_p) h_\perp^2, \quad (47)$$

$$-w^n \tilde{\xi} k_\mu h_\perp^\mu + (1+X)\dot{\alpha}_n = -\tilde{v}_n^\mu \nabla_\mu \alpha_n + w^n \tilde{\xi} \Omega_p h_\perp^2, \quad (48)$$

where

$$h_\perp^2 = h_\perp^\mu h_{\perp\mu} = h_\perp^\mu h_\mu, \quad (49)$$

and

$$\dot{\alpha}_n = k^\mu \nabla_\mu \alpha_n, \quad (50)$$

which is small but non-zero since k^μ is not an exact Killing vector. Note that we have used here the fact that $\mathcal{L}_{\tilde{h}} \alpha_n = h^\mu \nabla_\mu \alpha_n = 0$. The quantities $\tilde{\xi}$ and X appearing in equations (47) and (48) are defined as

$$\tilde{\xi} = \frac{\gamma_p}{\gamma_n} \xi \simeq \xi, \quad (51)$$

and

$$X = \frac{\gamma_p}{\gamma_n} \frac{n_p}{n_n} N_p \simeq \frac{n_p}{n_n} N_p, \quad (52)$$

where $\gamma_p \simeq \gamma_n$ to a very good approximation, since the neutron superfluid and the normal fluid are expected to be very close to corotation at any time, as suggested by the very small glitch amplitudes.

The determinant of the system (47)–(48) being non-zero, i.e. $(1+X)^2 + \tilde{\xi}^2 > 0$, this system of equations can be inverted, which leads to

$$-h_\perp^2 k_\mu h_\perp^\mu = (1-B') \left(\Omega_n + \Omega_p \frac{B'}{1-B'} + \Omega_- \right), \quad (53)$$

$$(w^n)^{-1} h_\perp^{-2} \dot{\alpha}_n = \mathcal{B} (\Omega_p - \Omega_n - \Omega_+), \quad (54)$$

where we have introduced the following mutual-friction coefficients

$$\mathcal{B} = \frac{\tilde{\xi}}{\tilde{\xi}^2 + (1+X)^2} \quad \text{and} \quad 1-B' = \frac{1+X}{\tilde{\xi}^2 + (1+X)^2}, \quad (55)$$

in a similar manner to what has been done in the Newtonian framework (see equation 57 of Sourie & Chamel 2020a) and the (small) non-circular contributions read

$$\Omega_- = h_\perp^{-2} \tilde{v}_n^\mu \left(h_{\perp\mu} - \tilde{\xi} (w^n)^{-1} (1+X)^{-1} \nabla_\mu \alpha_n \right), \quad (56)$$

$$\Omega_+ = h_\perp^{-2} \tilde{v}_n^\mu \left(h_{\perp\mu} + \tilde{\xi}^{-1} (w^n)^{-1} (1+X) \nabla_\mu \alpha_n \right). \quad (57)$$

The previous equations generalize those derived by Langlois et al. (1998) to allow for vortex pinning in the outer core of NSs. Indeed, taking $N_p = 0$ (or equivalently $X = 0$) in these equations leads to equations (84)–(87) of Langlois et al. (1998), $\tilde{\xi}$ reducing in this case to the drag-to-lift ratio in the absence of pinning, denoted by c_T in Langlois et al. (1998).

As shown in Appendix C, the terms on the left-hand side of equations (53) and (54) can be respectively interpreted as the mean angular velocity of vortices (modulo a small non-circular contribution) and the (inwards) ‘radial’ velocity of the vortex lines, i.e. along the unit vector $-\hat{r}^\mu$, where \hat{r}^μ is defined by

$$\hat{r}^\mu = (w^n)^{-1} h_\perp^{-1} \varpi^{n\mu\nu} h_\nu \quad \text{and} \quad \hat{r}^\mu \hat{r}_\mu = 1. \quad (58)$$

Note that $\hat{r}^\mu h_{\perp\mu} = 0$ and $\perp_\nu^\mu \hat{r}^\nu = \hat{r}^\mu$, from which we deduce that the unit vectors \hat{r}^μ and $\hat{h}_\perp^\mu = h_\perp^\mu / h_\perp$ form an orthonormal basis of the 2D surface orthogonal to the vortex worldsheet. We thus have

$$\perp_\nu^\mu = \hat{h}_\perp^\mu \hat{h}_{\perp\nu} + \hat{r}^\mu \hat{r}_\nu. \quad (59)$$

The projector tangential to the vortex worldsheet is thus given by

$$\eta_\nu^\mu \equiv \delta_\nu^\mu - \perp_\nu^\mu = \delta_\nu^\mu - \hat{h}_\perp^\mu \hat{h}_{\perp\nu} - \hat{r}^\mu \hat{r}_\nu. \quad (60)$$

The non-vanishing of the radial component of the mean vortex velocity (albeit very small) actually highlights the non-exact stationarity of the space–time considered here.

Finally, the Newtonian limit of equations (53) and (54) is found to match perfectly with results obtained from Sourie & Chamel (2020a), as shown in Appendix D.

3.3 Mutual-friction force in stationary rotating neutron stars

Since $\eta_\nu^\mu f_\mu^{\text{mf}} = 0$, see equations (17) and (20), the only non-zero components of the mutual-friction force (34) are those along the unit vectors \hat{h}_\perp^μ and \hat{r}^μ . Substituting equations (41) in (34) recalling $\nabla_\mu \alpha_n = w_{\mu\nu}^\nu h^\nu$ and using equations (53) and (54) leads to

$$\begin{aligned} \hat{h}_\perp^\mu f_\mu^{\text{mf}} &= h^\mu f_\mu^{\text{mf}} / h_\perp \\ &= \gamma_n n_n (k^\mu + \Omega_n h^\mu + \tilde{v}_n^\mu) \nabla_\mu \alpha_n / h_\perp \\ &= \gamma_n n_n (\dot{\alpha}_n + \tilde{v}_n^\mu \nabla_\mu \alpha_n) / h_\perp \\ &= \gamma_n n_n w^n h_\perp \mathcal{B} [\Omega_p - \Omega_n + X\Omega_+ - (1+X)\Omega_-], \end{aligned} \quad (61)$$

and

$$\begin{aligned}\hat{r}^\mu f_\mu^{\text{mf}} &= -\gamma_n n_n w^n (k^\mu + \Omega_n h^\mu + \tilde{v}_n^\mu) h_{\perp\mu} / h_\perp \\ &= -\gamma_n n_n w^n h_\perp (h_\perp^{-2} k_\mu h_\perp^\mu + \Omega_n + h_\perp^{-2} \tilde{v}_n^\mu h_{\perp\mu}) \\ &= \gamma_n n_n w^n h_\perp \mathcal{B}' \left[\Omega_p - \Omega_n - \frac{\tilde{\xi}^2 \Omega_+ + X(1+X)\Omega_-}{\tilde{\xi}^2 + X(1+X)} \right].\end{aligned}\quad (62)$$

Equations (61) and (62) establish the expression for the mutual-friction force as a function of the fluid velocities only. In the limit $X = 0$, equation (61) coincides with equation (89) from Langlois et al. (1998). The Newtonian limit of equations (61) and (62) are also found to be in perfect agreement with the expression (5) from Sourie & Chamel (2020b), as shown in Appendix D.

3.4 Fluid angular momenta

In order to illustrate the impact of mutual-friction forces on the superfluid dynamics of NSs, let us now focus on the angular momentum transfer that takes place during the spin-up stage of pulsar glitches.

The existence of an (exact) Killing vector associated with axisymmetry allows for a gauge-invariant definition of the stellar angular momentum using the so-called Komar definition. In the usual ‘3+1’ decomposition of the space–time, in which the space–time is foliated by a family $(\Sigma_t)_{t \in \mathbb{R}}$ of space-like hypersurfaces, the total angular momentum is thus defined as (see e.g. Gourgoulhon 2012)

$$J = - \int_{\Sigma_t} \left[T_{\mu\nu} n^\mu h^\nu - \frac{1}{2} g^{\mu\nu} T_{\mu\nu} n_\rho h^\rho \right] d\Sigma, \quad (63)$$

where $T_{\mu\nu}$ is the energy–momentum tensor of the two-fluid system (7), n^μ is the unit (future-oriented) vector normal to Σ_t and $d\Sigma$ is the volume element on the hypersurface Σ_t . In a coordinate system adapted to the foliation (of the kind $x^0 = t, x^1, x^2, x^3$), we have $d\Sigma = \sqrt{\gamma} dx^1 dx^2 dx^3$, where γ is the determinant of the spatial metric γ_{ij} (using Latin letters for space indices $i, j = 1, 2, 3$), defined from the 4-metric $\gamma_{\mu\nu}$ induced by $g_{\mu\nu}$ on Σ_t (i.e. $\gamma_{ij} = g_{ij}$). The second term in equation (63) can be dropped if the hypersurface is further required to be axisymmetric, i.e. if the Killing vector h^μ is tangent to Σ_t such that $h^\mu n_\mu = 0$. The angular momentum of the star thus simplifies as

$$J = \int_{\Sigma_t} j^\mu d\Sigma_\mu, \quad (64)$$

where $d\Sigma_\mu = -n_\mu d\Sigma$ and $j^\mu = T_\nu^\mu h^\nu$ is the local angular momentum current. In the presence of external forces ($f_\mu^{\text{ext}} \neq 0$), the time variation of the stellar angular momentum (as given from the time-translation generator k^μ) is non-zero and reads

$$\frac{dJ}{dt} = \Gamma^{\text{ext}}, \quad \Gamma^{\text{ext}} = \int_{\Sigma_t} h^\mu f_\mu^{\text{ext}} k^\nu d\Sigma_\nu, \quad (65)$$

where Γ^{ext} denotes the torque associated with the external force f_μ^{ext} (Langlois et al. 1998).

Using equation (7), the local angular momentum current can be decomposed into

$$j^\mu = j_n^\mu + j_p^\mu + \tilde{j}^\mu, \quad (66)$$

where we have introduced the following notations

$$j_n^\mu = n_n^\mu p_n^\nu h^\nu = \alpha_n n_n^\mu, \quad (67)$$

$$j_p^\mu = \left(n_p^\mu p_p^\nu + s \Theta u_p^\mu u_{p\nu} \right) h^\nu, \quad (68)$$

and

$$\tilde{j}^\mu = \Psi h^\mu. \quad (69)$$

Although the pressure term \tilde{j}^μ does not allow for an unambiguous decomposition of the total angular momentum current in terms of separate fluid contributions, the total angular momentum of the star can still be unambiguously decomposed into a neutron and a normal parts since $\tilde{j}^\mu n_\mu = 0$, yielding (Langlois et al. 1998; Sourie et al. 2016)

$$J = J_n + J_p, \quad (70)$$

where

$$J_n = \int_{\Sigma_t} j_n^\mu d\Sigma_\mu, \quad \text{and} \quad J_p = \int_{\Sigma_t} j_p^\mu d\Sigma_\mu. \quad (71)$$

3.5 Angular momentum transfer induced by mutual-friction forces

Due to mutual-friction forces, angular momentum is redistributed between the initially decoupled neutron superfluid and the rest of the star during the rise of pulsar glitches. The corresponding dynamical equations read (Langlois et al. 1998)

$$\frac{dJ_n}{dt} = \Gamma_{\text{mf}} \quad \text{and} \quad \frac{dJ_p}{dt} = \Gamma_{\text{ext}} - \Gamma_{\text{mf}}, \quad (72)$$

where

$$\Gamma_{\text{mf}} = \int_{\Sigma_t} h^\mu f_{n\mu} k^\nu d\Sigma_\nu = \int_{\Sigma_t} h^\mu f_\mu^{\text{mf}} k^\nu d\Sigma_\nu. \quad (73)$$

Using Equation (61), the mutual-friction torque is found to be given by

$$\Gamma_{\text{mf}} = \int_{\Sigma_t} \mathcal{B} \gamma_n n_n w^n h_\perp^2 \left[\Omega_p - \Omega_n + X \Omega_+ - (1+X) \Omega_- \right] k^\nu d\Sigma_\nu. \quad (74)$$

The pinning of proton fluxoids to neutron vortices does not only lead to a rescaling of the mutual-friction coefficient \mathcal{B} , but also introduces additional terms in the mutual-friction torque through the parameter X , which in turn is proportional to the number N_p of fluxoids attached to each vortex. These terms are still present in the Newtonian limit but were not taken into account in our previous analysis (Sourie & Chamel 2020b), see also Appendix D.

4 CONCLUSIONS

Following the seminal work of Langlois et al. (1998), we have adapted to the general-relativistic framework our recent Newtonian treatment (Sourie & Chamel 2020a) of the smooth-averaged mutual-friction force acting on the neutron superfluid and locally induced by the pinning of quantized neutron vortices to proton fluxoids in the outer core of superfluid NSs (or alternatively by the formation of vortex clusters of the kind proposed by Sedrakian & Sedrakian 1995). Quite generally, the force can be written as $f_\mu^{\text{mf}} = \mathcal{N}_n \mathcal{R} u_{p\perp\mu} + \mathcal{A} n_p^\nu \varpi_{\mu\nu}^n$. The apparent simplicity of this expression is deceptive because the microscopic coefficients \mathcal{R} and \mathcal{A} may depend on various physical parameters, such as the temperature (8), the currents through the scalars $n_n^\mu n_{n\mu}, n_p^\mu n_{p\mu}, n_n^\mu n_{p\mu}$, or the vorticity (36). Langlois et al. (1998) implicitly assumed $\mathcal{A} = 0$. However, this assumption turns out to be unrealistic whenever vortices interact with fluxoids.

Using the fully 4D covariant formulation of Newtonian dynamics of Carter & Chamel (2004, 2005a,b), we have shown how to relate the global general-relativistic dynamics of superfluid NSs to the local non-relativistic dynamics of individual vortices. Comparing our study of the non-relativistic motion of a single vortex (Sourie & Chamel 2020a), we have thus been able to identify \mathcal{R} with the drag coefficient and \mathcal{A} with the mean number N_p of fluxoids pinned to each vortex.

According to our recent Newtonian study (Sourie & Chamel 2020b), vortex pinning may have important implications for the dynamics of superfluid NSs. Considering quasi-stationary and axisymmetric rotating NSs, we have derived the general-relativistic dynamical equations describing the mean motion of vortices, equations (53) and (54), as well as the transfer of angular momentum between the different stellar components, equations (72) and (74). This set of equations generalizes the one that was obtained earlier by Langlois et al. (1998). The role of the pinning of fluxoids to vortices or the formation of vortex clusters on the global dynamics of NSs, as embedded in the new parameter X defined by equation (52), appears to be rather non-trivial.

This work provides all the necessary equations for carrying out realistic simulations of cold superfluid NSs in full general relativity allowing for vortices and fluxoids, as required for the detailed interpretation of pulsar frequency glitches.

ACKNOWLEDGEMENTS

This work was supported by the Fonds de la Recherche Scientifique (Belgium) under grants no. 1.B.410.18F, CDR J.0115.18, and PDR T.004320. Partial support comes also from the European Cooperation in Science and Technology (COST) action CA16214.

DATA AVAILABILITY

No new data were generated or analysed in support of this research.

REFERENCES

Alpar M. A., 2017, *J. Astrophys. Astron.*, 38, 44
 Alpar M. A., Pines D., Anderson P. W., Shaham J., 1984a, *ApJ*, 276, 325
 Alpar M. A., Langer S. A., Sauls J. A., 1984b, *ApJ*, 282, 533
 Anderson P. W., Itoh N., 1975, *Nature*, 256, 25
 Andersson N., Comer G. L. C., 2020, Relativistic Fluid Dynamics: Physics for Many Different Scales. preprint (arXiv:2008.12069)
 Andersson N., Sidery T., Comer G. L., 2006, *MNRAS*, 368, 162
 Andersson N., Glampedakis K., Ho W. C. G., Espinoza C. M., 2012, *Phys. Rev. Lett.*, 109, 241103
 Andersson N., Wells S., Vickers J. A., 2016, *Class. Quantum Gravity*, 33, 245010
 Ashton G., Lasky P. D., Graber V., Palfreyman J., 2019, *Nat. Astron.*, 3, 1143
 Baym G., Pethick C., Pines D., 1969, *Nature*, 224, 673
 Carter B., 1975, *Ann. Phys.*, 95, 53
 Carter B., 1989, in Anile A. M., Choquet-Bruhat Y., eds, *Lecture Notes in Mathematics, Vol. 1385, Relativistic Fluid Dynamics*. Springer, Berlin, Heidelberg, p. 1
 Carter B., 2001, in Blaschke D., Glendenning N. K., Sedrakian A., eds, *Lecture Notes in Physics, Vol. 578, Physics of Neutron Star Interiors*. Springer Verlag, Berlin, p. 54
 Carter B., Chamel N., 2004, *Int. J. Mod. Phys. D*, 13, 291
 Carter B., Chamel N., 2005a, *Int. J. Mod. Phys. D*, 14, 717
 Carter B., Chamel N., 2005b, *Int. J. Mod. Phys. D*, 14, 749
 Carter B., Langlois D., 1998, *Nucl. Phys. B*, 531, 478
 Carter B., Chachoua E., Chamel N., 2006, *Gen. Relat. Gravit.*, 38, 83
 Chamel N., 2008, *MNRAS*, 388, 737

Chamel N., 2013, *Phys. Rev. Lett.*, 110, 011101
 Chamel N., 2015, *Int. J. Mod. Phys. D*, 24, 1550018
 Chamel N., 2017a, *J. Astrophys. Astron.*, 38, 43
 Chamel N., 2017b, *J. Low. Temp. Phys.*, 189, 328
 Chamel N., Allard V., 2019, *Phys. Rev. C*, 100, 065801
 Chamel N., Carter B., 2006, *MNRAS*, 368, 796
 Delsate T., Chamel N., Gürlebeck N., Fantina A. F., Pearson J. M., Ducoin C., 2016, *Phys. Rev. D*, 94, 023008
 Gavassino L., Antonelli M., Pizzochero P. M., Haskell B., 2020, *MNRAS*, 494, 3562
 Glampedakis K., Andersson N., Samuelsson L., 2011, *MNRAS*, 410, 805
 Glendenning N. K., 1997, *Compact stars: Nucl. Phys., Particle Physics, and General Relativity*. Springer-Verlag, Berlin
 ourgoulhon E., 2012, *3+1 Formalism in General Relativity*. Springer-Verlag, Berlin
 Graber V., Andersson N., Hogg M., 2017, *Int. J. Mod. Phys. D*, 26, 1730015
 Gügercinoğlu E., 2017, *MNRAS*, 469, 2313
 Gügercinoğlu E., Alpar M. A., 2014, *ApJ*, 788, L11
 Gügercinoğlu E., Alpar M. A., 2016, *MNRAS*, 462, 1453
 Gügercinoğlu E., Alpar M. A., 2020, *MNRAS*, 496, 2506
 Gusakov M. E., Dommers V. A., 2016, *Phys. Rev. D*, 94, 083006
 Gusakov M. E., Haensel P., Kantor E. M., 2014, *MNRAS*, 439, 318
 Haskell B., Melatos A., 2015, *Int. J. Mod. Phys. D*, 24, 1530008
 Haskell B., Sedrakian A., 2018, in Rezzolla L., Pizzochero P., Jones D. I., Rea N., Vidana I., eds, *Astrophysics and Space Science Library, Vol. 457*. Springer, Cham, p. 401
 Ho W. C. G., Espinoza C. M., Antonopoulou D., Andersson N., 2015, *Sci. Adv.*, 1, e1500578
 Jahan-Miri M., 2002, *MNRAS*, 330, 279
 Langlois D., Sedrakian D. M., Carter B., 1998, *MNRAS*, 297, 1189
 Leinson L. B., 2018, *MNRAS*, 479, 3778
 Manchester R. N., 2017, *Proc. Int. Astron. Union*, 13, 197
 Muslimov A. G., Tsygan A. I., 1985, *Ap&SS*, 115, 43
 Palfreyman J., Dickey J. M., Hotan A., Ellingsen S., van Straten W., 2018, *Nature*, 556, 219
 Paschalidis V., Stergioulas N., 2017, *Living Rev. Relativ.*, 20, 7
 Peralta C., Melatos A., Giacobello M., Ooi A., 2006, *ApJ*, 651, 1079
 Pizzochero P. M., 2011, *ApJ*, 743, L20
 Pizzochero P. M., Montoli A., Antonelli M., 2020, *A&A*, 636, A101
 Prix R., 2004, *Phys. Rev. D*, 69, 043001
 Ruderman M., Zhu T., Chen K., 1998, *ApJ*, 492, 267
 Sauls J., 1989, in Ögelman H., van den Heuvel E. P. J., eds, *Proceedings of the NATO Advanced Study Institute on Timing Neutron Stars held April 4-15, 1988, in Çeşme, Izmir, Turkey*. Kluwer Academic/Plenum Publishers, New York, NY USA, p. 457
 Sauls J. A., Chamel N., Alpar M. A., 2020, Superfluidity in Disordered Neutron Stars Crusts. preprint (arXiv:2001.09959)
 Sedrakian A., Cordes J. M., 1999, *MNRAS*, 307, 365
 Sedrakian A. D., Sedrakian D. M., 1995, *ApJ*, 447, 305
 Shaw B. et al., 2018, *MNRAS*, 478, 3832
 Sourie A., Chamel N., 2020a, *MNRAS*, 493, 382
 Sourie A., Chamel N., 2020b, *MNRAS*, 493, L98
 Sourie A., Oertel M., Novak J., 2016, *Phys. Rev. D*, 93, 083004
 Sourie A., Chamel N., Novak J., Oertel M., 2017, *MNRAS*, 464, 4641
 Srinivasan G., Bhattacharya D., Muslimov A. G., Tsygan A. J., 1990, *Curr. Sci.*, 59, 31
 Sur A., Haskell B., Kuhn E., 2020, *MNRAS*, 495, 1360
 Watanabe G., Pethick C. J., 2017, *Phys. Rev. Lett.*, 119, 062701
 Yarmchuk E. J., Gordon M. J. V., Packard R. E., 1979, *Phys. Rev. Lett.*, 43, 214

APPENDIX A: NEWTONIAN LIMIT OF THE FORCE BALANCE EQUATION (32)

In this appendix, we use the covariant Newtonian formulation developed by Carter and Chamel (Carter & Chamel 2004, 2005a,b) to match the global general-relativistic description with the local

non-relativistic dynamics of individual vortices discussed in Sourie & Chamel (2020a). We will write the speed of light c explicitly since we are interested in the limit $c \rightarrow +\infty$.

A1 Covariant Newtonian formulation for superfluid neutron stars

Let $t_\mu \equiv \partial_\mu t$ be the gradient of the preferred Newtonian time coordinate t associated with the foliation of the space–time into flat 3D spaces with coordinates X^i (we shall use Latin letters for spatial indices). Introducing the 4D symmetric covariant tensor $\gamma_{\mu\nu}$, as obtained from pulling back the Euclidean spatial metric γ_{ij} , the (locally flat) Lorentzian metric of the relativistic description can be expressed as (Carter et al. 2006)

$$g_{\mu\nu} = \gamma_{\mu\nu} - c^2 t_\mu t_\nu. \quad (\text{A1})$$

The factor c^2 is introduced here because in an ‘Aristotelian’ coordinate system corresponding to the usual space–time decomposition, x^0 coincides with the Newtonian time t (not ct), and x^i with the space coordinates X^i . The tensor $\gamma_{\mu\nu}$ is not a space–time metric since it is degenerate

$$\gamma_{\mu\nu} e^\nu = 0, \quad (\text{A2})$$

thus defining the so-called ether-frame flow vector e^μ , normalized as

$$e^\mu t_\mu = 1. \quad (\text{A3})$$

Similarly, the contravariant metric tensor can be expressed as (Carter et al. 2006)

$$g^{\mu\nu} = \gamma^{\mu\nu} - \frac{1}{c^2} e^\mu e^\nu, \quad (\text{A4})$$

where the symmetric contravariant tensor $\gamma^{\mu\nu}$ is obtained from a pushforward of the 3D Euclidean spatial metric γ^{ij} . Note that $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ satisfy the following relation:

$$\gamma^{\mu\nu} \gamma_{\nu\rho} \equiv \gamma_\rho^\mu = \delta_\rho^\mu - e^\mu t_\rho, \quad (\text{A5})$$

where δ_ρ^μ denotes the Kronecker symbol. Like $\gamma_{\mu\nu}$, the tensor $\gamma^{\mu\nu}$ is also degenerate

$$\gamma^{\mu\nu} t_\nu = 0. \quad (\text{A6})$$

Therefore, the Newtonian space–time is characterized by the absence of any metric. This means that indices cannot be lowered or raised. In other words, covariant and contravariant indices of any space–time tensor are intrinsic. A special care is therefore needed when taking the Newtonian limit of relativistic expressions.

The fully antisymmetric 4D contravariant spatial measure tensor $\varepsilon^{\mu\nu\rho}$ is obtained by pushforward of the Euclidean measure tensor ε^{ijk} . The covariant spatial measure tensor is given by

$$\varepsilon_{\mu\nu\rho} = \gamma_{\mu\alpha} \gamma_{\nu\beta} \gamma_{\rho\gamma} \varepsilon^{\alpha\beta\gamma}. \quad (\text{A7})$$

The fully covariant space–time measure tensor $\varepsilon_{\mu\nu\rho\sigma}$ is defined by the relation

$$t_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\nu\rho\sigma}. \quad (\text{A8})$$

The contravariant counterpart $\varepsilon^{\mu\nu\rho\sigma}$ is obtained by the normalization condition

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} = -4!. \quad (\text{A9})$$

Note that we have

$$\varepsilon^{\mu\nu\rho} t_\rho = 0, \quad \varepsilon_{\mu\nu\rho} e^\rho = 0, \quad \text{and} \quad \varepsilon^{\mu\nu\rho\sigma} t_\sigma = \varepsilon^{\mu\nu\rho}. \quad (\text{A10})$$

A2 Force balance equation

Using equation (A1), the relativistic dissipative force (31) can be written as

$$f_{d\mu}^n = C_r (\gamma_{\mu\nu} - c^2 t_\mu t_\nu) \perp_\rho^v u_p^\rho. \quad (\text{A11})$$

Substituting equation (35) in equation (A11) using equation (A4) yields

$$f_{d\mu}^n = \frac{C_r}{(w^n)^2} (\gamma_{\mu\nu} - c^2 t_\mu t_\nu) \left(\gamma^{\nu\lambda} \gamma^{\sigma\tau} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n - \gamma^{\nu\lambda} \frac{e^\sigma e^\tau}{c^2} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n - \gamma^{\sigma\tau} \frac{e^\nu e^\lambda}{c^2} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n \right) u_p^\rho. \quad (\text{A12})$$

In the Newtonian limit $c \rightarrow +\infty$, this becomes

$$f_{d\mu}^n = \frac{C_r}{(w^n)^2} (\gamma_\mu^\lambda \gamma^{\sigma\tau} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n + t_\mu e^\lambda \gamma^{\sigma\tau} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n) u_p^\rho. \quad (\text{A13})$$

Let w_n^μ be the neutron vorticity vector, defined as (Carter & Chamel 2005b)

$$w_n^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho}^n t_\sigma = \frac{1}{2} \varepsilon^{\mu\nu\rho} \varpi_{\nu\rho}^n. \quad (\text{A14})$$

It follows from equation (A10) that

$$w_n^\mu t_\mu = 0, \quad (\text{A15})$$

meaning that the vorticity vector w_n^μ is purely spatial. Its norm is given by

$$\gamma_{\mu\nu} w_n^\mu w_n^\nu = \frac{1}{2} \gamma^{\mu\nu} \gamma^{\rho\sigma} \varpi_{\mu\rho}^n \varpi_{\nu\sigma}^n = (w^n)^2, \quad (\text{A16})$$

where the vorticity scalar w^n is the Newtonian limit ($c \rightarrow +\infty$) of equation (36).

From the degeneracy of the vorticity 2-form $w_{\mu\nu}^n$, we have

$$\varpi_{\mu\nu}^n w_n^\nu = 0, \quad (\text{A17})$$

see e.g. equation (75) from Carter & Chamel (2005b) or equation (41) from Chamel & Carter (2006). The unit spatial vector \hat{k}^μ , defined as

$$\hat{k}^\mu = \frac{w_n^\mu}{w^n}, \quad \gamma_{\mu\nu} \hat{k}^\mu \hat{k}^\nu = 1, \quad (\text{A18})$$

can thus be seen as the unit vector along the vortex line. Since $w_{\mu\nu}^n$ is of rank 2, we can introduce a vector u_L^μ satisfying (see equation 89 of Carter & Chamel 2005b)

$$\varpi_{\mu\nu}^n u_L^\nu = 0, \quad u_L^\nu t_\nu = 1, \quad \text{and} \quad \gamma_{\mu\nu} u_L^\mu \hat{k}^\nu = 0. \quad (\text{A19})$$

The vorticity surface-generating 4-vector u_L^μ can be interpreted as the local average 4-velocity of the vortices (in the sense that the vorticity 2-form $w_{\mu\nu}^n$ is Lie transported by u_L^μ) in a direction orthogonal to \hat{k}^μ , as illustrated in Fig. A1. We now define the spatial part of the vortex 4-velocity as

$$v_L^\mu = u_L^\mu - e^\mu. \quad (\text{A20})$$

Indeed, the normalization condition $u_L^\mu t_\mu = 1$, in combination with equation (A3), leads to

$$v_L^\mu t_\mu = 0. \quad (\text{A21})$$

Besides, inserting equation (A20) into the last relation in (A19) yields

$$\gamma_{\mu\nu} v_L^\mu \hat{k}^\nu = 0. \quad (\text{A22})$$

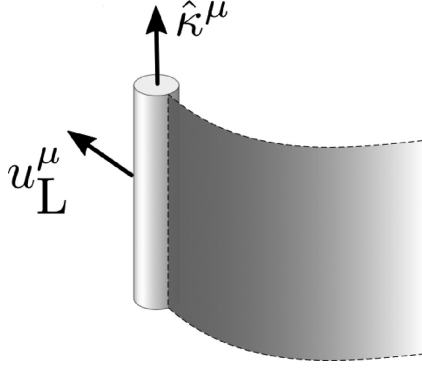


Figure A1. The 4-vectors u_L^μ and \hat{k}^μ form an orthogonal basis of the 2D surface swept by the quantized vortex. See the text for details.

From the previous considerations, the neutron vorticity 2-form $\varpi_{\mu\nu}^n$ is found to be expressible as (Carter & Chamel 2005b)

$$\varpi_{\mu\nu}^n = (\varepsilon_{\mu\nu\rho} + 2v_L^\sigma t_{[\nu} \varepsilon_{\mu]\rho\sigma}) w^n \hat{k}^\rho. \quad (\text{A23})$$

Defining the spatial part of the fluid 4-velocities by

$$v_n^\mu = u_n^\mu - e^\mu, \quad v_n^\mu t_\mu = 0, \quad (\text{A24})$$

and

$$v_p^\mu = u_p^\mu - e^\mu, \quad v_p^\mu t_\mu = 0, \quad (\text{A25})$$

the force balance equation (32) can be finally recast as

$$\begin{aligned} 0 = & n_n w^n \varepsilon_{\mu\nu\rho} (v_n^\nu - v_L^\nu) \hat{k}^\rho + n_n w^n t_\mu \varepsilon_{\nu\rho\sigma} v_n^\nu v_L^\rho \hat{k}^\sigma \\ & + \mathcal{N}_n \mathcal{R} \gamma_{\mu\nu} (v_p^\nu - \hat{k}^\nu \hat{k}_\rho v_p^\rho - v_L^\nu) \\ & + \mathcal{N}_n \mathcal{R} t_\mu \gamma_{\nu\rho} v_L^\nu (v_L^\rho - v_p^\rho) \\ & + \mathcal{A} n_p w^n \varepsilon_{\mu\nu\rho} (v_p^\nu - v_L^\nu) \hat{k}^\rho \\ & + \mathcal{A} n_p w^n t_\mu \varepsilon_{\nu\rho\sigma} v_p^\nu v_L^\rho \hat{k}^\sigma, \end{aligned} \quad (\text{A26})$$

where we have used equations (A2), (A3), (A5), (A10), (A13), and (A23). Contracting this last relation with the space projection tensor, defined as

$$P_\lambda^\mu = \delta_\lambda^\mu - e^\mu t_\lambda = \gamma_\lambda^\mu, \quad (\text{A27})$$

yields (only space indices i, j, k appear here)

$$\begin{aligned} 0 = & -\mathcal{N}_n \rho_n \kappa_n \varepsilon_{ijk} \hat{k}^j (v_n^k - v_L^k) - \mathcal{N}_n \mathcal{A} \rho_p \kappa_p \varepsilon_{ijk} \hat{k}^j (v_p^k - v_L^k) \\ & + \mathcal{N}_n \mathcal{R} \gamma_{ij} (v_p^j - \hat{k}^j \hat{k}_k v_p^k - v_L^j), \end{aligned} \quad (\text{A28})$$

where $\rho_n = m_n n_n$, $\rho_p = m_p n_p$, $\kappa_p = h/(2m_p)$ and we have used equation (37). This last expression coincides with the sum⁶ of equations (1), (2) and (3) from Sourie & Chamel (2020b) multiplied by the mean vortex surface density \mathcal{N}_n , as expected from the vanishing of the total force acting on individual vortex lines, provided that

$$\mathcal{R} = \rho_n \kappa_n \xi \quad \text{and} \quad \mathcal{A} = N_p, \quad (\text{A29})$$

where ξ is the (microscopic) ‘drag-to-lift’ ratio and N_p is the number of proton fluxoids pinned to each vortex line. Note that the time

⁶Let us remark that only velocities orthogonal to the direction \hat{k}^i of the vortex lines were considered in Sourie & Chamel (2020a,b), leading to $\hat{k}_k v_p^k = 0$.

component of the force balance equation (A26), as obtained by contracting with e^μ , yields the same equation as equation (A28; projected along v_L^i).

APPENDIX B: FLUID 4-VELOCITIES

Without any loss of generality, the fluid 4-velocities can be expressed as

$$u_n^\mu = u_n^1 k^\mu + u_n^2 h^\mu + u_n^1 V_n^\mu \quad \text{and} \quad u_p^\mu = u_p^1 k^\mu + u_p^2 h^\mu + u_p^1 V_p^\mu, \quad (\text{B1})$$

where each vector V_x^μ satisfies $V_x^\mu k_\mu = V_x^\mu h_\mu = 0$.

Assuming quasi-circular motion, one has $V_x \ll 1$ (recalling we set $c = 1$), where $V_x = \sqrt{V_{x\mu} V_x^\mu}$. Therefore, the strain-rate tensors of the fluids satisfy

$$\nabla_\mu V_{x\nu} + \nabla_\nu V_{x\mu} \ll \frac{1}{R}, \quad (\text{B2})$$

i.e. $\nabla_{(\mu} V_{\nu)}$ is $\mathcal{O}(L^{-1})$, where $L \gg R$ (R being the radius of the star). Since the quasi-Killing vector k^μ is also subject to a similar condition, see equation (38), one can now define $\tilde{k}^\mu = k^\mu + V_p^\mu$, which in turn satisfies

$$\nabla_\mu \tilde{k}_\nu + \nabla_\nu \tilde{k}_\mu = \mathcal{O}\left(\frac{1}{L}\right), \quad (\text{B3})$$

and $\tilde{k}^\mu \tilde{k}_\mu \simeq k^\mu k_\mu$ at lowest order in V_x . The 4-vector \tilde{k}^μ can thus be interpreted as another quasi-Killing vector associated with stationarity (this reflects the gauge freedom in the definition of k^μ as a ‘quasi’-Killing vector). The fluid 4-velocities can thus be rewritten as

$$u_p^\mu = \gamma_p (\tilde{k}^\mu + \Omega_p h^\mu) \quad \text{and} \quad u_n^\mu = \gamma_n (\tilde{k}^\mu + \Omega_n h^\mu + \tilde{v}_n^\mu), \quad (\text{B4})$$

where $\gamma_p = u_p^1$, $\Omega_p = u_p^2/u_p^1$, $\gamma_n = u_n^1$, $\Omega_n = u_n^2/u_n^1$, and $\tilde{v}_n^\mu = V_n^\mu - V_p^\mu$. The expression considered for the normal 4-velocity can be seen as a gauge fixing condition for the quasi-Killing vector associated with stationarity (absence of non-circular motion for the normal fluid). Considering cylindrical coordinates (t, r, θ, z) adapted to the space–time symmetries, i.e.

$$\tilde{k}^\mu = \partial_t^\mu \quad \text{and} \quad h^\mu = \partial_\theta^\mu, \quad (\text{B5})$$

we have $\Omega_p = u_p^\theta/u_p^t$ and $\Omega_n = u_n^\theta/u_n^t$, from which we deduce that Ω_p and Ω_n are to be interpreted as the angular velocities of the normal fluid and neutron superfluid as seen by a static observer located at spatial infinity.

APPENDIX C: MEAN VORTEX VELOCITY

The definition of a mean 4-velocity of vortices is not devoid of ambiguities, referring either to some given observers (as e.g. in Gavassino et al. 2020) or to the structure of the space–time under consideration (as in Appendix A, see equation A19). In either case, such a velocity should be so defined as to leave the vorticity 2-form $w_{\mu\nu}^n$ invariant by Lie-transport.

In what follows, the mean 4-velocity of the vortices is expressed in a similar form to that of the neutron superfluid (41), i.e.

$$u_L^\mu = \gamma_L (k^\mu + \Omega_L h^\mu + \tilde{v}_L^\mu), \quad (\text{C1})$$

where \tilde{v}_L^μ is a small non-circular contribution satisfying $\tilde{v}_{L\mu} k^\mu = \tilde{v}_{L\mu} h^\mu = 0$, Ω_L is the mean (non-uniform) angular velocity of the vortices and γ_L is some Lorentz-type factor obtained using the normalization condition $u_L^\mu u_{L\mu} = -1$.

Requiring⁷ $\varpi_{\mu\nu}^n u_L^\nu = 0$ (thus ensuring that the vorticity 2-form is Lie transported along the vortex 4-velocity) leads to

$$k^\mu \varpi_{\mu\nu}^n + \Omega_L h^\mu \varpi_{\mu\nu}^n + \tilde{v}_L^\mu \varpi_{\mu\nu}^n = 0. \quad (C2)$$

Contracting this relation with $w^{n\nu\rho} h_\rho$ now yields

$$-h_\perp^{-2} k_\mu h_\perp^\mu = \Omega_L + \tilde{v}_L^\mu h_{\perp\mu} h_\perp^{-2}, \quad (C3)$$

where we have used equations (35) and (49), from which we deduce that the term on the left-hand side of equation (53) is to be interpreted as the mean angular velocity of the vortices (plus a small non-circular contribution). On the other hand, contracting equation (C2) with h^ν yields

$$\dot{\alpha}_n = -\tilde{v}_L^\mu w_{\mu\nu}^n h^\nu. \quad (C4)$$

Introducing the unit space-like vector $\hat{r}^\mu = w_n^{-1} h_\perp^{-1} \varpi^{n\mu\nu} h_\nu$ (58), which is both orthogonal to the vortex worldsheet and to the vector h_\perp^μ , we get

$$w_n^{-1} h_\perp^{-2} \dot{\alpha}_n = -h_\perp^{-1} \tilde{v}_L^\mu \hat{r}_\mu. \quad (C5)$$

The left-hand side of equation (54) thus corresponds to the opposite of the vortex velocity along the unit vector \hat{r}^μ (i.e. to the ‘inwards radial’ velocity of the vortices), as divided by h_\perp .

APPENDIX D: NEWTONIAN LIMITS OF THE MEAN VORTEX VELOCITY AND THE MUTUAL-FRICTION FORCE

In this appendix, we will write the speed of light c explicitly since we will take the Newtonian limit $c \rightarrow +\infty$.

D1 Mean vortex velocity and equation of motion

The concept of Killing vectors remains relevant in the Newtonian space–time (Carter & Chamel 2005a; Chamel 2015). In this case, the (quasi-)Killing vector $k^\mu = \partial_t^\mu$ associated with stationarity coincides with the ether flow vector e^μ . In cylindrical coordinates (r, θ, z) adapted to the space–time symmetries, we have

$$h^\mu = \partial_\theta^\mu = r e_\theta^\mu \quad \text{and} \quad \hat{k}^\mu = e_z^\mu, \quad (D1)$$

where \hat{k}^μ introduced in equation (A18) is the unit vector along the vortex direction and $(e_r^\mu, e_\theta^\mu, e_z^\mu)$ is the right-handed orthonormal spatial vector basis associated with the cylindrical coordinates. Note that $h^\mu t_\mu = 0 = \hat{k}^\mu t_\mu$.

Using equation (C1), the mean 4-velocity of the vortices reads

$$u_L^\mu = \gamma_L (e^\mu + \Omega_L h^\mu + \tilde{v}_L^\mu). \quad (D2)$$

The condition $\tilde{v}_L^\mu k_\mu = 0$ translates into $t_\mu \tilde{v}_L^\mu = 0$, as demonstrated below:

$$0 = g_{\mu\nu} \tilde{v}_L^\mu k^\nu = (\gamma_{\mu\nu} - c^2 t_\mu t_\nu) \tilde{v}_L^\mu e^\nu = -c^2 t_\mu \tilde{v}_L^\mu, \quad (D3)$$

where we have used equations (A1) and (A2). The normalization condition $u_L^\mu t_\mu = 1$ together with equation (A3) imply that $\gamma_L = 1$. Since \tilde{v}_L^μ is purely spatial, the orthogonality condition $\tilde{v}_L^\mu h_\mu = 0$ becomes $\gamma_{\mu\nu} \tilde{v}_L^\mu h^\nu = 0$. By definition, \tilde{v}_L^μ is also orthogonal to \hat{k}^μ , see equation (A19), therefore $\tilde{v}_L^i = \tilde{v}_L^i e_r^i$ (using Latin letters for spatial indices). Finally, the spatial part (A20) of the mean 4-velocity of the vortices reads

$$v_L^i = r \Omega_L e_\theta^i + \tilde{v}_L^i e_r^i. \quad (D4)$$

⁷This condition can always be imposed since $\varpi_{\mu\nu}^n$ is of rank 2.

We shall now derive the Newtonian limit of equations (53) and (54) describing the average motion of vortices. Using equations (35), (A1), (A2), and (A4), we have

$$\begin{aligned} h_\perp^2 &= g_{\mu\nu} h_\perp^\mu h^\nu \\ &= g_{\mu\nu} \perp_\rho^\mu h^\rho h^\nu \\ &= g_{\mu\nu} \frac{1}{(w^n)^2} g^{\mu\lambda} g^{\sigma\tau} \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n h^\nu h^\rho \\ &= \gamma_{\mu\nu} \frac{1}{(w^n)^2} \left(\gamma^{\mu\lambda} - \frac{e^\mu e^\lambda}{c^2} \right) \left(\gamma^{\sigma\tau} - \frac{e^\sigma e^\tau}{c^2} \right) \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n h^\nu h^\rho \\ &= \gamma_{\mu\nu} \frac{1}{(w^n)^2} \gamma^{\mu\lambda} \left(\gamma^{\sigma\tau} - \frac{e^\sigma e^\tau}{c^2} \right) \varpi_{\tau\lambda}^n \varpi_{\sigma\rho}^n h^\nu h^\rho \\ &= \frac{1}{(w^n)^2} \left(\gamma^{\sigma\tau} - \frac{e^\sigma e^\tau}{c^2} \right) \varpi_{\tau\lambda}^n h^\lambda \varpi_{\sigma\rho}^n h^\rho. \end{aligned} \quad (D5)$$

Taking the Newtonian limit $c \rightarrow +\infty$ using equation (A23) yields

$$\begin{aligned} h_\perp^2 &= \frac{1}{(w^n)^2} \gamma^{\sigma\tau} \varpi_{\tau\lambda}^n h^\lambda \varpi_{\sigma\rho}^n h^\rho \\ &= \gamma^{\sigma\tau} \varepsilon_{\tau\lambda\delta} \hat{k}^\delta \varepsilon_{\sigma\rho\mu} \hat{k}^\mu h^\lambda h^\rho \\ &= \hat{k}^\delta \hat{k}^\mu h^\lambda h^\rho \varepsilon_{\lambda\delta} \varepsilon_{\sigma\rho\mu} \\ &= \hat{k}^\delta \hat{k}^\mu h^\lambda h^\rho (\gamma_{\lambda\rho} \gamma_{\delta\mu} - \gamma_{\lambda\mu} \gamma_{\delta\rho}) \\ &= \gamma_{\mu\nu} h^\mu h^\nu - (\gamma_{\mu\nu} h^\mu \hat{k}^\nu)^2. \end{aligned} \quad (D6)$$

It follows from equation (D1) that

$$h_\perp^2 = \gamma_{\mu\nu} h^\mu h^\nu = r^2. \quad (D7)$$

Similarly,

$$\begin{aligned} k_\mu h_\perp^\mu &= g_{\mu\nu} k^\nu h_\perp^\mu \\ &= g_{\mu\nu} e^\nu \perp_\rho^\mu h^\rho \\ &= (\gamma_{\mu\nu} - c^2 t_\mu t_\nu) e^\nu \perp_\rho^\mu h^\rho \\ &= -c^2 t_\mu \perp_\rho^\mu h^\rho \\ &= \frac{1}{(w^n)^2} e^\nu \gamma^{\sigma\mu} \varpi_{\nu\mu}^n \varpi_{\rho\sigma}^n h^\rho \\ &= \gamma^{\sigma\mu} \varepsilon_{\mu\nu\lambda} v_L^\nu \hat{k}^\lambda \varepsilon_{\rho\sigma\delta} \hat{k}^\delta h^\rho \\ &= \varepsilon^\sigma{}_{\nu\lambda} \varepsilon_{\rho\sigma\delta} v_L^\nu \hat{k}^\lambda \hat{k}^\delta h^\rho \\ &= -\varepsilon^\sigma{}_{\nu\lambda} \varepsilon_{\sigma\rho\delta} v_L^\nu \hat{k}^\lambda \hat{k}^\delta h^\rho \\ &= -(\gamma_{\rho\nu} \gamma_{\lambda\delta} - \gamma_{\nu\delta} \gamma_{\rho\lambda}) v_L^\nu \hat{k}^\lambda \hat{k}^\delta h^\rho \\ &= -\gamma_{\rho\nu} v_L^\nu h^\rho. \end{aligned} \quad (D8)$$

Collecting equations (D7) and (D8), the left-hand side of equation (53) thus reduces in the Newtonian limit to

$$-h_\perp^{-2} k_\mu h_\perp^\mu = \frac{v_L^\theta}{r} = \Omega_L. \quad (D9)$$

Using equations (A10), (A23), (C4), and (D1), we find

$$\begin{aligned} \dot{\alpha}_n &= -\tilde{v}_L^\mu w_{\mu\nu}^n h^\nu \\ &= -\tilde{v}_L^\mu w^n \varepsilon_{\mu\nu\rho} h^\nu \hat{k}^\rho \\ &= -\tilde{v}_L^i w^n r. \end{aligned} \quad (D10)$$

Therefore, the left-hand side of equation (54) is given by

$$w_n^{-1} h_{\perp}^{-2} \dot{\alpha}_n = -\frac{\tilde{v}_n^r}{r}, \quad (\text{D11})$$

where we have used equation (D7).

To find the Newtonian limit of equations (56) and (57), we need to evaluate the scalars $\tilde{v}_n^\mu h_{\perp\mu}$ and $\tilde{v}_n^\mu \nabla_\mu \alpha_n$. Using equation (A1) and the fact that $\tilde{v}_n^\mu t_\mu = 0$, we have

$$\begin{aligned} \tilde{v}_n^\mu h_{\perp\mu} &= \tilde{v}_n^\mu g_{\mu\nu} \perp_\rho^{\nu} h^\rho \\ &= \tilde{v}_n^\mu (\gamma_{\mu\nu} - c^2 t_\mu t_\nu) \perp_\rho^{\nu} h^\rho \\ &= \tilde{v}_n^\mu \gamma_{\mu\nu} \perp_\rho^{\nu} h^\rho. \end{aligned} \quad (\text{D12})$$

Using equations (35) and (A23), and taking the limit $c \rightarrow +\infty$, yield

$$\begin{aligned} \tilde{v}_n^\mu h_{\perp\mu} &= \tilde{v}_n^\mu \gamma_{\mu\nu} \frac{1}{(w_n)^2} \gamma^{\nu\lambda} \gamma^{\sigma\delta} \varpi_{\delta\lambda}^n \varpi_{\sigma\rho}^n h^\rho \\ &= \tilde{v}_n^\mu \gamma_{\mu}^{\lambda} \gamma^{\sigma\delta} \varepsilon_{\delta\lambda\nu} \hat{\kappa}^\nu \varepsilon_{\sigma\rho\gamma} \hat{\kappa}^\gamma h^\rho \\ &= \tilde{v}_n^\lambda \gamma^{\sigma\delta} \varepsilon_{\delta\lambda\nu} \hat{\kappa}^\nu \varepsilon_{\sigma\rho\gamma} \hat{\kappa}^\gamma h^\rho \\ &= \tilde{v}_n^\lambda \varepsilon^{\sigma\lambda\nu} \varepsilon_{\sigma\rho\gamma} \hat{\kappa}^\gamma \hat{\kappa}^\nu h^\rho \\ &= \tilde{v}_n^\lambda (\gamma_{\rho\lambda} \gamma_{\nu\gamma} - \gamma_{\lambda\gamma} \gamma_{\rho\nu}) \hat{\kappa}^\gamma \hat{\kappa}^\nu h^\rho \\ &= 0, \end{aligned} \quad (\text{D13})$$

where the last equality follows from $\gamma_{\mu\nu} \tilde{v}_n^\mu h^\nu = 0$ and $\gamma_{\mu\nu} \hat{\kappa}^\mu h^\nu = 0$. Similarly, using equations (A10), (A23), and (D1), we find

$$\tilde{v}_n^\mu \nabla_\mu \alpha_n = \tilde{v}_n^\mu w_{\mu\nu}^n h^\nu = \tilde{v}_n^r w_n^r. \quad (\text{D14})$$

Equations (56) and (57) thus become in the Newtonian limit

$$\Omega_- = -\frac{\tilde{\xi}}{1+X} \frac{\tilde{v}_n^r}{r}, \quad (\text{D15})$$

$$\Omega_+ = \frac{1+X}{\tilde{\xi}} \frac{\tilde{v}_n^r}{r}. \quad (\text{D16})$$

The Newtonian expressions for the dynamical equations (53) and (54) finally read

$$\frac{v_L^\theta}{r} = \Omega_L = (1 - \mathcal{B}') \left(\Omega_n + \Omega_p \frac{\mathcal{B}'}{1 - \mathcal{B}'} - \frac{\tilde{\xi}}{1+X} \frac{\tilde{v}_n^r}{r} \right), \quad (\text{D17})$$

$$-\frac{v_L^r}{r} = -\frac{\tilde{v}_L^r}{r} = \mathcal{B} \left(\Omega_p - \Omega_n - \frac{1+X}{\tilde{\xi}} \frac{\tilde{v}_n^r}{r} \right), \quad (\text{D18})$$

in perfect agreement with the expressions obtained from equation (56) of Sourie & Chamel (2020a) using

$$v_n^i = r \Omega_n e_\theta^i + \tilde{v}_n^r e_r^i, \quad \text{and} \quad v_p^i = r \Omega_p e_\theta^i. \quad (\text{D19})$$

D2 Mutual-friction force and torque

Let us first show that the unit vector \hat{r}^μ reduces to e_r^μ in the Newtonian limit. Starting from the definition (58) and substituting equation (A4), we find

$$\begin{aligned} \hat{r}^\mu &= w_n^{-1} h_{\perp}^{-1} g^{\mu\rho} g^{\nu\sigma} \varpi_{\rho\sigma}^n h_\nu \\ &= w_n^{-1} h_{\perp}^{-1} \left(\gamma^{\mu\rho} - \frac{e^\mu e^\rho}{c^2} \right) \left(\gamma^{\nu\sigma} - \frac{e^\nu e^\sigma}{c^2} \right) \varpi_{\rho\sigma}^n h_\nu. \end{aligned} \quad (\text{D20})$$

Taking the limit $c \rightarrow +\infty$ leads to

$$\begin{aligned} \hat{r}^\mu &= w_n^{-1} h_{\perp}^{-1} \gamma^{\mu\rho} \varpi_{\rho\sigma}^n h^\sigma \\ &= w_n^{-1} h_{\perp}^{-1} \gamma^{\mu\rho} \varepsilon_{\rho\sigma\delta} w^n \hat{\kappa}^\delta h^\sigma \\ &= e_r^\mu. \end{aligned} \quad (\text{D21})$$

We have used equation (D1) for the last equality. Using equation (37), the Newtonian limit of equations (61) and (62) thus reduces to

$$f^{\text{mf}\theta} = \rho_n \kappa_n \mathcal{N}_n \left[\mathcal{B} r (\Omega_p - \Omega_n) + \mathcal{B}' \tilde{v}_n^r \right], \quad (\text{D22})$$

$$f^{\text{mf}r} = \rho_n \kappa_n \mathcal{N}_n \left[\mathcal{B}' r (\Omega_p - \Omega_n) - \mathcal{B} \tilde{v}_n^r \right]. \quad (\text{D23})$$

Equations (D22) and (D23) agree perfectly with the radial and orthoradial components of the mutual-friction force obtained from equation (5) of Sourie & Chamel (2020b) with the prescription (D19). Using equation (74), the mutual-friction torque is given by the integral over the volume of the star

$$\Gamma_{\text{mf}} = \int n_n w^n r^2 \left[\mathcal{B} (\Omega_p - \Omega_n) + \mathcal{B}' \frac{\tilde{v}_n^r}{r} \right] dV. \quad (\text{D24})$$

recalling that $k^\mu = e^\mu$ and $e^\mu t_\mu = 1$, and using the fact that $d\Sigma_\mu = t_\mu dV$.

Assuming circular motion ($\tilde{v}_n^r = 0$), neglecting entrainment effects between the fluids ($w^n = 2 m_n \Omega_n$), and considering uniform mutual-friction coefficients and fluid angular velocities, equation (D24) reduces to (I_n is the moment of inertia of the neutron superfluid)

$$\Gamma_{\text{mf}} = 2 \mathcal{B} I_n \Omega_n (\Omega_p - \Omega_n), \quad (\text{D25})$$

in perfect agreement with equations (10) and (11) of Sourie & Chamel (2020b).

This paper has been typeset from a \LaTeX file prepared by the author.