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QUANTITATIVE STABILITY OF OPTIMAL TRANSPORT MAPS UNDER VARIATIONS OF THE TARGET MEASURE

ALEX DELALANDE AND QUENTIN MÉRIGOT

ABSTRACT. This work studies the quantitative stability of the quadratic optimal transport map between a fixed probability density ρ and a probability measure μ on \mathbb{R}^d , which we denote T_μ . Assuming that the source density ρ is bounded from above and below on a compact convex set, we prove that the map $\mu \mapsto T_\mu$ is bi-Hölder continuous on large families of probability measures, such as the set of probability measures whose moment of order $p > d$ is bounded by some constant. These stability estimates show that the *linearized optimal transport* metric $W_{2,\rho}(\mu, \nu) = \|T_\mu - T_\nu\|_{L^2(\rho, \mathbb{R}^d)}$ is bi-Hölder equivalent to the 2-Wasserstein distance on such sets, justifying its use in applications.

1. INTRODUCTION

Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures with finite second moment over \mathbb{R}^d and $\rho, \mu \in \mathcal{P}_2(\mathbb{R}^d)$. The optimal transport problem between ρ and μ with respect to the quadratic cost $c(x, y) = \|x - y\|^2$ is the following minimization problem, where the minimum is taken over the set $\Pi(\rho, \mu)$ of *transport plans* between ρ and μ , that is the set of probability measures over $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ and μ :

$$\min_{\gamma \in \Pi(\rho, \mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma(x, y).$$

The square root of the value of this problem is called the *2-Wasserstein distance* between ρ and μ and is denoted $W_2(\rho, \mu)$. A theorem of Brenier [8] asserts that if ρ is absolutely continuous with respect to the Lebesgue measure, the minimizer of the optimal transport problem is unique, and is induced by a map $T = \nabla\phi$, where ϕ is a convex function that verifies $\nabla\phi_{\#}\rho = \mu$. We recall that $T_{\#}\rho$ denotes the image measure of ρ under the map T . In our precise setting, where the density ρ is bounded from above and below on a compact convex set, the potential ϕ is uniquely defined in $L^2(\rho)$ up to an additional constant.

Definition 1.1 (Potentials and maps). We fix a probability measure $\rho \in \mathcal{P}_2(\mathbb{R}^d)$, which we assume to be absolutely continuous with respect to the Lebesgue measure and supported over a compact convex set \mathcal{X} . We assume that the density of ρ is bounded from above and below by positive constants on \mathcal{X} . Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we call

- *Brenier map* and denote T_μ the (unique) optimal transport map between ρ and μ ;
- *Brenier potential* the unique convex function $\phi_\mu \in L^2(\mathcal{X})$ such that $T_\mu = \nabla\phi_\mu$ and which satisfies $\int_{\mathcal{X}} \phi_\mu d\rho = 0$;

- *dual potential* the convex conjugate of ϕ_μ , denoted ψ_μ :

$$\forall y \in \mathbb{R}^d, \quad \psi_\mu(y) = \max_{x \in \mathcal{X}} \langle x|y \rangle - \phi_\mu(x).$$

Since μ is the image of ρ under T_μ , the mapping $\mu \in (\mathcal{P}_2(\mathbb{R}^d), W_2) \mapsto T_\mu \in L^2(\rho, \mathbb{R}^d)$ is obviously injective. Using that $(T_\mu, T_\nu)_{\#}\rho$ is a coupling between μ and ν , one can actually prove that this mapping increases distances, namely

$$\forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu, \nu) \leq \|T_\mu - T_\nu\|_{L^2(\rho)}.$$

This mapping is also continuous: if a sequence of probability measures $(\mu_n)_n$ converges to some μ in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$, then T_{μ_n} converges to T_μ in $L^2(\rho)$. This continuity property is for instance implied by Corollary 5.23 in [31], together with the dominated convergence theorem. However, we note that the arguments used to prove this general continuity result are non-quantitative.

These two properties of the map $\mu \mapsto T_\mu$ motivated its use to embed the metric space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ into the Hilbert space $L^2(\rho, \mathbb{R}^d)$ [32]. This approach is often referred to as the *Linearized Optimal Transport* framework and has shown great results in applications [32, 26, 4, 19, 10]. A practical benefit of the embedding is to enable the use of the classical Hilbertian statistical toolbox on families of probability measures while keeping some features of the Wasserstein geometry. Working with this embedding is equivalent to replacing the Wasserstein distance by the distance

$$W_{2,\rho}(\mu, \nu) = \|T_\mu - T_\nu\|_{L^2(\rho, \mathbb{R}^d)}.$$

We note that the geodesic curves with respect to the distance $W_{2,\rho}$ are called the *generalized geodesics* in the book of Ambrosio, Gigli, Savaré [1]. The choice of the Brenier map between a reference measure ρ and a measure μ as an embedding of μ may also be motivated by the Riemannian interpretation of the Wasserstein geometry [25, 1]. In this interpretation, the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at ρ is included in $L^2(\rho, \mathbb{R}^d)$. The Brenier map minus the identity, $T_\mu - \text{id}$, can be regarded as the vector in the tangent space at ρ which supports the Wasserstein geodesic from ρ to μ . In the Riemannian language again, the map $\mu \mapsto T_\mu - \text{id}$ would be called a *logarithm*, i.e. the inverse of the Riemannian exponential map: it sends a probability measure μ in the (curved) manifold $\mathcal{P}_2(\mathbb{R}^d)$ to a vector $T_\mu - \text{id}$ belonging to the linear space $L^2(\rho, \mathbb{R}^d)$. This establishes a connection between the linearized optimal transport framework idea and similar strategies used to extend statistical inference notions such as principal component analysis to manifold-valued data, e.g. [14, 11].

It is quite natural to expect that the embedding $\mu \mapsto T_\mu$ retains some of the geometry of the underlying space, or equivalently that the metric $W_{2,\rho}$ is comparable, in some coarse sense, to the Wasserstein distance. The main difficulty, which we study in this article, is to establish quantitative (e.g. Hölder) continuity properties for the mappings $\mu \mapsto T_\mu$ and $\mu \mapsto \phi_\mu$. We note that such stability estimates are also important in numerical analysis and in statistics, where a probability measure of interest $\mu \in \mathcal{P}_2(\mathbb{R})$ is often approximated by a sequence of finitely supported measures $(\mu_n)_n$: convergence rates of quantities related to the sequence $(T_{\mu_n})_n$ toward a quantity related to T_μ may then be directly deduced from quantitative stability estimates controlling $\|T_{\mu_n} - T_\mu\|_{L^2(\rho)}$ with $W_2(\mu_n, \mu)$.

Existing results. We focus here on the already known stability results on the mapping $\mu \mapsto T_\mu$, starting with negative results. We first note that explicit examples show that the mapping $\mu \mapsto \nabla\phi_\mu$ is in general not better than $\frac{1}{2}$ -Hölder, see §4 in [15] or Lemma 5.1 in [24]. A much stronger negative result comes from Andoni, Naor and Neiman [2, Theorem 7] showing that one cannot construct a bi-Hölder embedding of $(\mathcal{P}_2(\mathbb{R}^d), W_2)$, $d \geq 3$, into a Hilbert space:

Theorem (Andoni, Naor, Neiman). *$(\mathcal{P}_2(\mathbb{R}^3), W_2)$ does not admit a uniform, coarse or quasisymmetric embedding into any Banach space of nontrivial type.*

This theorem implies in particular that one cannot hope to prove that $\mu \mapsto T_\mu$ is bi-Hölder on the whole set $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment.

Existing quantitative stability results can be summed up under the two following statements. A first result due to Ambrosio and reported in [15], shows a local $1/2$ -Hölder behaviour near probability densities μ whose associated Brenier map T_μ is Lipschitz continuous. We quote here a variant of this statement, from [24]:

Theorem (Ambrosio). *Let $\mathcal{Y} \subset \mathbb{R}^d$ be a compact set and $\mu, \nu \in \mathcal{P}(\mathcal{Y})$. Assume that the Brenier map T_μ from ρ to μ is L -Lipschitz. Then,*

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \leq 2\sqrt{\text{diam}(\mathcal{X})LW_1(\mu, \nu)^{1/2}}.$$

We note that this regularity assumption on the Brenier map is rather strong. First, it implies that the support of μ is connected, so that the previous theorem cannot be applied when both μ and ν are finitely supported. In addition, to prove that T_μ is Lipschitz one has to invoke the regularity theory for optimal transport maps, which requires very strong assumptions on μ , in particular that its support should be convex. A more recent result, due to Berman [5], proves quantitative stability of the map $\mu \mapsto T_\mu$ under milder assumptions on the target probability measures. Berman assumes that the source measure ρ is the restriction of the Lebesgue measure on a compact convex set \mathcal{X} with unit volume. Under this assumption, he proves a stability result on the inverse transport maps when the target measure is bound to remain in a fixed compact set [5, Proposition 3.2]. This result implies quantitative stability of the Brenier maps; we refer to Corollary 2.4 in [24] for the precise statement.

Theorem (Berman). *Let \mathcal{X} be a compact convex subset with unit volume, let ρ be the restriction of the Lebesgue measure to \mathcal{X} , and let $\mathcal{Y} \subset \mathbb{R}^d$ be another compact set. Then there exists a constant C depending only on \mathcal{X} and \mathcal{Y} such that for any $\mu, \nu \in \mathcal{P}(\mathcal{Y})$,*

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \leq CW_1(\mu, \nu)^{\frac{1}{2^{(d-1)}(d+2)}}.$$

Unlike in Ambrosio's theorem, the Hölder behavior given does not depend on the regularity of the transport map T_μ . On the other hand, the Hölder exponent depends exponentially on the ambient dimension d . As we will see below, this is not optimal.

Contributions. In this article, we prove a quantitative stability results for quadratic optimal transport maps between a probability density ρ and target measure μ . We do not assume that μ is compactly supported. Introducing $M_p(\mu) = \int_{\mathbb{R}^d} \|x\|^p d\mu(x)$ the p -th moment of $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we prove in particular the following theorem. We denote by C_{a_1, \dots, a_n} a non-negative constant which depends on a_1, \dots, a_n .

Theorem (Corollary 3.4 and Theorems 4.2 and 4.3). *Let \mathcal{X} be a compact convex set and let ρ be a probability density on \mathcal{X} , bounded from above and below by positive constants. Let $p > d$ and $p \geq 4$. Assume that $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ have bounded p -th moment, i.e. $\max(M_p(\mu), M_p(\nu)) \leq M_p < +\infty$. Then*

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \leq C_{d,p,\mathcal{X},\rho,M_p} W_1(\mu, \nu)^{\frac{p}{6p+16d}},$$

$$\|\phi_\mu - \phi_\nu\|_{L^2(\rho)} \leq C_{d,p,\mathcal{X},\rho,M_p} W_1(\mu, \nu)^{1/2}.$$

If μ, ν are supported on a compact set \mathcal{Y} , we have an improved Hölder exponent for the Brenier map:

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \leq C_{d,p,\mathcal{X},\mathcal{Y},\rho} W_1(\mu, \nu)^{\frac{1}{6}}.$$

As noticed in Remark 3.1, a large class of probability measures verifies the moment assumption, such as sub-Gaussian or sub-exponential measures. A preliminary version of this theorem was announced in [24], with a different proof strategy, relying on the study of the case where both μ, ν are supported on the same finite set. The proof in [24] led to a worse Hölder exponent in the compact case, and couldn't deal with non-compactly supported measures.

To prove these stability estimates, we use the fact that the dual potentials solve a convex minimization problem involving the functional $\mathcal{K}(\psi) = \int \psi^* d\rho$, which we call *Kantorovich's functional*. We first prove in §2 a strong convexity estimate for Kantorovich's functional, relying in particular on the Brascamp-Lieb inequality, and which holds under the assumption that the Brenier potentials are bounded. This strong convexity estimate is then translated into a stability estimate concerning the dual and Brenier potentials (§3). The stability of Brenier maps is then obtained (§4), relying in particular on a Gagliardo-Nirenberg type inequality for the difference of convex functions (§5), which might be of independent interest.

2. STRONG CONVEXITY OF KANTOROVICH'S FUNCTIONAL

Let \mathcal{X} be a compact convex subset of \mathbb{R}^d , and let ρ be a probability density on \mathcal{X} . Given any measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we search for the coupling $\gamma \in \Pi(\rho, \mu)$ maximizing the correlation between ρ and μ , $\int \langle x|y \rangle d\gamma(x, y)$. This problem is equivalent to the standard quadratic optimal transport problem and in this setting Kantorovich duality reads

$$\max_{\gamma \in \Pi(\rho, \mu)} \int \langle x|y \rangle d\gamma(x, y) = \min_{\psi \in \mathcal{C}^0(\mathbb{R}^d)} \mathcal{K}(\psi) + \int_{\mathbb{R}^d} \psi d\mu,$$

where the functional \mathcal{K} , which we will call the *Kantorovich functional*, is defined by

$$\mathcal{K}(\psi) := \int_{\mathcal{X}} \psi^* d\rho.$$

This dual formulation of the maximal correlation problem can for instance be found as Particular Case 5.16 in [31]. The Kantorovich functional is convex because the convex conjugation $\psi \mapsto \psi^*$ is convex in ψ . Moreover, formal computations, which are justified in Proposition 2.2, show that

$$\nabla \mathcal{K}(\psi) = -(\nabla \psi^*)_{\#} \rho.$$

In particular, with ϕ_μ the Brenier potential associated to the optimal transport problem between ρ and μ and $\psi_\mu = \phi_\mu^*$ its convex conjugate, this gives the relation $\psi_\mu = (\nabla \mathcal{K})^{-1}(-\mu)$. Since \mathcal{K} is convex, its gradient is monotone, thus implying that for all probability measures $\mu^0, \mu^1 \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\langle \psi_{\mu^1} - \psi_{\mu^0} | \nabla \mathcal{K}(\psi_{\mu^1}) - \nabla \mathcal{K}(\psi_{\mu^0}) \rangle = \langle \psi_{\mu^1} - \psi_{\mu^0} | \mu^0 - \mu^1 \rangle \geq 0.$$

Our aim in this section is to prove Theorem 2.1, establishing strong convexity estimates for Kantorovich's functional \mathcal{K} , which we will later be able to translate into stability estimates for $\mu \mapsto \psi_\mu = (\nabla \mathcal{K})^{-1}(-\mu)$.

Theorem 2.1 (Strong convexity). *Let $\mu^0, \mu^1 \in \mathcal{P}_2(\mathbb{R}^d)$ and let ρ be a probability density over a compact convex set \mathcal{X} , satisfying $0 < m_\rho \leq \rho \leq M_\rho$. For $k \in \{0, 1\}$, denote $\phi^k = \phi_{\mu^k}$ the Brenier potential between ρ and μ^k (see Definition 1.1). Assume that*

$$\forall k \in \{0, 1\}, \quad 0 < m_\phi \leq \min_{\mathcal{X}} \phi^k \leq \max_{\mathcal{X}} \phi^k \leq M_\phi < +\infty. \quad (1)$$

Then the convex conjugates ψ^0 and ψ^1 of ϕ^0 and ϕ^1 verify:

$$\text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \leq C_d \frac{M_\rho^2}{m_\rho^2} (M_\phi - m_\phi) \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle, \quad (2)$$

where $C_d = e(d+1)2^d$.

Remark 2.1 (Variance). The left-hand side of (2) involves a variance instead of a simple squared L^2 norm because of the invariance of the Kantorovich's functional under addition of a constant. The choice of $\mu_0 + \mu_1$ as the reference measure for the variance term in inequality (2) may seem unnatural. This choice comes from the fact that there is no natural reference measure on the side of the target measures. However, this choice of reference proves relevant for establishing the stability of Brenier potentials in the next section, Proposition 3.1 especially asserts that $\text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \geq \text{Var}_\rho(\phi^1 - \phi^0)$. We also note that, as detailed in the proof of Theorem 2.1, the left-hand side of the inequality could actually be replaced by the quantity

$$C_d \frac{M_\rho}{m_\rho} \int_0^1 \text{Var}_{\mu^t}(\psi^1 - \psi^0) dt,$$

where for $t \in [0, 1]$, $\mu^t = \nabla((1-t)\psi^0 + t\psi^1)_{\#} \rho$ interpolates between μ^0 and μ^1 . This inequality is tighter, but this interpolation has no simple interpretation — for instance, $(\mu^t)_{t \in [0, 1]}$ is *not* a generalized geodesic in the sense of [1] — and is quite difficult to manipulate.

Remark 2.2 (Optimality of exponents). Estimate (2) is optimal in term of exponent of $\text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0)$. Indeed in dimension $d = 1$, for $\varepsilon \geq 0$, denote μ_ε the uniform probability measure on the segment $[\varepsilon, 1 + \varepsilon]$. Then for $\rho = \mu_0$, one can show that for $\varepsilon \leq 1$, both $\text{Var}_{\mu^0 + \mu^\varepsilon}(\psi^\varepsilon - \psi^0)$ and $\langle \psi^0 - \psi^\varepsilon | \mu^\varepsilon - \mu^0 \rangle$ are of the order of ε^2 .

The strong convexity estimate (2) is derived from a local estimate, a Poincaré-Wirtinger inequality for the second derivative of \mathcal{K} , which is in turn a consequence of the Brascamp-Lieb inequality (5). To make the connection with the Brascamp-Lieb inequality clearer, we first compute the first and second order derivatives of \mathcal{K} along the path $((1 - t)\psi^0 + t\psi^1)_{t \in [0,1]}$.

Proposition 2.2. *Let $\phi^0, \phi^1 \in \mathcal{C}^2(\mathbb{R}^d)$ be strongly convex functions. Define $\psi^0 = (\phi^0)^*$, $\psi^1 = (\phi^1)^*$ and $v = \psi^1 - \psi^0$. For $t \in [0, 1]$, define $\psi^t = \psi^0 + tv$ and finally $\phi^t = (\psi^t)^*$. Then, ϕ^t is a strongly convex function, belongs to $\mathcal{C}^2(\mathbb{R}^d)$, and*

$$\frac{d}{dt}\mathcal{K}(\psi^t) = - \int_{\mathcal{X}} v(\nabla\phi^t(x))d\rho(x), \quad (3)$$

$$\frac{d^2}{dt^2}\mathcal{K}(\psi^t) = \int_{\mathcal{X}} \langle \nabla v(\nabla\phi^t(x)) | D^2\phi^t(x) \cdot \nabla v(\nabla\phi^t(x)) \rangle d\rho(x). \quad (4)$$

We then find a positive lower-bound on the second order derivative expressed in equation (4) using the Brascamp-Lieb inequality [7]. We cite here a version of this inequality that is adapted to our context, i.e. that concerns log-concave probability measures supported on the compact and convex set \mathcal{X} . This statement is a special case of Corollary 1.3 of [20], where \mathcal{X} is a convex subset of a Riemannian manifold. We also refer to Section 3.1.1 of [18].

Theorem 2.3 (Brascamp-Lieb inequality). *Let $\phi \in \mathcal{C}^2(\mathcal{X})$ be a strictly convex function. Let $\tilde{\rho}$ be the probability measure defined by $d\tilde{\rho} = \frac{1}{Z_\phi} \exp(-\phi)dx$ with $Z_\phi = \int_{\mathcal{X}} \exp(-\phi)dx$. Then every smooth function s on \mathcal{X} verifies:*

$$\text{Var}_{\tilde{\rho}}(s) \leq \mathbb{E}_{\tilde{\rho}} \langle \nabla s | (D^2\phi)^{-1} \cdot \nabla s \rangle. \quad (5)$$

We now justify the computation of the derivatives presented in Proposition 2.2.

Proof of Proposition 2.2. We assume that ϕ^0, ϕ^1 are both α -strongly convex and belong to $\mathcal{C}^2(\mathbb{R}^d)$. Then, the convex conjugates $\psi^0 = (\phi^0)^*$, $\psi^1 = (\phi^1)^*$ have $1/\alpha$ -Lipschitz gradients and satisfy $D^2\psi^0 > 0, D^2\psi^1 > 0$ on \mathbb{R}^d . Hence their linear interpolates $\psi^t = (1 - t)\psi^0 + t\psi^1$ enjoy the same properties. This in turn implies that for all $t \in [0, 1]$, the convex conjugate ϕ^t of ψ^t belongs to $\mathcal{C}^2(\mathbb{R}^d)$ and is α -strongly convex.

We will now prove that the map $G : (t, x) \mapsto \nabla\phi^t(x)$ has class \mathcal{C}^1 . Let $F : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the continuously differentiable function defined by $F(t, x, y) = \nabla\psi^t(y) - x$. A well-known property of the convex conjugate is that $\nabla\phi^t$ is the inverse of $\nabla\psi^t$, implying that $G(t, x)$ is uniquely characterized by $F(t, x, G(t, x)) = 0$. Since $D^2\psi^t > 0$, the Jacobian $D_y f(t, x, y) = D^2\psi^t(y)$ is invertible and the implicit function theorem thus implies that G has class \mathcal{C}^1 . Differentiating the relation $F(t, x, G(t, x)) = 0$ with respect to time, we get

$$\frac{d}{dt}\nabla\phi^t(x) = -D^2\phi^t(x) \cdot \nabla v(\nabla\phi^t(x)). \quad (6)$$

By Fenchel-Young's equality case, one has for any $x \in \mathcal{X}$ and $t \in [0, 1]$,

$$\phi^t(x) = \langle x | \nabla \phi^t(x) \rangle - \psi^t(\nabla \phi^t(x)),$$

so that ϕ^t is at least \mathcal{C}^1 with respect to time. We can actually differentiate this equation with respect to time twice and using (6) we get

$$\begin{aligned} \frac{d}{dt} \phi^t(x) &= \langle x | \frac{d}{dt} \nabla \phi^t(x) \rangle - v(\nabla \phi^t(x)) - \langle \nabla \psi^t(\nabla \phi^t) | \frac{d}{dt} \nabla \phi^t(x) \rangle = -v(\nabla \phi^t(x)), \\ \frac{d^2}{dt^2} \phi^t(x) &= -\langle \nabla v(\nabla \phi^t(x)) | \frac{d}{dt} \nabla \phi^t(x) \rangle = \langle \nabla v(\nabla \phi^t(x)) | D^2 \phi^t(x) \cdot \nabla v(\nabla \phi^t(x)) \rangle. \end{aligned}$$

Since $\mathcal{K}(\psi^t) = \int_{\mathcal{X}} \phi^t(x) d\rho(x)$, we get the result by differentiating twice under the integral. \square

Proposition 2.4. *In addition to the assumptions of Theorem 2.1, assume that the Brenier potentials ϕ^0, ϕ^1 are strongly convex, belong to $\mathcal{C}^2(\mathbb{R}^d)$, and that $\nabla \phi^0$ and $\nabla \phi^1$ induce diffeomorphisms between \mathcal{X} and a closed ball \mathcal{Y} . Then, inequality (2) holds.*

Proof. Under the assumptions on ϕ^0, ϕ^1 , Proposition 2.2 ensures that the function ϕ^t it defines is strongly convex and belongs to $\mathcal{C}^2(\mathbb{R}^d)$ for any $t \in [0, 1]$. By the fundamental theorem of calculus, again with the notations of Proposition 2.2, we have:

$$\langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle = \frac{d}{dt} \mathcal{K}(\psi^t) \Big|_{t=1} - \frac{d}{dt} \mathcal{K}(\psi^t) \Big|_{t=0} = \int_0^1 \frac{d^2}{dt^2} \mathcal{K}(\psi^t) dt. \quad (7)$$

From Proposition 2.2, we have the following expression for the second derivative of \mathcal{K} :

$$\frac{d^2}{dt^2} \mathcal{K}(\psi^t) = \mathbb{E}_\rho \langle \nabla v(\nabla \phi^t) | (D^2 \phi^t) \cdot \nabla v(\nabla \phi^t) \rangle.$$

We introduce $\tilde{v}^t = v(\nabla \phi^t)$ for any $t \in [0, 1]$, which belongs to $\mathcal{C}^1(\mathbb{R}^d)$ as the composition of $v = \psi^1 - \psi^0 \in \mathcal{C}^2(\mathbb{R}^d)$ and $\nabla \phi^t$. We have $\nabla \tilde{v}^t = D^2 \phi^t \cdot \nabla v(\nabla \phi^t)$, where $(D^2 \phi^t)$ is invertible by strong convexity. Thus,

$$\frac{d^2}{dt^2} \mathcal{K}(\psi^t) = \mathbb{E}_\rho \langle \nabla \tilde{v}^t | (D^2 \phi^t)^{-1} \cdot \nabla \tilde{v}^t \rangle. \quad (8)$$

We now introduce $\tilde{\rho}^t = \exp(-\phi^t)/Z_t$ where $Z_t = \int_{\mathcal{X}} \exp(-\phi^t(x)) dx$, which is the density of a log-concave probability measure supported on \mathcal{X} . The Brascamp-Lieb inequality, recalled in Theorem 2.3, then ensures that

$$\text{Var}_{\tilde{\rho}^t}(\tilde{v}^t) \leq \mathbb{E}_{\tilde{\rho}^t} \langle \nabla \tilde{v}^t | (D^2 \phi^t)^{-1} \cdot \nabla \tilde{v}^t \rangle. \quad (9)$$

We assumed that for any $k \in \{0, 1\}$ and $x \in \mathcal{X}$, $m_\phi \leq \phi^k(x) \leq M_\phi$. We claim that this property is transferred to ϕ^t for any $t \in [0, 1]$. Indeed, on the one hand for all $t \in [0, 1]$,

$$\phi^t = \left((1-t)\psi^0 + t\psi^1 \right)^* \leq (1-t)(\psi^0)^* + t(\psi^1)^* = (1-t)\phi^0 + t\phi^1 \leq M_\phi,$$

where we used the convexity of the convex conjugation. On the other hand, for any $x \in \mathcal{X}$, we have by definition:

$$\phi^t(x) = \sup_{y \in \mathbb{R}^d} \langle x | y \rangle - \psi^t(y) \geq -\psi^t(0) = -(1-t)\psi^0(0) - t\psi^1(0).$$

But again, for $k \in \{0, 1\}$, $\psi^k(0) = \sup_{x \in \mathcal{X}} -\phi^k(x) \leq -m_\phi$, ensuring that $\phi^t \geq m_\phi$ for all $t \in [0, 1]$. The inequality $m_\phi \leq \phi^t \leq M_\phi$ allows us to compare the densities ρ and $\tilde{\rho}^t$:

$$\left(\frac{\exp(-M_\phi)}{M_\rho Z_t} \right) \rho \leq \tilde{\rho}^t \leq \left(\frac{\exp(-m_\phi)}{m_\rho Z_t} \right) \rho.$$

This comparison and equation (9) thus give:

$$\left(\frac{\exp(-M_\phi)}{M_\rho Z_t} \right) \text{Var}_\rho(\tilde{v}^t) \leq \left(\frac{\exp(-m_\phi)}{m_\rho Z_t} \right) \mathbb{E}_\rho \langle \nabla \tilde{v}^t | (D^2 \phi^t)^{-1} \cdot \nabla \tilde{v}^t \rangle,$$

that is, using $\tilde{v}^t = v(\nabla \phi^t)$, $\mu^t = (\nabla \phi^t)_\# \rho$, $v = \psi^1 - \psi^0$ and expression (8):

$$\text{Var}_{\mu^t}(\psi^1 - \psi^0) \leq \frac{M_\rho}{m_\rho} \exp(M_\phi - m_\phi) \frac{d^2}{dt^2} \mathcal{K}(\psi^t). \quad (10)$$

Recalling equation (7), this equation is similar to that of (2), except that we would like to replace μ^t by $\mu_0 + \mu_1$. For this purpose, we will prove that

$$\mu^t \geq \frac{m_\rho}{M_\rho} \min(t, 1-t)^d (\mu^0 + \mu^1). \quad (11)$$

This will be done using an explicit expression for μ^t . By smoothness and strong convexity of the function ϕ^t , the restriction of $\nabla \phi^t$ to \mathcal{X} is a diffeomorphism on its image. This implies that μ^t is absolutely continuous with respect to the Lebesgue measure. Moreover, by e.g. Villani [30, p.9], for any $x \in \mathcal{X}$ the density of μ^t with respect to Lebesgue, also noted μ^t , is given by $\mu^t(\nabla \phi^t(x)) \det(D^2 \phi^t(x)) = \rho(x)$. Setting $y = \nabla \phi^t(x)$ in this formula, we get

$$\forall y \in \nabla \phi^t(\mathcal{X}), \quad \mu^t(y) = \rho(\nabla \psi^t(y)) \det(D^2 \psi^t(y)).$$

By assumption, $\nabla \phi^k$ is a diffeomorphism from \mathcal{X} to \mathcal{Y} and so is $\nabla \psi^k$ from \mathcal{Y} to \mathcal{X} . Thus by convexity of \mathcal{X} , $\nabla \psi^t(\mathcal{Y}) \subset \mathcal{X}$, which entails $\mathcal{Y} \subset \nabla \phi^t(\mathcal{X})$. The equality above then gives

$$\forall k \in \{0, 1\}, \forall y \in \mathcal{Y}, \quad \mu^k(y) \leq M_\rho \det(D^2 \psi^k(y)).$$

On the other hand, the same equality gives

$$\forall t \in [0, 1], \forall y \in \mathcal{Y}, \quad \mu^t(y) \geq m_\rho \det(D^2 \psi^t(y)).$$

Using the two inequalities above and the concavity of $\det^{1/d}$ over the set of non-negative symmetric matrices, we get for every $y \in \mathcal{Y}$,

$$\begin{aligned} \mu^t(y) &\geq m_\rho \det(D^2 \psi^t(y)) \\ &\geq m_\rho \left((1-t) \det(D^2 \psi^0)^{1/d} + t \det(D^2 \psi^1)^{1/d} \right)^d \\ &\geq m_\rho \min(t, 1-t)^d (\det(D^2 \psi^0(y)) + \det(D^2 \psi^1(y))) \\ &\geq \frac{m_\rho}{M_\rho} \min(t, 1-t)^d (\mu^0(y) + \mu^1(y)) \end{aligned}$$

Using that $\text{spt}(\mu^0) = \text{spt}(\mu^1) = \mathcal{Y}$, this directly implies (11), which in turn gives us

$$\text{Var}_{\mu^t}(v) \geq \min(t, 1-t)^d \frac{m_\rho}{M_\rho} \text{Var}_{\mu^0+\mu^1}(v).$$

Combined with inequality (10), this gives after integrating over $t \in [0, 1]$:

$$\frac{1}{(d+1)2^d} \frac{m_\rho}{M_\rho} \text{Var}_{\mu^0+\mu^1}(v) \leq \frac{M_\rho}{m_\rho} \exp(M_\phi - m_\phi) \int_0^1 \frac{d^2}{dt^2} \mathcal{K}(\psi^t) dt.$$

Using (7), we obtain the inequality

$$\text{Var}_{\mu^0+\mu^1}(\psi^1 - \psi^0) \leq (d+1)2^d \frac{M_\rho^2}{m_\rho^2} \exp(M_\phi - m_\phi) \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle. \quad (12)$$

We finally leverage an in-homogeneity in the scale of the Brenier potentials in the last inequality in order to improve the dependence on $M_\phi - m_\phi$. For any $\lambda > 0$, introduce for $k \in \{0, 1\}$ the Brenier potential $\phi_\lambda^k = \lambda \phi^k$ and denote $\mu_\lambda^k = (\nabla \phi_\lambda^k)_{\#} \rho$ the corresponding probability measure and $\psi_\lambda^k = (\phi_\lambda^k)^*$ its dual potential. Then using the formula $\psi_\lambda^k = \lambda \psi^k(\cdot/\lambda)$, one can notice that for any $\lambda > 0$,

$$\begin{aligned} \text{Var}_{\mu_\lambda^0+\mu_\lambda^1}(\psi_\lambda^1 - \psi_\lambda^0) &= \lambda^2 \text{Var}_{\mu^0+\mu^1}(\psi^1 - \psi^0), \\ \langle \psi_\lambda^0 - \psi_\lambda^1 | \mu_\lambda^1 - \mu_\lambda^0 \rangle &= \lambda \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle, \\ \forall x \in \mathcal{X}, \forall k \in \{0, 1\}, \quad \lambda m_\phi &\leq \phi_\lambda^k(x) \leq \lambda M_\phi. \end{aligned}$$

Thus applying inequality (12) to $\mu_\lambda^0, \mu_\lambda^1$ and the associated potentials yields for any $\lambda > 0$

$$\text{Var}_{\mu^0+\mu^1}(\psi^1 - \psi^0) \leq (d+1)2^d \frac{M_\rho^2}{m_\rho^2} \frac{\exp(\lambda(M_\phi - m_\phi))}{\lambda} \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle.$$

Choosing $\lambda = \frac{1}{M_\phi - m_\phi}$ in the last inequality finally gives

$$\text{Var}_{\mu^0+\mu^1}(\psi^1 - \psi^0) \leq e(d+1)2^d \frac{M_\rho^2}{m_\rho^2} (M_\phi - m_\phi) \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle. \quad \square$$

To deduce the general case of Theorem 2.1, we need to approximate the convex potentials ϕ^0, ϕ^1 on \mathcal{X} with strongly convex potentials ϕ_n^0, ϕ_n^1 that belong to $\mathcal{C}^2(\mathcal{X})$ and that are such that their gradients $\nabla \phi_n^0, \nabla \phi_n^1$ induce diffeomorphisms between \mathcal{X} and a closed ball \mathcal{Y}_n . Although tempting, a regularization that uses a (standard) convolution does not seem directly feasible. This is because ϕ^k is defined on \mathcal{X} only, and its gradient explodes on the boundary of \mathcal{X} when μ^k has non-compact support, so that any convex extension of ϕ^k to \mathbb{R}^d has to take value $+\infty$.

Our strategy is as follows. First, we resort to Moreau-Yosida's regularization to approximate the functions ϕ^0, ϕ^1 by regular convex functions defined on \mathbb{R}^d . Then, we regularize the target probability measures associated to the approximated potentials and resort to Caffarelli's regularity theory to guarantee smoothness and strong convexity. Caffarelli's regularity theory results require smoothness assumptions on the source probability measure

and strong convexity and smoothness assumption on the domain. We make these assumptions in the next proposition, but we will later show that these can be relaxed to get the general case of Theorem 2.1.

Proposition 2.5. *Let $\mu^0, \mu^1 \in \mathcal{P}_2(\mathbb{R}^d)$. Let \mathcal{X} be a compact, smooth and strongly convex set, let ρ be a smooth probability density on \mathcal{X} and assume that ρ is bounded away from zero and infinity on this set. Denote ϕ^k the Brenier potentials for the quadratic optimal transport from ρ to μ^k , $\psi^k = (\phi^k)^*$, and assume that there exists $m_\phi, M_\phi \in \mathbb{R}$ such for $k \in \{0, 1\}$ and any $x \in \mathcal{X}$,*

$$m_\phi \leq \phi^k(x) \leq M_\phi.$$

Then there exists a sequence of strongly convex functions $(\phi_n^0)_{n \in \mathbb{N}}, (\phi_n^1)_{n \in \mathbb{N}}$ in $\mathcal{C}^2(\mathbb{R}^d)$ such that if one introduces $\mu_n^k = \nabla \phi_n^k \# \rho$ and $\psi_n^k = (\phi_n^k)^$, then:*

- (i) $\lim_{n \rightarrow +\infty} \langle \psi_n^0 - \psi_n^1 | \mu_n^1 - \mu_n^0 \rangle = \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle$,
- (ii) $\lim_{n \rightarrow +\infty} \text{Var}_{\mu_n^0 + \mu_n^1}(\psi_n^1 - \psi_n^0) = \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0)$,
- (iii) let $m_{\phi_n} = \min_{\mathcal{X}} \min_k \phi_n^k$, and $M_{\phi_n} = \max_{\mathcal{X}} \max_k \phi_n^k$. Then,

$$m_\phi \leq \liminf_{n \rightarrow +\infty} m_{\phi_n} \leq \limsup_{n \rightarrow +\infty} M_{\phi_n} \leq M_\phi$$

- (iv) *there exists a closed ball \mathcal{Y}_n such that for $k \in \{0, 1\}$, $\nabla \phi_n^k$ is a diffeomorphism between \mathcal{X} and \mathcal{Y}_n .*

Before proving this proposition, we recall some facts regarding Moreau-Yosida's regularization of convex functions. Quoting Section 3.4 of [3], the Moreau-Yosida regularization of parameter $\lambda > 0$ of a closed and proper convex function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined for all $x \in \mathbb{R}^d$ by infimum convolution of the function f with $\frac{1}{2\lambda} \|\cdot\|^2$:

$$f_\lambda(x) = \min_{u \in \mathbb{R}^d} f(u) + \frac{1}{2\lambda} \|u - x\|^2.$$

The next lemma gathers a few properties of the Moreau-Yosida regularisation.

Lemma 2.6. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed and proper convex function and let $\lambda > 0$. Then for all $x \in \mathbb{R}^d$,*

- (i) *for all $x \in \mathbb{R}^d$, $\lim_{\lambda \rightarrow 0} f_\lambda(x) = f(x)$,*
- (ii) *$f_\lambda \in \mathcal{C}^{1,1}(\mathbb{R}^d)$ and more precisely, ∇f_λ is $\frac{1}{\lambda}$ -Lipschitz,*
- (iii) *if f is differentiable at $x \in \mathbb{R}^d$, then $\lim_{\lambda \rightarrow 0} \nabla f_\lambda(x) = \nabla f(x)$,*
- (iv) *if f is differentiable at $x \in \mathbb{R}^d$, then $\|\nabla f_\lambda(x)\| \leq \|\nabla f(x)\|$.*

Proof. The first two points are found in Theorem 3.24 of [3] and the last two can be found in Proposition 2.6 of [9]. \square

Proof of Proposition 2.5. First regularization and truncation. We will first approximate and extend the Brenier potentials ϕ^0, ϕ^1 , which are defined on \mathcal{X} , with elements of $\mathcal{C}^{1,1}(\mathbb{R}^d)$. To do so, we extend ϕ^0, ϕ^1 by $+\infty$ outside of the set \mathcal{X} and for any $\alpha > 0$ we denote by ϕ_α^k the Moreau-Yosida regularization of ϕ^k with parameter α . We let $\mu_\alpha^k = (\nabla \phi_\alpha^k) \# \rho$ and define ψ_α^k as the convex conjugate of ϕ_α^k . By Lemma 2.6, $\nabla \phi_\alpha^k$ is Lipschitz on the

bounded domain \mathcal{X} , implying that the images of $\nabla\phi_\alpha^k(\mathcal{X})$ are contained in a closed ball $\mathcal{Y}_\alpha = B(0, R_\alpha)$. We now prove the claimed convergences (i)–(iii). We first note that if $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the growth condition $|f(x)| \leq C(1 + \|x\|^2)$ for some constant C ,

$$\langle f | \mu_\alpha^k \rangle = \int_{\mathcal{X}} f(\nabla\phi_\alpha^k) d\rho \xrightarrow{\alpha \rightarrow 0} \int_{\mathcal{X}} f(\nabla\phi^k) d\rho = \langle f | \mu^k \rangle. \quad (13)$$

Indeed, Lemma 2.6.(iii) ensures that for every point $x \in \mathcal{X}$ where ϕ^k is differentiable, thus for ρ -almost every point x , one has $\lim_{\alpha \rightarrow 0} \nabla\phi_\alpha^k(x) = \nabla\phi^k(x)$. Besides, for all such x , Lemma 2.6.(iv) gives

$$\left| f(\nabla\phi_\alpha^k(x)) \right| \leq C \left(1 + \|\nabla\phi_\alpha^k(x)\|^2 \right) \leq C \left(1 + \|\nabla\phi^k(x)\|^2 \right).$$

Moreover,

$$\int_{\mathcal{X}} \left(1 + \|\nabla\phi^k(x)\|^2 \right) d\rho(x) \leq 1 + M_2(\nabla\phi^k(x) \# \rho) = 1 + M_2(\mu^k) < +\infty.$$

Thus, the dominated convergence theorem ensures that (13) holds.

(i) Since ψ^0, ψ^1 are convex conjugates of functions defined on the compact set \mathcal{X} , the functions ψ^0 and ψ^1 are (globally) Lipschitz on \mathbb{R}^d . Thus $f = \psi_\alpha^0 - \psi_\alpha^1 = \psi^0 - \psi^1$ is also Lipschitz, and therefore satisfies a growth condition of the form $|f| \leq C(1 + \|x\|)$. By an application of (13), we get

$$\lim_{\alpha \rightarrow 0} \langle \psi_\alpha^0 - \psi_\alpha^1 | \mu_\alpha^1 - \mu_\alpha^0 \rangle = \lim_{\alpha \rightarrow 0} \langle \psi^0 - \psi^1 | \mu_\alpha^1 - \mu_\alpha^0 \rangle = \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle.$$

(ii) We use $\text{Var}_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$. Letting f as in the previous item, we get

$$\begin{aligned} \text{Var}_{\mu_\alpha^0 + \mu_\alpha^1}(\psi_\alpha^1 - \psi_\alpha^0) &= \langle f^2 | \mu_\alpha^0 + \mu_\alpha^1 \rangle - \langle f | \mu_\alpha^0 + \mu_\alpha^1 \rangle^2, \\ \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) &= \langle f^2 | \mu^0 + \mu^1 \rangle - \langle \psi^1 - \psi^0 | \mu^0 + \mu^1 \rangle^2. \end{aligned}$$

Since f is Lipschitz, both f and f^2 satisfy the growth condition allowing us to apply (13). We therefore get

$$\lim_{\alpha \rightarrow 0} \text{Var}_{\mu_\alpha^0 + \mu_\alpha^1}(\psi_\alpha^1 - \psi_\alpha^0) = \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0).$$

(iii) We note that for $k \in \{0, 1\}$ and $x \in \mathcal{X}$, $m_\phi \leq \phi_\alpha^k(x) \leq M_\phi$. This is a simple consequence of the definition of the Moreau-Yosida regularization ϕ_α^k as an infimum convolution. Indeed for any $x \in \mathcal{X}$, we have on one hand:

$$\phi_\alpha^k(x) = \inf_{x' \in \mathcal{X}} \left(\phi^k(x') + \frac{1}{2\alpha} \|x - x'\|^2 \right) \geq \inf_{x' \in \mathcal{X}} \phi^k(x') + \inf_{x' \in \mathcal{X}} \frac{1}{2\alpha} \|x - x'\|^2 \geq m_\phi.$$

On the other hand,

$$\phi_\alpha^k(x) = \inf_{x' \in \mathcal{X}} \left(\phi^k(x') + \frac{1}{2\alpha} \|x - x'\|^2 \right) \leq \phi^k(x) + \frac{1}{2\alpha} \|x - x\|^2 \leq M_\phi.$$

We have all the desired properties (i)–(iii) but the potentials ϕ_α^k are not strongly convex and \mathcal{C}^2 on \mathbb{R}^d : they are merely $\mathcal{C}^{1,1}$. Moreover, the property (iv) does not hold. These properties will be obtained thanks to a second regularization.

Second regularization. From now on, we fix some $\alpha > 0$, and we denote $\mathcal{Y}_\alpha = B(0, R_\alpha)$ a closed ball that contains the supports of μ_α^0 and μ_α^1 . To construct the regularization of ϕ_α^k we will regularize the measures μ_α^k and solve an optimal transport problem. We first note that it is straightforward, e.g. using a simple convolution and truncation, to approximate the probability measures μ_α^k on \mathcal{Y}_α by smooth probability densities $\mu_{\alpha,\beta}^k$ supported on \mathcal{Y}_α , bounded away from zero and infinity on \mathcal{Y}_α and such that $\lim_{\beta \rightarrow 0} W_2(\mu_{\alpha,\beta}^k, \mu_\alpha^k) = 0$. By Caffarelli's regularity theory (e.g. Theorem 3.3 in [13]), the optimal transport map $T_{\alpha,\beta}$ between ρ and $\mu_{\alpha,\beta}^k$ is the gradient of a strongly convex potential $\phi_{\alpha,\beta}^k$ belonging to $\mathcal{C}^2(\mathcal{X})$ and is actually a diffeomorphism between \mathcal{X} and \mathcal{Y}_α . By Theorem 4.4 in [33], the potential $\phi_{\alpha,\beta}^k$ can be extended into a \mathcal{C}^2 strongly convex function on \mathbb{R}^d . By stability of Kantorovich potentials (Theorem 1.51 in [27]), taking a subsequence if necessary, we can assume that $\phi_{\alpha,\beta}^k$ converges uniformly to ϕ_α^k on \mathcal{X} as $\beta \rightarrow 0$. Since $\nabla \phi_{\alpha,\beta}^k$ sends ρ to the measure $\mu_{\alpha,\beta}^k$, which is supported on $B(0, R_\alpha)$, we get $\|\nabla \phi_{\alpha,\beta}^k\| \leq R_\alpha$. Moreover, since the convex function $\phi_{\alpha,\beta}^k$ converges uniformly to ϕ_α^k as $\beta \rightarrow 0$, we get

$$\text{for a.e. } x \in \mathcal{X}, \lim_{\beta \rightarrow 0} \nabla \phi_{\alpha,\beta}^k(x) = \nabla \phi_\alpha^k(x).$$

From these two properties we get as above the desired convergence properties:

- (i) $\lim_{\beta \rightarrow 0} \langle (\phi_{\alpha,\beta}^0)^* - (\phi_{\alpha,\beta}^1)^* | \mu_{\alpha,\beta}^1 - \mu_{\alpha,\beta}^0 \rangle = \langle (\phi_\alpha^0)^* - (\phi_\alpha^1)^* | \mu_\alpha^1 - \mu_\alpha^0 \rangle$,
- (ii) $\lim_{\beta \rightarrow 0} \text{Var}_{\mu_{\alpha,\beta}^0 + \mu_{\alpha,\beta}^1}((\phi_{\alpha,\beta}^1)^* - (\phi_{\alpha,\beta}^0)^*) = \text{Var}_{\mu_\alpha^0 + \mu_\alpha^1}((\phi_\alpha^1)^* - (\phi_\alpha^0)^*)$,
- (iii) for $k \in \{0, 1\}$, $m_\phi \leq \liminf_{\beta \rightarrow 0} \min_{\mathcal{X}} \phi_{\alpha,\beta}^k(x) \leq \limsup_{\beta \rightarrow 0} \max_{\mathcal{X}} \phi_{\alpha,\beta}^k(x) \leq M_\phi$.

The sequence in the statement of the proposition is finally constructed using a diagonal argument. \square

Proposition 2.7. *In addition to the assumptions of Theorem 2.1, assume that \mathcal{X} is a smooth and strongly convex set and that the density ρ is smooth. Then, (2) holds.*

Proof. Let ϕ_n^0, ϕ_n^1 be the sequence of \mathcal{C}^2 and strongly convex potentials constructed by Proposition 2.5, converging respectively to ϕ^0 and ϕ^1 , and such that $\nabla \phi_n^0, \nabla \phi_n^1$ are diffeomorphisms from \mathcal{X} to a ball \mathcal{Y}_n . By Proposition 2.4, (2) holds for ϕ_n^0, ϕ_n^1 :

$$\text{Var}_{\mu_n^0 + \mu_n^1}(\psi_n^1 - \psi_n^0) \leq C_d \frac{M_\rho^2}{m_\rho^2} (M_{\phi_n} - m_{\phi_n}) \langle \psi_n^0 - \psi_n^1 | \mu_n^1 - \mu_n^0 \rangle.$$

By the claims (i)-(iv) in Proposition 2.5, all the terms in this inequality converge as $n \rightarrow +\infty$ and establish (2) in the limit. \square

Proof of Theorem 2.1. Let \mathcal{X} be a bounded convex set and assume that ρ is a probability density satisfying $m_\rho \leq \rho \leq M_\rho$. We extend ρ by m_ρ outside of \mathcal{X} . One can construct a sequence \mathcal{X}_n of smooth and strongly convex sets included in \mathcal{X} and converging to \mathcal{X} in the Hausdorff sense as $\alpha \rightarrow 0$ [28, §3.3]. Let K be a smooth, non-negative and compactly supported function, $K_n(x) = n^d K(nx)$ and define

$$\rho_n = \frac{1}{Z_n} (\rho * K_n)|_{\mathcal{X}_n}, m_{\rho_n} = \frac{m_\rho}{Z_n}, M_{\rho_n} = \frac{M_\rho}{Z_n},$$

where Z_n is a constant ensuring that ρ_n belongs to $\mathcal{P}(\mathcal{X}_n)$. We define $\mu_n^k = \nabla\phi_{\#}^k\rho_n$. Applying Proposition 2.7 to (\mathcal{X}_n, ρ_n) and (ϕ^0, ϕ^1) , we have:

$$\text{Var}_{\mu_n^0 + \mu_n^1}(\psi^1 - \psi^0) \leq C_d \frac{M_{\rho_n}^2}{m_{\rho_n}^2} (M_{\phi} - m_{\phi}) \langle \psi^0 - \psi^1 | \mu_n^1 - \mu_n^0 \rangle. \quad (14)$$

By construction, $\lim_{n \rightarrow +\infty} Z_n = 1$ and ρ_n converges to ρ in $L^1(\mathcal{X})$. Thus up to subsequences, ρ_n converges pointwise almost everywhere to ρ . Setting $f = \psi^0 - \psi^1$, we have

$$\langle \psi^0 - \psi^1 | \mu_n^0 \rangle = \int_{\mathcal{X}} f(\nabla\psi^0)\rho_n(x)dx \xrightarrow{n \rightarrow +\infty} \int_{\mathcal{X}} f(\nabla\psi^0)\rho(x)dx = \langle \psi^0 - \psi^1 | \mu^0 \rangle.$$

The limit in the above equation is proven as in Proposition 2.5, using that f is Lipschitz, that $M_2(\mu_0) < +\infty$ and applying the dominated convergence theorem. All the terms can be dealt with in a similar manner. Taking the limit $n \rightarrow +\infty$ in (14) gives the desired (2). \square

3. STABILITY OF POTENTIALS

A direct consequence of the strong convexity estimate of Theorem 2.1 is a quantitative stability result on the dual potentials with respect to the target measures. This estimate on dual potentials is readily transferred to the Brenier (primal) potentials thanks to the next proposition.

Proposition 3.1. *Let ρ be a probability density over a compact convex set \mathcal{X} , and let ϕ^0, ϕ^1 be convex functions on \mathcal{X} . Denote ψ^k the convex conjugate of ϕ^k and μ^k the image of ρ under $\nabla\phi^k$. Then for any $p > 0$,*

$$\|\phi^1 - \phi^0\|_{L^p(\rho)} \leq \|\psi^1 - \psi^0\|_{L^p(\mu^0 + \mu^1)}.$$

In particular,

$$\text{Var}_{\rho}(\phi^1 - \phi^0) \leq \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0).$$

The stability estimates resulting from Theorem 2.1 and this proposition are expressed in Corollary 3.2 in terms of L^2 -norms for the potentials and 1-Wasserstein distance for the target measures. Assuming that one of the target measures is absolutely continuous with respect to the other, these estimates can also be expressed in term of χ^2 or Kagan's divergence of the target measures, describing a stronger stability behavior since the χ^2 divergence can be interpreted as the *square of a divergence* (recall for instance that the total variation distance is only $\frac{1}{2}$ -Hölder stable w.r.t. the χ^2 divergence).

Corollary 3.2 (Stability of potentials). *Let ρ be a probability density over a compact convex set \mathcal{X} , satisfying $0 < m_{\rho} \leq \rho \leq M_{\rho}$ and let $\mu^0, \mu^1 \in \mathcal{P}_2(\mathbb{R}^d)$. For $k \in \{0, 1\}$, denote $\phi^k = \phi_{\mu^k}$ the Brenier potential between ρ and μ^k . Assume that ϕ^0, ϕ^1 satisfy (1) and denote ψ^0 and ψ^1 the convex conjugates of ϕ^0 and ϕ^1 . Then,*

$$\text{Var}_{\rho}(\phi^1 - \phi^0) \leq \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \leq \left(C_d \frac{M_{\rho}^2}{m_{\rho}^2} (M_{\phi} - m_{\phi}) \text{diam}(\mathcal{X}) \right) W_1(\mu^0, \mu^1), \quad (15)$$

with $C_d = e(d+1)2^d$. Assuming additionally that μ^1 is absolutely continuous w.r.t. μ^0 , then

$$\mathrm{Var}_\rho(\phi^1 - \phi^0) \leq \mathrm{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \leq \left(C_d \frac{M_\rho^2}{m_\rho^2} (M_\phi - m_\phi) \right)^2 D_{\chi^2}(\mu^1 | \mu^0), \quad (16)$$

where $D_{\chi^2}(\mu^1 | \mu^0)$ stands for the χ^2 or Kagan's divergence from μ^1 to μ^0 .

Proof. Proposition 3.1 combined with Theorem 2.1 give the inequalities

$$\mathrm{Var}_\rho(\phi^1 - \phi^0) \leq \mathrm{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \leq C_d \frac{M_\rho^2}{m_\rho^2} (M_\phi - m_\phi) \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle.$$

The estimate of equation (15) follows from Kantorovich-Rubinstein duality result and the fact that $\|\nabla\psi^1 - \nabla\psi^0\| \leq \mathrm{diam}(\mathcal{X})$, which allow to write

$$\langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle \leq \mathrm{diam}(\mathcal{X}) W_1(\mu^0, \mu^1).$$

Now notice that if μ^1 is absolutely continuous with respect to μ^0 , then we have for any constant $c \in \mathbb{R}$:

$$\begin{aligned} \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle &= \langle \psi^0 - \psi^1 - c | \mu^1 - \mu^0 \rangle \\ &= \int_{\mathbb{R}^d} (\psi^0 - \psi^1 - c) \left(\frac{d\mu^1}{d\mu^0} - 1 \right) d\mu^0 \\ &\leq \left(\int_{\mathbb{R}^d} (\psi^0 - \psi^1 - c)^2 d\mu^0 \right)^{1/2} \left(\int_{\mathbb{R}^d} \left(\frac{d\mu^1}{d\mu^0} - 1 \right)^2 d\mu^0 \right)^{1/2} \\ &= \|\psi^0 - \psi^1 - c\|_{L^2(\mu^0)} D_{\chi^2}(\mu^1 | \mu^0)^{1/2} \\ &\leq \|\psi^0 - \psi^1 - c\|_{L^2(\mu^0 + \mu^1)} D_{\chi^2}(\mu^1 | \mu^0)^{1/2}. \end{aligned}$$

The estimate of equation (16) comes after minimizing with respect to c in the last inequality:

$$\langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle \leq \mathrm{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0)^{1/2} D_{\chi^2}(\mu^1 | \mu^0)^{1/2}. \quad \square$$

Proof of Proposition 3.1. Let $A = \{x \in \mathcal{X} \mid \phi^1(x) \geq \phi^0(x)\}$ and let $x \in A$ where ϕ^1 is differentiable. The Fenchel-Young inequality (and equality) give:

$$\psi^0(\nabla\phi^1(x)) \geq \langle x | \nabla\phi^1(x) \rangle - \phi^0(x) = \psi^1(\nabla\phi^1(x)) + \phi^1(x) - \phi^0(x),$$

which thus ensures that for almost every $x \in A$,

$$\psi^0(\nabla\phi^1(x)) - \psi^1(\nabla\phi^1(x)) \geq \phi^1(x) - \phi^0(x) \geq 0.$$

Similarly, for almost every $x \in \mathcal{X} \setminus A$, we have

$$\psi^1(\nabla\phi^0(x)) - \psi^0(\nabla\phi^0(x)) \geq \phi^0(x) - \phi^1(x) \geq 0.$$

From this, we deduce the first statement of the proposition:

$$\begin{aligned} \|\psi^1 - \psi^0\|_{L^p(\mu^0 + \mu^1)}^p &= \int_{\mathcal{X}} |\psi^1(\nabla\phi^0) - \psi^0(\nabla\phi^0)|^p d\rho + \int_{\mathcal{X}} |\psi^1(\nabla\phi^1) - \psi^0(\nabla\phi^1)|^p d\rho \\ &\geq \int_{\mathcal{X} \setminus A} (\psi^1(\nabla\phi^0) - \psi^0(\nabla\phi^0))^p d\rho + \int_A (\psi^0(\nabla\phi^1) - \psi^1(\nabla\phi^1))^p d\rho \end{aligned}$$

$$\geq \int_{\mathcal{X} \setminus A} (\phi^0 - \phi^1)^p d\rho + \int_A (\phi^1 - \phi^0)^p d\rho = \|\phi^1 - \phi^0\|_{L^p(\rho)}^p.$$

Replacing ψ^0 by $\psi^0 + c$ in the above inequality, we see that for any $c \in \mathbb{R}$,

$$\|\phi^1 - \phi^0 + c\|_{L^p(\rho)} \leq \|\psi^1 - \psi^0 - c\|_{L^p(\mu^0 + \mu^1)}.$$

Taking c that achieves the minimum on the right-hand side, for $p = 2$, we get

$$\text{Var}_\rho(\phi^1 - \phi^0) \leq \|\phi^1 - \phi^0 + c\|_{L^2(\rho)}^2 \leq \|\psi^1 - \psi^0 - c\|_{L^2(\mu^0 + \mu^1)}^2 = \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0). \quad \square$$

All our stability estimates involve the oscillation of the Brenier potentials $M_\phi - m_\phi$. It is then natural to wonder under what assumption on a measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ can we control this oscillation. The next proposition, found in [6], shows that a sufficient condition is that μ admits a finite moment of order $p > d$. Remark 3.2 below shows that $M_p(\mu) < +\infty$ with $p < d$ does not imply boundedness of the Brenier potential.

Proposition 3.3. *Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and denote ϕ the Brenier potential for the quadratic optimal transport between ρ and μ . Assume that there exists $p > d$ and $M_p < +\infty$ such that*

$$M_p(\mu) = \int_{\mathbb{R}^d} \|y\|^p d\mu(y) \leq M_p.$$

Then ϕ is Hölder continuous and verifies for all $x, x' \in \mathcal{X}$:

$$|\phi(x) - \phi(x')| \leq C_{d,p,\mathcal{X}} \left(\frac{M_p}{m_\rho} \right)^{1/p} \|x - x'\|^{1 - \frac{d}{p}}.$$

In particular, there exists $m_\phi, M_\phi \in \mathbb{R}$ that can be chosen such that for any $x \in \mathcal{X}$, $m_\phi \leq \phi(x) \leq M_\phi$ and such that

$$M_\phi - m_\phi \leq C_{d,p,\mathcal{X}} \left(\frac{M_p}{m_\rho} \right)^{1/p} \text{diam}(\mathcal{X})^{1 - \frac{d}{p}}.$$

Corollary 3.2 and Proposition 3.3 together imply the following.

Corollary 3.4 (Stability with enough moments). *Let $p > d$. Then the restriction of the mapping $\mu \mapsto \phi_\mu$ to the set of probability measures with bounded p -th moment is $1/2$ -Hölder with respect to the W_1 distance. More precisely, if $\max(M_p(\mu^0), M_p(\mu^1)) \leq M_p < +\infty$, then*

$$\|\phi_{\mu^1} - \phi_{\mu^0}\|_{L^2(\rho)} \leq C_{d,p,\mathcal{X},\rho,M_p} W_1(\mu^0, \mu^1)^{1/2}.$$

Remark 3.1. A large class of probability distributions admit a finite moment of order $p > d$. For instance, sub-exponential measures, which encompass most of the commonly used heavy-tailed distributions fall into this class. We say that a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is sub-exponential with variance proxy σ^2 for $\sigma > 0$ if it has zero mean and if for all $r > 0$,

$$\mu(\{x \in \mathbb{R}^d \mid \|x\| \geq r\}) \leq 2e^{-2r/\sigma}.$$

We refer to Proposition 2.7.1 in [29] for equivalent characterization. The moments of such a measure are all bounded, and more precisely,

$$M_p(\mu) \leq 2p! \left(\frac{\sigma}{2}\right)^p.$$

Proof of Proposition 3.3. The gradient $\nabla\phi$ corresponds to the optimal transport map between ρ and μ . Using that μ is the image of ρ under $\nabla\phi$, the moment assumption gives,

$$\|\nabla\phi\|_{L^p(\mathcal{X})}^p = \int_{\mathcal{X}} \|\nabla\phi(x)\|^p dx \leq \frac{1}{m_\rho} \int_{\mathcal{X}} \|\nabla\phi(x)\|^p d\rho(x) \leq \frac{M_p}{m_\rho}. \quad (17)$$

We can add a constant to ϕ so that $\int_{\mathcal{X}} \phi(x) dx = 0$ without changing its modulus of continuity. The Poincaré–Wirtinger inequality then ensures that $\|\phi\|_{L^p(\mathcal{X})} \leq C_{p,\mathcal{X}} \|\nabla\phi\|_{L^p(\mathcal{X})}$. In particular, the potential ϕ belongs to the Sobolev space $W^{1,p}(\mathcal{X})$. Morrey’s inequality (Theorem 11.34 and Theorem 12.15 in [21]) ensures that ϕ is $(1 - \frac{d}{p})$ -Hölder and that there exists a constant depending only on d, p and \mathcal{X} such that

$$\forall x \neq x' \in \mathcal{X}, \quad \frac{|\phi(x) - \phi(x')|}{\|x - x'\|^{1-\frac{d}{p}}} \leq C_{d,p,\mathcal{X}} \|\phi\|_{W^{1,p}(\mathcal{X})} \leq C_{d,p,\mathcal{X}} \left(\frac{M_p}{m_\rho}\right)^{1/p}. \quad \square$$

Remark 3.2 (Morrey’s inequality for convex functions). Since the Brenier potentials ϕ are convex, one may wonder whether Morrey’s inequality and the resulting Sobolev embedding can be improved when restricted to the class of convex functions. However, one can show that for $\mathcal{X} = [0, 1]^d$ and $p < d$, for $\alpha \in \left(0, \frac{d}{p} - 1\right)$, the potential

$$\phi : \begin{cases} \mathcal{X} \rightarrow \mathbb{R} \\ (x_1, \dots, x_d) \mapsto (x_1 + \dots + x_d)^{-\alpha} \end{cases}$$

is convex, belongs to $W^{1,p}(\mathcal{X})$, but obviously neither Hölder continuous nor even bounded. In other words, assuming that $M_p(\mu) < +\infty$ for $p < d$ does not guarantee that the Brenier potential from ρ to μ is α -Hölder, or even bounded.

4. STABILITY OF OPTIMAL TRANSPORT MAPS

In this section, we derive quantitative stability estimates on optimal transport maps with respect to the target measures from the stability estimates on Brenier potentials given in the preceding section. This derivation relies on a Gagliardo–Nirenberg type inequality on the difference of Lipschitz convex functions. This inequality is proven in the next section.

Proposition 4.1. *Let K be a compact domain of \mathbb{R}^d with rectifiable boundary and let $u, v : K \rightarrow \mathbb{R}$ be two Lipschitz functions on K that are convex on any segment included in K . Then there exists a constant C_d depending only on d such that*

$$\|\nabla u - \nabla v\|_{L^2(K)}^2 \leq C_d \mathcal{H}^{d-1}(\partial K)^{2/3} (\|\nabla u\|_{L^\infty(K)} + \|\nabla v\|_{L^\infty(K)})^{4/3} \|u - v\|_{L^2(K)}^{2/3},$$

where \mathcal{H}^{d-1} denotes the $(d - 1)$ -dimensional Hausdorff measure.

We first show that the stability estimates on Brenier potentials given in Corollary 3.2 can be readily transferred to the corresponding optimal transport maps – i.e. to their gradients – when the target measures are compactly supported. Indeed, Proposition 4.1 together with Corollary 3.2 give directly:

Theorem 4.2 (Stability of the Brenier map, compact case). *Let \mathcal{X}, \mathcal{Y} be compact subsets of \mathbb{R}^d with \mathcal{X} convex, let ρ be a probability density over \mathcal{X} bounded from above and below by positive constants and let $\mu^0, \mu^1 \in \mathcal{P}(\mathcal{Y})$. Denoting T_{μ^k} the Brenier map from ρ to μ^k , we have*

$$W_2(\mu^0, \mu^1) \leq \|T_{\mu^0} - T_{\mu^1}\| \leq C_{d,\rho,\mathcal{X},\mathcal{Y}} W_1(\mu^0, \mu^1)^{\frac{1}{6}}.$$

In particular, the embedding $\mu \in \mathcal{P}_2(\mathcal{Y}) \rightarrow T_\mu \in L^2(\rho, \mathbb{R}^d)$ is bi-Hölder continuous.

Remark 4.1 (bi-Hölder embedding via potentials). The previous theorem and Proposition 4.1 together with Corollary 3.2 also ensure the following bi-Hölder behavior for the Brenier potentials (with zero mean against ρ on \mathcal{X}):

$$\forall \mu^0, \mu^1 \in \mathcal{P}(\mathcal{Y}), \quad W_2(\mu^0, \mu^1)^3 \lesssim \|\phi^1 - \phi^0\|_{L^2(\rho)} \lesssim W_1(\mu^0, \mu^1)^{\frac{1}{2}},$$

where the \lesssim notation hides multiplicative constants depending on $d, \rho, \mathcal{X}, \mathcal{Y}$.

We now phrase a similar stability result for probability measures whose Brenier potential is Hölder continuous and that admit a bounded fourth order moment. This includes a large class of probability measures, as noticed in Proposition 3.3 and Remark 3.1.

Theorem 4.3 (Stability of the Brenier map). *Let $\mu^0, \mu^1 \in \mathcal{P}_2(\mathbb{R})$ and denote ϕ^0, ϕ^1 the Brenier potentials for the quadratic optimal transport between ρ and μ^0, μ^1 respectively. Assume that there exists $M_\alpha > 0$ and $\alpha \in (0, 1)$ such that for all $x, x' \in \mathcal{X}$ and $k \in \{0, 1\}$,*

$$|\phi^k(x) - \phi^k(x')| \leq M_\alpha \|x - x'\|^\alpha.$$

Assume that there exists $0 < M < +\infty$ such that for $k \in \{0, 1\}$, $M_4(\mu^k) \leq M$. Then

$$W_2(\mu^0, \mu^1) \leq \|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho)} \leq C_{d,\rho,\mathcal{X},\alpha,M_\alpha,M} W_1(\mu^0, \mu^1)^{\frac{1}{2(11-8\alpha)}}. \quad (18)$$

Remark 4.2. The assumption $M_4(\mu^k) < +\infty$ comes from a use of the Cauchy-Schwarz inequality in the proof of Theorem 4.3. However, one could use Hölder's inequality instead, under different moment assumption and show that for any $q \geq 1$, assuming that $M_{2q}(\mu^k) \leq M_{2q} < +\infty$ for $k \in \{0, 1\}$, one has

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho)} \leq C_{d,\rho,\mathcal{X},M_\alpha,\alpha,M_{2q}} W_1(\mu^0, \mu^1)^{\frac{q-1}{2(q(7-4\alpha)-3)}}.$$

Since the exponent is an increasing function of q , a stronger stability can be obtained at the cost of stronger moment assumptions.

Theorem 4.3 and Proposition 3.3 directly imply the following.

Corollary 4.4 (Stability with enough moments). *For $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, denote $\nabla\phi_\mu$ the optimal transport map for the quadratic optimal transport between ρ and μ . Assume $p \geq 4$ and*

$p > d$. Then, the map $\mu \mapsto T_\mu$ is Hölder when restricted to the set of probability measures with bounded p -th moment. More precisely, if $\max(M_p(\mu^0), M_p(\mu^1)) \leq M_p < +\infty$, then

$$W_2(\mu^0, \mu^1) \leq \|\nabla\phi_{\mu^1} - \nabla\phi_{\mu^0}\|_{L^2(\rho)} \leq C_{d,p,\mathcal{X},\rho,M_p} W_1(\mu^0, \mu^1)^{\frac{p}{6p+16d}}.$$

To prove Theorem 4.3, we first show that whenever a Brenier potential defined on the compact and convex set \mathcal{X} is Hölder continuous, it is possible to control its Lipschitz constant on erosions of \mathcal{X} . We recall that for $\varepsilon > 0$, the ε -erosion of \mathcal{X} , denoted $\mathcal{X}_{-\varepsilon}$, corresponds to the set of points of \mathcal{X} that are at least at a distance ε from $\partial\mathcal{X}$.

Proposition 4.5 (Lipschitz behaviour on erosion). *Let $\mu \in \mathcal{P}_2(\mathbb{R})$ and denote ϕ the Brenier potential for the quadratic optimal transport between ρ and μ . Assume that there exists $M_\alpha > 0$ and $\alpha \in (0, 1)$ such that for all $x, x' \in \mathcal{X}$,*

$$|\phi(x) - \phi(x')| \leq M_\alpha \|x - x'\|^\alpha.$$

Then, ϕ is R -Lipschitz on the erosion $\mathcal{X}_{-\eta_R}$ with $\eta_R = \left(\frac{M_\alpha}{R}\right)^{\frac{1}{1-\alpha}}$.

The proof of this proposition is inspired by the proof of Proposition 3.3 in [17].

Proof. Let $x \in \mathcal{X}$ be such that $d(x, \partial\mathcal{X}) \geq \eta_R$, and let $g \in \partial\phi(x)$. We will show that $\|g\| \leq R$, thus implying the statement. Denoting $\psi = (\phi)^*$, the Fenchel-Young equality and inequality ensures that

$$\begin{cases} \psi(g) = \langle g|x \rangle - \phi(x), \\ \psi(g) \geq \langle g|x' \rangle - \phi(x') \quad \text{for all } x' \in \mathcal{X} \end{cases}$$

Putting these equation, we get that for any $x' \in \mathcal{X}$,

$$\langle g|x' - x \rangle \leq \phi(x') - \phi(x) \leq M_\alpha \|x' - x\|^\alpha, \quad (19)$$

where we used the Hölder continuity assumption on ϕ . We now choose x' to be the unique point in the intersection between the ray $x + \mathbb{R}^+g$ and $\partial\mathcal{X}$, so that

$$\langle g|x' - x \rangle = \|x - x'\| \|g\| \geq d(x, \partial\mathcal{X}) \|g\| \geq \eta_R \|g\|.$$

This choice of x' in equation (19) gives us as desired

$$\|g\| \leq \frac{M_\alpha}{\|x - x'\|^{1-\alpha}} \leq \frac{M_\alpha}{d(x, \partial\mathcal{X})^{1-\alpha}} = R. \quad \square$$

Proposition 4.5 allows to control the Lipschitz constant of the restriction ϕ^k to $\mathcal{X}_{-\eta}$ assuming that ϕ^k is α -Hölder. Combining it with the inequality of Proposition 4.1, we get a stability estimate for the restriction of the transport map to $\mathcal{X}_{-\eta}$. To conclude the proof of the theorem, we will rely on an upper bound on the volume of the symmetric difference between \mathcal{X} and its erosion $\mathcal{X}_{-\varepsilon}$ given in the next proposition.

Proposition 4.6 (Volume of boundary slices). *Let $\mathcal{X} \subset \mathbb{R}^d$ be a compact convex set containing the origin, and denote $r_{\mathcal{X}} > 0$ and $R_{\mathcal{X}} > 0$ the largest and smallest radii such that $B(0, r_{\mathcal{X}}) \subseteq \mathcal{X} \subseteq B(0, R_{\mathcal{X}})$. Then, for all $\varepsilon \in [0, r_{\mathcal{X}})$,*

$$\text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\varepsilon}) \leq S_{d-1}(R_{\mathcal{X}} + r_{\mathcal{X}})^{d-1} \frac{R_{\mathcal{X}}}{r_{\mathcal{X}}} \varepsilon,$$

where S_{d-1} the surface area of the $(d-1)$ -dimensional unit sphere.

We quote a lemma extracted from [22] that allows to control the volume of the difference between a convex \mathcal{X} and its ε -erosion $\mathcal{X}_{-\varepsilon}$ using the volume of ε -dilation of \mathcal{X} , denoted $\mathcal{X}_{+\varepsilon} = \{x \in \mathbb{R}^d \mid d(x, \mathcal{X}) \leq \varepsilon\}$.

Lemma 4.7 (Lemma 1 in [22]). *For all $\varepsilon \leq r_{\mathcal{X}}$, $\text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\varepsilon}) \leq \text{vol}^d(\mathcal{X}_{+\varepsilon} \setminus \mathcal{X})$.*

This lemma, together with Steiner's formula already implies that $\text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\varepsilon})$ grows linearly in ε for small values of ε . We provide a direct proof below.

Proof of Proposition 4.6. This result is proven using the radial function ρ of \mathcal{X} , $\rho_{\mathcal{X}}(x) = \max\{\lambda \geq 0 \mid \lambda x \in \mathcal{X}\}$. Since $x \in \mathcal{S}^{d-1} \mapsto \rho(x)x$ is a radial parametrization of $\partial\mathcal{X}$, we have:

$$\text{vol}^d(\mathcal{X}) = \int_{\mathcal{X}} 1 dx = \int_{\mathcal{S}^{d-1}} \int_0^{\rho_{\mathcal{X}}(u)} r^{d-1} dr du = \frac{1}{d} \int_{\mathcal{S}^{d-1}} \rho_{\mathcal{X}}(u)^d du.$$

Combined with Lemma 4.7, this implies that for any $0 \leq \varepsilon \leq r_{\mathcal{X}}$,

$$\begin{aligned} \text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\varepsilon}) &\leq \text{vol}^d(\mathcal{X}_{+\varepsilon} \setminus \mathcal{X}) = \frac{1}{d} \int_{\mathcal{S}^{d-1}} (\rho_{\mathcal{X}_{+\varepsilon}}(u)^d - \rho_{\mathcal{X}}(u)^d) du \\ &\leq \frac{1}{d} \int_{\mathcal{S}^{d-1}} (\rho_{\mathcal{X}_{+\varepsilon}}(u) - \rho_{\mathcal{X}}(u)) d(R_{\mathcal{X}} + r_{\mathcal{X}})^{d-1} du. \end{aligned}$$

Using the inclusions $B(0, r_{\mathcal{X}}) \subseteq \mathcal{X} \subseteq B(0, R_{\mathcal{X}})$, one can prove that for any $\varepsilon > 0$ and for any unit vector u , $0 \leq \rho_{\mathcal{X}_{+\varepsilon}}(u) - \rho_{\mathcal{X}}(u) \leq \frac{R_{\mathcal{X}}}{r_{\mathcal{X}}} \varepsilon$. This finally gives, for $\varepsilon \in [0, r_{\mathcal{X}}]$,

$$\text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\varepsilon}) \leq \int_{\mathcal{S}^{d-1}} \frac{R_{\mathcal{X}}}{r_{\mathcal{X}}} \varepsilon (R_{\mathcal{X}} + r_{\mathcal{X}})^{d-1} du = S_{d-1}(R_{\mathcal{X}} + r_{\mathcal{X}})^{d-1} \frac{R_{\mathcal{X}}}{r_{\mathcal{X}}} \varepsilon.$$

One can easily check that in the case $\varepsilon \geq r_{\mathcal{X}}$ the inequality also holds. \square

Proof of Theorem 4.3. In the following, the \lesssim notation hides multiplicative constants that might depend on $d, \rho, \mathcal{X}, \alpha, M_{\alpha}, M$. We get the left inequality of (18) by recalling that

$$W_2(\mu^0, \mu^1)^2 = \min_{\gamma \in \Pi(\mu^0, \mu^1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma(x, y),$$

and by noticing that the optimal transport maps $\nabla\phi^0, \nabla\phi^1$ between ρ and μ^0, μ^1 yield an admissible coupling $\gamma^{0,1} := (\nabla\phi^0, \nabla\phi^1)_{\#}\rho \in \Pi(\mu^0, \mu^1)$, which leads to:

$$W_2(\mu^0, \mu^1)^2 \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma^{0,1}(x, y) = \int_{\mathcal{X}} \|\nabla\phi^1 - \nabla\phi^0\|^2 d\rho.$$

We now prove the right inequality of (18). We recall that $\eta_R = \left(\frac{M_\alpha}{R}\right)^{\frac{1}{1-\alpha}}$. Then,

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho)}^2 = \|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X}_{-\eta_R})}^2 + \|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2. \quad (20)$$

On $\mathcal{X}_{-\eta_R}$, Proposition 4.5 ensures that $\|\nabla\phi^k\|_\infty \leq R$ for $k \in \{0, 1\}$. This fact thus ensures with Proposition 4.1 that for any $c \in \mathbb{R}$:

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X}_{-\eta_R})}^2 \lesssim R^{4/3} \|\phi^1 - \phi^0 - c\|_{L^2(\mathcal{X}_{-\eta_R})}^{2/3} \lesssim R^{4/3} \|\phi^1 - \phi^0 - c\|_{L^2(\rho)}^{2/3}.$$

Minimizing over c in the last inequality thus ensures

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X}_{-\eta_R})}^2 \lesssim R^{4/3} \text{Var}_\rho(\phi^1 - \phi^0)^{1/3} \lesssim R^{4/3} W_1(\mu^0, \mu^1)^{1/3}, \quad (21)$$

where we used Corollary 3.2 to get the second inequality. On the other hand, notice that

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2 \leq 2\|\nabla\phi^1\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2 + 2\|\nabla\phi^0\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2.$$

By the Cauchy-Schwartz inequality we have for $k \in \{0, 1\}$

$$\begin{aligned} \|\nabla\phi^k\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2 &= \int_{\mathcal{X} \setminus \mathcal{X}_{-\eta_R}} \|\nabla\phi^k\|^2 d\rho \leq \left(\int_{\mathcal{X} \setminus \mathcal{X}_{-\eta_R}} \|\nabla\phi^k\|^4 d\rho \right)^{1/2} \left(\int_{\mathcal{X} \setminus \mathcal{X}_{-\eta_R}} 1^2 d\rho \right)^{1/2} \\ &\lesssim M_4(\mu^k)^{1/2} \text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\eta_R})^{1/2}. \end{aligned}$$

Proposition 4.6 ensures that for any $R \geq 0$, we have

$$\text{vol}^d(\mathcal{X} \setminus \mathcal{X}_{-\eta_R}) \lesssim \eta_R = \left(\frac{M_\alpha}{R}\right)^{1/(1-\alpha)}.$$

This gives thus the estimation

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho, \mathcal{X} \setminus \mathcal{X}_{-\eta_R})}^2 \lesssim R^{-1/2(1-\alpha)} \quad (22)$$

Injecting estimations (21) and (22) into equation (20) thus give for $R \geq 0$

$$\|\nabla\phi^1 - \nabla\phi^0\|_{L^2(\rho)}^2 \lesssim R^{4/3} W_1(\mu^0, \mu^1)^{1/3} + R^{-1/2(1-\alpha)}. \quad (23)$$

Solving for $R^{4/3} W_1(\mu^0, \mu^1)^{1/3} = R^{-1/2(1-\alpha)}$ yields $R = W_1(\mu^0, \mu^1)^{\frac{-2(1-\alpha)}{11-8\alpha}}$. Injecting this value of R in (23) yields the desired estimate. \square

In the previous results, if we assume that the target measures μ^0, μ^1 are supported on a compact set $\mathcal{Y} \subset \mathbb{R}^d$ and are absolutely continuous with densities bounded away from zero and infinity, then one can slightly improve the different stability estimates.

Corollary 4.8. *Let \mathcal{X}, \mathcal{Y} be compact subsets of \mathbb{R}^d , and assume that \mathcal{X} is convex and that \mathcal{Y} has a rectifiable boundary. Let ρ be a probability density over \mathcal{X} satisfying $0 < m_\rho \leq \rho \leq M_\rho < +\infty$ and let μ^0, μ^1 be probability densities over \mathcal{Y} satisfying*

$$\forall k \in \{0, 1\}, \quad 0 < c_\mu \leq \mu^k \leq C_\mu < +\infty.$$

Then, if ϕ^k (resp. T^k) is the Brenier potential (resp. Brenier map) from ρ to μ^k , we have

$$W_2(\mu^0, \mu^1)^6 \lesssim \text{Var}_\rho(\phi^1 - \phi^0) \leq \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \lesssim W_2(\mu^0, \mu^1)^{\frac{6}{5}},$$

$$W_2(\mu^0, \mu^1) \leq \|\nabla\phi^1 - \nabla\phi^0\| \lesssim W_2(\mu^0, \mu^1)^{\frac{1}{5}},$$

where the \lesssim notation hides multiplicative constants depending on $d, \rho, \mathcal{X}, \mathcal{Y}, c_\mu, C_\mu$.

This corollary will be a consequence of the following lemma from [23], which we will use as a replacement of the Kantorovich-Rubinstein inequality.

Lemma 4.9 (Lemma 3.5 in [23]). *Assume that μ^0 and μ^1 are absolutely continuous measures on the compact \mathcal{Y} , whose densities are bounded by a same constant C_μ . Then, for all function $f \in H^1(\mathcal{Y})$, we have the following inequality:*

$$\int_{\mathcal{Y}} f d(\mu^1 - \mu^0) \leq \sqrt{C_\mu} \|\nabla f\|_{L^2(\mathcal{Y})} W_2(\mu^0, \mu^1).$$

Proof of Corollary 4.8. Setting $\psi^0 = (\phi^0)^*$, $\psi^1 = (\phi^1)^*$, we have by (2) in Theorem 2.1:

$$\text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \lesssim \langle \psi^0 - \psi^1 | \mu^1 - \mu^0 \rangle. \quad (24)$$

For $c \in \mathbb{R}$ such that $\|\psi^1 - \psi^0 - c\|_{L^2(\mu^0 + \mu^1)}^2 = \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0)$, estimation (24) and Lemma 4.9 ensure that:

$$\|\psi^1 - \psi^0 - c\|_{L^2(\mu^0 + \mu^1)}^2 \lesssim \|\nabla\psi^1 - \nabla\psi^0\|_{L^2(\mathcal{Y})} W_2(\mu^0, \mu^1). \quad (25)$$

But Proposition 4.1 applied to the convex and Lipschitz functions $\psi^0 + c, \psi^1$ ensures that

$$\|\nabla\psi^1 - \nabla\psi^0\|_{L^2(\mathcal{Y})} \lesssim \|\psi^1 - \psi^0 - c\|_{L^2(\mu^0 + \mu^1)}^{1/3}.$$

Injecting this estimation into (25) yields

$$\|\psi^1 - \psi^0 - c\|_{L^2(\mu^0 + \mu^1)}^2 \lesssim W_2(\mu^0, \mu^1)^{6/5}.$$

This gives thus with Proposition 3.1

$$\text{Var}_\rho(\phi^1 - \phi^0) \leq \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0) \lesssim W_2(\mu^0, \mu^1)^{6/5}.$$

Finally, a last use of Proposition 4.1 also ensures that under these assumptions on the targets μ^0, μ^1 we have

$$W_2(\mu^0, \mu^1) \leq \|\nabla\phi^1 - \nabla\phi^0\| \lesssim \text{Var}_\rho(\phi^1 - \phi^0)^{\frac{1}{6}} \lesssim \text{Var}_{\mu^0 + \mu^1}(\psi^1 - \psi^0)^{\frac{1}{6}} \lesssim W_2(\mu^0, \mu^1)^{\frac{1}{5}}. \quad \square$$

5. GAGLIARDO–NIRENBERG TYPE INEQUALITY FOR DIFFERENCE OF CONVEX FUNCTIONS

We prove here Proposition 4.1, which allows to control the L^2 distance between the gradients of Lipschitz convex functions with the L^2 distance between the functions. It is a refinement of Theorem 3.5 in [12], in which the upper bound uses the uniform distance between the function instead of the L^2 distance. Proposition 4.1 is first proven in dimension $d = 1$ and on a segment (Lemma 5.1) and then generalized to higher dimensions using arguments from integral geometry (Lemma 5.2).

Remark 5.1 (Relation to the Gagliardo–Nirenberg inequality). Although the estimate of Proposition 4.1 resembles to a Gagliardo–Nirenberg inequality, the former cannot be deduced from it because the involved parameters are not admissible. More precisely, we note that without convexity of u and v , the inequality in (4.1) does not hold. One can see this by taking $u = 0$ and $v_n(x) = 1/n \sin(nx)$ on $K = [0, 1]$.

Remark 5.2 (Optimality of exponents). The inequality proposed in Proposition 4.1 is sharp in term of the exponents of the norms $\|\nabla u\|_{L^\infty(K)} + \|\nabla v\|_{L^\infty(K)}$ and $\|u - v\|_{L^2(K)}$ in the right-hand side. In the case $d = 1$, let $L > 0, \varepsilon > 0$ and define on $K = [0, 1]$, $u(x) = L|x - \frac{1}{2}|$ and $v = \max(u, \varepsilon)$. Then u, v are convex and L -Lipschitz and we have:

$$\|u - v\|_{L^2([0,1])}^2 = \frac{\varepsilon^3}{L} \quad \text{and} \quad \|u' - v'\|_{L^2([0,1])}^2 = 2L\varepsilon.$$

so that $\|u' - v'\|_{L^2([0,1])}^2 = 2L^{4/3}\|u - v\|_{L^2([0,1])}^{2/3}$.

Lemma 5.1. *Let $I \subset \mathbb{R}$ be a segment and let $u, v : I \rightarrow \mathbb{R}$ be two convex functions with uniformly bounded gradients on I . Then*

$$\|u' - v'\|_{L^2(I)}^2 \leq 8(\|u'\|_{L^\infty(I)} + \|v'\|_{L^\infty(I)})^{4/3}\|u - v\|_{L^2(I)}^{2/3}. \quad (26)$$

Proof. Using a simple approximation, we may assume that u, v are C^2 on I . We also first assume that $I = [0, 1]$. The convexity hypothesis allows to get first an L^∞ estimate:

$$\|u' - v'\|_{L^2([0,1])}^2 = \int_{[0,1]} (u' - v')^2 = [(u - v)(u' - v')]_0^1 - \int_{[0,1]} (u - v)(u'' - v'').$$

But $|[(u - v)(u' - v')]_0^1| \leq 2(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})\|u - v\|_{L^\infty}$, and by convexity

$$\begin{aligned} \left| \int_{[0,1]} (u - v)(u'' - v'') \right| &\leq \|u - v\|_{L^\infty} \left(\int_{[0,1]} |u''| + \int_{[0,1]} |v''| \right) \\ &= \|u - v\|_{L^\infty} \left(\int_{[0,1]} u'' + \int_{[0,1]} v'' \right) \\ &\leq 2(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})\|u - v\|_{L^\infty}. \end{aligned}$$

This gives

$$\|u' - v'\|_{L^2([0,1])}^2 \leq 4(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})\|u - v\|_{L^\infty}. \quad (27)$$

We now bound the L^∞ norm of $u - v$ with its L^2 norm using that the Lipschitz constant of $u - v$ is less than $L = \|u'\|_{L^\infty} + \|v'\|_{L^\infty}$. Let $\varepsilon = \|u - v\|_{L^\infty}$ and let $x^* \in [0, 1]$ where the maximum of $|u - v|$ is attained. Since $\text{Lip}(u - v) \leq L$, one gets $|u(x) - v(x)| \geq \frac{\varepsilon}{2}$ on the interval $I_* = I \cap [x^* - \frac{\varepsilon}{2L}, x^* + \frac{\varepsilon}{2L}]$. The length of I_* is at least $\min(\frac{\varepsilon}{2L}, 1)$, so that

$$\|u - v\|_{L^2([0,1])}^2 \geq \frac{1}{4} \min\left(\frac{\varepsilon}{2L}, 1\right)\varepsilon^2. \quad (28)$$

Assume first that $\varepsilon \leq 2L$. Then, equation (28) gives $\varepsilon = \|u - v\|_\infty^3 \leq 8L\|u - v\|_{L^2([0,1])}^2$, thus implying

$$\|u - v\|_{L^\infty} \leq 2(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})^{1/3}\|u - v\|_{L^2([0,1])}^{2/3}.$$

This gives, with equation (27):

$$\|u' - v'\|_{L^2([0,1])}^2 \leq 8(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})^{4/3}\|u - v\|_{L^2([0,1])}^{2/3}. \quad (29)$$

On the other hand, if $\varepsilon \geq 2L$, then $\|u - v\|_{L^2([0,1])} \geq \frac{\varepsilon}{2}$ by equation (28), so that

$$8(\|u'\|_{L^\infty} + \|v'\|_{L^\infty})^{4/3}\|u - v\|_{L^2([0,1])}^{2/3} \geq 8L^{4/3} \left(\frac{\varepsilon}{2}\right)^{2/3} \geq L^{4/3+2/3} \geq \|u' - v'\|_{L^2([0,1])}^2.$$

We get inequality (26) for a general interval $I = [a, b]$ by an affine change of variable. \square

The one-dimensional result from Lemma 5.1 is generalized to higher dimensions thanks to two formulas from integral geometry that allow to rewrite the L^2 norms of the scalar-field $u - v$ and vector-field $\nabla u - \nabla v$ over set $K \subset \mathbb{R}^d$ using integrals over lines intersecting K .

Integral geometry. Denote V_d the volume of the unit d -ball and S_{d-1} the area of the unit $(d-1)$ -sphere. Let \mathcal{L}^d be the set of oriented affine lines in \mathbb{R}^d , seen as the submanifold of \mathbb{R}^{2d} consisting of pairs $(e, p) \in \mathbb{R}^d \times \mathbb{R}^d$ with $e \in \mathcal{S}^{d-1}$ and p in the hyperplane $\{e\}^\perp$, and endowed with the induced Riemannian metric. The volume measure $d\mathcal{L}^d$ is invariant under rigid motions. Denoting \mathcal{H}^k the k -dimensional Hausdorff measure, the usual Crofton formula – see for instance the first paragraph of Chapter 5 in [16] – states that for any \mathcal{H}^{d-1} -rectifiable subset S of \mathbb{R}^d ,

$$\mathcal{H}^{d-1}(S) = \frac{1}{4V_{d-1}} \int_{\ell \in \mathcal{L}^d} \#(\ell \cap S) d\mathcal{L}^d(\ell), \quad (30)$$

where $\#X$ is the cardinality of the set X . We denote \mathcal{L}_e^d the set of oriented lines with a fixed direction $e \in \mathcal{S}^{d-1}$, endowed with the $(d-1)$ -dimensional Lebesgue measure $d\mathcal{L}_e^d$ on $\{e\}^\perp$, so that

$$d\mathcal{L}^d = \int_{\mathcal{S}^{d-1}} d\mathcal{L}_e^d \otimes d\mathcal{S}^{d-1}(e).$$

We will also use the following formula, which easily follows from Fubini's theorem: if K is a measurable subset of \mathbb{R}^d , then for any fixed direction $e \in \mathcal{S}^{d-1}$,

$$\mathcal{H}^d(K) = \int_{\ell \in \mathcal{D}_e^d} \mathcal{H}^1(\ell \cap K) d\mathcal{L}_e^d(\ell). \quad (31)$$

We begin with an elementary lemma.

Lemma 5.2. *Let K be a compact subset of \mathbb{R}^d and $f \in L^2(K)$. Then,*

$$\|f\|_{L^2(K)}^2 = \frac{1}{S_{d-1}} \int_{\ell \in \mathcal{L}^d} \int_{y \in \ell \cap K} f(y)^2 dy d\mathcal{L}^d(\ell). \quad (32)$$

Similarly, for any vector field $F \in L^2(K, \mathbb{R}^d)$, one has

$$\|F\|_{L^2(K, \mathbb{R}^d)}^2 = C_d \int_{\ell \in \mathcal{L}^d} \int_{y \in \ell \cap K} \langle F(y) | e(\ell) \rangle^2 dy d\mathcal{L}^d(\ell), \quad (33)$$

where for $\ell \in \mathcal{L}^d$, $e(\ell) \in \mathcal{S}^{d-1}$ is the oriented direction of ℓ , and C_d depends only on d .

Proof. Piecewise constant functions (resp. vector fields) are dense in $L^2(K)$ (resp. $L^2(K, \mathbb{R}^d)$). Using this fact and the continuity of equations (32), (33), it is therefore enough to prove these equations when f and F are of the form $f = \chi_K$ and $F = x\chi_K$ for some fixed $x \in \mathcal{S}^{d-1}$. We have for $f = \chi_K$, using formula (31):

$$\begin{aligned} \int_{\ell \in \mathcal{L}^d} \int_{y \in \ell \cap K} f(y)^2 dy d\mathcal{L}^d(\ell) &= \int_{e \in \mathcal{S}^{d-1}} \int_{\ell \in \mathcal{D}_e^d} \int_{y \in \ell \cap K} dy d\mathcal{L}_e^d(\ell) de \\ &= \int_{e \in \mathcal{S}^{d-1}} \int_{\ell \in \mathcal{D}_e^d} \mathcal{H}^1(\ell \cap K) d\mathcal{L}_e^d(\ell) de \\ &= S_{d-1} \mathcal{H}^d(K) \\ &= S_{d-1} \|f\|_{L^2(K)}^2, \end{aligned}$$

which proves equation (32). Now for $F = x\chi_K$, we get for $e \in \mathcal{S}^{d-1}$:

$$\begin{aligned} \int_{\ell \in \mathcal{D}_e^d} \int_{y \in \ell \cap K} \langle F(y) | e \rangle^2 dy d\mathcal{L}_e^d(\ell) &= \langle x | e \rangle^2 \int_{\ell \in \mathcal{D}_e^d} \mathcal{H}^1(\ell \cap K) d\mathcal{L}_e^d(\ell) \\ &= \langle x | e \rangle^2 \|F\|_{L^2(K, \mathbb{R}^d)}^2 \end{aligned}$$

Hence we get:

$$\int_{\ell \in \mathcal{L}^d} \int_{y \in \ell \cap K} \langle F(y) | e(\ell) \rangle^2 dy d\mathcal{L}^d(\ell) = \|F\|_{L^2(K, \mathbb{R}^d)}^2 \int_{e \in \mathcal{S}^{d-1}} \langle x | e \rangle^2 de.$$

The last integral does not depend on x , thus establishing the result. \square

We are now ready to prove the Gagliardo-Nirenberg type inequality of Proposition 4.1.

Proof of Proposition 4.1. We apply formula (33) from Lemma 5.2 to $(\nabla u - \nabla v)$:

$$\|\nabla u - \nabla v\|_{L^2(K)}^2 = C_d \int_{\ell \in \mathcal{L}^d} \int_{y \in \ell \cap K} \langle (\nabla u - \nabla v)(y) | e(\ell) \rangle^2 dy d\mathcal{L}^d(\ell).$$

For any $\ell \in \mathcal{L}^d$, denote $u_\ell = u|_{\ell \cap K}$, $v_\ell = v|_{\ell \cap K}$, and notice that the last equation reads:

$$\|\nabla u - \nabla v\|_{L^2(K)}^2 = C_d \int_{\ell \in \mathcal{L}^d} \|u'_\ell - v'_\ell\|_{L^2(\ell \cap K)}^2 d\mathcal{L}^d(\ell).$$

Given any oriented line $\ell \in \mathcal{D}^d$, denote $n_\ell \in \mathbb{N} \cup \{+\infty\}$ the number of connected components of $\ell \cap K$. Then, $n_\ell \leq \#(\ell \cap \partial K)$ so that by Crofton formula's,

$$\int_{\ell \in \mathcal{D}^d} n_\ell d\mathcal{L}(\ell) \leq \int_{\ell \in \mathcal{D}^d} \#(\ell \cap \partial K) d\mathcal{L}(\ell) < +\infty.$$

This implies that for almost every $\ell \in \mathcal{L}^d$, the set $\ell \cap K$ may be decomposed as a finite union of segments, i.e. $\ell \cap K = \bigcup_{i=1}^{n_\ell} I_\ell^i$ of n_ℓ . This gives

$$\|u'_\ell - v'_\ell\|_{L^2(\ell \cap K)}^2 = \sum_{i=1}^{n_\ell} \|u'_\ell - v'_\ell\|_{L^2(I_\ell^i)}^2 \quad \text{and} \quad \|u_\ell - v_\ell\|_{L^2(\ell \cap K)}^2 = \sum_{i=1}^{n_\ell} \|u_\ell - v_\ell\|_{L^2(I_\ell^i)}^2.$$

Lemma 5.1 combined with Jensen’s inequality then ensure that we have for any $l \in \mathcal{L}^d$:

$$\begin{aligned} \|u'_\ell - v'_\ell\|_{L^2(\ell \cap K)}^2 &\leq 8(\|u'_\ell\|_{L^\infty(\ell \cap K)} + \|v'_\ell\|_{L^\infty(\ell \cap K)})^{4/3} \sum_{i=1}^{n_\ell} \|u_\ell - v_\ell\|_{L^2(I_\ell^i)}^{2/3} \\ &\leq 8(\|u'_\ell\|_{L^\infty(\ell \cap K)} + \|v'_\ell\|_{L^\infty(\ell \cap K)})^{4/3} n_\ell^{2/3} \|u_\ell - v_\ell\|_{L^2(\ell \cap K)}^{2/3}. \end{aligned}$$

This leads to the inequality

$$\|\nabla u - \nabla v\|_{L^2(K)}^2 \leq 8C_d(\|\nabla u\|_{L^\infty(K)} + \|\nabla v\|_{L^\infty(K)})^{4/3} \int_{\ell \in \mathcal{L}^d} n_\ell^{2/3} \|u_\ell - v_\ell\|_{L^2(\ell \cap K)}^{2/3} d\mathcal{L}^d(\ell).$$

But Hölder’s inequality together with formula (32) give

$$\int_{\ell \in \mathcal{L}^d} n_\ell^{2/3} \|u_\ell - v_\ell\|_{L^2(\ell \cap K)}^{2/3} d\mathcal{L}^d(\ell) \leq \left(\int_{\ell \in \mathcal{L}^d} n_\ell d\mathcal{L}^d(\ell) \right)^{2/3} \left(S_{d-1} \|u - v\|_{L^2(K)}^2 \right)^{1/3}.$$

The conclusion comes after using again that $n_\ell \leq \#(\ell \cap \partial K)$ and Crofton’s formula (30)

$$\int_{\ell \in \mathcal{L}^d} n_\ell d\mathcal{L}^d(\ell) \leq \int_{\ell \in \mathcal{L}^d} \#(\ell \cap \partial K) d\mathcal{L}^d(\ell) = 4V_{d-1} \mathcal{H}^{d-1}(\partial K). \quad \square$$

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