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# ALGEBRAIC COHERENT CONFLUENCE AND HIGHER-DIMENSIONAL GLOBULAR KLEENE ALGEBRAS

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CAMERON CALK    ERIC GOUBAULT    PHILIPPE MALBOS    GEORG STRUTH

**Abstract** – We extend the formalisation of confluence results in Kleene algebras to a formalisation of coherent proofs by confluence. To this end, we introduce the structure of modal higher-dimensional globular Kleene algebra, a higher-dimensional generalisation of modal and concurrent Kleene algebra. We give a calculation of a coherent Church-Rosser theorem and Newman’s lemma in higher-dimensional Kleene algebras. We interpret these results in the context of higher-dimensional rewriting systems described by polygraphs.

**Keywords** – Modal Kleene algebras, confluence, coherence, higher-dimensional rewriting.

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# 1. INTRODUCTION

Rewriting is a model of computation widely used in algebra, computer science, and logic. Computation rules or algebraic laws are described by *rewrite relations* on symbolic or algebraic expressions. Rewriting theory is strongly based on diagrammatic intuitions. Indeed, a central theme of rewriting is that of completing certain *branching shapes* with *confluence shapes*, thus obtaining *confluence diagrams*. Traditionally, the rewriting machinery was formalised in terms of algebras of binary relations: confluence properties are described by union, composition and iteration operations. A natural generalisation of this is given by the structure of Kleene algebra, in which proofs of classical confluence results such as the Church-Rosser theorem or Newman's lemma have been calculated [5, 28, 29]. Beyond that, Kleene algebras and similar structures are well known for their ability to capture complex computational properties by simple equational reasoning [7, 20, 30, 32] and unify various semantics of computational interest, including formal languages, binary relations, path algebras and execution traces of automata [17].

Rewriting provides constructive procedures for proving some coherence properties in categorical algebra as well. Coherence properties in this setting are formulated via a notion of contractibility for higher-dimensional categories. The rewriting approach consists in proving a coherence property from a given set of higher-dimensional witnesses for (local) confluence diagrams. It was initiated by Squier [25] in the context of homotopical finiteness conditions in string rewriting, which have more recently been expressed in the setting of higher dimensional rewriting [14]. By contrast to the standard diagrammatic or relational approach, witnesses for confluence proofs are provided, in the sense that traditional confluence diagrams are filled with higher dimensional cells. Such a method has been applied, for instance, in [8, 15] to give constructive proofs for coherence in monoids, and in [12] for coherence theorems in monoidal categories.

In this work we combine these two branches of research on Kleene-algebraic and higher-dimensional rewriting into a coherent framework. We show how some classical calculational confluence proofs using Kleene algebra can be extended to coherent confluence proofs. To achieve this, we introduce higher-dimensional Kleene algebras, with many compositions and domain and codomain operations, which generalise both modal Kleene algebras [6] and concurrent Kleene algebras [16]. This structure algebraically captures the semantics of higher dimensional abstract rewriting. As applications of this formalism we provide calculational proofs for both the coherent Church-Rosser theorem and the coherent Newman's lemma. We also relate these generalised results to the point-wise approach given by higher-dimensional rewriting systems described by polygraphs. The main contribution of this work is therefore the provision of a point-free, algebraic approach to coherence in higher dimensional rewriting which seems of general interest in categorical algebra.

## Abstract coherent reduction

Coherence proofs by rewriting are based on coherent formulations of confluence results such as Church-Rosser's theorem and Newman's lemma. We present the coherent extension of the Church-Rosser theorem as an example. Recall that an *abstract rewriting system*  $\rightarrow$  on a set  $X$  is a binary relation on  $X$ , and that the *confluence* of such a relation is characterised by the inclusion

$$\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow,$$

where  $\rightarrow^*$  denotes the reflexive, transitive closure of the relation  $\rightarrow$ , the relation  $\leftarrow$  its converse and  $\cdot$  stands for relational composition. The relation  $\rightarrow$  has the *Church-Rosser property* if the inclusion

$$\leftrightarrow^* \subseteq \rightarrow^* \cdot \leftarrow^*$$

holds, where  $\leftrightarrow^* = (\leftarrow \cup \rightarrow)^*$  is the reflexive, symmetric and transitive closure of  $\rightarrow$ . The Church-Rosser theorem for  $\rightarrow$ , which states that these two properties are equivalent, can be formulated along similar lines in Kleene algebra using the *Kleene star* operation, an abstraction of the notion of reflexive, transitive closure as in Theorem 4 of [28], recalled in (4.1.3), which states that for any  $x, y$  in a Kleene algebra  $K$ , the following equivalence holds:

$$x^* \cdot y^* \leq y^* \cdot x^* \iff (x + y)^* \leq y^* \cdot x^*.$$

The Church-Rosser theorem for the relation  $\rightarrow$  is the special case where  $K$  is the algebra of binary relations on  $X$ ,  $x = \leftarrow$  and  $y = \rightarrow$ . This is justified by the fact that the binary relations over any set  $X$  form a Kleene algebra with respect to relational composition, relational union, the reflexive transitive closure operation, the empty relation and the unit (or diagonal) relation.

The diagrammatic interpretation of the relation  $\rightarrow$  states that there is an arrow  $u \rightarrow v$ , whenever  $(u, v)$  belongs to  $\rightarrow$ . When  $(u, v)$  is an element of  $\rightarrow^*$  (resp.  $\leftrightarrow^*$ ), we say that  $u$  is related to  $v$  by a *rewriting sequence* (resp. *zig-zag sequence*). Diagrammatically, the Church-Rosser theorem states that, for any branching  $(f, g)$  of rewriting sequences, there exists an associated confluence  $(f', g')$ , if and only if, for any zig-zag sequence  $h$ , there exists an associated confluence  $(h', k')$ :



A coherent extension of this result can be formulated in the context of higher-dimensional rewriting theory. Roughly, it states that if there exists a set  $\Gamma$  of 2-dimensional cells (of globular shape) such that every branching can be completed to a confluence diagram filled by elements of  $\Gamma$  pasted together along their 1-dimensional borders, then every zig-zag sequence may be completed to a Church-Rosser diagram filled by elements of  $\Gamma$  pasted along their 1-dimensional borders. Pictorially, these statements are represented, respectively, by:



where  $\alpha$  and  $\beta$  are built from 2-cells in  $\Gamma$ . This result constitutes one step in the proof of Squier's theorem for higher-dimensional rewriting systems, which provides a constructive approach to coherence results akin to the coherence condition satisfied by associativity and units in monoidal categories: if certain diagrams of natural isomorphisms commute, then all the diagrams built from the corresponding natural isomorphisms are commutative. A key issue is therefore to reduce the infinite requirement "every diagram

## 1. Introduction

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commutes”, to a finite requirement “if a specified finite set of diagrams each commute then every diagram commute”, [21, 26]. The notion of coherent confluence provides a constructive way of proving such coherence results.

### Organisation and main results of the article

**Higher-dimensional rewriting.** In Section 2 we recall notions from higher-dimensional rewriting. We first recall the structure of *polygraph*, which represents a system of generators and relations for higher-dimensional categories. Polygraphs were introduced by Street and Burroni, [3, 27], and are widely used as rewriting systems presenting higher-dimensional algebraic structures [11, 23]. Furthermore, polygraphs are used to formulate homotopical properties of rewriting systems through polygraphic resolutions, [13, 22], as well as coherence properties for monoids, [8], higher categories, [11], and monoidal categories, [12]. The latter are inspired by Squier’s approach to proving coherence results for monoids using convergent string rewriting systems [25]. Explicitly, an  $n$ -polygraph  $P$  is a higher dimensional rewriting system made of globular cells of dimension  $0, 1, \dots, n$ , such that, for any  $0 \leq k \leq n$ , its set of  $k$ -cells  $P_k$  consists in  $k$ -dimensional rewriting rules of globular shape:

$$\begin{array}{ccc}
 & s_{k-1}(\alpha) & \\
 & \curvearrowright & \\
 s_{k-2}(\alpha) & \Downarrow \alpha & t_{k-2}(\alpha) \\
 & \curvearrowleft & \\
 & t_{k-1}(\alpha) & 
 \end{array}$$

A *cellular extension* of the free  $n$ -category  $P_n^*$  (resp. the free  $(n, n-1)$ -category  $P_n^\top$ ) generated by  $P_n$  is a set of globular  $(n+1)$ -cells that relate  $n$ -cells of  $P_n^*$  (resp.  $P_n^\top$ ).

**Coherent confluence.** A *branching* in an  $n$ -polygraph  $P$  is a pair  $(f, g)$  of  $n$ -cells of the free  $n$ -category  $P_n^*$  which have the same  $(n-1)$ -source. A branching is *local* when  $f$  and  $g$  are *rewriting steps*, i.e. generating elements for the rewriting system given by  $P$ , see Section 2.2.1. A cellular extension  $\Gamma$  of the free  $(n, n-1)$ -category  $P_n^\top$  is a *confluence filler* of the branching  $(f, g)$  if there exist  $n$ -cells  $f', g'$  in the free  $n$ -category  $P_n^*$  and two  $(n+1)$ -cells

$$\begin{array}{ccc}
 f^- & \xrightarrow{\quad} & u & \xrightarrow{\quad} & g \\
 & \searrow & \Downarrow \alpha & \swarrow & \\
 u_1 & & & & v_1 \\
 & \swarrow & & \searrow & \\
 f' & \xrightarrow{\quad} & u' & \xrightarrow{\quad} & (g')^-
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & u & & \\
 & \swarrow & & \swarrow & \\
 f & & & & g^- \\
 & \searrow & \Downarrow \alpha' & \swarrow & \\
 u_1 & & & & v_1 \\
 & \swarrow & & \swarrow & \\
 (f')^- & \xrightarrow{\quad} & u' & \xrightarrow{\quad} & g'
 \end{array}$$

in the free  $(n+1)$ -category  $P_n^\top[\Gamma]$  generated by  $\Gamma$  over  $P_n^\top$ . We say that the cellular extension  $\Gamma$  is a (*local*) *confluence filler* for  $P$  if it is a confluence filler for each of its (*local*) branchings. In the case of zig-zag sequences, we say that  $\Gamma$  is a *confluence filler* of an  $n$ -cell  $f$  in  $P_n^\top$  if there exist  $n$ -cells  $f'$  and  $g'$  in  $P_n^*$  and

an  $(n + 1)$ -cell  $\alpha$  in the free  $(n + 1)$ -category  $P_n^\top[\Gamma]$  of the form

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ & \searrow f' & \nearrow (g')^- \\ & u' & \end{array}$$

The cellular extension  $\Gamma$  is a *Church-Rosser filler* for an  $n$ -polygraph  $P$  when it is a confluence filler for every  $n$ -cell in  $P_n^\top$ . Theorem 2.3.4 states that for an  $n$ -polygraph  $P$ , a cellular extension  $\Gamma$  of  $P_n^\top$  is a confluence filler for  $P$  if, and only if,  $\Gamma$  is a Church-Rosser filler for  $P$ . Theorem 2.3.7 states that, when  $P$  is *terminating*, then  $\Gamma$  is a local confluence filler, if, and only if,  $\Gamma$  is a confluence filler for  $P$ . These are the coherent, higher-dimensional extensions of the Church-Rosser theorem and Newman's lemma, respectively. In Subsection 2.4, we relate this confluence filler property to the coherent confluence property already defined in higher-dimensional rewriting [10].

**Higher-dimensional globular Kleene algebras.** Section 3 contains the definitions of the various algebraic structures we employ. We first recall the notion of *modal Kleene algebra* [6]. These are Kleene algebras with forward and backward modal operators defined via domain and codomain operations. Modal Kleene algebras provide an algebraic framework for computational relational models such as abstract rewriting systems, and beyond that for dynamics logics and predicate transformers.

In Section 3.2, we introduce a notion of globular higher-dimensional modal Kleene algebra. First, we define a *0-doid* as a bounded distributive lattice, and for  $n \geq 1$ , an *n-doid* as a family  $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$  of doids, or additively idempotent semirings, satisfying lax interchange laws between the multiplication operations, akin to those of concurrent Kleene algebra [16]. We then extend this structure with domain and codomain operators  $d_i, r_i : S \rightarrow S$  for  $0 \leq i < n$ , satisfying  $d_{i+1} \circ d_i = d_i$ , and  $r_{i+1} \circ r_i = r_i$  for any  $i$ .

The domain and codomain operations induce forward and backward diamond operators  $|\_i, \langle \_i$  in the sense that for any  $x \in S$ ,  $|\_i, \langle \_i$  are modal operators on the *i-dimensional domain algebra*  $S_i := d_i(S)$ . In 3.2.11 we define conditions for *globularity*, conducing to the notion of *globular modal n-doid*.

Finally, we equip these structures with Kleene star operations  $(-)^{*i} : K \rightarrow K$  for each  $0 \leq i < n$ . These are *lax morphisms* with respect to the  $i$ -multiplication of  $j$ -dimensional elements on the right (resp. left), that is for all  $0 \leq i < j < n$ , all elements  $A \in K$  and all  $\phi \in K_j$  in the  $j$ -dimensional domain algebra, we have

$$\phi \odot_i A^{*j} \leq (\phi \odot_i A)^{*j} \quad (\text{resp. } A^{*j} \odot_i \phi \leq (A \odot_i \phi)^{*j}).$$

These structures are called *globular modal n-Kleene algebras*.

In Section 3.3 we relate this structure to polygraphs by providing a model for higher dimensional Kleene algebras in the form of a higher dimensional path algebra  $K(P, \Gamma)$  induced by an  $n$ -polygraph  $P$  and a cellular extension  $\Gamma$ .

**Algebraic coherent confluence.** Section 4 contains the main results of this article. After recalling the formulation of Church-Rosser's theorem and Newman's lemma in modal Kleene algebras in Section 4.1, the notion of *fillers* in a globular modal  $n$ -Kleene algebra  $K$  is defined in Section 4.2.1. Given  $j$ -dimensional elements  $\phi, \psi \in K_j := d_j(K)$ , we say that  $A \in K$  is an *i-confluence filler* (resp. *i-Church-Rosser filler* for  $(\phi, \psi)$ ) if

$$|\_j(\psi^{*i} \odot_i \phi^{*i}) \geq \phi^{*i} \odot_i \psi^{*i} \quad (\text{resp. } |\_j(\psi^{*i} \odot_i \phi^{*i}) \geq (\psi + \phi)^{*i}).$$

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We similarly define a notion of *local i-confluence filler*. A notion of *whiskering* in  $n$ -Kleene algebras is introduced in Section 4.2.3 and some of its properties are made explicit. We also define, for  $\phi, \psi \in K_j$  and an  $i$ -confluence filler  $A \in K$  of  $(\phi, \psi)$ , the  $j$ -dimensional  $i$ -whiskering of  $A$ :

$$\hat{A} := (\phi + \psi)^{*i} \odot_i A \odot_i (\phi + \psi)^{*i}.$$

We then prove two results interpreting the coherent Church-Rosser theorem in the setting of  $n$ -Kleene algebras. The first, Proposition 4.2.7, uses an inductive argument external to the  $n$ -Kleene structure. Given  $0 \leq i < j < n$ , it states that for  $\phi, \psi \in K_j$ , any  $i$ -confluence filler  $A$  of  $(\phi, \psi)$ , and any natural number  $k \geq 0$ , there exists an  $A_k \leq \hat{A}^{*j}$  such that

$$r_j(A_k) \leq \psi^{*i} \phi^{*i} \quad \text{and} \quad d_j(A_k) \geq (\phi + \psi)^{k_i},$$

where  $(\phi + \psi)^{0_i} = 1_i$  and  $(\phi + \psi)^{k_i} = (\phi + \psi) \odot_i (\phi + \psi)^{k_i - 1}$ .

The second interpretation of the Church-Rosser theorem, with a proof relying only on the internal fixpoint induction of the Kleene star, constitutes our first main result:

**Theorem 4.2.8.** *Let  $K$  be a globular  $n$ -modal Kleene algebra and  $0 \leq i < j < n$ . Given  $\phi, \psi \in K_j$  and any  $i$ -confluence filler  $A \in K$  of  $(\phi, \psi)$ , we have*

$$|\hat{A}^{*j}\rangle_j(\psi^{*i} \phi^{*i}) \geq (\phi + \psi)^{*i}.$$

*Thus  $\hat{A}^{*j}$  is an  $i$ -Church-Rosser filler for  $(\phi, \psi)$ .*

In Section 4.3, we introduce notions of termination and well-foundedness in  $n$ -Kleene algebras in which the domain algebras  $K_i$  have a Boolean structure for all  $i \leq p < n$ . This leads to our second main result, a coherent formulation of Newman's lemma in such algebras:

**Theorem 4.3.2.** *Let  $0 \leq i \leq p < j < n$ , and let  $K$  be a globular  $p$ -Boolean modal Kleene algebra such that*

- i)  $(K_i, +, 0, \odot_i, 1_i, \neg_i)$  is a complete Boolean algebra,
- ii)  $K_j$  is continuous with respect to  $i$ -restriction, i.e. for all  $\psi, \psi' \in K_j$  and every family  $(p_\alpha)_{\alpha \in I}$  of elements of  $K_i$  such that  $\sup_I(p_\alpha)$  exists, we have

$$\psi \odot_i \sup_I(p_\alpha) \odot_i \psi' = \sup_I(\psi \odot_i p_\alpha \odot_i \psi').$$

*For any  $\psi \in K_j$   $i$ -Noetherian, and  $\phi \in K_j$   $i$ -well-founded, if  $A$  is a local  $i$ -confluence filler for  $(\phi, \psi)$ , we have*

$$|\hat{A}^{*i}\rangle_j(\psi^{*i} \phi^{*i}) \geq \phi^{*i} \psi^{*i}.$$

*Thus  $\hat{A}^{*i}$  is an  $i$ -confluence filler for  $(\phi, \psi)$ .*

Finally, in Section 4.4, we interpret these results in the context of higher dimensional abstract rewriting, using the higher dimensional path model defined in Section 3.3.

**Toward an algebraic Squier’s theorem.** In this work we provide a formal proofs of the coherent Church-Rosser’s theorem and the coherent Newman’s lemma in higher-dimensional Kleene algebras. These results are the main ingredients in the proof of Squier’s coherence theorem, [25] used in constructive proofs of coherence in categorical algebra. A main objective remains to formalise this result within our algebraic framework. The first main obstacle towards it is the formalisation of the coherent critical branching lemma. This requires taking the algebraic and syntactic nature of terms of the rewriting system into account, which is currently an open problem in the formalism of Kleene algebras. The second difficulty is to capture normalisation strategies algebraically in higher-dimensional Kleene algebras. Squier’s coherence theorem is the first step in the construction of cofibrant replacements of algebraic structures using convergent presentations. We expect that the material introduced in this article will enable us to obtain an algebraic formalisation of acyclicity, which could in turn provide an algebraic criterion for cofibrance.

## 2. PRELIMINARIES ON HIGHER-DIMENSIONAL REWRITING

In this preliminary section, we recall the notions of higher-dimensional rewriting relevant to this article. In its two subsections we recall the definition of polygraphs and their properties as rewriting systems presenting higher-dimensional categories. In Subsection 2.3 we introduce the notion of *confluence filler* for a polygraph with respect to a cellular extension. We then formulate and prove the coherent versions of the Church-Rosser theorem and Neman’s lemma in the point-wise, polygraphic setting. Finally, in its last subsection, we relate the confluence filler property to the coherent confluence property previously introduced in [10].

### 2.1. Polygraphs

**2.1.1. Notations.** Let  $n$  be a natural number or  $\infty$ . For a (strict and globular)  $n$ -category  $\mathcal{C}$ , and  $0 \leq k < n$  we denote by  $\mathcal{C}_k$  the  $k$ -category of  $k$ -cells of  $\mathcal{C}$ . As an abuse of notation, we also write  $\mathcal{C}_k$  for the set of  $k$ -cells of  $\mathcal{C}$ . For a  $k$ -cell  $f$  of  $\mathcal{C}$ , and  $0 \leq i < k$ , we denote by  $s_i(f)$  (resp.  $t_i(f)$ ) the  $i$ -source (resp.  $i$ -target), and by  $1_f$  its identity  $(k+1)$ -cell of  $f$ . The source and target maps  $s_i, t_i : \mathcal{C}_k \rightarrow \mathcal{C}_i$  satisfy the *globular relations*

$$s_i \circ s_{i+1} = s_i \circ t_{i+1}, \quad t_i \circ s_{i+1} = t_i \circ t_{i+1}. \quad (2.1.2)$$

When  $f$  and  $g$  are  $i$ -composable  $k$ -cells, that is when  $t_i(f) = s_i(g)$ , we denote by  $f \star_i g$  their  $i$ -composite. We recall that the composition operations satisfy the exchange relation

$$(f \star_i f') \star_j (g \star_i g') = (f \star_j g) \star_i (f' \star_j g'), \quad (2.1.3)$$

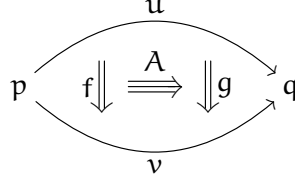
for any  $0 \leq i < j < n$  and whenever the compositions are defined. The  $(k-1)$ -composition of  $k$ -cells  $f$  and  $g$  is denoted by juxtaposition  $fg$ , and the  $(k-1)$ -source  $s_{k-1}(f)$  and the  $(k-1)$ -target  $t_{k-1}(f)$  of a  $k$ -cell  $f$  are denoted by  $s(f)$  and  $t(f)$ , respectively. If we denote by  $f : u \Rightarrow v$  a  $k$ -cell in  $\mathcal{C}$ , we denote by  $u : p \rightarrow q$  the  $(k-1)$ -cells of  $\mathcal{C}$  and by  $A : f \Rrightarrow g$  the  $(k+1)$ -cells of  $\mathcal{C}$  in order to notationally



## 2. Preliminaries on higher-dimensional rewriting

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distinguish their respective dimensions. These globular cells are depicted as follows:



**2.1.4. Identities and whiskers.** Given a  $k$ -cell  $f$ , the  $l$ -dimensional identity on  $f$  for  $k \leq l \leq n$  is denoted by  $\iota_k^l(f)$  and defined by induction, setting  $\iota_k^k(f) := f$  and  $\iota_k^l(f) := 1_{\iota_{k-1}^l}$  for  $k < l \leq n$ . In this way, for  $0 \leq k < l \leq n$ , we associate to a  $k$ -cell  $f$  a unique identity cell  $\iota_k^l(f)$  of dimension  $l$ , called the *l-dimensional identity* on  $f$ .

In higher category theory, the use of such iterated identities is of great importance for defining compositions. Given  $0 \leq i < k < l \leq n$ , a  $k$ -cell  $f$ , and a  $l$ -cell  $g$ , such that  $t_i(f) = s_i(g)$ , the  $i$ -composite of  $f$  and  $g$ , is defined by

$$f \star_i g = \iota_k^l(f) \star_i g,$$

and if  $t_i(g) = s_i(f)$ , we define  $g \star_i f = g \star_i \iota_k^l(f)$ .

For  $0 \leq i < j \leq k$ , an  $(i, j)$ -whiskering of a  $k$ -cell  $f$  is a  $k$ -cell  $\iota_j^k(u) \star_i f \star_i \iota_j^k(v)$ , where  $u$  and  $v$  are  $j$ -cells. To simplify notation, we denote this  $k$ -cell by  $u \star_i f \star_i v$ . A  $(k-1, k-1)$ -whiskering  $1_u \star_{k-1} f \star_{k-1} 1_v$  of a  $k$ -cell  $f$  will be called a *whiskering* of  $f$  and is denoted by  $ufv$ .

**2.1.5.  $(n, p)$ -categories.** If  $\mathcal{C}$  is an  $n$ -category, for  $0 \leq i < k \leq n$ , a  $k$ -cell  $f$  of  $\mathcal{C}$  is  *$i$ -invertible* if there exists a  $k$ -cell  $g$  in  $\mathcal{C}$ , with  $i$ -source  $t_i(f)$  and  $i$ -target  $s_i(f)$  in  $\mathcal{C}$ , called the  *$i$ -inverse* of  $f$ , which satisfies

$$f \star_i g = 1_{s_i(f)} \quad \text{and} \quad g \star_i f = 1_{t_i(f)}.$$

The  $i$ -inverse of a  $k$ -cell is necessarily unique. When  $i = k-1$ , we say that  $f : u \rightarrow v$  is *invertible* and we denote by  $f^- : v \rightarrow u$  its  $(k-1)$ -inverse, simply called *inverse* for short. If moreover the  $(k-1)$ -cells  $u$  and  $v$  are invertible, then there exist  $k$ -cells

$$u^- \star_{k-2} f^- \star_{k-2} v^- : u^- \rightarrow v^-, \quad v^- \star_{k-2} f^- \star_{k-2} v^- : u^- \rightarrow v^-$$

in  $\mathcal{C}$ . For a natural number  $p \leq n$ , or for  $p = n = \infty$ , an  $(n, p)$ -category is an  $n$ -category whose  $k$ -cells are invertible for every  $k > p$ . When  $n < \infty$ , this is a  $p$ -category enriched in  $(n-p)$ -groupoids and, when  $n = \infty$ , a  $p$ -category enriched in  $\infty$ -groupoids.

**2.1.6. Spheres, asphericity and cellular extensions.** Let  $\mathcal{C}$  be an  $n$ -category. A *0-sphere* of  $\mathcal{C}$  is a pair of 0-cells of  $\mathcal{C}$ . For  $1 \leq k \leq n$ , a *k-sphere* of  $\mathcal{C}$  is a pair  $(f, g)$  of  $k$ -cells such that  $s_{k-1}(f) = s_{k-1}(g)$  and  $t_{k-1}(f) = t_{k-1}(g)$ . We denote by  $\text{Sph}_k(\mathcal{C})$  the set of  $k$ -spheres of  $\mathcal{C}$ .

When  $n < \infty$ , the  $n$ -category  $\mathcal{C}$  is *aspherical* if any  $n$ -sphere of  $\mathcal{C}$  is of the form  $(f, f)$ , with  $f$  in  $\mathcal{C}_n$ . A *cellular extension* of  $\mathcal{C}$  is a set  $\Gamma$  equipped with a map  $\partial : \Gamma \rightarrow \text{Sph}_n(\mathcal{C})$ . For  $\alpha \in \Gamma$ , the *boundary* of the sphere  $\partial(\alpha)$  is denoted  $(s_n(\alpha), t_n(\alpha))$ , defining in this way two maps  $s_n, t_n : \Gamma \rightarrow \mathcal{C}_n$  satisfying the following globular relations

$$s_{n-1} \circ s_n = s_{n-1} \circ t_n \quad \text{and} \quad t_{n-1} \circ s_n = t_{n-1} \circ t_n.$$

The free  $(n + 1)$ -category generated by  $\Gamma$  over  $\mathcal{C}$  is the  $(n + 1)$ -category, denoted by  $\mathcal{C}[\Gamma]$ , whose underlying  $n$ -category is  $\mathcal{C}$  and whose  $(n + 1)$ -cells are built as formal  $i$ -compositions, for  $0 \leq i \leq n$ , of elements of  $\Gamma$  and  $k$ -cells of  $\mathcal{C}$ , seen as  $(n + 1)$ -cells with source and target in  $\mathcal{C}_n$ . The free  $(n + 1, n)$ -category generated by  $\Gamma$  over  $\mathcal{C}$  is denoted by  $\mathcal{C}(\Gamma)$ . We refer the reader to [22] for explicit free constructions on cellular extensions over an  $n$ -category.

**2.1.7.  $n$ -polygraph.** An  $n$ -polygraph  $P$  consists of a set  $P_0$  and for every  $0 \leq k < n$  a cellular extension  $P_{k+1}$  of the free  $k$ -category

$$P_k^* = P_0[P_1] \dots [P_k].$$

For  $0 \leq k \leq n$ , the elements of  $P_k$  are called the *generating  $k$ -cells* of  $P$ . The free  $n$ -category  $P_0[P_1] \dots [P_{n-1}][P_n]$  (resp. the free  $(n, n - 1)$ -category  $P_0[P_1] \dots [P_{n-1}](P_n)$ ) generated by  $P$  is denoted by  $P_n^*$  (resp.  $P_n^\top$ ). We refer to [22] for the details of the free constructions on an  $n$ -polygraph. Note that a 0-polygraph is a set and an 1-polygraph corresponds to a directed graph, whose set of vertices is  $P_0$  and  $P_1$  is the set of arrows  $f$  with source  $s_0(f)$  and target  $t_0(f)$ .

**2.1.8. Examples of low-dimensional polygraphs.** Low-dimensional polygraphs describe abstract rewriting systems. Recall that an *abstract rewriting system* consists of a set  $X$  and a family  $\rightarrow = \{\rightarrow_i\}_{i \in I}$  of binary relations on  $X$ , i.e.  $\rightarrow_i \subseteq X \times X$  for all  $i \in I$ . We refer the reader to [31] for a complete treatment on abstract rewriting. An abstract rewriting system  $\mathcal{A} = (X, \{\rightarrow_i\}_{i \in I})$ , can be described by a 1-polygraph, denoted  $P(\mathcal{A})$ , whose set of 0-cells is  $X$ , and whose set of 1-cells consists of

$$u_{(x,y,i)} : x \rightarrow y$$

for any  $x, y \in X$  and  $i \in I$  such that  $(x, y) \in \rightarrow_i$ . When  $I$  is a singleton, the 1-cells of the free 1-category  $P(\mathcal{A})^*$  correspond to the elements of the reflexive and transitive closure  $\xrightarrow{*}$  of the relation  $\rightarrow$ . Moreover the 1-cells of the free  $(1, 0)$ -category  $P(\mathcal{A})^\top$  correspond to the elements of the symmetric closure of the relation  $\rightarrow^*$ .

A *string rewriting system* is an abstract rewriting system on a free monoid [2], and can be described by a 2-polygraph with a single 0-cell. Finally, 3-polygraphs describe term rewriting systems [9] or three-dimensional rewriting [23].

## 2.2. Rewriting properties of polygraphs

**2.2.1. Polygraphic rewriting.** A *rewriting step* for an  $n$ -polygraph  $P$  is an  $n$ -cell of the  $n$ -category  $P_n^*$  of the form

$$u_{n-1} \star_{n-2} (u_{n-2} \star_{n-3} \dots \star_2 (u_2 \star_1 (u_1 \star_0 f \star_0 v_1) \star_1 v_2) \star_2 \dots \star_{n-3} v_{n-2}) \star_{n-2} v_{n-1},$$

for a generating  $n$ -cell  $f \in P_n$  and  $i$ -cells  $u_i, v_i$  with  $1 \leq i < n$ . We denote by  $P_n^c$  the set of rewriting steps of  $P$ . A *rewriting path in  $P$  of length  $k$*  is an  $(n - 1)$ -composition

$$f_1 \star_{n-1} f_2 \star_{n-1} \dots \star_{n-1} f_k$$

of rewriting steps of  $P$ . A *zig-zag in  $P$  of length  $k$*  is an  $(n - 1)$ -composition

$$f_1^{\epsilon_1} \star_{n-1} f_2^{\epsilon_2} \star_{n-1} \dots \star_{n-1} f_k^{\epsilon_k}$$

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of rewriting steps for  $P$ , where  $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$ , and which is reduced with respect the reduction  $f \star_{n-1} f^- \rightarrow 1$ .

The set of rewriting steps induces an abstract rewriting system on the set of  $(n-1)$ -cells of  $P_n^*$  denoted by  $\rightarrow_{P_n}$ , and defined by  $u \rightarrow_{P_n} u'$  if there exists a rewriting step for  $P$  that reduces  $u$  to  $u'$ . In this case, we say that  $u$  *rewrites* to  $u'$ . An  $(n-1)$ -cell  $u$  in  $P_n$  is *irreducible* with respect to  $P$  if there is no rewriting step for  $P$  that reduces  $u$ .

**2.2.2. Remark.** Given a cellular extension  $\Gamma$  of an  $n$ -category  $\mathcal{C}$ , we also denote by  $\Gamma^c$  the set of *cells of  $\Gamma$  in context*, that is the set of  $(n+1)$ -cells of the form

$$f_n \star_{n-1} \dots \star_2 (f_2 \star_1 (f_1 \star_0 \alpha \star_0 g_1) \star_1 g_2) \star_2 \dots \star_{n-1} g_n,$$

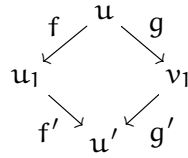
where  $f_i, g_i$  are  $i$ -cells of  $\mathcal{C}$  for  $0 \leq i \leq n$ , and  $\alpha \in \Gamma$ . Any  $(n+1)$ -cell in the free  $(n+1)$ -category  $\mathcal{C}[\Gamma]$  can be written as an  $n$ -composition of elements of  $\Gamma^c$  using the algebraic laws of higher categories, most notably the exchange relation. In particular, this means that the  $n$ -cells of  $P_n^*$  correspond to the reflexive, transitive closure of  $\rightarrow_{P_n}$  in the sense that given an  $n$ -cell  $f$  of  $P_n^*$ , we have

$$f = f_1 \star_{n-1} f_2 \star_{n-1} \dots \star_{n-1} f_k,$$

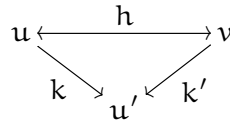
where  $f_i \in P_n^c$ .

**2.2.3. Rewriting properties of an  $n$ -polygraph.** Let  $\mathcal{P}$  be a rewriting property defined on abstract rewriting systems. A polygraph  $P$  *has the property  $\mathcal{P}$*  if the abstract rewriting system  $\rightarrow_{P_n}$  has the property  $\mathcal{P}$ . In particular,  $P$  is *terminating* if there is no infinite rewriting path for  $P$ .

A *branching* in  $P$  is an unordered pair  $(f, g)$  of rewriting paths of  $P$  such that  $s_{n-1}(f) = s_{n-1}(g)$ . Such a branching is *local* when  $f$  and  $g$  are rewriting steps. We say that  $P$  is *confluent* (resp. *locally confluent*) if for any branching (resp. local branching)  $(f, g)$ , there exist rewriting paths  $f', g'$  of  $P$  with  $t_{n-1}(f') = t_{n-1}(g')$  such that the compositions  $f \star_{n-1} f'$  and  $g \star_{n-1} g'$  are defined, as illustrated in the following diagram:



The *source* of a branching  $(f, g)$  is the common  $(n-1)$ -source of  $f$  and  $g$ . We say that  $P$  is *Church-Rosser* if for any zig-zag  $h$  of  $P$ , there exist rewriting paths  $k, k'$  of  $P$  as in the following diagram:



## 2.3. Coherent confluence

**2.3.1. Coherent confluence.** Let  $P$  be an  $n$ -polygraph and  $\Gamma$  a cellular extension of  $P_n^\top$ .

The cellular extension  $\Gamma$  is a *confluence filler* of a branching  $(f, g)$  of  $P$  if there exist rewriting paths  $f', g'$  of  $P$  as in (2.3.2), and two  $(n+1)$ -cells  $\alpha, \alpha'$  in the free  $(n+1)$ -category  $P_n^\top[\Gamma]$  of the form  $\alpha : f^- \star_{n-1} g \rightarrow f' \star_{n-1} (g')^-$  and  $\alpha' : g^- \star_{n-1} f \rightarrow g' \star_{n-1} (f')^-$ :

$$\begin{array}{ccc}
 \begin{array}{ccc} f & u & g \\ \swarrow & & \searrow \\ u_1 & & v_1 \\ \searrow & & \swarrow \\ f' & u' & g' \end{array} & 
 \begin{array}{ccc} f^- & u & g \\ \swarrow & \Downarrow \alpha & \searrow \\ u_1 & & v_1 \\ \searrow & & \swarrow \\ f' & u' & (g')^- \end{array} & 
 \begin{array}{ccc} f & u & g^- \\ \swarrow & & \searrow \\ u_1 & & v_1 \\ \searrow & \Downarrow \alpha' & \swarrow \\ (f')^- & u' & g' \end{array}
 \end{array} \quad (2.3.2)$$

In this case,  $\alpha$  and  $\alpha'$  are  $n$ -compositions of  $(n+1)$ -cells of  $\Gamma^c$  as recalled in Remark 2.2.2. We say that the cellular extension  $\Gamma$  is a *confluence filler* for the polygraph  $P$  if  $\Gamma$  is a confluence filler for each of its branchings.

More generally, the cellular extension  $\Gamma$  is a *confluence filler* of an  $n$ -cell  $f$  in  $P_n^\top$  if there exist  $n$ -cells  $f'$  and  $g'$  in  $P_n^*$  and an  $(n+1)$ -cell  $\alpha$  in the free  $(n+1)$ -category  $P_n^\top[\Gamma]$  of the form  $\alpha : f \rightarrow f' \star_{n-1} g'^-$ :

$$\begin{array}{ccc}
 \begin{array}{ccc} u & \xrightarrow{f} & v \\ \searrow & & \swarrow \\ f' & u' & g' \end{array} & 
 \begin{array}{ccc} u & \xrightarrow{f} & v \\ \searrow & \Downarrow \alpha & \swarrow \\ f' & u' & (g')^- \end{array}
 \end{array} \quad (2.3.3)$$

The cellular extension  $\Gamma$  is a *Church-Rosser filler* for an  $n$ -polygraph  $P$  when it is a confluence filler of every  $n$ -cell in  $P_n^\top$ .

**2.3.4. Theorem (Church-Rosser coherent filler lemma).** *Let  $P$  be an  $n$ -polygraph. A cellular extension  $\Gamma$  of  $P_n^\top$  is a confluence filler for  $P$  if, and only if,  $\Gamma$  is a Church-Rosser filler for  $P$ .*

*Proof.* First suppose that  $\Gamma$  is a Church-Rosser filler for  $P$ . Given a branching  $(f, g)$ , we have that  $f^- \star_{n-1} g$  and  $g^- \star_{n-1} f$  are elements of  $P_n^\top$  and thus  $\Gamma$  is a confluence filler for these  $n$ -cells. This gives us the cells  $\alpha$  and  $\alpha'$  as in (2.3.2), and so  $\Gamma$  is a confluence filler for  $P$ .

Conversely, suppose that  $\Gamma$  is a confluence filler for  $P$ , and let  $f \in P_n^\top$  be an  $n$ -cell. We prove by induction on the length of  $f$  that  $\Gamma$  is a Church-Rosser filler for  $P$ . For  $f$  of length 0 or 1, we clearly have that  $f$  is  $\Gamma$ -confluent, since it suffices to take an identity  $(n+1)$ -cell. Suppose that every  $n$ -cell of length  $k \geq 2$  is  $\Gamma$ -confluent and that  $f$  is of length  $k+1$ . Then  $f = f_1 \star_{n-1} f_2$  with  $f_1 : u \rightarrow u_1$  in  $P_n^\top$  of length  $k$  and  $f_2$  is of length 1 in  $P_n^*$  either of the form  $v \rightarrow u_1$  or  $u_1 \rightarrow v$ . By the induction hypothesis there exist rewriting paths  $h$  and  $k$  and an  $(n+1)$ -cell  $\alpha$  such that  $\alpha : f \Rightarrow hk^-$ . If  $f_2 : u_1 \rightarrow v$ , there exist rewriting paths  $k'$  and  $f''$  and an  $(n+1)$ -cell  $\beta$  as depicted in diagram (2.3.5) since  $\Gamma$  is a confluence filler for  $P$ . Thus  $(\alpha f_2) \star_n (h\beta)$  is a confluence filler for  $f$ .

$$\begin{array}{ccccc}
 u & \xleftarrow{f_1} & u_1 & \xrightarrow{f_2} & v \\
 \searrow h & & \swarrow k^- & & \swarrow f''^- \\
 & & u' & \xrightarrow{k'} & u'' \\
 & & & & \swarrow f''^-
 \end{array} \quad (2.3.5)$$

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If  $f_2 : v \rightarrow u_1$ , the  $(n + 1)$ -cell  $\alpha f_2^- \star_n h 1_{k^-(f_2)^-} = \alpha f_2^-$  is a confluence filler for  $f$ .

$$\begin{array}{ccccc}
 & & f_1 & & (f_2)^- \\
 & & \longrightarrow & & \longrightarrow \\
 u & & & u_1 & \longrightarrow v \\
 & \searrow h & \Downarrow \alpha & \nearrow k^- & \Downarrow 1_{k^-(f_2)^-} \\
 & & u' & & u'' \\
 & & \xrightarrow{(f_2)^-} & & 
 \end{array} \quad (2.3.6)$$

□

**2.3.7. Theorem (Coherent Newman filler lemma).** *Let  $P$  be a terminating  $n$ -polygraph, and  $\Gamma$  a cellular extension of  $P_n^\top$ . Then  $\Gamma$  is a local confluence filler, if, and only if,  $\Gamma$  is a confluence filler for  $P$ .*

*Proof.* Firstly, observe that if  $\Gamma$  is a confluence filler for  $P$ , then it is also a local confluence filler for  $P$  since local branchings are also branchings.

Now suppose that  $\Gamma$  is a local confluence filler for  $P$ . We prove by Noetherian induction that, for every  $(n - 1)$ -cell  $u$  of  $P_n^*$ ,  $\Gamma$  is a confluence filler for every branching of  $P$  with source  $u$ . For the base case, if  $u$  is irreducible for  $P$ , then  $(1_u, 1_u)$  is the only branching with source  $u$ , and it is  $\Gamma$ -confluent, taking the  $(n + 1)$ -cell  $1_{1_u}$ .

Suppose now the induction hypothesis, namely that  $u$  is a reducible  $(n - 1)$ -cell of  $P_n^*$  and that  $\Gamma$  is a confluence filler for every branching with source an  $(n - 1)$ -cell  $u'$  such that  $u$  rewrites to  $u'$ . Let  $(f, g)$  be a branching of  $P$  with source  $u$ . If one of  $f$  or  $g$  is an identity, say  $f$ , then  $\Gamma$  is a confluence filler for  $(f, g)$  by considering the  $(n + 1)$ -cells  $1_g$  and  $1_{g^-}$ . We may now suppose that the  $n$ -cells  $f$  and  $g$  are not identities, thus we write  $f = f_1 \star_{n-1} f_2$  and  $g = g_1 \star_{n-1} g_2$ , where  $g_1, f_1$  are rewriting steps and  $g_2, f_2$  are  $n$ -cells of  $P_n^*$ . Since  $\Gamma$  is a local confluence filler for  $P$ , there exist  $n$ -cells  $f'_1, g'_1$  in  $P_n^*$ , and an  $(n + 1)$ -cell  $\alpha$  in  $P_n^*[\Gamma]$  as in the diagram (2.3.8). We apply the induction hypothesis to the branching  $(f_2, f'_1)$ , which yields  $n$ -cells  $f'_2, h$  in  $P_n^*$  and an  $(n + 1)$ -cell  $\beta$  in  $P_n^*[\Gamma]$  as in the diagram (2.3.8). Finally, we apply the induction hypothesis again to the branching  $(g'_1 \star_{n-1} h, g_2)$  yielding  $n$ -cells  $k$  and  $g'_2$  and an  $(n + 1)$ -cell  $\gamma$  in  $P_n^*[\Gamma]$  as in (2.3.8).

$$\begin{array}{ccccccc}
 & & & & u & & \\
 & & & & \nearrow f_1^- & & \searrow g_1 \\
 & & & & u_1 & & v_1 \\
 & \nearrow f_2 & & \searrow f'_1 & \Downarrow \alpha & \nearrow (g'_1)^- & \searrow g_2 \\
 u_2 & & & & u' & & v_2 \\
 & \searrow f'_2 & & \nearrow h^- & \Downarrow \beta & \nearrow \gamma & \\
 & & & & u'_2 & & \\
 & & & & \searrow k & & \nearrow (g'_2)^- \\
 & & & & u'' & & 
 \end{array} \quad (2.3.8)$$

The  $n$ -composition

$$\delta = (((f_2^- \star_{n-1} \alpha) \star_n (\beta \star_{n-1} (g'_1)^-)) \star_{n-1} g_2) \star_n (f'_2 \star_{n-1} \gamma) \quad (2.3.9)$$

is an  $(n + 1)$ -cell in  $P_n^*[\Gamma]$  with source  $f \star_{n-1} g$  and target  $f'_2 \star_{n-1} k \star_{n-1} (g'_2)^-$ . We can similarly find an  $(n + 1)$ -cell  $\delta'$  with source  $g \star_{n-1} f$  and with target a confluence.  $\Gamma$  is thus a confluence filler for  $P$ , which proves the result.  $\square$

**2.3.10. Remark.** Readers more familiar with abstract rewriting than higher dimensional rewriting may notice that the proofs of Theorems 2.3.4 and 2.3.7 are similar to the classical proofs of these results for abstract rewriting systems. Indeed, if we forget the  $(n + 1)$ -dimensional coherence cells and look only at their  $n$ -dimensional borders in (2.3.5), (2.3.6) and (2.3.8), we obtain precisely the diagrams used to prove the analogous 1-dimensional results for abstract rewriting systems. This shows that the higher dimensional approach is consistent with the abstract case while providing several advantages. Firstly, using explicit witnesses for confluence allows for a constructive formulation of classical results in the form of *normalisation strategies*. Furthermore, since the higher-dimensional cells may be considered as rewriting systems in their own right, and since the procedures describe the above work in any dimension, higher-dimensional rewriting provides a constructive method for calculating resolutions and cofibrant replacements of algebraic structures.

## 2.4. $\Gamma$ -confluence and filling

Recall from [10] that, given an  $n$ -polygraph  $P$  and a cellular extension of  $P_n^*$ , we say that  $P$  is  $\Gamma$ -confluent (resp.  $\Gamma$ -locally confluent) if for any branching (resp. local branching)  $(f, g)$  of  $P$  there exist  $n$ -cells  $f', g'$  in the free  $n$ -category  $P_n^*$  as in (2.3.2), and an  $(n + 1)$ -cell  $\alpha$  in the free  $(n + 1, n)$ -category  $P_n^*(\Gamma)$  of the form  $\alpha : f \star_{n-1} f' \rightarrow g \star_{n-1} g'$ . Similarly, we say that  $P$  is  $\Gamma$ -Church-Rosser if for any  $n$ -cell  $f$  of  $P_n^\top$  there exist  $n$ -cells  $f', g'$  in the free  $n$ -category  $P_n^*$  as in 2.3.3, and an  $(n + 1)$ -cell  $\alpha$  in the free  $(n + 1, n)$ -category  $P_n^*(\Gamma)$  of the form  $\alpha : f \star_{n-1} f' \rightarrow g'$ . Note that when  $\Gamma = \text{Sph}(P_n^*)$ , the property of (local)  $\Gamma$ -confluence coincides with the property of (local) confluence of  $P$  as defined in (2.2.1), and the property of  $\Gamma$ -Church-Rosser coincides with the Church-Rosser property of  $P$ .

Theorems 2.3.4 and 2.3.7, formulated above in terms of fillers, are expressed in terms of  $\Gamma$ -confluence as follows:

**2.4.1. Theorem (Church-Rosser coherent lemma).** *Let  $P$  be an  $n$ -polygraph, and  $\Gamma$  be a cellular extension of  $P_n^*$ . If  $P$  is  $\Gamma$ -confluent, then  $P$  is  $\Gamma$ -Church-Rosser.*

*Proof.* The proof is similar to that of Theorem 2.3.4, but with the  $(n + 1)$ -cells oriented horizontally in the induction step, as pictured in the following diagram:

$$\begin{array}{ccccc}
 u & \xleftarrow{f} & & v & \xrightarrow{f'} \\
 & \xrightarrow{\alpha} & k & \xrightarrow{\beta} & \\
 & \searrow h & & & \searrow f'' \\
 & & u' & \xrightarrow{k'} & v'
 \end{array} \tag{2.4.2}$$

The composite  $(\alpha \star_{n-1} k') \star_n (f \star_{n-1} \beta)$  makes the  $n$ -cell  $f$   $\Gamma$ -confluent.  $\square$

**2.4.3. Theorem (Coherent Newman lemma).** *Let  $P$  be a terminating  $n$ -polygraph, and  $\Gamma$  a cellular extension of  $P_n^*$ . If  $P$  is locally  $\Gamma$ -confluent, then  $P$  is  $\Gamma$ -confluent.*

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*Proof.* The proof is similar to that of Theorem 2.3.7, but with the following induction diagram:

$$\begin{array}{ccccc}
 & & u & & \\
 & f_1 \swarrow & & \searrow g_1 & \\
 & u_1 & \xrightarrow{\alpha} & v_1 & \\
 f_2 \swarrow & & f'_1 \searrow & & g_2 \searrow \\
 u_2 & \xrightarrow{\beta} & u' & & v_2 \\
 f'_2 \swarrow & & h \searrow & & \gamma \xrightarrow{\quad} \\
 & u'_2 & & & \\
 & k \searrow & & & g'_2 \searrow \\
 & u'' & & & 
 \end{array} \tag{2.4.4}$$

Then the following  $n$ -composition

$$\delta = (((f_1 \star_{n-1} \beta) \star_n (\alpha \star_{n-1} h)) \star_{n-1} k) \star_n (g_1 \star_{n-1} \gamma) \tag{2.4.5}$$

is an  $(n + 1)$ -cell in  $P_n^*(\Gamma)$  with source  $f \star_{n-1} (f'_2 \star_{n-1} k)$  and target  $g \star_{n-1} g'_2$ , proving the result.  $\square$

Note that for  $\Gamma = \text{Sph}(P_n^*)$ , Theorems 2.4.3 and 2.4.1 correspond to Newman's lemma and the Church-Rosser theorem [24], see also [18].

**2.4.6. Remarks.** In this section, we have defined two approaches to coherence properties of an  $n$ -polygraph  $P$  with respect to a cellular extension  $\Gamma$ :

- i) A "vertical" approach in which the coherence cells, *i.e.* the  $(n + 1)$ -cells generated by  $\Gamma$ , have a branching as  $n$ -source and a confluence as  $n$ -target. This necessitates having inverses of  $n$ -cells, that is  $\Gamma$  is a cellular extension of  $P_n^\top$ . However, in the proofs of Theorems 2.3.4 and 2.3.7, we do not need inverses of  $(n + 1)$ -cells.
- ii) A "horizontal" approach in which coherence cells have rewriting paths for both source and target, and we do not need inverses of  $n$ -cells, *i.e.* we consider cellular extensions of  $P_n^*$ , only inverses of  $(n + 1)$ -cells in order to prove Theorems 2.4.1 and 2.4.3.

These differences can be summed up by saying that, in the first approach, the proofs take place in  $P_n^\top[\Gamma]$ , whereas, in the second one, the proofs take place in  $P_n^*(\Gamma)$ .

Furthermore, it is worth noting that, in the first approach, we specify two filler cells  $\alpha$  and  $\alpha'$  as depicted in diagram 2.3.2 for each branching  $(f, g)$ . This is due to the fact that branchings are unordered pairs, so we must account for both cases. This equally constitutes the reason we require inverses of  $(n + 1)$ -cells in the second approach.

In the rest of this article, we will exclusively consider the "vertical" approach to paving diagrams with higher dimensional cells. The motivation of this choice lies in the fact that with Kleene algebras, we pave diagrams from a *relational* rather than a *polygraphic* point of view. We thus follow the direction of the  $n$ -cells in branchings and confluences—*i.e.* "vertically". This is a consequence of the quantification on branchings and confluences: we quantify *universally* over branchings and *existentially* over confluences. In the polygraphic approach, this quantification is hidden by specifying the  $(n + 1)$ -cells filling confluence diagrams.

### 3. HIGHER DIMENSIONAL MODAL KLEENE ALGEBRAS

In this section we introduce the notion of higher-dimensional globular modal Kleene algebra. In its first subsection, we recall the axioms of *modal Kleene algebra* [6]. We then define  $n$ -dimensional dioids and equip these with domain and star operations, thus obtaining *modal  $n$ -Kleene algebras*. Finally, we provide a model of this structure in the form of a higher-dimensional path algebra associated to an  $n$ -polygraph with a cellular extension  $\Gamma$ .

#### 3.1. Modal Kleene algebras

**3.1.1. Semirings.** Recall that a *semiring* is a tuple  $(S, +, 0, \cdot, 1)$  made of a set  $S$  and two binary operations  $+$  and  $\cdot$  such that  $(S, +, 0)$  is a commutative monoid,  $(S, \cdot, 1)$  is a monoid whose *multiplication operation*  $\cdot$  distributes on the left and the right over the *addition operation*  $+$ , and  $0$  is a left and right zero for multiplication.

A *dioid* is a semiring  $S$  in which addition is idempotent, *i.e.* for all  $x \in S$ , we have  $x + x = x$ . In this case, the relation defined by

$$x \leq y \iff x + y = y, \tag{3.1.2}$$

for all  $x, y \in S$ , is a partial order on  $S$ , with respect to which addition and multiplication are monotone, and  $0$  is minimal. Where there is no possible confusion, we will often denote multiplication simply by juxtaposition. A *bounded distributive lattice* is a dioid  $(S, +, 0, \cdot, 1)$ , whose multiplication  $\cdot$  is commutative and idempotent, and  $x \leq 1$ , for every  $x \in S$ .

**3.1.3. Domain semirings.** Recall from [6] that a *domain semiring* is a dioid  $(S, +, \cdot, 0, 1)$  equipped with a *domain operation*  $d : S \rightarrow S$  that satisfies the following five axioms for all  $x, y \in S$ :

- i)  $x \leq d(x)x$ ,
- ii)  $d(xy) = d(xd(y))$ ,
- iii)  $d(x) \leq 1$ ,
- iv)  $d(0) = 0$ ,
- v)  $d(x + y) = d(x) + d(y)$ ,

These structures are called *domain semirings* and not *domain dioids* because a semiring equipped with a domain operation is automatically idempotent. Consequences of the axioms of domain semirings include the fact that the image of  $S$  under  $d$  is precisely the set of fixpoints of  $d$ , *i.e.*

$$S_d := \{x \in S \mid d(x) = x\} = d(S),$$

and that  $S_d$  forms a distributive lattice with  $+$  as join and  $\cdot$  as meet, bounded by  $0$  and  $1$ . It contains the largest Boolean subalgebra of  $S$  bounded by  $0$  and  $1$ . We henceforth write  $p, q, r, \dots$  for elements of  $S_d$  and refer to  $S_d$  as the *domain algebra* of  $S$ . Moreover,  $S_d$  is a subsemiring of  $S$  in the sense that its element satisfy the semiring axioms,  $0$  and  $1$  are in the set, and the set is closed with respect to  $\cdot$  and  $+$ . Further properties include

$$d(0) = 0, \quad d(px) = pd(x), \quad x \leq y \Rightarrow d(x) \leq d(y),$$

for all  $x, y \in S_d$ , and  $d$  commutes with all existing sups [6].



### 3. Higher dimensional modal Kleene algebras

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**3.1.4. Boolean domain semirings.** A limitation of domain semirings is that complementation in  $S_d$  cannot be expressed. This requires a notion of antidomain that abstractly describes those elements that are *not* in the domain of a particular element. Recall from [6] that a *Boolean domain semiring* is a dioid  $(S, +, \cdot, 0, 1)$  equipped with an *antidomain operation*  $ad : S \rightarrow S$  that satisfies the following three axioms, for all  $x, y \in S$ :

- i)  $ad(x)x = 0$ ,
- ii)  $ad(xy) \leq ad(x ad^2(y))$ ,
- iii)  $ad^2(x) + ad(x) = 1$ .

Setting  $d = ad^2$ , we recover a domain semiring, that is,  $d$  satisfies the domain semiring axioms. In the presence of the operation  $ad$ , the subalgebra  $S_d$  of all fixpoints of  $d$  in  $S$  is now the greatest Boolean algebra in  $S$  bounded by 0 and 1; we have that  $S_d = ad(S)$  and  $ad$  acts as Boolean complementation on  $S_d$ . We therefore denote the restriction of  $ad$  to  $S_d$  by  $\neg$ .

**3.1.5. Modal semirings.** We denote the *opposite* of a semiring  $S$ , in which the order of multiplication has been reversed, by  $S^{op}$ . It is once again a semiring. A *codomain* (resp. *Boolean codomain*) *semiring* is a semiring equipped with a map  $r : S \rightarrow S$  (resp.  $ar : S \rightarrow S$ ) such that  $(S^{op}, r)$  (resp.  $(S^{op}, ar)$ ) is a domain (resp. Boolean domain) semiring.

Consider a semiring equipped with a domain operator and a codomain operation. The domain and range axioms alone do not imply that  $S_d = S_r$ , let alone the compatibility properties

$$d(r(x)) = r(x), \quad r(d(x)) = d(x), \quad (3.1.6)$$

for every  $x$  in  $S$ . Indeed, consider the domain and range semiring  $S = (\{a\}, +, \cdot, 0, 1, d, r)$  with addition defined by  $0 < a < 1$ , multiplication by  $a^2 = a$ , domain by  $d a = 1$  and codomain by  $r a = a$ . We have  $S_d = \{0, 1\} \neq \{0, a, 1\} = S_r$  and  $d(r a) = 1 \neq a = r a$ . The identity  $r \circ d = d$  fails in the opposite semiring.

Recall from [6] that a *modal semiring*  $S$  is a domain semiring that is also a codomain semiring, and satisfying the compatibility properties (3.1.6). Boolean domain semirings that are also Boolean codomain semirings are called *Boolean modal semirings*. In this case, maximality of  $S_d$  and  $S_r = \{x \in S \mid r(x) = x\}$  forces the domain and range algebra of  $S$  to coincide, so that the extra axioms (3.1.6) are not necessary. We provide a formal proof, as this fact has so far been overlooked.

**3.1.7. Lemma.** *In every Boolean modal semiring the compatible properties (3.1.6) hold.*

*Proof.* Let  $S$  be a Boolean modal semiring, and  $x$  in  $S$ . Then

$$\begin{aligned} d(r(x)) &= (ar(x) + r(x))d(r(x)) \\ &= ar(x)d(r(x)) + r(x)d(r(x))(ar(x) + r(x)) \\ &= 0 + r(x)d(r(x))ar(x) + r(x)d(r(x))r(x) \\ &= 0 + r(x)r(x) = r(x), \end{aligned}$$

proving the first equality in (3.1.6). In the third step,  $ar(x)d(r(x)) = 0$  because  $ar(x)r(x) = 0$  and  $yz = 0 \Leftrightarrow yd(z) = 0$  hold in any Boolean modal semiring. In the fourth step,  $r(x)d(r(x))ar(x) = 0$

because  $d(r(x)) \leq 1$  and again  $ar(x)r(x) = 0$ . Moreover  $r(x)d(r(x))r(x) = r(x)r(x)$  because  $d(y)y = y$  holds in any modal semiring. The proof of the second equality in (3.1.6) follows by opposition.

It then follows that  $S_d = S_r$ : if  $d(x) = x$ , then  $r(x) = r(d(x)) = d(x) = x$ ; and  $r(x) = x$  implies  $d(x) = x$  by opposition. This forces that  $S_d = S_r = \{x \in S \mid r(x) = x\}$ .  $\square$

**3.1.8. Modal Kleene algebras.** A *Kleene algebra* is a dioid  $K$  equipped with an operation  $(-)^* : K \rightarrow K$  called *Kleene star*, satisfying the following axioms. For all  $x, y, z \in K$ ,

i) (*unfold axioms*)  $1 + xx^* \leq x^*$  and  $1 + x^*x \leq x^*$ ,

ii) (*induction axioms*)  $z + xy \leq y \Rightarrow x^*z \leq y$  and  $z + yx \leq y \Rightarrow zx^* \leq y$ .

Note that the axioms on the left are the opposites of those on the right. Useful consequences of Axioms i) and ii) include the following identities for all  $x, y \in K$ , and  $i \in \mathbb{N}$ ,

$$x^i \leq x^* \quad x^*x^* = x^* \quad x^{**} = x^* \quad x(yx)^* = (xy)x^* \quad (x + y)^* = x^*(yx^*)^* = (x^*y^*)^*,$$

where  $x^i$  denotes the  $i$ -fold multiplication of  $x$  with itself, as well as the quasi-identities

$$x \leq 1 \Rightarrow x^* = 1 \quad x \leq y \Rightarrow x^* \leq y^* \quad xz \leq zy \Rightarrow x^*z \leq zy^* \quad zx \leq yz \Rightarrow zx^* \leq y^*z.$$

The *Kleene plus* is the operation  $(-)^+ : K \rightarrow K$  defined by  $x^+ = xx^*$ .

The above notions of domain and codomain extend to Kleene algebras without having to add any further axioms. We thus define a (*Boolean*) *modal Kleene algebra* as a Kleene algebra that is also a (*Boolean*) modal-semiring.

**3.1.9. Modal Operators.** Let  $(S, +, \cdot, 0, 1, d, r)$  be a modal semiring. For  $x \in S$  and  $p \in S_d$ , we define the modal diamond operators:

$$|x\rangle p = d(xp), \quad \langle x|p = r(px). \quad (3.1.10)$$

When  $S$  is a Boolean modal semiring, we additionally define modal box operators:

$$[x]p = \neg|x\rangle(\neg p), \quad [x|p = \neg\langle x|(\neg p). \quad (3.1.11)$$

These are modal operators in the sense of Boolean algebras with operators [19] because the following identities hold:

$$|x\rangle(p + q) = |x\rangle p + |x\rangle q, \quad |x\rangle 0 = 0, \quad \langle x|(p + q) = \langle x|p + \langle x|q, \quad \langle x|0 = 0,$$

and

$$[x](pq) = [x]p + [x]q, \quad [x]1 = 1, \quad [x|(pq) = [x|p + [x|q, \quad [x|1 = 1.$$

It is easy to see that  $|-\rangle$  and  $\langle -|$ , as well as  $|-\rceil$  and  $\lceil -|$  are related by opposition. In a (*Boolean*) modal Kleene algebra, this can be expressed by the conjugation laws

$$|x\rangle p \cdot q = 0 \Leftrightarrow p \cdot \langle x|q = 0 \quad \text{and} \quad |x\rangle p + q = 1 \Leftrightarrow p + [x|q = 1.$$

### 3. Higher dimensional modal Kleene algebras

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In a Boolean modal semiring, boxes and diamonds are related by De Morgan duality by their definition (3.1.11) and additionally by

$$|x\rangle p = \neg[x](\neg p), \quad \langle x|p = \neg[x](\neg p). \quad (3.1.12)$$

Finally, boxes and diamonds are adjoints in Galois connections, expressed by the following relations

$$|x\rangle p \leq q \Leftrightarrow p \leq [x]q \quad \text{and} \quad \langle x|p \leq q \Leftrightarrow p \leq [x]q.$$

As a consequence, diamonds preserve all existing sups in  $S$ , whereas boxes reverse all existing infs to sups, and all modal operators are order preserving. Finally, we mention the properties  $|xy\rangle = |x\rangle \circ |y\rangle$ ,  $\langle xy| = \langle y| \circ \langle x|$ ,  $|xy| = |x| \circ |y|$  and  $[xy] = [y] \circ [x]$ .

**3.1.13. Example: relation Kleene algebra.** For any set  $X$ , the structure

$$(\mathcal{P}(X \times X), \cup, \cdot, \emptyset_X, Id_X, (-)^*)$$

forms a Kleene algebra, called the *full relation Kleene algebra* over  $X$ . The operation  $\cdot$  is the relational composition defined by  $(a, b) \in R \cdot S$  if and only if  $(a, c) \in R$  and  $(c, b) \in S$ , for some  $c \in X$ . The relation  $Id_X = \{(a, a) \mid a \in X\}$  is the identity relation on  $X$  and  $(-)^*$  the reflexive transitive closure operation defined, for  $R^0 = Id_X$  and  $R^{i+1} = R \cdot R^i$ , by

$$R^* = \bigcup_{i \in \mathbb{N}} R^i,$$

The subidentity relations below  $Id_X$  form its greatest Boolean subalgebra between  $\emptyset_X$  and  $Id_X$ , which is isomorphic to the power set algebra  $\mathcal{P}(X)$ . Every subalgebra of a full relation Kleene algebra is a *relation Kleene algebra*.

The full relation Kleene algebra over  $X$  extends to a *full relation Boolean modal Kleene algebra* over  $X$  by defining

$$d(R) = \{(a, a) \mid \exists b \in X. (a, b) \in R\} \quad \text{and} \quad r(R) = \{(a, a) \mid \exists b. (b, a) \in R\}.$$

The antidomain and anticodomain maps are then given by complementation  $ad(R) = Id_X \setminus d(R)$  and  $ar(R) = Id_X \setminus r(R)$ , and it is straightforward to check that

$$\begin{aligned} |R\rangle P &= \{(a, a) \mid \exists b \in X. (a, b) \in R \wedge (b, b) \in P\}, \\ [R]P &= \{(a, a) \mid \forall b \in X. (a, b) \in R \Rightarrow (b, b) \in P\}, \end{aligned}$$

which corresponds to the standard relational Kripke semantics of boxes and diamonds. Similar expressions for the backward modalities are obtained by swapping  $(a, b)$  to  $(b, a)$  in the above expressions.

**3.1.14. Example: path Kleene algebras.** Let  $P^*$  be the free 1-category generated by the 1-polygraph  $P = (P_0, P_1)$ . Then  $(\mathcal{P}(P_1^*), \cup, \odot, \emptyset, \mathbb{1}, (-)^*)$  forms a Kleene algebra, the *full path (Kleene) algebra*  $K(P)$  over  $P$ . Here, composition is defined as

$$\phi \odot \psi = \{ u \star_0 v \mid u \in \phi \wedge v \in \psi \wedge t_0(u) = s_0(v) \}$$

### 3.2. n-Dimensional globular Kleene algebras

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for any  $\phi, \psi \in \mathbb{P}(P_1^*)$ , and  $\mathbb{1}$  is the set of all identity arrows of  $P$ . The Kleene star can be defined as

$$\phi^* = \bigcup_{i \in \mathbb{N}} \phi^i$$

where  $\phi^0 = \mathbb{1}$  and  $\phi^{i+1} = \phi \odot \phi^i$ . Every subalgebra of the full path Kleene algebra over  $P$  is a *path Kleene algebra*.

The full path algebra over  $P$  can be extended to a *full path Boolean modal Kleene algebra* over  $P$  by defining

$$d(\phi) = \{1_{s(u)} \mid u \in \phi\} \quad \text{and} \quad r(\phi) = \{1_{t(u)} \mid u \in \phi\}$$

where  $1_x$  denotes the identity arrow on the object  $x \in P_0$ . The antidomain and anticodomain maps are then given by complementation  $ad(\phi) = \mathbb{1} \setminus d(\phi)$  and  $ar(\phi) = \mathbb{1} \setminus r(\phi)$ . It is then easy to check that

$$|\phi\rangle p = \{1_{s(u)} \mid u \in \phi \wedge t(u) \in p\} \quad \text{and} \quad |\phi]p = \{1_{s(u)} \mid u \in \phi \Rightarrow t(u) \in p\}$$

where  $p \subseteq \mathbb{1}$  is some set of identity arrows. Once again, similar expressions for backward modalities can be obtained by swapping source and target functions in the right places.

The relational and the path model are very similar. In fact the relational model can be obtained from the path model by applying a suitable homomorphism of modal Kleene algebras.

### 3.2. n-Dimensional globular Kleene algebras

We now extend the definitions in the previous sections to a notion of globular  $n$ -dimensional modal Kleene algebra. First, we define a notion of  $n$ -dimensional dioid satisfying lax interchange laws between multiplication operations of different dimensions, similar to those of concurrent Kleene algebra [16]. We then extend it with domain operations and add further axioms that capture globularity. Finally we equip these algebras with star operations for each dimension and impose novel lax interchange laws between compositions and stars of different dimensions.

**3.2.1. n-Dioid.** We define a *0-dioid* as a bounded distributive lattice and a *1-dioid* as a dioid. More generally, for  $n \geq 1$ , an *n-dioid* is a structure  $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$  satisfying the following conditions:

- i)  $(S, +, 0, \odot_i, 1_i)$  is a dioid for  $0 \leq i < n$ ,
- ii) the following *lax interchange laws* hold for all  $0 \leq i < j < n$ :

$$(x \odot_j x') \odot_i (y \odot_j y') \leq (x \odot_i y) \odot_j (x' \odot_i y') \tag{3.2.2}$$

- iii) Higher dimensional units are idempotents of lower dimensional multiplications, *i.e.*

$$1_j \odot_i 1_j = 1_j \tag{3.2.3}$$

for  $0 \leq i < j < n$ .

With lax interchange laws, by contrast to the equational case, we need not worry about an Eckmann-Hilton collapse.

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**3.2.4. Domain  $n$ -semirings.** For  $n = 0$ , we stipulate that a *domain 0-semiring* is a 0-diod. For  $n \geq 1$ , a *domain  $n$ -semiring* is an  $n$ -diod  $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$  equipped with  $n$  domain maps  $d_i : S \rightarrow S$ , for all  $0 \leq i < n$ , satisfying the following conditions:

- i)  $(S, +, 0, \odot_i, 1_i, d_i)$  is a domain semiring,
- ii)  $d_{i+1} \circ d_i = d_i$ .

For  $0 \leq i < n$ , the set  $S_{d_i} = d_i(S)$  will be called the  *$i$ -dimensional domain algebra*, and will be denoted by  $S_i$ . Furthermore, to distinguish elements of distinct dimensions  $0 \leq i < j < n$ , we henceforth denote elements of  $S_i$  by  $p, q, r, \dots$ , elements of  $S_j$  by  $\phi, \psi, \xi, \dots$ , and other elements of  $S$  by  $A, B, C, \dots$ . This notation simplifies the reading of proofs when elements of different dimensions are interacting. For a natural number  $k \geq 0$ , the  *$k$ -fold  $i$ -multiplication* of an element  $A$  of  $S$ , for  $0 \leq i < n$ , is defined by

$$A^{0_i} = 1_i, \quad A^{k_i} = A \odot_i A^{(k-1)_i}.$$

The axioms **ii**) and **iii**) from Section 3.2.1 for  $n$ -dioids provide the basic algebraic structure for reasoning about higher-dimensional rewriting systems. Indeed, the dependencies between multiplications of different dimensions expressed by the lax interchange laws capture the lifting of the equational interchange law for  $n$ -categories, while the idempotence of  $i$ -multiplication for the  $j$ -unit expresses completeness of the set of  $j$ -dimensional cells in an  $n$ -category with respect to  $i$ -composition. In this way, these axioms begin to capture the higher dimensional character of polygraphs, as is made clear in Section 3.3.1, in which we provide a model of this structure based on polygraphs. The domain axiom **ii**) from Section 3.2.4 further captures characteristics of dimension, which are expressed abstractly in the following proposition.

**3.2.5. Proposition.** *For  $n \geq 1$ , in any domain  $n$ -semiring  $S$ , for all  $0 \leq i < j < n$ , the following conditions hold:*

- i)  $d_j \circ d_i = d_i$ ,
- ii)  $d_j(1_i) = 1_i$ ,
- iii)  $1_i \leq 1_j$ ,
- iv)  $S_i \subseteq S_j$ ,
- v)  $(S_j, +, 0, \odot_i, 1_i, d_i)$  is a domain sub semiring of  $(S, +, 0, \odot_i, 1_i, d_i)$  and  $d_i(S_j) = S_i$ ,
- vi)  $(S_j, +, 0, \odot_k, 1_k, d_k)_{0 \leq k \leq i}$  is a domain sub  $(i+1)$ -semiring of  $(S, +, 0, \odot_k, 1_k, d_k)_{0 \leq k \leq i}$ .
- vii)  $(S_i, +, 0, \odot_i, 1_i)$  is a 0-diod.

*Proof.* The first identity is proved by a simple induction on axiom **ii**) in (3.2.4). The second one quickly follows, since  $d_i(1_i) = 1_i$  follows from the domain semiring axioms, and thus  $d_j(1_i) = 1_i$  using **i**). The third identity is again a direct consequence, since by **ii**) we know that  $1_i \in S_j$ , and that  $1_j$  is the greatest element of  $S_j$ . The fourth one follows since  $x \in S_i$  if, and only if,  $d_i(x) = x$ , which is equivalent to  $d_j(x) = x$  by **ii**). The fifth identity is verified by noticing that the inclusion  $S_j \hookrightarrow S$  is a morphism of domain semirings with the operation  $\odot_i$ . Furthermore, since  $d_i(S_j) \subseteq S_i$  and  $S_i \subseteq S_j$ , we have  $d_i(S_j) = S_i$ . Noticing that, in fact,  $S_j \hookrightarrow S$  is a morphism of domain semirings with the operation  $\odot_k$  for any  $0 \leq k \leq i$  gives us **vi**). The final result follows from basic properties of domain semirings.  $\square$

Given an  $n$ -semiring  $S$ , we denote by  $S^{\text{op}}$  the  $n$ -semiring in which the order of each multiplication operation has been reversed. An  $n$ -semiring  $S$  is a *codomain  $n$ -semiring* if  $S^{\text{op}}$  is a domain  $n$ -semiring. The codomain operators are denoted by  $r_i$ . A *modal  $n$ -semiring* is an  $n$ -semiring with domains and codomains, in which the coherence conditions  $d_i \circ r_i = r_i$  and  $r_i \circ d_i = d_i$  hold for all  $0 \leq i < n$ .

**3.2.6. Remarks.** Section 3.1.14 recalls that the path algebra  $K(P)$  defined as the power set of 1-cells in the free category generated by a 1-polygraph  $P = (P_0, P_1)$  is a model of modal 1-semiring. The domain algebra  $K(P)_d$  is isomorphic to the power set of  $P_0$ . As recalled in Section 3.1.3, in the general case of a domain semiring  $(S, +, 0, \cdot, 1, d)$ , the domain algebra  $S_d$  forms a bounded distributive lattice with  $+$  as join,  $\cdot$  as meet,  $0$  as bottom and  $1$  as top. It is for this reason that we consider a 0-doid as a bounded distributive lattice. Indeed, the idempotence and commutativity of the multiplication operation simulate the properties of a set of identity 1-cells.

Note also that, in Section 3.3, we will construct higher-dimensional path algebras over  $n$ -polygraphs and show that these form models of modal  $n$ -semirings. In this case it makes sense that  $(S_i, +, 0, \odot_i, 1_i)$  is a 0-doid, since an  $i$ -cell  $f : u \rightarrow v$  of an  $n$ -category  $\mathcal{C}$  is a 0-cell in the hom-category  $\mathcal{C}(u, v)$ .

**3.2.7. Diamond operators.** Let  $S$  be a modal  $n$ -semiring. We introduce *forward* and *backward  $i$ -diamond* operators defined via (co-)domain operators in each dimension by analogy to (3.1.5). For any  $0 \leq i < n$ ,  $A \in S$  and  $\phi \in S_i$ , we define

$$\langle A \rangle_i(\phi) = d_i(A \odot_i \phi), \quad \text{and} \quad \langle A \rangle_i(\phi) = r_i(\phi \odot_i A). \quad (3.2.8)$$

In the absence of antidomains, box operators cannot be expressed in this setting. These diamond operators have all of the properties recalled in Section 3.1.9 with respect to  $i$ -multiplication and elements of  $S_i$ .

**3.2.9.  $p$ -Boolean domain semirings.** For  $0 \leq p < n$ , a domain  $n$ -semiring  $(S, +, 0, \odot_i, 1_i, d_i)_{0 \leq i < n}$  is called  *$p$ -Boolean* if it is augmented with  $(p + 1)$  maps

$$(ad_i : S \rightarrow S)_{0 \leq i \leq p}$$

such that for all  $0 \leq i \leq p$ , the following conditions are satisfied:

- i)  $(S, +, 0, \odot_i, 1_i, ad_i)$  is a Boolean domain semiring,
- ii)  $d_i = ad_i^2$ .

By definition, a 0-Boolean domain 1-semiring is a Boolean domain semiring, and by convention we define a *0-Boolean domain 0-semiring* as a Boolean algebra.

We define a  *$p$ -Boolean codomain semiring* as an  $n$ -semiring such that its opposite  $n$ -semiring is a  $p$ -semiring with antidomains. In this case the anticodomain operators are denoted  $ar_i$ .

**3.2.10. Remark.** The key difference between modal  $n$ -semirings and their  $p$ -Boolean counterparts is that the latter are equipped with negation operations in their lower dimensions. Indeed, in a  $p$ -Boolean modal Kleene algebra  $K$ , for every  $0 \leq i \leq p$ , the tuple

$$(K_i, +, 0, \odot_i, 1_i, ad_i)$$

is a Boolean algebra. For this reason, we denote the restriction of  $ad_i$  to  $K_i$  by  $\neg_i$ . Furthermore, as recalled in (3.1.9), for  $0 \leq j \leq p$ ,  $A \in K$  and  $\phi \in K_j$  we can define *forward* (resp. *backward*) *box operators*

$$[A]_j(\phi) := \neg_j(\langle A \rangle_j(\neg_j \phi)) \quad (\text{resp. } [A]_j(\phi) := \neg_j(\langle A \rangle_j(\neg_j \phi)))$$

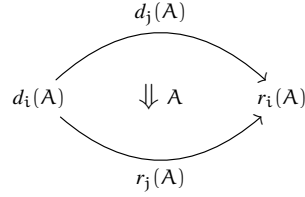
### 3. Higher dimensional modal Kleene algebras

**3.2.11. Globular modal  $n$ -semiring.** A modal semiring  $S$  is called *globular* if the following *globular relations* hold for  $0 \leq i < j < n$  and  $A, B \in K$ :

$$d_i \circ d_j = d_i \text{ and } d_i \circ r_j = d_i, \quad (3.2.12) \quad d_j(A \odot_i B) = d_j(A) \odot_i d_j(B), \quad (3.2.14)$$

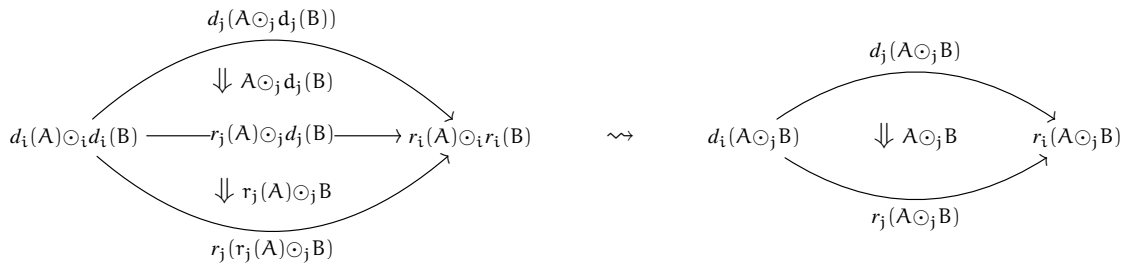
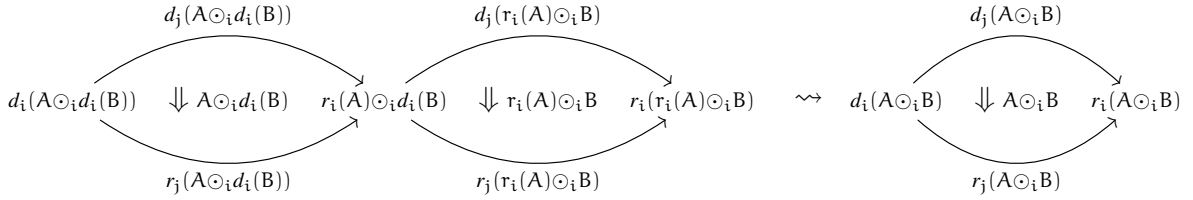
$$r_i \circ d_j = r_i, \text{ and } r_i \circ r_j = r_i, \quad (3.2.13) \quad r_j(A \odot_i B) = r_j(A) \odot_i r_j(B). \quad (3.2.15)$$

An element  $A$  of  $S$  will be represented graphically by the following diagram with respect to its  $i$ - and  $j$ -borders, when  $i < j$ :



The intuition here is that  $A$  is a *collection* of cells and that for  $k \in \{i, j\}$ ,  $d_k(A)$  (resp.  $r_k(A)$ ) is a *collection* of  $k$ -cells each of which is the  $k$ -source (resp.  $k$ -target) of some cell belonging to  $A$ . In Section 3.3 this intuition is elucidated via the polygraphic model.

Below are graphical representations of  $i$ - and  $j$ -multiplication with respect to  $i$ - and  $j$ -borders:



The illustrations underline the fact that multiplication of elements in a Kleene algebra is equivalent to multiplying their restrictions to the appropriate domain or range, as below:

$$A \odot_i B = (A \odot_i r_i(A)) \odot_i (d_i(B) \odot_i B) = (A \odot_i d_i(B)) \odot_i (r_i(A) \odot_i B),$$

where we have used properties of domain semirings given in (3.1.3), and that these restrictions are compatible with the globular relations.

### 3.3. A model of higher-dimensional modal Kleene algebras

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**3.2.16. Modal  $n$ -Kleene algebra.** An  $n$ -Kleene algebra is an  $n$ -dioid  $K$  equipped with operations  $(-)^{*i} : K \rightarrow K$  satisfying the following conditions:

- i)  $(K, +, 0, \odot_i, 1_i, (-)^{*i})$  is a Kleene algebra for  $0 \leq i < n$ ,
- ii) For  $0 \leq i < j < n$ , the Kleene star operation  $(-)^{*j}$  is a *lax morphism* with respect to the  $i$ -whiskering of  $j$ -dimensional elements on the right (resp. left), that is for all  $A \in K$  and  $\phi \in K_j$ ,

$$\phi \odot_i A^{*j} \leq (\phi \odot_i A)^{*j}, \quad \text{and} \quad (\text{resp. } A^{*j} \odot_i \phi \leq (A \odot_i \phi)^{*j}). \quad (3.2.17)$$

As in the case of 1-Kleene algebras, recalled in (3.1.8), the notions of ( $p$ -Boolean)  $n$ -semiring structures with (co)domains are compatible with the notion of  $n$ -Kleene algebra. Hence, a  $n$ -Kleene algebra with domains (resp. codomains) is a  $n$ -Kleene algebra such that the underlying semiring has domains (resp. codomains). When the underlying  $n$ -semiring is modal, we have a *modal  $n$ -Kleene algebra*. If it is  $p$ -Boolean, we have a  *$p$ -Boolean modal  $n$ -Kleene algebra*. We say that these are *globular* when the underlying modal  $n$ -semiring is globular.

Finally, note that for  $n = 2$ , we obtain the standard concurrent Kleene algebra axioms [16], except that  $1_0 = 1_1$  is normally assumed in this case.

### 3.3. A model of higher-dimensional modal Kleene algebras

**3.3.1. Polygraphic model.** Let  $P$  be an  $n$ -polygraph and  $\Gamma$  a cellular extension of the free  $(n, n - 1)$ -category  $P_n^\top$ . We define an  $(n + 1)$ -modal Kleene algebra  $K(P, \Gamma)$ , the *full  $(n + 1)$ -path algebra* over  $P_n^\top[\Gamma]$ , as follows

- i) The carrier set of  $K(P, \Gamma)$  is the power set  $\mathcal{P}(P_n^\top[\Gamma])$ , whose elements, denoted by  $A, B, C, \dots$ , are sets of  $(n + 1)$ -cells. We denote these  $(n + 1)$ -cells by  $\alpha, \beta, \gamma, \dots$  in what follows.
- ii) Recall that for  $\alpha$  a  $k$ -cell, the elements  $s_i(\alpha)$ ,  $t_i(\alpha)$ ,  $l_k^i(\alpha)$  were defined for  $0 \leq i \leq k \leq l \leq n + 1$  in Sections 2.1.1 and 2.1.4. When  $k \leq i$ , we define  $s_i(\alpha) = t_i(\alpha) = l_k^i(\alpha)$ ,
- iii) Recall that the  $i$ -composition of a  $k$ -cell  $\alpha$  and an  $l$ -cell  $\beta$  for  $0 \leq i < k \leq l \leq n + 1$  was defined in Sections 2.1.1 and 2.1.4. For  $0 \leq k \leq l < n + 1$ , we define

$$\alpha \star_i \beta = \begin{cases} l_k^{i+1}(\alpha) \star_i \beta & \text{for } k \leq i < l, \\ l_k^{i+1}(\alpha) \star_i l_l^{i+1}(\beta) & \text{for } l \leq i. \end{cases}$$

- iv) For  $0 \leq i < n + 1$ , the binary operation  $\odot_i$  on  $K(P, \Gamma)$  corresponds to the lifting of the composition operations of  $P_n^\top[\Gamma]$  to the power-set, *i.e.* for any  $A, B \in K(P, \Gamma)$ ,

$$A \odot_i B := \{\alpha \star_i \beta \mid \alpha \in A \wedge \beta \in B \wedge t_i(\alpha) = s_i(\beta)\}.$$

- v) For  $0 \leq i < n + 1$ , denote by  $\mathbb{1}_i$  the set

$$\mathbb{1}_i = \{l_i^{n+1}(u) \mid u \in P_n^\top[\Gamma]_i\},$$



### 3. Higher dimensional modal Kleene algebras

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These sets are the units for the multiplication operations, that is we have

$$A \odot_i \mathbb{1}_i = \mathbb{1}_i \odot_i A = A.$$

Furthermore, when  $i < j$ , the inclusion  $\mathbb{1}_i \subseteq \mathbb{1}_j$  holds. Indeed, in that case  $\iota_i^{n+1}(u) = \iota_j^{n+1}(\iota_i^j(u))$  by uniqueness of identity cells, and  $\iota_i^j(u) \in P_n^\top(\Gamma)_j$  is a  $j$ -cell.

vi) The addition in  $K(P, \Gamma)$  is given by set union  $\cup$ . The ordering is therefore given by set inclusion.

vii) The  $i$ -domain and  $i$ -codomain maps  $d_i$  and  $r_i$  are defined by

$$d_i(A) := \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \quad \text{and} \quad r_i(A) := \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

These are thus given by lifting the source and target maps of  $P_n^\top[\Gamma]$  to the power set. The  $i$ -antidomain and  $i$ -anticodomain maps are then given by complementation with respect to the set of  $i$ -cells:

$$ad_i(A) := \mathbb{1}_i \setminus \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \quad \text{and} \quad ar_i(A) := \mathbb{1}_i \setminus \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

viii) The  $i$ -star is given by

$$A^{*i} = \bigcup_{k \in \mathbb{N}} A^{k_i},$$

where in the above,  $A^{0_i} := \mathbb{1}_i$  and  $A^{k_i} := A \odot_i A^{(k-1)_i}$ .

**3.3.2. Proposition.** *For any  $n$ -polygraph  $P$  and cellular extension  $\Gamma$  of  $P_n^\top$ ,  $K(P, \Gamma)$  is an  $n$ -Boolean  $(n+1)$ -modal Kleene algebra.*

*Additionally, the set  $\Gamma^c$  of rewriting steps generated by  $\Gamma$  as defined in Remark 2.2.2, is represented in  $n$ -Kleene algebra by*

$$\Gamma^c = \mathbb{1}_n \odot_{n-1} (\cdots \odot_2 (\mathbb{1}_2 \odot_1 (\mathbb{1}_1 \odot_0 \Gamma \odot_0 \mathbb{1}_1) \odot_1 \mathbb{1}_2) \odot_2 \cdots) \odot_{n-1} \mathbb{1}_n.$$

*Therefore,  $\alpha$  is an  $(n+1)$ -cell of  $P_n^\top[\Gamma]$  if, and only if,  $\alpha \in (\Gamma^c)^{*n}$ .*

*Proof.* It is easy to check that, for  $0 \leq i < n+1$ , the tuple  $(\mathcal{P}((P_n^\top[\Gamma])_{n+1}), \cup, \emptyset, \odot_i, \mathbb{1}_i, (-)^{*i}, d_i, r_i)$  is a modal semiring. The fact that it is  $n$ -Boolean is a result of it being a power-set algebra.

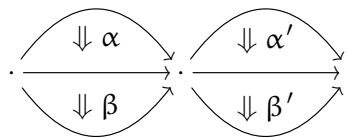
Let  $A, A', B, B' \in K(P, \Gamma)$  and  $0 \leq i < j < n+1$ . We want to show that the lax interchange law holds, *i.e.*

$$(A \odot_j B) \odot_i (A' \odot_j B') \subseteq (A \odot_i A') \odot_j (B \odot_i B'). \quad (3.3.3)$$

This is the case since, given  $(n+1)$ -cells  $\alpha \in A, \alpha' \in A', \beta \in B, \beta' \in B'$ , if  $(\alpha \star_j \beta) \star_i (\alpha' \star_j \beta')$  is defined, then as a consequence of the exchange law for  $(n+1)$  categories, we have

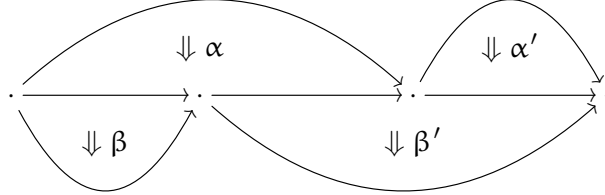
$$(\alpha \star_j \beta) \star_i (\alpha' \star_j \beta') = (\alpha \star_i \alpha') \star_j (\beta \star_i \beta') \in (A \odot_i A') \odot_j (B \odot_i B')$$

which gives the desired inclusion (3.3.3). This situation is illustrated by the following diagram:



### 3.3. A model of higher-dimensional modal Kleene algebras

The lax interchange law is not reduced to an equality due to composition of diagrams of the following shape:



where  $\alpha \in A$ ,  $\alpha' \in A'$ ,  $\beta \in B$ ,  $\beta' \in B'$ . Indeed, the composition  $(\alpha \star_i \alpha') \star_j (\beta \star_i \beta') \in (A \odot_i A') \odot_j (B \odot_i B')$  is defined, whereas neither  $\alpha$  and  $\beta$  nor  $\alpha'$  and  $\beta'$  are  $j$ -composable, meaning that in general the inclusion (3.3.3) is strict.

Further, given  $0 \leq i < j < n + 1$ , we have  $\mathbb{1}_j \subseteq \mathbb{1}_j \odot_i \mathbb{1}_j$ . Indeed, for any  $j$ -cell  $\alpha$ , we have  $\alpha \star_i \iota_i^{n+1}(t_i(\alpha)) = \alpha$  because  $\iota_i^{n+1}(t_i(\alpha))$  is the  $(n + 1)$ -dimensional identity cell on the  $i$ -dimensional target of  $\alpha$ . Furthermore,  $\iota_i^{n+1}(t_i(\alpha)) \in \mathbb{1}_i \subseteq \mathbb{1}_j$ , proving the inclusion. Thus  $\mathbb{1}_j = \mathbb{1}_j \odot_i \mathbb{1}_j$  since  $(P_n^\top(\Gamma))_j$  is closed under  $i$ -composition.

Given  $0 \leq i < n$ , we have  $d_{i+1} \circ d_i = d_i$  since the  $(i + 1)$ -dimensional border of an identity cell on an  $i$ -cell  $u$  is  $u$  itself. Since  $d_i(\mathbb{1}_i) = \mathbb{1}_i$ , we equally have  $d_{i+1}(\mathbb{1}_i) = \mathbb{1}_i$ .

The first two globularity axioms are immediate consequences of the globularity conditions on the source and target maps of  $P_n^\top(\Gamma)$ . Furthermore, for  $0 \leq i < j < n + 1$  and  $A, B \in K(P, \Gamma)$ , we have  $u \in d_j(A \odot_i B)$  if, and only if, there exist  $\alpha \in A$  and  $\beta \in B$  such that  $u = s_j(\alpha \star_i \beta) = s_j(\alpha) \star_i s_j(\beta)$ , which is equivalent to  $u \in d_j(A) \odot_i d_j(B)$ . Similarly, we show that  $r_j(A \star_i B) = r_j(A) \odot_i r_j(B)$ .

Finally, we consider the Kleene star axioms. It is easy to check that, given a family  $(B_k)_{k \in I}$  of elements of  $K(P, \Gamma)$  and another element  $A$ , we have, for all  $0 \leq i < n + 1$ ,

$$A \odot_i \left( \bigcup_{k \in I} B_k \right) = \bigcup_{k \in I} (A \odot_i B_k) \quad \text{and} \quad \left( \bigcup_{k \in I} B_k \right) \odot_i A = \bigcup_{k \in I} (B_k \odot_i A).$$

It then follows by routine calculations that the element  $A^{*i}$  defined above satisfies, for each  $i$ , the Kleene star axioms, recalled in (3.1.8). It only remains to check that for  $0 \leq i < j < n + 1$ , the  $j$ -star is a lax morphism for  $i$ -whiskering of  $j$ -dimensional elements on the left (the right case being symmetric), that is  $\phi \odot_i A^{*j} \subseteq (\phi \odot_i A)^{*j}$  for  $\phi \in K(P, \Gamma)_j$  and  $A \in K(P, \Gamma)$ . By construction,  $K(P, \Gamma)_j$  is in bijective correspondence with  $(P_n^\top(\Gamma))_j$ , the set of  $j$ -cells of  $P_n^\top[\Gamma]$ . Considering such elements  $\phi$  and  $A$ , we have  $\beta \in \phi \odot_i A^{*j}$  in the following two cases:

- i) There exist  $u \in \phi$  and  $\alpha \in A^{+j}$ , where we recall that  $A^{+j} := A \odot_j A^{*j}$  is the Kleene plus operation, such that  $\beta = u \star_i \alpha$ . Since  $\alpha \in A^{+j}$ , there exist a  $k > 0$  and cells  $\alpha_1, \alpha_2, \dots, \alpha_k \in A$  such that

$$\alpha = \alpha_1 \star_j \alpha_2 \star_j \dots \star_j \alpha_k.$$

Since  $i < j$ , the following is a consequence of the exchange law for  $n$ -categories:

$$u \star_i (\alpha_1 \star_j \alpha_2 \star_j \dots \star_j \alpha_k) = (u \star_i \alpha_1) \star_j (u \star_i \alpha_2) \star_j \dots \star_j (u \star_i \alpha_k),$$

and thus we have  $\beta \in (\phi \odot_i A)^{+j}$ .

#### 4. Algebraic coherent confluence

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- ii) There exist  $u \in \phi$ , and  $v \in (P_n^\top(\Gamma))_j$  with  $v \notin A$  such that  $\beta = u \star_i v$ . This is due to the fact that  $A^{*j} = \mathbb{1}_j + A^{+j}$ . In that case, we have  $\beta \in (P_n^\top(\Gamma))_j$ , i.e.  $\beta \in \mathbb{1}_j$ . By the unfold axiom, we have  $\mathbb{1}_j \subseteq (\phi \odot_i A)^{*j}$ , and thus  $\beta \in (\phi \odot_i A)^{*j}$ .

The fact that  $\alpha$  is an  $(n+1)$ -cell of  $P_n^\top[\Gamma]$  if, and only if,  $\alpha \in (\Gamma^c)^{*n}$ , follows by definition of  $\Gamma^c$  and the fact that any  $(n+1)$ -cell of  $P_n^\top[\Gamma]$  is an  $n$ -composition of rewriting steps.  $\square$

## 4. ALGEBRAIC COHERENT CONFLUENCE

In this section, we present proofs of the coherent Church-Rosser theorem and coherent Newman's lemma in the setting of higher-dimensional globular Kleene algebras. These constitute the main results of this article. First, we recall from [5, 28, 29] abstract rewriting properties formulated in modal Kleene algebras. We then formalise notions from higher-dimensional rewriting needed to prove our results, introducing *fillers* in the setting of globular modal  $n$ -Kleene algebras, which correspond to the notion of fillers for polygraphs defined in Section 2.3.1. We also define the notion of *whiskering* in modal  $n$ -Kleene algebras, analogous to the polygraphic definition in Section 2.1.4 and describe the properties thereof needed for our proofs. The coherent Church-Rosser theorem is dealt with in Section 4.2, first in Proposition 4.2.7 using classical induction and then in Theorem 4.2.8 using only the induction axioms provided by the Kleene star. In Section 4.3, we define notions of termination and well-foundedness in globular modal  $p$ -Boolean Kleene algebras and prove Theorem 4.3.2, the coherent Newman's lemma.

### 4.1. Rewriting properties formulated in modal Kleene algebra

Let  $K$  be a modal Kleene algebra.

**4.1.1. Termination.** An element  $x \in K$  *terminates*, or is *Noetherian*, [5] if

$$p \leq |x\rangle p \Rightarrow p = 0$$

holds for all  $p \in K_d$ . The set of Noetherian elements of  $K$  is denoted by  $\mathcal{N}(K)$ . Using the Galois connections (3.1.9) yields the following equivalent characterisation:  $x \in K$  is Noetherian if and only if

$$|x\rangle p \leq p \Rightarrow p = 1$$

holds for all  $p \in K_d$ .

**4.1.2. Semi-commutation.** The notions of local confluence, confluence and the Church-Rosser property for rewriting systems can be captured in Kleene algebras as follows. Given elements  $x, y \in K$ , we say that the ordered pair  $(x, y)$  *semi-commutes* (resp. *semi-commutes locally*) if

$$x^* \cdot y^* \leq y^* \cdot x^* \quad (\text{resp. } x \cdot y \leq y^* \cdot x^*).$$

We say that the ordered pair  $(x, y)$  *semi-commutes modally* (resp. *semi-commutes locally modally*) if

$$|x^*\rangle \circ |y^*\rangle \leq |y^*\rangle \circ |x^*\rangle \quad (\text{resp. } |x\rangle \circ |y\rangle \leq |y^*\rangle \circ |x^*\rangle).$$

It is obvious that (local) commutation implies (local) modal commutation, but not vice versa. Finally we say that  $(x, y)$  has the *Church-Rosser property* if

$$(x + y)^* \leq y^* x^*.$$

**4.1.3. Confluence results in Kleene algebras.** The Church-Rosser theorem and Newman's lemma for abstract rewriting systems are instances of the following formulations in modal Kleene algebras. In the following subsections we prove higher-dimensional generalisations of these results.

The Church-Rosser theorem in  $K$  [28, Thm. 4] states that, for any  $x, y \in K$ , the following holds:

$$x^*y^* \leq y^*x^* \iff (x + y)^* \leq y^*x^*.$$

Newman's Lemma in  $K$ , with  $K_d$  a complete Boolean algebra [5], states that for any  $x, y \in K$  such that  $(x + y) \in \mathcal{N}(K)$ , the following holds:

$$|x\rangle \circ |y\rangle \leq |y^*\rangle \circ |x^*\rangle \iff |x^*\rangle \circ |y^*\rangle \leq |y^*\rangle \circ |x^*\rangle.$$

## 4.2. A coherent Church-Rosser theorem

Let  $K$  be a globular  $n$ -modal Kleene algebra and  $0 \leq i < j < n$ . Before defining fillers in globular modal  $n$ -Kleene algebras, we first recall the intuition behind the forward diamond operators in  $n$ -modal Kleene algebras, defined in Section 3.2.7. Given  $A \in K$  and  $\phi, \phi' \in K_j$ , recall that by definition

$$|A\rangle_j(\phi) \geq \phi' = d_j(A \odot_j \phi) \geq \phi'.$$

In terms of quantification over sets of cells, as for example in the polygraphic model, this signifies that *for every* element  $u$  of  $\phi'$ , *there exist* elements  $v$  of  $\phi$  and  $\alpha$  of  $A$  such that the  $j$ -source (resp.  $j$ -target) of  $\alpha$  is  $u$  (resp.  $v$ ). This observation motivates the definitions in the following paragraph.

**4.2.1. Confluence fillers.** Given elements  $\phi$  and  $\psi$  of  $K_j$ , we say that an element  $A$  in  $K$  is a

**i) local  $i$ -confluence filler** for  $(\phi, \psi)$  if

$$|A\rangle_j(\psi^{*i} \odot_i \phi^{*i}) \geq \phi \odot_i \psi,$$

**ii) left (resp. right) semi- $i$ -confluence filler** for  $(\phi, \psi)$  if

$$|A\rangle_j(\psi^{*i} \odot_i \phi^{*i}) \geq \phi \odot_i \psi^{*i}, \quad (\text{resp. } |A\rangle_j(\psi^{*i} \odot_i \phi^{*i}) \geq \phi^{*i} \odot_i \psi),$$

**iii)  $i$ -confluence filler** for  $(\phi, \psi)$  if

$$|A\rangle_j(\psi^{*i} \odot_i \phi^{*i}) \geq \phi^{*i} \odot_i \psi^{*i},$$

**iv)  $i$ -Church-Rosser filler** for  $(\phi, \psi)$  if

$$|A\rangle_j(\psi^{*i} \odot_i \phi^{*i}) \geq (\psi + \phi)^{*i}.$$

In any  $n$ -Kleene algebra, the following inequalities hold:

$$(\psi + \phi)^{*i} \geq \phi^{*i} \odot_i \psi^{*i} \geq \phi \odot_i \psi.$$

We may therefore deduce that an  $i$ -Church-Rosser filler for  $(\phi, \psi)$  is an  $i$ -confluence filler for  $(\phi, \psi)$  and that an  $i$ -confluence filler for  $(\phi, \psi)$  is a local  $i$ -confluence filler for  $(\phi, \psi)$ .

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**4.2.2. Remarks.** Conditions on the domain and codomain in the above definitions imply an  $i$ -dimensional globular character of the pair  $(\phi, \psi)$  in the sense that we have the relation

$$|\phi^{*i} \odot_i \psi^{*i}\rangle_i(\mathfrak{p}) \leq |\psi^{*i} \odot_i \phi^{*i}\rangle_i(\mathfrak{p})$$

for all  $\mathfrak{p} \in K_i$ . Indeed, writing  $A' = A \odot_j (\psi^{*i} \odot_i \phi^{*i})$ , we have

$$\begin{aligned} |\phi^{*i} \odot_i \psi^{*i}\rangle_i(\mathfrak{p}) &= d_i(\phi^{*i} \odot_i \psi^{*i} \odot_i \mathfrak{p}) \leq d_i(d_j(A') \odot_i \mathfrak{p}) \\ &= d_i(d_j(A' \odot_i \mathfrak{p})) \\ &= d_i(r_j(A' \odot_i \mathfrak{p})) \\ &= d_i(r_j(A') \odot_i \mathfrak{p}) \\ &\leq d_i((\psi^{*i} \odot_i \phi^{*i}) \odot_i \mathfrak{p}) = |\psi^{*i} \odot_i \phi^{*i}\rangle_i(\mathfrak{p}), \end{aligned}$$

where the first step holds by definition of diamonds, the second by the fact that  $A$  is an  $i$ -confluence filler and by monotonicity of  $d_i$ , the third, fourth and fifth by the globularity relations (3.2.14), (3.2.12) and (3.2.15) respectively. The final inequality follows because  $d(\mathfrak{p} \cdot \mathfrak{x}) = \mathfrak{p} \cdot d(\mathfrak{x})$  holds in modal Kleene algebra (see the end of Section 3.1.3). In the case of codomains, its dual implies that

$$r_j(A') = r_j(A \odot_j (\psi^{*i} \odot_i \phi^{*i})) = r_j(A) \odot_j r_j(\psi^{*i} \odot_i \phi^{*i}) \leq r_j(\psi^{*i} \odot_i \phi^{*i}).$$

The final step is again by definition of the diamond operators. Similar results hold in the case of local and semi-confluence fillers. Thus,  $\phi$  and  $\psi$  commute modally (resp. locally modally) with respect to  $i$ -multiplication. For this reason, the confluence filler (local confluence filler) defined in (4.2.1) can be represented graphically as follows



**4.2.3. Whiskers.** Let  $K$  be a globular modal  $n$ -Kleene algebra. Given  $0 \leq i < j < n$  and  $\phi \in K_j$ , the *right* (resp. *left*)  $i$ -whiskering of an element  $A \in K$  by  $\phi$  is the element

$$A \odot_i \phi \quad (\text{resp. } \phi \odot_i A)$$

In what follows, we list properties of whiskering and define completions.

- i) Firstly, it holds that  $i$ -whiskering by  $j$ -dimensional cells commutes with  $j$ -modalities. Indeed, for all  $A \in K$  and  $0 \leq i < j < n$  and all  $\phi, \psi, \phi', \psi', \gamma \in K_j$  such that  $\phi' \leq \phi$ ,  $\psi' \leq \psi$ , and  $d_j(A) \leq \gamma$ , we have:

$$\phi' \odot_i \langle A \rangle_j(\gamma) \odot_i \psi' = \langle \phi' \odot_i A \odot_i \psi' \rangle_j(\phi \odot_i \gamma \odot_i \psi) \quad (4.2.4)$$

To see this, consider the deductions

$$\begin{aligned} & d_j(\phi' \odot_i A \odot_i \psi') = \phi' \odot_i d_j(A) \odot_i \psi' \leq \phi \odot_i \gamma \odot_i \psi \\ \Rightarrow & \phi' \odot_i A \odot_i \psi' = (\phi \odot_i \gamma \odot_i \psi) \odot_j (\phi' \odot_i A \odot_i \psi') \\ \Rightarrow & r_j(\phi' \odot_i A \odot_i \psi') = \langle \phi' \odot_i A \odot_i \psi' \rangle_j(\phi \odot_i \gamma \odot_i \psi), \end{aligned}$$

where in the first line we have used the globularity axiom from (3.2.14), as well as the hypothesis  $d_j(A) \leq \gamma$ , and in the second one the fact that  $d(x) \leq p \Rightarrow px = x$ , a consequence of axiom **i**) from Section 3.1.3. Applying  $r_j$  to the second line and using the definition of modalities (3.2.8) yields the third. Again, since  $d_j(A) \leq \gamma$ , we have  $A = \gamma \odot_j A$ , whereby we deduce

$$r_j(\phi' \odot_i A \odot_i \psi') = \phi' \odot_i r_j(\gamma \odot_j A) \odot_i \psi' = \phi' \odot_i \langle A \rangle_j(\gamma) \odot_i \psi',$$

using Axiom (3.2.15). This gives identity (4.2.4), and we have a dual identity for the forward diamond.

- ii)** Secondly, we define *completions* of elements by whiskering. Let  $A$  be an  $i$ -confluence filler of a pair  $(\phi, \psi)$  of elements in  $K_j$ . The  $j$ -dimensional  $i$ -whiskering of  $A$  is the following element of  $K$ :

$$(\phi + \psi)^{*i} \odot_i A \odot_i (\phi + \psi)^{*i}. \quad (4.2.5)$$

The  $j$ -star of this element is called the  $i$ -whiskered  $j$ -completion of  $A$ .

- iii)** Finally, we have that the  $i$ -whiskered  $j$ -completion of a confluence filler  $A$ , which in the following paragraph we denote by  $\hat{A}$ , absorbs whiskers, *i.e.* for any  $\xi \leq (\phi + \psi)^{*i}$

$$\xi \odot_i \hat{A}^{*j} \leq \hat{A}^{*j} \quad \text{and} \quad \hat{A}^{*j} \odot_i \xi \leq \hat{A}^{*j}. \quad (4.2.6)$$

Indeed, by definition of  $\hat{A}$ , we have

$$\xi \odot_i \hat{A} \leq \hat{A} \geq \hat{A} \odot_i \xi$$

for any  $\xi \leq (\phi + \psi)^{*i}$ . Using the fact that  $(-)^{*j}$  is a lax morphism with respect to  $i$ -whiskering by  $j$ -dimensional elements, see Section 3.1.8, we deduce

$$\xi \odot_i \hat{A}^{*j} \leq (\xi \odot_i \hat{A})^{*j} \leq \hat{A}^{*j},$$

where the last inequality holds by monotonicity of  $(-)^{*j}$ . A similar proof shows that  $\hat{A}^{*j} \odot_i \xi \leq \hat{A}^{*j}$ .

**4.2.7. Proposition (Coherent Church-Rosser theorem in globular  $n$ -MKA (by induction)).** *Let  $K$  be a globular modal  $n$ -Kleene algebra and  $0 \leq i < j < n$ . Given  $\phi, \psi \in K_j$ , an  $i$ -confluence filler  $A$  of  $(\phi, \psi)$  and any natural number  $k \geq 0$ , there exists an  $A_k \leq \hat{A}^{*j}$  such that*

- i)**  $r_j(A_k) \leq \psi^{*i} \phi^{*i}$ ,
- ii)**  $d_j(A_k) \geq (\phi + \psi)^{k_i}$ ,

where  $\hat{A}$  is the  $j$ -dimensional  $i$ -whiskering of  $A$ .

*Proof.* In this proof, juxtaposition of elements denotes  $i$ -multiplication. We reason by induction on  $k \geq 0$ . For  $k = 0$ , we may take  $A_0 = 1_i$ . Indeed,

$$1_i \leq 1_j \leq \hat{A}^{*j}.$$

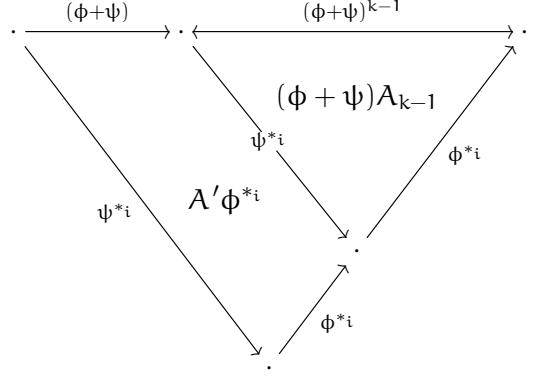
Furthermore, we have  $d_j(A_0) = 1_i = (\phi + \psi)^{0_i}$  and  $r_j(A_0) = 1_i \leq \psi^{*i} \phi^{*i}$ . Supposing that  $A_{k-1}$  is constructed, we set

$$A_k = ((\phi + \psi)A_{k-1}) \odot_j (A' \phi^{*i}),$$

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where  $A' = A \odot_j (\psi^{*i} \phi^{*i})$ . We first show that  $d_j(A_k) \geq (\phi + \psi)^{k_i}$  as follows

$$\begin{aligned}
d_j(A_k) &= d_j(((\phi + \psi)A_{k-1}) \odot_j (A' \phi^{*i})), \\
&= d_j(((\phi + \psi)A_{k-1}) \odot_j d_j(A' \phi^{*i})), \\
&= d_j(((\phi + \psi)A_{k-1}) \odot_j d_j(A') \phi^{*i}), \\
&\geq d_j(((\phi + \psi)A_{k-1}) \odot_j \phi^{*i} \psi^{*i} \phi^{*i}), \\
&= d_j((\phi + \psi)A_{k-1}), \\
&= (\phi + \psi) d_j(A_{k-1}), \\
&= (\phi + \psi) (\phi + \psi)^{(k-1)_i}, \\
&= (\phi + \psi)^{k_i},
\end{aligned}$$



where the first step is given by definition of  $A_k$ , the second by axiom **ii**) from (3.1.3), the third by globularity (3.2.14). The inequality in the fourth step is by hypothesis that  $A$  is an  $i$ -confluence filler, and the fifth is a consequence of the fact that

$$((\phi + \psi)A_{k-1}) \odot_1 (\phi^{*i} \psi^{*i} \phi^{*i}) = (\phi + \psi)A_{k-1},$$

which is in turn a consequence of the following:

$$r_j((\phi + \psi)A_{k-1}) = (\phi + \psi)r_j(A_{k-1}) \leq \phi^{*i} \psi^{*i} \phi^{*i}.$$

The sixth step is again a consequence of globularity (3.2.14), the seventh follows from the induction hypothesis, and the last equality is by definition of the  $k$ -fold  $i$ -multiplication.

Now we show  $r_j(A_k) \leq \psi^{*i} \phi^{*i}$ :

$$\begin{aligned}
r_j(A_k) &= r_j(((\phi + \psi)A_{k-1}) \odot_j (A' \phi^{*i})) \\
&= r_j(r_j((\phi + \psi)A_{k-1}) \odot_j (A' \phi^{*i})) \\
&\leq r_j((\phi + \psi) \psi^{*i} \phi^{*i} \odot_j (A' \phi^{*i})) \\
&\leq r_j((\phi^{*i} \psi^{*i} \phi^{*i}) \odot_j (A' \phi^{*i})) \\
&= r_j(d_j(A' \phi^{*i}) \odot_j (A' \phi^{*i})) \\
&= r_j(A') \phi^{*i} \\
&\leq \psi^{*i} \phi^{*i} \phi^{*i} \\
&= \psi^{*i} \phi^{*i}.
\end{aligned}$$

The first equality holds by definition of  $A_k$ , the second by axiom **ii**) from Section 3.1.3 (for codomains), the third by the induction hypothesis, the fourth by  $\phi \leq \phi^{*i}$  and  $\psi \psi^{*i} = \psi^{*i}$ . The fifth step holds since  $A$  is an  $i$ -confluence filler, the sixth by the fact that  $d(x) \cdot x = x$ , a consequence of axiom **i**) from Section 3.1.3. Finally, as recalled in Section 4.2.2,

$$r_j(A') = r_j(A \odot_j (\psi^{*i} \odot_i \phi^{*i})) = r_j(A) \odot_j r_j(\psi^{*i} \odot_i \phi^{*i}) \leq r_j(\psi^{*i} \odot_i \phi^{*i}),$$

which gives step seven since  $\psi^{*i} \odot_i \phi^{*i} \in K_j$ . The final step is due to  $\phi^{*i} \odot_i \phi^{*i} = \phi^{*i}$ , a consequence of the Kleene star axioms.

To conclude, we must also show that  $A_k \leq \hat{A}^{*j}$ . By whisker absorption, described in (4.2.3), and the fact that  $A' \leq A \leq \hat{A}$ , we have

$$A' \phi^{*i} \leq \hat{A} \phi^{*i} = \hat{A}, \quad \text{and} \quad (\phi + \psi)A_{k-1} \leq (\phi + \psi)\hat{A}^{*j} \leq \hat{A}^{*j}.$$

Thus,  $A_k = ((\phi + \psi)A_{k-1}) \odot_j (A\phi^{*i}) \leq \hat{A}^{*j} \odot_j \hat{A}^{*j} = \hat{A}^{*j}$ , which completes the proof.  $\square$

We now reprove this theorem using the implicit induction of Kleene algebra.

**4.2.8. Theorem (Coherent Church-Rosser in globular  $n$ -MKA).** *Let  $K$  be a globular  $n$ -modal Kleene algebra and  $0 \leq i < j < n$ . Given  $\phi, \psi \in K_j$  and an  $i$ -confluence filler  $A \in K$  of  $(\phi, \psi)$ , we have*

$$|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \geq (\phi + \psi)^{*i},$$

where  $\hat{A}$  is the  $j$ -dimensional  $i$ -whiskering of  $A$ . Thus  $\hat{A}^{*j}$  is an  $i$ -Church-Rosser filler for  $(\phi, \psi)$ .

*Proof.* As in the previous proof,  $i$ -multiplication will be denoted by juxtaposition. Let  $\phi, \psi$  be in  $K_j$ , for  $0 < j < n$ , and  $A$  in  $K$  be an  $i$ -confluence filler of  $(\phi, \psi)$ , with  $0 \leq i < j$ . By the left  $i$ -star induction axiom, see Section 3.1.8, we have

$$1_i + (\phi + \psi)|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \Rightarrow (\phi + \psi)^{*i} \leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})$$

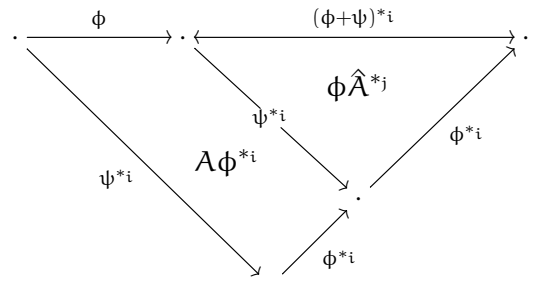
The inequality  $1_i \leq \psi^{*i}\phi^{*i} \leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})$  holds. Indeed, by the unfold axiom from Section 3.1.8, we have  $1_i \leq \psi^{*i}$ ,  $1_i \leq \phi^{*i}$ , giving the first inequality, and  $1_j \leq \hat{A}^{*j}$ . The latter implies that  $\text{id}_{S_{d_j}} = |1_j\rangle_j \leq |\hat{A}^{*j}\rangle_j$ , which gives  $\psi^{*i}\phi^{*i} \leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})$ . It then remains to show that

$$(\phi + \psi)|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}).$$

By distributivity, we may prove this for each of the summands:

- In the case of whiskering by  $\phi$  on the left:

$$\begin{aligned} \phi|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) &\leq |\phi\hat{A}^{*j}\rangle_j(\phi\psi^{*i}\phi^{*i}) \\ &\leq |\phi\hat{A}^{*j}\rangle_j(|A\rangle_j(\psi^{*i}\phi^{*i})\phi^{*i}) \\ &\leq |\phi\hat{A}^{*j}\rangle_j(|A\phi^{*i}\rangle_j(\psi^{*i}\phi^{*i}\phi^{*i})) \\ &\leq |\phi\hat{A}^{*j}\rangle_j(\phi\hat{A}^{*j} \odot_j A\phi^{*i})\rangle_j(\psi^{*i}\phi^{*i}) \\ &\leq |\hat{A}^{*j} \odot_j \hat{A}\rangle_j(\psi^{*i}\phi^{*i}) \\ &\leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \end{aligned}$$



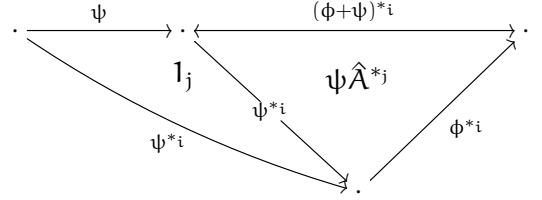
The first step is given by whiskering properties from (4.2.3), the second by the hypothesis that  $A$  is an  $i$ -confluence filler and that  $\phi\psi^{*i} \leq \phi^{*i}\psi^{*i}$ . The third step is again by whiskering, and the fourth follows by definition of diamonds and axiom **ii**) from (3.1.3). The fifth follows by whisker absorption, (4.2.3), and the last step follows from the unfold axiom from (3.1.8), since it implies that  $x \cdot x^* \leq x^*$ .



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— In the case of whiskering by  $\psi$  on the right:

$$\begin{aligned} \psi|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) &\leq |\psi\hat{A}^{*j}\rangle_j(\psi\psi^{*i}\phi^{*i}) \\ &\leq |\psi\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \\ &\leq |\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}). \end{aligned}$$



The first step is again by whiskering properties from Section 4.2.3, the second by the fact that  $\psi\psi^{*i} \leq \psi^{*i}$  which as explained above is a consequence of the unfold axiom recalled in Section 3.1.8. Finally, whisker absorption justifies the last inequality.  $\square$

**4.2.9. Remarks.** Note that in Theorem 4.2.7, the elements  $A_k$  verify  $|A_k\rangle_j(\psi^{*i}\phi^{*i}) \geq (\phi + \psi)^{k_i}$ , meaning that scanning backward along  $A_k$  from  $\psi^{*i}\phi^{*i}$ , we see *at least* all of the "zig-zags" in  $\phi$  and  $\psi$  of length  $k$ , whereas in Theorem 4.2.8, the inequality  $|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \geq (\phi + \psi)^{*i}$  means that scanning back from  $\psi^{*i}\phi^{*i}$ , we see *at least* all of the zig-zags in  $\phi$  and  $\psi$  of any length. However, the elements  $A_k$  from Theorem 4.2.7 satisfy in addition

$$\langle A_k|_j((\phi + \psi)^{k_i}) \leq \psi^{*i}\phi^{*i}.$$

This formulation is of interest, since it coincides with the intuition of paving *from* zigzags  $(\phi + \psi)^{k_i}$  to the confluences  $\psi^{*i}\phi^{*i}$ . However, this sort of inequality cannot be expected of the  $j$ -dimensional  $i$ -completion of  $A$ , since in general, using the path algebra intuition,  $\hat{A}^{*j}$  contains cells which go from zigzags to zigzags. In conclusion, the fact that the diamonds scan *all* possible future or past states means that we must formulate as in Theorem 4.2.8 when considering completions, or construct the elements paving precisely what we would like as in Theorem 4.2.7.

**4.2.10. Corollary.** Let  $K$  be a globular modal  $n$ -Kleene algebra. Given  $\phi, \psi \in K_j$ , for  $i < j < n$ , for any semi- $i$ -confluence filler  $A \in K$  we have

$$|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \geq (\phi + \psi)^{*i},$$

where  $\hat{A}$  is the  $j$ -dimensional  $i$ -whiskering of  $A$ .

*Proof.* In the case of a left semi-confluence filler, the proof is identical. If  $A$  is a right semi-confluence filler, we use the right  $i$ -star axiom and the proof is given by symmetry.  $\square$

## 4.3. Newman's lemma in globular modal $n$ -Kleene algebra

**4.3.1. Termination in  $n$ -semirings.** We define the notion of termination, or Noethericity, in a modal  $n$ -semiring  $K$  as an extension of the notion of termination in modal Kleene algebras, recalled in (4.1.1). Given  $0 \leq i < j < n$ , an element  $\phi \in K_j$  is said to be  *$i$ -Noetherian* or  *$i$ -terminating* if

$$p \leq |\phi\rangle_i p \Rightarrow p \leq 0$$

### 4.3. Newman's lemma in globular modal $n$ -Kleene algebra

holds for all  $p \in K_i$ . The set of  $i$ -Noetherian elements of  $K$  is denoted by  $\mathcal{N}_i(K)$ . When  $K$  is a modal  $p$ -Boolean semiring, we recall that as a consequence of the adjunction between diamonds and boxes, see Section 3.1.9, we obtain an equivalent formulation of Noethericity in terms of the forward box operator:

$$\phi \in \mathcal{N}_i(K) \quad \iff \quad \forall p \in K_i, |\phi|_i p \leq p \Rightarrow 1_i \leq p.$$

We also define a notion of *well-foundedness*;  $\phi$  is said to be  $i$ -well-founded if it is  $i$ -Noetherian in the opposite  $n$ -semiring of  $K$ .

**4.3.2. Theorem (Coherent Newman's lemma for globular  $p$ -Boolean MKA).** *Let  $K$  be a globular  $p$ -Boolean modal Kleene algebra, and  $0 \leq i \leq p < j < n$ , such that*

- i)  $(K_i, +, 0, \odot_i, 1_i, \neg_i)$  is a complete Boolean algebra,
- ii)  $K_j$  is continuous with respect to  $i$ -restriction, i.e. for all  $\psi, \psi' \in K_j$  and every family  $(p_\alpha)_{\alpha \in I}$  of elements of  $K_i$  such that  $\sup_I(p_\alpha)$  exists, we have

$$\psi \odot_i \sup_I(p_\alpha) \odot_i \psi' = \sup_I(\psi \odot_i p_\alpha \odot_i \psi').$$

Let  $\psi \in K_j$  be  $i$ -Noetherian and  $\phi \in K_j$   $i$ -well-founded. If  $A$  is a local  $i$ -confluence filler for  $(\phi, \psi)$ , then

$$|\hat{A}^{*j}|_j(\psi^{*i} \phi^{*i}) \geq \phi^{*i} \psi^{*i},$$

i.e.  $\hat{A}^{*j}$  is a confluence filler for  $(\phi, \psi)$ .

*Proof.* We denote  $i$ -multiplication by juxtaposition. First, we define a predicate expressing restricted  $j$ -paving. Given  $p \in K_i$ , let

$$\text{RP}(p) \quad \iff \quad |\hat{A}^{*1}|_j(\psi^{*i} \phi^{*i}) \geq \phi^{*i} p \psi^{*i}.$$

By completeness of  $K_i$ , we may set  $r := \sup\{p \mid \text{RP}(p)\}$ . By continuity of  $i$ -restriction, we may infer  $\text{RP}(r)$ . Furthermore, by downward closure of  $\text{RP}$ , we have the following equivalence:

$$\text{RP}(p) \quad \iff \quad p \leq r.$$

This in turn allows us to make the following deductions:

$$\begin{aligned} \forall p. (\text{RP}(|\phi|_i p) \wedge \text{RP}(\langle \psi |_i p) \Rightarrow \text{RP}(p)) &\iff \forall p. (|\phi|_i p \leq r \wedge \langle \psi |_i p \leq r \Rightarrow p \leq r) \\ &\iff \forall p. (p \leq [|\phi|_i r \wedge \langle \psi |_i r] \Rightarrow p \leq r) \\ &\iff [|\phi|_i r \leq r \wedge \langle \psi |_i r] \leq r \end{aligned}$$

Thus, it suffices to show  $\forall p. (\text{RP}(|\phi|_i p) \wedge \text{RP}(\langle \psi |_i p) \Rightarrow \text{RP}(p))$  in order to conclude that  $r = 1_i$ , by Noethericity (resp. well-foundedness) of  $\psi$  (resp.  $\phi$ ).

Let  $p \in K_i$ , set  $|\phi|_i(p) = p_\phi$  and  $\langle \psi |_i(p) = p_\psi$  and suppose that  $\text{RP}(p_\phi)$  and  $\text{RP}(p_\psi)$  hold. Note that we have

$$\phi p = d_i(\phi p) \phi p = |\phi|_i(p) \phi p \leq p_\phi \phi,$$

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since  $d(x)x = x$  by axiom **i**) from Section 3.1.3 and  $p \leq 1_i$ . We have a similar inequality for  $\psi$ , that is  $p\psi \leq \psi p\psi$ . These inequalities, along with the unfold axioms from Section 3.1.8, give

$$\begin{aligned} \phi^{*i} p \psi^{*i} &\leq \phi^{*i} p + \phi^{*i} \phi p \psi \psi^{*i} + p \psi \psi^{*i} \\ &\leq \phi^{*i} p + \phi^{*i} p \phi \phi \psi p \psi \psi^{*i} + p \psi \psi^{*i}. \end{aligned}$$

The outermost summands are below  $|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})$ . Indeed,  $\text{id}_{S_j} = |1_j\rangle_j \leq |\hat{A}^{*j}\rangle_j$  since  $1_j \leq \hat{A}^{*j}$ ,  $p \leq 1_i$  and  $\phi^{*i}, \psi^{*i} \leq \psi^{*i}\phi^{*i}$ .

For the middle summand, we calculate

$$\begin{aligned} \phi^{*i} p \phi \phi \psi p \psi \psi^{*i} &\leq \phi^{*i} p \phi |A\rangle_j(\psi^{*i}\phi^{*i}) p \psi \psi^{*i} \\ &\leq |\phi^{*i} p \phi A p \psi \psi^{*i}\rangle_j(\phi^{*i} p \psi \psi^{*i} \phi^{*i} p \psi \psi^{*i}) \\ &\leq |\phi^{*i} p \phi \hat{A} p \psi \psi^{*i}\rangle_j(|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})\phi^{*i} p \psi \psi^{*i}) \\ &\leq |\hat{A}\rangle_j(|\hat{A}^{*j}\phi^{*i} p \psi \psi^{*i}\rangle_j(\psi^{*i}\phi^{*i} p \psi \psi^{*i})) \\ &\leq |\hat{A} \odot_j \hat{A}^{*j} \phi^{*i} p \psi \psi^{*i}\rangle_j(\psi^{*i}\phi^{*i} p \psi \psi^{*i}) \\ &\leq |\hat{A} \odot_j \hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i} p \psi \psi^{*i}), \end{aligned}$$

The first step is by the local  $i$ -confluence filler hypothesis, the second by whiskering properties from Section 4.2.3 and the third by  $\text{RP}(p_\phi)$ . The fourth step is again by whiskering properties, and the fifth follows from axiom **ii**) in Section 3.1.3 and the definition of diamond operators. The final step is by whisker absorption, see Section 4.2.3. By similar arguments, we have

$$\begin{aligned} |\hat{A} \odot_j \hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i} p \psi \psi^{*i}) &\leq |\hat{A} \odot_j \hat{A}^{*j}\rangle_j(\psi^{*i}|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})) \\ &\leq |\hat{A} \odot_j \hat{A}^{*j}\rangle_j(|\psi^{*i}\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i})) \\ &\leq |\hat{A} \odot_j \hat{A}^{*j} \odot_j \psi^{*i}\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \\ &\leq |\hat{A} \odot_j \hat{A}^{*j} \odot_j \hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \end{aligned}$$

Indeed, the first step follows from  $\text{RP}(p_\psi)$ , and the second by whiskering properties. The third step follows from axiom **ii**) in Section 3.1.3 and the definition of diamond operators as in the preceding calculation. The final step follows from whisker absorption. Finally, we observe that

$$\hat{A} \odot_j \hat{A}^{*j} \odot_j \hat{A}^{*j} \leq \hat{A}^{*j},$$

and thus by monotonicity of the diamond operator we may conclude that

$$\phi^{*i} p_\phi \phi \psi p_\psi \psi^{*i} \leq |\hat{A}^{*j}|_j (\psi^{*i} \phi^{*i}).$$

We have thereby shown that  $\forall p (RP(p_\phi) \wedge RP(p_\psi) \Rightarrow RP(p))$  and thus that  $r = 1_i$ , concluding the proof.  $\square$

**4.3.3. Remark.** Similarly to the discussion from Remark 2.3.10 in the context of polygraphs, we remark here that the proofs of Theorems 4.2.8 and 4.3.2 are similar to those of the analogous 1-dimensional results for modal Kleene algebra found in [5, 28]. Indeed, if we look exclusively at the induction axioms and deductions applied to  $j$ -dimensional cells, we obtain the same proof structures as in the case of modal Kleene algebras. This indicates that the structure of globular modal  $n$ -Kleene algebra is a natural higher dimensional generalisation of modal Kleene algebras in which proofs of coherent confluence may be calculated. The consistency of the abstract, algebraic results from the previous sections with the point-wise, polygraphic results from Section 2.3 are made explicit in the next section.

## 4.4. Application in rewriting

In this section we interpret the theorems from the preceding section in terms of polygraphs. We fix an  $n$ -polygraph  $P$  and a cellular extension  $\Gamma$  of  $P_n^\top$ .

**4.4.1. Converses.** Recall from [1] that a Kleene algebra with converse is a Kleene algebra  $K$  equipped with an involution  $(-)^{\vee} : K \rightarrow K$  that distributes through addition, acts contravariantly on multiplication, commutes with the Kleene star, *i.e.*

$$\begin{aligned} (a + b)^{\vee} &= a^{\vee} + b^{\vee}, & (a \cdot b)^{\vee} &= b^{\vee} \cdot a^{\vee}, \\ (a^*)^{\vee} &= (a^{\vee})^*, & (a^{\vee})^{\vee} &= a, \end{aligned}$$

and satisfies the inequality  $a \leq a a^{\vee} a$ . When the underlying Kleene algebra is a modal Kleene algebra, we say that it is a modal Kleene algebra with converse, see also [4].

**4.4.2.  $(n, p)$ -Kleene algebra.** A modal  $(n, p)$ -Kleene algebra  $K$  is a modal  $n$ -Kleene algebra equipped with operations  $(-)^{\vee_j} : K_{j+1} \rightarrow K_{j+1}$  for  $p \leq j < n - 1$  and an operation  $(-)^{\vee_{n-1}} : K \rightarrow K$ , satisfying the axioms listed above relative to the appropriate multiplication operation, *i.e.* for all  $\phi, \psi \in K_{j+1}$ ,

$$\begin{aligned} (\phi + \psi)^{\vee_j} &= \phi^{\vee_j} + \psi^{\vee_j}, & (\phi \odot_j \psi)^{\vee_j} &= \psi^{\vee_j} \odot_j \phi^{\vee_j}, \\ (\phi^{*j})^{\vee_j} &= (\phi^{\vee_j})^{*j}, & (\phi^{\vee_j})^{\vee_j} &= \phi, & \phi &\leq \phi \odot_j \phi^{\vee_j} \odot_j \phi, \end{aligned}$$

and  $(-)^{\vee_{n-1}}$  satisfies the above axioms with  $j = n - 1$  and for any elements of  $K$ . Note that for  $\phi \in K_i$  with  $i < j$ , we have  $\phi^{\vee_j} = \phi$ . This is a consequence of the fact that  $\odot_j$  is idempotent for elements of  $K_i$ .

## 4. Algebraic coherent confluence

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**4.4.3. Conversion in  $K(P, \Gamma)$ .** The modal  $(n + 1)$ -Kleene algebra  $K(P, \Gamma)$ , as defined in Section 3.3.1, is a modal  $(n + 1, n - 1)$ -Kleene algebra. Indeed, for any  $\phi \in K(P, \Gamma)_n$  and  $A \in K$ , let

$$\phi^{\vee_{n-1}} := \{ u^- \mid u \in \phi \} \quad \text{and} \quad A^{\vee_n} = \{ \alpha^- \mid \alpha \in A \}.$$

This operation is well defined in the following sense: If  $\phi \in K(P, \Gamma)_n$ , then  $\phi$  is a set of cells of dimension less than or equal to  $n$ . Given a cell  $v$  of dimension  $i < n$ , its  $n$ -inverse is itself, since we consider it as an identity. Given a cell  $u$  of dimension  $n$ , we know that  $u^-$  is well defined since if  $u \in P_n^\top$  then  $u^- \in P_n^\top$ . Similarly for the case of  $(-)^{\vee_n}$ .

**4.4.4.  $\Gamma$ -coherence properties as fillers.** Recall that  $\Gamma$  and  $P_n^*$  are themselves elements of  $K(P, \Gamma)$ , and that in Proposition 3.3.2 we observed that

$$\Gamma^c = (1_n \odot_{n-1} (\cdots \odot_2 (1_2 \odot_1 (1_1 \odot_0 \Gamma \odot_0 1_1) \odot_1 1_2) \odot_2 \cdots) \odot_{n-1} 1_n),$$

where  $\Gamma^c$  is the set of cells of  $\Gamma$  in context. In the following, we will denote by  $P_n^c$  the set of rewriting steps generated by  $P_n$ , which can be expressed in  $K(P, \Gamma)$  as

$$P_n^c = (1_{n-1} \odot_{n-2} (\cdots \odot_2 (1_2 \odot_1 (1_1 \odot_0 P_n \odot_0 1_1) \odot_1 1_2) \odot_2 \cdots) \odot_{n-2} 1_{n-1}).$$

The construction of  $K(P, \Gamma)$  is compatible with  $\Gamma$ -coherence properties in the following sense:

**4.4.5. Proposition.** *With  $\Gamma' := (\Gamma^c)^{*n}$ , the following equivalences hold:*

- i)  $\Gamma$  is a (local) confluence filler for  $P \iff \Gamma'$  is a (local)  $(n - 1)$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ ,
- ii)  $\Gamma$  is a Church-Rosser filler for  $P \iff \Gamma'$  is an  $(n - 1)$ -Church-Rosser filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ .

*Proof.* Let us prove the equivalence in the case of (global) confluence.

Suppose that  $\Gamma$  is a confluence filler for  $P$ . An element  $f^- \star_{n-1} g \in (P_n^c)^{\vee_{n-1}} \odot_{n-1} P_n^c$  corresponds to a branching  $(f, g)$ . By hypothesis, there exists an  $\alpha \in P_n^\top[\Gamma]$  such that  $s_n(\alpha) = f^- \star_{n-1} g$  and  $\alpha$  is an  $n$ -composition of rewriting steps so  $\alpha \in \Gamma'$ . Furthermore, the  $n$ -target of  $\alpha$  is a confluence, so  $\alpha \in \Gamma' \odot_n (P_n^c \odot_{n-1} (P_n^c)^{\vee_{n-1}})$ . In equations, this means that

$$(P_n^c)^{\vee_{n-1}} \odot_{n-1} P_n^c \subseteq d_n \left( \Gamma' \odot_n (P_n^c \odot_{n-1} (P_n^c)^{\vee_{n-1}}) \right) = |\Gamma'|_n \left( P_n^c \odot_{n-1} (P_n^c)^{\vee_{n-1}} \right),$$

*i.e.*  $\Gamma'$  is an  $(n - 1)$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ .

Conversely, if  $\Gamma'$  is an  $(n - 1)$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ , then given some branching  $(f, g)$ , we know that  $f^- \star_{n-1} g \in d_i \Gamma' \odot_n (P_n^c \odot_{n-1} (P_n^c)^{\vee_{n-1}})$ . This means there exists some cell  $\alpha \in \Gamma'$  with  $n$ -source  $f^- \star_{n-1} g$  and whose  $n$ -target is a confluence. Since  $\alpha \in \Gamma'$ , we know that it is a composition of rewriting steps of  $\Gamma$ . With this we conclude that  $P$  is  $\Gamma$ -confluent.

The other cases are similarly deduced. □

Due to this compatibility, we may deduce the following theorems, that is Theorems 2.3.4 and 2.3.7, as corollaries of our main results:

**4.4.6. Theorem (Church Rosser for  $n$ -polygraphs).** *Let  $P$  be an  $n$ -polygraph and  $\Gamma$  a cellular extension of  $P_n^\top$ . Then  $\Gamma$  is a confluence filler for  $P$  if, and only if,  $\Gamma$  is a Church-Rosser filler for  $P$ .*

*Proof.* Suppose first that  $\Gamma$  is a confluence filler for  $P$ . Using the result and notations from Proposition 4.4.5, we know that  $\Gamma'$  is an  $(n-1)$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . We apply Theorem 4.2.8 to  $K(P, \Gamma)$  for  $i = n-1$  and  $j = n$ , obtaining that  $\widehat{\Gamma}'^{*n}$  is an  $(n-1)$ -Church-Rosser filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . Observing that  $(P_n^c + (P_n^c)^{\vee_{n-1}})^{*_{n-1}} = P_n^\top$ , we have

$$\widehat{\Gamma}'^{*n} = \left( P_n^\top \odot_{n-1} (\Gamma^c)^{*n} \odot_{n-1} P_n^\top \right)^{*n} \subseteq \left( (P_n^\top \odot_{n-1} \Gamma^c \odot_{n-1} P_n^\top)^{*n} \right)^{*n} = \Gamma',$$

where the first step is by definition, the second uses the fact that the  $n$ -star is a lax morphism for  $(n-1)$ -multiplication, see Section 3.2.16, and the third uses the fact that  $\Gamma^c$  absorbs whiskers and that  $(A^{*n})^{*n} = A^{*n}$ . Since additionally,  $\Gamma' \subseteq \widehat{\Gamma}'^{*n}$ ,  $\Gamma'$  is an  $(n-1)$ -Church-Rosser filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . By Proposition 4.4.5, this allows us to conclude that  $\Gamma$  is a Church-Rosser filler for  $P$ .

For the trivial direction, suppose that  $\Gamma$  is a Church-Rosser filler for  $P$ . We deduce by Proposition 4.4.5 that  $\Gamma'$  is an  $(n-1)$ -Church-Rosser filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . As pointed out at the end of Section 4.2.1, this means that  $\Gamma'$  is an  $i$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ , by which we conclude that  $\Gamma$  is a confluence filler for  $P$ .  $\square$

**4.4.7. Theorem (Newman for  $n$ -polygraphs).** *Let  $P$  be a terminating  $n$ -polygraph and  $\Gamma$  a cellular extension of  $P_n^\top$ . Then  $\Gamma$  is a local confluence filler for  $P$  if, and only if,  $\Gamma$  is a confluence filler for  $P$ .*

*Proof.* Suppose that  $\Gamma$  is a local confluence filler for  $P$ . Using the result and notations from Proposition 4.4.5, we know that  $\Gamma'$  is an  $(n-1)$ -local confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . We apply Theorem 4.3.2 to  $K(P, \Gamma)$  for  $i = n-1$  and  $j = n$ , obtaining that  $\widehat{\Gamma}'^{*n}$  is an  $(n-1)$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . As in the proof of the previous theorem, we have that  $\widehat{\Gamma}'^{*n} = \Gamma'$ , allowing us to conclude that  $\Gamma$  is a confluence filler for  $P$ , again by Proposition 4.4.5.

For the trivial direction, suppose that  $\Gamma$  is a confluence filler for  $P$ . As above, we deduce that  $\Gamma'$  is an  $(n-1)$ -Church-Rosser filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ . Again, as pointed out at in Section 4.2.1, this means that  $\Gamma'$  is a local  $i$ -confluence filler for  $((P_n^c)^{\vee_{n-1}}, P_n^c)$ , by which we conclude that  $\Gamma$  is a local confluence filler for  $P$  via Proposition 4.4.5.  $\square$

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