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Shuffle polygraphic resolutions for operads

Philippe Malbos - Isaac Ren

Abstract – Shuffle operads were introduced to forget the symmetric group actions on symmetric operads while preserving all possible operadic compositions. Rewriting methods were then applied to symmetric operads via shuffle operads: in particular, a notion of Gröbner basis was introduced for shuffle operads with respect to a total order on tree monomials. In this article, we introduce the structure of shuffle polygraphs as a categorical model for rewriting in shuffle operads, which generalizes the Gröbner bases approach by removing the constraint of a monomial order for the orientation of the rewriting rules. We define ω-operads as internal ω-categories in the category of shuffle operads. We show how to extend a convergent shuffle polygraph into a shuffle polygraphic resolution generated by the overlapping branchings of the original polygraph. Finally, we prove that a shuffle operad presented by a quadratic convergent shuffle polygraph is Koszul.

Keywords – Shuffle operads, higher-dimensional rewriting, Gröbner bases, Koszulness.

M.S.C. 2020 – 18M70, 68Q42, 18N30.

1 Introduction **2**
2 Higher shuffle operads **6**
2.1 Shuffle operads . . . 6
2.2 Compositions in shuffle operads . . . 11
2.3 Higher shuffle operads . . . 13
2.4 Higher operads as globular bimodules . . . 16
3 Shuffle operadic polygraphs **23**
3.1 Shuffle polygraphs . . . 23
3.2 Shuffle polygraphic resolutions . . . 27
4 Shuffle operadic rewriting **29**
4.1 Rewriting in shuffle operads . . . 29
4.2 Confluence of shuffle polygraphs . . . 33
4.3 Monomial-ordered shuffle polygraphs . . . 37
5 Shuffle polygraphic resolutions from convergence **39**
5.1 Polygraphic resolutions and contractions . . . 39
5.2 Polygraphic resolution from a convergent presentation . . . 44
5.3 Bimodule resolutions from polygraphic resolutions . . . 51
5.4 Confluence and Koszulness . . . 55

1. Introduction

Algebraic rewriting theory provides methods to compute cofibrant replacements of algebraic structures from presentations that take into account computational properties of these structures. This rewriting approach gives algebraic algorithmic methods to solve decidability and computational problems, such as the ideal membership problem, and the computation of linear bases and of (co)homological properties. Abelian resolutions for monoids [13, 33, 46], groups [18], small categories [26], associative algebras [2, 25], and linear operads [19, 20] have been constructed using rewriting methods. The machinery at the heart of these constructions consists in presenting an algebraic structure by a system of *generators* and *rewriting rules*, and producing a cofibrant replacement that involves the overlappings occurring in the applications of the rewriting rules. Rewriting approaches for linear structures were developed in many algebraic algorithmic contexts, notably by Janet and Buchberger for commutative algebras [14, 30], Shirshov, Bokut, and Bergman for associative algebras [10, 11, 45], Dotsenko-Khoroshkin for linear operads [19, 31]. In all of these works, the rewriting systems are formulated in terms of *Gröbner bases*, and thus are defined with respect to a given monomial order. Rewriting approaches have also been used in the categorical context to present higher categories by higher-dimensional rewriting systems, called *polygraphs* (or *computads*) [16, 48]. In this context, the cofibrant replacements of a higher category are generated by *polygraphic resolutions* introduced in [26, 34, 40].

An important issue when studying algebras or operads is the automatic construction of small abelian resolutions. There exist some inductive constructions that start from presentations with certain computational properties. In particular, Anick introduced a general machinery that computes a resolution for an associative algebra, whose n-dimensional generators correspond to overlappings of applications of n defining relations. This resolution is a compromise between the bar resolution, which is easy to compute but very large, and the minimal resolution, which is difficult to make explicit in general. Moreover, it is difficult to determine conditions for which Anick's resolution is minimal [49]. Some conditions have been shown to be sufficient: as an immediate consequence of its construction, Anick's resolution is minimal for monomial algebras, and a quadratic algebra with a convergent presentation has a minimal resolution given by its Koszul dual. For N-homogeneous algebras, minimality is harder, and Berger introduces an extra condition in [9]. Dotsenko and Khoroshkin constructed resolutions for shuffle monomial operads by the inclusion-exclusion principle and for operads presented by a Gröbner basis by deformation of the monomial case [20]. In this operadic case, the question of minimality of the resolution is even more difficult due to the combinatorial complexity of the underlying tree structure of shuffle operad terms. In particular, unlike for algebras the constructed resolution for a monomial operad is not necessarily minimal.

In this work, we combine the polygraphic and the Gröbner bases approaches in order to compute higher-dimensional presentations of shuffle operads using the polygraphic machinery. We define *shuffle ω-operads* as internal ω-categories in the category of shuffle operads. We introduce the structure of *shuffle ω-polygraphs* as systems of generators and relations for shuffle ω-operads. Unlike the Gröbner bases approach, the orientation of the relations in a shuffle polygraph does not depend on a given monomial order. The main construction of this article extends a confluent and terminating shuffle polygraph presenting a shuffle operad into a shuffle polygraphic resolution generated by the overlapping branchings of the original polygraph. In order to address the question of minimal resolutions, we make explicit these overlappings in all dimensions of the polygraphic resolution. We then give an inductive

method to compute a bimodule resolution that allows us to state a minimality result for shuffle operads, as well as a condition for Koszulness.

Now we present the organization and the main results of this article.

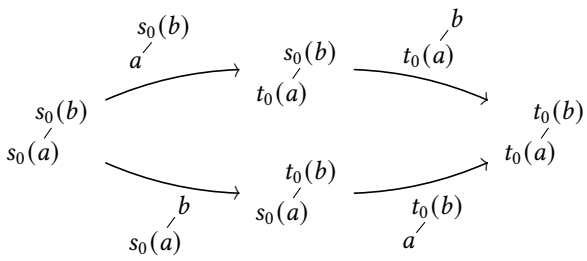
Higher operads

The notion of a *symmetric operad* appears in many situations to describe operations in several arguments, with symmetric group actions, acting on topological or algebraic objects [35, 39]. *Shuffle operads* were introduced by Dotsenko and Khoroshkin in [19] to forget the symmetric group actions on the arguments while preserving all possible operadic compositions. The shuffle version allows us to define *monomials* and *oriented relations* in order to present symmetric operads by rewriting systems. Symmetric and shuffle operads are defined as internal monoids in the monoidal presheaf categories of *collections* and *symmetric collections* respectively, as recalled in Section 2.1. Explicitly, a *collection* is a presheaf on the category Ord of nonempty finite ordered sets and order-preserving bijections, with values in the category Vect of vector spaces. The monoidal product on collections is the *shuffle composition* recalled in § 2.1.3. A *symmetric collection* is a presheaf on the category Fin of nonempty finite sets and bijections, with values in the category Vect. The functor $-^u : \mathsf{Ord} \to \mathsf{Fin}$ that forgets the order induces a functor $-^u : \mathfrak{S}\mathsf{Coll} \to \mathsf{Coll}$ from the category of symmetric collections to the category of collections. Since we restrict to nonempty sets, none of the operads considered in this work have operations of arity 0.

In Section 2.3, we introduce the notion of a (strict) higher shuffle operad. We define a *shuffle ω-operad* as an internal ω-category in the category $\sqcup\!\sqcup\mathsf{Op}$ of shuffle operads. Shuffle ω-operads, with internal ω-functors, form a category denoted by $\sqcup\!\sqcup\mathsf{Op}_\omega$. In Section 2.4, we study the interaction between the higher-categorical structure of ω-operads and its underlying linear structure. The object of n-cells of a shuffle ω-operad has a shuffle operad structure, and the n-cells can be $\star_k$*-composed* along k-dimensional cells for $0 \leqslant k < n$. Due to the linear structure, the $\star_k$-composition of two n-cells a and b in a shuffle ω-operad can be written as the following linear combination:

$$a \star_k b = a - t_k(a) + b,$$

where $t_k(a)$ denotes the k-dimensional target of a, which coincides with the k-dimensional source of b. In particular, every n-cell in a shuffle ω-operad is invertible. Moreover, for $n \geqslant 1$, the compatibility between the shuffle composition and the $\star_0$-composition implies that the *elementary composition* $a \circ_{i,\tau} b$ of n-cells a and b, as defined in § 2.2.1, can be seen either one of two *orthogonal reduction paths* from $s_0(a) \circ_{i,\tau} s_0(b)$ to $t_0(a) \circ_{i,\tau} t_0(b)$, pictured as follows:



The *linear exchange relation* introduced in § 2.4.2 states that these two reductions paths are equal. With these remarkable relations, the axioms of shuffle ω-operads can be simplified. We deduce a characteri-

zation of the structure of ω-operad in terms of bimodules over shuffle operads. Our first result, Theorem 2.4.8, proves that the category $\sqcup\!\sqcup\mathsf{Op}_\omega$ is isomorphic to the full subcategory of $\mathsf{RGlob}(\mathsf{Bimod}(\sqcup\!\sqcup\mathsf{Op}))$, whose objects are pairs (P, A) where P is a shuffle operad and $A = (A_n)_{n\geqslant 0}$ is a reflexive globular P-bimodule such that $A_0 = P$, and A_n satisfies the linear exchange relation for all $n \geqslant 1$.

Shuffle operadic polygraphs and rewriting

The notion of a *polygraph* was introduced in the set-theoretical context by Street and Burroni as systems of generators and relations for presentations of higher (strict) categories [15, 48]. A linear version of polygraphs was introduced in [25] for the presentation of associative ω-algebras. In Section 3.1, we define an analogous notion for shuffle ω-operads, which we call *shuffle polygraphs*. Explicitly, for $n \geqslant 0$, a *shuffle n-polygraph* is a data $X = (X_0, \dots X_n)$ defined by induction, where X_k, the set of *k-generators*, forms a globular extension of the free shuffle $(k-1)$-operad generated by the shuffle $(k-1)$-polygraph $(X_0, \dots X_{k-1})$. Such a data can be pictured as a diagram

$$\begin{tikzcd}
X_0^{\sqcup\!\sqcup} \arrow[r, shift right, "t_0^{\sqcup\!\sqcup}"'] \arrow[r, shift left, "s_0^{\sqcup\!\sqcup}"] & X_1^{\sqcup\!\sqcup} \arrow[l, shift right, "t_0"'] \arrow[l, shift left, "s_0"] \arrow[r, shift right, "t_1^{\sqcup\!\sqcup}"'] \arrow[r, shift left, "s_1^{\sqcup\!\sqcup}"] & \cdots \arrow[l, shift right, "t_1"'] \arrow[l, shift left, "s_1"] \arrow[r, shift right, "t_{n-2}^{\sqcup\!\sqcup}"'] \arrow[r, shift left, "s_{n-2}^{\sqcup\!\sqcup}"] & X_{n-1}^{\sqcup\!\sqcup} \arrow[l, shift right, "t_{n-2}"'] \arrow[l, shift left, "s_{n-2}"] \arrow[r, shift right, "t_{n-1}^{\sqcup\!\sqcup}"'] \arrow[r, shift left, "s_{n-1}^{\sqcup\!\sqcup}"] & X_n^{\sqcup\!\sqcup} \arrow[l, shift right, "t_{n-1}"'] \arrow[l, shift left, "s_{n-1}"] \\
X_0 \arrow[u, hook, "\iota_0"] & X_1 \arrow[u, hook, "\iota_1"] & \cdots & X_{n-1} \arrow[u, hook, "\iota_{n-1}"] & X_n \arrow[u, hook, "\iota_n"]
\end{tikzcd}$$

where s_i and t_i denote the source and target maps of the globular extensions, and the horizontal diagram corresponds to the underlying globular operad of the free n-operad generated by the n-polygraph X, denoted by $X_n^{\sqcup\!\sqcup}$. As for set-theoretical polygraphs, in Section 3.1 we define the category $\sqcup\!\sqcup\mathsf{Pol}_n$ of *shuffle n-polygraphs* and the free n-operad functor $(-)^{\sqcup\!\sqcup} : \sqcup\!\sqcup\mathsf{Pol}_n \to \sqcup\!\sqcup\mathsf{Op}_n$ by induction on the dimension n, and the category of shuffle ω-polygraphs as the limit of the forgetful functors $\sqcup\!\sqcup\mathsf{Pol}_n \to \sqcup\!\sqcup\mathsf{Pol}_{n-1}$ for $n \geqslant 1$.

The shuffle polygraphic approach lets us present shuffle operads by oriented presentations, called *rewriting systems*: the shuffle operad $\overline{X}$ presented by a shuffle 1-polygraph X is defined as the coequalizer of the source and target morphisms $s_0^{\sqcup\!\sqcup}, t_0^{\sqcup\!\sqcup} : X_1^{\sqcup\!\sqcup} \rightrightarrows X_0^{\sqcup\!\sqcup}$ in the category $\sqcup\!\sqcup\mathsf{Op}$. Note that, in this work, we consider operads with only one color. The 0-generators correspond to the generators of the shuffle operads, and the 1-generators correspond to the oriented relations. For presentations of multicolored shuffle operads, we need to consider shuffle two-dimensional polygraphs, whose 0-generators correspond to colors, 1-generators to generators, and 2-generators to oriented relations.

We use rewriting theory on shuffle 1-polygraphs to deduce global rewriting properties, such as *confluence* and *termination* from local properties of the 1-generators, also called *rewriting rules*. Without loss of generality, in Section 4 we will consider 1-polygraphs with *left-monomial* rules, reducing a single monomial into a linear combination of monomials. A *rewriting step* of a left-monomial 1-polygraph X is a 1-cell f of the free shuffle 1-operad $X_1^{\sqcup\!\sqcup}$ of size 1, and of the form $f = \lambda g + 1_c$, where λ is a nonzero scalar, g is a 1-monomial of $X_1^{\sqcup\!\sqcup}$, and c is a 0-cell of the free shuffle operad $X_0^{\sqcup\!\sqcup}$ such that the 0-monomial $s_0(u) \notin \mathrm{Supp}(c)$. A 1-cell of the free 1-operad $X_1^{\sqcup\!\sqcup}$ is *positive* if it is the $\star_0$-composition of rewriting steps. A polygraph is *terminating* if there is no infinite sequence of $\star_0$-composition of rewriting steps.

This shuffle polygraphic approach generalizes that of Gröbner bases introduced by Dotsenko and Khoroshkin in [19]. Indeed, the orientation of the polygraphic rules does not depend on a given mono-

mial order. However, termination is not ensured by a monomial order, so it must be proven by considering the rewriting rules themselves. Beyond the property of termination, the *confluence* property of a 1-polygraph X states that for every *branching* of two positive 1-cells f, g of $X_1^{\amalg}$ with the same source a, there exist two positive 1-cells h and k of $X_1^{\amalg}$ as in the following *confluence diagram*:

$$\begin{tikzcd} & b \arrow[dr, bend left, "h"] & \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & & d \\ & c \arrow[ur, bend right, "k"'] & \end{tikzcd}$$

When the system is terminating, confluence can be deduced from *local confluence*, that is, when all the branchings of rewriting steps are confluent [29, 41]. Local confluence can be proven by the confluence of all branchings involving minimal overlappings of the rules, called *critical branchings*. This is the *critical branching theorem* proved in many algebraic contexts [17, 32, 42]. Coherent versions of this result where introduced in [23, 25]. In § 4.2.2 we introduce *essential branchings* that refine the notion of a critical branching, and generate all critical branchings by transitivity. Theorem 4.2.4 proves a coherent essential branching theorem for shuffle polygraphs. It states that we can extend a terminating, convergent, left-monomial shuffle 1-polygraph X into an *acyclic shuffle* 2*-polygraph* by considering confluences of essential branchings.

In Section 4.3, we give several algebraic interpretations of the confluence property of a terminating, left-monomial shuffle 1-polygraph X. Proposition 4.2.6 proves that the confluence of X is equivalent to having a decomposition of the free shuffle operad $X_0^{\amalg}$ into a direct sum of the ideal generated by the 1-generators of X and the collection of normal forms with respect to this 1-generators. Proposition 4.3.2 proves that the notion of a convergent shuffle 1-polygraph, where the rules are oriented with respect to a given monomial order, is equivalent to the notion of Gröbner bases introduced in [19]. Proposition 4.3.4 gives a polygraphic interpretation of the Poincaré-Birkhoff-Witt (PBW) criterion introduced by Hoffbeck in [28] as a generalization of Priddy's PBW criterion for associative algebras [43].

The overlapping polygraphic resolution and Koszulness

An ω-polygraph X is *acyclic* if, for every $n \geqslant 1$, the quotient of the free shuffle n-operad $X_n^{\amalg}$ by the ideal generated by the cellular extension X_{n+1} is aspherical, that is all parallel n-cells are equal. We say that an acyclic ω-polygraph X is a *shuffle polygraphic resolution* of the shuffle operad it presents. Section 5 presents the main result of this article, Theorem 5.2.6, which extends a reduced convergent left-monomial shuffle 1-polygraph X into a polygraphic resolution, denoted by $Ov(X)$, of the presented shuffle operad. The generators of this polygraphic resolution correspond to higher-dimensional overlappings induced by the rewriting rules of X, defined in § 5.2.1: an n-generator of $Ov(X)$, called an *n-overlapping*, is a sequence of monomials, written

$$u_0 \Lsh \vec{v}_1 \Lsh \vec{v}_2 \Lsh \cdots \Lsh \vec{v}_n,$$

where, when seen as planar trees, each sequence of monomials $\vec{v}_i = (v_{i,1}, \ldots, v_{i,k})$ is attached to the leaves of the $(i-1)$-overlapping $u_0 \Lsh \vec{v}_1 \Lsh \cdots \Lsh \vec{v}_{i-1}$ in a manner that adds exactly enough to apply a new rewriting rule. Explicitly, for low dimensions, the 0-overlappings correspond to the 0-generators X_0,

the 1-overlappings correspond to the sources of the rewriting rules of X_1, and the 2-overlappings correspond to critical branchings of X.

The acyclicity of the shuffle ω-polygraph $\mathcal{O}v(X)$ is proven by the construction of a *homotopical contraction*. Thus the shuffle polygraphic resolution $\mathcal{O}v(X)$ provides an alternative construction to the differential-graded shuffle operads constructed by Dotsenko and Khoroshkin [20]. Moreover, since associative algebras are particular cases of shuffle operads, the construction of $\mathcal{O}v(X)$ is another way to obtain the polygraphic resolutions for associative algebras introduced in [25].

The Quillen homology of a symmetric operad can be computed from its associated shuffle operad. Indeed, the reduced bar complex of a symmetric operad P is isomorphic, as a shuffle differential-graded cooperad, to the reduced bar complex of the associated shuffle operad P^u, that is, $\overline{B}(P)^u \simeq \overline{B}(P^u)$, [20, Prop. 1.4]. In Section 5.3, we associate to an acyclic shuffle ω-polygraph X that presents a shuffle operad P a P-bimodule resolution $P\langle X\rangle$ of the *trivial P-bimodule* Ω_P. Thus shuffle polygraphic resolutions provide a constructive way to compute the homology of symmetric operads. Proposition 5.3.10 proves that when there exists an increasing function $w : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that the n-generators of X are concentrated in weight $w(n)$, then the resolution $P\langle X\rangle$ is minimal. Finally, we define in Section 5.4 a criterion of Koszulness in terms of quadratic convergence: Theorem 5.4.2 states that shuffle operads presented by quadratic convergent 1-polygraphs are Koszul. This result generalizes those obtained by Dotsenko and Khoroshkin in [19] for shuffle operads with quadratic Gröbner bases defined with respect to a given monomial order. This new rewriting-based sufficient condition for Koszulness does not depend on a monomial order, which is required to define Gröbner bases.

Conventions and notations

Throughout this article $\mathbf{k}$ denotes a field of characteristic zero. All vector spaces are over this field $\mathbf{k}$, and we denote by Vect the category of vector spaces and linear maps as morphisms. We denote by $\mathbb{N}$ the set of nonnegative integers. We denote by Ord the category of nonempty finite ordered sets, whose morphisms are order-preserving bijections. We denote by Fin the category of nonempty finite sets, whose morphisms are bijections.

2. Higher shuffle operads

In this section we introduce the notion of a higher shuffle operad. We first recall the structure of shuffle operads from [19] and we decompose the shuffle composition into elementary compositions. We then define the category of shuffle ω-operads and characterize it as a certain subcategory of globular bimodules over shuffle operads.

2.1. Shuffle operads

In this preliminary subsection we recall from [19] the definitions on shuffle operads used in this article. We refer the reader to [36] or [12] for a complete account on symmetric and shuffle operads.

2.1.1. Presheaves on finite sets. In general, a presheaf X on Ord or Fin with values in a category C are determined by the family of objects $(X(k))_{k\geqslant 1}$, where $X(k) := X(\{1,\dots,k\})$. We will adopt this notation for the following definitions.

An *set indexed by the category* Ord, or *indexed set* for short, is a presheaf on Ord with values in the category Set. We denote by Ind the category of indexed sets with natural transformations as morphisms. A *collection* is a presheaf on Ord with values in the category Vect. We denote by Coll the category of collections and their natural transformations. A *basis* of V is a indexed set $B = (B(k))_{k\geqslant 1}$ such that, for each k, $B(k)$ is a basis of the space $V(k)$.

A *symmetric collection* is a presheaf on Fin with values in the category Vect. We denote by $\mathfrak{S}$Coll the category of symmetric collections and their natural transformations. The functor $-^u : \mathsf{Ord} \to \mathsf{Fin}$ that forgets the order induces a functor

$$-^u : \mathfrak{S}\mathsf{Coll} \to \mathsf{Coll}. \tag{2.1.2}$$

In addition, we denote by $\mathbf{k}\langle - \rangle : \mathsf{Ind} \to \mathsf{Coll}$ the left adjoint functor of the forgetful functor $\mathsf{Coll} \to \mathsf{Ind}$.

2.1.3. Operads ([19]). The categories Coll and $\mathfrak{S}$Coll are monoidal with the following products:

i) the *shuffle composition* on Coll denoted by $\circ_{\text{Ш}}$, and defined for $V, W \in \mathsf{Coll}$ by

$$(V \circ_{\text{Ш}} W)(I) := \bigoplus_{k=1}^{\infty} V(k) \otimes \left(\bigoplus_{\substack{f : I \twoheadrightarrow \{1,\dots,k\} \\ (*)}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right),$$

where $I \in \mathbf{Ord}^o$ and the sum $(*)$ is taken on *shuffle surjections*, *i.e.*, surjections $f : I \twoheadrightarrow \{1,\dots,k\}$ such that

$$\min f^{-1}\{1\} < \ldots < \min f^{-1}\{k\}.$$

ii) the *symmetric composition* on $\mathfrak{S}$Coll, denoted by $\circ_{\mathfrak{S}}$, is defined for $V, W \in \mathfrak{S}\mathsf{Coll}$ by

$$(V \circ_{\mathfrak{S}} W)(I) := \bigoplus_{k=1}^{\infty} V(k) \otimes_{\mathbf{k}[\mathfrak{S}_k]} \left(\bigoplus_{f : I \twoheadrightarrow \{1,\dots,k\}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right),$$

where $I \in \mathbf{Fin}^o$, and the sum is taken on all surjections.

In both case, the unit is the collection $\mathbb{I}$ concentrated in arity 1 with $\mathbb{I}(1) = \mathbf{k}$.

A *shuffle* (resp. *symmetric*) *operad* is an internal monoid (P, μ_P, η_P) in $(\mathsf{Coll}, \circ_{\text{Ш}}, \mathbb{I})$ (resp. $(\mathfrak{S}\mathsf{Coll}, \circ_{\mathfrak{S}}, \mathbb{I})$), where μ_P is the multiplication morphism and η_P is the unit morphism. We denote respectively by $\mathfrak{S}$Op and ШOp the category of symmetric operads and shuffle operads and their morphisms.

The free operad functors $-^{*\text{Ш}} : \mathsf{Coll} \to \text{Ш}\mathsf{Op}$ and $-^{*\mathfrak{S}} : \mathfrak{S}\mathsf{Coll} \to \mathfrak{S}\mathsf{Op}$ are defined using the free monoid functor on left distributive categories as detailed in [7, Appendix B]. For an indexed set X, we denote by $X^{\text{Ш}}$ the free shuffle operad on X given by the composite of free functor

$$\mathsf{Ind} \xrightarrow{\mathbf{k}} \mathsf{Coll} \xrightarrow{-^{*\text{Ш}}} \text{Ш}\mathsf{Op}. \tag{2.1.4}$$

Recall from [19], see also [12], that the forgetful functor $-^u$ is monoidal in the sense that for all symmetric collections V, W, we have $(V \circ_{\mathfrak{S}} W)^u = V^u \circ_{\text{Ш}} W^u$ in Coll, and in particular that it commutes with free operad functors $-^{*\mathfrak{S}}$ and $-^{*\text{Ш}}$, in the sense that for every symmetric collection V, we have the isomorphism

$$(V^{*\mathfrak{S}})^u = (V^u)^{*\text{Ш}}. \tag{2.1.5}$$

2.1.6. Shuffle composition on indexed sets. We define a monoidal *shuffle composition* on Ind, also denoted by $\times_{\text{Ш}}$, by setting, for indexed sets X, Y

$$(X \times_{\text{Ш}} Y)(I) := \coprod_{k=1}^{\infty} X(k) \times \left(\coprod_{\substack{f : I \twoheadrightarrow \{1,\dots,k\} \\ (*)}} Y(f^{-1}\{1\}) \times \cdots \times Y(f^{-1}\{k\}) \right)$$

where the coproduct $(*)$ is taken on shuffle surjections. The composition $\times_{\text{Ш}}$ has for unit the indexed set concentrated in arity 1, denoted by $\mathbf{1}$, and such that $\mathbf{1}(1)$ is a singleton, whose only element is denoted by ε. The functor $\mathbf{k}$ is compatible with product and coproduct, hence the following diagram commutes:

$$\begin{tikzcd} \mathsf{Ind} \times \mathsf{Ind} \arrow[r, "\mathbf{k} \times \mathbf{k}"] \arrow[d, "\times_{\text{Ш}}"'] & \mathsf{Coll} \times \mathsf{Coll} \arrow[d, "\circ_{\text{Ш}}"] \\ \mathsf{Ind} \arrow[r, "\mathbf{k}"'] & \mathsf{Coll} \end{tikzcd} \tag{2.1.7}$$

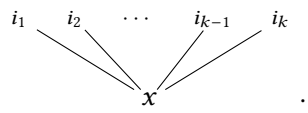
Note that the adjunction between the monoidal categories $(\mathsf{Set}, \times, \{*\})$ and $(\mathsf{Vect}, \otimes, \mathbf{k})$ is compatible with the canonical isomorphisms of units, associativity, and distributivity, so the induced functors between $(\mathsf{Ind}, \times_{\text{Ш}}, \mathbb{I})$ and $(\mathsf{Coll}, \circ_{\text{Ш}}, \mathbb{I})$ make a lax monoidal adjunction.

2.1.8. Tree monomials. The shuffle composition is monoidal, and we denote by ШTree the category of internal monoids in Ind with respect to this composition. The functor $\mathbf{k}$ preserves colimits as a left adjoint, and sends $\times_{\text{Ш}}$ to $\circ_{\text{Ш}}$ as a consequence of commutativity of (2.1.7). Free internal monoids in Ind and Coll are constructed by colimits and shuffle composition, thus the linearization functor $\mathbf{k}$ induces a linearization functor ШTree $\to$ ШOp such that the following square commutes:

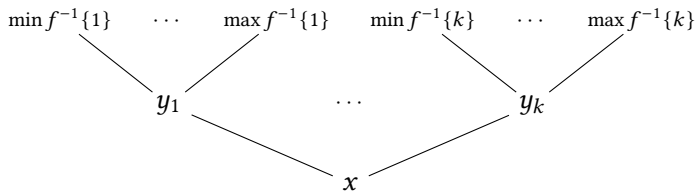
$$\begin{tikzcd} \mathsf{Ind} \arrow[r, "\mathbf{k}"] \arrow[d, "-^{*\text{Ш}}"'] & \mathsf{Coll} \arrow[d, "-^{*\text{Ш}}"] \\ \text{ШTree} \arrow[r, "\mathbf{k}"'] & \text{ШOp} \end{tikzcd}$$

For an indexed set X, the elements of the free internal monoid $X^{*\text{Ш}}$ in ШTree are called *tree monomials on* X. We have $X^{*\text{Ш}} = (X_k^{*\text{Ш}})_{k \geqslant 0}$, and elements of $X_k^{*\text{Ш}}$ are said to be of *arity* k. In particular the unit in $X^{*\text{Ш}}$ corresponding to the indexed set $\mathbf{1}$ is called the *trivial tree monomial*. Elements of the free operad $X^{\text{Ш}}$ are linear combination of tree monomials having a same given arity k, and called *terms on* X *of arity* k.

2.1.9. Graphical representation of tree monomials. The elements of an indexed set can be represented graphically as (planar rooted) trees. For an indexed set X and an ordered set $I = \{i_1 < i_2 < \cdots < i_k\}$, an element $x \in X(I)$ is depicted by a *corolla*, that is, a tree with only one vertex:

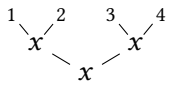


For indexed sets X, Y, elements of $(X \times_{\text{Ш}} Y)(I)$ have the form



where $k \geqslant 1$, $f : I \twoheadrightarrow \{1, \ldots, k\}$ is a shuffle surjection, $x \in X(k)$ and $y_i \in Y(f^{-1}\{i\})$ for all $i \in \{1, \ldots, k\}$. In this way, a tree monomial u on X can be represented by a planar tree $T(u)$, whose vertices are elements of X, and its arity is the number of its leaves. More generally, for V, W two collections, $V \circ W$ has a basis of tree monomials.

The *weight* of a tree monomial u on an indexed set X is the number of vertices of $T(u)$. A tree monomial v is a (resp. *rooted*) *submonomial* of u if $T(v)$ is a (resp. rooted) subtree of $T(u)$. If v is a rooted submonomial of u, we write $v \subseteq u$. When listing submonomials of a tree monomial u, we distinguish the different occurrences of a subtree of $T(u)$: for instance, the tree monomial



contains three distinct occurrences of the submonomial

, one of which is rooted.

2.1.10. Inline notation for tree monomials. For $n_1, \ldots, n_k \geqslant 1$, a *shuffle surjection of type* $(n_1, \ldots, n_k)$ is a surjection $f : \{1, \ldots, n_1 + \cdots + n_k\} \twoheadrightarrow \{1, \ldots, k\}$ such that, for all $i \in \{1, \ldots, k\}$, $|f^{-1}(\{i\})| = n_i$, and

$$\min f^{-1}\{1\} < \cdots < \min f^{-1}\{k\}.$$

Denote by $S(n_1, \ldots, n_k)$ the set of shuffle surjections of type $(n_1, \ldots, n_k)$.

Let X be an indexed set. The *inline notation* for the indexed set of tree monomials $X^{*\text{Ш}}$ is the term algebra in indexed sets given by the *Backus-Naur form*

$$X^{*\text{Ш}} ::= \mathbf{1} \,\big|\, (X(k) \mid_f X^{*\text{Ш}}(n_1) \cdots X^{*\text{Ш}}(n_k)),$$

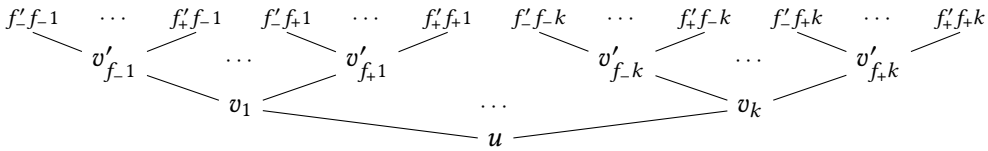
where $\mathbf{1}$ is the indexed set defined in 2.1.6, and f is a shuffle surjection of type $(n_1, \ldots, n_k)$. When possible, we omit subscript f, and we write

$$(u \mid \vec{v}) := (u \mid v_1 \cdots v_k).$$

We will use also the notation $\vec{v}$ for the list of tree monomials $v_1, \dots, v_k$. Finally, note that for two indexed sets X, Y the indexed set $X \times_{\text{Ш}} Y$ can be written in an explicit way as, for $n \geqslant 1$,

$$(X \times_{\text{Ш}} Y)(n) = \left\{ (x \mid_f y_1 \cdots y_k) \;\middle|\; \begin{array}{l} n_1, \dots, n_k \geqslant 1,\ n_1 + \cdots + n_k = n, \\ x \in X(k), y_1 \in Y(n_1), \dots, y_k \in Y(n_k), \\ f \in S(n_1, \dots, n_k) \end{array} \right\}.$$

2.1.11. Explicit associativity of shuffle composition. Let $u, \vec{v}, \vec{v}'$ be monomials, and f, f' shuffle surjections such that the tree monomial $((u \mid_f \vec{v}) \mid_{f'} \vec{v}')$ is well defined. Denoting the minimal and maximal elements of $f^{-1}\{i\}$ by f_-i and f_+i respectively for all $i \in \{1, \dots, k\}$, this monomial is represented graphically as



For $i \in \{1, \dots, k\}$, the restriction of f' to $f'^{-1}\{f^{-1}\{i\}\}$ is a shuffle surjection of domain $f^{-1}\{i\}$. We introduce the notation $(\vec{v} \mid_{f,f'} \vec{v}')$ for the list of monomials $\{(v_i \mid_{f'_{|f'^{-1}(f^{-1}\{i\})}} v'_{\min f^{-1}\{i\}} \cdots v'_{\max f^{-1}\{i\}})\}_{1 \leqslant i \leqslant k}$. Then

$$((u \mid_f \vec{v}) \mid_{f'} \vec{v}') = (u \mid_{ff'} (\vec{v} \mid_{f,f'} \vec{v}')),$$

where ff' is the composition of shuffle surjections.

2.1.12. Bimodules and ideals. Recall that a P-*bimodule* over a shuffle operad P, called a *linear module over* P in [7, Def. 2.13], is a collection A equipped with two families of morphisms of collections

$$\lambda : P(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes A(f^{-1}\{i\}) \otimes \cdots P(f^{-1}\{k\}) \to A(I),$$
$$\rho : A(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \to A(I),$$

for all shuffling surjections $f : I \twoheadrightarrow \{1, \dots, k\}$, defining a *left crossed action* and a *right action* respectively, satisfying compatibility axioms with each other, and associativity and unit axioms with the product of P. A morphism of P-modules is a morphism of collections compatible with the left and right actions. We denote by $\mathsf{Bimod}(P)$ the category of P-bimodules and morphisms of P-bimodules, and by $\mathsf{Bimod}(\text{Ш}\mathsf{Op}) := \coprod_{P \in \text{Ш}\mathsf{Op}} \mathsf{Bimod}(P)$ the category of pairs (P, A) composed of an operad P and a P-bimodule A. We denote by $\mathcal{L}^P : \mathsf{Coll} \to \mathsf{Bimod}(P)$ the *free bimodule functor* defined in [7, Prop. 2.11], see also [38], and given for every $V \in \mathsf{Coll}$ and $I \in \mathsf{Ord}$ by

$$\mathcal{L}^P(V)(I) := \bigoplus_{k \geqslant 1} P(k) \otimes \left(\bigoplus_{\substack{f : I \twoheadrightarrow \{1, \dots, k\} \\ 1 \leqslant i \leqslant k}} P(f^{-1}\{1\}) \otimes \cdots \otimes (A \circ_{\text{Ш}} P)(f^{-1}\{i\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \right)$$

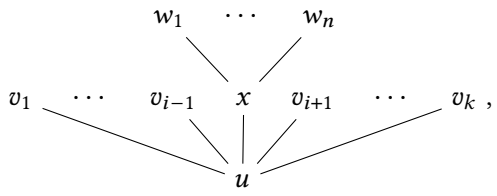
For an indexed set X, we denote by

$$P\langle X \rangle := \mathcal{L}^P(\mathbf{k}X)$$

the free P-bimodule on X.

An *ideal* of an operad P is a P-bimodule $\mathcal{I}$ equipped with an inclusion of P-bimodules $\mathcal{I} \hookrightarrow P$.

2.1.13. Graphical description of bimodules. Let X be an indexed set an P a shuffle operad. The free P-bimodule $P\langle X\rangle$ is the collection generated by tree monomials of the form



where $n, k \geqslant 1$, $x \in X(n)$, $i \in \{1, \dots, k\}$, $u, v_1, \dots, \check{v}_i, \dots, v_k, w_1, \dots, w_n \in P$, and the inputs are omitted. We use the check notation $\check{v}_i$ to indicate that we omit v_i, as opposed to the usual hat notation $\hat{v}_i$, in order to avoid confusion with a notation in Section 5.

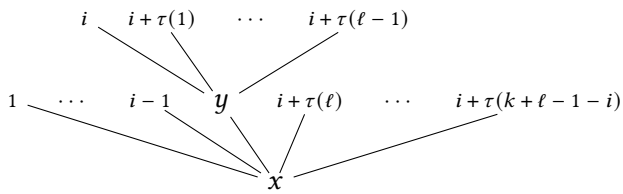
2.2. Compositions in shuffle operads

In this subsection, we decompose shuffle composition into partial compositions, and we introduce notations for composition and terms in an operad.

2.2.1. Shuffle partial composition. Recall from [19, Prop. 2], that for $k, \ell \geqslant 1$, a *shuffle permutation of type* (k, ℓ) is a permutation $\tau \in \mathfrak{S}_{k+\ell}$ such that

$$\tau(1) < \cdots < \tau(k), \quad \text{and} \quad \tau(k+1) < \cdots < \tau(k+\ell).$$

Denote by $\text{Ш}(k, \ell)$ the set of shuffle permutations of type (k, ℓ). Given indexed sets X, Y, and $x \in X(k)$, $y \in Y(\ell)$, for $i \in \{1, \dots, k\}$ and $\tau \in \text{Ш}(\ell - 1, k - i)$, we define the *elementary composition* $x \circ_{i,\tau} y$ as the following tree



Elementary compositions are extended to collections by linearity and bidistributivity. We denote by $V(k) \circ_{i,\tau} W(\ell)$ the collection composed by elementary compositions of the form $v \circ_{i,\tau} w$, for $v \in V(k)$ and $w \in W(\ell)$. Then the *shuffle partial composition* of collections V, W is defined by

$$(V \odot_{\text{Ш}} W)(n) := \bigoplus_{\substack{k,\ell,i \geqslant 1 \\ k+\ell-1=n \\ \tau \in \text{Ш}(\ell-1,k-i)}} V(k) \circ_{i,\tau} W(\ell).$$

2.2.2. Properties of partial compositions. Note that there are isomorphisms

$$(V \odot_{\text{Ш}} W)(n) \overset{\textbf{(i)}}{\simeq} \bigoplus_{k=1}^{\infty} V(k) \otimes \left(\bigoplus_{\substack{i,\ell \geqslant 1 \\ k+\ell-1=n \\ \tau \in \text{Ш}(\ell-1,k-i)}} \mathbb{I}(\{1\}) \otimes \cdots \otimes \mathbb{I}(\{i-1\}) \otimes W(\{i, i+\tau(1), \ldots, i+\tau(\ell-1)\}) \right.$$
$$\left. \otimes\, \mathbb{I}(\{i+\tau(\ell)\}) \otimes \cdots \otimes \mathbb{I}(\{i+\tau(k+\ell-1-i)\}) \right)$$
$$\overset{\textbf{(ii)}}{\simeq} \bigoplus_{\substack{k,\ell,i \geqslant 1 \\ k+\ell-1=n \\ \tau \in \text{Ш}(\ell-1,k-i)}} V(k) \otimes W(\ell).$$

The isomorphism **(i)** implies that there is an injection of collections

$$V \odot_{\text{Ш}} W \hookrightarrow V \circ_{\text{Ш}} (\mathbb{I} \oplus W).$$

The isomorphism **(ii)** implies that partial composition a bidistributive bifunctor $\odot_{\text{Ш}} : \mathsf{Coll} \times \mathsf{Coll} \to \mathsf{Coll}$. The partial composition $\odot_{\text{Ш}}$ is not associative. However, if there is no possible confusion, we will use the left bracket rule, that is, $U \odot V \odot W := (U \odot V) \odot W$.

2.2.3. Decomposition of shuffle compositions. Let A be a collection equipped with a morphism $\eta_A : \mathbb{I} \to A$, that is, η_A is an object of $\mathbb{I}/\mathsf{Coll}$. We can express the shuffle composition $A \circ_{\text{Ш}} A$ in terms of partial compositions. There exists a natural transformation φ from the functor $A \mapsto \bigoplus A \odot_{\text{Ш}} A^{\odot_{\text{Ш}} p}$ to the functor $A \mapsto A \circ_{\text{Ш}} A$ defined as follows:

$$\varphi_A : \bigoplus_{p=1}^{\infty} A \odot_{\text{Ш}} A^{\odot_{\text{Ш}} p} \to \bigoplus_{p=1}^{\infty} A \circ_{\text{Ш}} (\mathbb{I} \oplus A)^{\circ_{\text{Ш}} p} \xrightarrow{\sum 1 \circ_{\text{Ш}} (\eta_A + 1)^{\circ_{\text{Ш}} p}} A \circ_{\text{Ш}} A.$$

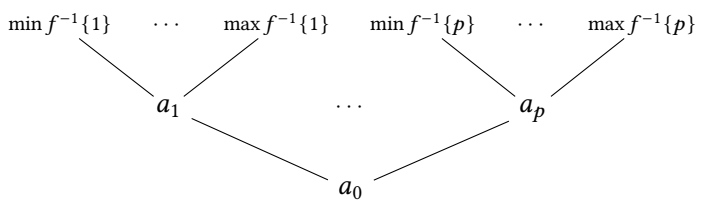
In order to express $A \circ_{\text{Ш}} A$ in terms of partial compositions, it suffices to define a right inverse to φ, that is, a natural transformation σ from the functor $A \mapsto A \circ_{\text{Ш}} A$ to the functor $A \mapsto \bigoplus A \odot_{\text{Ш}} A^{\odot_{\text{Ш}} p}$ such that $\varphi_A \sigma_A = id_{A \circ_{\text{Ш}} A}$ for all A.

Define the morphism

$$\sigma_A : A \circ_{\text{Ш}} A \to \bigoplus_{p=1}^{\infty} A \odot_{\text{Ш}} A^{\odot_{\text{Ш}} p},$$

natural in A, as follows. An element a of $(A \circ_{\text{Ш}} A)(n)$ can be written

$a =$



where $f : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, p\}$ is a shuffle surjection. Set $\sigma_A(a) := a_0 \circ_{p,\tau_p} a_p \circ_{p-1,\tau_{p-1}} \cdots \circ_{1,\tau_1} a_1$, where

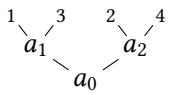
$$\begin{aligned} \tau_p &\in \text{Ш}\left(|f^{-1}\{p\}| - 1, 0\right) = \{id_{|f^{-1}\{p\}|-1}\}, \\ \tau_{p-1} &\in \text{Ш}\left(|f^{-1}\{p-1\}| - 1, |f^{-1}\{p\}|\right), \\ &\vdots \\ \tau_1 &\in \text{Ш}\left(|f^{-1}\{1\}| - 1, |f^{-1}\{2, \dots, p\}|\right), \end{aligned}$$

are the appropriate shuffle permutations. We check that, for every morphism $f : A \to B$ of collections, the square

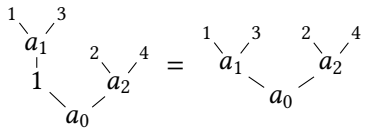
$$\begin{tikzcd} A \circ_{\text{Ш}} A \arrow[r, "\sigma_A"] \arrow[d, "f \circ_{\text{Ш}} f"'] & \bigoplus_{p \geqslant 0} A \odot_{\text{Ш}} A^{\odot_{\text{Ш}} p} \arrow[d, "\sum_{p \geqslant 0} f \odot_{\text{Ш}} f^{\odot_{\text{Ш}} p}"] \\ B \circ_{\text{Ш}} B \arrow[r, "\sigma_B"'] & \bigoplus_{p \geqslant 0} B \odot_{\text{Ш}} B^{\odot_{\text{Ш}} p} \end{tikzcd}$$

commutes. This defines the natural transformation σ, and we check that it is a right inverse to the natural transformation φ.

2.2.4. Example. Let A be an object of $\mathbb{I}/\mathrm{Coll}$ and

$a =$ 

and element of $A \circ_{\text{Ш}} A$, where $a_0, a_1, a_2 \in A(2)$. Then we have

$\sigma_A(a) = a_0 \circ_{2,id_1} a_2 \circ_{1,(1\,2)} a_1$ and $\varphi_A\sigma_A(a) =$  .

2.3. Higher shuffle operads

In this subsection, we introduce the structure of (strict) shuffle ω-operads.

2.3.1. Globular objects. We denote by $\mathbb{RO}$ the *reflexive globe category*, whose objects are natural numbers, denoted by $\underline{n}$, for $n \in \mathbb{N}$, and morphisms are generated by

$$\sigma_n : \underline{n} \to \underline{n+1}, \qquad \tau_n : \underline{n} \to \underline{n+1}, \qquad \iota_{n+1} : \underline{n+1} \to \underline{n},$$

for all n in $\mathbb{N}$, and submitted to the following *globular* and *identities relations*:

$$\begin{aligned} \sigma_{n+1} \circ \sigma_n &= \tau_{n+1} \circ \sigma_n, & \sigma_{n+1} \circ \tau_n &= \tau_{n+1} \circ \tau_n, \\ \iota_n \circ \sigma_n &= id_{\underline{n}}, & \iota_n \circ \tau_n &= id_{\underline{n}}, \end{aligned}$$

for all n in $\mathbb{N}$. Omitting the identity maps ι_n gives the definition of the *globe category* $\mathbb{O}$. We denote by $\mathbb{RO}_n$ (resp. $\mathbb{O}_n$) the full subcategory of $\mathbb{RO}$ (resp. $\mathbb{O}$) whose objects are $\underline{0}, \underline{1}, \ldots, \underline{n}$.

A *reflexive globular object* in a category C is a functor $\mathbb{RO}^{op} \to C$, whose restriction to the category $\mathbb{RO}_n^{op}$ is called a *reflexive n-globular object.* Explicitly, a reflexive globular object is given by a sequence $A = (A_k)_{k\in\mathbb{N}}$ of objects of C, equipped with indexed morphisms

$$s = (s_k : A_{k+1} \to A_k)_{k\in\mathbb{N}}, \qquad t = (t_k : A_{k+1} \to A_k)_{k\in\mathbb{N}}, \qquad i = (i_k : A_{k-1} \to A_k)_{k\in\mathbb{N}},$$

of degree -1, -1 and 1 respectively, and satisfying the following globular and identities relations

$$s^2 = st, \quad t^2 = ts, \quad si = id_A, \quad ti = id_A. \tag{2.3.2}$$

The elements of A_k are called *k-cells* of A. A *morphism of reflexive globular objects* is an indexed morphism of degree 0 that commutes with morphisms s, t and i. We denote by $\mathsf{RGlob}(C)$ (resp. $\mathsf{Glob}(C)$) the category of *reflexive globular objects* (resp. *globular objects*) in C and their morphisms. We denote by $\mathsf{RGlob}_n(C)$ (resp. $\mathsf{Glob}_n(C)$) the full subcategory of $\mathsf{RGlob}(\amalg\mathsf{Op})$ of reflexive n-globular objects (resp. n-globular objects) in C. We will denote by

$$\mathcal{V}_n(C) : \mathsf{Glob}_{n+1}(C) \to \mathsf{Glob}_n(C)$$

the functor that forgets the $(n+1)$-cells. For A a globular object and $\ell \geqslant k \geqslant 0$, the *$(\ell, k)$-source* and *$(\ell, k)$-target morphisms*

$$s_k^\ell : A_\ell \to A_k, \qquad t_k^\ell : A_\ell \to A_k,$$

are respectively defined as the following iterated composition of source and target morphisms:

$$s_k^\ell := s_k \circ \ldots s_{\ell-2} \circ s_{\ell-1}, \qquad t_k^\ell := t_k \circ \ldots t_{\ell-2} \circ t_{\ell-1}.$$

We denote by $i_\ell^k : A_k \to A_\ell$ the iterated identity $i_l^k = i_\ell \circ i_{\ell-1} \ldots \circ i_{k+1}$. When there is no ambiguity, we will write s_k and t_k for source and target maps respectively, and we will omit i_ℓ^k entirely, since i_ℓ^k is injective by (2.3.2). For $k \geqslant 0$, we denote by $A \star_k A$ the following pullback of globular operads

$$\begin{tikzcd} A \star_k A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "s_k"] \\ A \arrow[r, "t_k"'] & A_k \end{tikzcd}$$

Let A be a globular object of some category C. For $n \geqslant 1$, two n-cells a, b of A are *parallel* if $s(a) = s(b)$ and $t(a) = t(b)$. An *n-sphere of A* is a pair (a, b) of parallel n-cells.

2.3.3. Higher categories for operads. Recall that, for $n \geqslant 0$, an (internal strict) *n-category* in C is a

i) reflexive n-globular object, that is a diagram in C of the form

$$A_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} A_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \cdots \underset{t_{n-2}}{\overset{s_{n-2}}{\leftleftarrows}} A_{n-1} \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} A_n$$

whose morphisms satisfy globular and identity relations (2.3.2),

ii) equipped with a structure of category in C on

$$A_k \underset{t_k}{\overset{s_k}{\leftleftarrows}} A_\ell$$

for all $k < \ell$, whose $\star_k$*-composition morphism of* ℓ*-cells* is denoted by $\star_k^\ell : A_\ell \star_k A_\ell \to A_\ell$,

iii) such that the 2-globular object

$$A_j \underset{t_j}{\overset{s_j}{\leftleftarrows}} A_k \underset{t_k}{\overset{s_k}{\leftleftarrows}} A_\ell$$

is a 2-category in C for all $j < k < \ell$.

We denote by $n\mathsf{Cat}(C)$ the category of n-categories in C and their n-functors. The category $\omega\mathsf{Cat}(C)$ of ω*-categories in* C is the limit of

$$0\mathsf{Cat}(C) \leftarrow 1\mathsf{Cat}(C) \leftarrow \cdots \leftarrow n\mathsf{Cat}(C) \leftarrow \cdots$$

where each arrow forgets the cells of highest dimension.

For $n \in \mathbb{N} \cup \{\omega\}$, a *shuffle* (resp. *symmetric*) n*-operad* is an n-category in $\text{Ш}\mathsf{Op}$ (resp. $\mathfrak{S}\mathsf{Op}$). We denote by $\text{Ш}\mathsf{Op}_n$ (resp. $\mathfrak{S}\mathsf{Op}_n$) the corresponding category with internal n-functors as morphisms. We denote by $\mathcal{U}_n^{\text{Ш}} : \text{Ш}\mathsf{Op}_n \to \mathsf{Glob}_n(\mathsf{Ind})$ (resp. $\mathcal{U}_n^{\mathfrak{S}} : \mathfrak{S}\mathsf{Op}_n \to \mathsf{Glob}_n(\mathfrak{S}\mathsf{Coll})$) the forgetful functor that forgets the operadic structure.

Note that, in an ω-shuffle operad P, the composition $\star_\ell^k : P_k \star_\ell P_k \to P_k$ is a morphism of shuffle operads. As a consequence, the composition satisfies the following *exchange relation between* $\circ_{\text{Ш}}$ *and* $\star_k$

$$\begin{tikzcd}
(P_k \star_\ell P_k) \circ_{\text{Ш}} (P_k \star_\ell P_k) \arrow[rr, "(\pi_1 \circ_{\text{Ш}} \pi_1) \star_\ell (\pi_2 \circ_{\text{Ш}} \pi_2)"] \arrow[dr, "\mu_k((1 \star_\ell 1) \circ_{\text{Ш}} (1 \star_\ell 1))"'] & & (P_k \circ_{\text{Ш}} P_k) \star_\ell (P_k \circ_{\text{Ш}} P_k) \arrow[dl, "\mu_k(1 \circ_{\text{Ш}} 1) \star_\ell \mu_k(1 \circ_{\text{Ш}} 1)"] \\
& P_n &
\end{tikzcd} \tag{2.3.4}$$

As for associative ω-algebras, [25, Prop. 1.2.3], the interaction between the categorical and linear structures gives useful expressions:

2.3.5. Lemma. *Let P be a shuffle (resp. symmetric) ω-operad.*

i) *For every $0 \leqslant k < n$ and $\star_k$-composable pair (a, b) of P_n, we have $a \star_k b = a - t_k(a) + b$.*

ii) *For all $n \geqslant 1$, every n-cell a of P is invertible with inverse $a^- := s_{n-1}(a) - a + t_{n-1}(a)$.*

We deduce the following proposition:

2.3.6. Proposition. *The category $\text{Ш}\mathsf{Op}_\omega$ (resp. $\mathfrak{S}\mathsf{Op}_\omega$) is isomorphic to the category $\mathsf{Gpd}_\omega(\text{Ш}\mathsf{Op})$ (resp. $\mathsf{Gpd}_\omega(\mathfrak{S}\mathsf{Op})$) of internal ω-groupoids in $\text{Ш}\mathsf{Op}$ (resp. $\mathfrak{S}\mathsf{Op}$).*

Finally, we give some categorical properties of the categories of shuffle operads.

2.3.7. Proposition. *The forgetful functor* $\text{Ш}\mathsf{Op} \to \mathsf{Coll}$ *(resp.* $\mathfrak{S}\mathsf{Op} \to \mathfrak{S}\mathsf{Coll}$*) reflects all limits, filtered colimits, and reflexive coequalizers.*

Proof. The statement for the functor $\mathfrak{S}\mathsf{Op} \to \mathfrak{S}\mathsf{Coll}$ is proven in [22, Prop. 1.2.4], where the limits, filtered colimits, and reflexive coequalizers of symmetric operads are equipped with unique monoidal structures. In particular, it is shown that the monoidal product $\circ_{\mathfrak{S}}$ in $\mathfrak{S}\mathsf{Coll}$ preserves all limits, filtered colimits and reflexive coequalizers. This comes from the fact that the tensor product of Vect preserves these limits and colimits. Given the similarities between the monoidal products $\circ_{\mathfrak{S}}$ and $\circ_{\text{Ш}}$, the same arguments apply to the monoidal structure of Coll, and so we conclude that the functor $\text{Ш}\mathsf{Op} \to \mathsf{Coll}$ preserves limits, filtered colimits, and reflexive coequalizers. □

2.3.8. Proposition. *The category* $\text{Ш}\mathsf{Op}_\omega$ *is locally finitely presentable. In particular, it is complete and cocomplete.*

Proof. Let us first show that ШOp is locally finitely presentable by viewing it as the category of algebras over an accessible monad. Recall that there exists an adjunction $\mathsf{Coll} \dashv \text{Ш}\mathsf{Op}$ where the left adjoint is $-^{\text{Ш}} : \mathsf{Coll} \to \text{Ш}\mathsf{Op}$. Therefore, ШOp is the category of algebras of the monad of free shuffle operads $T : \mathsf{Coll} \to \mathsf{Coll}$. By the Proposition 2.3.7, the forgetful functor $\text{Ш}\mathsf{Op} \to \mathsf{Coll}$ preserves filtered colimits, *i.e.* it is finitary, making the monad T finitary. Moreover, the category Vect of vector spaces is locally finitely presentable, and the category Ord is a small category, so Coll is also locally finitely presentable. Thus T is an accessible monad on a locally finitely presentable category. Following [1, § 2.78] the category of T-algebras ШOp is locally finitely presentable.

The category $\text{Ш}\mathsf{Op}_\omega$ of ω-categories internal in ШOp is the category of models of a finite limit sketch [3], in the locally finitely presentable category ШOp. By [1, Prop. 1.53], we conclude that $\text{Ш}\mathsf{Op}_\omega$ is also locally finitely presentable. □

2.4. Higher operads as globular bimodules

In Theorem 2.4.8, we show that the axioms of the definition of the category $\text{Ш}\mathsf{Op}_\omega$ of shuffle ω-operads are redundant by proving that it is isomorphic to a category with fewer axioms thanks to the linear exchange relation. Throughout this section, the compositions $\circ$ and $\odot$ designate $\circ_{\text{Ш}}$ and $\odot_{\text{Ш}}$.

2.4.1. Partial multiplication. For (P, μ, η) a shuffle operad, denote by ι_P the morphism

$$\iota_P : P \odot P \hookrightarrow P \circ (\mathbb{I} \oplus P) \xrightarrow{1 \circ (\eta + 1)} P \circ P.$$

We equip the operad P with a morphism called *partial multiplication*

$$\mu^{\odot} : P \odot P \xrightarrow{\iota_P} P \circ P \xrightarrow{\mu} P.$$

As a consequence, we have the equality of morphisms

$$\mu\varphi_P = \sum_p (\mu^{\odot})^{\odot p} : \bigoplus_{p=1}^{\infty} P \odot P^{\odot p} \to P.$$

2.4.2. Linear exchange relation. Let $(P_n, \mu_n, \eta_n)_{n\geqslant 0}$ be an ω-operad. By the exchange relation between compositions $\circ$ and $\star_0$, we observe that, for every $n \geqslant 1$,

$$\begin{aligned}
\mu_n^{\odot} &= \mu_n((1 \star_0 t_0) \circ (s_0 \star_0 1))\iota_{P_n} \\
&= (\mu_n(1 \circ s_0) \star_0 \mu_n(t_0 \circ 1))\iota_{P_n} && (2.3.4)\\
&= (\mu_n(1 \circ s_0) + \mu_n(t_0 \circ 1) - \mu_n(t_0 \circ s_0))\iota_{P_n} && (\text{Lemma } 2.3.5)\\
&= \mu_n^{\odot}(1 \odot s_0) + \mu_n^{\odot}(t_0 \odot 1) - \mu_n^{\odot}(t_0 \odot s_0).
\end{aligned}$$

Similarly, we calculate

$$\mu_n^{\odot} = \mu_n^{\odot}(s_0 \odot 1) + \mu_n^{\odot}(1 \odot t_0) - \mu_n^{\odot}(s_0 \odot t_0).$$

Regarding P_n as a P_0-bimodule, these equations still hold, although we need to introduce new notations for the partial left and right actions of P_0 on P_n. This motivates the following definitions.

Let (P, μ, η) be a shuffle operad and (A, λ, ρ) be a P-bimodule such that (P, A) is a reflexive 1-globular P-bimodule. More explicitly, there are morphisms $s, t : A \to P$ and $i : P \to A$. We equip A with morphisms called *partial actions*

$$\begin{aligned}
\lambda^{\odot} &: P \odot A \xrightarrow{\varphi} P \circ (\mathbb{I} \oplus A) \xrightarrow{1\circ(\eta\oplus 1)} P \circ (P \oplus A) \xrightarrow{\lambda} A,\\
\rho^{\odot} &: A \odot P \xrightarrow{\varphi} A \circ (\mathbb{I} \oplus P) \xrightarrow{1\circ(\eta+1)} A \circ P \xrightarrow{\rho} A.
\end{aligned}$$

We also define the morphisms

$$\begin{aligned}
\mu_A^{\uparrow} &:= \rho^{\odot}(1 \odot s) + \lambda^{\odot}(t \odot 1) - i\mu^{\odot}(t \odot s),\\
\mu_A^{\downarrow} &:= \lambda^{\odot}(s \odot 1) + \rho^{\odot}(1 \odot t) - i\mu^{\odot}(s \odot t)
\end{aligned}$$

and the multiplication

$$\mu_A : A \circ A \xrightarrow{\sigma_A} \bigoplus_p A \odot A^{\odot p} \xrightarrow{\sum(\mu_A^{\uparrow})^p} A.$$

We say that a reflexive 1-globular P-bimodule (P, A) satisfies the *linear exchange relation* if the following relation holds:

$$\mu_A^{\uparrow} = \mu_A^{\downarrow}. \tag{2.4.3}$$

2.4.4. Interpretation of morphisms $\mu_A^{\uparrow}$ and $\mu_A^{\downarrow}$. If (P, A) is a reflexive 1-globular P-bimodule, we can interpret the elements of A as rewriting rules that relate elements of P: an element $a \in A$ rewrites $s(a)$ as $t(a)$, which we denote by $a : s(a) \to t(a)$. Via the injection i, an element x of P can also seen as a trivial rewriting rule $i(x) : x \to x$.

Let $a, b \in A$. For every compatible elementary composition $\circ_{i,\tau}$, we would like to interpret the

composition $a \circ_{i,\tau} b$ as a pair of orthogonal reductions:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & s(b) \arrow[r, bend left] & s(b) \arrow[r, bend left] & b \\
s(b) & & & t(b) \\
s(a) \arrow[u, dash] \arrow[r, bend right] & b \arrow[r, bend right] & t(b) & t(a) \arrow[u, dash] \\
 & s(a) \arrow[u, dash] & t(a) \arrow[u, dash] & t(b) \arrow[u, dash]
\end{tikzcd}$$

where $\begin{smallmatrix} s(b) \\ s(a) \end{smallmatrix}$ is a graphical representation of $s(a) \circ_{i,\tau} s(b)$, and so on. While $a \circ_{i,\tau} b$ is not necessarily an element of A, we find that

$$\mu_A^{\uparrow}\left(\begin{smallmatrix} & b \\ a & \end{smallmatrix} \right) = \begin{smallmatrix} s(b) \\ a \end{smallmatrix} - \begin{smallmatrix} s(b) \\ t(a) \end{smallmatrix} + \begin{smallmatrix} b \\ t(a) \end{smallmatrix},$$
$$\mu_A^{\downarrow}\left(\begin{smallmatrix} & b \\ a & \end{smallmatrix} \right) = \begin{smallmatrix} b \\ s(a) \end{smallmatrix} - \begin{smallmatrix} t(b) \\ s(a) \end{smallmatrix} + \begin{smallmatrix} t(b) \\ a \end{smallmatrix}.$$

We see that $\mu_A^{\uparrow}$ applies the rule a first, and b second, while $\mu_A^{\downarrow}$ does the opposite; this motivates the upwards and downwards arrow notations.

2.4.5. Example. Let $a = \begin{smallmatrix} a_1 & & a_2 \\ & a_0 & \end{smallmatrix}$ be the element of $A \circ A$ from Example 2.2.4 with inputs omitted. Then

$$\mu_A(a) = \begin{smallmatrix} s(a_1) & & s(a_2) \\ & a_0 & \end{smallmatrix} - \begin{smallmatrix} s(a_1) & & s(a_2) \\ & t(a_0) & \end{smallmatrix} + \begin{smallmatrix} s(a_1) & & a_2 \\ & t(a_0) & \end{smallmatrix} - \begin{smallmatrix} s(a_1) & & t(a_2) \\ & t(a_0) & \end{smallmatrix} + \begin{smallmatrix} a_1 & & t(a_2) \\ & t(a_0) & \end{smallmatrix}.$$

2.4.6. Lemma. *Let (A, λ, ρ) be a P-bimodule such that (P, A) is a reflexive 1-globular P-bimodule satisfying the linear exchange relation. Then $(A, \mu_A, i\eta)$ is an operad.*

Proof. Write $\mu_A^{\odot} := \mu_A^{\uparrow} = \mu_A^{\downarrow}$. It suffices to check the associativity and unit axioms of internal monoidal objects. The unit axioms are clearly satisfied, by definition of σ_A and by the unit axioms of P-bimodules.

To show the associativity axiom, we need to calculate and compare $\mu_A(\mu_A \circ 1)$ and $\mu_A(1 \circ \mu_A)$. The key calculation is the following, which generalizes the previous example: for all $p \geqslant 0$ and $\tau \in \mathfrak{S}_p$, we have the equality of morphisms

$$\mu_A^{\odot}(1^{\odot p}) = \sum_{i=1}^{p} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{1} \odot \cdots \odot f_{i,p}^{\tau} - \sum_{i=1}^{p-1} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{t} \odot \cdots \odot f_{i,p}^{\tau}, \tag{2.4.7}$$

where

$$f_{i,j}^{\tau} = \begin{cases} t \text{ if } \tau^{-1}(j) < i, \\ s \text{ if } \tau^{-1}(j) > i, \end{cases}$$

and $-\odot-$ represents $\lambda^{\odot}(-\odot-)$, $\rho^{\odot}(-\odot-)$, or $\mu^{\odot}(-\odot-)$ depending on the types of arguments, always with bracketing to the left.

We show this equality by induction on p. For $p = 0, 1$, the result is trivial. For $p = 2$, for $\tau = id$,

$$\mu_A^{\odot} = \mu_A^{\uparrow} = \rho^{\odot}(1 \odot s) + \lambda^{\odot}(t \odot 1) - i_n\mu^{\odot}(t \odot s),$$

by definition, and for $\tau = (1\,2)$,

$$\mu_A^{\odot} = \mu_A^{\downarrow} = \rho^{\odot}(1 \odot t) + \lambda^{\odot}(s \odot 1) - i_n\mu^{\odot}(s \odot t)$$

by hypothesis on (P, A).

Let $p \geqslant 2$ and suppose that we have shown the equality for p. Let $\tau \in \mathfrak{S}_{p+1}$, and denote $i_0 = \tau^{-1}(p+1)$ and $\tau' = \tau\,(i_0\ p+1)$. If $i_0 < p+1$, then

$$\begin{aligned}
&(\mu_A^{\odot})^{p+1}(1^{\odot p+1}) = \mu_A^{\odot}((\mu_A^{\odot})^{p}(1^{\odot p}) \odot 1)\\
&= \mu_A^{\odot}\left(\sum_{i=1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 - \sum_{i=1}^{p-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1\right)\\
&= \Bigg(\sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 + \sum_{i=i_0}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{s} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1\\
&\quad + \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot s + \sum_{i=i_0}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot t\\
&\quad - \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot s - \sum_{i=i_0}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{s} \odot \cdots \odot f_{i,p}^{\tau'} \odot t\Bigg)\\
&\quad - \left(\sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 + \sum_{i=i_0+1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{s} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1\right)\\
&= f_{i_0,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i_0)}{s} \odot \cdots \odot f_{i_0,p}^{\tau'} \odot 1\\
&\quad + \sum_{\substack{i=1\\ i\neq i_0}}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot f_{i,p+1}^{\tau} + f_{i_0,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i_0)}{1} \odot \cdots \odot f_{i_0,p}^{\tau'} \odot t\\
&\quad - \sum_{\substack{i=1\\ i\neq i_0}}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot f_{i,p+1}^{\tau} - f_{i_0,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i_0)}{s} \odot \cdots \odot f_{i_0,p}^{\tau'} \odot t\\
&= \sum_{i=1}^{p+1} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{1} \odot \cdots \odot f_{i,p+1}^{\tau} - \sum_{i=1}^{p} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{t} \odot \cdots \odot f_{i,p+1}^{\tau}.
\end{aligned}$$

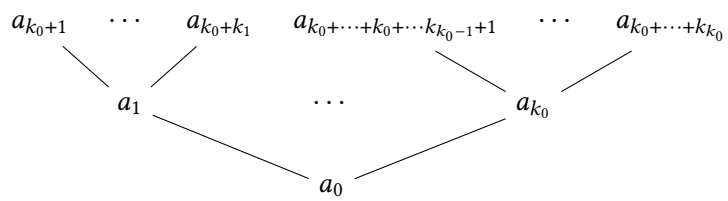
If $i_0 = p+1$, then

$$
\begin{aligned}
&(\mu_A^{\odot})^{p+1}(1^{\odot p+1}) = \mu_A^{\odot}((\mu_A^{\odot})^p(1^{\odot p}) \odot 1) \\
&= \mu_A^{\odot}\left(\sum_{i=1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 - \sum_{i=1}^{p-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1\right) \\
&= \sum_{i=1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 + \sum_{i=1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{1} \odot \cdots \odot f_{i,p}^{\tau'} \odot s \\
&\quad - \sum_{i=1}^{p} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot s - \sum_{i=1}^{p-1} f_{i,1}^{\tau'} \odot \cdots \odot \underset{\tau'(i)}{t} \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 \\
&= \sum_{i=1}^{p+1} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{1} \odot \cdots \odot f_{i,p+1}^{\tau} - \sum_{i=1}^{p} f_{i,1}^{\tau} \odot \cdots \odot \underset{\tau(i)}{t} \odot \cdots \odot f_{i,p+1}^{\tau}.
\end{aligned}
$$

This proves (2.4.7) for $p+1$. Next, let a be an arbitrary tree monomial of $A \circ A \circ A \simeq (A \circ A) \circ A \simeq A \circ (A \circ A)$. We write

$$
\begin{aligned}
a &= ((a_0 \mid a_1 \cdots a_{k_0}) \mid a_{k_0+1} \cdots a_{k_0+k_1} \cdots a_{k_0+\cdots+k_{k_0-1}+1} \cdots a_{k_0+\cdots+k_{k_0}}) \\
&= (a_0 \mid (a_1 \mid a_{k_0+1} \cdots a_{k_0+k_1}) \cdots (a_{k_0} \mid a_{k_0+\cdots+k_0+\cdots k_{k_0-1}+1} \cdots a_{k_0+\cdots+k_{k_0}})),
\end{aligned}
$$

where k_0 is the arity of a_0 and, for $i \in \{1, \ldots, k_0\}$, k_i is the arity of a_i. We can also understand a via its planar tree $T(a)$:



For the rest of this proof, we write $\cdot$ for all elementary compositions $\circ_{i,\tau}$. On the one hand, calculating $\mu_A(\mu_A \circ 1)(a)$ is equal to calculating $(\mu_A^{\odot})(a_0 \cdot a_{k_0} \cdots a_1 \cdot a_{k_0+\cdots+k_{k_0}} \cdots a_{k_0+1})$ as in the previous calculation, with $p = k_0 + \cdots + k_{k_0}$ and the identity permutation. On the other hand, calculating $\mu_A(1 \circ \mu_A)(a)$ is equal to calculating $(\mu_A^{\odot})(a_0 \cdot a_{k_0} \cdots a_1 \cdot a_{k_0+\cdots+k_{k_0}} \cdots a_{k_0+1})$ as in the previous calculation, with $p = k_0 + \cdots + k_{k_0}$ and the permutation

$$
[0, k_0, k_0 + \cdots + k_{k_0}, \ldots, k_0 + \cdots + k_{k_0-1} + 1, \ldots, 1, k_0 + k_1, \ldots, k_0 + 1],
$$

where each integer i represents the position of a_i in the argument of $\mu_A^{\odot}$. Thus, by the previous calculation, $\mu_A(\mu_A \circ 1) = \mu_A(1 \circ \mu_A)$. We conclude that $(A, \mu_A, i\eta)$ is an operad.

As an aside, calculating $\mu_A(\mu_A \circ 1)$ corresponds to a breadth-first traversal of a tree, while calculating $\mu_A(1 \circ \mu_A)$ corresponds to a depth-first traversal. □

2.4.8. Theorem. *The following categories are isomorphic:*

i) *the category* $\text{Ш}\mathrm{Op}_{\omega}$,

ii) *the full subcategory of* $\mathsf{RGlob}(\mathsf{Bimod}(Ш\mathsf{Op}))$ *whose objects are pairs* (P, A) *where* (P, μ, η) *is a shuffle operad and* $A = (A_n, \lambda_n, \rho_n)_{n\geqslant 0}$ *is a reflexive globular* P*-bimodule such that* $A_0 = P$ *and* (P, A_n) *satisfies the linear exchange relation* (2.4.3) *for all* $n \geqslant 1$.

Proof. We show that each category is a full subcategory of the other.

(**i** $\subseteq$ **ii**) Let $P = (P_n, \mu_n, \eta_n)_{n\geqslant 0}$ be an ω-operad. Forgetting the $\star_k$-compositions and operadic multiplications μ_n, P is equipped with a reflexive globular P_0-bimodule structure. By the calculations and discussion of 2.4.2, for all $n \geqslant 1$, P_n seen as a P_0-bimodule satisfies the linear exchange relation (2.4.3). Thus $Ш\mathsf{Op}_\omega$ is a full subcategory of the second category.

(**ii** $\subseteq$ **i**) Let (P, μ, η) be an operad and $(A_n, \lambda_n, \rho_n)_{n\geqslant 0}$ a globular reflexive P-bimodule satisfying the linear exchange relation (2.4.3) and such that $A_0 = P$. We proceed in two steps: first we equip A with a globular reflexive operad structure, then we equip it with a ω-operad structure.

First, let $n \geqslant 0$. Equip A_n with the partial multiplication $\mu_n^{\odot} := \mu_{A_n}^{\uparrow} = \mu_{A_n}^{\downarrow}$ and then define the operadic multiplication

$$\mu_n : A_n \circ A_n \xrightarrow{\sigma_{A_n}} \bigoplus_{p\geqslant 1} A_n \odot A_n^{\odot p} \xrightarrow{\sum(\mu_n^{\odot})^p} A_n.$$

The multiplication μ_n satisfies the associativity and unit axioms by Lemma 2.4.6. Moreover, by construction, $\mu_n^{\odot}$ satisfies the relations

$$\mu_n^{\odot} = \mu_n^{\odot}(1 \odot s_0) + \mu_n^{\odot}(t_0 \odot 1) - \mu_n^{\odot}(t_0 \odot s_0) = \mu_n^{\odot}(s_0 \odot 1) + \mu_n^{\odot}(1 \odot t_0) - \mu_n^{\odot}(s_0 \odot t_0). \tag{2.4.9}$$

This gives A a globular reflexive operad structure. Next, for the ω-operad structure on A, we define the $\star_k$-compositions as follows: for all $\star_k$-composable n-cells a, b, define

$$a \star_k b := a - t_k(a) + b.$$

Let $0 \leqslant k < \ell < n$ be three integers. The target morphism $t_\ell : A_n \longrightarrow A_\ell$ is linear, so it commutes with $\star_k$. For all $\star_\ell$-composable pairs (a, a') and (b, b') of A_n such that (a, b) and (a', b') are $\star_k$-composable, we calculate

$$\begin{aligned}(a \star_\ell a') \star_k (b \star_\ell b') &= (a - t_\ell(a) + a') \star_k (b - t_\ell(b) + b') \\ &= a \star_k b - t_\ell(a) \star_k t_\ell(b) + a' \star_k b' \\ &= a \star_k b - t_\ell(a \star_k b) + a' \star_k b' \\ &= (a \star_k b) \star_\ell (a' \star_k b').\end{aligned}$$

Thus the $\star_k$-compositions satisfy exchange relations. To show that A is an ω-operad, it suffices to show that the $\star_k$-compositions are morphisms of operads. $A_n \star_k A_n$ is equipped with an operad structure given by the multiplication

$$(A_n \star_k A_n) \circ (A_n \star_k A_n) \xrightarrow{(\pi_1\circ\pi_1)\star_k(\pi_2\circ\pi_2)} (A_n \circ A_n) \star_k (A_n \circ A_n) \xrightarrow{\mu_k\star_k\mu_k} A_n \star_k A_n,$$

where π_1, π_2 are the projections of the fiber product $A_n \star_k A_n$. Therefore it suffices to check the exchange relation (2.3.4):

$$\begin{tikzcd}
(A_n \star_k A_n) \circ (A_n \star_k A_n) \arrow[rr, "(\pi_1 \circ \pi_1) \star_k (\pi_2 \circ \pi_2)"] \arrow[dr, "\mu_n((1 \star_k 1) \circ (1 \star_k 1))"'] & & (A_n \circ A_n) \star_k (A_n \circ A_n) \arrow[dl, "\mu_n(1 \circ 1) \star_k \mu_n(1 \circ 1)"] \\
& A_n &
\end{tikzcd}$$

Writing

$$\mu_n = \mu_n \varphi_{A_n \star_k A_n} \sigma_{A_n \star_k A_n} = \left(\sum_p (\mu_n^{\odot})^{\odot p} \right) \sigma_{A_n \star_k A_n},$$

we get the diagram

$$\begin{tikzcd}
(A_n \star_k A_n) \circ (A_n \star_k A_n) \arrow[rr, "(\pi_1 \circ \pi_1) \star_k (\pi_2 \circ \pi_2)"] \arrow[d, "\sigma_{A_n \star_k A_n}"'] & & (A_n \circ A_n) \star_k (A_n \circ A_n) \arrow[d, "\sigma_{A_n} \star_k \sigma_{A_n}"] \\
\bigoplus (A_n \star_k A_n) \odot (A_n \star_k A_n)^{\odot p} \arrow[rr, "(\pi_1 \odot \pi_1^{\odot p}) \star_k (\pi_2 \odot \pi_2^{\odot p})"] \arrow[dr, "\mu_n^{\odot}((1 \star_k 1) \odot (1 \star_k 1)^{\odot p})"'] & & \left(\bigoplus A_n \odot A_n^{\odot p} \right) \star_k \left(\bigoplus A_n \odot A_n^{\odot p} \right) \arrow[dl, "\mu_n^{\odot}(1 \odot 1^{\odot p}) \star_k \mu_n^{\odot}(1 \odot 1^{\odot p})"] \\
& A_n &
\end{tikzcd}$$

The upper square commutes by naturality of σ. To show that the lower triangle commutes, it suffices to show that, for all $\star_k$-composable pairs (a, a') and (b, b') of A_n and all elementary compositions $\circ_{i,\tau}$ such that $a \circ_{i,\tau} b$ and $a' \circ_{i,\tau} b'$ are well defined,

$$(a \star_k a') \circ_{i,\tau} (b \star_k b') = (a \circ_{i,\tau} b) \star_k (a' \circ_{i,\tau} b').$$

Write $\cdot$ for $\circ_{i,\tau}$. Let us begin with the case $k = 0$. By definition of $\star_k$-composition and bidistributivity of $\circ_{i,\tau}$, we have

$$\begin{aligned}
(a + a' - t_0(a)) \cdot (b + b' - t_0(b)) = a \cdot b + a \cdot b' + a' \cdot b + a' \cdot b' - t_0(a) \cdot b - t_0(a) \cdot b' \\
- a \cdot t_0(b) - a' \cdot t_0(b) + t_0(a) \cdot t_0(b)
\end{aligned}$$

By applying (2.4.9) to $a \cdot b'$ and $a' \cdot b$, we get

$$\begin{aligned}
a \cdot b' &= t_0(a) \cdot b' + a \cdot s_0(b') - t_0(a) \cdot s_0(b'), \\
a' \cdot b &= s_0(a') \cdot b + a' \cdot t_0(b) - s_0(a') \cdot t_0(b).
\end{aligned}$$

Since $t_0(a) = s_0(a')$ and $t_0(b) = s_0(b')$, we conclude that

$$\begin{aligned}
(a + a' - t_0(a)) \cdot (b + b' - t_0(b)) &= a \cdot b + a' \cdot b' - t_0(a) \cdot t_0(b) \\
&= (a \cdot b) \star_0 (a' \cdot b).
\end{aligned}$$

Now, let $k \geqslant 1$. In this case, $n \geqslant 2$, so by globularity, $t_0(a) = t_0(a')$ and $s_0(b) = s_0(b')$. Write $c := t_0(a) = t_0(a')$ and $d := s_0(b) = s_0(b')$. Using the exchange relations between $\odot$ and $\star_0$, and between $\star_0$ and $\star_k$, we get

$$\begin{aligned}(a \cdot b) \star_k (a' \cdot b') &= ((a \star_0 c) \cdot (d \star_0 b)) \star_k ((a' \star_0 c) \cdot (d \star_0 b')) \\ &= ((a \cdot d) \star_0 (c \cdot b)) \star_k ((a' \cdot d) \star_0 (c \cdot b')) \\ &= ((a \cdot d) \star_k (a' \cdot d)) \star_0 ((c \cdot b) \star_k (c \cdot b')).\end{aligned}$$

By definition of $\star_k$ and by bidistributivity of $\circ_{i,\tau}$, we get

$$\begin{aligned}(a \cdot d) \star_k (a' \cdot d) &= a \cdot d + a' \cdot d - t_k(a) \cdot d = (a \star_k a') \cdot d, \\ (c \cdot b) \star_k (c \cdot b') &= c \cdot b + c \cdot b' - c \cdot t_k(b) = c \cdot (b \star_k b').\end{aligned}$$

Thus

$$\begin{aligned}(a \cdot b) \star_k (a' \cdot b') &= ((a \star_k a') \cdot d) \star_0 (c \cdot (b \star_k b')) \\ &= (a \star_k a') \cdot (b \star_k b').\end{aligned}$$

Thus the exchange relation is satisfied, and we conclude that A is an ω-operad. □

3. Shuffle operadic polygraphs

In this section we introduce the notion of a shuffle polygraph that defines systems of generators and oriented relations for higher shuffle operads.

3.1. Shuffle polygraphs

The structure of polygraph was introduced independently by Street and Burroni as a system of generators for free higher categories [16, 48]. This subsection introduces a version of this structure for higher shuffle operads.

3.1.1. Cellular extensions. For $n \geqslant 0$, a *cellular extension* of a shuffle n-operad P is an indexed set X equipped with two morphisms

$$P_n \underset{t_n}{\overset{s_n}{\leftleftarrows}} X$$

in Ind satisfying the globular relations $s_{n-1}s_n = s_{n-1}t_n$ and $t_{n-1}s_n = t_{n-1}t_n$, for $n \geqslant 1$, and whose elements are called $(n+1)$*-generators*. Note that every n-operad P has two canonical cellular extensions: the empty one, and the one denoted by $\mathrm{Sph}(P_n)$ that consists of a $(n+1)$-generator $a \longrightarrow b$ for every n-sphere (a, b) of P.

3.1.2. Extended higher operads. For $n \geqslant 0$, the category of *extended n-operads* $\text{Ш}\mathsf{Op}_n^+$ is defined by the following pullback of forgetful functors

$$\begin{tikzcd} \text{Ш}\mathsf{Op}_n^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Glob}_{n+1}(\mathsf{Ind}) \arrow[d, "\mathcal{V}_n"] \\ \text{Ш}\mathsf{Op}_n \arrow[r, "\mathcal{U}_n^{\text{Ш}}"'] & \mathsf{Glob}_n(\mathsf{Ind}) \end{tikzcd}$$

where the functor $\mathcal{U}_n$ forgets the shuffle composition, and the functor $\mathcal{V}_n$ forgets the $(n+1)$-cells. Explicitly, an extended shuffle n-operad is a pair (P, X) where P is an n-operad and X a cellular extension of P_n. A morphism of extended n-operads $(P, X) \to (P', X')$ is a morphism of shuffle n-operads $f : M \to M'$ and a morphism $g : X \to X'$ in Ind such that the two following square diagrams commute in Ind:

$$\begin{tikzcd} P_n \arrow[d, "f_n"'] & X \arrow[l, shift right, "s_n"'] \arrow[l, shift left, "t_n"] \arrow[d, "g"] \\ P'_n & X' \arrow[l, shift right, "s_n"'] \arrow[l, shift left, "t_n"] \end{tikzcd}$$

3.1.3. Proposition. *Let P be a shuffle $(n-1)$-operad and X be a cellular extension of P. Let $X^{\text{Ш}}$ denote the coequalizer of the two morphisms*

$$(P_0\langle X\rangle \oplus P_{n-1}) \odot (P_0\langle X\rangle \oplus P_{n-1}) \underset{\mu^{\downarrow}_{P_0\langle X\rangle \oplus P_{n-1}}}{\overset{\mu^{\uparrow}_{P_0\langle X\rangle \oplus P_{n-1}}}{\rightrightarrows}} P_0\langle X\rangle \oplus P_{n-1}$$

in the category $\mathsf{Bimod}(P)$*, where the morphisms are defined relative to the pair* $(P_0, P_0\langle X\rangle \oplus P_{n-1})$*. Then* $(P, X^{\text{Ш}})$ *is the free shuffle n-operad on* (P, X)*.*

Proof. We will progressively enrich the cellular extension X with more and more structure in order to get a reflexive globular bimodule satisfying the linear exchange relation (2.4.3). First, let us define categories of "enriched cellular extensions" of P. Let $\mathbb{D}$ be a category such that P is a reflexive $(n-1)$-globular object of $\mathbb{D}$. Define

- $\mathsf{Glob}_P(\mathbb{D})$ the full subcategory of $\mathsf{Glob}_n(\mathbb{D})$ whose objects X satisfy $(X_0, \ldots, X_{n-1}) = P$, and
- $\mathsf{RGlob}_P(\mathbb{D})$ the full subcategory of $\mathsf{RGlob}_n(\mathbb{D})$ whose objects X satisfy $(X_0, \ldots, X_{n-1}) = P$.

Finally, denote by $\mathsf{RGlob}_P^{\odot}(\mathsf{Bimod}(P_0))$ the full subcategory of $\mathsf{RGlob}_P(\mathsf{Bimod}(P_0))$ whose objects satisfy the linear exchange relation (2.4.3).

Following Theorem 2.4.8, given an extended $(n-1)$-operad (P, X), in order to construct the free n-operad, it suffices to construct the free object on $(P, X) \in \mathsf{Glob}_P(\mathsf{Ind})$ in $\mathsf{RGlob}_P^{\odot}(\mathsf{Bimod}(P_0))$. Therefore, it suffices to construct the sequence of free functors

$$\mathsf{Glob}_P(\mathsf{Ind}) \to \mathsf{Glob}_P(\mathsf{Bimod}(P_0)) \to \mathsf{RGlob}_P(\mathsf{Bimod}(P_0)) \to \mathsf{RGlob}_P^{\odot}(\mathsf{Bimod}(P_0)).$$

- Let (P, X) be and extended $(n-1)$-operad. The first free functor is induced by the free functors $\mathsf{Ind} \to \mathsf{Coll} \to \mathsf{Bimod}(P_0)$, so it sends X to $P_0\langle X\rangle$.
- Let (P, X) be an object of $\mathsf{Glob}_P(\mathsf{Bimod}(P_0))$. Since P is already a reflexive $(n-1)$-globular object, the second free functor is induced by the free functor $\mathsf{Bimod}(P_0) \to P_{n-1}/\mathsf{Bimod}(P_0)$, so it sends X to $X \oplus P_{n-1}$.
- Let (P, X) be an object of $\mathsf{RGlob}_P(\mathsf{Bimod}(P_0))$. The third free functor sends X to the coequalizer of

$$X \odot X \underset{\mu_X^{\downarrow}}{\overset{\mu_X^{\uparrow}}{\rightrightarrows}} X$$

  where the morphisms are defined relative to the pair (P_0, X).

By composing these functors, we get

$$X^{\amalg} := \mathrm{coeq}\left(\mu^{\uparrow}_{P_0\langle X\rangle \oplus P_{n-1}}, \mu^{\downarrow}_{P_0\langle X\rangle \oplus P_{n-1}} : (P_0\langle X\rangle \oplus P_{n-1}) \odot (P_0\langle X\rangle \oplus P_{n-1}) \rightrightarrows P_0\langle X\rangle \oplus P_{n-1}\right)$$

and we conclude that $(P, X^{\amalg})$ is the free n-operad on (P, X). □

3.1.4. Free shuffle n-operad. For $n \geqslant 1$, the forgetful functor $\mathcal{W}_n^{\amalg} : \amalg\mathsf{Op}_n \to \amalg\mathsf{Op}_{n-1}^{+}$ that forgets the composition of n-cells admits a left adjoint

$$\mathcal{L}_n^{\amalg} : \amalg\mathsf{Op}_{n-1}^{+} \to \amalg\mathsf{Op}_n \tag{3.1.5}$$

that associates to an extended $(n-1)$-operad (P, X) the *free n-operad over* (P, X) given by $\mathcal{L}_n^{\amalg}(P, X) = (P, X^{\amalg})$. In the sequel, the n-operad $(P, X^{\amalg})$ will be denoted by $P[X]$, and its k-source and k-target maps will be denoted by s_k and t_k respectively.

3.1.6. Shuffle polygraphs. We define the category $\amalg\mathsf{Pol}_n$ of *n-polygraphs* and the free functor

$$\mathcal{F}_n : \amalg\mathsf{Pol}_n \to \amalg\mathsf{Op}_n,$$

by induction on $n \geqslant 0$. For $n = 0$, we define $\amalg\mathsf{Pol}_0$ as the category Ind. The free 0-monoid functor

$$\mathcal{F}_0 : \amalg\mathsf{Pol}_0 \to \amalg\mathsf{Op}_0$$

is the composite of free functors (2.1.4). We suppose that for $n \geqslant 1$ the category $\amalg\mathsf{Pol}_{n-1}$ of $(n-1)$-polygraphs is defined and that the free $(n-1)$-operad functor

$$\mathcal{F}_{n-1} : \amalg\mathsf{Pol}_{n-1} \to \amalg\mathsf{Op}_{n-1}$$

is constructed. The category $\amalg\mathsf{Pol}_n$ of n-polygraphs is defined as the following pullback in Cat

$$\begin{tikzcd} \amalg\mathsf{Pol}_n \arrow[r, "\widetilde{\mathcal{U}}_{n-1}"] \arrow[d, "\widetilde{\mathcal{V}}_{n-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \amalg\mathsf{Op}_{n-1}^{+} \arrow[d, "\mathcal{W}_n^{\amalg}"] \\ \amalg\mathsf{Pol}_{n-1} \arrow[r, "\mathcal{F}_{n-1}"'] & \amalg\mathsf{Op}_{n-1} \end{tikzcd} \tag{3.1.7}$$

where the vertical functor on the right forgets the cellular extension of an extended monoid. The free symmetric n-operad functor is defined as the composite

$$\text{ШPol}_n \xrightarrow{\widetilde{\mathcal{U}}_{n-1}} \text{ШOp}^+_{n-1} \xrightarrow{\mathcal{L}^{\text{Ш}}_n} \text{ШOp}_n,$$

where $\widetilde{\mathcal{U}}_{n-1}$ is the functor defined by the pullback (3.1.7) and $\mathcal{L}^{\text{Ш}}_n$ is the free functor defined in (3.1.5).

The category ШPol_ω of ω-polygraphs and the free ω-operad functor $\mathcal{F}_\omega : \text{ШPol}_\omega \to \text{ШOp}_\omega$ are defined as the limit of the functors:

$$\cdots \to \text{ШPol}_n \xrightarrow{\widetilde{\mathcal{V}}_{n-1}} \text{ШPol}_{n-1} \to \cdots \to \text{ШPol}_1 \xrightarrow{\widetilde{\mathcal{V}}_0} \text{ШPol}_0,$$

in the category of categories, where the functors $\widetilde{\mathcal{V}}_{n-1}$ are defined by (3.1.7).

In this way, an n-polygraph X is defined inductively as a data $(X_0, \ldots, X_n)$, where X_0 is an indexed set and for every $0 < k < n$, X_k is a cellular extension of the free $(k-1)$-operad generated by $(X_0, \ldots, X_{k-1})$, denoted by

$$X_k^{\text{Ш}} = X_0^{\text{Ш}}[X_1] \cdots [X_k].$$

For $0 \leqslant p < n$, we will denote by $X_{\leqslant p}$ the underlying p-polygraph $(X_0, \ldots, X_p)$.

3.1.8. Higher-dimensional monomials. Let X be an ω-polygraph. Tree monomials in $X_0^{*\text{Ш}}$ are called *0-monomials of* $X^{\text{Ш}}$, and they form a linear basis of the collection $X_0^{\text{Ш}}$, which means that every 0-cell a of $X_0^{\text{Ш}}$ can be uniquely written as a (possibly empty) linear combination

$$a = \sum_{i=1}^{p} \lambda_i u_i + \lambda\varepsilon$$

of pairwise distinct 0-monomials $u_1, \ldots, u_p$ of $X_0^{\text{Ш}}$, with $\lambda_i \in \mathbf{k} \setminus \{0\}$, $\lambda \in \mathbf{k}$, and ε denotes the trivial monomial. This expression is called *the canonical decomposition of* a, and we define the *support of* a as the set $\mathrm{Supp}(a) = \{u_1, \ldots, u_p\}$.

For $n \geqslant 1$, if α is an n-cell of X, and $\vec{v}$ is a list of 0-monomials, we will denote by $(\alpha \mid \vec{v})$ the n-cell of $X^{\text{Ш}}$ with source $(s(\alpha) \mid \vec{v})$ and target $(t(\alpha) \mid \vec{v})$. An *n-monomial of* $X^{\text{Ш}}$ is an n-cell of $X^{\text{Ш}}$ of the form $u \circ_{i,\tau} (\alpha \mid \vec{v})$, where α is an n-cell of X, and u and $\vec{v}$ are monomials of $X^{\text{Ш}}$. By construction of the free n-operad over $(X_{n-1}^{\text{Ш}}, X_n)$, and by freeness of $X_{n-1}^{\text{Ш}}$, every n-cell a of $X^{\text{Ш}}$ can be written as a linear combination

$$a = \sum_{i=1}^{p} \lambda_i a_i + 1_c \tag{3.1.9}$$

of pairwise distinct n-monomials $a_1, \ldots, a_p$ and of an identity n-cell 1_c of $X^{\text{Ш}}$, and this decomposition is unique up to the linear exchange relation (2.4.3). The *size* of an n-cell a of $X^{\text{Ш}}$ is the minimal number of n-monomials of $X^{\text{Ш}}$ required to write a as in (3.1.9).

3.1.10. Graded shuffle polygraps. In order to define in Section 5 minimality and Koszulness properties with respect to a polygraphic resolution, we introduce the notion of a graded shuffle ω-polygraph. Just as we defined shuffle operads as internal monoids in the presheaf category $\mathrm{Vect}^{\mathrm{Ord}^o}$ in Section 2.1, we define *graded shuffle operads* as internal monoids in the presheaf category $\mathrm{grVect}^{\mathrm{Ord}^o}$. For $n \in$

$\mathbb{N} \cup \{\omega\}$, a *graded shuffle n-operad* is an n-category in grШOp, and we denote by $\mathrm{grШOp}_n$ the corresponding category with internal n-functors as morphisms. In particular, the source, target and composition morphisms of graded shuffle n-operad are graded.

The category $\mathrm{grШOp}_n^+$ of *graded extended n-operads* is defined similarly to $\mathrm{ШOp}_n^+$: its objects are pairs (P, X), where P is a graded n-operad, and X is a graded cellular extension of A, meaning that $X = \amalg_{i \geqslant 0} X^{(i)}$ and that the source and target of each x in $X^{(i)}$ are homogeneous of degree i. In that case, the free $(n+1)$-operad $P[X]$, defined as in the nongraded case, is also graded.

A *graded ω-polygraph* is an ω-polygraph X such that each set X_n is graded, for $n \geqslant 0$. This notion restricts to n-polygraphs, and a 1-polygraph X is called *quadratic* if X_0 is concentrated in degree 1 and X_1 is concentrated in degree 2. We say that a graded ω-polygraph X is *concentrated on the superdiagonal* if each graded set X_n, for $n \geqslant 0$, is concentrated in degree $n+1$. In that case, because the source and target maps are graded, for $n \geqslant 1$, the source and target of every n-cell of $X^{\mathrm{Ш}}$ are homogeneous $(n-1)$-cells of $X^{\mathrm{Ш}}$ of degree $n+1$.

3.2. Shuffle polygraphic resolutions

In this subsection we introduce the notion of a polygraphic resolution for shuffle operads.

3.2.1. Presentation of a shuffle operad. The shuffle operad *presented* by a shuffle 1-polygraph X is the coequalizer in the category ШOp of the following source and target morphisms, denoted by $\overline{X}$,

$$X_1^{\mathrm{Ш}} \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0^{\mathrm{Ш}} \xrightarrow{\pi_X} \overline{X}. \tag{3.2.2}$$

Following Proposition (2.3.7), the category ШOp preserves reflexive coequalizers and so the construction is well defined. We say that a shuffle operad P is *presented by a polygraph* X, or that X is a *presentation of* P, if P is isomorphic to $\overline{X}$ in the category ШOp.

3.2.3. Shuffle polygraphic resolutions. For $n \geqslant 1$, let P be a shuffle n-operad. A cellular extension X of P is *acyclic* if for every n-sphere (f, g) in P_n, there exists a shuffle $(n+1)$-cell F in the $(n+1)$-operad $P[X]$ with source f and target g. A *coherent presentation* of a shuffle polygraph P is a shuffle 2-polygraph X, that presents P, and whose cellular extension X_2 is acyclic. A *shuffle polygraphic resolution* of a shuffle operad P is a shuffle ω-polygraph X that presents P and whose cellular extensions X_k are acyclic for $k \geqslant 2$.

3.2.4. Example: standard polygraphic resolution. Let P be a shuffle operad, and B a basis of P seen as a collection. We then define the *standard polygraphic resolution* $\mathrm{Std}(P)$ by induction as follows. For $n = 0$, we define the indexed set $\mathrm{Std}(P)_0 := B$, and for $u \in B$ we denote by $[u]$ the corresponding element in $\mathrm{Std}(P)_0$. Any element $a = \sum_{u \in B} \lambda_u u$ of P then corresponds to a linear combination of elements $[a] := \sum_{u \in B} \lambda_u [u]$ in $\mathrm{Std}(P)_0^{\mathrm{Ш}}$. Next, for $n = 1$, we set

$$\mathrm{Std}(P)_1 := \{\ [u] \circ_{i,\tau} [v] \longrightarrow [u \circ_{i,\tau} v] \ \mid\ u, v \in B\ \},$$

so that the pair $(B, \mathrm{Std}(P)_1)$ forms a 1-polygraph that presents the shuffle operad P. Now, suppose that $\mathrm{Std}(P)_n$ is defined for $n \geqslant 1$. Then we set $\mathrm{Std}(P)_{n+1} := \mathrm{Sph}(\mathrm{Std}(P)_n^{\mathrm{Ш}})$. By construction, the ω-polygraph $\mathrm{Std}(P)$ is a polygraphic resolution of the shuffle operad P.

3.2.5. Tietze equivalence of ω-polygraphs. We define the notion of a *weak-equivalence* of ω-operads as for ω-categories, defined in [34]. For $n \geqslant 0$, two n-cells a, b of an ω-operad P are *ω-equivalent* if there exists an $(n+1)$-cell $f : a \to b$ in P. In that case, we write $a \sim_\omega b$. A morphism of ω-operads $F : P \to Q$ is a *weak equivalence* if it satisfies the following properties:

i) For every 0-cell a of Q, there exists a 0-cell $\widehat{a}$ in P such that $F(\widehat{a}) \sim_\omega a$.

ii) For every pair of 0-cells a, b of P and every 1-cell $f : F(a) \to F(b)$ of Q, there exists a 1-cell $\widehat{f} : a \to b$ of P such that $F(\widehat{f}) \sim_\omega f$.

iii) For $n \geqslant 1$ and every pair of parallel n-cells a, b of P and every $(n+1)$-cell $f : F(a) \to F(b)$ of Q, there exists an $(n+1)$-cell $\widehat{f} : a \to b$ of P such that $F(\widehat{f}) \sim_\omega f$.

We say that two shuffle ω-polygraphs X and Y are *Tietze equivalent* if the induced free ω-operads X^{III} and Y^{III} are weakly equivalent. The original notion of Tietze equivalence for 1-polygraphs is a particular case of this notion for ω-polygraphs. Two 1-polygraphs X and Y are *Tietze equivalent* if the presented shuffle operads $\overline{X}$ and $\overline{Y}$ are isomorphic. In that case, extending X and Y into ω-polygraphs with identities in higher dimensions gives two Tietze equivalent ω-polygraphs. Tietze equivalence also generalizes the notion of Tietze equivalence between $(3,1)$-polygraphs introduced in [23].

3.2.6. Proposition. *Let X and Y be two acyclic ω-polygraphs. Then the presented shuffle operads $\overline{X}$ and $\overline{Y}$ are isomorphic if, and only if, the polygraphs X and Y are Tietze equivalent.*

Proof. ($\Rightarrow$) Denote by $\varphi : \overline{X} \to \overline{Y}$ the isomorphism. We define a morphism of ω-operads $F : X^{\text{III}} \to Y^{\text{III}}$ such that $\pi_Y F = \varphi \pi_X$ on 0-cells and show that it is a weak equivalence simultaneously. Since X^{III} is a free ω-operad, it suffices to define F on the n-generators of X for all $n \geqslant 0$. We proceed by induction on $n \geqslant 0$.

For $n = 0$, define linear maps $i_X : \overline{X} \to X^{\text{III}}$ and $i_Y : \overline{Y} \to Y^{\text{III}}$, which are sections of π_X and π_Y, respectively. For $x \in X_0$, we set $F(x_0) := i_Y(\varphi(\pi_X(x_0)))$, and we check that $\pi_Y F = \varphi \pi_X$ on 0-cells. Now, for a 0-cell a of Y^{III}, let $\hat{a} := \varphi^{-1} i_Y(a) \in X^{\text{III}}$. Then $\pi_Y(F(\hat{a})) = \pi_Y(a)$, so $F(\hat{a}) \sim_\omega a$.

For $n = 1$, for $\alpha : a \to b$ a 1-generator of X, $\pi_Y(F(a)) = \pi_Y(F(b))$, so there exists a 1-cell $f : F(a) \to F(b)$ in Y^{III}. We set $F(\alpha) := f$. Then, for every pair of 0-cells a, b of X^{III} and every 1-cell $f : F(a) \to F(b)$ of Y^{III}, $\pi_Y(F(a)) = \pi_Y(F(b))$, which is equivalent to $\pi_X(a) = \pi_X(b)$ via the isomorphism φ. Therefore there exists $\hat{f} : a \to b$ in X^{III}, and $F(\hat{f}) : F(a) \to F(b)$ is parallel to f. Since Y is acyclic, $F(\hat{f}) \sim_\omega f$.

Let $n \geqslant 1$ and suppose that F is defined on n-cells of X^{III}. For $\alpha : a \to b$ an $(n+1)$-generator of X, the n-cells a and b of X^{III} are parallel, so the n-cells $F(a)$ and $F(b)$ of Y^{III} are parallel. By acyclicity of Y, there exists an $(n+1)$-cell $f : F(a) \to F(b)$ in Y^{III}, so we set $F(\alpha) := f$. Now, let a, b be two parallel n-cells of X^{III} and $f : F(a) \to F(b)$ an $(n+1)$-cell of Y^{III}. By acyclicity of X, there exists an $(n+1)$-cell $\hat{f} : a \to b$, so $F(\hat{f})$ and f are parallel in Y^{III}, so by acyclicity of Y we have $F(\hat{f}) \sim_\omega f$.

We conclude that $F : X^{\text{III}} \to Y^{\text{III}}$ is a weak equivalence, so X and Y are Tietze equivalent.

($\Leftarrow$) Let $F : X^{\text{III}} \to Y^{\text{III}}$ be a weak equivalence. By condition **i)**, $\pi_Y F : X^{\text{III}} \to \overline{Y}$ is surjective. Moreover, if $a \sim_\omega b$ in X^{III}, then $\pi_Y F(a) = \pi_Y F(b)$, so F induces a morphism of ω-operads $\overline{F} : \overline{X} \to \overline{Y}$. By condition **ii)**, $\overline{F}$ is injective. Thus $\overline{F}$ is an isomorphism between $\overline{X}$ and $\overline{Y}$. □

3.2.7. Reduced polygraphs. Let X be a left-monomial 1-polygraph. Recall from [3] that a 1-generator $\alpha \in X_1$ is *right* (resp. *left*) *reduced* if $t_0(\alpha) \in \mathrm{Red}(X_1)$ (resp. $s_0(\alpha) \in \mathrm{Red}(X_1 \setminus \{\alpha\})$). We say that X is *reduced* when each of its 1-generators is left and right reduced. We prove that every (finite) convergent left-monomial 1-polygraph is Tietze-equivalent to a reduced (finite) convergent left-monomial 1-polygraph.

4. Shuffle operadic rewriting

The first part of this section presents the main rewriting properties of shuffle 1-polygraphs. We relate the notion of a convergent shuffle polygraph, whose 1-generators are oriented with respect to a given monomial order, with the notion of Gröbner bases introduced in [19], and with the notion of Poincaré-Birkhoff-Witt bases introduced in [28]. Throughout this section, all operads and polygraphs are shuffle.

4.1. Rewriting in shuffle operads

We introduce a concept of rewriting in the context of shuffle operads.

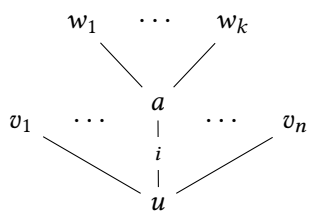
4.1.1. The terminal indexed set. Denote by $\square$ the terminal object of Ind, that is, the indexed set that is a singleton $\square(k) = \{\square_k\}$ for each arity $k \geqslant 1$. Denote by $\iota_k : \mathsf{Set} \to \mathsf{Ind}$ the inclusion functor defined by $\iota_k(X_0)(k) = X_0$ and $\iota_k(X_0) = \varnothing$ for the other arities.

4.1.2. One-hole contexts of indexed sets. A *one-hole context* of an indexed set X_0 is an element Γ of the free X_0^{III}-bimodule $X_0^{\mathrm{III}}\langle\square\rangle$. We say that Γ is of *inner arity* k if it is an element of $X_0^{\mathrm{III}}\langle\square(k)\rangle$.

Let A be an X_0^{III}-bimodule and $a \in A(k)$. Identifying $A(k)$ with $\mathrm{Hom}_{\mathsf{Set}}(\square(k), A(k))$, a induces a morphism

$$\varphi_a : X_0^{\mathrm{III}}\langle\square(k)\rangle \to X_0^{\mathrm{III}}\langle A(k)\rangle.$$

via the the functor $X_0^{\mathrm{III}}\langle\iota_k\rangle : \mathsf{Set} \to \mathsf{Bimod}(X_0^{\mathrm{III}})$. For Γ a one-hole context of $X_0^{*\mathrm{III}}$ of inner arity k, we write $\Gamma[a] := \varphi_a(\Gamma)$. Explicitly, $\Gamma[a]$ is a tree of the form



where $k, n \geqslant 1$, $i \in \{1, \ldots, n\}$, $u \in X_0^{\mathrm{III}}(n)$, $v_1, \ldots, \check{v}_i, \ldots, v_n, w_1, \ldots, w_k \in X_0^{\mathrm{III}}$ and $\square_k$ appears in the ith position. The notation $\check{v}_i$ means that we omit v_i. In this way, every one-hole context Γ on X_0 can be written $\Gamma := w \circ_{i,\tau} (\square_k \mid \vec{w})$ with $w, \vec{w} \in X_0^{\mathrm{III}}$. In this work, we will only consider *monomial* one-hole contexts, that is when $w, \vec{w}$ are monomials of $X_0^{*\mathrm{III}}$, so we will omit the word monomial.

4.1.3. Two-hole contexts of indexed sets. Let X_0 be an indexed set. There exists a bifunctor $C_2^{X_0} : \mathsf{Ind} \times \mathsf{Ind} \to \mathsf{Ind}$ which sends a pair of indexed sets Y, Y' to the indexed set of elements of $(X_0 \sqcup Y \sqcup Y')^{*\text{Ш}}$ with one occurrence of both Y and Y'. A *two-hole context of* $X_0^{*\text{Ш}}$ is an element Γ of $C_2^{X_0}(\square, \square)$. We say that Γ is of *inner arities* (k, ℓ) if it is an element of $C_2^{X_0}(\square(k), \square(\ell))$.

Let P be an operad equipped with a morphism $\pi : X_0^{\text{Ш}} \to P$, and $a \in P(k), a' \in P(\ell)$. Identifying

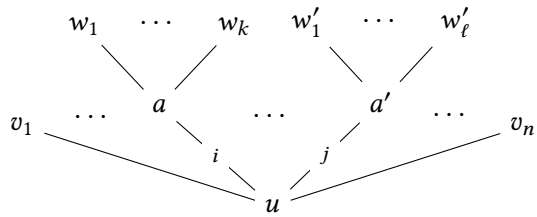
$$\begin{aligned} P(k) \times P(\ell) &\simeq \mathrm{Hom}_{\mathsf{Set}}(\square(k), P(k)) \times \mathrm{Hom}_{\mathsf{Set}}(\square(\ell), P(\ell)) \\ &\simeq \mathrm{Hom}_{\mathsf{Set}\times\mathsf{Set}}((\square(k), \square(\ell)), (P(k), P(\ell))), \end{aligned}$$

the pair (a, a') induces a morphism

$$\varphi_{a,a'} : C_2^{X_0}(\square(k), \square(\ell)) \to C_2^{X_0}(P(k), P(\ell))$$

via the bifunctor $C_2^{X_0}(\iota_k, \iota_\ell) : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Ind}$. Moreover, f induces a morphism $\pi_* : C_2^{X_0}(P(k), P(\ell)) \to P$. For Γ a two-hole context of $X_0^{*\text{Ш}}$ of inner arities (k, ℓ), we write $\Gamma(a, a') := i_* \varphi_{a,a'}(\Gamma)$. Explicitly, $\Gamma(a, a')$ is a tree of one of the following two forms, where the application of i is implicit:

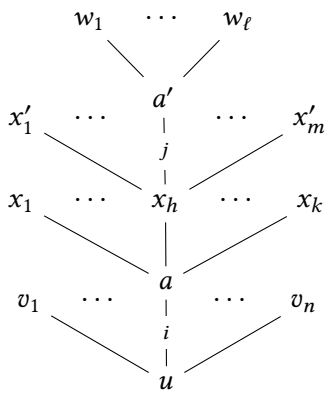
i)



where

$$n \geqslant 2, \quad i, j \in \{1, \dots, n\}, \quad u \in X_0^{\text{Ш}}(n), \text{ and}$$
$$v_1, \dots, \check{v}_i, \dots, \check{v}_j, \dots, v_n, w_1, \dots, w_k, w'_1, \dots, w'_\ell \in X_0^{\text{Ш}},$$

ii)



where

$$m, n \geqslant 1, \quad i \in \{1, \dots, n\}, \quad j \in \{1, \dots, m\}, \quad h \in \{1, \dots, k\}, \text{ and}$$
$$u, v_1, \dots, \check{v}_i, \dots, v_n, w_1, \dots, w_\ell, x_1, \dots, x_k, x'_1, \dots, \check{x}'_j, \dots, x'_m \in X_0^{\text{Ш}}.$$

In this work, we will only consider *monomial* two-hole contexts, that is, when in **i)** and **ii)** the u, $\vec{v}$, $\vec{x}$, $\vec{x}'$, $\vec{w}$, $\vec{w}'$ are monomials of $X_0^{*\sqcup\!\sqcup}$.

4.1.4. Left-monomiality and homogeneity. A cellular extension X_1 of $X_0^{\sqcup\!\sqcup}$ is *left-monomial* if, for every 1-generator α in X_1 the source $s_0(\alpha)$ is a non-trivial 0-monomial, and $s_0(\alpha) \notin \mathrm{Supp}(t_0(\alpha))$. A 1-polygraph is *left-monomial* if X_1 is so. We prove that every 1-polygraph is Tietze equivalent to a left-monomial one. For $N \geqslant 1$, a cellular extension X_1 of $X_0^{\sqcup\!\sqcup}$ is *homogeneous* if, for every 1-generator α in X_1 the weight of $s_0(\alpha)$ and $t_0(\alpha)$ are equal to N. A 1-polygraph is N*-homogeneous* if X_1 is so. When $N = 2$ we say *quadratic* for N-homogeneous.

4.1.5. Rewriting step. A *rewriting step* of a left-monomial 1-polygraph X is a 1-cell f of $X_1^{\sqcup\!\sqcup}$ of size 1 of the form

$$f = \lambda g + 1_c,$$

where $\lambda \in \mathbf{k} \setminus \{0\}$, g is a 1-monomial of $X_1^{\sqcup\!\sqcup}$, and c is a 0-cell of $X_0^{\sqcup\!\sqcup}$ such that the 0-monomial $s_0(u) \notin \mathrm{Supp}(c)$. A 1-cell of $X_1^{\sqcup\!\sqcup}$ if *positive* if it is the $\star_0$-composition of rewriting steps.

A 0-cell a of $X_0^{\sqcup\!\sqcup}$ is *reduced* if there is no rewriting step with source a. We denote by $\mathrm{Red}(X)$ the indexed submodule of reduced 0-cells. The indexed set $\mathrm{Red}_m(X)$ of reduced 0-monomials of $X_0^{\sqcup\!\sqcup}$ forms a basis of $\mathrm{Red}(X)$. A *normal form* of a is a reduced 0-cell b such that there is a positive 1-cell with source a and target b.

4.1.6. Monomial orders and termination. An *indexed poset* $(A, \prec)$ is an indexed set A, such that each $A(k)$ is equipped with a partial order $\prec_k$; we will omit the index on $\prec$. An indexed poset $(A, \prec)$ is *well-founded* if each $A(k)$ is a well-founded poset.

Let X_0 be an indexed set. An order relation $\prec$ on the free monoid $X_0^{*\sqcup\!\sqcup}$ of tree monomials is *stable by product* if, for all $u, u' \in X_0^{*\sqcup\!\sqcup}(k)$, $v, v' \in X_0^{*\sqcup\!\sqcup}(\ell)$, $i \in \{1, \dots, k\}$, and $\tau \in \sqcup\!\sqcup(\ell - 1, k - i)$, $u \prec u', v \prec v'$ implies $u \circ_{i,\tau} v \prec u' \circ_{i,\tau} v'$. A total order relation stable by product is called a *monomial order* on $X_0^{*\sqcup\!\sqcup}$. Note that this notion also appears in [28] and [19].

For Y a left-monomial cellular extension of $X_0^{\sqcup\!\sqcup}$, an order relation $\prec$ on $X_0^{*\sqcup\!\sqcup}$ is *compatible with* Y if, for every 1-cell $\alpha : u \longrightarrow a$ of Y and every monomial $v \in \mathrm{Supp}(a)$, $v \prec u$. The relation $\prec$ can be extended to the free shuffle operad $X_0^{\sqcup\!\sqcup}$ as follows: for two 0-cells a, b of $X_0^{\sqcup\!\sqcup}$, we have $b \prec a$ if the two following conditions are satisfied

i) $\mathrm{Supp}(a) \setminus \mathrm{Supp}(b) \neq \varnothing$,

ii) for all $v \in \mathrm{Supp}(b) \setminus \mathrm{Supp}(a)$, there exists $u \in \mathrm{Supp}(a) \setminus \mathrm{Supp}(b)$ such that $v \prec u$.

For a left-monomial 1-polygraph X, we denote by $\prec_{X_1}$ the smallest partial order relation on $X_0^{*\sqcup\!\sqcup}$ stable by product and compatible with X_1. A 1-polygraph X is *terminating* if the relation $\prec_{X_1}$ is well-founded. In that case, for every rewriting step f of X, we have $t_0(f) \prec_{X_1} s_0(f)$, and thus there does not exist infinite sequence of rewriting steps of X.

4.1.7. Path-lexicographic order on 1-monomials. Let $(X_0, \prec)$ be a totally ordered indexed set, and $\prec_{\mathfrak{S}}$ a total order on permutations. Let us recall from [19, 28] the *path-lexicographic* monomial order on $X_0^{*\sqcup\!\sqcup}$. Given a 0-monomial u of arity k, there exists a unique path from its root to each of its inputs. Write such a path as a word $a = x_1 \dots x_n$ in the alphabet X_0 made of labels of the vertices of the path. To the 0-monomial u we associate the pair (L, f) where L is the sequence $(p_1, \dots, p_k)$ of path

words from the root to the inputs of u, and f is the permutation of the inputs of u. Then we define the *path-lexicographic order* induced by the orders $\prec$ and $\prec_{\mathfrak{S}}$, denoted by $\prec_{pl}$, by setting

$$(L, f) \prec_{pl} (L', f') \text{ if } \big(L \prec_{lex} L' \text{ or } \big(L = L' \text{ and } f \prec_{\mathfrak{S}} f'\big)\big),$$

where $\prec_{lex}$ denotes the lexicographic order on words on X_0 induced by $\prec$.

Now, let X be a left-monomial 1-polygraph. Consider a total order $\prec$ on $X_0 \sqcup X_1$, such that $\alpha \prec x$, for every $x \in X_0$ and $\alpha \in X_1$, and $\prec_{\mathfrak{S}}$ a total order on permutations. We will denote by $\prec_{pl}$ the induced path-lexicographic order on $(X_0 \sqcup X_1)^{*\text{ш}}$, which induces a *path-lexicographic order on* 1*-monomials.*

4.1.8. Proposition. *Let X be a left-monomial 1-polygraph. If $X_0^{*\text{ш}}$ admits a well-founded monomial order $\prec$ compatible with X_1, then X is terminating.*

Proof. We have $\prec_{X_1} \subseteq \prec$, so $\prec_{X_1}$ is well-founded, so X is terminating. □

However, the converse implication is not true. In general, in order to prove termination when no monomial order is known, it is necessary to use a proof strategy appropriated to the set of rules. The following gives an illustration for one of the simplest strategies.

4.1.9. Proposition. *A left-monomial 1-polygraph X terminates if, and only if, there exists a well-founded indexed poset $(W, <)$ and a morphism of indexed sets $\Phi : X_0^{*\text{ш}} \to W$ such that $\Phi(\Gamma[v]) < \Phi(\Gamma[s(\alpha)])$ holds for every 1-generator $\alpha \in X_1$, one-hole context Γ, and $v \in \mathrm{Supp}(t(\alpha))$.*

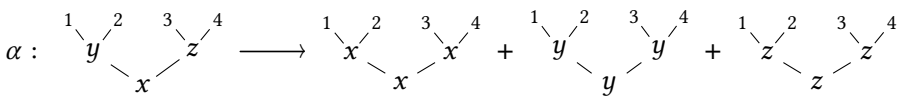
Proof. Suppose that the polygraph X terminates. Then $X_0^{*\text{ш}}$ is equipped with a well-founded partial order $\prec_{X_1}$, and We set Φ to be the identity morphism on $X_0^{*\text{ш}}$.

Conversely, let $\prec$ be the partial order generated by $v \prec u$ if there exists a rewriting rule $\alpha \in X_1$ and a one-hole context Γ such that $u = \Gamma[s(\alpha)]$ and $v \in \mathrm{Supp}(\Gamma[t(\alpha)])$. The order $\prec$ is stable by product by considering

$$u \circ_{i,\tau} v \;\prec\; u' \circ_{i,\tau} v \;\prec\; u' \circ_{i,\tau} v',$$

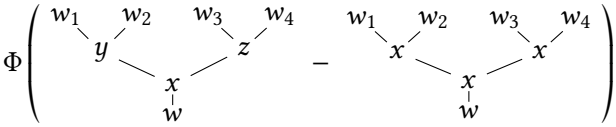
for all tree monomials u, v, u', v' in $X_0^{*\text{ш}}$, and is compatible with X_1 by definition. Thus $\prec_{X_1} \subseteq \prec$, and so the map $\Phi : (X_0^{*\text{ш}}, \prec_{X_1}) \to W$ is a strictly monotone morphism of indexed posets. Since W is well-founded, $(X_0^{*\text{ш}}, \prec_{X_1})$ is as well, and so the 1-polygraph X terminates. □

4.1.10. Example. We consider the polygraph X with three 0-generators x, y, z and the following 1-generator:

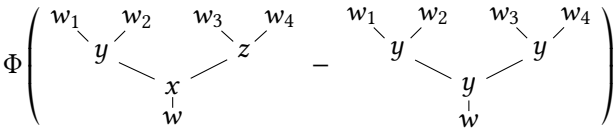


For $u \in X_0^{*\text{ш}}$, we set $\Phi(u) := |u|_x + 3|u|_{y-z}$, where $|u|_x$ denotes the number of occurrences of x in $T(u)$ and $|u|_{y-z}$ the number of inner vertices of $T(u)$ whose two children are, from left to right, y and z.

Then $\Phi(\Gamma[s(\alpha)]) > \Phi(\Gamma[v])$ for all contexts Γ of inner arity 4 and every $v \in \mathrm{Supp}(t(\alpha))$. Indeed, for every $\Gamma = w \circ_{i,\tau} (\square_4 \mid w_1\, w_2\, w_3\, w_4)$, we have

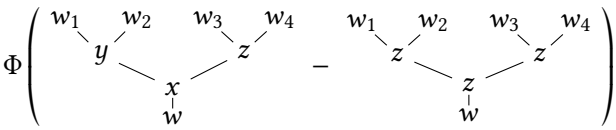


$$= (1-1) + (3-0) = 1,$$



$$= 1 + 3\left|\begin{array}{ccc} y & & z \\ & x & \end{array}\right|_{y-z} + 3|w|_{y-z} - 3|w \circ_{i,\tau} y|$$

$$= \begin{cases} 4 & \text{if } |w \circ_{i,\tau} y|_{y-z} = |w|_{y-z}, \\ 1 & \text{if } |w \circ_{i,\tau} y|_{y-z} = |w|_{y-z} + 1, \end{cases}$$



$$= \begin{cases} 4 & \text{if } |w \circ_{i,\tau} z|_{y-z} = |w|_{y-z}, \\ 1 & \text{if } |w \circ_{i,\tau} z|_{y-z} = |w|_{y-z} + 1. \end{cases}$$

Following Proposition 4.1.9 the polygraph X terminates. Note that, there is no monomial order that orients this rule in this way. Indeed, every orientation compatible with a monomial order reduces first one of the term of right hand side.

4.2. Confluence of shuffle polygraphs

In this subsection we define and algebraically characterize the property of confluence of a shuffle polygraph. We prove the coherent critical branching theorem for shuffle polygraphs involving a restricted notion of critical branchings. The definitions and results of this section do not differ much from the case of associative algebras in [25], and indeed associative algebras can be seen as operads concentrated in arity 1, but since the notion of contexts is not made explicit in previous works, we restate all of the definitions and results.

4.2.1. Branchings. A *branching* of a left-monomial 1-polygraph X is a pair (f, g) of positive 1-cells of X_1^{III} where f and g have the same source $s_0(f) = s_0(g)$, which we denote by $s_0(f, g)$. The branching (f, g) is said to be *local* if f and g are both rewriting steps.

Let X be a 2-polygraph such that $X_{\leqslant 1}$ is left-monomial. A branching (f, g) of the 1-polygraph $X_{\leqslant 1}$ is X_2*-coherently confluent*, or (f, g) is coherently confluent for short, if there exist positive 1-cells h and k of X_1^{III} and a 2-cell F of X_2^{III} as in the following diagram

$$\begin{tikzcd} & t_0(f) \arrow[dr, "h"] \arrow[dd, phantom, "\Downarrow F"] & \\ s_0(f) = s_0(g) \arrow[ur, "f"] \arrow[dr, "g"'] & & t_0(h) = t_0(k) \\ & t_0(g) \arrow[ur, "k"'] & \end{tikzcd}$$

If u is a 0-cell of X_0^{III}, the 2-polygraph X is *coherently confluent* (resp. *locally coherently confluent*) *at* u if every branching (resp. local branching) of X of source u is coherently confluent. The 2-polygraph X is *coherently confluent* (resp. *locally coherently confluent*) if it is so at every 0-cell of X_0^{III}, and that X is *coherently convergent* if it is terminating and coherently confluent. A left-monomial 1-polygraph X is *confluent* (resp. *locally confluent*) if the 2-polygraph $(X_0, X_1, \mathrm{Sph}(X_1^{\mathrm{III}}))$ has the corresponding coherent property, and it is *convergent* if it is both terminating and confluent.

4.2.2. Classification of local branchings. We distinguish the following four types of local branchings of a left-monomial 1-polygraph X:

i) *aspherical branchings*: (f, f), where f is a rewriting step of X,

ii) *additive branchings*: $(\lambda f + \mu 1_v + 1_c, \lambda 1_u + \mu g + 1_c)$, where $f : u \to a$ and $g : v \to b$ are 1-monomials of X_1^{III}, λ and μ are nonzero scalars, c is a 0-cell of X_0^{III}, $u \neq v$, and $u, v \notin \mathrm{Supp}(c)$.

iii) *multiplicative branchings*: $(\lambda\Gamma[f, 1_v] + 1_c, \lambda\Gamma[1_u, g] + 1_c)$, where Γ is a two-hole context of $X_0^{*\mathrm{III}}$, $f : u \to a$ and $g : v \to b$ are 1-monomials of X_1^{III}, λ is a nonzero scalar, c is a 0-cell of X_0^{III}, and $\Gamma[u, v] \notin \mathrm{Supp}(c)$.

iv) *intersecting branchings*: local branchings that are neither aspherical, additive, nor multiplicative.

We define a well-founded partial order $\sqsubseteq$ on branchings of X as follows: for every one-hole context Γ of $X_0^{*\mathrm{III}}$ and every 0-cell c of X_0^{III}, we set

$$(f, g) \sqsubseteq (\Gamma[f] + 1_c, \Gamma[g] + 1_c).$$

The *critical branchings* are the minimal intersecting branchings for this order. We denote the intersecting branchings by $(\Gamma[\alpha] + 1_c, \Delta[\beta] + 1_c)$, where α, β are 1-generators of X, Γ, Δ are one-hole contexts of $X_0^{*\mathrm{III}}$, and $s_0(\Gamma[\alpha]) = s_0(\Delta[\beta])$. An *essential branching* is a critical branching $(\Gamma[\alpha], \Delta[\beta])$ where $\Gamma[\alpha]$ and $\Delta[\beta]$ are consecutive 1-monomials for the path-lexicographic monomial order $\prec_{pl}$ defined in § 4.1.7.

Let X be a 2-polygraph such that $X_{\leqslant 1}$ is left-monomial, and u be a 0-cell of X_1^{III}. We say that X is *essentially coherently confluent at* u if every essential branching of X of source u is coherently confluent, and that X is *essentially coherently confluent* if it is so at every 0-cell of X_0^{III}. A left-monomial 1-polygraph X is *essentially confluent* if the 2-polygraph $(X_0, X_1, \mathrm{Sph}(X_1^{\mathrm{III}}))$ is essentially coherently confluent.

As for polygraphs of associative algebras, we have:

4.2.3. Lemma ([25, Lemmata 3.1.3 and 4.1.2.]). *Let X be a 2-polygraph such that $X_{\leqslant 1}$ is left-monomial, and a 0-cell a in X_0^{III} such that X is coherently confluent at b for any $b \prec_{X_1} a$. If f is a 1-cell of X_1^{III} that decomposes*

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_p} a_p$$

into 1-cells of size 1, with $a_i \prec_{X_1} a$ for all $i \in \{1, \ldots, p-1\}$, then there exists a 0-cell a', 1-cells g, h, and a 2-cell F in X_2^{III} such that

$$\begin{tikzcd} & a_p \arrow[dr, bend left, "h"] \arrow[d, phantom, "\Downarrow F"] & \\ a_0 \arrow[ur, bend left, "f"] \arrow[rr, bend right, "g"'] & {} & a' \end{tikzcd}$$

When $p = 1$, then F is an identity 2-cell.

4.2.4. Theorem (Coherent essential branchings theorem). *Let X be a 2-polygraph such that $X_{\leqslant 1}$ is terminating and left-monomial. If X is essentially coherently confluent, then it is coherently confluent.*

Proof. The structure of the proof is the same as for the similar result for associative algebras given in [25]. The primary difference is that we prove that we can restrict the hypotheses to the critical branchings that are essential. Suppose that X is an essentially coherently confluent 2-polygraph. We proceed by well-founded induction on the sources of the branchings of $X_{\leqslant 1}$, with respect to the order $\prec_{X_1}$, to prove that X is coherently confluent at every 0-cell of X_0^{III}. For each source, we first prove local coherent confluence and then deduce coherent confluence by Newman's lemma, exactly as for associative algebras in [25, Thm. 4.2.1].

A reduced 0-cell cannot be the source of a local branching, so X is coherently confluent at reduced 0-cells. Now, fix a nonreduced 0-cell a_0 of X_0^{III}, and assume that X is coherently confluent at every $b \prec_{X_1} a_0$. Then we proceed by case analysis on the type of the local branchings. If we show that critical branchings are coherently confluent, then the cases of aspherical, additive, multiplicative, and non-critical intersecting branchings are handled exactly as in [25, Thm. 4.2.1]. There remains to show the coherent confluence of critical branchings.

Let $(\Gamma[\alpha], \Delta[\beta])$ be a critical branching of source a_0 as in § 4.2.2. We proceed by induction on the size of the source of the branching and the number of 1-monomials $\Lambda[\gamma]$ of X_1^{III} such that $\Gamma[\alpha] \prec_{pl} \Lambda[\gamma] \prec_{pl} \Delta[\beta]$. If the branching is essential, then it is coherently confluent by hypothesis. Otherwise, there exists a 1-monomial $\Lambda[\gamma]$ of X_1^{III} between $\Gamma[\alpha]$ and $\Delta[\beta]$. We get two branchings $(\Gamma[\alpha], \Lambda[\gamma])$ and $(\Lambda[\gamma], \Delta[\beta])$.

The branching $(\Gamma[\alpha], \Lambda[\gamma])$ is either multiplicative or intersecting. If it is multiplicative, then it is coherently confluent by the multiplicative case. Otherwise, it is either non-minimal with respect to the order $\sqsubseteq$ or a critical branching. In the non-minimal case, there exists a factorisation

$$\begin{tikzcd}
 & \Gamma_1\Gamma_0[t(\alpha)] \arrow[dr, bend left, "\Gamma_1[f_0]"] \arrow[dd, Rightarrow, "F"] & \\
\Gamma_1\Gamma_0[s(\alpha)] = \Gamma_1\Lambda_0[s(\gamma)] \arrow[ur, bend left, "\Gamma_1\Gamma_0[\alpha]"] \arrow[dr, bend right, "\Gamma_1\Lambda_0[\gamma]"'] & & \Gamma_1[b_0] \\
 & \Gamma_1\Lambda_0[t(\gamma)] \arrow[ur, bend right, "\Gamma_1[g_0]"'] &
\end{tikzcd}$$

where $(\Gamma_0[\alpha], \Lambda_0[\gamma])$ is a critical branching, and by induction on the size of its source, Γ_1 is a one-hole context of $X_0^{*\mathrm{III}}$, and f_0, g_0 are positive 1-cells of X_1^{III} making this critical branching coherently confluent. In the critical case, the branching is either essential, or not. If it is essential, then it is coherently confluent by hypothesis. Otherwise, it is coherently confluent by induction hypothesis.

We proceed similarly for the branching $(\Lambda[\gamma], \Delta[\beta])$: in every case, we write

- the branching $(\Gamma[\alpha], \Lambda[\gamma])$ as $(\Gamma_1\Gamma_0[\alpha], \Gamma_1\Lambda_0[\gamma])$, with Γ_1 a one-hole context of $X_0^{*\mathrm{III}}$ and $(\Gamma_0[\alpha], \Lambda_0[\gamma])$ a confluent branching,

- the branching $(\Lambda[\gamma], \Delta[\beta])$ as $(\Delta_1\Lambda'_0[\gamma], \Delta_1\Delta_0[\beta])$, with Δ_1 a one-hole context of $X_0^{*\mathrm{III}}$ and $(\Lambda'_0[\gamma], \Delta_0[\beta])$ a confluent branching.

We then construct the following coherently confluent diagram

$$\begin{tikzcd}
 & \Gamma_1\Gamma_0[t(\alpha)] \arrow[rr, bend left, "\Gamma_1[f_0]"] \arrow[d, phantom, "\Downarrow F"] & & \Gamma_1[b_0] \arrow[dr, bend left, "h_1"] \arrow[d, phantom, "\Downarrow H"] & \\
\Gamma[s(\alpha)] \arrow[ur, bend left, "\Gamma[\alpha]"] \arrow[r, equal] \arrow[dr, bend right, "\Delta[\beta]"'] & \Lambda[\gamma] \arrow[r] \arrow[d, phantom, "\Downarrow G"] & \Lambda[t(\gamma)] \arrow[ur, bend left, "\Gamma_1[g_0]"] \arrow[dr, bend left, "\Delta_1[f_0']"] & & c \\
 & \Delta_1\Delta_0[t(\beta)] \arrow[rr, bend right, "\Delta_1[g_0']"'] & & \Delta_1[b_0'] \arrow[ur, bend right, "h_1'"'] &
\end{tikzcd}$$

where the 2-cells F and G are defined by the aforementioned coherently confluent branchings and the 2-cell H is given by induction hypothesis. □

4.2.5. Operad presented by an ideal. Let X_0 be an indexed set and I an ideal of the free shuffle operad X_0^{III}. We equip the collection $\mathbb{I} \oplus I$ with a shuffle operad structure, with unit $\eta : \mathbb{I} \hookrightarrow \mathbb{I} \oplus I$ and multiplication given by the following composition

$$\begin{aligned}
\mu_I : (\mathbb{I} \oplus I) \circ_{\mathrm{III}} (\mathbb{I} \oplus I) &\simeq (\mathbb{I} \oplus I) \oplus (I \circ_{\mathrm{III}} (\mathbb{I} \oplus I)) \\
&\to (\mathbb{I} \oplus I) \oplus (I \circ_{\mathrm{III}} X_0^{\mathrm{III}}) \\
&\xrightarrow{1 \oplus \rho} (\mathbb{I} \oplus I) \oplus I \\
&\to \mathbb{I} \oplus I,
\end{aligned}$$

where ρ is the right action of I as an X_0^{III}-bimodule. Denote by X_0^{III}/I the coequalizer of the morphisms of shuffle operads

$$\mathbb{I} \oplus I \underset{\eta \oplus 0}{\overset{\eta \oplus 1}{\rightrightarrows}} X_0^{\mathrm{III}}$$

in $\mathrm{III}\mathrm{Op}$. Note that the underlying collection of X_0^{III}/I is the cokernel of the inclusion $I \hookrightarrow X_0^{\mathrm{III}}$ in Coll.

Let X be a 1-polygraph. The *boundary* of a 1-generator α in X is the 1-cell $\partial(\alpha) := s_0(\alpha) - t_0(\alpha)$, and we set $\partial(X_1) := \{\, \partial(\alpha) \mid \alpha \in X_1 \,\}$. We denote by $I(X)$ the ideal of the free operad X_0^{III} generated by the set of boundaries of the 1-generators of X, that is the free X_0^{III}-bimodule generated by $\partial(X_1)$. Explicitly, the ideal $I(X)$ is made of all the linear combinations

$$\sum_{i=1}^{p} \lambda_i \Gamma_i[\partial(\alpha_i)]$$

where λ_i is a scalar and Γ_i is a one-hole context. Note that the operad $\overline{X}$ presented by X is isomorphic to $X_0^{\mathrm{III}}/I(X)$.

We also have the result corresponding to [25, Prop. 3.3.4]:

4.2.6. Proposition. *For a terminating left-monomial* 1*-polygraph* X*, the following assertions are equivalent :*

i) *X is confluent.*

ii) $\mathrm{Red}(X) \cap I(X) = 0$.

iii) $X_0^{\mathrm{III}} = \mathrm{Red}(X) \oplus I(X)$.

4.3. Monomial-ordered shuffle polygraphs

In this subsection, we consider 1-polygraphs whose orientations of 1-generators are compatible with a fixed monomial order. We relate these polygraphs to the notion of Gröbner bases for operads introduced in [19]. From Theorem 4.2.4 we deduce a completion procedure for these polygraphs as in [19], but by resolving only essential branchings instead of all critical branchings.

4.3.1. Gröbner bases [19]. Let X_0 be an indexed set and $\preccurlyeq$ be a monomial order on the free operad X_0^{III}. If a is a nonzero 0-cell of X_0^{III}, the *leading monomial of a* is the maximum element $\mathrm{lm}(a)$ of $\mathrm{Supp}(a)$ with respect to $\preccurlyeq$, and 0 when $\mathrm{Supp}(a)$ is empty. The *leading coefficient of a* is the coefficient $\mathrm{lc}(a)$ of $\mathrm{lm}(a)$ in a, and the *leading term of a* is the element $\mathrm{lt}(a) := \mathrm{lc}(a)\,\mathrm{lm}(a)$ of X_0^{III}. Observe that, for a, b in X_0^{III}, we have $a \prec b$ if, and only if, either $\mathrm{lm}(a) \prec \mathrm{lm}(b)$ or $(\mathrm{lt}(a) = \mathrm{lt}(b)$ and $a - \mathrm{lt}(a) \prec b - \mathrm{lt}(b))$. For Y an indexed subset of X_0^{III}, we denote by $\mathrm{lm}(Y)$ the indexed set of leading monomials of elements of Y.

Let I be an ideal of the free operad X_0^{III}. A *Gröbner basis for I with respect to* $\preccurlyeq$ is an indexed subset $\mathcal{G}$ of I such that the ideals of X_0^{III} generated by $\mathrm{lm}(I)$ and by $\mathrm{lm}(\mathcal{G})$ coincide.

4.3.2. Proposition. *If X is a convergent left-monomial 1-polygraph, and $\preccurlyeq$ is a monomial order on X_0^{III} that is compatible with X_1, then the indexed set $\partial(X_1)$ forms a Gröbner basis of $I(X)$.*

Conversely, let X_0 be an indexed set, let $\preccurlyeq$ be a monomial order on X_0^{III}, let I be an ideal of X_0^{III} and $\mathcal{G}$ be a subset of I. Define $X(\mathcal{G})$ as the 1-polygraph with 0-generators X_0 and a 1-generator

$$\alpha_a \,:\, \mathrm{lm}(a) \longrightarrow \mathrm{lm}(a) - \frac{1}{\mathrm{lc}(a)}a$$

for each a in $\mathcal{G}$. If $\mathcal{G}$ is a Gröbner basis for I, then $X(\mathcal{G})$ is a convergent left-monomial presentation of X_0^{III}/I, such that $I(X(\mathcal{G})) = I$, and $\preccurlyeq$ is compatible with $X(\mathcal{G})_1$.

Proof. Suppose that X is convergent. For every 1-generator α of X, $\partial(\alpha)$ is in $I(X)$. Since $\preccurlyeq$ is compatible with X_1, we have $\mathrm{lm}(\partial(\alpha)) = s(\alpha)$ for every 1-cell α of X. If a is a nonzero 0-cell of $I(X)$, then by Proposition 4.2.6, there exists a positive 1-cell $a \longrightarrow 0$. By compatibility between X_1 and $\preccurlyeq$, any rewriting rule that does not reduce the leading monomial $\mathrm{lm}(a)$ will reduce a into a 0-cell with the same leading monomial. Thus, in order to rewrite to 0, we must apply a rewriting rule to $\mathrm{lm}(a)$ at some point, and so $\mathrm{lm}(a)$ belongs to the ideal generated by the leading monomials of ∂X_1. Thus $\partial(X_1)$ is a Gröbner basis for $(I(X), \preccurlyeq)$.

Conversely, assume that $\mathcal{G}$ is a Gröbner basis for $(I, \preccurlyeq)$. The monomial order $\preccurlyeq$ is compatible with $X(\mathcal{G})_1$, hence by Proposition 4.1.8, the polygraph $X(\mathcal{G})$ terminates. Moreover, we have $I(X(\mathcal{G})) = I$, so the algebra presented by $X(\mathcal{G})$ is indeed isomorphic to X_0^{III}/I. Moreover, the reduced monomials of $X(\mathcal{G})^{\mathrm{III}}$ are the monomials of X_0^{III} that cannot be decomposed as $\Gamma[\mathrm{lm}(a)]$ with a in $\mathcal{G}$ and Γ a one-hole context of $X_0^{*\mathrm{III}}$. Thus, if a reduced 0-cell a of X_0^{III} is in I, its leading monomial must be 0, because $\mathcal{G}$ is a Gröbner basis of $(I, \preccurlyeq)$. By proposition 4.2.6, we get that the polygraph $X(\mathcal{G})$ is confluent. □

4.3.3. Poincaré-Birkhoff-Witt bases [28]. Let P be a operad, let X_0 be a generating indexed set of P, and let $\preccurlyeq$ be a monomial order of X_0^{III}. A *Poincaré-Birkhoff-Witt (PBW) basis for* $(P, X_0, \preccurlyeq)$ is an indexed subset $\mathcal{B}$ of $X_0^{*\text{III}}$ such that:

i) $\mathcal{B}$ is a linear basis of P, for $u \in X_0^{*\text{III}}$, we write $[u]_{\mathcal{B}} := \sum_i \lambda_i w_i$ its decomposition in P on the basis $\mathcal{B}$,

ii) for all u, v in $\mathcal{B}$ and all compatible elementary compositions $\circ_{i,\tau}$, either $u \circ_{i,\tau} v$ belongs to $\mathcal{B}$ or $u \circ_{i,\tau} v \succ [u \circ_{i,\tau} v]_{\mathcal{B}}$,

iii) a tree monomial u of $X_0^{*\text{III}}$ is in $\mathcal{B}$ if, and only if, for every decomposition $u = \Gamma(x \circ_{i,\tau} x')$ of u where $x, x' \in X_0$ and Γ is a one-hole context of X_0^{III}, $x \circ_{i,\tau} x' \in \mathcal{B}$.

4.3.4. Proposition. *If X is a convergent left-monomial quadratic presentation of an operad P, and $\preccurlyeq$ is a monomial order on X_0^{III} compatible with X_1, then the indexed set* $\mathrm{Red_m}(X)$ *is a PBW basis for* $(P, X_0, \preccurlyeq)$.

Conversely, let P be a quadratic operad, X a generating indexed set of P, $\preccurlyeq$ a monomial order on X_0^{III}, and $\mathcal{B}$ a PBW basis of $(A, X_0, \preccurlyeq)$. Define $X(\mathcal{B})$ as the 1-polygraph with 0-genrators X_0 and with a 1-generator

$$x \circ_{i,\tau} x' \xrightarrow{\alpha_{x \circ_{i,\tau} x'}} [x \circ_{i,\tau} x']_{\mathcal{B}}$$

for all x, x' in $X_0 \cap \mathcal{B}$ such that $x \circ_{i,\tau} x' \neq [x \circ_{i,\tau} x']_{\mathcal{B}}$ in X_0^{III}. Then $X(\mathcal{B})$ is a quadratic convergent left-monomial presentation of P such that $\mathrm{Red_m}(X(\mathcal{B})) = \mathcal{B}$ *and $\preccurlyeq$ is compatible with $X(\mathcal{B})_1$.*

Proof. Suppose that X is a quadratic convergent left-monomial presentation of an operad P. By proposition 4.2.6, we have the following exact sequence of collections:

$$0 \to I(X) \to X_0^{\text{III}} \to \mathrm{Red}(X) \to 0.$$

Since P is isomorphic to $X_0^{\text{III}}/I(X)$ as an operad, it is also isomorphic to $\mathrm{Red}(X)$ as a collection, and therefore $\mathrm{Red_m}(X)$ is a basis of P. The fact that $\preccurlyeq$ is compatible with X_1 implies axiom **(ii)** of PBW bases. Axiom **(iii)** comes from the definition of a reduced monomial for a quadratic left-monomial 1-polygraph.

Conversely, assume that $\mathcal{B}$ is a PBW basis for $(P, X, \preccurlyeq)$. By definition, $X(\mathcal{B})$ is quadratic and left-monomial, and axiom **(iii)** of PBW bases implies $\mathrm{Red_m}(X(\mathcal{B})) \cap I(X(\mathcal{B})) = 0$. Termination of $X(\mathcal{B})$ is given by axiom **(ii)** of PBW bases because $\preccurlyeq$ is well-founded. By proposition 4.2.6, it is sufficient to prove that $\mathrm{Red}(X(\mathcal{B})) \cap I(X(\mathcal{B})) = 0$ to get confluence: on the one hand, a reduced 0-cell u of $\mathrm{Red}(X(\mathcal{B}))$ is a linear combination of 0-cells of $\mathcal{B}$, so that u is its only normal form; and, on the other hand, if u belongs to $I(X(\mathcal{B}))$, then u admits 0 as a normal form. Finally, the operad presented by $X(\mathcal{B})$ is isomorphic to $\mathrm{Red}(X(\mathcal{B}))$, that is to $\mathbf{k}\mathcal{B}$, hence to P by the previous exact sequence and because $\mathcal{B}$ is a linear basis of P. □

4.3.5. Completion procedure. For a 2-polygraph X where $X_2 = \mathrm{Sph}(X_1^{\text{III}})$, Theorem 4.2.4 leads to a completion procedure for 1-polygraphs that reaches a convergent polygraph by resolving essential branchings. Given a terminating 1-polygraph X, and a monomial order $\prec$ on $X_0^{*\text{III}}$ compatible with X_1, the procedure works as follows

(i) for every essential branching (f, g) of X, the 0-cells $t_0(f)$ and $t_0(g)$ are reduced to some normal forms $\widehat{t_0(f)}$ and $\widehat{t_0(g)}$. If $\widehat{t_0(f)} \neq \widehat{t_0(g)}$:

$$\begin{tikzcd} & t_0(f) \arrow[r] & \widehat{t_0(f)} \arrow[d, dotted, leftrightarrow, "h"] \\ u \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & & \\ & t_0(g) \arrow[r] & \widehat{t_0(g)} \end{tikzcd}$$

a 1-generator $h : \mathrm{lm}(a) \to a - \mathrm{lm}(a)$, where $a = \widehat{t_0(f)} - \widehat{t_0(g)}$, is added to reach confluence of the branching ;

(ii) the addition of 1-generators in the step **(i)** can create new essential branchings, whose confluence must also be completed as in **(i)** ;

(iii) Repeat the previous steps until there are no non-confluent essential branchings.

As a consequence of Theorem 4.2.4, we have

4.3.6. Proposition. *The procedure § 4.3.5 on a 1-polygraph X produces a (possibly infinite) convergent polygraph that presents the operad $\overline{X}$.*

An analogue completion procedure for non-symmetric operads has been described in detail with an explicit handling of the critical branchings in [37, Algorithm 2].

5. Shuffle polygraphic resolutions from convergence

In this section, unless otherwise specified, all operads and polygraphs are shuffle. We recall from [25] the characterization of the property of acyclicity for an ω-polygraph through the existence of a homotopical contraction. Subsection 5.2, presents the main result of this article, Theorem 5.2.6, that extends a reduced convergent left-monomial 1-polygraph into a polygraphic resolution of the presented operad. In Subsection 5.3, given a polygraphic resolution of an operad, we construct a bimodule resolution for the operad. Finally, in Subsection 5.4 we prove a criterion of Koszulness in terms of quadratic convergence.

5.1. Polygraphic resolutions and contractions

In this first subsection, we extend to ω-operads the notion of homotopy developed in [25] for ω-algebras, see also [4] and [24]. Then we introduce the notion of a *contraction* of a polygraph, which allow us to characterize acyclic ω-polygraphs.

5.1.1. Homotopies. Let P and Q be ω-operads and $F, G : P \to Q$ be morphisms of ω-operads. A *homotopy* from F to G is a graded linear map

$$\eta : P \to Q$$

of degree 1, *i.e.*, η sends n-cells to $n+1$-cells), such that, writing η_a for $\eta(a)$,

i) for every $n \geqslant 0$, for every n-cell a of P,

$$s_n(\eta_a) = F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(a)} \tag{5.1.2}$$
$$t_n(\eta_a) = \eta_{s_{n-1}(a)} \star_{n-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(a), \tag{5.1.3}$$

ii) for all $0 \leqslant k < n$ and every $\star_k$-composable pair (a, b) of n-cells of P,

$$\begin{aligned}\eta_{a \star_k b} = {} & F(s_{k+1}(a)) \star_0 \eta_{t_0(b)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(b)} \star_k \eta_b \\ & \star_{k+1} \eta_a \star_k \eta_{s_{k-1}(a)} \star_{k-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(t_{k-1}(b)),\end{aligned}$$

iii) for all $n \geqslant 0$ and every n-cell a of P,

$$\eta_{1_a} = 1_{\eta_a}.$$

In order for this definition to be licit, we need to check that the $\star_k$-compositions of **(i)** are well defined. See [4, Appendix B.8] or [25, § 5.1.1] for the verification. Note that the mappings $a \mapsto s(\eta_a)$ and $a \mapsto t(\eta_a)$ are operads morphisms because both are composites of operad morphisms. The globularity of η_a follows from

$$\begin{aligned} & ss(\eta_a) = s(F(a)) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-2} \eta_{t_{n-2}(a)} = s(\eta_{s(a)}) = st(\eta_a) \\ \text{and} \quad & ts(\eta_a) = t(\eta_{t(a)}) = \eta_{s_{n-2}(a)} \star_{n-2} \cdots \star_1 \eta_{s_0(a)} \star_0 t(G(a)) = tt(\eta_a). \end{aligned}$$

5.1.4. Let us expand the homotopy η in low dimension. It maps a 1-cell $f : a \to a'$ of P to a 2-cell

$$\begin{tikzcd} & F(a') \arrow[dr, bend left, "\eta_{a'}"] \arrow[d, phantom, "\Downarrow \eta_f"] & \\ F(a) \arrow[ur, bend left, "F(f)"] \arrow[r, bend right, "\eta_a"'] & G(a) \arrow[r, bend right, "G(f)"'] & G(a') \end{tikzcd}$$

of Q, and a 2-cell $A : f \Rightarrow f' : a \to a'$ of P to the following 3-cell of Q

$$\begin{tikzcd} & F(a') \arrow[dr, bend left, "\eta_{a'}"] \arrow[d, phantom, "\Downarrow \eta_{f'}"] & \\ F(a) \arrow[ur, bend left, "F(f)"] \arrow[ur, bend right, "F(f')"'] \arrow[r, bend right, "\eta_a"'] \arrow[ur, phantom, bend left=60, "\Downarrow F(A)"] & G(a) \arrow[r, bend right, "G(f')"'] & G(a') \end{tikzcd} \overset{\eta_A}{\Rrightarrow} \begin{tikzcd} & F(a') \arrow[dr, bend left, "\eta_{a'}"] \arrow[d, phantom, "\Downarrow \eta_f"] & \\ F(a) \arrow[ur, bend left, "F(f)"] \arrow[r, bend right, "\eta_a"'] & G(a) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(f')"'] \arrow[r, phantom, "\Downarrow G(A)"] & G(a') \end{tikzcd}$$

5.1.5. Unital sections and contractions. Let X be a ω-polygraph. A *unital section* of X is a morphism of ω-operads $\iota : \overline{X} \to X^{\mathrm{III}}$, which is a section of the canonical projection $\pi : X^{\mathrm{III}} \twoheadrightarrow \overline{X}$, and such that $\iota_1 = 1$, where $1 \in \mathbf{k} \subseteq X^{\mathrm{III}}(1)$. The morphism ι assigns to every 0-cell a of $\overline{X}$ a representative 0-cell ι_a in X^{III}, in such a way that is the identity on the unit $\mathbf{k}$. Note that a unital section is not necessarily compatible with shuffle composition. For a an n-cell of X^{III}, we will write $\widehat{a}$ for $\iota\pi(a)$. Note that $\widehat{a} = 1_{\widehat{s_0(a)}}$ for $n \geqslant 1$.

Fix ι a unital section of X. An *ι-contraction* of X is a homotopy $\sigma : id_{X^{\mathrm{III}}} \to \iota\pi$ such that $\sigma_a = 1_a$ for every n-cell a of X^{III} that belongs to the image of ι or σ. We say that σ is a *right* ι-contraction if, for all $n \geqslant 0$, n-cells f, g of X^{III}, and compatible elementary composition $\circ_{i,\tau}$,

$$\sigma_{f \circ_{i,\tau} g} = (s_0(f) \circ_{i,\tau} \sigma_g) \star_0 \sigma_{f \circ_{i,\tau} \widehat{g}}. \tag{5.1.6}$$

5.1.7. Lemma. *Let σ be a ι-contraction. For $n \geqslant 1$ and every n-cell a of X^{III},*

$$s_n(\sigma_a) = a - t_{n-1}(a) + \sigma_{t_{n-1}(a)} \quad \textit{and} \quad t_n(\sigma_a) = \sigma_{s_{n-1}(a)}. \tag{5.1.8}$$

Note that for a a 0-cell of X^{III}, $s_0(\sigma_a) = a$ and $t_0(\sigma_a) = \widehat{a}$.

Proof. Let us first prove

$$a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_k \sigma_{t_k(a)} = a - t_k(a) + \sigma_{t_k(a)}$$

by induction on $k \in \{0, \dots, n-1\}$. The result is clear for $k = 0$. For $k \geqslant 1$, we calculate

$$\begin{aligned} a \star_0 \eta_{t_0(a)} \star_1 \cdots \star_k \sigma_{t_k(a)} &= a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_{k-1} \sigma_{t_{k-1}(a)} \\ &\quad - t_k(a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_{k-1} \sigma_{t_{k-1}(a)}) \\ &\quad + \sigma_{t_k(a)} \\ &= (a - t_{k-1}(a) + \sigma_{t_{k-1}(a)}) - t_k(a - t_{k-1}(a) + \sigma_{t_{k-1}(a)}) + \sigma_{t_k(a)} \\ &= a - t_k(a) + \sigma_{t_k(a)}, \end{aligned}$$

the last equality coming from the fact that $t_k t_{k-1}(a) = t_{k-1}(a)$ and $t_k(\sigma_{t_{k-1}(a)}) = \sigma_{t_{k-1}(a)}$. Applying $k = n-1$ and (5.1.2) with $F = id_{X^{\mathrm{III}}}$, we conclude that

$$s_n(\sigma_a) = a - t_{n-1}(a) + \sigma_{t_{n-1}(a)}.$$

For the second equation, we proceed similarly to show that, for all $k \in \{0, \dots, n-1\}$,

$$\sigma_{s_k(a)} \star_k \cdots \star_1 \sigma_{s_0(a)} \star_0 \widehat{a} = \widehat{a} - s_k(\widehat{a}) + \sigma_{s_k(a)} = \sigma_{s_k(a)}$$

because $\widehat{a} = 1_{\widehat{s_0(a)}}$. Applying $k = n-1$ and (5.1.3) with $G = \iota\pi$, we conclude that

$$t_n(\sigma_a) = \sigma_{s_0(a)}.$$

□

5.1.9. Reduced and essential monomials. Let ι be an unital section of X, and σ an ι-contraction of an ω-polygraph X. A 0-monomial u of X^{III} is *ι-reduced* if $\widehat{u} = u$. A non-ι-reduced 0-monomial u of the free ω-operad X^{III} is *ι-essential* if $u = (x \mid \vec{v})$ where x is a 0-generator of X and $v_1, \dots, v_k$ are ι-reduced 0-monomials of X^{III}. When the underlying 1-polygraph $X_{\leqslant 1}$ is convergent, and the section ι sends a 0-monomial on its unique normal form with respect to X_1, the ι-reduced 0-monomials coincide with reduced ones.

For $n \geqslant 0$, an n-monomial a of X^{III} is *σ-reduced* if it is an identity or in the image of σ. If σ is a right ι-contraction of X and $n \geqslant 0$, then a non-σ-reduced n-monomial a of X^{III} is *σ-essential* if $a = (\alpha \mid \vec{v})$, where α is a n-generator of X and $v_1, \dots, v_k$ are ι-reduced 0-monomials of the ω-operad X^{III}.

5.1.10. Lemma. *Let X be an ω-polygraph and ι a unital section of X. A right ι-contraction σ of X is uniquely and entirely determined by its values on the ι-essential 0-monomials and, for $n \geqslant 1$, on the σ-essential n-monomials of X^{III}.*

Proof. The proof follows the same arguments as in the case of associative algebras given in [25, Section 5.2], and it is divided in two steps:

i) First, we prove that a homotopy $\eta : F \to G$ between morphisms of ω-operads $F, G : X^{\mathrm{III}} \to X^{\mathrm{III}}$ is uniquely and entirely determined by its values on n-monomials for all $n \geqslant 0$, provided it satisfies the following relation:

$$\eta_{\mu^{\uparrow}_{X^{\mathrm{III}}_n}} = \eta_{\mu^{\downarrow}_{X^{\mathrm{III}}_n}}, \tag{5.1.11}$$

where $\mu^{\uparrow}_{X^{\mathrm{III}}_n}$ and $\mu^{\downarrow}_{X^{\mathrm{III}}_n}$ are defined considering X^{III}_n as a X^{III}_0-bimodule.

ii) Next, we prove that the values of a right ι-contraction on n-monomials are uniquely and entirely determined by the values on ι-essential and σ-essential monomials, and that the resulting values satisfy (5.1.11).

(i) Proceed by induction on $n \geqslant 0$. For $n = 0$, assume that $\eta_u : F(u) \to G(u)$ is a fixed 1-cell of X^{III} for every 0-monomial u of X^{III}. Extend η uniquely to every 0-cell a of X^{III} by linearity.

Now fix $n \geqslant 1$ and assume that an $(n+1)$-cell η_u of X^{III} has been chosen for every n-monomial u of X^{III}, with source and target given by the definition of homotopies, such that (5.1.11) holds for n-monomials. By construction, the n-cells of X^{III} are linear combinations of n-monomials of X^{III} and of identities of $(n-1)$-cells of X^{III} up to the relation

$$\mu^{\uparrow}_{X^{\mathrm{III}}_0\langle X_n\rangle \oplus X^{\mathrm{III}}_{n-1}} = \mu^{\downarrow}_{X^{\mathrm{III}}_0\langle X_n\rangle \oplus X^{\mathrm{III}}_{n-1}}.$$

Thus we can extend η to all n-cells a of X^{III} by choosing a decomposition of a into a linear combination of n-monomials and an identity, and using (5.1.11) to ensure that the resulting cell does not depend on the choice of decomposition. We check that the source and target of the resulting $(n+1)$-cell η_a match the definition of homotopies by linearity of F, G and the $\star_k$-compositions.

(ii) First, we construct σ as a graded linear map by induction on n. For $n = 0$, if u is a non-ι-essential monomial, then either $u = \widehat{u}$, or $u = (x \mid \vec{v})$ where x is a 0-cell of X and some v_i is a non-ι-reduced monomial. In the former case, $\sigma_u = 1_u$ is forced because u is ι-reduced. In the latter case, take i maximal. Writing $(x \mid \vec{v}) = (x \mid v_1 \cdots 1 \cdots v_k) \circ_{i,\tau} v_i$ for some shuffle permutation τ, (5.1.6) imposes

$$\sigma_{(x|\vec{v})} = ((x \mid v_1 \cdots 1 \cdots v_k) \circ_{i,\tau} \sigma_{v_i}) \star_0 \sigma_{(x|v_1 \cdots \widehat{v_i} \cdots v_k)}.$$

Then proceed by induction on the weight of the v_i to define σ_{v_i} from the values of σ on ι-reduced monomials.

Now let $n \geqslant 1$. For every n-monomial $\Gamma[\alpha]$, with α a n-generator of X and Γ a one-hole context of $X^{*\mathrm{III}}_0$, writing

$$\Gamma[\alpha] = u \circ_{i,\tau} (\alpha \mid \vec{v})$$

and

$$(\alpha \mid \vec{v}) = \alpha \circ_{k,\tau_k} v_k \circ_{k-1,\tau_{k-1}} \cdots \circ_{1,\tau_1} v_1,$$

the equation (5.1.6) imposes that we set

$$\begin{aligned}\sigma_{\Gamma[\alpha]} := {} & (u \circ_{i,\tau} (s_0(\alpha) \mid \sigma_{v_1} v_2 \cdots v_k)) \star_0 \cdots \star_0 (u \circ_{i,\tau} (s_0(\alpha) \mid \widehat{v_1} \cdots \widehat{v_{k-1}} \sigma_{v_k})) \\ & \star_0 (u \circ_{i,\tau} \sigma_{(\alpha|\vec{v})}) \star_0 \sigma_{u \circ_{i,\tau} \widehat{(\alpha|\vec{v})}},\end{aligned}$$

where $(\alpha \mid \vec{\widehat{v}})$ is a shortcut for $(\alpha \mid \widehat{v_1} \cdots \widehat{v_k})$. Let us check that this definition is well-founded. The σ_{v_i} are defined by induction on the weight of the v_i, and $\sigma_{u \circ_{i,\tau} \widehat{(\alpha \mid \vec{v})}} = \sigma_{u \circ_{i,\tau} \widehat{(s_0(\alpha) \mid \vec{v})}}$ is defined by induction on n. It remains to check that $\sigma_{(\alpha \mid \vec{\widehat{v}})}$ is defined. If $(\alpha \mid \vec{\widehat{v}})$ is σ-essential, then it is defined by hypothesis. Otherwise, $(\alpha \mid \vec{\widehat{v}})$ is σ-reduced, in which case $(\alpha \mid \vec{\widehat{v}}) = \sigma_b$ for some $(n-1)$-cell b of X^{III}, which imposes $\sigma_{(\alpha \mid \vec{\widehat{v}})} := 1_{\sigma_b}$.

Now it remains only to show (5.1.11) and then apply the first point. More explicitly, we need to show

$$\sigma_{u \circ_{i,\tau} s_0(v)} + \sigma_{t_0(u) \circ_{i,\tau} v} - \sigma_{t_0(u) \circ_{i,\tau} s_0(v)} = \sigma_{u \circ_{i,\tau} t_0(v)} + \sigma_{s_0(u) \circ_{i,\tau} v} - \sigma_{s_0(u) \circ_{i,\tau} t_0(v)}$$

for all u, v two n-monomials of X^{III} and compatible elementary composition $\circ_{i,\tau}$. Write $a = s_0(u)$, $a' = t_0(u)$, $b = s_0(v)$, $b' = t_0(v)$, and $\cdot = \circ_{i,\tau}$. On the one hand,

$$\begin{aligned}\sigma_{u \cdot b} + \sigma_{a' \cdot v} - \sigma_{a' \cdot b} &= (a \cdot \sigma_b) \star_0 \sigma_{u \cdot \widehat{b}} + (a' \cdot \sigma_v) \star_0 \sigma_{a' \cdot \widehat{v}} - (a' \cdot \sigma_b) \star_0 \sigma_{a' \cdot \widehat{b}} \\ &= a \cdot \sigma_b + \sigma_{u \cdot \widehat{b}} - a \cdot \widehat{b} + a' \cdot \sigma_v - a' \cdot \sigma_b,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sigma_{u \cdot b'} + \sigma_{a \cdot v} - \sigma a \cdot b' &= (a \cdot \sigma_{b'}) \star_0 \sigma_{u \cdot \widehat{b'}} + (a \cdot \sigma_v) \star_0 \sigma_{a \cdot \widehat{v}} - (a \cdot \sigma_{b'}) \star_0 \sigma_{a \cdot \widehat{b'}} \\ &= \sigma_{u \cdot \widehat{b'}} + a \cdot \sigma_v - a \cdot \widehat{v} \\ &= \sigma_{u \cdot \widehat{b}} + a \cdot \sigma_v - a \cdot \widehat{b}\end{aligned}$$

Therefore it remains to prove

$$a \cdot \sigma_b + a' \cdot \sigma_v = a \cdot \sigma_v + a' \cdot \sigma_b. \tag{5.1.12}$$

Since $\star_0$-composition in X^{III} is a morphism of ω-operads, we have

$$u \cdot \sigma_b = u \cdot b \star_0 a' \cdot \sigma_b = a \cdot \sigma_b \star_0 u \cdot \widehat{b}.$$

Using the linear expression of $\star_0$-composition, we get

$$u \cdot b + a' \cdot \sigma_b - a' \cdot b = a \cdot \sigma_b + u \cdot \widehat{b} - a \cdot \widehat{b}$$

Similarly, considering $u \cdot \sigma_v$, we get

$$u \cdot b + a' \cdot \sigma_v - a' \cdot b = a \cdot \sigma_v + u \cdot \widehat{b} - a \cdot \widehat{b}.$$

Taking the difference of the two previous equations gives us (5.1.12). □

5.1.13. Proposition. *Let X be an ω-polygraph with a fixed unital section ι. Then X is a polygraphic resolution of the ω-operad $\overline{X}$ if, and only if, X admits a right ι-contraction.*

Proof. Suppose that X is a polygraphic resolution of the operad $\overline{X}$, and define a right ι-contraction σ of X. Using Lemma 5.1.10, we shall define σ on ι- and σ-essential n-monomials of X^{III} by induction on $n \geqslant 0$. If $(x \mid \vec{v})$ is an ι-essential 0-monomial, then $\pi_X(x \mid \vec{v}) = \pi_X(\widehat{(x \mid \vec{v})})$ in $\overline{X}$, hence there exists a 1-cell $\sigma_{(x \mid \vec{v})} : (x \mid \vec{v}) \to \widehat{(x \mid \vec{v})}$ in X^{III}. Now assume that σ is defined on the n-cells of X^{III} for $n \geqslant 0$ and let $(\alpha \mid \vec{v})$ be a σ-essential $(n+1)$-monomial of X^{III}. The n-cells defining $s(\sigma_{(\alpha \mid \vec{v})})$ and $t(\sigma_{(\alpha \mid \vec{v})})$ as

in (5.1.8) are parallel, so, by acyclicity of X, there exists an $(n+2)$-cell $\sigma_{(\alpha|\vec{v})}$ with this source and target in X^{III}.

Conversely, let σ be a right ι-contraction of the polygraph X, and let a, b be parallel n-cells of X^{III} for $n \geqslant 1$. We have $t(\sigma_a) = \sigma_{s(a)} = \sigma_{s(b)} = t(\sigma_b)$ by (5.1.8), so the $(n+1)$-cell $\sigma_a \star_n \sigma_b^-$ is well defined, with source $s(\sigma_a)$ and target $s(\sigma_b)$. Since $t_k(a) = t_k(b)$ for $k \in \{0, \dots, n-1\}$, we find that

$$(\sigma_a \star_n \sigma_b^-) \star_{n-1} \sigma^-_{t_{n-1}(a)} \star_{n-2} \cdots \star_1 \sigma^-_{t_0(a)}$$

is a well defined $(n+1)$-cell of X^{III} of source a and target b, thus proving that X_{n+1} is an acyclic extension of X_n^{III}. Thus X is a polygraphic resolution of $\overline{X}$. □

5.2. Polygraphic resolution from a convergent presentation

This subsection contains the main result of this article. We show how to extend a reduced left-monomial convergent shuffle 1-polygraph into a shuffle polygraphic resolution of its presented operad. The n-generators of the resolution correspond to certain overlappings of the 1-generators of the polygraph.

5.2.1. Higher-dimensional overlappings. Let X be a left-monomial 1-polygraph, and consider the path-lexicographic order $\prec_{pl}$ on 1-monomials of X defined in § 4.1.7. We define the family of indexed sets $\mathcal{O}v(X) = (\mathcal{O}v(X)_n)_{n\geqslant 0}$ by induction on $n \geqslant 0$. The elements of $\mathcal{O}v(X)_n$ are called *n-overlappings of X*, and for an n-overlapping u_n we will also define its *source* $s_0(u_n)$ and its set of *branches* $B(u_n)$.

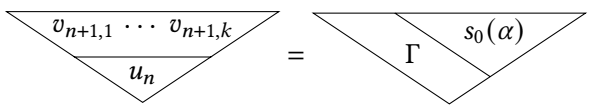
For $n = 0$, define a 0-overlapping u_0 as a 0-generator in X_0. Define its source as $s_0(u_0) := u_0$ and its set of *branches* as $B(u_0) := \varnothing$.

Now suppose that n-overlappings are defined for $n \geqslant 0$. Let u_n be an n-overlapping and $B(u_n) = \{\Gamma_1[\alpha_1] \prec_{pl} \cdots \prec_{pl} \Gamma_n[\alpha_n]\}$ its set of branches, where each Γ_k is a one-hole context and α_k is a 1-generator in X_1. Given 0-monomials $\vec{v}_{n+1}$, we define

$$E(u_n, \vec{v}_{n+1}) := \left\{ \Gamma[\alpha] \;\middle|\; \begin{array}{l} \Gamma \text{ one-hole context}, \alpha \in X_1, \\ \Gamma[s_0(\alpha)] = (s_0(u_n) \mid \vec{v}_{n+1}), \\ \Gamma[\alpha] \succ_{pl} (\Gamma_n[\alpha_n] \mid \vec{v}_{n+1}) \end{array} \right\}.$$

An $(n+1)$*-overlapping* is a tuple $(u_n, \vec{v}_{n+1})$, denoted by $u_n \Lleftarrow \vec{v}_{n+1}$, where u_n is an n-overlapping, and $\vec{v}_{n+1}$ is a list of reduced 0-monomials such that, for any list of rooted submonomials $\vec{w}_{n+1} \subsetneq \vec{v}_{n+1}$, $\#E(u_n, \vec{w}_{n+1}) < \#E(u_n, \vec{v}_{n+1})$. We then define its source as $s_0(u_n \Lleftarrow \vec{v}_{n+1}) := (s_0(u_n) \mid \vec{v}_{n+1})$ and its set of branches as $B(u_n \Lleftarrow \vec{v}_{n+1}) := \{(\Gamma_k[\alpha_k] \mid \vec{v}_{n+1}) \mid 1 \leqslant k \leqslant n\} \cup \{\max E(u_n, \vec{v}_{n+1})\}$.

5.2.2. Crowns. An $(n+1)$-overlapping $u_n \Lleftarrow \vec{v}_{n+1}$ can be represented graphically as



where $\Gamma[\alpha]$ is the maximal element of $E(u_n, \vec{v}_{n+1})$. We call the list $\vec{v}_{n+1}$ of reduced 0-monomial a *crown on u_n*. Given an n-overlapping u_n and a list of 0-monomials $\vec{v}_{n+1}$, we define

$$C(u_n, \vec{v}_{n+1}) := \{\vec{w}_{n+1} \subseteq \vec{v}_{n+1} \mid u_n \Lleftarrow \vec{w}_{n+1} \in \mathcal{O}v(X)_{n+1}\},$$

that is, the set of crowns $\vec{w}_{n+1}$ on u_n included in $\vec{v}_{n+1}$. This set is equipped with the total order defined by $\vec{w}_{n+1} \prec \vec{w}'_{n+1}$ if $\max_{\prec_{pl}} B(u_n \Lleftarrow \vec{w}_{n+1}) \prec_{pl} \max_{\prec_{pl}} B(u_n \Lleftarrow \vec{w}'_{n+1})$. Note that $C(u_n, \vec{v}_{n+1})$ is empty if, and only if, $E(u_n, \vec{v}_{n+1})$ is empty.

5.2.3. Description in low dimensions. Let us look at the definitions of n-overlappings in low dimensions. A 1-overlapping $u_0 \Lleftarrow \vec{v}_1$ is associated to a single branch $\Gamma[\alpha]$. Since the 0-monomials $\vec{v}_1$ are reduced and minimal, the context C must be trivial. Thus $\mathcal{O}v(X)_1$ is in bijection with X_1, and this bijection is given by taking the unique branch of the 1-overlapping. Next, a 2-overlapping $u_0 \Lleftarrow \vec{v}_1 \Lleftarrow \vec{v}_2$ corresponds to a pair of branches $(\Gamma_1[\alpha_1], \Gamma_2[\alpha_2])$ which form an critical branching in context. Since the crown $\vec{v}_2$ must be minimal, this context is trivial. Thus $\mathcal{O}v(X)_2$ is in bijection with the set of critical branchings.

5.2.4. Overlappings as paths of crowns. Given a left-monomial 1-polygraph X, the n-overlappings can be defined inductively as certain paths of length n in the directed graph $\mathcal{G}(X)$ defined as follows. Its vertices are the 0-monomials of $X_0^{*\text{Ш}}$, and its edges are

$$u \xrightarrow{\Lleftarrow \vec{v}} (u|\vec{v}) \ ,$$

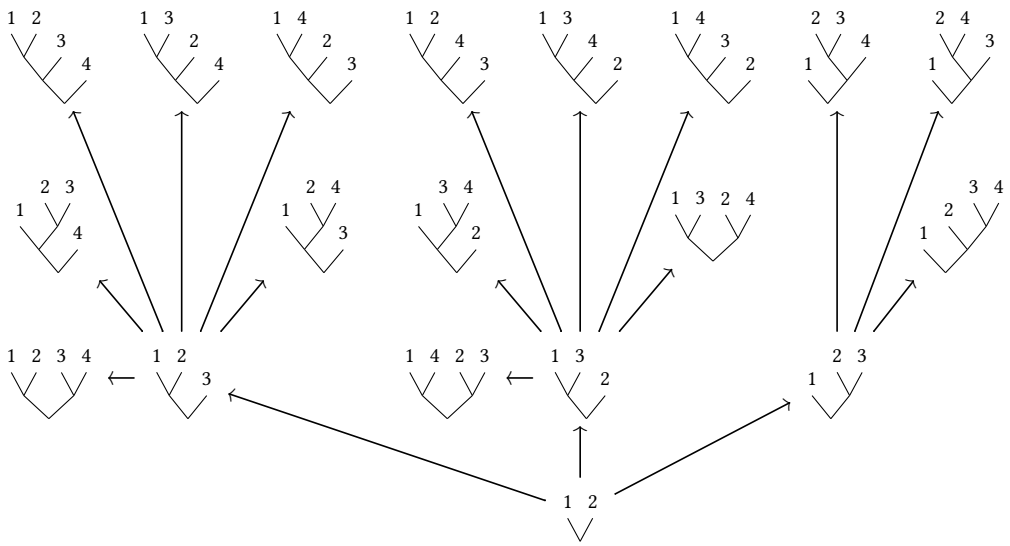
such that u is the source of an overlapping and $\vec{v}$ is a crown. Then the indexed set of n-overlappings of X corresponds to a subset of paths of $\mathcal{G}(X)$, starting in X_0 and of length n, where each step of the path corresponds to the addition of a crown. That is $u_0 \Lleftarrow \vec{v}_1 \Lleftarrow \cdots \Lleftarrow \vec{v}_n$ corresponds to a path

$$u_0 \longrightarrow (u_0 \mid \vec{v}_1) \longrightarrow \cdots \longrightarrow (u_0 \mid \vec{v}_1 \mid \cdots \mid \vec{v}_n).$$

5.2.5. Examples. (i) Consider the following binary quadratic 1-polygraph

$$X := \left\langle x \in X_0(2) \;\middle|\; \begin{array}{ccc} 1 & & 2 \\ & x & \\ & & 3 \\ & x & \end{array} \longrightarrow 0, \quad \begin{array}{ccc} 1 & & 3 \\ & x & \\ & & 2 \\ & x & \end{array} \longrightarrow 0, \quad \begin{array}{ccc} & 2 & 3 \\ 1 & & x \\ & x & \end{array} \longrightarrow 0 \right\rangle.$$

It has 15 critical branchings, which correspond to all possible critical branchings in the quadratic binary case. Let us draw the part of the directed graph $\mathcal{G}(X)$ corresponding to $\mathcal{O}v(X)_2$:

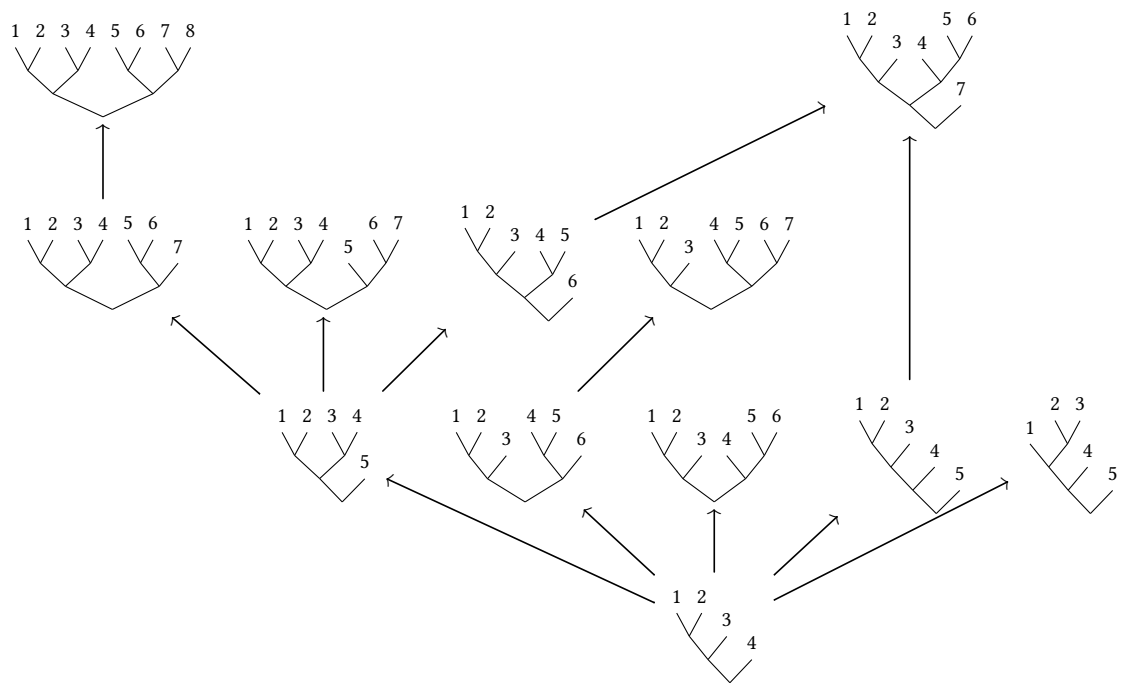


Every internal vertex of every tree monomial is x, so we omit them.

(ii) Next, consider the following binary cubic 1-polygraph

$$X := \left\langle x \in X_0(2) \;\middle|\; \begin{array}{ccccc} 1 & & 2 & & \\ & x & & 3 & \\ & & x & & 4 \\ & & & x & \end{array} \xrightarrow{\alpha} 0,\; \begin{array}{ccccc} & 2 & & 3 & \\ 1 & & x & & \\ & x & & 4 & \\ & & x & & \end{array} \xrightarrow{\beta} 0,\; \begin{array}{ccccc} 2 & & 3 & & \\ & x & & 4 & \\ 1 & & x & & \\ & x & & & \end{array} \xrightarrow{\gamma} 0,\; \begin{array}{ccccc} & 3 & & 4 & \\ 2 & & x & & \\ 1 & & x & & \\ & x & & & \end{array} \xrightarrow{\delta} 0 \right\rangle,$$

and consider the path-lexicographic order $\prec_{pl}$ on 1-monomials where $\alpha \prec \beta \prec \gamma \prec \delta$ (see § 4.1.7). Let us draw a part of the directed graph $\mathcal{G}(X)$ around $s_0(\alpha)$:



This drawing is not exhaustive, but presents some interesting phenomena. For example, due to our choice of order $\prec_{pl}$, the top left 4-overlapping can only be obtained in one way, by adding α, β, γ, and δ in order. In addition, the top right monomial can be reached from $s_0(\alpha)$ by a path of length 2 or 3, and so corresponds to a 3-overlapping and a 4-overlapping, depending on if β is present or not.

5.2.6. Theorem (Overlapping polygraphic resolution). *Let X be a reduced, convergent, left-monomial 1-polygraph and ι the unital section sending every monomial to its reduced form. Then there exist a unique ω-polygraph structure on $\mathcal{O}v(X)$ and a unique right ι-contraction σ of $\mathcal{O}v(X)$ such that, for all n-overlappings u_n of $\mathcal{O}v(X)$ and reduced 0-monomials $\vec{v}_{n+1}$ of $X_0^{*\text{Ш}}$,*

$$\sigma(u_n \mid \vec{v}_{n+1}) = \begin{cases} u_n \Lleftarrow \vec{v}_{n+1} & \text{if } u_n \Lleftarrow \vec{v}_{n+1} \in \mathcal{O}v(X)_{n+1}, \\ \text{an identity} & \text{if } C(u_n, \vec{v}_{n+1}) = \varnothing, \\ \sigma(u_n \mid \vec{v}_{n+1}) & \text{otherwise (tautological condition).} \end{cases} \tag{5.2.7}$$

As a consequence, $\mathcal{O}v(X)$ is a polygraphic resolution of the operad $\overline{X}$.

Proof. By induction on $n \geqslant 0$, we simultaneously construct the source and target maps of the ω-polygraph structure on the $(n+1)$-generators of $Ov(X)$ and the right ι-contraction $\sigma : Ov(X)_n^{\text{III}} \to Ov(X)_{n+1}^{\text{III}}$. By Lemma 5.1.10, it suffices to define σ on the ι- and σ-essential n-monomials of the ω-operad $Ov(X)^{\text{III}}$.

Let $n = 0$. The ι-essential 0-monomials of $Ov(X)^{\text{III}}$ are the $(u_0 \mid \vec{v}_1)$ where u_0 is a 0-generator of X and the $v_{1,i}$ are reduced 0-monomials of X_0^{III} such that $(u_0 \mid \vec{v}_1)$ is not reduced. By (5.1.8), it suffices to define $\sigma(u_0 \mid \vec{v}_1)$ such that $s_0\sigma(u_0 \mid \vec{v}_1) = (u_0 \mid \vec{v}_1)$ and $t_0\sigma(u_0 \mid \vec{v}_1) = \widehat{(u_0 \mid \vec{v}_1)}$. If $u_0 \Lleftarrow v_1$ is a 1-overlapping, then we set

$$s_0(u_0 \Lleftarrow \vec{v}_1) := (u_0 \mid \vec{v}_1), \qquad t_0(u_0 \Lleftarrow \vec{v}_1) := \widehat{(u_0 \mid \vec{v}_1)},$$

and the first case of (5.2.7) imposes $\sigma(u_0 \mid \vec{v}_1) := u_0 \Lleftarrow \vec{v}_1$. Otherwise, since we have supposed $(u_0 \mid \vec{v}_1)$ reducible, $C(u_0, \vec{v}_1)$ is nonempty. Let $\vec{w}_1 = \max C(u_0, \vec{v}_1)$ and write $(u_0 \mid \vec{w}_1 \mid \vec{w}_2) = (u_0 \mid \vec{v}_1)$. Then $u_0 \Lleftarrow \vec{v}_1 \in Ov(X)_1$ and $E(u_0 \Lleftarrow \vec{w}_1, \vec{w}_2) = \varnothing$, so by the second case of (5.2.7), the (not yet defined) 2-cell $\sigma(u_0 \Lleftarrow \vec{w}_1 \mid \vec{w}_2)$ is an identity. By (5.1.8), we know that the target of $\sigma(u_0 \Lleftarrow \vec{w}_1 \mid \vec{w}_2)$ is $\sigma(s_0(u_0 \Lleftarrow \vec{w}_1) \mid \vec{w}_2) = \sigma(u_0 \mid \vec{v}_1)$. Thus we set

$$\sigma(u_0 \mid \vec{v}_1) := t_0\sigma(u_0 \Lleftarrow \vec{w}_1 \mid \vec{w}_2) = s_0\sigma(u_0 \Lleftarrow \vec{w}_1 \mid \vec{w}_2) = (u_0 \Lleftarrow \vec{w}_1 \mid \vec{w}_2) \star_0 \sigma(\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2).$$

Since X is terminating, we define $\sigma(\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2) : (\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2) \to (u_0 \mid \vec{w}_1 \mid \vec{w}_2)$ by well-founded induction on $\prec_{X_1}$, so this definition is licit.

Now let $n \geqslant 1$. The essential n-cells of $Ov(X)^{\text{III}}$ are the $(u_n \mid \vec{v}_{n+1})$ where u_n is an n-overlapping and the $v_{n+1,i}$ are reduced 0-monomials of X_0^{III} such that $(u_n \mid \vec{v}_{n+1})$ is not σ-reduced. Denote the branches of u_n by $(\Gamma_1[\alpha_1], \ldots, \Gamma_n[\alpha_n])$. We distinguish the three cases of (5.2.7). The induction step for the ω-polygraph structure on $Ov(X)$ is entirely contained within the first case.

First case. First, suppose that $u_n \Lleftarrow \vec{v}_{n+1}$ is an $(n+1)$-overlapping. Since condition (5.2.7) imposes $u_n \Lleftarrow \vec{v}_{n+1} = \sigma(u_n \mid \vec{v}_{n+1})$, and (5.1.8) gives us the source and target of the (not yet defined) $(n+1)$-cell $\sigma(u_n \mid \vec{v}_{n+1})$, we set

$$\begin{aligned} s_n(u_n \Lleftarrow \vec{v}_{n+1}) &:= (u_n \mid \vec{v}_{n+1}) - (t_{n-1}(u_n) \mid \vec{v}_{n+1}) + \sigma(t_{n-1}(u_n) \mid \vec{v}_{n+1}), \\ t_n(u_n \Lleftarrow \vec{v}_{n+1}) &:= \sigma(s_{n-1}(u_n) \mid \vec{v}_{n+1}), \end{aligned}$$

which are indeed globular, and define $\sigma(u_n \mid \vec{v}_{n+1}) := u_n \Lleftarrow \vec{v}_{n+1}$. This gives us the polygraphic structure on the $(n+1)$-overlappings.

Second case. Next, suppose that $C(u_n, \vec{v}_{n+1}) = \varnothing$. Writing $u_n = u_{n-1} \Lleftarrow \vec{v}_n$, we make the following observations:

- the pair $(u_{n-1}, (\vec{v}_n \mid \vec{v}_{n+1}))$ is not an n-overlapping.
- $C(u_{n-1}, (\vec{v}_n \mid \vec{v}_{n+1}))$ is nonempty, since it includes $\vec{v}_n$.
- The 0-monomials of $(\vec{v}_n \mid \vec{v}_{n+1})$ are reduced. Indeed, if not, then there would exist $\Gamma_{n+1}[\alpha_{n+1}] \succ_{pl} (\Gamma_n[\alpha_n] \mid \vec{v}_{n+1})$ in $E(u_n, \vec{v}_{n+1})$, which contradicts the fact that $C(u_n, \vec{v}_{n+1})$ is empty.

In particular, the third observation says that $(u_{n-1} \mid \vec{v}_n \mid \vec{v}_{n+1})$ is an essential $(n-1)$-monomial. Thus we are in the third case of the induction hypothesis. Following the calculations of the induction hypothesis in the third case below, let $\vec{w}_n$ be the maximal element of $C(u_{n-1}, (\vec{v}_n \mid \vec{v}_{n+1}))$ and let $\vec{w}_{n+1}$ be 0-monomials such that $(u_{n-1} \mid \vec{w}_n \mid \vec{w}_{n+1}) = (u_{n-1} \mid \vec{v}_n \mid \vec{v}_{n+1})$. Then, by induction, the source and target of $\sigma(u_{n-1} \Lleftarrow \vec{w}_n \mid \vec{w}_{n+1})$ are equal.

Suppose by contradiction that $\vec{v}_n \neq \vec{w}_n$. Let $\Gamma'_n[\alpha'_n]$ be the last branch associated to $\vec{w}_n$. Then there exists a $(n+1)$-overlapping $u_{n-1} \Lleftarrow \vec{v}_n \Lleftarrow \vec{w}'_{n+1} \in Ov(X)_{n+1}$, whose branches are

$$\{(\Gamma_1[\alpha_1] \mid \vec{w}'_{n+1}), \dots, (\Gamma_n[\alpha_n] \mid \vec{w}'_{n+1}), (\Gamma'_n[\alpha'_n] \mid \vec{v}'_n)\}$$

where $\vec{v}'_n$ is the appropriate list of 0-monomials. Thus $\vec{w}'_{n+1} \in C(u_n, \vec{v}_{n+1})$, which contradicts the hypothesis that $C(u_n, \vec{v}_{n+1})$ is empty. Therefore $\vec{v}_n = \vec{w}_n$, and we conclude that the source and target of $\sigma(u_{n-1} \Lleftarrow \vec{v}_n \mid \vec{v}_{n+1}) = \sigma(u_n \mid \vec{v}_{n+1})$ are equal, allowing us to define $\sigma(u_n \mid \vec{v}_{n+1})$ as an identity.

Third case. Otherwise, $C(u_n, \vec{v}_{n+1})$ is nonempty. Let $\vec{w}_{n+1}$ be its maximal element, $\Gamma_{n+1}[\alpha_{n+1}]$ the associated 1-monomial, and write $(u_n \mid \vec{v}_{n+1}) = (u_n \mid \vec{w}_{n+1} \mid \vec{w}_{n+2})$. Then

$$E(u_n \Lleftarrow \vec{w}_{n+1}, \vec{w}_{n+2}) \subseteq \{\Gamma'[\alpha'] \in E(u_n, \vec{v}_{n+1}) \mid \Gamma'[\alpha'] \succ_{pl} \Gamma_{n+1}[\alpha_{n+1}]\} = \varnothing.$$

In addition, the monomials $\vec{w}_{n+2}$ are reduced, so this is exactly the condition of the second case, so we have the constraint that the source and target of the (not yet defined) $(n+2)$-cell $\sigma(u_n \Lleftarrow \vec{w}_{n+1} \mid \vec{w}_{n+2})$ are equal.

The rest of this case is rather technical, so we summarize our strategy here. We prove and use Lemma 5.2.8 in order to get an explicit expression of $\partial\sigma(u_n \Lleftarrow \vec{w}_{n+1} \mid \vec{w}_{n+2})$, which must be equal to 0, as we have observed. This expression consists of many terms, including $\sigma(u_n \mid \vec{v}_{n+1})$, which is the term that we have to define. We then proceed by well-founded induction on the terms of $\partial\sigma(u_n \Lleftarrow \vec{w}_{n+1} \mid \vec{w}_{n+2})$ to define $\sigma(u_n \mid \vec{v}_{n+1})$ using the other terms.

Let $k \geqslant 1$ and $\vec{v}_0, \vec{v}_1 \dots, \vec{v}_k$ 0-cells of X^{III} such that $\{(v_{0,i} \mid \vec{v}^i_1 \mid \cdots \mid \vec{v}^i_n)\}_i$ is a well-defined list of 0-monomials of X^{III} where $\vec{v}_0$ is the list of roots and, for $\ell \in \{1, \dots, k\}$, $\vec{v}^i_\ell$ is the sublist of $\vec{v}_\ell$ of ancestor $v_{0,i}$. We denote this list by $(\vec{v}_0 \mid \cdots \mid \vec{v}_n)$. Similarly, we denote by $\widetilde{(\vec{v}_0 \mid \vec{v}_1)}$ the list of reduced 0-cells $\widetilde{(v_{0,i} \mid \vec{v}^i_1)}$. Finally, we denote by $(\vec{v}_0 \| \cdots \| \vec{v}_k)$ the list of k-cells

$$\underbrace{\sigma(\sigma(\cdots \sigma(\sigma}_{k}(v_{0,i} \mid \vec{v}^i_1) \mid \vec{v}^i_2) \mid \cdots \vec{v}^i_{k-1}) \mid \vec{v}^i_k).$$

Note that, if $u_0 \Lleftarrow \vec{v}_1 \Lleftarrow \cdots \Lleftarrow \vec{v}_k$ is an k-overlapping, then $u_0 \Lleftarrow \vec{v}_1 \Lleftarrow \cdots \Lleftarrow \vec{v}_k = (u_0 \| \vec{v}_1 \| \cdots \| \vec{v}_k)$.

5.2.8. Lemma. *For $n \geqslant 2$ and $\vec{v}_0, \dots, \vec{v}_n$ 0-monomials of X^{III}_0, we have the equality of $(n-1)$-cells*

$$\begin{aligned}\partial(\vec{v}_0 \| \cdots \| \vec{v}_n) = {} & ((\vec{v}_0 \| \cdots \| \vec{v}_{n-1}) \mid \vec{v}_n) \\ & + \sum_{k=1}^{n} (-1)^k (\vec{v}_0 \| \cdots \| \widetilde{(\vec{v}_{n-k} \mid \vec{v}_{n-k+1})} \| \cdots \| \vec{v}_n) \\ & + (-1)^{n+1} (\vec{v}_0 \mid (\vec{v}_1 \| \cdots \| \vec{v}_n)) + 1_c,\end{aligned}$$

where $\partial = s - t$ and c is some $(n-2)$-cell.

Proof. Proceed by induction on $n \geqslant 2$. According to (5.1.8), for every n-cell a,

$$\partial\sigma(a) = a - \sigma(\partial a) + 1_c,$$

where $c = -t_{n-1}(a)$ is an $(n-1)$-cell. For $n = 2$, applying this equality to $(\vec{v}_0 \| \vec{v}_1 \| \vec{v}_2)$ gives

$$\begin{aligned}
\partial(\vec{v}_0 \| \vec{v}_1 \| \vec{v}_2) &= \partial\sigma(\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) \\
&= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) - \sigma(\partial\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + 1_c \\
&= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + \sigma(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \mid \vec{v}_2) - \sigma(\vec{v}_0 \mid \vec{v}_1 \mid \vec{v}_2) + 1_c \\
&= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + \sigma(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \mid \vec{v}_2) - (\vec{v}_0 \mid \sigma(\vec{v}_1 \mid \vec{v}_2)) - \sigma(\vec{v}_0 \mid \widehat{(\vec{v}_1 \mid \vec{v}_2)}) + 1_{c'} + 1_c \\
&= ((\vec{v}_0 \| \vec{v}_1) \mid \vec{v}_2) + (-1)^1(\vec{v}_0 \| \widehat{(\vec{v}_1 \mid \vec{v}_2)}) + (-1)^2(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \| \vec{v}_2) + (-1)^3(\vec{v}_0 \mid (\vec{v}_1 \| \vec{v}_2)) + 1_{c+c'}.
\end{aligned}$$

Let $n \geqslant 2$. Recall that, for all $(n-1)$-cells u and 0-cells $\vec{v}$,

$$\sigma(u \mid \vec{v}) = (s_0(u) \mid \sigma(\vec{v})) \star_0 (u \mid \widehat{\vec{v}}) = \sigma(u \mid \widehat{\vec{v}}) + 1_c$$

with c an $(n-1)$-cell. We calculate

$$\begin{aligned}
\partial(\vec{v}_0 \| \cdots \| \vec{v}_{n+1}) &= \partial\sigma((\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) \\
&= ((\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma(\partial(\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) + 1_c \\
&= ((\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma((\vec{v}_0 \| \cdots \| \vec{v}_{n-1}) \mid \vec{v}_n \mid \vec{v}_{n+1}) \\
&\quad - \sum_{k=1}^{n} (-1)^k \sigma((\vec{v}_0 \| \cdots \| \widehat{(\vec{v}_{n-k} \mid \vec{v}_{n-k+1})} \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) \\
&\quad - (-1)^{n+1} \sigma(\vec{v}_0 \mid (\vec{v}_1 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) + \sigma(1_{c'}) + 1_c \\
&= ((\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma((\vec{v}_0 \| \cdots \| \vec{v}_{n-1}) \mid \widehat{(\vec{v}_n \mid \vec{v}_{n+1})}) - 1_{c''} \\
&\quad - \sum_{k=1}^{n} (-1)^k (\vec{v}_0 \| \cdots \| \widehat{(\vec{v}_{n-k} \mid \vec{v}_{n-k+1})} \| \cdots \| \vec{v}_{n+1}) \\
&\quad - (-1)^{n+1} (\vec{v}_0 \mid \sigma((\vec{v}_1 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1})) + 1_{\sigma(c')} + 1_c \\
&= ((\vec{v}_0 \| \cdots \| \vec{v}_n) \mid \vec{v}_{n+1}) \\
&\quad + \sum_{k=1}^{n+1} (-1)^k (\vec{v}_0 \| \cdots \| \widehat{(\vec{v}_{n-k+1} \mid \vec{v}_{n-k+2})} \| \cdots \| \vec{v}_{n+1}) \\
&\quad + (-1)^{n+2} (\vec{v}_0 \mid (\vec{v}_1 \| \cdots \| \vec{v}_{n+1})) + 1_{c+\sigma(c')-c''},
\end{aligned}$$

which concludes the induction step, and the proof of the lemma. □

Writing $u_n = u_0 \Lleftarrow \vec{v}_1 \Lleftarrow \cdots \Lleftarrow \vec{v}_n$, we apply the lemma to $\sigma(u_n \Lleftarrow \vec{w}_{n+1} \mid \vec{w}_{n+2}) = (u_0 \| \vec{v}_1 \| \cdots \| \vec{v}_n \| \vec{w}_{n+1} \| \vec{w}_{n+2})$ to get the equation of $(n+1)$-cells

$$\begin{aligned}
\partial\sigma(u_n \Lleftarrow \vec{w}_{n+1} \mid \vec{w}_{n+2}) = 0 &= ((u_0 \| \vec{v}_1 \| \cdots \| \vec{v}_n \| \vec{w}_{n+1}) \mid \vec{w}_{n+2}) \\
&\quad + \sum_{k=1}^{n+2} (-1)^k (u_0 \| \vec{v}_1 \| \cdots \| \widehat{(\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})} \| \cdots \| \vec{v}_n \| \vec{w}_{n+1} \| \vec{w}_{n+2}) \\
&\quad + (-1)^{n+3} (u_0 \mid (\vec{v}_1 \| \cdots \| \vec{v}_n \| \vec{w}_{n+1} \| \vec{w}_{n+2})) + 1_c,
\end{aligned}$$

where c is an n-cell. On the righthand side, the $(n+1)$-cell

$$(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n\|(\widehat{\vec{w}_{n+1} \mid \vec{w}_{n+2}})) = \sigma(u_n \mid \vec{v}_{n+1})$$

appears. We want to define this $(n+1)$-cell using the other $(n+1)$-cells that appear, that is,

$$\begin{gathered}(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n\|w_{n+1}), \quad (\vec{v}_1\|\cdots\|\vec{v}_n\|\vec{w}_{n+1}\|\vec{w}_{n+2}),\\ (u_0\|\vec{v}_1\|\cdots\|(\widehat{\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3}})\|\cdots\|\vec{v}_n\|\vec{w}_{n+1}\|\vec{w}_{n+2}), \quad k \in \{2,\dots,n+2\}.\end{gathered} \tag{5.2.9}$$

We define a well-founded order $\prec$ on $(n+1)$-cells of the form $(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n)$ by setting $(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n) \prec (u'_0\|\vec{v}'_1\|\cdots\|\vec{v}'_n)$ if

i) $T(u_0 \mid \vec{v}_1 \mid \cdots \mid \vec{v}_n)$ is a proper submonomial of $T(u'_0 \mid \vec{v}'_1 \mid \cdots \mid \vec{v}'_n)$, or

ii) $T(u_0 \mid \vec{v}_1 \mid \cdots \mid \vec{v}_n) = T(u'_0 \mid \vec{v}'_1 \mid \cdots \mid \vec{v}'_n)$ and there exist i, j such that $u_0 = u'_0, \vec{v}_1 = \vec{v}'_1, \dots, \vec{v}_{i-1} = \vec{v}'_{i-1}, v_{i,1} = v'_{i,1}, \dots, v_{i,j-1} = v'_{i,j-1}$, and the weight of $v_{i,j}$ is less than that of $v'_{i,j}$, or

iii) there exists a positive 1-cell $f : (u'_0\|\vec{v}'_1\|\cdots\|\vec{v}'_n) \to b$ of X^{III} such that $(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n) \in \mathrm{Supp}(b)$.

The relation $\prec$ is an order because the 1-polygraph X is supposed reduced (so we cannot rewrite a 0-monomial into a larger 0-monomial). The relation $\prec$ is well-founded because every sequence $((u^i_0\|\vec{v}^i_1\|\cdots\|\vec{v}^i_n))_{i\geqslant 0}$ that decreases for $\prec$ can be rearranged into the concatenation of a decreasing sequence for **iii)** followed by a decreasing sequence for the lexicographic order induced by **i)** and **ii)** (if we can rewrite a submonomial of a 0-monomial, then we can rewrite the 0-monomial following the same rule).

We initialize our well-founded induction on the $(n+1)$-overlappings, since $u_0 \lessdot \vec{v}_1 \lessdot \cdots \lessdot \vec{v}_{n+1} = (u_0\|\vec{v}_1\|\cdots\|\vec{v}_{n+1})$ is already defined. We then check that all of the $(n+1)$-cells of (5.2.9) are smaller than $\sigma(u_n \mid \vec{v}_{n+1})$ for the order $\prec$: $(u_0\|\vec{v}_1\|\cdots\|\vec{v}_n\|\vec{w}_{n+1})$ and $(\vec{v}_1\|\cdots\|\vec{v}_n\|\vec{w}_{n+1}\|\vec{w}_{n+2})$ satisfy **i)**, and $(u_0\|\vec{v}_1\|\cdots\|(\widehat{\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3}})\|\cdots\|\vec{v}_n\|\vec{w}_{n+1}\|\vec{w}_{n+2})$ satisfies **ii)** if $(\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})$ is reduced, and **iii)** otherwise, by confluence of X. Thus we can define $\sigma(u_n \mid \vec{v}_{n+1})$ by well-founded induction.

Finally, by Proposition 5.1.13, the ω-polygraph $Ov(X)$ is acyclic. Since X is reduced, by the discussion of § 5.2.3, $Ov(X)_{\leqslant 1}$ coincides with X. Therefore $Ov(X)$ is a polygraphic resolution of the operad $\overline{X}$. □

5.2.10. Corollary. *Let X be an essentially confluent reduced, terminating, left-monomial 1-polygraph. Then there exists a ω-polygraph structure on $Ov(X)$ making it a polygraphic resolution of $\overline{X}$.*

Proof. By Theorem 4.2.4, the polygraph X is convergent. Thus, following Theorem 5.2.6, $Ov(X)$ is equipped with a ω-polygraph structure and is a polygraphic resolution of $\overline{X}$. □

5.2.11. Coherent presentations from convergence. In [47], Squier showed how to compute a coherent presentation of a monoid from a convergent one. This construction is described in the case of associative algebras in [25, Thm 4.3.2], and in the case of shuffle operads by using the following result. For a convergent left-monomial 1-polygraph X, and a cellular extension Y of X^{III}_1 that contains a 2-generator $A_{f,g}$ of shape

$$\begin{tikzcd} & b \arrow[dr, bend left, "h"] & \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & \Downarrow A_{f,g} & d \\ & c \arrow[ur, bend right, "k"'] & \end{tikzcd}$$

with h and k positive 1-cells of X_1^{III}, for every critical branching (f,g) of X, then the 2-polygraph (X,Y) is acyclic. The 2-generator $A_{f,g}$ is called a *generating confluence* associated to the critical branching (f,g). Note that such a generating confluence depends on the choice of the positives cells h and k and the orientation of the 2-cell $A_{f,g}$. The proof of this result is done in two steps. First, we show that the 2-polygraph (X,Y) is coherently confluent, then we prove acyclicity of the cellular extension Y, see [27, Thm. 4.3.2].

The proof of Theorem 5.2.6 makes explicit the generating confluences, which can be represented as follows:

$$\begin{tikzcd}
(u_0 \lessdot \vec{v}_1 \,|\, \vec{v}_2) \arrow[rr, bend left] & & \widehat{((u_0 \,|\, \vec{v}_1) \,|\, \vec{v}_2)} \arrow[dd, Rightarrow, "u_0 \lessdot \vec{v}_1 \lessdot \vec{v}_2"] \arrow[dr, "\sigma_{(\widehat{(u_0|\vec{v}_1)}|\vec{v}_2)}"] & \\
(u_0 \,|\, \vec{v}_1 \,|\, \vec{v}_2) \arrow[ur, bend left] \arrow[dr, bend right] & & & \widehat{(u_0 \,|\, \vec{v}_1 \,|\, \vec{v}_2)} \\
(u_0 \,|\, \sigma_{(\vec{v}_1|\vec{v}_2)}) \arrow[rr] & & \widehat{(u_0 \,|\, (\vec{v}_1 \,|\, \vec{v}_2))} \arrow[ur, "\sigma_{(u_0|\widehat{(\vec{v}_1|\vec{v}_2)})}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(u_0 \lessdot \vec{v}_1 \,|\, \vec{v}_2) \arrow[rr, bend left] & & \widehat{((u_0 \,|\, \vec{v}_1) \,|\, \vec{v}_2)} \arrow[dd, Rightarrow, "u_0 \lessdot \vec{v}_1 \lessdot \vec{v}_2"] \arrow[dr, "\sigma_{(\widehat{(u_0|\vec{v}_1)}|\vec{v}_2)}"] & \\
(u_0 \,|\, \vec{v}_1 \,|\, \vec{v}_2) \arrow[ur, bend left] \arrow[dr, bend right] & & & \widehat{(u_0 \,|\, \vec{v}_1 \,|\, \vec{v}_2)} \\
(u_0 \lessdot \vec{w}_1 \,|\, \vec{w}_2) \arrow[rr] & & \widehat{((u_0 \,|\, \vec{w}_1) \,|\, \vec{w}_2)} \arrow[ur, "\sigma_{(\widehat{(u_0|\vec{w}_1)}|\vec{w}_2)}"'] &
\end{tikzcd}$$

where $u_0 \lessdot \vec{v}_1 \lessdot \vec{v}_2$ is a 2-overlapping of X. If $(\vec{v}_1 \mid \vec{v}_2)$ is reducible, we have the left diagram. If $(\vec{v}_1 \mid \vec{v}_2)$ is reduced, we take $\vec{w}_1 = \max C(u_0, (\vec{v}_1 \mid \vec{v}_2))$ and get the right diagram.

5.2.12. Examples.

i) The terminating reduced 1-polygraph X defined in Example 4.1.10 does not have critical branchings. As a consequence, it is convergent and can be extended into a polygraphic resolution $\mathcal{O}v(X)$, with $\mathcal{O}v(X)_n$ empty for $n \geqslant 2$.

ii) Following Proposition 3.2.6, if X is an acyclic ω-polygraph whose underlying 1-polygraph $X_{\leqslant 1}$ is left-monomial and convergent, then X is Tietze equivalent to the ω-polygraph $\mathcal{O}v(X_{\leqslant 1})$. In particular, for every operad P, the ω-polygraphs $\mathrm{Std}(P)$ and $\mathcal{O}v(\mathrm{Std}(P)_{\leqslant 1})$ are Tietze equivalent.

5.3. Bimodule resolutions from polygraphic resolutions

In this subsection, we show how to deduce the homology of a shuffle operad with coefficients in bimodules from a shuffle polygraphic resolution of the operad.

5.3.1. Construction of a chain complex. Let X be a shuffle ω-polygraph, and denote P the shuffle operad presented by X. Denote by $\overline{u}$ the image of u by the canonical projection π_X defined in (3.2.2). Consider the chain complex $(P\langle X\rangle_n)_{n\geqslant -1}$ in the category $\mathsf{Bimod}(P)$ generated by X, that is, for all $n \geqslant -1$, $P\langle X\rangle_n := P\langle X_n\rangle$ is the free P-bimodule on X_n, where X_{-1} is the unit indexed set $\mathbf{1}$ defined in§ 2.1.6. The generators of $P\langle X\rangle_n$, are denoted $[x]$ for x in X_n. The boundary map $\delta_n : P\langle X\rangle_{n+1} \longrightarrow P\langle X\rangle_n$ is defined as follows

i) For $x \in X_0(k)$, we set

$$\delta_{-1}([x]) = (\varepsilon \mid \overline{x}) - \sum_{i=1}^{k} (\overline{x} \mid \overline{1} \cdots \underset{i}{\varepsilon} \cdots \overline{1}). \tag{5.3.2}$$

ii) Consider the derivation $[\ ] : X_0^{\mathrm{III}} \to P\langle X_0\rangle$ defined by induction on the weight of monomials u in X_0^{III}, by setting $[1] := 0$, $[x] := x$ and

$$[u \mid \vec{v}] := ([u] \mid \overline{v}_1 \cdots \overline{v}_k) + \sum_{i=1}^{k} (\overline{u} \mid \overline{v}_1 \cdots [v_i] \cdots \overline{v}_k).$$

We set δ_0 the *Fox differential* defined for every 1-generator α in X_1 by

$$\delta_0([\alpha]) := [s_0(\alpha)] - [t_0(\alpha)].$$

iii) For $n \geqslant 1$, we define the map $[\ ] : X_n^{\mathrm{III}} \to P\langle X_n\rangle$ by setting, for

$$f = \sum_{i=1}^{p} \lambda_i \Gamma_i [\alpha_i] + 1_c$$

an n-cell of X^{III}, where $\alpha_i \in X_n$ and Γ_i is a one-hole context of $X_0^{*\mathrm{III}}$,

$$[f] = \sum_{i=1}^{p} \lambda_i \overline{\Gamma}_i \big[[\alpha_i]\big],$$

where $\overline{\Gamma}_i$ is the one-hole context of P induced by the context Γ_i. Note that $[f]$ does not depend on the choice of decomposition, so $[\ ]$ is well defined. We set for every $(n+1)$-generator A in X_{n+1}

$$\delta_n([A]) := [s_n(A)] - [t_n(A)].$$

As a consequence of the globularity of the polygraph X, for all $n \geqslant -1$, we have $\delta_{n+1}\delta_n = 0$ and thus $P\langle X\rangle$ forms a chain complex.

5.3.3. Lemma. *For every* 0*-monomial* $u \in X_0^{*\mathrm{III}}(k)$*, we have*

$$\delta_{-1}([u]) = (\varepsilon \mid \overline{u}) - \sum_{i=1}^{k} (\overline{u} \mid \overline{1} \cdots \underset{i}{\varepsilon} \cdots \overline{1}).$$

Proof. Proceed by induction on the depth of the 0-monomial u. The equality is true by definition for $x \in X_0$. For the induction step, consider $(u \mid \vec{v})$ with $u \in X_0^{\ell}(k)$, $v_i \in X_0^{\ell}(\ell_i)$ for all $1 \leqslant i \leqslant k$:

$$\begin{aligned}
\delta_{-1}([u \mid \vec{v}]) &= (\delta_{-1}([u]) \mid \overline{v}_1 \cdots \overline{v}_k) + \sum_{i=1}^{k} (\overline{u} \mid \overline{v}_1 \cdots \delta_{-1}([v_i]) \cdots \overline{v}_k) \\
&= (\varepsilon \mid \overline{u} \mid \overline{v}_1 \cdots \overline{v}_k) - \sum_{i=1}^{k} (\overline{u} \mid (\overline{1} \mid \overline{v}_1) \cdots (\varepsilon \mid v_i) \cdots (\overline{1} \mid \overline{v}_k)) \\
&\quad + \sum_{i=1}^{k} (\overline{u} \mid \overline{v}_1 \cdots (\varepsilon \mid \overline{v}_i) \cdots \overline{v}_k) - \sum_{i=1}^{k} \sum_{j=1}^{\ell_i} (\overline{u} \mid \overline{v}_1 \cdots (\overline{v}_i \mid \overline{1} \cdots \underset{j}{\varepsilon} \cdots \overline{1}) \cdots \overline{v}_k) \\
&= (\varepsilon \mid \overline{(u \mid \vec{v})}) - \sum_{i=1}^{\ell_1 + \cdots + \ell_k} (\overline{(u \mid \vec{v})} \mid \overline{1} \cdots \underset{i}{\varepsilon} \cdots \overline{1}).
\end{aligned}$$

□

5.3.4. Trivial P-bimodule. Define the *trivial P-bimodule*, denoted by Ω_P, as the free P-bimodule generated by the unit indexed set $\mathbf{1}$ quotiented by the relations

$$(\varepsilon \mid \overline{u}) = \sum_{i=1}^{k} \overline{u} \circ_i \varepsilon \tag{5.3.5}$$

for every $k \geqslant 1$ and $\overline{u} \in P(k)$. Every element of the P-bimodule Ω_P can be written as a linear combination of monomials of the form $\overline{u} \circ_i \varepsilon$ where $k \geqslant 1$, $\overline{u} \in P(k)$, and $1 \leqslant i \leqslant k$.

5.3.6. Proposition. *Let X be an acyclic shuffle ω-polygraph and P the shuffle operad presented by X. Then the chain complex $P\langle X\rangle$ is a resolution of Ω_P in the category* $\mathrm{Bimod}(P)$.

Proof. Note that Ω_P is exactly the cokernel of δ_{-1}. Thus it suffices to show that the chain complex $P\langle X\rangle$ is exact.

Let us fix ι a unital section of X. Following Proposition 5.1.13, the acyclicity of the polygraph X implies that it admits a right ι-contraction. Let σ be such a right ι-contraction. We define the linear map $i_0 : P\langle \mathbf{1}\rangle \to P\langle X_0\rangle$ by

$$i_0(u \mid v_1 \cdots (\varepsilon \underset{i}{\mid} w) \cdots v_k) := (u \mid v_1 \cdots [\widehat{w}] \underset{i}{\cdots} v_k),$$

for $u, v_1, \ldots, \check{v}_i, \ldots, v_n, w \in P$, and, for $n \geqslant 1$, the linear map $i_n : P\langle X_{n-1}\rangle \to P\langle X_n\rangle$ by

$$i_n(u \mid v_1 \cdots ([x] \underset{i}{\mid} w_1 \cdots w_\ell) \cdots v_k) := (u \mid v_1 \cdots [\sigma_{(x \mid \widehat{w}_1 \cdots \widehat{w}_\ell)}] \underset{i}{\cdots} v_k),$$

and $u, v_1, \ldots, \check{v}_i, \ldots, v_k, w_1, \ldots, w_\ell \in P$. Note that the linear maps i_n are compatible with the left action of P. Hence, we prove that the maps i_n define a contracting homotopy of the complex $P\langle X\rangle$, by showing that the identity $i_n\delta_{n-1} + \delta_n i_{n+1} = id_{P\langle X_n\rangle}$ holds on generators of the P-bimodule $P\langle X_n\rangle$ as follows.

For $n = 0$, on the one hand, we have

$$\begin{aligned} i_0\delta_{-1}([x] \mid w_1 \cdots w_n) &= i_0(\varepsilon \mid \overline{x} \mid \vec{w}) - \sum_{i=1}^{n} i_0(\overline{x} \mid (\overline{1} \mid w_1) \cdots (\varepsilon \mid w_i) \cdots (\overline{1} \mid w_n)) \\ &= [\widehat{(\overline{x} \mid \vec{w})}] - \sum_{i=1}^{n} (\overline{x} \mid w_1 \cdots [\widehat{w}_i] \cdots w_n). \end{aligned}$$

On the other, we have

$$\begin{aligned} \delta_0 i_1([x] \mid w_1 \cdots w_n) &= \delta_0[\sigma_{(x \mid \widehat{w}_1 \cdots \widehat{w}_n)}] \\ &= \delta_0[(x \mid \widehat{w}_1 \cdots \widehat{w}_n)] - \delta_0[\widehat{(x \mid \vec{w})}] \\ &= ([x] \mid w_1 \cdots w_n) + \sum_{i=1}^{n} (\overline{x} \mid w_1 \cdots [\widehat{w}_i] \cdots w_n) - [\widehat{(x \mid \vec{w})}], \end{aligned}$$

proving the equality $\delta_0 i_1 + i_0\delta_{-1} = id_{P\langle X_0\rangle}$.

For $n \geqslant 1$, by definition of the right ι-contraction σ, we show that, for every $(n-1)$-cells $u, w_1, \ldots, w_n$ of X^{III},

$$i_n[(u \mid w_1 \cdots w_n)] = [\sigma_{(u \mid \widehat{w}_1 \cdots \widehat{w}_n)}].$$

Therefore, for every n-generator $A : a \to b$ in X_n, we have

$$\begin{aligned} i_n\delta_{n-1}([A] \mid w_1 \cdots w_n) &= i_n[(a \mid \vec{w})] - i_n[(b \mid \vec{w})] = [\sigma_{(a|\widehat{w}_1\cdots\widehat{w}_n)}] - [\sigma_{(b|\widehat{w}_1\cdots\widehat{w}_n)}], \\ \delta_n i_{n+1}([A] \mid w_1 \cdots w_n) &= [(A \mid \widehat{w}_1 \cdots \widehat{w}_n) \star_0 \sigma_{(b|\widehat{w}_1\cdots\widehat{w}_n)}] - [\sigma_{(a|\widehat{w}_1\ldots\widehat{w}_n)}] \\ &= ([A] \mid w_1 \cdots w_n) + [\sigma_{(b|\widehat{w}_1\cdots\widehat{w}_n)}] - [\sigma_{(a|\widehat{w}_1\cdots\widehat{w}_n)}], \end{aligned}$$

proving that $i_n\delta_{n-1} + \delta_n i_{n+1} = id_{P\langle X_n\rangle}$. □

5.3.7. Homology of shuffle operads. Recall that the Cartan-Eilenberg homology of a shuffle operad P with coefficients in a P-bimodule A is defined by

$$H^{CE}_\bullet(P, A) := \mathrm{Tor}^{\mathsf{Bimod}(P)}_\bullet(\Omega_P, A).$$

In addition, the Quillen homology of P is defined with coefficients in $\mathsf{Ab}(\text{Ш}\mathsf{Op}/P)$, the category of abelian groups internal to $\text{Ш}\mathsf{Op}/P$ [44]. The category $\mathsf{Ab}(\text{Ш}\mathsf{Op}/P)$ is equivalent to the category $\mathsf{Bimod}(P)$ of P-bimodules [7], and we define the Quillen homology of P with coefficients in a P-bimodule A by setting

$$H^Q_\bullet(P, A) := H_\bullet(\mathsf{Ab}(\mathcal{X}) \otimes_P A),$$

where $\mathcal{X}$ is a simplicial cofibrant resolution of the operad P in the category $\text{Ш}\mathsf{Op}/P$, and $\mathsf{Ab}(-) : \text{Ш}\mathsf{Op}/P \to \mathsf{Ab}(\text{Ш}\mathsf{Op}/P) \approx \mathsf{Bimod}(P)$ is the abelianization functor. Following [5, Thm. 4.1], see also [6, Thm. 6.2.1], these two homologies are isomorphic up to shift in degree:

$$H^Q_\bullet(P, A) \simeq H^{CE}_{\bullet+1}(P, A).$$

5.3.8. Finite homological type. From Theorem 5.2.6 we deduce a generalization of Squier's homological finiteness condition [46], for finite convergence in the case of operads. We say that a shuffle operad P has *finite homological type*, FP_∞ for short, if the P-bimodule Ω_P has a resolution in $\mathsf{Bimod}(P)$ by finitely generated projective bimodules. If P admits a finite convergent presentation X, then by Theorem 5.2.6, the overlapping polygraphic resolution $\mathcal{O}v(X)$ is finite and the complex $P\langle\mathcal{O}v(X)\rangle$ is a finitely generated free resolution of Ω_P. Thus, P has homological type FP_∞.

5.3.9. Minimal resolutions. A *minimal bimodule resolution* of an operad P is a minimal free P-bimodule resolution $(A_\bullet, \delta)$ of its trivial P-bimodule Ω_P. The minimal condition means that the sequence $(A_\bullet \otimes_P \mathbf{k}, \delta \otimes_P id)$ has a null differential, where $\mathbf{k}$ denotes the P-bimodule concentrated in degree 0, whose left and right actions vanish.

5.3.10. Proposition. *Let X be an acyclic shuffle ω-polygraph and $w : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ an increasing function such that X_n is concentrated in weight $w(n)$. Then $P\langle X\rangle$ is a minimal P-bimodule resolution of the operad P presented by X.*

Proof. The 1-generators of X are of homogeneous weight, so P is equipped with a weight grading. Given an n-generator u_n in X_n, we have $\delta_{n-1}[u_n] = \sum_i \lambda_i\overline{\Gamma}_i[u_{n-1,i}] + 1_c$, where the λ_i are scalars, the Γ_i are one-hole contexts of $X_0^{*\text{Ш}}$, the $u_{n-1,i}$ are $(n-1)$-overlappings, and c is an $(n-2)$-cell in $X^{\text{Ш}}_{n-2}$. The map δ_{n-1} preserves weight, and since the $(n-1)$-overlappings $u_{n-1,i}$ are of strictly smaller weight than u_n, it follows that the $\overline{\Gamma}_i$ are nontrivial. As a consequence, tensoring over P by $\mathbf{k}$ sends $\delta_{n-1}(u_n)$ to 0, so $P\langle X\rangle$ is minimal. □

5.3.11. Examples. Consider the convergent 1-polygraph X with one 1-generator $x \in X(2)$ of weight 1 and one 2-generator

$$\begin{tikzcd} x \arrow[dr] & & x \arrow[dl] \\ & x & \end{tikzcd} \longrightarrow 0$$

Then $\mathcal{O}v(X)_n$ is a polygraphic resolution concentrated in degree $2n+1$, so by Proposition 5.3.10 $P\langle X\rangle$ is a minimal bimodule resolution of the operad P presented by X. Note however that, for the convergent 1-polygraph with one 1-generator $x \in X(1)$ of weight 1 and one 2-generator $x^3 \longrightarrow 0$, the resolution induced by $\mathcal{O}v(X)$ is not minimal.

5.4. Confluence and Koszulness

In this subsection, we show that shuffle operads presented by a quadratic convergent 1-polygraph are Koszul. Our result does not suppose that the rewriting rules are oriented with respect to a monomial order. In this way, it generalizes the result obtained by Dotsenko and Khoroshkin in [19] for shuffle operads with quadratic Gröbner bases. We first begin by recalling the Koszul property for operads.

5.4.1. Koszul operads. Let P be a (connected and graded) symmetric operad. We denote by $\overline{B}(P)$ for the reduced bar complex on P. Recall from [21, Def. 5.2.3] that the *Koszul complex* on P is defined by

$$K(P)_{(s)} := H_s(\overline{B}(P)_{(s)}, \delta) = \ker(\delta : \overline{B}_s(P)_{(s)} \longrightarrow \overline{B}_{s-1}(P)_{(s)}),$$

the second equality coming from the fact that $\overline{B}_n(P)_{(s)} = 0$ when $n > s$, and where (s) denotes the degree of P. By definition, the complex $K(P)_{(s)}$ is concentrated in degree s. The operad P is *Koszul* if the inclusion morphism $K(P) \hookrightarrow \overline{B}(P)$ is a quasi-isomorphism [21, Def. 5.2.8], or equivalently the homology of the reduced bar complex of P is concentrated on the diagonal [21, Thm. 5.3.3], that is,

$$\mathrm{H}_n(\overline{B}(P)_{(s)}) = 0, \qquad \text{for } n \neq s.$$

Recall that the bar-cobar construction on P is a resolution, whose abelianization is the reduced bar complex [20, § 1.1], so that the operad P is Koszul if, and only if, its Quillen homology is concentrated on the diagonal. Finally, recall from [20, Cor. 1.5] that for a symmetric operad P, there is an isomorphism

$$\mathrm{H}_\bullet(\overline{B}(P))^u \simeq \mathrm{H}_\bullet(\overline{B}(P^u)).$$

As a consequence, the Koszulness of a symmetric operad can be proved via its shuffle version as follows.

5.4.2. Theorem. *Let P be a quadratic symmetric operad. If the associated shuffle operad P^u has a quadratic convergent presentation, then P is Koszul.*

Proof. Let X be a quadratic convergent 1-polygraph presenting P^u. By definition, the 0-generators in X_0 are concentrated in degree 1 and the 1-generators in X_1 in degree 2. By construction, the polygraphic resolution $\mathcal{O}v(X)$, constructed in Theorem 5.2.6, is concentrated on the superdiagonal, that is for $n \geqslant 2$, the n-generators in $\mathcal{O}v(X)_n$ are of degree $n+1$; these cells are the generators of the P-bimodules of the resolution $P\langle \mathcal{O}v(X)\rangle$ of Ω_P of Theorem 5.3.6. Thus, the Cartan-Eilenberg homology of P is concentrated in degree $n+1$, and so the Quillen homology is concentrated on the diagonal. Following 5.4.1, we conclude that P is Koszul. □

5.4.3. Remark. If we consider a quadratic symmetric operad whose generators are all of arity one, using Theorem 5.4.2 we recover the similar result for quadratic associative algebras: every algebra having a quadratic convergent presentation is Koszul, as proved in [25, Prop. 7.2.2] by a polygraphic construction, see also [36, Sec. 4.3], and [8] for a such a criterion with the rewriting rules ordered with respect to a monomial order.

5.4.4. Koszul associative algebra without monomial order. Let A be the associative algebra presented by

$$\left\langle\ w, x, y, z \ \middle|\ w^2 = wx,\ x^2 = yx,\ y^2 = yz,\ z^2 = wz\ \right\rangle.$$

If we orient the relations according to a monomial order, say the order generated by $w < x < y < z$, this gives a 1-polygraph with two critical branching that are non-confluent

$$\begin{tikzcd} & y^2z \arrow[dr, bend left] & \\ yz^2 \arrow[ur, bend left] \arrow[dr, bend right] & & y^3 \\ & ywz & \end{tikzcd} \qquad \begin{tikzcd} & wzz \arrow[dr, bend left] & \\ zzz \arrow[ur, bend left] \arrow[dr, bend right] & & wwz \\ & zwz & \end{tikzcd}$$

Moreover, we show that any alphabetic order conduces to a similar situation of non-confluent critical branching. Instead, consider the following 1-polygraph:

$$X := \left\langle\ w, x, y, z \ \middle|\ wx \to w^2,\ yx \to x^2,\ yz \to y^2,\ wz \to z^2\ \right\rangle.$$

The termination of X is equivalent to the termination of the following 1-polygraph

$$\langle\ w, x, y, z \mid wx \to w,\ yx \to x,\ yz \to y,\ wz \to z\ \rangle,$$

and this second 1-polygraph clearly terminates by considering the lengths of words, so X terminates. Moreover, X has no critical branchings, so it is confluent. Thus X is a convergent quadratic 1-polygraph, so by [25, Prop. 7.2.2], thus the algebra A is Koszul.

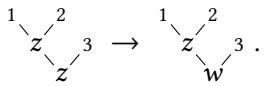
5.4.5. Koszul operad without monomial order. Following the previous example, let P be the symmetric operad presented by

$$\left\langle\ w, x, y, z \in P(2)\ \middle|\ \begin{array}{l} w(1\,2) = w,\ x(1\,2) = x,\ y(1\,2) = y,\ z(1\,2) = z, \\ w \circ_{1,id} w = w \circ_{1,id} x,\ x \circ_{1,id} x = y \circ_{1,id} x, \\ y \circ_{1,id} y = y \circ_{1,id} z,\ z \circ_{1,id} z = w \circ_{1,id} z, \end{array}\ \right\rangle.$$

Consider the associated shuffle operad P^u, which is presented by

$$\left\langle\ w, x, y, z \in P^u(2)\ \middle|\ \begin{array}{l} \overset{1}{w}\overset{2}{w}\overset{3}{w} = \overset{1}{x}\overset{2}{w}\overset{3}{w},\ \overset{1}{w}\overset{3}{w}\overset{2}{w} = \overset{1}{x}\overset{3}{w}\overset{2}{w},\ \overset{1}{w}\overset{2}{w}\overset{3}{w} = \overset{1}{w}\overset{2}{x}\overset{3}{w}, \\ \overset{1}{x}\overset{2}{x}\overset{3}{x} = \overset{1}{x}\overset{2}{y}\overset{3}{x},\ \overset{1}{x}\overset{3}{x}\overset{2}{x} = \overset{1}{x}\overset{3}{y}\overset{2}{x},\ \overset{1}{x}\overset{2}{x}\overset{3}{x} = \overset{1}{x}\overset{2}{y}\overset{3}{x}, \\ \overset{1}{y}\overset{2}{y}\overset{3}{y} = \overset{1}{z}\overset{2}{y}\overset{3}{y},\ \overset{1}{y}\overset{3}{y}\overset{2}{y} = \overset{1}{z}\overset{3}{y}\overset{2}{y},\ \overset{1}{y}\overset{2}{y}\overset{3}{y} = \overset{1}{y}\overset{2}{z}\overset{3}{y}, \\ \overset{1}{z}\overset{2}{z}\overset{3}{z} = \overset{1}{z}\overset{2}{w}\overset{3}{z},\ \overset{1}{z}\overset{3}{z}\overset{2}{z} = \overset{1}{z}\overset{3}{w}\overset{2}{z},\ \overset{1}{z}\overset{2}{z}\overset{3}{z} = \overset{1}{z}\overset{2}{w}\overset{3}{z} \end{array}\ \right\rangle.$$

If we orient the induced relations according to a monomial order, say an order where $w < x < y < z$, then in particular we get the rewriting rule



Comparing the this rewriting rule with those of the previous example, we find that this rule creates a non-confluent critical pair. By Proposition 4.3.2, this also means that this presentation of P does not admit a quadratic Gröbner basis.

Instead, if we orient every relation from right to left, we get a shuffle 1-polygraph X with 0-generators $w, x, y, z \in X_0(2)$ and with twelve 1-generators. With arguments similar to previous example, we show that X is terminating. Moreover, there are no critical branchings, so by Theorem 4.2.4 the 1-polygraph X is confluent. In this way, X is a convergent quadratic 1-polygraph, so by Theorem 5.4.2, the operad P is Koszul.

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