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# SHUFFLE POLYGRAPHIC RESOLUTIONS FOR OPERADS

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PHILIPPE MALBOS - ISAAC REN

**Abstract** – Shuffle operads were introduced to make explicit the actions of symmetric groups on symmetric operads. Rewriting methods were then applied to symmetric operads via shuffle operads: in particular, a notion of Gröbner basis was introduced for shuffle operads with respect to a total order on tree monomials. In this article, we introduce the structure of shuffle polygraphs as a general framework for rewriting in shuffle operads, which generalizes the Gröbner bases approach by removing the constraint of a monomial order for the orientation of the rewriting rules. We define  $\omega$ -operads as internal  $\omega$ -categories in the category of shuffle operads. We show how to extend a convergent shuffle polygraph into a shuffle polygraphic resolution generated by the overlapping branchings of the original polygraph. Finally, we prove that a shuffle operad presented by a quadratic convergent shuffle polygraph is Koszul.

**Keywords** – Shuffle operads, higher-dimensional rewriting, Gröbner bases, Koszulness.

**M.S.C. 2020** – 18M70, 68Q42, 18N30.

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# 1. INTRODUCTION

Algebraic rewriting theory provides methods to compute cofibrant replacements of algebraic structures from presentations that take into account computational properties of these structures. This rewriting approach gives algebraic algorithmic methods to solve decidability and computational problems, such as the ideal membership problem, and the computation of linear bases and of (co)homological properties. In particular, resolutions for monoids [12, 31, 42], groups [17], small categories [25], associative algebras [2, 24], and linear operads [18, 19] have been constructed using rewriting methods. The machinery at the heart of these constructions consists in presenting an algebraic structure by a system of *generators* and *rewriting rules*, and producing a cofibrant replacement that involves the overlappings occurring in the applications of the rewriting rules. Rewriting approaches for linear structures were developed in many algebraic algorithmic contexts, notably by Janet and Buchberger for commutative algebras [13, 29], Shirshov, Bokut, and Bergman for associative algebras [9, 10, 41], Dotsenko-Khoroshkin for linear operads [18]. In all of these works, the rewriting systems are formulated in terms of *Gröbner bases*, and thus are defined with respect to a given monomial order. Rewriting approaches have also been used in the categorical context to present higher-dimensional categories by higher-dimensional rewriting systems, called *polygraphs* (or *computads*) [15, 44]. In this context, the cofibrant replacements of a higher-dimensional category are generated by *polygraphic resolutions* introduced by Métayer [32, 37].

In this work, we combine the polygraphic and the Gröbner bases approaches in order to compute higher-dimensional presentations of shuffle operads using the polygraphic machinery. We define *shuffle  $\omega$ -operads* as internal  $\omega$ -categories in the category of shuffle operads. We introduce the structure of *shuffle  $\omega$ -polygraphs* as systems of generators and relations for shuffle  $\omega$ -operads. Unlike the Gröbner bases approach, the orientation of the relations in a shuffle polygraph does not depend on a given monomial order. The main construction of this article extends a confluent and terminating shuffle polygraph presenting a shuffle operad into a shuffle polygraphic resolution generated by the overlapping branchings of the original polygraph. We deduce some homological properties on shuffle operads.

Now we present the organization and the main results of this article.

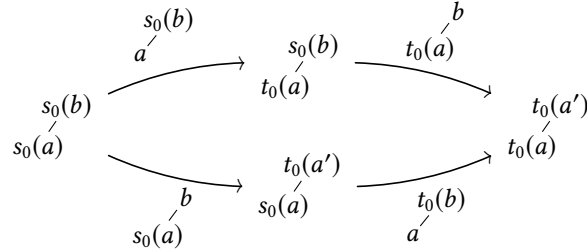
## Higher-dimensional operads

The notion of a *symmetric operad* appears in many situations to describe operations in several arguments, with symmetric group actions, acting on topological or algebraic objects [33, 36]. *Shuffle operads* were introduced by Dotsenko and Khoroshkin in [18] to make explicit the action of symmetric groups on the arguments. The shuffle version allows us to define *monomials* and *oriented relations* in order to present symmetric operads by rewriting systems. Symmetric and shuffle operads are defined as internal monoids in the monoidal presheaf categories of *collections* and *symmetric collections* respectively, as recalled in Section 2.1. Explicitly, a *collection* is a presheaf on the category  $\text{Ord}$  of nonempty finite ordered sets and order-preserving bijections, with values in the category  $\text{Vect}$  of vector spaces. The monoidal product on collections is the *shuffle composition* recalled in § 2.1.3. A *symmetric collection* is a presheaf on the category  $\text{Fin}$  of nonempty finite sets and bijections, with values in the category  $\text{Vect}$ . The functor  $-^u : \text{Ord} \rightarrow \text{Fin}$  that forgets the order induces a functor  $-^u : \mathfrak{S}\text{Coll} \rightarrow \text{Coll}$  from the category of symmetric collections to the category of collections.

In Section 2.3, we introduce the notion of a (strict) higher-dimensional shuffle operad. We define a *shuffle  $\omega$ -operad* as an internal  $\omega$ -category in the category  $\text{III}\text{Op}$  of shuffle operads. Shuffle  $\omega$ -operads, with internal  $\omega$ -functors, form a category denoted by  $\text{III}\text{Op}_\omega$ . In Section 2.4, we study the interaction between the higher-categorical structure of  $\omega$ -operads and its underlying linear structure. The object of  $n$ -cells of a shuffle  $\omega$ -operad has a shuffle operad structure, and the  $n$ -cells can be  $\star_k$ -composed along  $k$ -dimensional cells for  $0 \leq k < n$ . Due to the linear structure, the  $\star_k$ -composition of two  $n$ -cells  $a$  and  $b$  in a shuffle  $\omega$ -operad can be written as the following linear combination:

$$a \star_k b = a - t_k(a) + b,$$

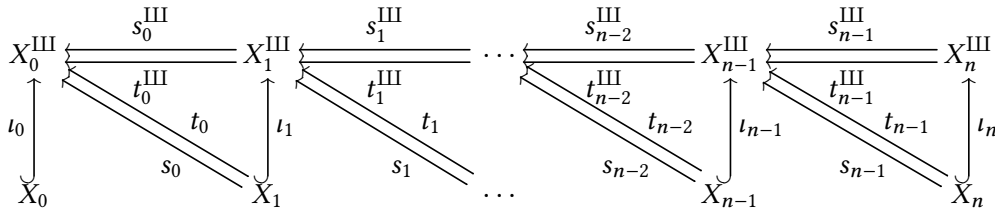
where  $t_k(a)$  denotes the  $k$ -dimensional target of  $a$ , which coincides with the  $k$ -dimensional source of  $b$ . In particular, every  $n$ -cell in a shuffle  $\omega$ -operad is invertible. Moreover, for  $n \geq 1$ , the compatibility between the shuffle composition and the  $\star_0$ -composition implies that the *elementary composition*  $a \circ_{i,\tau} a'$  of  $n$ -cells  $a$  and  $b$ , as defined in § 2.2.1, can be seen either one of two *orthogonal reduction paths* from  $s_0(a) \circ_{i,\tau} s_0(b)$  to  $t_0(a) \circ_{i,\tau} t_0(b)$ , pictured as follows:



The *exchange linear relation* introduced in § 2.4.2 states that these two reductions paths are equal. With these remarkable relations, the axioms of shuffle  $\omega$ -operads can be simplified. We deduce a characterization of the structure of  $\omega$ -operad in terms of bimodules over shuffle operads. Our first result, Theorem 2.4.8, proves that the category  $\text{III}\text{Op}_\omega$  is isomorphic to the full subcategory of  $\text{RGlob}(\text{Bimod}(\text{III}\text{Op}))$ , whose objects are pairs  $(P, A)$  where  $P$  is a shuffle operad and  $A = (A_n)_{n \geq 0}$  is a reflexive globular  $P$ -bimodule such that  $A_0 = P$ , and  $A_n$  satisfies the linear exchange relation for all  $n \geq 1$ .

### Shuffle operadic polygraphs and rewriting

The notion of a *polygraph* was introduced in the set-theoretical context by Street and Burroni as systems of generators and relations for presentations of higher-dimensional (strict) categories [14, 44]. A linear version of polygraphs was introduced in [24] for the presentation of associative  $\omega$ -algebras. In Section 3.1, we define an analogous notion for shuffle  $\omega$ -operads, which we call *shuffle polygraphs*. Explicitly, for  $n \geq 0$ , a *shuffle  $n$ -polygraph* is a data  $X = (X_0, \dots, X_n)$  constructed by induction, where  $X_k$ , the set of  $k$ -generators, forms a globular extension of the free shuffle  $(k - 1)$ -operad generated by the shuffle  $(k - 1)$ -polygraph  $(X_0, \dots, X_{k-1})$ . Such a data can be pictured as a diagram



## 1. Introduction

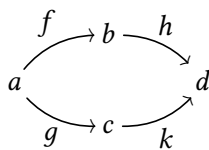
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where  $s_i$  and  $t_i$  denote the source and target maps of the globular extensions, and the horizontal diagram corresponds to the underlying globular operad of the free  $n$ -operad generated by the  $n$ -polygraph  $X$ , denoted by  $X_n^{\text{III}}$ . As for set-theoretical polygraphs, in Section 3.1 we define the category  $\text{III}Pol_n$  of *shuffle  $n$ -polygraphs* and the free  $n$ -operad functor  $(-)^{\text{III}} : \text{III}Pol_n \rightarrow \text{III}Op_n$  by induction on the dimension  $n$ , and the category of shuffle  $\omega$ -polygraphs as the limit of the forgetful functors  $\text{III}Pol_n \rightarrow \text{III}Pol_{n-1}$  for  $n \geq 1$ .

The shuffle polygraphic approach lets us present shuffle operads by oriented presentations, called *rewriting systems*: the shuffle operad  $\bar{X}$  presented by a shuffle 1-polygraph  $X$  is defined as the coequalizer of the source and target morphisms  $s_0^{\text{III}}, t_0^{\text{III}} : X_1^{\text{III}} \rightrightarrows X_0^{\text{III}}$  in the category  $\text{III}Op$ . Note that, in this work, we consider operads with only one color. The 0-generators of  $X$  correspond to the generators of the shuffle operads, and the 1-generators of  $X$  correspond to the oriented relations. For presentations of multicolored shuffle operads, we need to consider shuffle 2-polygraphs, whose 0-generators correspond to the colors, 1-generators to generators, and 2-generators to oriented relations.

The goal of rewriting theory on a shuffle 1-polygraph  $X$  is to deduce global rewriting properties, such as *confluence* and *termination* of  $X$  from local properties of the 1-generators of  $X$ , also called *rewriting rules*. Without loss of generality, in Section 4 we will consider 1-polygraphs with *left-monomial* rules, reducing a single monomial into a linear combination of monomials. A *rewriting step* of a left-monomial 1-polygraph  $X$  is a 1-cell  $f$  of the free shuffle 1-operad  $X_1^{\text{III}}$  of size 1, and of the form  $f = \lambda g + 1_c$ , where  $\lambda$  is a nonzero scalar,  $g$  is a 1-monomial of  $X_1^{\text{III}}$ , and  $c$  is a 0-cell of the free shuffle operad  $X_0^{\text{III}}$  such that the 0-monomial  $s_0(u) \notin \text{Supp}(c)$ . A 1-cell of the free 1-operad  $X_1^{\text{III}}$  is *positive* if it is the  $\star_0$ -composition of rewriting steps. A polygraph is *terminating* if there is no infinite sequence of  $\star_0$ -composition of rewriting steps.

This polygraphic approach to shuffle operads generalizes that of Gröbner bases introduced by Dot-senko and Khoroshkin in [18]. Indeed, the orientation of the polygraphic rules does not depend on a given monomial order. However, termination is not ensured by a monomial order, so it must be proven by considering the rewriting rules themselves. Beyond the property of termination, the *confluence* property of a 1-polygraph  $X$  states that for every *branching* of two positive 1-cells  $f, g$  of  $X_1^{\text{III}}$  with the same source  $a$ , there exist two positive 1-cells  $h$  and  $k$  of  $X_1^{\text{III}}$  as in the following *confluence diagram*:



When the system is terminating, confluence can be deduced from *local confluence*, that is, when all the branchings of rewriting steps are confluent [28, 38]. Local confluence can be proven by the confluence of all branchings involving minimal overlappings of the rules, called *critical branchings*. This is the *critical branching theorem* proved in many algebraic contexts [16, 30, 39]. Coherent versions of this result were introduced in [22, 24]. Theorem 4.2.4 proves a coherent critical branching theorem for shuffle polygraphs. It states that we can extend a terminating, convergent, left-monomial shuffle 1-polygraph  $X$  into an *acyclic shuffle 2-polygraph* by considering *generating confluences* of critical branchings, which are minimal in the sense that they cannot be decomposed into smaller critical branchings.

In Section 4.3, we also give several algebraic interpretations of the confluence property of a terminating, left-monomial shuffle 1-polygraph  $X$ . Proposition 4.2.7 proves that the confluence of  $X$  is

equivalent to having a decomposition of the free shuffle operad  $X_0^{\text{III}}$  into a direct sum of the ideal generated by the 1-generators of  $X$  and the collection of normal forms with respect to this 1-generators. Proposition 4.3.2 proves that the notion of a convergent shuffle 1-polygraph, where the rules are oriented with respect to a given monomial order, is equivalent to the notion of Gröbner bases introduced in [18]. Proposition 4.3.4 gives a polygraphic interpretation of the Poincaré-Birkhoff-Witt (PBW) criterion introduced by Hoffbeck in [27] as a generalization of Priddy's PBW criterion for associative algebras [40].

### The overlapping polygraphic resolution and Koszulness

An  $\omega$ -polygraph  $X$  is *acyclic* if, for every dimension  $n \geq 1$ , the quotient of the free shuffle  $n$ -operad  $X_n^{\text{III}}$  by the ideal generated by the cellular extension  $X_{n+1}$  is aspherical, that is all parallel  $n$ -cells are equal. We say that an acyclic  $\omega$ -polygraph  $X$  is a *shuffle polygraphic resolution* of the shuffle operad it presents. Section 5 presents the main result of this article, Theorem 5.2.9, which extends a reduced convergent left-monomial shuffle 1-polygraph  $X$  into a polygraphic resolution, denoted by  $\mathcal{O}v(X)$ , of the presented shuffle operad. The generators of this polygraphic resolution correspond to higher-dimensional overlappings induced by the rewriting rules of  $X$ : an  $n$ -generator of  $\mathcal{O}v(X)$ , called an  *$n$ -overlapping*, is a sequence of monomials, written

$$u_0 \ll \vec{v}_1 \ll \vec{v}_2 \ll \cdots \ll \vec{v}_n,$$

where, when seen as planar trees, each sequence of monomials  $\vec{v}_i = (v_{i,1}, \dots, v_{i,k})$  is attached to the leaves of the  $(i-1)$ -overlapping  $u_0 \ll \vec{v}_1 \ll \cdots \ll \vec{v}_{i-1}$  in a manner that adds exactly enough to apply a new rewriting rule. Explicitly, for low dimensions, the 0-overlappings correspond to the 0-generators  $X_0$ , the 2-overlappings correspond to the sources of the rewriting rules of  $X_1$ , and the 2-overlappings correspond to minimal critical branchings of  $X$ .

The acyclicity of the shuffle  $\omega$ -polygraph  $\mathcal{O}v(X)$  is proven by the construction of a *homotopical contraction*, using results on contractions of polygraphs recalled from [24] in Section 5.1. Thus the shuffle polygraphic resolution  $\mathcal{O}v(X)$  generalizes Anick's resolution [2] and the linear polygraphic resolution [24] for associative associative algebras, and provides an alternative construction to the differential-graded shuffle operads constructed by Dotsenko and Khoroshkin [19].

It is known that the Quillen (co)homology of a symmetric operad can be computed from its associated shuffle operad: indeed, the reduced bar complex of a symmetric operad  $P$  is isomorphic, as a shuffle differential-graded cooperad, to the reduced bar complex of the associated shuffle operad  $P^u$ , that is,  $B^*(P)^u \simeq B^*(P^u)$ , [19, Prop. 1.4]. Proposition 3.3.12 proves a similar result in the polygraphic setting. We introduce the notion of a *symmetric polygraphic resolution* for a symmetric operad  $P$  and we establish isomorphisms between symmetric polygraphic resolutions of  $P$  and shuffle polygraphic resolutions of the associated shuffle operad  $P^u$ . Then, in Section 5.3, we associate to an acyclic shuffle  $\omega$ -polygraph  $X$  that presents a shuffle operad  $P$  a  $P$ -bimodule resolution of the *trivial  $P$ -bimodule*  $\Omega_P$ . Thus shuffle polygraphic resolutions provide a constructive way to compute the (co)homology of symmetric operads. As an application, we define in Section 5.4 a criterion of Koszulness in terms of quadratic convergence: Theorem 5.4.3 states that shuffle operads presented by quadratic convergent 1-polygraphs are Koszul. This result generalizes those obtained by Dotsenko and Khoroshkin in [18] for shuffle operads with quadratic Gröbner bases defined with respect to a given monomial order. This new rewriting-based

## 2. Higher-dimensional shuffle operads

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sufficient condition for Koszulness does not depend on a monomial order, which is required to define Gröbner bases.

### Conventions and notations

Throughout this article  $\mathbf{k}$  denotes a field of characteristic zero. All vector spaces are over this field  $\mathbf{k}$ , and we denote by  $\mathbf{Vect}$  the category of vector spaces and linear maps as morphisms. We denote by  $\mathbb{N}$  the set of nonnegative integers. We denote by  $\mathbf{Ord}$  the category of nonempty finite ordered sets, whose morphisms are order-preserving bijections. We denote by  $\mathbf{Fin}$  the category of nonempty finite sets, whose morphisms are bijections. We denote by  $\mathbf{k}[\mathfrak{S}]$  the graded vector space  $\bigoplus_{k=0}^{\infty} \mathbf{k}[\mathfrak{S}_k]$  whose component in degree  $k$  is provided by the free action of the symmetric group  $\mathfrak{S}_k$ . In abelian categories, we will use the matrix notation for morphisms between biproducts, given morphisms  $f_{ij} : A_i \rightarrow B_j$ , with  $1 \leq i \leq n$  and  $1 \leq j \leq p$ , we denote by

$$A_1 \oplus \cdots \oplus A_n \xrightarrow{\begin{bmatrix} f_{11} & \cdots & f_{1p} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{np} \end{bmatrix}} B_1 \oplus \cdots \oplus B_p$$

the induced morphism.

## 2. HIGHER-DIMENSIONAL SHUFFLE OPERADS

In this section we introduce the notion of a higher-dimensional shuffle operad. We first recall the structure of shuffle operads from [18] and we decompose the shuffle composition into elementary compositions. We then define the category of shuffle  $\omega$ -operads and characterize it as a certain subcategory of globular bimodules over shuffle operads.

### 2.1. Shuffle operads

In this preliminary subsection we recall from [18] the definitions on shuffle operads used in this article. We refer the reader to [34] or [11] for a complete account on symmetric and shuffle operads.

**2.1.1. Presheaves on finite sets.** In general, a presheaf  $X$  on  $\mathbf{Ord}$  or  $\mathbf{Fin}$  with values in a category  $\mathcal{C}$  are determined by the family of objects  $(X(k))_{k \geq 1}$ , where  $X(k) := X(\{1, \dots, k\})$ . We will adopt this notation for the following definitions.

An *set indexed by the category*  $\mathbf{Ord}$ , or *indexed set* for short, is a presheaf on  $\mathbf{Ord}$  with values in the category  $\mathbf{Set}$ . We denote by  $\mathbf{Ind}$  the category of indexed sets with natural transformations as morphisms. A *collection* is a presheaf on  $\mathbf{Ord}$  with values in the category  $\mathbf{Vect}$ . We denote by  $\mathbf{Coll}$  the category of collections and their natural transformations. A *basis* of  $V$  is a indexed set  $B = (B(k))_{k \geq 1}$  such that, for each  $k$ ,  $B(k)$  is a basis of the space  $V(k)$ .

A *symmetric collection* is a presheaf on  $\mathbf{Fin}$  with values in the category  $\mathbf{Vect}$ . We denote by  $\mathfrak{S}\mathbf{Coll}$  the category of symmetric collections and their natural transformations. The functor  $-^u : \mathbf{Ord} \rightarrow \mathbf{Fin}$  that forgets the order induces a functor

$$-^u : \mathfrak{S}\mathbf{Coll} \rightarrow \mathbf{Coll}. \tag{2.1.2}$$

In addition, we denote by  $\mathbf{k}\langle - \rangle : \text{Ind} \rightarrow \text{Coll}$  the left adjoint functor of the forgetful functor  $\text{Coll} \rightarrow \text{Ind}$ .

**2.1.3. Operads ([18]).** The categories  $\text{Coll}$  and  $\mathfrak{S}\text{Coll}$  are monoidal with the following products:

i) the *shuffle composition* on  $\text{Coll}$  denoted by  $\circ_{\text{III}}$ , and defined for  $V, W \in \text{Coll}$  by

$$(V \circ_{\text{III}} W)(I) := \bigoplus_{k=0}^{\infty} V(k) \otimes \left( \bigoplus_{\substack{f:I \twoheadrightarrow \{1, \dots, k\} \\ (*)}} W(f^{-1}\{1\}) \otimes \dots \otimes W(f^{-1}\{k\}) \right),$$

where  $I \in \text{Ord}^{\circ}$  and the sum (\*) is taken on *shuffle surjections*, i.e., surjections  $f : I \twoheadrightarrow \{1, \dots, k\}$  such that

$$\min f^{-1}\{1\} < \dots < \min f^{-1}\{k\}.$$

ii) the *symmetric composition* on  $\mathfrak{S}\text{Coll}$ , denoted by  $\circ_{\mathfrak{S}}$ , is defined for  $V, W \in \mathfrak{S}\text{Coll}$  by

$$(V \circ_{\mathfrak{S}} W)(I) := \bigoplus_{k=0}^{\infty} V(k) \otimes_{\mathbf{k}[\mathfrak{S}_k]} \left( \bigoplus_{f:I \twoheadrightarrow \{1, \dots, k\}} W(f^{-1}\{1\}) \otimes \dots \otimes W(f^{-1}\{k\}) \right),$$

where  $I \in \text{Fin}^{\circ}$ , and the sum is taken on all surjections.

In both case, the unit is the collection  $\mathbb{I}$  concentrated in arity 1 with  $\mathbb{I}(1) = \mathbf{k}$ .

A *shuffle* (resp. *symmetric*) *operad* is an internal monoid  $(P, \mu_P, \eta_P)$  in  $(\text{Coll}, \circ_{\text{III}}, \mathbb{I})$  (resp.  $(\mathfrak{S}\text{Coll}, \circ_{\mathfrak{S}}, \mathbb{I})$ ), where  $\mu_P$  is the multiplication morphism and  $\eta_P$  is the unit morphism. We denote respectively by  $\mathfrak{S}\text{Op}$  and  $\text{III}\text{Op}$  the category of symmetric operads and shuffle operads and their morphisms.

The free operad functors  $-^{*\text{III}} : \text{Coll} \rightarrow \text{III}\text{Op}$  and  $-^{*\mathfrak{S}} : \mathfrak{S}\text{Coll} \rightarrow \mathfrak{S}\text{Op}$  are defined using the free monoid functor on left distributive categories as detailed in [7, Appendix B]. For an indexed set  $X$ , we denote by  $X^{\text{III}}$  the free shuffle operad on  $X$  given by the composite of free functor

$$\text{Ind} \xrightarrow{\mathbf{k}} \text{Coll} \xrightarrow{-^{*\text{III}}} \text{III}\text{Op}. \quad (2.1.4)$$

Recall from [18], see also [11], that the forgetful functor  $-^u$  is monoidal in the sense that for all symmetric collections  $V, W$ , we have  $(V \circ_{\mathfrak{S}} W)^u = V^u \circ_{\text{III}} W^u$  in  $\text{Coll}$ , and in particular that it commutes with free operad functors  $-^{*\mathfrak{S}}$  and  $-^{*\text{III}}$ , in the sense that for every symmetric collection  $V$ , we have the isomorphism

$$(V^{*\mathfrak{S}})^u = (V^u)^{*\text{III}}. \quad (2.1.5)$$

**2.1.6. Shuffle composition on indexed sets.** We define a monoidal *shuffle composition* on  $\text{Ind}$ , also denoted by  $\times_{\text{III}}$ , by setting, for indexed sets  $X, Y$

$$(X \times_{\text{III}} Y)(I) := \prod_{k=0}^{\infty} X(k) \times \left( \prod_{\substack{f:I \twoheadrightarrow \{1, \dots, k\} \\ (*)}} Y(f^{-1}\{1\}) \times \dots \times Y(f^{-1}\{k\}) \right)$$



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where the coproduct ( $*$ ) is taken on shuffle surjections. The composition  $\times_{\text{III}}$  has for unit the indexed set concentrated in arity 1, denoted by  $\mathbf{1}$ , and such that  $\mathbf{1}(1)$  is a singleton, whose only element is denoted by  $\varepsilon$ . The functor  $\mathbf{k}$  is compatible with product and coproduct, hence the following diagram commutes:

$$\begin{array}{ccc} \text{Ind} \times \text{Ind} & \xrightarrow{\mathbf{k} \times \mathbf{k}} & \text{Coll} \times \text{Coll} \\ \times_{\text{III}} \downarrow & & \downarrow \circ_{\text{III}} \\ \text{Ind} & \xrightarrow{\mathbf{k}} & \text{Coll} \end{array} \quad (2.1.7)$$

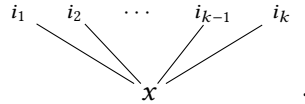
Note that the adjunction between the monoidal categories  $(\text{Set}, \times, \{*\})$  and  $(\text{Vect}, \otimes, \mathbf{k})$  is compatible with the canonical isomorphisms of units, associativity, and distributivity, so the induced functors between  $(\text{Ind}, \times_{\text{III}}, \mathbb{I})$  and  $(\text{Coll}, \circ_{\text{III}}, \mathbb{I})$  make a lax monoidal adjunction.

**2.1.8. Tree monomials.** The shuffle composition is monoidal, and we denote by  $\text{IIITree}$  the category of internal monoids in  $\text{Ind}$  with respect to this composition. The functor  $\mathbf{k}$  preserves colimits as a left adjoint, and sends  $\times_{\text{III}}$  to  $\circ_{\text{III}}$  as a consequence of commutativity of (2.1.7). Free internal monoids in  $\text{Ind}$  and  $\text{Coll}$  are constructed by colimits and shuffle composition, thus the linearization functor  $\mathbf{k}$  induces a linearization functor  $\text{IIITree} \rightarrow \text{IIIOP}$  such that the following square commutes:

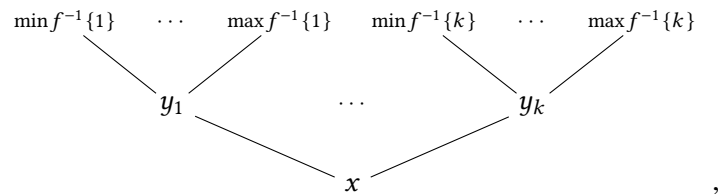
$$\begin{array}{ccc} \text{Ind} & \xrightarrow{\mathbf{k}} & \text{Coll} \\ -*_{\text{III}} \downarrow & & \downarrow -*_{\text{III}} \\ \text{IIITree} & \xrightarrow{\mathbf{k}} & \text{IIIOP} \end{array}$$

For an indexed set  $X$ , the elements of the free internal monoid  $X^{*\text{III}}$  in  $\text{IIITree}$  are called *tree monomials on  $X$* . We have  $X^{*\text{III}} = (X_k^{*\text{III}})_{k \geq 0}$ , and elements of  $X_k^{*\text{III}}$  are said to be of *arity  $k$* . In particular the unit in  $X^{*\text{III}}$  corresponding to the indexed set  $\mathbf{1}$  is called the *trivial tree monomial*. Elements of the free operad  $X^{\text{III}}$  are linear combination of tree monomials having a same given arity  $k$ , and called *terms on  $X$  of arity  $k$* .

**2.1.9. Graphical representation of tree monomials.** The elements of an indexed set can be represented graphically as (planar rooted) trees. For an indexed set  $X$  and an ordered set  $I = \{i_1 < i_2 < \dots < i_k\}$ , an element  $x \in X(I)$  is depicted by a *corolla*, that is, a planar tree with only one vertex:

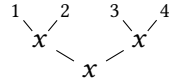


For indexed sets  $X, Y$ , elements of  $(X \times_{\text{III}} Y)(I)$  have the form



where  $k \geq 1$ ,  $f : I \rightarrow \{1, \dots, k\}$  is a shuffle surjection,  $x \in X(k)$  and  $y_i \in Y(f^{-1}\{i\})$  for all  $i \in \{1, \dots, k\}$ . In this way, a tree monomial  $u$  on  $X$  can be represented by a planar tree  $T(u)$ , whose vertices are elements of  $X$ , and its arity is the number of its leaves. More generally, for  $V, W$  two collections,  $V \circ W$  has a basis of tree monomials.

The *weight* of a tree monomial  $u$  on an indexed set  $X$  is the number of vertices of  $T(u)$ . A tree monomial  $v$  is a (resp. *proper*, resp. *rooted*) *submonomial* of  $u$  if  $T(v)$  is a (resp. proper, resp. rooted) subtree of  $T(u)$ . When listing submonomials of a tree monomial  $u$ , we distinguish the different occurrences of a subtree of  $T(u)$ : for instance, the tree monomial



contains three distinct occurrences of the proper submonomial  $\begin{matrix} 1 \\ \diagdown \\ x \\ \diagup \\ 2 \end{matrix}$ , one of which is rooted.

**2.1.10. Inline notation for tree monomials.** For  $n_1, \dots, n_k \geq 1$ , a *shuffle surjection of type*  $(n_1, \dots, n_k)$  is a surjection  $f : \{1, \dots, n_1 + \dots + n_k\} \rightarrow \{1, \dots, k\}$  such that, for all  $i \in \{1, \dots, k\}$ ,  $|f^{-1}\{i\}| = n_i$ , and

$$\min f^{-1}\{1\} < \dots < \min f^{-1}\{k\}.$$

Denote by  $S(n_1, \dots, n_k)$  the set of shuffle surjections of type  $(n_1, \dots, n_k)$ .

Let  $X$  be an indexed set. The *inline notation* for the indexed set of tree monomials  $X^{*\text{III}}$  is the term algebra in indexed sets given by the *Backus-Naur form*

$$X^{*\text{III}} ::= \mathbf{1} \mid (X(k) \mid_f X^{*\text{III}}(n_1) \cdots X^{*\text{III}}(n_k)),$$

where  $\mathbf{1}$  is the indexed set defined in 2.1.6, and  $f$  is a shuffle surjection of type  $(n_1, \dots, n_k)$ . When possible, we omit subscript  $f$ , and we write

$$(u \mid \vec{v}) := (u \mid v_1 \cdots v_k).$$

We will use also the notation  $\vec{v}$  for the list of tree monomials  $v_1, \dots, v_k$ . Finally, note that for two indexed sets  $X, Y$  the indexed set  $X \times_{\text{III}} Y$  can be written in an explicit way as, for  $n \geq 1$ ,

$$(X \times_{\text{III}} Y)(n) = \left\{ (x \mid_f y_1 \cdots y_k) \left| \begin{array}{l} n_1, \dots, n_k \geq 1, n_1 + \dots + n_k = n, \\ x \in X(k), y_1 \in Y(n_1), \dots, y_k \in Y(n_k), \\ f \in S(n_1, \dots, n_k) \end{array} \right. \right\}.$$

**2.1.11. Explicit associativity of shuffle composition.** Let  $u, \vec{v}, \vec{v}'$  be monomials, and  $f, f'$  shuffle surjections such that  $((u \mid_f \vec{v}) \mid_{f'} \vec{v}')$  is defined. Let  $k$  be the arity of  $u$ ,  $n_i$  the arity of  $v_i$ , and  $p_j$  the arity of  $v'_j$ . For  $i \in \{1, \dots, k\}$ , the restriction of  $f'$  to  $f'^{-1}(f^{-1}\{i\})$  is a shuffle surjection of domain  $f^{-1}\{i\}$ . We introduce the notation  $(\vec{v} \mid_{f, f'} \vec{v}')$  for the list of monomials  $\{(v_i \mid_{f'^{-1}(f^{-1}\{i\})} v'_{\min f^{-1}\{i\}} \cdots v'_{\max f^{-1}\{i\}})\}_{i \in \{1, \dots, k\}}$ . Then

$$((u \mid_f \vec{v}) \mid_{f'} \vec{v}') = (u \mid_{ff'} (\vec{v} \mid_{f, f'} \vec{v}')),$$

where  $ff'$  is the composition of shuffle surjections.

## 2. Higher-dimensional shuffle operads

**2.1.12. Bimodules and ideals.** Recall that a  $P$ -bimodule over a shuffle operad  $P$ , called a *linear module over  $P$*  in [7, Def. 2.13], is a collection  $A$  equipped with two families of morphisms of collections

$$\begin{aligned}\lambda &: P(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes A(f^{-1}\{i\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \rightarrow A(I), \\ \rho &: A(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \rightarrow A(I),\end{aligned}$$

for all shuffling surjections  $f : I \twoheadrightarrow \{1, \dots, k\}$ , defining a *left crossed action* and a *right action* respectively, satisfying compatibility axioms with each other, and associativity and unit axioms with the product of  $P$ . A morphism of  $P$ -modules is a morphism of collections compatible with the left and right actions. We denote by  $\text{Bimod}(P)$  the category of  $P$ -bimodules and morphisms of  $P$ -bimodules, and by  $\text{Bimod}(\text{III Op}) := \coprod_{P \in \text{III Op}} \text{Bimod}(P)$  the category of pairs  $(P, A)$  composed of an operad  $P$  and a  $P$ -bimodule  $A$ . We denote by  $\mathcal{L}^P : \text{Coll} \rightarrow \text{Bimod}(P)$  the *free bimodule functor* defined in [7, Prop. 2.11], see also [35], and given for every  $V \in \text{Coll}$  and  $I \in \text{Ord}$  by

$$\mathcal{L}^P(V)(I) := \bigoplus_{k \geq 1} P(k) \otimes \left( \bigoplus_{\substack{f: I \twoheadrightarrow \{1, \dots, k\} \\ 1 \leq i \leq k}} P(f^{-1}\{1\}) \otimes \cdots \otimes (A \circ_{\text{III Op}} P)(f^{-1}\{i\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \right)$$

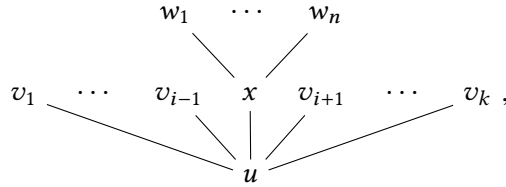
For an indexed set  $X$ , we denote by

$$P\langle X \rangle := \mathcal{L}^P(\mathbf{k}X)$$

the free  $P$ -bimodule on  $X$ .

An *ideal* of an operad  $P$  is a  $P$ -bimodule  $\mathcal{I}$  equipped with an inclusion of  $P$ -bimodules  $\mathcal{I} \hookrightarrow P$ .

**2.1.13. Graphical description of bimodules.** Let  $X$  be an indexed set and  $P$  a shuffle operad. The free  $P$ -bimodule  $P\langle X \rangle$  is the collection generated by tree monomials of the form



where  $n, k \geq 1$ ,  $x \in X(n)$ ,  $i \in \{1, \dots, k\}$ ,  $u, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k, w_1, \dots, w_n \in P$ , and the inputs are omitted.

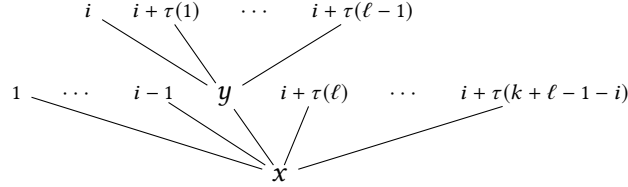
## 2.2. Compositions in shuffle operads

In this subsection, we decompose shuffle composition into partial compositions, and we introduce notations for composition and terms in an operad.

**2.2.1. Shuffle partial composition.** Recall from [18, Prop. 2], that for  $k, \ell \geq 1$ , a *shuffle permutation of type  $(k, \ell)$*  is a permutation  $\tau \in \mathfrak{S}_{k+\ell}$  such that

$$\tau(1) < \cdots < \tau(k), \quad \text{and} \quad \tau(k+1) < \cdots < \tau(k+\ell).$$

Denote by  $\text{III}(k, \ell)$  the set of shuffle permutations of type  $(k, \ell)$ . Given indexed sets  $X, Y$ , and  $x \in X(k)$ ,  $y \in Y(\ell)$ , for  $i \in \{1, \dots, k\}$  and  $\tau \in \text{III}(\ell - 1, k - i)$ , we define the *elementary composition*  $x \circ_{i, \tau} y$  as the following tree



Elementary compositions are extended to collections by linearity and bidistributivity. We denote by  $V(k) \circ_{i, \tau} W(\ell)$  the collection composed by elementary compositions of the form  $v \circ_{i, \tau} w$ , for  $v \in V(k)$  and  $w \in W(\ell)$ . Then the *shuffle partial composition* of collections  $V, W$  is defined by

$$(V \circ_{\text{III}} W)(n) := \bigoplus_{\substack{k, \ell, i \geq 1 \\ k + \ell - 1 = n \\ \tau \in \text{III}(\ell - 1, k - i)}} V(k) \circ_{i, \tau} W(\ell).$$

**2.2.2. Properties of partial compositions.** Note that there are isomorphisms

$$(V \circ_{\text{III}} W)(n) \stackrel{(i)}{\cong} \bigoplus_{k=1}^{\infty} V(k) \otimes \left( \bigoplus_{\substack{i, \ell \geq 1 \\ k + \ell - 1 = n \\ \tau \in \text{III}(\ell - 1, k - i)}} \mathbb{I}(\{1\}) \otimes \dots \otimes \mathbb{I}(\{i - 1\}) \otimes W(\{i, i + \tau(1), \dots, i + \tau(\ell - 1)\}) \right. \\ \left. \otimes \mathbb{I}(\{i + \tau(\ell)\}) \otimes \dots \otimes \mathbb{I}(\{i + \tau(k + \ell - 1 - i)\}) \right) \\ \stackrel{(ii)}{\cong} \bigoplus_{\substack{k, \ell, i \geq 1 \\ k + \ell - 1 = n \\ \tau \in \text{III}(\ell - 1, k - i)}} V(k) \otimes W(\ell).$$

The isomorphism (i) implies that there is an injection of collections

$$V \circ_{\text{III}} W \hookrightarrow V \circ_{\text{III}} (\mathbb{I} \oplus W).$$

The isomorphism (ii) implies that partial composition is a bidistributive bifunctor  $\circ_{\text{III}} : \text{Coll} \times \text{Coll} \rightarrow \text{Coll}$ . The partial composition  $\circ_{\text{III}}$  is not associative. However, if there is no possible confusion, we will use the left bracket rule, that is,  $U \circ V \circ W := (U \circ V) \circ W$ .

**2.2.3. Decomposition of shuffle compositions.** Let  $A$  be a collection equipped with a morphism  $\eta_A : \mathbb{I} \rightarrow A$ , that is,  $\eta_A$  is an object of  $\mathbb{I}/\text{Coll}$ . We can express the shuffle composition  $A \circ_{\text{III}} A$  in terms of partial compositions. There exists a natural transformation  $\varphi$  from the functor  $A \mapsto \bigoplus A \circ_{\text{III}} A^{\circ_{\text{III}} p}$  to the functor  $A \mapsto A \circ_{\text{III}} A$  defined as follows:

$$\varphi_A : \bigoplus_{p=1}^{\infty} A \circ_{\text{III}} A^{\circ_{\text{III}} p} \rightarrow \bigoplus_{p=1}^{\infty} A \circ_{\text{III}} (\mathbb{I} \oplus A)^{\circ_{\text{III}} p} \xrightarrow{\sum 1^{\circ_{\text{III}}(\eta_A + 1)^{\circ_{\text{III}} p}}} A \circ_{\text{III}} A.$$



### 2.3. Higher-dimensional shuffle operads

In this subsection, we introduce the structure of (strict) shuffle  $\omega$ -operads.

**2.3.1. Globular objects.** We denote by  $\mathbb{R}\mathbb{O}$  the *reflexive globe category*, whose objects are natural numbers, denoted by  $\underline{n}$ , for  $n \in \mathbb{N}$ , and morphisms are generated by

$$\sigma_n : \underline{n} \rightarrow \underline{n+1}, \quad \tau_n : \underline{n} \rightarrow \underline{n+1}, \quad \iota_{n+1} : \underline{n+1} \rightarrow \underline{n},$$

for all  $n$  in  $\mathbb{N}$ , and submitted to the following *globular* and *identities relations*:

$$\begin{aligned} \sigma_{n+1} \circ \sigma_n &= \tau_{n+1} \circ \sigma_n, & \sigma_{n+1} \circ \tau_n &= \tau_{n+1} \circ \tau_n, \\ \iota_n \circ \sigma_n &= id_{\underline{n}}, & \iota_n \circ \tau_n &= id_{\underline{n}}, \end{aligned}$$

for all  $n$  in  $\mathbb{N}$ . Omitting the identity maps  $\iota_n$  gives the definition of the *globe category*  $\mathbb{O}$ . We denote by  $\mathbb{R}\mathbb{O}_n$  (resp.  $\mathbb{O}_n$ ) the full subcategory of  $\mathbb{R}\mathbb{O}$  (resp.  $\mathbb{O}$ ) whose objects are  $\underline{0}, \underline{1}, \dots, \underline{n}$ .

A *reflexive globular object* in a category  $C$  is a functor  $\mathbb{R}\mathbb{O}^{op} \rightarrow C$ , whose restriction to the category  $\mathbb{R}\mathbb{O}_n^{op}$  is called a *reflexive  $n$ -globular object*. Explicitly, is given by a sequence  $A = (A_k)_{k \in \mathbb{N}}$  of objects of  $C$ , equipped with indexed morphisms

$$s = (s_k : A_{k+1} \rightarrow A_k)_{k \in \mathbb{N}}, \quad t = (t_k : A_{k+1} \rightarrow A_k)_{k \in \mathbb{N}}, \quad i = (i_k : A_{k-1} \rightarrow A_k)_{k \in \mathbb{N}},$$

of degree  $-1, -1$  and  $1$  respectively, and satisfying the following globular and identities relations

$$s^2 = st, \quad t^2 = ts, \quad si = id_A, \quad ti = id_A. \quad (2.3.2)$$

The elements of  $A_k$  are called  $k$ -cells of  $A$ . A *morphism of reflexive globular objects* is an indexed morphism of degree 0 that commutes with morphisms  $s, t$  and  $i$ . We denote by  $\text{RGlob}(C)$  (resp.  $\text{Glob}(C)$ ) the category of *reflexive globular objects* (resp. *globular objects*) in  $C$  and their morphisms. We denote by  $\text{RGlob}_n(C)$  (resp.  $\text{Glob}_n(C)$ ) the full subcategory of  $\text{RGlob}(\text{III}\mathbb{O}p)$  of reflexive  $n$ -globular objects (resp.  $n$ -globular objects) in  $C$ . We will denote by

$$\mathcal{V}_n(C) : \text{Glob}_{n+1}(C) \rightarrow \text{Glob}_n(C)$$

the functor that forgets the  $(n+1)$ -cells. For  $A$  a globular object and  $\ell \geq k \geq 0$ , the  $(\ell, k)$ -source and  $(\ell, k)$ -target morphisms

$$s_k^\ell : A_\ell \rightarrow A_k, \quad t_k^\ell : A_\ell \rightarrow A_k,$$

are respectively defined as the following iterated composition of source and target morphisms:

$$s_k^\ell := s_k \circ \dots \circ s_{\ell-2} \circ s_{\ell-1}, \quad t_k^\ell := t_k \circ \dots \circ t_{\ell-2} \circ t_{\ell-1}.$$

We denote by  $i_\ell^k : A_k \rightarrow A_\ell$  the iterated identity  $i_\ell^k = i_\ell \circ i_{\ell-1} \dots \circ i_{k+1}$ . When there is no ambiguity, we will write  $s_k$  and  $t_k$  for source and target maps respectively, and we will omit  $i_\ell^k$  entirely, since  $i_\ell^k$  is injective by (2.3.2). For  $k \geq 0$ , we denote by  $A \star_k A$  the following pullback of globular operads

$$\begin{array}{ccc} A \star_k A & \xrightarrow{\pi_1} & A \\ \pi_2 \downarrow & \lrcorner & \downarrow s_k \\ A & \xrightarrow{t_k} & A_k \end{array}$$

Let  $A$  be a globular object of some category  $C$ . For  $n \geq 1$ , two  $n$ -cells  $a, b$  of  $A$  are *parallel* if  $s(a) = s(b)$  and  $t(a) = t(b)$ . An  $n$ -sphere of  $A$  is a pair  $(a, b)$  of parallel  $n$ -cells.

## 2. Higher-dimensional shuffle operads

**2.3.3. Higher-dimensional operads.** Recall that, for  $n \geq 0$ , an (internal strict)  $n$ -category in  $C$  is a

i) reflexive  $n$ -globular object, that is a diagram in  $C$  of the form

$$A_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{i_1} \\ \xleftarrow{t_0} \end{array} A_1 \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{i_2} \\ \xleftarrow{t_1} \end{array} \cdots \begin{array}{c} \xleftarrow{s_{n-2}} \\ \xrightarrow{i_{n-1}} \\ \xleftarrow{t_{n-2}} \end{array} A_{n-1} \begin{array}{c} \xleftarrow{s_{n-1}} \\ \xrightarrow{i_n} \\ \xleftarrow{t_{n-1}} \end{array} A_n$$

whose morphisms satisfy globular and identity relations (2.3.2),

ii) equipped with a structure of category in  $C$  on

$$A_k \begin{array}{c} \xleftarrow{s_k} \\ \xrightarrow{t_k} \end{array} A_\ell$$

for all  $k < \ell$ , whose  $\star_k$ -composition morphism of  $\ell$ -cells is denoted by  $\star_k^\ell : A_\ell \star_k A_\ell \rightarrow A_\ell$ ,

iii) such that the 2-globular object

$$A_j \begin{array}{c} \xleftarrow{s_j} \\ \xrightarrow{t_j} \end{array} A_k \begin{array}{c} \xleftarrow{s_k} \\ \xrightarrow{t_k} \end{array} A_\ell$$

is a 2-category in  $C$  for all  $j < k < \ell$ .

We denote by  $n\text{Cat}(C)$  the category of  $n$ -categories in  $C$  and their  $n$ -functor. The category  $\omega\text{Cat}(C)$  of  $\omega$ -categories in  $C$  is the limit of

$$0\text{Cat}(C) \leftarrow 1\text{Cat}(C) \leftarrow \cdots \leftarrow n\text{Cat}(C) \leftarrow \cdots$$

where each arrow forgets the cells of highest dimension.

For  $n \in \mathbb{N} \cup \{\omega\}$ , a *shuffle* (resp. *symmetric*)  $n$ -operad is an  $n$ -category in  $\text{III}\text{Op}$  (resp.  $\mathfrak{S}\text{Op}$ ). We denote by  $\text{III}\text{Op}_n$  (resp.  $\mathfrak{S}\text{Op}_n$ ) the corresponding category with internal  $n$ -functors as morphisms. We denote by  $\mathcal{U}_n^{\text{III}} : \text{III}\text{Op}_n \rightarrow \text{Glob}_n(\text{Ind})$  (resp.  $\mathcal{U}_n^{\mathfrak{S}} : \mathfrak{S}\text{Op}_n \rightarrow \text{Glob}_n(\mathfrak{S}\text{Coll})$ ) the forgetful functor that forgets the operadic structure.

Note that, in an  $\omega$ -shuffle operad  $P$ , the composition  $\star_\ell^k : P_k \star_\ell P_k \rightarrow P_k$  is a morphism of shuffle operads. As a consequence, the composition satisfies the following *exchange relation between  $\circ_{\text{III}}$  and  $\star_k$*

$$\begin{array}{ccc} (P_k \star_\ell P_k) \circ_{\text{III}} (P_k \star_\ell P_k) & \xrightarrow{(\pi_1 \circ_{\text{III}} \pi_1) \star_\ell (\pi_2 \circ_{\text{III}} \pi_2)} & (P_k \circ_{\text{III}} P_k) \star_\ell (P_k \circ_{\text{III}} P_k) \\ & \searrow & \swarrow \\ & \mu_k((1 \star_\ell 1) \circ_{\text{III}} (1 \star_\ell 1)) & \mu_k(1 \circ_{\text{III}} 1) \star_\ell \mu_k(1 \circ_{\text{III}} 1) \\ & & P_n \end{array} \quad (2.3.4)$$

**2.3.5. Lemma.** *Let  $P$  be a shuffle (resp. symmetric)  $\omega$ -operad. For every  $0 \leq k < n$ , and  $\star_k$ -composable pair  $(a, b)$  of  $P_n$ , we have*

$$a \star_k b = a - t_k(a) + b.$$

## 2.4. Higher-dimensional operads as globular bimodules

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*Proof.* The  $\star_k$ -composition being linear, by the neutrality axiom and the fact that  $t_k(a) = s_k(b)$ , we have

$$\begin{aligned} a \star_k b &= (a - t_k(a) + t_k(a)) \star_k (s_k(b) - s_k(b) + b), \\ &= a \star_k t_k(a) - s_k(b) \star_k t_k(a) + s_k(b) \star_k b, \\ &= a - t_k(a) + b. \end{aligned}$$

□

As for associative  $\omega$ -algebras, [24, Prop. 1.2.3], this lemma implies that, for  $P$  an  $\omega$ -operad and  $n \geq 1$ , every  $n$ -cell  $a$  of  $P$  is invertible with inverse  $a^- := s_{n-1}(a) - a + t_{n-1}(a)$ . We then deduce the following proposition:

**2.3.6. Proposition.** *The category  $\text{IIIOp}_\omega$  (resp.  $\text{Op}_\omega$ ) is isomorphic to the category  $\text{Gpd}_\omega(\text{IIIOp})$  (resp.  $\text{Gpd}_\omega(\text{Op})$ ) of internal  $\omega$ -groupoids in  $\text{IIIOp}$  (resp.  $\text{Op}$ ).*

Finally, we give some categorical properties of the categories of shuffle operads.

**2.3.7. Proposition.** *The forgetful functor  $\text{IIIOp} \rightarrow \text{Coll}$  (resp.  $\text{Op} \rightarrow \text{Coll}$ ) reflects all limits, filtered colimits, and reflexive coequalizers.*

*Proof.* The statement for the functor  $\text{Op} \rightarrow \text{Coll}$  is proven in [21, Prop. 1.2.4], where the limits, filtered colimits, and reflexive coequalizers of symmetric operads are equipped with unique monoidal structures. In particular, it is shown that the monoidal product  $\circ_\varepsilon$  in  $\text{Coll}$  preserves all limits, filtered colimits and reflexive coequalizers. This comes from the fact that the tensor product of  $\text{Vect}$  preserves these limits and colimits. Given the similarities between the monoidal products  $\circ_\varepsilon$  and  $\circ_{\text{III}}$ , the same arguments apply to the monoidal structure of  $\text{Coll}$ , and so we conclude that the functor  $\text{IIIOp} \rightarrow \text{Coll}$  preserves limits, filtered colimits, and reflexive coequalizers. □

**2.3.8. Proposition.** *The category  $\text{IIIOp}_\omega$  is locally finitely presentable. In particular, it is complete and cocomplete.*

*Proof.* Let us first show that  $\text{IIIOp}$  is locally finitely presentable by viewing it as the category of algebras over an accessible monad. Recall that there exists an adjunction  $\text{Coll} \dashv \text{IIIOp}$  where the left adjoint is  $-^{\text{III}} : \text{Coll} \rightarrow \text{IIIOp}$ . Therefore,  $\text{IIIOp}$  is the category of algebras of the monad of free shuffle operads  $T : \text{Coll} \rightarrow \text{Coll}$ . By the Proposition 2.3.7, the forgetful functor  $\text{IIIOp} \rightarrow \text{Coll}$  preserves filtered colimits, i.e. it is finitary, making the monad  $T$  finitary. Moreover, the category  $\text{Vect}$  of vector spaces is locally finitely presentable, and the category  $\text{Ord}$  is a small category, so  $\text{Coll}$  is also locally finitely presentable. Thus  $T$  is an accessible monad on a locally finitely presentable category. Following [1, § 2.78] the category of  $T$ -algebras  $\text{IIIOp}$  is locally finitely presentable.

The category  $\text{IIIOp}_\omega$  of  $\omega$ -categories internal in  $\text{IIIOp}$  is the category of models of a finite limit sketch [3], in the locally finitely presentable category  $\text{IIIOp}$ . By [1, Prop. 1.53], we conclude that  $\text{IIIOp}_\omega$  is also locally finitely presentable. □

## 2.4. Higher-dimensional operads as globular bimodules

In Theorem 2.4.8, we show that the axioms of the category  $\text{IIIOp}_\omega$  of shuffle  $\omega$ -operads are redundant by proving that it is isomorphic to a category with less axioms thanks to the linear exchange relation. Throughout this section, the compositions  $\circ$  and  $\odot$  designate  $\circ_{\text{III}}$  and  $\odot_{\text{III}}$ .



## 2. Higher-dimensional shuffle operads

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**2.4.1. Partial multiplication.** For  $(P, \mu, \eta)$  a shuffle operad, denote by  $\iota_P$  the morphism

$$\iota_P : P \odot P \hookrightarrow P \circ (\mathbb{I} \oplus P) \xrightarrow{1 \circ (\eta \oplus 1)} P \circ P.$$

We equip the operad  $P$  with a morphism called *partial multiplication*

$$\mu^\odot : P \odot P \xrightarrow{\iota_P} P \circ P \xrightarrow{\mu} P.$$

As a consequence, we have the equality of morphisms

$$\mu\varphi_P = \sum_p (\mu^\odot)^{\odot p} : \bigoplus_{p=1}^{\infty} P \odot P^{\odot p} \rightarrow P.$$

**2.4.2. Linear exchange relation.** Let  $(P_n, \mu_n, \eta_n)_{n \geq 0}$  be an  $\omega$ -operad. By the exchange relation between compositions  $\circ$  and  $\star_0$ , we observe that, for every  $n \geq 1$ ,

$$\begin{aligned} \mu_n^\odot &= \mu_n((1 \star_0 t_0) \circ (s_0 \star_0 1))\iota_{P_n} \\ &= (\mu_n(1 \circ s_0) \star_0 \mu_n(t_0 \circ 1))\iota_{P_n} && (2.3.4) \\ &= (\mu_n(1 \circ s_0) + \mu_n(t_0 \circ 1) - \mu_n(t_0 \circ s_0))\iota_{P_n} && (\text{Lemma 2.3.5}) \\ &= \mu_n^\odot(1 \odot s_0) + \mu_n^\odot(t_0 \odot 1) - \mu_n^\odot(t_0 \odot s_0). \end{aligned}$$

Similarly, we calculate

$$\mu_n^\odot = \mu_n^\odot(s_0 \odot 1) + \mu_n^\odot(1 \odot t_0) - \mu_n^\odot(s_0 \odot t_0).$$

Regarding  $P_n$  as a  $P_0$ -bimodule, these equations still hold, although we need to introduce new notations for the partial left and right actions of  $P_0$  on  $P_n$ . This motivates the following definitions.

Let  $(P, \mu, \eta)$  be a shuffle operad and  $(A, \lambda, \rho)$  be a  $P$ -bimodule such that  $(P, A)$  is a reflexive 1-globular  $P$ -bimodule. More explicitly, there are morphisms  $s, t : A \rightarrow P$  and  $i : P \rightarrow A$ . We equip  $A$  with morphisms called *partial actions*

$$\begin{aligned} \lambda^\odot : P \odot A &\xrightarrow{\varphi} P \circ (\mathbb{I} \oplus A) \xrightarrow{1 \circ (\eta \oplus 1)} P \circ (P \oplus A) \xrightarrow{\lambda} A, \\ \rho^\odot : A \odot P &\xrightarrow{\varphi} A \circ (\mathbb{I} \oplus P) \xrightarrow{1 \circ (\eta \oplus 1)} A \circ P \xrightarrow{\rho} A. \end{aligned}$$

We also define the morphisms

$$\begin{aligned} \mu_A^\uparrow &:= \rho^\odot(1 \odot s) + \lambda^\odot(t \odot 1) - i\mu^\odot(t \odot s), \\ \mu_A^\downarrow &:= \lambda^\odot(s \odot 1) + \rho^\odot(1 \odot t) - i\mu^\odot(s \odot t) \end{aligned}$$

and the multiplication

$$\mu_A : A \circ A \xrightarrow{\sigma_A} \bigoplus_p A \odot A^{\odot p} \xrightarrow{\sum (\mu_A^\uparrow)^p} A.$$

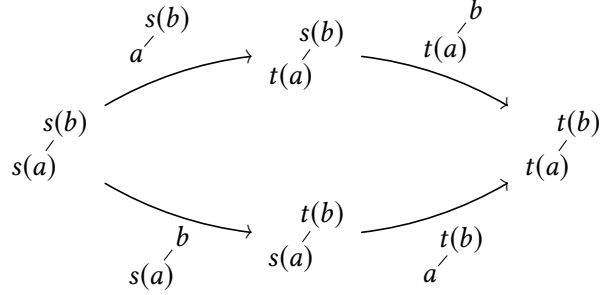
We say that a reflexive 1-globular  $P$ -bimodule  $(P, A)$  satisfies the *linear exchange relation* if the following relation holds:

$$\mu_A^\uparrow = \mu_A^\downarrow. \quad (2.4.3)$$

## 2.4. Higher-dimensional operads as globular bimodules

**2.4.4. Interpretation of morphisms  $\mu_A^\uparrow$  and  $\mu_A^\downarrow$ .** If  $(P, A)$  is a reflexive 1-globular  $P$ -bimodule, we can interpret the elements of  $A$  as rewriting rules that relate elements of  $P$ : an element  $a \in A$  rewrites  $s(a)$  as  $t(a)$ , which we denote by  $a : s(a) \rightarrow t(a)$ . Via the injection  $i$ , an element  $x$  of  $P$  can also be seen as a trivial rewriting rule  $i(x) : x \rightarrow x$ .

Let  $a, b \in A$ . For every compatible elementary composition  $\circ_{i,\tau}$ , we would like to interpret the composition  $a \circ_{i,\tau} b$  as a pair of orthogonal reductions:



where  $\begin{matrix} s(b) \\ \diagdown \\ s(a) \end{matrix}$  is a graphical representation of  $s(a) \circ_{i,\tau} s(b)$ , and so on. While  $a \circ_{i,\tau} b$  is not necessarily an element of  $A$ , we find that

$$\begin{aligned} \mu_A^\uparrow \left( \begin{matrix} s(b) \\ \diagdown \\ s(a) \end{matrix} \right) &= \begin{matrix} s(b) \\ \diagdown \\ a' \end{matrix} - \begin{matrix} s(b) \\ \diagdown \\ t(a) \end{matrix} + \begin{matrix} s(b) \\ \diagdown \\ t(a) \end{matrix}, \\ \mu_A^\downarrow \left( \begin{matrix} s(b) \\ \diagdown \\ s(a) \end{matrix} \right) &= \begin{matrix} s(b) \\ \diagdown \\ s(a) \end{matrix} - \begin{matrix} t(b) \\ \diagdown \\ s(a) \end{matrix} + \begin{matrix} t(b) \\ \diagdown \\ a' \end{matrix}. \end{aligned}$$

We see that  $\mu_A^\uparrow$  applies the rule  $a$  first, and  $b$  second, while  $\mu_A^\downarrow$  does the opposite; this motivates the upwards and downwards arrow notations.

**2.4.5. Example.** Let  $a = \begin{matrix} a_1 & a_2 \\ \diagdown & \diagup \\ a_0 \end{matrix}$  be the element of  $A \circ A$  from Example 2.2.4 with inputs omitted. Then

$$\mu_A(a) = \begin{matrix} s(a_1) & s(a_2) \\ \diagdown & \diagup \\ a_0 \end{matrix} - \begin{matrix} s(a_1) & s(a_2) \\ \diagdown & \diagup \\ t(a_0) \end{matrix} + \begin{matrix} s(a_1) & a_2 \\ \diagdown & \diagup \\ t(a_0) \end{matrix} - \begin{matrix} s(a_1) & t(a_2) \\ \diagdown & \diagup \\ t(a_0) \end{matrix} + \begin{matrix} a_1 & t(a_2) \\ \diagdown & \diagup \\ t(a_0) \end{matrix}.$$

**2.4.6. Lemma.** Let  $(A, \lambda, \rho)$  be a  $P$ -bimodule such that  $(P, A)$  is a reflexive 1-globular  $P$ -bimodule satisfying the linear exchange relation. Then  $(A, \mu_A, i\eta)$  is an operad.

*Proof.* Write  $\mu_A^\circ := \mu_A^\uparrow = \mu_A^\downarrow$ . It suffices to check the associativity and unit axioms of internal monoidal objects. The unit axioms are clearly satisfied, by definition of  $\sigma_A$  and by the unit axioms of  $P$ -bimodules.

To show the associativity axiom, we need to calculate and compare  $\mu_A(\mu_A \circ 1)$  and  $\mu_A(1 \circ \mu_A)$ . The key calculation is the following, which generalizes the previous example: for all  $p \geq 0$  and  $\tau \in \mathfrak{S}_p$ , we have the equality of morphisms

$$\mu_A^\circ(1^{\circ p}) = \sum_{i=1}^p f_{i,1}^\tau \circ \cdots \circ 1_{\tau(i)} \circ \cdots \circ f_{i,p}^\tau - \sum_{i=1}^{p-1} f_{i,1}^\tau \circ \cdots \circ t_{\tau(i)} \circ \cdots \circ f_{i,p}^\tau, \quad (2.4.7)$$

## 2. Higher-dimensional shuffle operads

where

$$f_{i,j}^\tau = \begin{cases} t & \text{si } \tau^{-1}(j) < i, \\ s & \text{si } \tau^{-1}(j) > i, \end{cases}$$

and  $- \odot -$  represents  $\lambda^\odot(- \odot -)$ ,  $\rho^\odot(- \odot -)$ , or  $\mu^\odot(- \odot -)$  depending on the types of arguments, always with bracketing to the left.

We show this equality by induction on  $p$ . For  $p = 0, 1$ , the result is trivial. For  $p = 2$ , for  $\tau = id$ ,

$$\mu_A^\odot = \mu_A^\uparrow = \rho^\odot(1 \odot s) + \lambda^\odot(t \odot 1) - i_n \mu^\odot(t \odot s),$$

by definition, and for  $\tau = (1\ 2)$ ,

$$\mu_A^\odot = \mu_A^\downarrow = \rho^\odot(1 \odot t) + \lambda^\odot(s \odot 1) - i_n \mu^\odot(s \odot t)$$

by hypothesis on  $(P, A)$ .

Let  $p \geq 2$  and suppose that we have shown the equality for  $p$ . Let  $\tau \in \mathfrak{S}_{p+1}$ , and denote  $i_0 = \tau^{-1}(p+1)$  and  $\tau' = \tau(i_0\ p+1)$ . Then

$$\begin{aligned} (\mu_A^\odot)^{p+1}(1^{\odot p+1}) &= \mu_A^\odot((\mu_A^\odot)^p(1^{\odot p}) \odot 1) \\ &= \mu_A^\odot \left( \sum_{i=1}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} 1 \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 - \sum_{i=1}^{p-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} t \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 \right) \\ &= \left( \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} t \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 + \sum_{i=i_0}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} s \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 \right. \\ &\quad + \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} 1 \odot \cdots \odot f_{i,p}^{\tau'} \odot s + \sum_{i=i_0}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} 1 \odot \cdots \odot f_{i,p}^{\tau'} \odot t \\ &\quad \left. - \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} t \odot \cdots \odot f_{i,p}^{\tau'} \odot s - \sum_{i=i_0}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} s \odot \cdots \odot f_{i,p}^{\tau'} \odot t \right) \\ &\quad - \left( \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} t \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 + \sum_{i=i_0+1}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} s \odot \cdots \odot f_{i,p}^{\tau'} \odot 1 \right) \\ &= f_{i_0,1}^{\tau'} \odot \cdots \odot_{\tau'(i_0)} t \odot \cdots \odot f_{i_0,p}^{\tau'} \odot 1 + \sum_{\substack{i=1 \\ i \neq i_0}}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} 1 \odot \cdots \odot f_{i,p}^{\tau'} \odot f_{i,p+1}^\tau \\ &\quad - \sum_{i=1}^{i_0-1} f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} t \odot \cdots \odot f_{i,p}^{\tau'} \odot f_{i,p+1}^\tau - f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i_0)} s \odot \cdots \odot f_{i,p}^{\tau'} \odot t \\ &\quad - \sum_{i=i_0+1}^p f_{i,1}^{\tau'} \odot \cdots \odot_{\tau'(i)} s \odot \cdots \odot f_{i,p}^{\tau'} \odot f_{i,p+1}^\tau \\ &= \sum_{i=0}^{p+1} f_{i,1}^\tau \odot \cdots \odot_{\tau(i)} 1 \odot \cdots \odot f_{i,p+1}^\tau - \sum_{i=1}^p f_{i,1}^\tau \odot \cdots \odot_{\tau(i)} t \odot \cdots \odot f_{i,p+1}^\tau. \end{aligned}$$

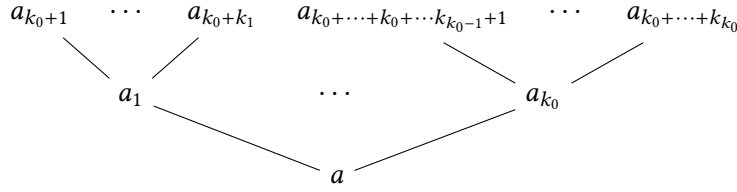
## 2.4. Higher-dimensional operads as globular bimodules

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This proves (2.4.7). Next, let  $a$  be an arbitrary tree monomial of  $A \circ A \circ A \simeq (A \circ A) \circ A \simeq A \circ (A \circ A)$ . We write

$$\begin{aligned} a &= ((a_0 \mid a_1 \cdots a_{k_0}) \mid a_{k_0+1} \cdots a_{k_0+k_1} \cdots a_{k_0+\cdots+k_{k_0-1}+1} \cdots a_{k_0+\cdots+k_{k_0}}) \\ &= (a_0 \mid (a_1 \mid a_{k_0+1} \cdots a_{k_0+k_1}) \cdots (a_{k_0} \mid a_{k_0+\cdots+k_0+\cdots+k_{k_0-1}+1} \cdots a_{k_0+\cdots+k_{k_0}})), \end{aligned}$$

where  $k_0$  is the arity of  $a_0$  and, for  $i \in \{1, \dots, k_0\}$ ,  $k_i$  is the arity of  $a_i$ . We can also understand  $a$  via its planar tree  $T(a)$ :



For the rest of this proof, we write  $\cdot$  for all elementary compositions  $\circ_{i, \tau}$ . On the one hand, calculating  $\mu_A(\mu_A \circ 1)(a)$  is equal to calculating  $(\mu_A^\circ)(a_0 \cdot a_{k_0} \cdots a_1 \cdot a_{k_0+\cdots+k_{k_0}} \cdots a_{k_0+1})$  as in the previous calculation, with  $p = k_0 + \cdots + k_{k_0}$  and the identity permutation. On the other hand, calculating  $\mu_A(1 \circ \mu_A)(a)$  is equal to calculating  $(\mu_A^\circ)(a_0 \cdot a_{k_0} \cdots a_1 \cdot a_{k_0+\cdots+k_{k_0}} \cdots a_{k_0+1})$  as in the previous calculation, with  $p = k_0 + \cdots + k_{k_0}$  and the permutation

$$[0, k_0, k_0 + \cdots + k_{k_0}, \dots, k_0 + \cdots + k_{k_0-1} + 1, \dots, 1, k_0 + k_1, \dots, k_0 + 1],$$

where each integer  $i$  represents the position of  $a_i$  in the argument of  $\mu_A^\circ$ . Thus, by the previous calculation,  $\mu_A(\mu_A \circ 1) = \mu_A(1 \circ \mu_A)$ . We conclude that  $(A, \mu_A, i\eta)$  is an operad.

As an aside, calculating  $\mu_A(\mu_A \circ 1)$  corresponds to a breadth-first traversal of a tree, while calculating  $\mu_A(1 \circ \mu_A)$  corresponds to a depth-first traversal.  $\square$

**2.4.8. Theorem.** *The following categories are isomorphic:*

- i) *the category  $\text{III}\text{Op}_\omega$ ,*
- ii) *the full subcategory of  $\text{RGlob}(\text{Bimod}(\text{III}\text{Op}))$  whose objects are pairs  $(P, A)$  where  $(P, \mu, \eta)$  is a shuffle operad and  $A = (A_n, \lambda_n, \rho_n)_{n \geq 0}$  is a reflexive globular  $P$ -bimodule such that  $A_0 = P$ , and  $(P, A_n)$  satisfies the linear exchange relation (2.4.3) for all  $n \geq 1$ .*

*Proof.* We show that each category is a full subcategory of the other.

(i  $\subseteq$  ii) Let  $P = (P_n, \mu_n, \eta_n)_{n \geq 0}$  be an  $\omega$ -operad. Forgetting the  $\star_k$ -compositions and operadic multiplications  $\mu_n$ ,  $P$  is equipped with a reflexive globular  $P_0$ -bimodule structure. By the calculations and discussion of 2.4.2, for all  $n \geq 1$ ,  $P_n$  seen as a  $P_0$ -bimodule satisfies the linear exchange relation (2.4.3). Thus  $\text{III}\text{Op}_\omega$  is a full subcategory of the second category.

(ii  $\subseteq$  i) Let  $(P, \mu, \eta)$  be an operad and  $(A_n, \lambda_n, \rho_n)_{n \geq 0}$  a globular reflexive  $P$ -bimodule satisfying the linear exchange relation (2.4.3) and such that  $A_0 = P$ . We proceed in to steps: first, we equip  $A$  with a globular reflexive operad structure, then we equip it with a  $\omega$ -operad structure.

## 2. Higher-dimensional shuffle operads

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First, let  $n \geq 0$ . Equip  $A_n$  with the partial multiplication  $\mu_n^\circ := \mu_{A_n}^\uparrow = \mu_{A_n}^\downarrow$  and then define the operadic multiplication

$$\mu_n : A_n \circ A_n \xrightarrow{\sigma_{A_n}} \bigoplus_{p \geq 1} A_n \circ A_n^{\circ p} \xrightarrow{\sum (\mu_n^\circ)^p} A_n.$$

The multiplication  $\mu_n$  satisfies the associativity and unit axioms by Lemma 2.4.6. Moreover, by construction,  $\mu_n^\circ$  satisfies the relations

$$\mu_n^\circ = \mu_n^\circ(1 \circ s_0) + \mu_n^\circ(t_0 \circ 1) - \mu_n^\circ(t_0 \circ s_0) = \mu_n^\circ(s_0 \circ 1) + \mu_n^\circ(1 \circ t_0) - \mu_n^\circ(s_0 \circ t_0). \quad (2.4.9)$$

This gives  $A$  a globular reflexive operad structure. Next, for the  $\omega$ -operad structure on  $A$ , we define the  $\star_k$ -compositions as follows: for all  $\star_k$ -composable  $n$ -cells  $a, b$ , define

$$a \star_k b := a - t_k(a) + b.$$

Let  $0 \leq k < \ell < n$  be three integers. The target morphism  $t_\ell : A_n \rightarrow A_\ell$  is linear, so it commutes with  $\star_k$ . For all  $\star_\ell$ -composable pairs  $(a, a')$  and  $(b, b')$  of  $A_n$  such that  $(a, b)$  and  $(a', b')$  are  $\star_k$ -composable, we calculate

$$\begin{aligned} (a \star_\ell a') \star_k (b \star_\ell b') &= (a - t_\ell(a) + a') \star_k (b - t_\ell(b) + b') \\ &= a \star_k b - t_\ell(a) \star_k t_\ell(b) + a' \star_k b' \\ &= a \star_k b - t_\ell(a \star_k b) + a' \star_k b' \\ &= (a \star_k b) \star_\ell (a' \star_k b'). \end{aligned}$$

Thus the  $\star_k$ -compositions satisfy exchange relations. To show that  $A$  is an  $\omega$ -operad, it suffices to show that the  $\star_k$ -compositions are morphisms of operads.  $A_n \star_k A_n$  is equipped with an operad structure given by the multiplication

$$(A_n \star_k A_n) \circ (A_n \star_k A_n) \xrightarrow{(\pi_1 \circ \pi_1) \star_k (\pi_2 \circ \pi_2)} (A_n \circ A_n) \star_k (A_n \circ A_n) \xrightarrow{\mu_k \star_k \mu_k} A_n \star_k A_n,$$

where  $\pi_1, \pi_2$  are the projections of the fiber product  $A_n \star_k A_n$ . Therefore it suffices to check the exchange relation (2.3.4):

$$\begin{array}{ccc} (A_n \star_k A_n) \circ (A_n \star_k A_n) & \xrightarrow{(\pi_1 \circ \pi_1) \star_k (\pi_2 \circ \pi_2)} & (A_n \circ A_n) \star_k (A_n \circ A_n) \\ & \searrow \mu_n((1 \star_k 1) \circ (1 \star_k 1)) & \swarrow \mu_n(1 \circ 1) \star_k \mu_n(1 \circ 1) \\ & & A_n \end{array}$$

Writing

$$\mu_n = \mu_n \varphi_{A_n \star_k A_n} \sigma_{A_n \star_k A_n} = \left( \sum_p (\mu_n^\circ)^{\circ p} \right) \sigma_{A_n \star_k A_n},$$

## 2.4. Higher-dimensional operads as globular bimodules

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we get the diagram

$$\begin{array}{ccc}
 (A_n \star_k A_n) \circ (A_n \star_k A_n) & \xrightarrow{(\pi_1 \circ \pi_1) \star_k (\pi_2 \circ \pi_2)} & (A_n \circ A_n) \star_k (A_n \circ A_n) \\
 \sigma_{A_n \star_k A_n} \downarrow & & \downarrow \sigma_{A_n \star_k \sigma_{A_n}} \\
 \bigoplus (A_n \star_k A_n) \circ (A_n \star_k A_n)^{\circ p} & \xrightarrow{(\pi_1 \circ \pi_1^{\circ p}) \star_k (\pi_2 \circ \pi_2^{\circ p})} & \left( \bigoplus A_n \circ A_n^{\circ p} \right) \star_k \left( \bigoplus A_n \circ A_n^{\circ p} \right) \\
 \mu_n^{\circ}((1 \star_k 1) \circ (1 \star_k 1)^{\circ p}) \searrow & & \swarrow \mu_n^{\circ}(1 \circ 1^{\circ p}) \star_k \mu_n^{\circ}(1 \circ 1^{\circ p}) \\
 & A_n & 
 \end{array}$$

The upper square commutes by naturality of  $\sigma$ . To show that the lower triangle commutes, it suffices to show that, for all  $\star_k$ -composable pairs  $(a, a')$  and  $(b, b')$  of  $A_n$  and all elementary compositions  $\circ_{i,\tau}$  such that  $a \circ_{i,\tau} b$  and  $a' \circ_{i,\tau} b'$  are well defined,

$$(a \star_k a') \circ_{i,\tau} (b \star_k b') = (a \circ_{i,\tau} b) \star_k (a' \circ_{i,\tau} b').$$

Write  $\cdot$  for  $\circ_{i,\tau}$ . Let us begin with the case  $k = 0$ . By definition of  $\star_k$ -composition and bidistributivity of  $\circ_{i,\tau}$ , we have

$$\begin{aligned}
 (a + a' - t_0(a)) \cdot (b + b' - t_0(b)) &= a \cdot b + a \cdot b' + a' \cdot b + a' \cdot b' - t_0(a) \cdot b - t_0(a) \cdot b' \\
 &\quad - a \cdot t_0(b) - a' \cdot t_0(b) + t_0(a) \cdot t_0(b)
 \end{aligned}$$

By applying (2.4.9) to  $a \cdot b'$  and  $a' \cdot b$ , we get

$$\begin{aligned}
 a \cdot b' &= t_0(a) \cdot b' + a \cdot s_0(b') - t_0(a) \cdot s_0(b'), \\
 a' \cdot b &= s_0(a') \cdot b + a' \cdot t_0(b) - s_0(a') \cdot t_0(b).
 \end{aligned}$$

Since  $t_0(a) = s_0(a')$  and  $t_0(b) = s_0(b')$ , we conclude that

$$\begin{aligned}
 (a + a' - t_0(a)) \cdot (b + b' - t_0(b)) &= a \cdot b + a' \cdot b' - t_0(a) \cdot t_0(b) \\
 &= (a \cdot b) \star_0 (a' \cdot b').
 \end{aligned}$$

Now, let  $k \geq 1$ . In this case,  $n \geq 2$ , so by globularity,  $t_0(a) = t_0(a')$  and  $s_0(b) = s_0(b')$ . Write  $c := t_0(a) = t_0(a')$  and  $d := s_0(b) = s_0(b')$ . Using the exchange relations between  $\circ$  and  $\star_0$ , and between  $\star_0$  and  $\star_k$ , we get

$$\begin{aligned}
 (a \cdot b) \star_k (a' \cdot b') &= ((a \star_0 c) \cdot (d \star_0 b)) \star_k ((a' \star_0 c) \cdot (d \star_0 b')) \\
 &= ((a \cdot d) \star_0 (c \cdot b)) \star_k ((a' \cdot d) \star_0 (c \cdot b')) \\
 &= ((a \cdot d) \star_k (a' \cdot d)) \star_0 ((c \cdot b) \star_k (c \cdot b')).
 \end{aligned}$$

By definition of  $\star_k$  and by bidistributivity of  $\circ_{i,\tau}$ , we get

$$\begin{aligned}
 (a \cdot d) \star_k (a' \cdot d) &= a \cdot d + a' \cdot d - t_k(a) \cdot d = (a \star_k a') \cdot d, \\
 (c \cdot b) \star_k (c \cdot b') &= c \cdot b + c \cdot b' - c \cdot t_k(b) = c \cdot (b \star_k b').
 \end{aligned}$$

Thus

$$\begin{aligned}
 (a \cdot b) \star_k (a' \cdot b') &= ((a \star_k a') \cdot d) \star_0 (c \cdot (b \star_k b')) \\
 &= (a \star_k a') \cdot (b \star_k b').
 \end{aligned}$$

Thus the exchange relation is satisfied, and we conclude that  $A$  is an  $\omega$ -operad.  $\square$

### 3. SHUFFLE OPERADIC POLYGRAPHS

In this section we introduce the notion of a shuffle polygraph that defines systems of generators and relations for higher-dimensional shuffle operads.

#### 3.1. Shuffle polygraphs

The structure of polygraph was introduced independently by Street and Burroni as a system of generators for free higher-dimensional categories [15, 44]. This subsection introduces a version of this structure for higher-dimensional shuffle operads.

**3.1.1. Cellular extensions.** For  $n \geq 0$ , a *cellular extension* of a shuffle  $n$ -operad  $P$  is an indexed set  $X$  equipped with two morphisms

$$P_n \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} X$$

in  $\text{Ind}$  satisfying the globular relations  $s_{n-1}s_n = s_{n-1}t_n$  and  $t_{n-1}s_n = t_{n-1}t_n$ , for  $n \geq 1$ , and whose elements are called  $(n+1)$ -generators. Note that every  $n$ -operad  $P$  has two canonical cellular extensions: the empty one, and the one denoted by  $\text{Sph}(P_n)$  that consists of a  $(n+1)$ -generator  $a \rightarrow b$  for every  $n$ -sphere  $(a, b)$  of  $P$ .

**3.1.2. Extended higher-dimensional operads.** For  $n \geq 0$ , the category of *extended  $n$ -operads*  $\text{III}\text{Op}_n^+$  is defined by the following pullback of forgetful functors

$$\begin{array}{ccc} \text{III}\text{Op}_n^+ & \longrightarrow & \text{Glob}_{n+1}(\text{Ind}) \\ \downarrow \lrcorner & & \downarrow \mathcal{V}_n \\ \text{III}\text{Op}_n & \xrightarrow{\mathcal{U}_n^{\text{III}}} & \text{Glob}_n(\text{Ind}) \end{array}$$

where the functor  $\mathcal{U}_n$  forgets the shuffle composition, and the functor  $\mathcal{V}_n$  forgets the  $(n+1)$ -cells. Explicitly, an extended shuffle  $n$ -operad is a pair  $(P, X)$  where  $P$  is an  $n$ -operad and  $X$  a cellular extension of  $P_n$ . A morphism of extended  $n$ -operads  $(P, X) \rightarrow (P', X')$  is a morphism of shuffle  $n$ -operads  $f : M \rightarrow M'$  and a morphism  $g : X \rightarrow X'$  in  $\text{Ind}$  such that the two following square diagrams commute in  $\text{Ind}$ :

$$\begin{array}{ccc} P_n & \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} & X \\ f_n \downarrow & & \downarrow g \\ P'_n & \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} & X' \end{array}$$

**3.1.3. Proposition.** *Let  $P$  be a shuffle  $(n-1)$ -operad and  $X$  be a cellular extension of  $P$ . Let  $X^{\text{III}}$  denote the coequalizer of the two morphisms*

$$(P_0\langle X \rangle \oplus P_{n-1}) \odot (P_0\langle X \rangle \oplus P_{n-1}) \begin{array}{c} \xrightarrow{\mu_{P_0\langle X \rangle \oplus P_{n-1}}^\uparrow} \\ \xrightarrow{\mu_{P_0\langle X \rangle \oplus P_{n-1}}^\downarrow} \end{array} P_0\langle X \rangle \oplus P_{n-1}$$

in the category  $\text{Bimod}(P)$ , where the morphisms are defined relative to the pair  $(P_0, P_0\langle X \rangle \oplus P_{n-1})$ . Then  $(P, X^{\text{III}})$  is the free shuffle  $n$ -operad on  $(P, X)$ .

*Proof.* We will progressively enrich the cellular extension  $X$  with more and more structure in order to get a reflexive globular bimodule satisfying the linear exchange relation (2.4.3). First, let us define categories of “enriched cellular extensions” of  $P$ . Let  $\mathbb{D}$  be a category such that  $P$  is a reflexive  $(n-1)$ -globular object of  $\mathbb{D}$ . Define

- $\text{Glob}_P(\mathbb{D})$  the full subcategory of  $\text{Glob}_n(\mathbb{D})$  whose objects  $X$  satisfy  $(X_0, \dots, X_{n-1}) = P$ , and
- $\text{RGlob}_P(\mathbb{D})$  the full subcategory of  $\text{RGlob}_n(\mathbb{D})$  whose objects  $X$  satisfy  $(X_0, \dots, X_{n-1}) = P$ .

Finally, denote by  $\text{RGlob}_P^\odot(\text{Bimod}(P_0))$  the full subcategory of  $\text{RGlob}_P(\text{Bimod}(P_0))$  whose objects satisfy the linear exchange relation (2.4.3).

Following Theorem 2.4.8, given an extended  $(n-1)$ -operad  $(P, X)$ , in order to construct the free  $n$ -operad, it suffices to construct the free object on  $(P, X) \in \text{Glob}_P(\text{Ind})$  in  $\text{RGlob}_P^\odot(\text{Bimod}(P_0))$ . Therefore, it suffices to construct the sequence of free functors

$$\text{Glob}_P(\text{Ind}) \rightarrow \text{Glob}_P(\text{Bimod}(P_0)) \rightarrow \text{RGlob}_P(\text{Bimod}(P_0)) \rightarrow \text{RGlob}_P^\odot(\text{Bimod}(P_0)).$$

- Let  $(P, X)$  be an extended  $(n-1)$ -operad. The first free functor is induced by the free functors  $\text{Ind} \rightarrow \text{Coll} \rightarrow \text{Bimod}(P_0)$ , so it sends  $X$  to  $P_0\langle X \rangle$ .
- Let  $(P, X)$  be an object of  $\text{Glob}_P(\text{Bimod}(P_0))$ . Since  $P$  is already a reflexive  $(n-1)$ -globular object, the second free functor is induced by the free functor  $\text{Bimod}(P_0) \rightarrow P_{n-1}/\text{Bimod}(P_0)$ , so it sends  $X$  to  $X \oplus P_{n-1}$ .
- Let  $(P, X)$  be an object of  $\text{RGlob}_P(\text{Bimod}(P_0))$ . The third free functor sends  $X$  to the coequalizer of

$$X \odot X \begin{array}{c} \xrightarrow{\mu_X^\uparrow} \\ \xrightarrow{\quad} \\ \xrightarrow{\mu_X^\downarrow} \end{array} X$$

where the morphisms are defined relative to the pair  $(P_0, X)$ .

By composing these functors, we get

$$X^{\text{III}} := \text{coeq}(\mu_{P_0\langle X \rangle \oplus P_{n-1}}^\uparrow, \mu_{P_0\langle X \rangle \oplus P_{n-1}}^\downarrow) : (P_0\langle X \rangle \oplus P_{n-1}) \odot (P_0\langle X \rangle \oplus P_{n-1}) \rightrightarrows P_0\langle X \rangle \oplus P_{n-1}$$

and we conclude that  $(P, X^{\text{III}})$  is the free  $n$ -operad on  $(P, X)$ . □

**3.1.4. Free shuffle  $n$ -operad.** For  $n \geq 1$ , the forgetful functor  $\mathcal{W}_n^{\text{III}} : \text{III}\text{Op}_n \rightarrow \text{III}\text{Op}_{n-1}^+$  that forgets the composition of  $n$ -cells admits a left adjoint

$$\mathcal{L}_n^{\text{III}} : \text{III}\text{Op}_{n-1}^+ \rightarrow \text{III}\text{Op}_n \tag{3.1.5}$$

that associates to an extended  $(n-1)$ -operad  $(P, X)$  the free  $n$ -operad over  $(P, X)$  given by  $\mathcal{L}_n^{\text{III}}(P, X) = (P, X^{\text{III}})$ . In the sequel, the  $n$ -operad  $(P, X^{\text{III}})$  will be denoted by  $P[X]$ , and its  $k$ -source and  $k$ -target maps will be denoted by  $s_k$  and  $t_k$  respectively.



### 3. Shuffle operadic polygraphs

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**3.1.6. Shuffle polygraphs.** We define the category  $\text{III}Pol_n$  of  $n$ -polygraphs and the free functor

$$\mathcal{F}_n : \text{III}Pol_n \rightarrow \text{III}Op_n,$$

by induction on  $n \geq 0$ . For  $n = 0$ , we define  $\text{III}Pol_0$  as the category  $\text{Ind}$ . The free 0-monoid functor

$$\mathcal{F}_0 : \text{III}Pol_0 \rightarrow \text{III}Op_0$$

is the composite of free functors (2.1.4). We suppose that for  $n \geq 1$  the category  $\text{III}Pol_{n-1}$  of  $(n-1)$ -polygraphs is defined and that the free  $(n-1)$ -operad functor

$$\mathcal{F}_{n-1} : \text{III}Pol_{n-1} \rightarrow \text{III}Op_{n-1}$$

is constructed. The category  $\text{III}Pol_n$  of  $n$ -polygraphs is defined as the following pullback in  $\text{Cat}$

$$\begin{array}{ccc} \text{III}Pol_n & \xrightarrow{\tilde{\mathcal{U}}_{n-1}} & \text{III}Op_{n-1}^+ \\ \tilde{\mathcal{V}}_{n-1} \downarrow & \lrcorner & \downarrow \mathcal{W}_n^{\text{III}} \\ \text{III}Pol_{n-1} & \xrightarrow{\mathcal{F}_{n-1}} & \text{III}Op_{n-1} \end{array} \quad (3.1.7)$$

where the vertical functor on the right forgets the cellular extension of an extended monoid. The free symmetric  $n$ -operad functor is defined as the composite

$$\text{III}Pol_n \xrightarrow{\tilde{\mathcal{U}}_{n-1}} \text{III}Op_{n-1}^+ \xrightarrow{\mathcal{L}_n^{\text{III}}} \text{III}Op_n,$$

where  $\tilde{\mathcal{U}}_{n-1}$  is the functor defined by the pullback (3.1.7) and  $\mathcal{L}_n^{\text{III}}$  is the free functor defined in (3.1.5).

The category  $\text{III}Pol_\omega$  of  $\omega$ -polygraphs and the free  $\omega$ -operad functor  $\mathcal{F}_\omega : \text{III}Pol_\omega \rightarrow \text{III}Op_\omega$  are defined as the limit of the functors:

$$\cdots \rightarrow \text{III}Pol_n \xrightarrow{\tilde{\mathcal{V}}_{n-1}} \text{III}Pol_{n-1} \rightarrow \cdots \rightarrow \text{III}Pol_1 \xrightarrow{\tilde{\mathcal{V}}_0} \text{III}Pol_0,$$

in the category of categories, where the functors  $\tilde{\mathcal{V}}_{n-1}$  are defined by (3.1.7).

In this way, an  $n$ -polygraph  $X$  is defined inductively as a data  $(X_0, \dots, X_n)$ , where  $X_0$  is an indexed set and for every  $0 < k < n$ ,  $X_k$  is a cellular extension of the free  $(k-1)$ -operad generated by  $(X_0, \dots, X_{k-1})$ , denoted by

$$X_{k-1}^{\text{III}} = X_0^{\text{III}}[X_1] \cdots [X_k].$$

For  $0 \leq p < n$ , we will denote by  $X_{\leq p}$  the underlying  $p$ -polygraph  $(X_0, \dots, X_p)$ .

**3.1.8. Higher-dimensional monomials.** Let  $X$  be an  $\omega$ -polygraph. Tree monomials in  $X_0^{*\text{III}}$  are called 0-monomials of  $X^{\text{III}}$ , and they form a linear basis of the collection  $X_0^{\text{III}}$ , which means that every 0-cell  $a$  of  $X_0^{\text{III}}$  can be uniquely written as a (possibly empty) linear combination

$$a = \sum_{i=1}^p \lambda_i u_i + \lambda \varepsilon$$

of pairwise distinct 0-monomials  $u_1, \dots, u_p$  of  $X_0^{\text{III}}$ , with  $\lambda_i \in \mathbf{k} \setminus \{0\}$ ,  $\lambda \in \mathbf{k}$ , and  $\varepsilon$  denotes the trivial monomial. This expression is called *the canonical decomposition of  $a$* , and we define the *support of  $a$*  as the set  $\text{Supp}(a) = \{u_1, \dots, u_p\}$ .

For  $n \geq 1$ , if  $\alpha$  is an  $n$ -cell of  $X$ , and  $\vec{v}$  is a list of 0-monomials, we will denote by  $(\alpha \mid \vec{v})$  the  $n$ -cell of  $X^{\text{III}}$  with source  $(s(\alpha) \mid \vec{v})$  and target  $(t(\alpha) \mid \vec{v})$ . An  $n$ -monomial of  $X^{\text{III}}$  is an  $n$ -cell of  $X^{\text{III}}$  of the form  $u \circ_{i,\tau} (\alpha \mid \vec{v})$ , where  $\alpha$  is an  $n$ -cell of  $X$ , and  $u$  and  $\vec{v}$  are monomials of  $X^{\text{III}}$ . By construction of the free  $n$ -operad over  $(X_{n-1}^{\text{III}}, X_n)$ , and by freeness of  $X_{n-1}^{\text{III}}$ , every  $n$ -cell  $a$  of  $X^{\text{III}}$  can be written as a linear combination

$$a = \sum_{i=1}^p \lambda_i a_i + 1_c \quad (3.1.9)$$

of pairwise distinct  $n$ -monomials  $a_1, \dots, a_p$  and of an identity  $n$ -cell  $1_c$  of  $X^{\text{III}}$ , and this decomposition is unique up to the linear exchange relation (2.4.3). The *size* of an  $n$ -cell  $a$  of  $X^{\text{III}}$  is the minimal number of  $n$ -monomials of  $X^{\text{III}}$  required to write  $a$  as in (3.1.9).

### 3.2. Shuffle polygraphic resolutions

In this subsection we introduce the notion of a polygraphic resolution for shuffle operads.

**3.2.1. Presentation of a shuffle operad.** The shuffle operad *presented* by a shuffle 1-polygraph  $X$  is the coequalizer in the category  $\text{III Op}$  of the following source and target morphisms, denoted by  $\bar{X}$ ,

$$X_1^{\text{III}} \begin{array}{c} \xrightarrow{s_0} \\ \xrightarrow[t_0]{} \end{array} X_0^{\text{III}} \xrightarrow{\pi_X} \bar{X}. \quad (3.2.2)$$

Following Proposition (2.3.7), the category  $\text{III Op}$  preserves reflexive coequalizers and so the construction is well defined. We say that a shuffle operad  $P$  is *presented by a polygraph  $X$* , or that  $X$  is a *presentation of  $P$* , if  $P$  is isomorphic to  $\bar{X}$  in the category  $\text{III Op}$ .

**3.2.3. Shuffle polygraphic resolutions.** For  $n \geq 1$ , let  $P$  be a shuffle  $n$ -operad. A cellular extension  $X$  of  $P$  is *acyclic* if for every  $n$ -sphere  $(f, g)$  in  $P_n$ , there exists a shuffle  $(n+1)$ -cell  $F$  in the  $(n+1)$ -operad  $P[X]$  with source  $f$  and target  $g$ . A *coherent presentation* of a shuffle polygraph  $P$  is a shuffle 2-polygraph  $X$ , that presents  $P$ , and whose cellular extension  $X_2$  is acyclic. A *shuffle polygraphic resolution* of a shuffle operad  $P$  is a shuffle  $\omega$ -polygraph  $X$  that presents  $P$  and whose cellular extensions  $X_k$  are acyclic for  $k \geq 2$ .

**3.2.4. Tietze equivalence of  $\omega$ -polygraphs.** We define the notion of a *weak-equivalence* of  $\omega$ -operads as for  $\omega$ -categories, defined in [32]. For  $n \geq 0$ , two  $n$ -cells  $a, b$  of an  $\omega$ -operad  $P$  are  *$\omega$ -equivalent* if there exists an  $(n+1)$ -cell  $f : a \rightarrow b$  in  $P$ . In that case, we write  $a \sim_\omega b$ . A morphism of  $\omega$ -operads  $F : P \rightarrow Q$  is a *weak equivalence* if it satisfies the following properties:

- i) For every 0-cell  $a$  of  $Q$ , there exists a 0-cell  $\widehat{a}$  in  $P$  such that  $F(\widehat{a}) \sim_\omega a$ .
- ii) For every pair of 0-cells  $a, b$  of  $P$  and every 1-cell  $f : F(a) \rightarrow F(b)$  of  $Q$ , there exists a 1-cell  $\widehat{f} : a \rightarrow b$  of  $P$  such that  $F(\widehat{f}) \sim_\omega f$ .

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iii) For  $n \geq 1$  and every pair of parallel  $n$ -cells  $a, b$  of  $P$  and every  $(n + 1)$ -cell  $f : F(a) \rightarrow F(b)$  of  $Q$ , there exists an  $(n + 1)$ -cell  $\hat{f} : a \rightarrow b$  of  $P$  such that  $F(\hat{f}) \sim_\omega f$ .

We say that two shuffle  $\omega$ -polygraphs  $X$  and  $Y$  are *Tietze equivalent* if the induced free  $\omega$ -operads  $X^{\text{III}}$  and  $Y^{\text{III}}$  are weakly equivalent. The original notion of Tietze equivalence for 1-polygraphs is a particular case of this notion for  $\omega$ -polygraphs. Two 1-polygraphs  $X$  and  $Y$  are *Tietze equivalent* if the presented shuffle operads  $\bar{X}$  and  $\bar{Y}$  are isomorphic. In that case, extending  $X$  and  $Y$  into  $\omega$ -polygraphs with identities in higher dimensions gives two Tietze equivalent  $\omega$ -polygraphs. Tietze equivalence also generalizes the notion of Tietze equivalence between (3, 1)-polygraphs introduced in [22].

**3.2.5. Proposition.** *Let  $X$  and  $Y$  be two acyclic  $\omega$ -polygraphs. Then the presented shuffle operads  $\bar{X}$  and  $\bar{Y}$  are isomorphic if, and only if,  $X$  and  $Y$  are Tietze equivalent.*

*Proof.* ( $\Rightarrow$ ) Denote by  $\varphi : \bar{X} \rightarrow \bar{Y}$  the isomorphism. We define a morphism of  $\omega$ -operads  $F : X^{\text{III}} \rightarrow Y^{\text{III}}$  such that  $\pi_Y F = \varphi \pi_X$  on 0-cells and show that it is a weak equivalence simultaneously. Since  $X^{\text{III}}$  is a free  $\omega$ -operad, it suffices to define  $F$  on the  $n$ -generators of  $X$  for all  $n \geq 0$ . We proceed by induction on  $n \geq 0$ .

For  $n = 0$ , define linear maps  $i_X : \bar{X} \rightarrow X^{\text{III}}$  and  $i_Y : \bar{Y} \rightarrow Y^{\text{III}}$ , which are sections of  $\pi_X$  and  $\pi_Y$ , respectively. For  $x \in X_0$ , we set  $F(x_0) := i_Y(\varphi(\pi_X(x_0)))$ , and we check that  $\pi_Y F = \varphi \pi_X$  on 0-cells. Now, for a 0-cell  $a$  of  $Y^{\text{III}}$ , let  $\hat{a} := \varphi^{-1} i_Y(a) \in X^{\text{III}}$ . Then  $\pi_Y(F(\hat{a})) = \pi_Y(a)$ , so  $F(\hat{a}) \sim_\omega a$ .

For  $n = 1$ , for  $\alpha : a \rightarrow b$  a 1-generator of  $X$ ,  $\pi_Y(F(\alpha)) = \pi_Y(F(b))$ , so there exists a 1-cell  $f : F(a) \rightarrow F(b)$  in  $Y^{\text{III}}$ . We set  $F(\alpha) := f$ . Then, for every pair of 0-cells  $a, b$  of  $X^{\text{III}}$  and every 1-cell  $f : F(a) \rightarrow F(b)$  of  $Y^{\text{III}}$ ,  $\pi_Y(F(\alpha)) = \pi_Y(F(b))$ , which is equivalent to  $\pi_X(a) = \pi_X(b)$  via the isomorphism  $\varphi$ . Therefore there exists  $\hat{f} : a \rightarrow b$  in  $X^{\text{III}}$ , and  $F(\hat{f}) : F(a) \rightarrow F(b)$  is parallel to  $f$ . Since  $Y$  is acyclic,  $F(\hat{f}) \sim_\omega f$ .

Let  $n \geq 1$  and suppose that  $F$  is defined on  $n$ -cells of  $X^{\text{III}}$ . For  $\alpha : a \rightarrow b$  an  $(n + 1)$ -generator of  $X$ , the  $n$ -cells  $a$  and  $b$  of  $X^{\text{III}}$  are parallel, so the  $n$ -cells  $F(a)$  and  $F(b)$  of  $Y^{\text{III}}$  are parallel. By acyclicity of  $Y$ , there exists an  $(n + 1)$ -cell  $f : F(a) \rightarrow F(b)$  in  $Y^{\text{III}}$ , so we set  $F(\alpha) := f$ . Now, let  $a, b$  be two parallel  $n$ -cells of  $X^{\text{III}}$  and  $f : F(a) \rightarrow F(b)$  an  $(n + 1)$ -cell of  $Y^{\text{III}}$ . By acyclicity of  $X$ , there exists an  $(n + 1)$ -cell  $\hat{f} : a \rightarrow b$ , so  $F(\hat{f})$  and  $f$  are parallel in  $Y^{\text{III}}$ , so by acyclicity of  $Y$  we have  $F(\hat{f}) \sim_\omega f$ .

We conclude that  $F : X^{\text{III}} \rightarrow Y^{\text{III}}$  is a weak equivalence, so  $X$  and  $Y$  are Tietze equivalent.

( $\Leftarrow$ ) Let  $F : X^{\text{III}} \rightarrow Y^{\text{III}}$  be a weak equivalence. By condition **i**),  $\pi_Y F : X^{\text{III}} \rightarrow \bar{Y}$  is surjective. Moreover, if  $a \sim_\omega b$  in  $X^{\text{III}}$ , then  $\pi_Y F(a) = \pi_Y F(b)$ , so  $F$  induces a morphism of  $\omega$ -operads  $\bar{F} : \bar{X} \rightarrow \bar{Y}$ . By condition **ii**),  $\bar{F}$  is injective. Thus  $\bar{F}$  is an isomorphism between  $\bar{X}$  and  $\bar{Y}$ .  $\square$

**3.2.6. Standard polygraphic resolution.** The *standard polygraphic resolution* of a shuffle operad  $P$  is the  $\omega$ -polygraph, denoted by  $\text{Std}(P)$ , and defined by induction as follows. For  $n = 0$ , we set  $\text{Std}(P)_0 := \mathcal{U}_0(P)$ . For every 0-cell  $a$  in  $P$ , we denote  $[a]$  the associated element in  $\mathcal{U}_0(P)$ . For  $n = 1$ , we set

$$\text{Std}(P)_1 := \{ [a] \circ_{i,\tau} [b] \rightarrow [a \circ_{i,\tau} b] \mid a, b \in P \},$$

so that the pair  $(\mathcal{U}_0(P), \text{Std}(P)_1)$  forms a 1-polygraph that presents the shuffle operad  $P$ . Suppose that  $\text{Std}(P)_n$  is defined for  $n \geq 1$ . Then we set  $\text{Std}(P)_{n+1} := \text{Sph}(\text{Std}(P)_n^{\text{III}})$ . By construction, the  $\omega$ -polygraph  $\text{Std}(P)$  is a polygraphic resolution of the shuffle operad  $P$ .

### 3.3. Symmetric polygraphic resolutions

In this subsection we define the structure of polygraphs for symmetric operads. This structure is similar to that of shuffle polygraphs, the main difference being the additional linear structure on the cellular extensions. We define a polygraphic resolution of a symmetric operad as an acyclic symmetric polygraph that presents the operad. We show that, for a symmetric operad, there is no loss of information when we resolve the associated shuffle operad by a shuffle polygraphic resolutions instead of resolving the symmetric operad by a symmetric polygraphic resolution.

**3.3.1. Bimodules over a symmetric operad.** Let  $P$  be a symmetric operad. Similarly to the shuffle case, a  $P$ -bimodule is a symmetric collection  $A$  equipped with two families of morphisms of symmetric collections:

$$\begin{aligned}\lambda &: P(k) \otimes P(n_1) \otimes \cdots \otimes A(n_i) \otimes \cdots \otimes P(n_k) \rightarrow A(n_1 + \cdots + n_k), \\ \rho &: A(k) \otimes P(n_1) \otimes \cdots \otimes P(n_k).\end{aligned}$$

We also define the *free  $P$ -bimodule* over a symmetric collection  $A$ , denoted by  $\mathcal{L}^P(A)$ , by setting

$$\mathcal{L}^P(A)(I) := \bigoplus_{k \geq 1} P(k) \otimes_{\mathfrak{S}_k} \left( \bigoplus_{f: I \rightarrow \{1, \dots, k\}} P(f^{-1}\{1\}) \otimes \cdots \otimes (A \circ_{\mathfrak{S}} P)(f^{-1}\{i\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \right).$$

**3.3.2. Symmetric partial composition.** The *symmetric partial composition* of symmetric collections  $V$  and  $W$  in  $\mathfrak{S}\text{Coll}$  is defined by

$$(V \circ_{\mathfrak{S}} W)(n) := \bigoplus_{\substack{k, \ell, i \geq 1 \\ k + \ell - 1 = n \\ \tau \in \text{III}(\ell - 1, k - i)}} V(k) \circ_{i, \tau} W(\ell),$$

where each vector space  $V(k) \circ_{i, \tau} W(\ell)$  is equipped with the following symmetric action: for  $x \circ_{i, \tau} y \in V \circ_{i, \tau} W$  and  $\sigma \in \mathfrak{S}_n$ ,

$$(x \circ_{i, \tau} y) \cdot \sigma = (x \cdot \tau_x) \circ_{j, \tau'} (y \cdot \tau_y),$$

where  $\sigma \in \mathfrak{S}_n$ ,  $j = \min\{\sigma^{-1}(i), \sigma^{-1}(i + \tau(1)), \dots, \sigma^{-1}(1 + \tau(\ell - 1))\}$ ,  $\tau' \in \mathfrak{S}_{n-j}$ ,  $\tau_x \in \mathfrak{S}_k$  and  $\tau_y \in \mathfrak{S}_\ell$  are appropriate permutations.

From the definitions of partial compositions, we have

$$(V \circ_{\mathfrak{S}} W)^u = V^u \circ_{\text{III}} W^u.$$

The partial composition  $\circ_{\mathfrak{S}}$  is not associative. However, as for  $\circ_{\text{III}}$ , if there is no possible confusion, we will use the left bracket rule for  $\circ_{\mathfrak{S}}$ .

Similarly to the shuffle case in 2.2.3, we define a natural transformation  $\varphi^{\mathfrak{S}}$  from the functor  $A \mapsto \bigoplus A \circ_{\mathfrak{S}} A^{\circ_{\mathfrak{S}} P}$  to the functor  $A \mapsto A \circ_{\mathfrak{S}} A$ .

**3.3.3. Symmetric  $\omega$ -operads as bimodules.** Theorem 2.4.8 can be restated in the symmetric case. In particular, we prove as in the shuffle case that the category  $\mathfrak{S}\text{Op}_{\omega}$  is isomorphic to the full subcategory of  $\text{RGlob}(\text{Bimod}(\mathfrak{S}\text{Op}))$  whose objects are pairs  $(P, A)$  where  $(P, \mu, \eta)$  is a symmetric operad and  $A = (A_n, \lambda_n, \rho_n)_{n \geq 0}$  is a reflexive globular  $P$ -bimodule such that  $A_0 = P$  and, for all  $n$ ,

$$\mu_{A_n}^{\uparrow} = \mu_{A_n}^{\downarrow}.$$

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**3.3.4. Cellular extensions.** Let us denote  $\mathfrak{S}\text{Coll}_B$  the category whose objects are pairs  $(V, B)$ , where  $V$  is a symmetric collection and  $B$  is an linear basis of  $V$ , and whose morphism  $(V, B) \rightarrow (V', B')$  are morphisms of symmetric collections  $V \rightarrow V'$ . For  $n \geq 0$ , a *cellular extension* of a symmetric  $n$ -operad  $P$  is an object  $(V, B)$  of  $\mathfrak{S}\text{Coll}_B$ , equipped with morphisms of symmetric collections

$$P_n \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} V$$

satisfying the globular relations  $s_{n-1}s_n = s_{n-1}t_n$  and  $t_{n-1}s_n = t_{n-1}t_n$ .

The following result constructs the free operad on a cellular extension. Its proof is the same as in the shuffle case of Proposition 3.1.3.

**3.3.5. Proposition.** *Let  $P$  be a symmetric  $(n-1)$ -operad and  $(V, B)$  be a cellular extension of  $P$ . Denote by  $V^\ominus$  the coequalizer of the two morphisms*

$$(P_0\langle V \rangle \oplus P_{n-1}) \odot (P_0\langle V \rangle \oplus P_{n-1}) \begin{array}{c} \xrightarrow{\mu_{P_0\langle V \rangle \oplus P_{n-1}}^\uparrow} \\ \xrightarrow{\mu_{P_0\langle V \rangle \oplus P_{n-1}}^\downarrow} \end{array} P_0\langle V \rangle \oplus P_{n-1}$$

in the category  $\text{Bimod}(P)$ . Then  $(P, V^\ominus)$  is the free symmetric  $n$ -operad on  $(P, (V, B))$ .

**3.3.6. Extended higher-dimensional symmetric operads.** For  $n \geq 0$ , the category of *extended symmetric  $n$ -operads*  $\mathfrak{S}\text{Op}_n^+$  is the category defined as follows:

- i) The objects are the triplets  $(P, (V, B))$  where  $P$  is a symmetric  $n$ -operad and  $(V, B)$  is a cellular extension of  $P$ .
- ii) A morphism  $(P, (V, B)) \rightarrow (P', (V', B'))$  is the data of a morphism of symmetric  $n$ -operads  $f : P \rightarrow P'$  and a morphism of symmetric collections  $g : V \rightarrow V'$  such that  $g$  sends  $B$  to  $B'$  increasingly, and such that the following squares commute in  $\mathfrak{S}\text{Coll}$ :

$$\begin{array}{ccc} P_n & \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} & V \\ f \downarrow & & \downarrow g \\ P'_n & \begin{array}{c} \xleftarrow{s_n} \\ \xleftarrow{t_n} \end{array} & V' \end{array}$$

**3.3.7. Free symmetric  $n$ -operad.** For  $n \geq 1$ , the forgetful functor  $\mathcal{W}_n^\ominus : \mathfrak{S}\text{Op}_n \rightarrow \mathfrak{S}\text{Op}_{n-1}^+$  that forgets the composition of  $n$ -cells admits a left adjoint

$$\mathcal{L}_n^\ominus : \mathfrak{S}\text{Op}_{n-1}^+ \rightarrow \mathfrak{S}\text{Op}_n \tag{3.3.8}$$

that associates to an extended  $(n-1)$ -operad  $(P, (V, B))$  the *free  $n$ -operad over  $(P, (V, B))$*  given by

$$\mathcal{L}_n^{\text{III}}(P, (V, B)) = (P, V^\ominus),$$

that we will denote by  $P[[V]]$ .

**3.3.9. Symmetric polygraphs.** Just as shuffle polygraphs in (3.1.6) are defined, we define a *symmetric  $n$ -polygraph*  $(V, B)$  inductively as a data  $((V_0, B_0), \dots, (V_n, X_n))$ , where for every  $0 \leq k < n$ ,  $(V_k, B_k)$  is a cellular extension of the free symmetric  $(k - 1)$ -operad generated by  $((V_0, B_0), \dots, (V_{k-1}, B_{k-1}))$  and denoted by

$$V_{k-1}^{\mathfrak{S}} = V_0^{\mathfrak{S}} \llbracket V_1 \rrbracket \cdots \llbracket V_k \rrbracket.$$

We denote by  $\mathfrak{S}\text{Pol}_n$  the category of symmetric  $n$ -polygraphs, and we define the category  $\mathfrak{S}\text{Pol}_\omega$  of symmetric  $\omega$ -polygraphs as limit of the forgetful functors  $\mathfrak{S}\text{Pol}_n \rightarrow \mathfrak{S}\text{Pol}_{n-1}$ .

**3.3.10. Presentation of a symmetric operad.** The symmetric operad presented by a symmetric 1-polygraph  $(V, B)$  is the coequalizer in the category  $\mathfrak{S}\text{Op}$  of the following source and target morphisms, denoted by  $\bar{V}$ ,

$$V_1^{\mathfrak{S}} \begin{array}{c} \xrightarrow{s_0} \\ \xrightarrow{t_0} \end{array} V_0^{\mathfrak{S}} \xrightarrow{\pi_V} \bar{V}.$$

**3.3.11. Symmetric polygraphic resolutions.** As in the shuffle case in (3.2.3), we define a *symmetric polygraphic resolutions* of a symmetric operad  $P$  as a symmetric  $\omega$ -polygraph  $(V, B)$  that presents the operad  $P$  and whose cellular extensions  $(V_k, B_k)$  are acyclic for  $k \geq 2$ .

The functor  $-^u : \mathfrak{S}\text{Coll} \rightarrow \text{Coll}$  defined in (2.1.2) extends into a functor

$$-^u : \mathfrak{S}\text{Pol}_\omega \rightarrow \text{III}\text{Pol}_\omega$$

sending a symmetric  $\omega$ -polygraph  $(V, B)$  on the shuffle  $\omega$ -polygraph  $(V, B)^u$ , whose set of  $k$ -generators is defined by  $((V, B)^u)_k := B_k$ . We shall write  $B$  for  $(V, B)^u$ . The following results relate shuffle and symmetric polygraphic resolutions.

**3.3.12. Proposition.** *Let  $P$  be a symmetric operad. Let  $(V, B)$  be a symmetric polygraphic resolution of  $P$ , then  $B$  is a shuffle polygraphic resolution of the shuffle operad  $P^u$  and there is an isomorphism of shuffle  $\omega$ -operads*

$$(V^{\mathfrak{S}})^u \simeq B^{\text{III}}.$$

*Proof.* By definition, the symmetric operad  $P$  is the coequalizer of  $V_1^{\mathfrak{S}} \rightrightarrows V_0^{\mathfrak{S}}$  in  $\mathfrak{S}\text{Op}$ . Note that this coequalizer is reflexive, so, by Proposition 2.3.7, it is calculated in  $\mathfrak{S}\text{Coll}$ . Moreover, the functor  $-^u : \mathfrak{S}\text{Coll} \rightarrow \text{Coll}$  preserves reflexive coequalizers because it is left adjoint to the functor  $\text{Hom}_{\text{Coll}}(\mathbf{k}[\mathfrak{S}], -)$ , so the shuffle operad presented by  $P^u$ , which is calculated as the reflexive coequalizer of  $B_1^{\text{III}} \rightrightarrows B_0^{\text{III}}$ , is isomorphic to  $P^u$ . Moreover, the functor  $-^u : \mathfrak{S}\text{Pol}_\omega \rightarrow \text{III}\text{Pol}_\omega$  preserves acyclicity by definition, proving that  $B$  is a shuffle polygraphic resolution of  $P^u$ .

Now let us show that the following square commutes

$$\begin{array}{ccc} \mathfrak{S}\text{Pol}_n & \xrightarrow{-^{\mathfrak{S}}} & \mathfrak{S}\text{Op}_n \\ -^u \downarrow & & \downarrow -^u \\ \text{III}\text{Pol}_n & \xrightarrow{-^{\text{III}}} & \text{III}\text{Op}_n \end{array}$$

by showing that the constructions of Propositions 3.1.3 and 3.3.5 commute with  $-^u$ .

## 4. Shuffle operadic rewriting

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Following isomorphism (2.1.5), we have an isomorphism  $(V_0^\varepsilon)^u \simeq (B_0)^{\text{III}}$ , for every vector space  $V_0$  and basis  $B_0$  of  $V_0$ . For every symmetric operad  $P$ , the functor  $(-)^u : \mathfrak{S}\text{Coll} \rightarrow \text{Coll}$  also commutes with  $\mathcal{L}^P$  and  $\mathcal{L}^{P^u}$ , respectively the free symmetric  $P$ -bimodule and shuffle  $P^u$ -bimodule functors. Indeed, the arguments showing that  $-^u$  sends  $\circ_\varepsilon$  to  $\circ_{\text{III}}$ , also apply to the explicit construction of free bimodules in (2.1.12). Finally, as mentioned above,  $-^u$  preserves reflexive coequalizers, and the morphism pairs  $(\mu^\uparrow, \mu^\downarrow)$  that appear in Propositions 3.1.3 and 3.3.5 are reflexive.

Thus every step of the construction of free symmetric and shuffle  $\omega$ -operads commutes with the functor  $-^u$ , so for every symmetric  $\omega$ -polygraph  $(V, B)$ , we have an isomorphism  $(V^\varepsilon)^u \simeq (B^u)^{\text{III}}$ .  $\square$

## 4. SHUFFLE OPERADIC REWRITING

The first part of this section presents the main rewriting properties of shuffle 1-polygraphs. We relate the notion of a convergent shuffle polygraph, whose 1-generators are oriented with respect to a given monomial order, with the notion of Gröbner bases introduced in [18], and with the notion of Poincaré-Birkhoff-Witt bases introduced in [27]. Throughout this section, all operads and polygraphs are shuffle.

### 4.1. Rewriting in shuffle operads

We introduce a concept of rewriting in the context of shuffle operads.

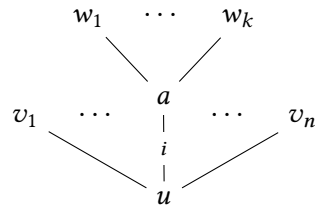
**4.1.1. The terminal indexed set.** Denote by  $\square$  the terminal object of  $\text{Ind}$ , that is, the indexed set that is a singleton  $\square(k) = \{\square_k\}$  for each arity  $k \geq 1$ . Denote by  $\iota_k : \text{Set} \rightarrow \text{Ind}$  the inclusion functor defined by  $\iota_k(X_0)(k) = X_0$  and  $\iota_k(X_0) = \emptyset$  for the other arities.

**4.1.2. One-hole contexts of indexed sets.** A *one-hole context* of an indexed set  $X_0$  is an element  $\Gamma$  of the free  $X_0^{\text{III}}$ -bimodule  $X_0^{\text{III}}\langle \square \rangle$ . We say that  $\Gamma$  is of *inner arity*  $k$  if it is an element of  $X_0^{\text{III}}\langle \square(k) \rangle$ .

Let  $A$  be an  $X_0^{\text{III}}$ -bimodule and  $a \in A(k)$ . Identifying  $A(k)$  with  $\text{Hom}_{\text{Set}}(\square(k), A(k))$ ,  $a$  induces a morphism

$$\varphi_a : X_0^{\text{III}}\langle \square(k) \rangle \rightarrow X_0^{\text{III}}\langle A(k) \rangle.$$

via the the functor  $X_0^{\text{III}}\langle \iota_k \rangle : \text{Set} \rightarrow \text{Bimod}(X_0^{\text{III}})$ . For  $\Gamma$  a one-hole context of  $X_0$  of inner arity  $k$ , we write  $\Gamma[a] := \varphi_a(\Gamma)$ . Explicitly,  $\Gamma[a]$  is a tree of the form



where  $k, n \geq 1$ ,  $i \in \{1, \dots, n\}$ ,  $u \in X_0^{\text{III}}(n)$ ,  $v_1, \dots, \check{v}_i, \dots, v_n, w_1, \dots, w_k \in X_0^{\text{III}}$  and  $\square_k$  appears in the  $i^{\text{th}}$  position. The notation  $\check{v}_i$  means that we omit  $v_i$ . In this way, every one-hole context  $\Gamma$  on  $X_0$  can be written  $\Gamma := w \circ_{i, \tau} (\square_k \mid \vec{w})$  with  $w, \vec{w} \in X_0^{\text{III}}$ . In this work, we will only consider *monomial* one-hole contexts, that is when  $w, \vec{w}$  are monomials of  $X_0^{*\text{III}}$ , so we will omit the word monomial.

**4.1.3. Two-hole contexts of indexed sets.** Let  $X_0$  be an indexed set. There exists a bifunctor  $C_2^{X_0} : \text{Ind} \times \text{Ind} \rightarrow \text{Ind}$  which sends a pair of indexed sets  $Y, Y'$  to the indexed set of elements of  $(X_0^{\text{III}} \sqcup Y \sqcup Y')^{*\text{III}}$  with one occurrence of both  $Y$  and  $Y'$ . A *two-hole context* of  $X_0$  is an element  $\Gamma$  of  $C_2^{X_0}(\square, \square)$ . We say that  $\Gamma$  is of *inner arities*  $(k, \ell)$  if it is an element of  $C_2^{X_0}(\square(k), \square(\ell))$ .

Let  $P$  be an operad equipped with a morphism  $\pi : X_0^{\text{III}} \rightarrow P$ , and  $a \in P(k), a' \in P(\ell)$ . Identifying

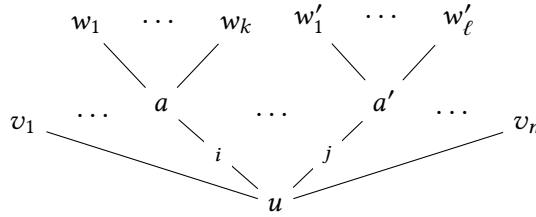
$$\begin{aligned} P(k) \times P(\ell) &\simeq \text{Hom}_{\text{Set}}(\square(k), P(k)) \times \text{Hom}_{\text{Set}}(\square(\ell), P(\ell)) \\ &\simeq \text{Hom}_{\text{Set} \times \text{Set}}((\square(k), \square(\ell)), (P(k), P(\ell))), \end{aligned}$$

the pair  $(a, a')$  induces a morphism

$$\varphi_{a, a'} : C_2^{X_0}(\square(k), \square(\ell)) \rightarrow C_2^{X_0}(P(k), P(\ell))$$

via the bifunctor  $C_2^{X_0}(\iota_k, \iota_\ell) : \text{Set} \times \text{Set} \rightarrow \text{Ind}$ . Moreover,  $f$  induces a morphism  $\pi_* : C_2^{X_0}(P(k), P(\ell)) \rightarrow P$ . For  $\Gamma$  a two-hole context of  $X_0$  of inner arities  $(k, \ell)$ , we write  $\Gamma(a, a') := i_* \varphi_{a, a'}(\Gamma)$ . Explicitly,  $\Gamma(a, a')$  is a tree of one of the following two forms, where the application of  $i$  is implicit:

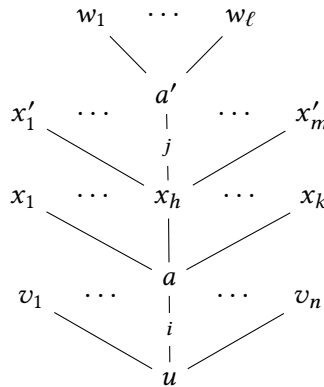
i)



where

$$\begin{aligned} n \geq 2, \quad i, j \in \{1, \dots, n\}, \quad u \in X_0^{\text{III}}(n), \text{ and} \\ v_1, \dots, \check{v}_i, \dots, \check{v}_j, \dots, v_n, w_1, \dots, w_k, w'_1, \dots, w'_\ell \in X_0^{\text{III}}, \end{aligned}$$

ii)



where

$$\begin{aligned} m, n \geq 1, \quad i \in \{1, \dots, n\}, \quad j \in \{1, \dots, m\}, \quad h \in \{1, \dots, k\}, \text{ and} \\ u, v_1, \dots, \check{v}_i, \dots, v_n, w_1, \dots, w_\ell, x_1, \dots, \check{x}_h, \dots, x_k, x'_1, \dots, \check{x}'_j, \dots, x'_m \in X_0^{\text{III}}. \end{aligned}$$



## 4. Shuffle operadic rewriting

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In this work, we will only consider *monomial* two-hole contexts, that is, when in **i)** and **ii)** the  $u, \vec{v}, \vec{x}, \vec{x}', \vec{w}, \vec{w}'$  are monomials of  $X_0^{*\text{III}}$ .

**4.1.4. left-monomiality and homogeneity.** A cellular extension  $X_1$  of  $X_0^{\text{III}}$  is *left-monomial* if, for every 1-generator  $\alpha$  in  $X_1$  the source  $s_0(\alpha)$  is a non-trivial 0-monomial, and  $s_0(\alpha) \notin \text{Supp}(t_0(\alpha))$ . A 1-polygraph is *left-monomial* if  $X_1$  is so. We prove that every 1-polygraph is Tietze equivalent to a left-monomial one. For  $N \geq 1$ , a cellular extension  $X_1$  of  $X_0^{\text{III}}$  is *homogeneous* if, for every 1-generator  $\alpha$  in  $X_1$  the weight of  $s_0(\alpha)$  and  $t_0(\alpha)$  are equals to  $N$ . A 1-polygraph is *N-homogeneous* if  $X_1$  is so. When  $N = 2$  we say *quadratic* for  $N$ -homogeneous.

**4.1.5. Rewriting step.** A *rewriting step* of a left-monomial 1-polygraph  $X$  is a 1-cell  $f$  of  $X_1^{\text{III}}$  of size 1 of the form

$$f = \lambda g + 1_c,$$

where  $\lambda \in \mathbf{k} \setminus \{0\}$ ,  $g$  is a 1-monomial of  $X_1^{\text{III}}$ , and  $c$  is a 0-cell of  $X_0^{\text{III}}$  such that the 0-monomial  $s_0(u) \notin \text{Supp}(c)$ . A 1-cell of  $X_1^{\text{III}}$  is *positive* if it is the  $\star_0$ -composition of rewriting steps.

A 0-cell  $a$  of  $X_0^{\text{III}}$  is *reduced* if there is no rewriting step with source  $a$ . We denote by  $\text{Red}(X)$  the indexed submodule of reduced 0-cells. The indexed set  $\text{Red}_m(X)$  of reduced 0-monomials of  $X_0^{\text{III}}$  forms a basis of  $\text{Red}(X)$ . A *normal form* of  $a$  is a reduced 0-cell  $b$  such that there is a positive 1-cell with source  $a$  and target  $b$ .

**4.1.6. Monomial orders and termination.** An *indexed poset*  $(X, <)$  is an indexed set  $X$ , such that each  $X(k)$  is equipped with a partial order  $<_k$ ; we will omit the index on  $<$ . An indexed poset  $(X, <)$  is *well-founded* if each  $X(k)$  is a well-founded poset.

Let  $X_0$  be an indexed set. An order relation  $<$  on the free monoid  $X_0^{*\text{III}}$  of tree monomials is *stable by product* if, for all  $u, u' \in X_0^{*\text{III}}(k)$ ,  $v, v' \in X_0^{*\text{III}}(\ell)$ ,  $i \in \{1, \dots, k\}$ , and  $\tau \in \text{III}(\ell - 1, k - i)$ ,  $u < u', v < v'$  implies  $u \circ_{i, \tau} v < u' \circ_{i, \tau} v'$ . A total order relation stable by product is called a *monomial order* on  $X_0^{*\text{III}}$ . Note that this notion also appears in [27] and [18].

For  $Y$  a left-monomial cellular extension of  $X_0^{\text{III}}$ , an order relation  $<$  on  $X_0^{*\text{III}}$  is *compatible with  $Y$*  if, for every 1-cell  $\alpha : u \rightarrow a$  of  $Y$  and every monomial  $v \in \text{Supp}(a)$ ,  $v < u$ . The relation  $<$  can be extended to the free shuffle operad  $X_0^{\text{III}}$  as follows: for two 0-cells  $a, b$  of  $X_0^{\text{III}}$ , we have  $b < a$  if the two following conditions are satisfied

- i)  $\text{Supp}(a) \setminus \text{Supp}(b) \neq \emptyset$ ,
- ii) for all  $v \in \text{Supp}(b) \setminus \text{Supp}(a)$ , there exists  $u \in \text{Supp}(a) \setminus \text{Supp}(b)$  such that  $v < u$ .

For a left-monomial 1-polygraph  $X$ , we denote by  $<_{X_1}$  the smallest partial order relation on  $X_0^{*\text{III}}$  stable by product and compatible with  $X_1$ . A 1-polygraph  $X$  is *terminating* if the relation  $<_{X_1}$  is well-founded. In that case, for every rewriting step  $f$  of  $X$ , we have  $t_0(f) <_{X_1} s_0(f)$ , and thus there does not exist infinite sequence of rewriting steps of  $X$ .

**4.1.7. Proposition.** *Let  $X$  be a left-monomial 1-polygraph. If  $X_0^{*\text{III}}$  admits a well-founded monomial order  $<$  compatible with  $X_1$ , then  $X$  is terminating.*

*Proof.* We have  $<_{X_1} \subseteq <$ , so  $<_{X_1}$  is well-founded, so  $X$  is terminating. □

However, the converse implication is not true. In general, in order to prove termination when no monomial order is known, it is necessary to use a proof strategy appropriated to the set of rules. The following gives an illustration for one of the simplest strategies.

**4.1.8. Proposition.** *A left-monomial 1-polygraph  $X$  terminates if, and only if, there exists a well-founded indexed poset  $(W, <)$  and a morphism of indexed sets  $\Phi : X_0^{*\text{III}} \rightarrow W$  such that  $\Phi(\Gamma[v]) < \Phi(\Gamma[s(\alpha)])$  holds for every 1-generator  $\alpha \in X_1$ , one-hole context  $\Gamma$ , and  $v \in \text{Supp}(t(\alpha))$ .*

*Proof.* Suppose that the polygraph  $X$  terminates. Then  $X_0^{*\text{III}}$  is equipped with a well-founded partial order  $<_{X_1}$ , and we set  $\Phi$  to be the identity morphism on  $X_0^{*\text{III}}$ .

Conversely, let  $<$  be the partial order generated by  $v < u$  if there exists a rewriting rule  $\alpha \in X_1$  and a one-hole context  $\Gamma$  such that  $u = \Gamma[s(\alpha)]$  and  $v \in \text{Supp}(\Gamma[t(\alpha)])$ . The order  $<$  is stable by product by considering

$$u \circ_{i,\tau} v < u' \circ_{i,\tau} v < u' \circ_{i,\tau} v',$$

for all tree monomials  $u, v, u', v'$  in  $X_0^{*\text{III}}$ , and is compatible with  $X_1$  by definition. Thus  $<_{X_1} \subseteq <$ , and so the map  $\Phi : (X_0^{*\text{III}}, <_{X_1}) \rightarrow W$  is a strictly monotone morphism of indexed posets. Since  $W$  is well-founded,  $(X_0^{*\text{III}}, <_{X_1})$  is as well, and so the 1-polygraph  $X$  terminates.  $\square$

**4.1.9. Example.** We consider the polygraph  $X$  with three 0-generators  $x, y, z$  and the following 1-generator:

$$\alpha : \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \\ y \quad x \\ \diagup \quad \diagdown \\ x \end{array} \longrightarrow \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \\ x \quad x \\ \diagup \quad \diagdown \\ x \end{array} + \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \\ y \quad y \\ \diagup \quad \diagdown \\ y \end{array} + \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \\ z \quad z \\ \diagup \quad \diagdown \\ z \end{array}$$

For  $u \in X_0^{*\text{III}}$ , we set  $\Phi(u) := |u|_x + 3|u|_{y-z}$ , where  $|u|_x$  denotes the number of occurrences of  $x$  in  $T(u)$  and  $|u|_{y-z}$  the number of inner vertices of  $T(u)$  whose two children are, from left to right,  $y$  and  $z$ .

Then  $\Phi(\Gamma[s(\alpha)]) > \Phi(\Gamma[v])$  for all contexts  $\Gamma$  of inner arity 4 and every  $v \in \text{Supp}(t(\alpha))$ . Indeed, for every  $\Gamma = w \circ_{i,\tau} (\square_4 \mid w_1 w_2 w_3 w_4)$ , we have

$$\begin{aligned} \Phi \left( \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ y \quad z \\ \diagup \quad \diagdown \\ x \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ x \quad x \\ \diagup \quad \diagdown \\ x \\ \diagup \quad \diagdown \\ w \end{array} \right) &= \left| \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} - \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} \right|_x + 3 \left| \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} - \begin{array}{c} x \quad x \\ \diagdown \quad \diagup \\ x \end{array} \right|_{y-z} \\ &= (1 - 1) + (3 - 0) = 1, \end{aligned}$$

$$\begin{aligned} \Phi \left( \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ y \quad z \\ \diagup \quad \diagdown \\ x \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ y \quad y \\ \diagup \quad \diagdown \\ y \\ \diagup \quad \diagdown \\ w \end{array} \right) &= \left| \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} - \begin{array}{c} y \quad y \\ \diagdown \quad \diagup \\ y \end{array} \right|_x + 3 \left| \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} y \quad y \\ \diagdown \quad \diagup \\ y \\ \diagup \quad \diagdown \\ w \end{array} \right|_{y-z} \\ &= 1 + 3 \left| \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} \right|_{y-z} + 3|w|_{y-z} - 3|w \circ_{i,\tau} y| \end{aligned}$$

$$= \begin{cases} 4 & \text{if } |w \circ_{i,\tau} y|_{y-z} = |w|_{y-z}, \\ 1 & \text{if } |w \circ_{i,\tau} y|_{y-z} = |w|_{y-z} + 1, \end{cases}$$

$$\Phi \left( \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ y \quad z \\ \diagup \quad \diagdown \\ x \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \diagdown \quad \diagup \\ z \quad z \\ \diagup \quad \diagdown \\ z \\ \diagup \quad \diagdown \\ w \end{array} \right) = \begin{cases} 4 & \text{if } |w \circ_{i,\tau} z|_{y-z} = |w|_{y-z}, \\ 1 & \text{if } |w \circ_{i,\tau} z|_{y-z} = |w|_{y-z} + 1. \end{cases}$$

## 4. Shuffle operadic rewriting

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Following Proposition 4.1.8 the polygraph  $X$  terminates. Note that, there is no monomial order that orients this rule in this way. Indeed, every orientation compatible with a monomial order reduces first one of the term of right hand side.

### 4.2. Confluence of shuffle polygraphs

In this subsection we define the property of confluence of a shuffle polygraph, and we give an algebraic characterization of this property.

**4.2.1. Branchings.** A *branching* of a 1-polygraph  $X$  is a pair  $(f, g)$  of positive 1-cells of  $X_1^{\text{III}}$  where  $f$  and  $g$  have the same source  $s_0(f) = s_0(g)$ , which we denote by  $s_0(f, g)$ . The branching  $(f, g)$  is said to be *local* if  $f$  and  $g$  are both rewriting steps.

Let  $X$  be a 2-polygraph. A branching  $(f, g)$  of the 1-polygraph  $X_{\leq 1}$  is  *$X_2$ -coherently confluent*, or  $(f, g)$  is *coherently confluent* for short, if there exist positive 1-cells  $h$  and  $k$  in  $X_1^{\text{III}}$  and a 2-cell  $F$  in  $X_2^{\text{III}}$  as in the following diagram

$$\begin{array}{ccccc}
 & f & \rightarrow & t_0(f) & \xleftarrow{h} \\
 s_0(f) = s_0(g) & \searrow & & \Downarrow F & \nearrow t_0(h) = t_0(k) \\
 & g & \rightarrow & t_0(g) & \xleftarrow{k}
 \end{array}$$

If  $u$  is a 0-cell of  $X_1^{\text{III}}$ , we say that the 2-polygraph  $X$  is *coherently confluent* (resp. *locally coherently confluent*) at  $u$  if every branching (resp. local branching) of  $X$  of source  $u$  is coherently confluent. We say that the 2-polygraph  $X$  is *coherently confluent* (resp. *locally coherently confluent*) if it is so at every 0-cell of  $X_0^{\text{III}}$ , and that  $X$  is *coherently convergent* if it is terminating and coherently confluent. A 1-polygraph  $X$  is *confluent* (resp. *locally confluent*) if the 2-polygraph  $(X_0, X_1, \text{Sph}(X_1^{\text{III}}))$  has the corresponding coherent property. A 1-polygraph is *convergent* if it is both terminating and confluent.

**4.2.2. Classification of local branchings.** We classify local branchings into four types:

- i) *aspherical branchings*:  $(f, f)$ , where  $f$  is a rewriting step.
- ii) *additive branchings*:  $(\lambda f + \mu 1_v + 1_c, \lambda 1_u + \mu g + 1_c)$ , where  $f : u \rightarrow a$  and  $g : v \rightarrow b$  are 1-monomials,  $\lambda$  and  $\mu$  are nonzero scalars,  $c$  is a 0-cell,  $u \neq v$ , and  $u, v \notin \text{Supp}(c)$ .
- iii) *multiplicative branchings*:  $(\lambda \Gamma[f, 1_v] + 1_c, \lambda \Gamma[1_u, g] + 1_c)$ , where  $\Gamma$  is a two-hole context,  $f : u \rightarrow a$  and  $g : v \rightarrow b$  are 1-monomials,  $\lambda$  is a nonzero scalar,  $c$  is a 0-cell, and  $\Gamma[u, v] \notin \text{Supp}(c)$ .
- iv) *intersecting branchings*: the rest of the local branchings.

We define a well-founded partial order  $\sqsubseteq$  on branchings as follows: for every one-hole context  $\Gamma$  and every 0-cell  $c$ ,

$$(f, g) \sqsubseteq (\Gamma[f] + 1_c, \Gamma[g] + 1_c).$$

The *critical branchings* are the minimal intersecting branchings for this order. We denote the intersecting branchings by  $(\Gamma[\alpha] + 1_c, \Delta[\beta] + 1_c)$ , where  $\alpha, \beta$  are 1-generators of  $X$ ,  $\Gamma, \Delta$  are one-hole contexts,

and  $s_0(\Gamma[\alpha]) = s_0(\Delta[\beta])$ . A critical pair  $(\Gamma[\alpha], \Delta[\beta])$  is *minimal* if, for any 1-monomial  $\Lambda[\gamma]$  with  $\gamma$  a 1-generator, such that  $\Lambda[\gamma] = s_0(\Gamma[\alpha], \Delta[\beta])$ ,  $T(s_0(\alpha)) \setminus T(s_0(\beta)) \subseteq T(s_0(\gamma))$  or  $T(s_0(\beta)) \setminus T(s_0(\alpha)) \subseteq T(s_0(\gamma))$ .

Let  $X$  be a 2-polygraph. If  $u$  is a 0-cell of  $X_1^{\text{III}}$ , we say that the 2-polygraph  $X$  is *critically coherently confluent at  $u$*  if every minimal critical branching of  $X$  of source  $u$  is coherently confluent. We say that the 2-polygraph  $X$  is *critically coherently confluent* if it is so at every 0-cell of  $X_0^{\text{III}}$ .

**4.2.3. Lemma ([24, Lemmata 3.1.3 and 4.1.2.]).** *Let  $X$  be a 2-polygraph such that  $X_{\leq 1}$  is left-monomial, and a 0-cell  $a$  in  $X_0^{\text{III}}$  such that  $X$  is coherently confluent at  $b$  for any  $b <_{X_1} a$ . If  $f$  is a 1-cell of  $X_1^{\text{III}}$  that decomposes*

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_p} a_p$$

*into 1-cells of size 1, with  $a_i <_{X_1} a$  for all  $i \in \{1, \dots, p-1\}$ , then there exists a 0-cell  $a'$ , 1-cells  $g, h$ , and a 2-cell  $F$  in  $X_2^{\text{III}}$  such that*

$$\begin{array}{ccc} & a_p & \\ f \nearrow & & \searrow h \\ a_0 & & a' \\ & \Downarrow F & \\ & a_0 & \xrightarrow{g} a' \end{array}$$

*When  $p = 1$ , then  $F$  is an identity 2-cell.*

*Proof.* The proof of this result for shuffle polygraphs is the same as for polygraphs of associative algebras given in [24].  $\square$

**4.2.4. Theorem (Coherent critical branchings theorem).** *Let  $X$  be a 2-polygraph such that  $X_{\leq 1}$  is terminating and left-monomial. If  $X$  is critically coherently confluent, then it is coherently confluent.*

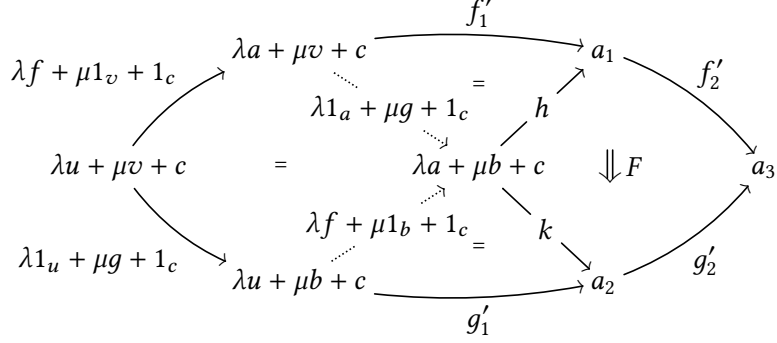
*Proof.* The structure of the proof is the same as for the similar result for associative algebras given in [24]. The primary difference is that we prove that we can restrict the hypotheses to the critical branchings that are minimal.

Suppose that  $X$  is a critically coherently confluent 2-polygraph. We proceed by noetherian induction on the sources of the branchings of  $X_{\leq 1}$ , with the order  $<_{X_1}$ , to prove that  $X$  is coherently confluent at every 0-cell of  $X_0^{\text{III}}$ . A reduced 0-cell cannot be the source of a local branching, so  $X$  is coherently confluent at reduced 0-cells. Now, fix a nonreduced 0-cell  $a_0$  of  $X_0^{\text{III}}$ , and assume that  $X$  is coherently confluent at every  $b <_{X_1} a_0$ . Then we proceed by case analysis on the type of the local branchings. First, an aspherical branching is always coherently confluent.

**Additive branchings.** Let  $(\lambda f + \mu 1_v + 1_c, \lambda 1_u + \mu g + 1_c)$  be an additive branching of source  $a_0$ , where  $f : u \rightarrow a$  and  $g : v \rightarrow b$  are 1-monomials of  $X_1^{\text{III}}$ ,  $\lambda, \mu$  are nonzero scalar, and 0-cell  $c$  of  $X_0^{\text{III}}$ , with

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$u \neq v$  and  $u, v \notin \text{Supp}(c)$ . We construct the following coherent confluent diagram:

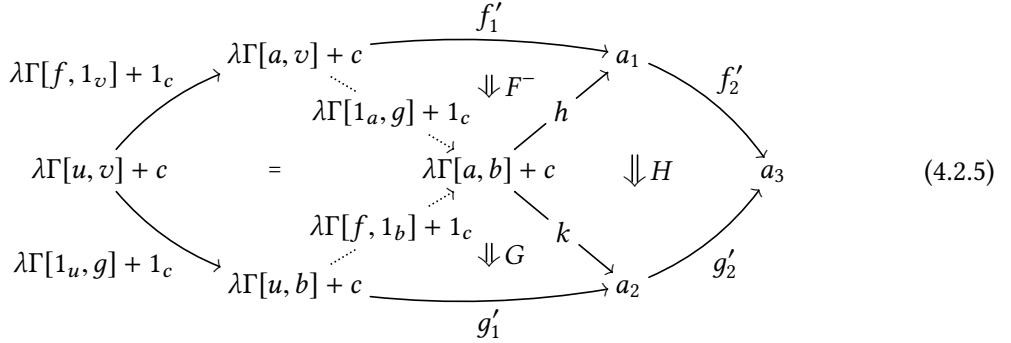


By linearity of the 0-composition, the square on the left is aspherical of the form  $(\lambda f + \mu g + 1_c, \lambda f + \mu g + 1_c)$ . The dotted 1-cells  $\lambda 1_a + \mu g + 1_c$  and  $\lambda f + \mu 1_b + 1_c$  are not positive in general, since possibly  $u \in \text{Supp}(b)$  or  $v \in \text{Supp}(a)$ . However, those 1-cells are of size 1, and by Lemma 4.2.3 there exist positive 1-cells  $f'_1, g'_1, h$  and  $k$  that satisfy

$$f'_1 = (\lambda 1_a + \mu g + 1_c) \star_0 h \quad \text{and} \quad g'_1 = (\lambda f + \mu 1_b + 1_c) \star_0 k.$$

Now,  $a <_{X_1} u, b <_{X_1} v$ , and  $\lambda, \mu \in \mathbf{k} \setminus \{0\}$  imply  $\lambda a + \mu b + c <_{X_1} \lambda u + \mu v + c$ . Thus, the branching  $(h, k)$  is coherently confluent by induction hypothesis, yielding the positive 1-cells  $f'_2$  and  $g'_2$  and the 2-cell  $F$  in  $X_2^{\text{III}}$ .

**Multiplicative branchings.** Let  $(\lambda \Gamma[f, 1_v] + 1_c, \lambda \Gamma[1_u, g] + 1_c)$  be a multiplicative branching of source  $a_0$ , where  $\Gamma$  is a two-hole context,  $f : u \rightarrow a$  and  $g : v \rightarrow b$  are 1-monomials of  $X_1^{\text{III}}$ ,  $\lambda$  is a nonzero scalar,  $c$  is a 0-cell of  $X_0^{\text{III}}$ , and  $\Gamma[u, v] \notin \text{Supp}(c)$ , we construct the following coherent confluence diagram



By linearity of the 0-composition, the square on the left is aspherical of the form  $(\lambda \Gamma[f, g] + 1_c, \lambda \Gamma[f, g] + 1_c)$ . The dotted 1-cells  $\lambda \Gamma[1_a, g] + 1_c$  and  $\lambda \Gamma[f, 1_b] + 1_c$  are not positive in general, since possibly either  $\text{Supp}(\Gamma[u, b]) \cap \text{Supp}(c) \neq \emptyset$  or  $\text{Supp}(\Gamma[a, v]) \cap \text{Supp}(c) \neq \emptyset$ . Let  $a = \sum_{i=1}^p \mu_i m_i$  be the monomial decomposition of  $a$ . The 1-cell  $\lambda \Gamma[1_a, g] + 1_c$  can be written

$$\lambda \Gamma[1_a, g] + 1_c = g_1 \star_0 \cdots \star_0 g_p,$$

where the 1-cells  $g_i$  are of size 1, and of the form

$$g_j = \sum_{1 \leq i < j} \lambda \mu_i \Gamma[m_i, b] + \lambda \mu_j \Gamma[m_j, g] + \sum_{j < i \leq p} \lambda \mu_i \Gamma[m_i, u'] + 1_c.$$

We have  $m_i \prec_{X_1} u$  for every  $i$ , and  $b \prec_{X_1} v$ , giving  $s_0(g_j) \prec_{X_1} \lambda\Gamma[u, v] + c$ , for every  $1 \leq j \leq p$ . By Lemma 4.2.3 applied on the 1-cell  $\lambda\Gamma[u, v] + c$ , there exist positive 1-cells  $f'_1, h$  and a 2-cell  $F$  in  $X_2^{\text{III}}$  as in the diagram (4.2.5). The positive 1-cells  $g'_1, k$  and the 2-cell  $G$  in  $X_2^{\text{III}}$  are constructed similarly. Having  $\lambda\Gamma[a, b] + c \prec_{X_1} \lambda\Gamma[u, v] + c$ , by induction hypothesis, one deduces that the branching  $(h, k)$  is coherently confluent, giving the positive 1-cells  $f'_2, g'_2$  and the 2-cell  $H$  in  $X_2^{\text{III}}$ .

**Critical branchings.** Let  $(f, g) = (\Gamma[\alpha], \Delta[\beta])$  be a critical branching of source  $a_0$ , where  $\Gamma$  and  $\Delta$  are one-hole contexts and  $\alpha : u \rightarrow a$  and  $\beta : v \rightarrow b$  are 1-generators of  $X$ . We proceed by noetherian induction on the size of  $\Gamma[u] = \Delta[v]$ . If  $(f, g)$  is minimal, then by hypothesis it is coherently confluent, so there exist positive 1-cells  $f', g'$  and a 2-cell  $F$  in  $X_2^{\text{III}}$  as in the following diagram

$$\begin{array}{ccccc} & \Gamma[\alpha] & \rightarrow & \Gamma[a] & \xrightarrow{f'} & d \\ & \searrow & & \Downarrow F & \nearrow & \\ \Gamma[u] = \Delta[v] & & & & & \\ & \Delta[\beta] & \rightarrow & \Delta[b] & \xrightarrow{g'} & d \end{array}$$

Otherwise, by non-minimality of  $(f, g)$ , there exists a 1-generator  $\gamma : w \rightarrow c$  of  $X$  and a one-hole context  $\Lambda$  such that  $\Lambda[w] = \Gamma[u] = \Delta[v]$  and such that neither  $T(u) \setminus T(v)$  nor  $T(v) \setminus T(u)$  are contained in  $T(w)$ . We also write  $h = \Lambda[\gamma]$ . Thus the weights of  $T(u) \cup T(w)$  and  $T(w) \cup T(v)$  are less than the weight of  $T(u) \cup T(v) = T(s_0(f, g))$ . Let  $\Gamma_0, \Gamma_1, \Delta_0, \Delta_1, \Lambda_0, \Lambda'_0$  be non-trivial one-hole contexts such that

$$\begin{aligned} f &= \Gamma_1\Gamma_0[\alpha] = \Gamma[\alpha], & g &= \Delta_1\Delta_0[\beta] = \Delta[\beta], \\ h &= \Gamma_1\Lambda_0[\gamma] = \Lambda[\gamma] = \Delta_1\Lambda'_0[\gamma], \end{aligned}$$

and  $(\Gamma_0[\alpha], \Lambda_0[\gamma])$  and  $(\Lambda'_0[\gamma], \Delta_0[\beta])$  are minimal branchings for  $\sqsubseteq$ . By monomiality of contexts these branchings are either multiplicative, or critical. In the multiplicative case, we have constructed coherent confluences in the previous cases. In critical case, the weights of the sources of these two branchings are less than the weight of  $s_0(f, g)$ , so we apply the induction hypothesis. In all cases, we get the coherent confluences

$$\begin{array}{ccc} \Gamma_0[\alpha] \rightarrow \Gamma_0[a] \xrightarrow{f_0} d_0 & & \Lambda'_0[\gamma] \rightarrow \Lambda'_0[c] \xrightarrow{f'_0} d'_0 \\ \Gamma_0[u] = \Lambda_0[w] \searrow \Downarrow F_0 \nearrow & & \Lambda'_0[w] = \Delta_0[v] \searrow \Downarrow F'_0 \nearrow \\ \Lambda_0[\gamma] \rightarrow \Lambda_0[c] \xrightarrow{g_0} d_0 & & \Delta_0[\beta] \rightarrow \Delta_0[b] \xrightarrow{g'_0} d'_0 \end{array}$$

Applying the contexts  $\Gamma_1$  to the left diagram and  $\Delta_1$  to the right diagram, we obtain the coherent confluence

$$\begin{array}{ccccc} & \Gamma_1[f_0] & \rightarrow & \Gamma_1[d_0] & \xrightarrow{h_1} & d \\ & \searrow f & & \Downarrow \Gamma_1[F_0] & \nearrow & \\ s_0(f, g) & \xrightarrow{h} & \Lambda[c] & \xrightarrow{\Gamma_1[g_0]} & \Gamma_1[d_0] & \\ & \searrow g & & \Downarrow \Delta_1[F'_0] & \nearrow & \\ & \Delta[b] & \rightarrow & \Delta_1[d'_0] & \xrightarrow{h'_1} & d \\ & & & \Delta_1[g'_0] & & \end{array}$$

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where  $J$  is given by the global induction hypothesis.

**Non-critical intersecting branchings.** Finally, let  $(f, g) = (\lambda\Gamma[\alpha] + 1_c, \lambda\Delta[\beta] + 1_c)$  be an intersecting branching of source  $a_0$  that is not critical, where  $\alpha : u \rightarrow a, \beta : v \rightarrow b$  are 1-generators of  $X$ ,  $\Gamma, \Delta$  are one-hole contexts,  $\lambda$  is a nonzero scalar, and  $c$  is a 0-cell of  $X_0^{\text{III}}$ , such that  $s_0(f, g) \notin \text{Supp}(c)$ . Let  $\Gamma_0, \Gamma_1, \Delta_0$  be one-hole contexts such that  $\Gamma_1\Gamma_0[\alpha] = \Gamma[\alpha]$ ,  $\Gamma_1\Delta_0[\beta] = \Delta[\beta]$ , and  $(\Gamma_0[\alpha], \Delta_0[\beta])$  is a critical branching. In other words,  $(\Gamma_0[\alpha], \Delta_0[\beta])$  is the smallest branching smaller than  $(f, g)$  for the partial order  $\sqsubseteq$ . By the previous case, we have the coherent confluence diagram

$$\begin{array}{ccccc} & \Gamma_0[\alpha] & \xrightarrow{\quad} & \Gamma_0[a] & \xrightarrow{f'_0} & d_0 \\ \Gamma_0[u] = \Delta_0[v] & & & \Downarrow F_0 & & \\ & \Delta_0[\beta] & \xrightarrow{\quad} & \Delta_0[b] & \xrightarrow{g'_0} & d_0 \end{array}$$

We then construct the following coherent confluence diagram

$$\begin{array}{ccccccc} & & & & f'_1 & & \\ & & & & \searrow & & \\ \lambda\Gamma[\alpha] + 1_c & \xrightarrow{\quad} & \lambda\Gamma[a] + c & \xrightarrow{\quad} & a_1 & \xrightarrow{f'_2} & a_3 \\ & & \lambda\Gamma_1[f'_0] + 1_c & \xrightarrow{\quad} & h & & \\ & & \Downarrow F & & \lambda\Gamma_1[d_0] + c & \xrightarrow{\quad} & \Downarrow I \\ \lambda\Gamma[u] + c = \lambda\Delta[v] + c & & & & & & \\ & & \lambda\Gamma_1[g'_0] + 1_c & \xrightarrow{\quad} & k & & \\ & & \Downarrow H & & a_2 & \xrightarrow{g'_2} & a_3 \\ \lambda\Delta[\beta] + 1_c & \xrightarrow{\quad} & \lambda\Delta[b] + c & \xrightarrow{\quad} & a_2 & \xrightarrow{g'_1} & a_3 \end{array}$$

The dotted 1-cells are not positive in general, in particular when  $\text{Supp}(c)$  intersects  $\text{Supp}(\Gamma[a])$  or  $\text{Supp}(\Delta[b])$ . However, the 1-cell  $\Gamma_1[f'_0]$  is positive because  $f'_0$  is positive, hence a 0-composite  $\Gamma_1[f'_0] = l_1 \star_0 \cdots \star_0 l_p$  of rewriting steps. Thus we have the chain of inequalities

$$\Gamma[u] > \Gamma[a] = s(l_1) > \cdots > s(l_p) > \Gamma_1[d_0].$$

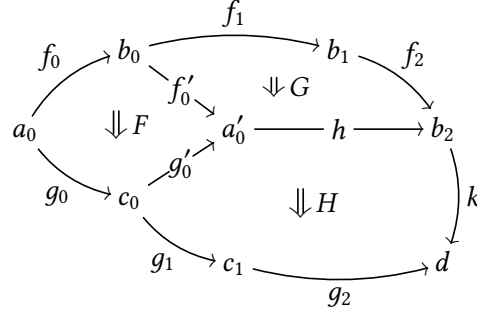
Since we have  $\lambda \neq 0$  and  $\Gamma[u] \notin \text{Supp}(c)$  by hypothesis, the inequality  $\lambda\Gamma[u] + c > \lambda s(l_i) + c$  holds for every  $i$ , so that the following decomposition of the 1-cell  $\lambda\Gamma_1[f'_0] + 1_c$  satisfies the hypotheses of Lemma 4.2.3, thus we have a decomposition

$$\lambda\Gamma_1[f'_0] + 1_c = (\lambda l_1 + 1_c) \star_0 \cdots \star_0 (\lambda l_p + 1_c).$$

This gives positive 1-cells  $f'_1, h$ , and the 2-cell  $G$  in  $X_2^{\text{III}}$ . We proceed similarly to get the positive 1-cells  $g'_1, k$  and the 2-cell  $H$  in  $X_2^{\text{III}}$ . Finally, apply the induction hypothesis on  $(h, k)$ , since  $\lambda\Gamma_1[d_0] + c <_{X_1} \lambda\Gamma[u] + c$ , to get  $f'_2, g'_2$ , and  $I$  as in the diagram.

**Newman's lemma [28].** Thus we have shown that  $X$  is locally coherently confluent at  $a_0$ . To conclude the induction step, let us show that  $X$  is coherently confluent at  $a_0$ . Let  $(f, g)$  be a non-local branching of source  $a_0$ , and write  $f = f_0 \star_0 f_1, g = g_0 \star_0 g_1$ , where  $f_0 : a_0 \rightarrow b_0, g_0 : a_0 \rightarrow c_0$  are rewriting steps

and  $f_1 : b_0 \rightarrow b_1, g_1 : c_0 \rightarrow c_1$  are positive 1-cells of  $X^{\text{III}}$ . We then construct the coherent confluence diagram



By hypothesis, the local branching  $(f_0, g_0)$  is confluent, yielding the positive 1-cells  $f'_0$  and  $g'_0$  and the 2-cell  $F$ . Since both  $a_0 > b_0$  and  $a_0 > c_0$ , the induction hypothesis applies to the branching  $(f_1, f'_0)$  to give  $f_2, h$ , and  $G$ , and, then, to the branching  $(g'_0 \star_0 h, g_1)$  to give  $k, g_2$ , and  $H$ .  $\square$

**4.2.6. Operad presented by an ideal.** Let  $X_0$  be an indexed set and  $I$  an ideal of the free shuffle operad  $X_0^{\text{III}}$ . We equip the collection  $\mathbb{I} \oplus I$  with a shuffle operad structure, with unit  $\eta : \mathbb{I} \hookrightarrow \mathbb{I} \oplus I$  and multiplication given by the following composition

$$\begin{aligned} \mu_I : (\mathbb{I} \oplus I) \circ_{\text{III}} (\mathbb{I} \oplus I) &\simeq (\mathbb{I} \oplus I) \oplus (I \circ_{\text{III}} (\mathbb{I} \oplus I)) \\ &\rightarrow (\mathbb{I} \oplus I) \oplus (I \circ_{\text{III}} X_0^{\text{III}}) \\ &\xrightarrow{1 \oplus \rho} (\mathbb{I} \oplus I) \oplus I \\ &\rightarrow \mathbb{I} \oplus I, \end{aligned}$$

where  $\rho$  is the right action of  $I$  as an  $X_0^{\text{III}}$ -bimodule. Denote by  $X_0^{\text{III}}/I$  the coequalizer of the morphisms of shuffle operads

$$\mathbb{I} \oplus I \begin{array}{c} \xrightarrow{[1 \ 1]} \\ \xrightarrow{[1 \ 0]} \end{array} X_0^{\text{III}}$$

in  $\text{III}Op$ . Note that the underlying collection of  $X_0^{\text{III}}/I$  is the cokernel of the inclusion  $I \hookrightarrow X_0^{\text{III}}$  in  $\text{Coll}$ .

Let  $X$  be a 1-polygraph. The *boundary* of a 1-generator  $\alpha$  in  $X$  is the 1-cell  $\partial(\alpha) := s_0(\alpha) - t_0(\alpha)$ , and we set  $\partial(X_1) := \{ \partial(\alpha) \mid \alpha \in X_1 \}$ . We denote by  $I(X)$  the ideal of the free operad  $X_0^{\text{III}}$  generated by the set of boundaries of the 1-generators of  $X$ , that is the free  $X_0^{\text{III}}$ -bimodule generated by  $\partial(X_1)$ . Explicitly, the ideal  $I(X)$  is made of all the linear combinations

$$\sum_{i=1}^p \lambda_i \Gamma_i[\partial(\alpha_i)]$$

where  $\lambda_i$  is a scalar and  $\Gamma_i$  is a one-hole context. Note that the operad  $\overline{X}$  presented by  $X$  is isomorphic to  $X_0^{\text{III}}/I(X)$ .

**4.2.7. Proposition.** *For a terminating left-monomial 1-polygraph  $X$ , the following assertions are equivalent :*



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i)  $X$  is confluent.

ii)  $\text{Red}(X) \cap I(X) = 0$ .

iii)  $X_0^{\text{III}} = \text{Red}(X) \oplus I(X)$ .

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $X$  is confluent and prove that  $\text{Red}(X) \cap I(X)$  is reduced to 0. Let  $a$  be in  $\text{Red}(X) \cap I(X)$ . On the one hand,  $a$  is reduced and, thus, admits itself as only normal form. On the other hand, if  $a$  is in  $I(X)$ , then

$$a = \sum_{i=1}^p \lambda_i \Gamma_i [\partial(\alpha_i)],$$

where  $\lambda_i \in \mathbf{k}$  and  $\Gamma_i$  is a one-hole context. Hence the following 1-cell  $f$  of  $X_1^{\text{III}}$  has source  $a$  and target 0:

$$f = \sum_{i=1}^p \lambda_i \Gamma_i [\alpha_i - t_0(\alpha_i)].$$

Since  $X$  is confluent, this implies that  $a$  and 0 have the same normal form, if any. And since 0 is reduced, this implies that 0 is a normal form of  $a$ .

(ii)  $\Rightarrow$  (iii) Suppose that  $\text{Red}(X) \cap I(X) = 0$  and prove that  $X_0^{\text{III}} = \text{Red}(X) + I(X)$ . Since the 1-polygraph  $X$  terminates, every 0-cell  $a$  of  $X_0^{\text{III}}$  admits at least a normal form  $b$  in  $\text{Red}(X)$ . Let  $f$  be a positive 1-cell of  $X_1^{\text{III}}$  that reduces  $a$  into  $b$ . There is decomposition of  $f$  into rewriting steps:

$$f = f_1 \star_0 \cdots \star_0 f_p,$$

where  $f_i$  applies the 1-generator  $\alpha_i$ . Since  $t(f_i) = s(f_{i+1})$ , we have  $a - b = \partial(f_1) + \cdots + \partial(f_p)$ . Moreover, by definition  $\partial(f_i)$  belongs to  $I(X)$ , and thus so does  $a - b$ . Finally, writing  $a = b + (a - b)$  we have the decomposition

$$X_0^{\text{III}} = \text{Red}(X) + I(X).$$

(iii)  $\Rightarrow$  (i) Suppose that there is a decomposition  $X_0^{\text{III}} = \text{Red}(X) \oplus I(X)$ , and consider a branching  $(f, g)$  of  $X$ , with  $f : a \rightarrow b$  and  $g : a \rightarrow c$ . Since  $X$  terminates, there exist positive 1-cells  $h : b \rightarrow b'$  and  $k : c \rightarrow c'$  in  $X_1^{\text{III}}$  with  $b'$  and  $c'$  reduced, and thus  $b' - c'$  is reduced. Having a 1-cell  $(f \star_0 h)^- \star_0 (g \star_0 k)$  with source  $b'$  and target  $c'$ , we prove as above that  $b' - c'$  belongs to  $I(X)$ . The hypothesis gives  $b' - c' = 0$ , so that  $(f, g)$  is confluent.  $\square$

**4.2.8. Completion procedure.** We define a completion procedure for polygraphs analogously to the completion procedure for Gröbner bases of operads [18]. Given a terminating 1-polygraph  $X$ ,

(i) for every critical branching  $(f, g)$  of  $X$ , we reduce the 0-cells  $t_0(f)$  and  $t_0(g)$  to some normal forms. If  $\widehat{t_0(f)} \neq \widehat{t_0(g)}$ :

$$\begin{array}{ccccc} & f & \longrightarrow & t_0(f) & \longrightarrow & \widehat{t_0(f)} \\ & \searrow & & & & \uparrow \text{---} h \text{---} \\ u & & & & & \downarrow \text{---} h \text{---} \\ & \swarrow & & & & \downarrow \text{---} h \text{---} \\ & g & \longrightarrow & t_0(g) & \longrightarrow & \widehat{t_0(g)} \end{array}$$

and we add a 1-generator  $h$  in order to reach confluence of the branching ;

(ii) the addition of 1-cells in the step (i) can create new critical branchings, whose confluence must also be completed as in (i) ;

(iii) Repeat the previous steps until there are no non-confluent critical branchings.

The orientation of the 1-cell  $h$  in Step (i) can be determined by a reduction strategy with respect to the relative positions of the source of reductions  $f$  and  $g$  on  $u$ . Moreover, the orientation of  $h$  must preserve termination. In general we cannot ensure that an orientation can be chosen to preserve termination. In that case, the procedure fails.

**4.2.9. Proposition.** *When the procedure (4.2.8) on a 1-polygraph  $X$  does not fail, it produces a (possibly infinite) convergent polygraph that presents the operad  $\overline{X}$ .*

### 4.3. Monomial-ordered shuffle polygraphs

**4.3.1. Gröbner bases [18].** Let  $X_0$  be an indexed set and  $\leq$  be a monomial order on the free operad  $X_0^{\text{III}}$ . If  $a$  is a nonzero 0-cell of  $X_0^{\text{III}}$ , the *leading monomial* of  $a$  is the maximum element  $\text{lm}(a)$  of  $\text{Supp}(a)$  with respect to  $\leq$ , and 0 when  $\text{Supp}(a)$  is empty. The *leading coefficient* of  $a$  is the coefficient  $\text{lc}(a)$  of  $\text{lm}(a)$  in  $a$ , and the *leading term* of  $a$  is the element  $\text{lt}(a) := \text{lc}(a)\text{lm}(a)$  of  $X_0^{\text{III}}$ . Observe that, for  $a, b$  in  $X_0^{\text{III}}$ , we have  $a < b$  if, and only if, either  $\text{lm}(a) < \text{lm}(b)$  or ( $\text{lt}(a) = \text{lt}(b)$  and  $a - \text{lt}(a) < b - \text{lt}(b)$ ). For  $Y$  an indexed subset of  $X_0^{\text{III}}$ , we denote by  $\text{lm}(Y)$  the indexed set of leading monomials of elements of  $Y$ .

Let  $I$  be an ideal of the free operad  $X_0^{\text{III}}$ . A *Gröbner basis* for  $I$  with respect to  $\leq$  is an indexed subset  $\mathcal{G}$  of  $I$  such that the ideals of  $X_0^{\text{III}}$  generated by  $\text{lm}(I)$  and by  $\text{lm}(\mathcal{G})$  coincide.

**4.3.2. Proposition.** *If  $X$  is a convergent left-monomial 1-polygraph, and  $\leq$  is a monomial order on  $X_0^{\text{III}}$  that is compatible with  $X_1$ , then the indexed set  $\partial(X_1)$  forms a Gröbner basis of  $I(X)$ .*

*Conversely, let  $X_0$  be an indexed set, let  $\leq$  be a monomial order on  $X_0^{\text{III}}$ , let  $I$  be an ideal of  $X_0^{\text{III}}$  and  $\mathcal{G}$  be a subset of  $I$ . Define  $X(\mathcal{G})$  as the 1-polygraph with 0-cells  $X$  and a 1-cell*

$$\alpha_a : \text{lm}(a) \rightarrow \text{lm}(a) - \frac{1}{\text{lc}(a)}a$$

*for each  $a$  in  $\mathcal{G}$ . If  $\mathcal{G}$  is a Gröbner basis for  $I$ , then  $X(\mathcal{G})$  is a convergent left-monomial presentation of  $X_0^{\text{III}}/I$ , such that  $I(X(\mathcal{G})) = I$ , and  $\leq$  is compatible with  $X(\mathcal{G})_1$ .*

*Proof.* Suppose that  $X$  is convergent. For every 1-cell  $\alpha$  of  $X$ ,  $\partial(\alpha)$  is in  $I(X)$ . Since  $\leq$  is compatible with  $X_1$ , we have  $\text{lm}(\partial(\alpha)) = s(\alpha)$  for every 1-cell  $\alpha$  of  $X$ . Now, if  $a$  is in  $I(X)$ , it is a linear combination

$$a = \sum_i \lambda_i \Gamma_i[\partial(\alpha_i)]$$

of 1-cells  $\Gamma_i[\partial(\alpha_i)]$ , where  $\alpha_i$  is a 1-generator of  $X$  and  $\Gamma_i$  is a one-hole context of  $X$ . This implies that

$$\text{lm}(a) = \Gamma_i[s(\alpha_i)] = \Gamma_i[\text{lm}(\partial(\alpha_i))]$$

holds for some  $i$ . Thus  $\partial(X_1)$  is a Gröbner basis for  $(I(X), \leq)$ .

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Conversely, assume that  $\mathcal{G}$  is a Gröbner basis for  $(I, \leq)$ . The monomial order  $\leq$  is compatible with  $X(\mathcal{G})_1$ , hence by Proposition 4.1.7, the polygraph  $X(\mathcal{G})$  terminates. Moreover, we have  $I(X(\mathcal{G})) = I$ , so the algebra presented by  $X(\mathcal{G})$  is indeed isomorphic to  $X_0^{\text{III}}/I$ . Moreover, the reduced monomials of  $X(\mathcal{G})^{\text{III}}$  are the monomials of  $X_0^{\text{III}}$  that cannot be decomposed as  $\Gamma[\text{lm}(a)]$  with  $a$  in  $\mathcal{G}$  and  $\Gamma$  a one-hole context of  $X(\mathcal{G})$ . Thus, if a reduced 0-cell  $a$  of  $X(\mathcal{G})^{\text{III}}$  is in  $I$ , its leading monomial must be 0, because  $\mathcal{G}$  is a Gröbner basis of  $(I, \leq)$ . By proposition 4.2.7, we get that  $X(\mathcal{G})$  is confluent.  $\square$

**4.3.3. Poincaré-Birkhoff-Witt bases [27].** Let  $P$  be a operad, let  $X_0$  be a generating indexed set of  $P$ , and let  $\leq$  be a monomial order of  $X_0^{\text{III}}$ . A *Poincaré-Birkhoff-Witt (PBW) basis* for  $(P, X_0, \leq)$  is an indexed subset  $\mathcal{B}$  of  $X_0^{\text{III}}$  such that:

- i)  $\mathcal{B}$  is a linear basis of  $P$ , for  $u \in X_0^{\text{III}}$ , we write  $[u]_{\mathcal{B}} := \sum_i \lambda_i w_i$  its decomposition in  $P$  on the basis  $\mathcal{B}$ ,
- ii) for all  $u, v$  in  $\mathcal{B}$  and all compatible elementary compositions  $\circ_{i,\tau}$ , either  $u \circ_{i,\tau} v$  belongs to  $\mathcal{B}$  or  $u \circ_{i,\tau} v > [u \circ_{i,\tau} v]_{\mathcal{B}}$ ,
- iii) a tree monomial  $u$  of  $X_0^{\text{III}}$  is in  $\mathcal{B}$  if, and only if, for every decomposition  $u = \Gamma(x \circ_{i,\tau} x')$  of  $u$  where  $x, x' \in X_0$  and  $\Gamma$  is a one-hole context of  $X_0^{\text{III}}$ ,  $x \circ_{i,\tau} x' \in \mathcal{B}$ .

**4.3.4. Proposition.** *If  $X$  is a convergent left-monomial quadratic presentation of an operad  $P$ , and  $\leq$  is a monomial order on  $X_0^{\text{III}}$  compatible with  $X_1$ , then the indexed set  $\text{Red}_m(X)$  is a PBW basis for  $(P, X_0, \leq)$ .*

*Conversely, let  $P$  be a quadratic operad,  $X$  a generating indexed set of  $P$ ,  $\leq$  a monomial order on  $X_0^{\text{III}}$ , and  $\mathcal{B}$  a PBW basis of  $(A, X_0, \leq)$ . Define  $X(\mathcal{B})$  as the 1-polygraph with 0-cells  $X_0$  and with a 1-cell*

$$x \circ_{i,\tau} x' \xrightarrow{\alpha_{x \circ_{i,\tau} x'}} [x \circ_{i,\tau} x']_{\mathcal{B}}$$

*for all  $x, x'$  in  $X_0 \cap \mathcal{B}$  such that  $x \circ_{i,\tau} x' \neq [x \circ_{i,\tau} x']_{\mathcal{B}}$  in  $X_0^{\text{III}}$ . Then  $X(\mathcal{B})$  is a quadratic convergent left-monomial presentation of  $P$  such that  $\text{Red}_m(X(\mathcal{B})) = \mathcal{B}$  and  $\leq$  is compatible with  $X(\mathcal{B})_1$ .*

*Proof.* Suppose that  $X$  is a quadratic convergent left-monomial presentation of an operad  $P$ . By proposition 4.2.7, we have the following exact sequence of collections:

$$0 \rightarrow I(X) \rightarrow X_0^{\text{III}} \rightarrow \text{Red}(X) \rightarrow 0.$$

Since  $P$  is isomorphic to  $X_0^{\text{III}}/I(X)$  as an operad, it is also isomorphic to  $\text{Red}(X)$  as a collection, and therefore  $\text{Red}_m(X)$  is a basis of  $P$ . The fact that  $\leq$  is compatible with  $X_1$  implies axiom (ii) of PBW bases. Axiom (iii) comes from the definition of a reduced monomial for a quadratic left-monomial 1-polygraph.

Conversely, assume that  $\mathcal{B}$  is a PBW basis for  $(P, X, \leq)$ . By definition,  $X(\mathcal{B})$  is quadratic and left-monomial, and axiom (iii) of PBW bases implies  $\text{Red}_m(X(\mathcal{B})) \cap I(X(\mathcal{B})) = 0$ . Termination of  $X(\mathcal{B})$  is given by axiom (ii) of PBW bases because  $\leq$  is well-founded. By proposition 4.2.7, it is sufficient to prove that  $\text{Red}(X(\mathcal{B})) \cap I(X(\mathcal{B})) = 0$  to get confluence: on the one hand, a reduced 0-cell  $u$  of  $\text{Red}(X(\mathcal{B}))$  is a linear combination of 0-cells of  $\mathcal{B}$ , so that  $u$  is its only normal form; and, on the other hand, if  $u$  belongs to  $I(X(\mathcal{B}))$ , then  $u$  admits 0 as a normal form. Finally, the operad presented by  $X(\mathcal{B})$  is isomorphic to  $\text{Red}(X(\mathcal{B}))$ , that is to  $\mathbf{k}\mathcal{B}$ , hence to  $P$  by the previous exact sequence and because  $\mathcal{B}$  is a linear basis of  $P$ .  $\square$

## 5. SHUFFLE POLYGRAPHIC RESOLUTIONS FROM CONVERGENCE

In this section, all operads and polygraphs are shuffle. We recall from [24] the characterization of the property of acyclicity for an  $\omega$ -polygraph through the existence of a homotopical contraction. Subsection 5.2, presents the main result of this article, Theorem 5.2.9, that extends a reduced convergent left-monomial 1-polygraph into a polygraphic resolution of the presented operad. In Subsection 5.3, given a polygraphic resolution of an operad, we construct a bimodule resolution for the operad. Finally, in Subsection 5.4 we prove a criterion of Koszulness in terms of quadratic convergence.

### 5.1. Polygraphic resolutions and contractions

In this first subsection, we extend to  $\omega$ -operads the notion of homotopy developed in [24] for  $\omega$ -algebras, see also [4] and [23]. Then we introduce the notion of a *contraction* of a polygraph, which allow us to characterize acyclic  $\omega$ -polygraphs.

**5.1.1. Homotopies.** Let  $P$  and  $Q$  be  $\omega$ -operads and  $F, G : P \rightarrow Q$  be morphisms of  $\omega$ -operads. A *homotopy* from  $F$  to  $G$  is a graded linear map

$$\eta : P \rightarrow Q$$

of degree 1, *i.e.*,  $\eta$  sends  $n$ -cells to  $n + 1$ -cells), such that, writing  $\eta_a$  for  $\eta(a)$ ,

**i)** for every  $n \geq 0$ , for every  $n$ -cell  $a$  of  $P$ ,

$$s_n(\eta_a) = F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(a)} \tag{5.1.2}$$

$$t_n(\eta_a) = \eta_{s_{n-1}(a)} \star_{n-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(a), \tag{5.1.3}$$

**ii)** for all  $0 \leq k < n$  and every  $\star_k$ -composable pair  $(a, b)$  of  $n$ -cells of  $P$ ,

$$\begin{aligned} \eta_{a \star_k b} &= F(s_{k+1}(a)) \star_0 \eta_{t_0(b)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(b)} \star_k \eta_b \\ &\quad \star_{k+1} \eta_a \star_k \eta_{s_{k-1}(a)} \star_{k-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(t_{k-1}(b)), \end{aligned}$$

**iii)** for all  $n \geq 0$  and every  $n$ -cell  $a$  of  $P$ ,

$$\eta_{1_a} = 1_{\eta_a}.$$

In order for this definition to be licit, we need to check that the  $\star_k$ -compositions of **(i)** are well defined. See [4, appendix B.8.] or [24, 5.1.1] for the verification. Note that the mappings  $a \mapsto s(\eta_a)$  and  $a \mapsto t(\eta_a)$  are operads morphisms because both are composites of operads morphisms. The globularity of  $\eta_a$  follows from

$$ss(\eta_a) = s(F(a)) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-2} \eta_{t_{n-2}(a)} = s(\eta_{s(a)}) = st(\eta_a)$$

$$\text{and } ts(\eta_a) = t(\eta_{t(a)}) = \eta_{s_{n-2}(a)} \star_{n-2} \cdots \star_1 \eta_{s_0(a)} \star_0 t(G(a)) = tt(\eta_a).$$

## 5. Shuffle polygraphic resolutions from convergence

5.1.4. Let us expand the homotopy  $\eta$  in low dimension. It maps a 1-cell  $f : a \rightarrow a'$  of  $P$  to a 2-cell

$$\begin{array}{ccccc} & & F(f) & \rightarrow & F(a') & \xrightarrow{\eta_{a'}} & G(a') \\ & & \searrow & & \Downarrow \eta_f & & \nearrow \\ F(a) & & & & & & \\ & & \nearrow & & \eta_a & \rightarrow & G(a) & \xrightarrow{G(f)} & G(a') \end{array}$$

of  $Q$ , and a 2-cell  $A : f \Rightarrow f' : a \rightarrow a'$  of  $P$  to the following 3-cell of  $Q$

$$\begin{array}{ccc} \begin{array}{ccccc} F(f) & \xrightarrow{F(A)} & F(a') & \xrightarrow{\eta_{a'}} & G(a') \\ \Downarrow & & \Downarrow \eta_{f'} & & \\ F(a) & \xrightarrow{F(f')} & G(a) & \xrightarrow{G(f')} & G(a') \end{array} & \xrightarrow{\eta_A} & \begin{array}{ccccc} F(f) & \xrightarrow{F(A)} & F(a') & \xrightarrow{\eta_{a'}} & G(a') \\ \Downarrow & & \Downarrow \eta_f & & \Downarrow G(A) \\ F(a) & \xrightarrow{F(f')} & G(a) & \xrightarrow{G(f')} & G(a') \end{array} \end{array}$$

5.1.5. **Unital sections and contractions.** Let  $X$  be a  $\omega$ -polygraph. A *unital section* of  $X$  is a morphism of  $\omega$ -operads  $\iota : \bar{X} \rightarrow X^{\text{III}}$  that is a section of the canonical projection  $\pi : X^{\text{III}} \rightarrow \bar{X}$  and such that  $\iota_1 = 1$ , where  $1 \in \mathbf{k} \subseteq X^{\text{III}}(1)$ . Concretely,  $\iota$  assigns to every 0-cell  $a$  of  $\bar{X}$  a representative 0-cell  $\iota_a$  in  $X^{\text{III}}$ , in such a way that is the identity on the unit  $\mathbf{k}$ . Note that a unital section is not necessarily compatible with shuffle composition. For  $a$  an  $n$ -cell of  $X^{\text{III}}$ , we write  $\widehat{a}$  for  $\iota\pi(a)$ . Note that  $\widehat{a} = 1_{\overline{s_0(a)}}$  for  $n \geq 1$ .

Fix  $\iota$  a unital section of  $X$ . An  $\iota$ -*contraction* of  $X$  is a homotopy  $\sigma : id_{X^{\text{III}}} \rightarrow \iota\pi$  such that  $\sigma_a = 1_a$  for every  $n$ -cell  $a$  of  $X^{\text{III}}$  that belongs to the image of  $\iota$  or  $\sigma$ .

We say that  $\sigma$  is a *right*  $\iota$ -contraction if, for all  $n \geq 0$ ,  $f, g$   $n$ -cells of  $X^{\text{III}}$ , and compatible elementary composition  $\circ_{i,\tau}$ ,

$$\sigma_{f \circ_{i,\tau} g} = (s_0(f) \circ_{i,\tau} \sigma_g) \star_0 \sigma_{f \circ_{i,\tau} \widehat{g}}. \quad (5.1.6)$$

5.1.7. **Lemma.** Let  $\sigma$  be a  $\iota$ -contraction. For  $n \geq 1$  and every  $n$ -cell  $a$  of  $X^{\text{III}}$ ,

$$s_n(\sigma_a) = a - t_{n-1}(a) + \sigma_{t_{n-1}(a)} \quad \text{and} \quad t_n(\sigma_a) = \sigma_{s_{n-1}(a)}. \quad (5.1.8)$$

Note that for  $a$  a 0-cell of  $X^{\text{III}}$ ,  $s_0(\sigma_a) = a$  and  $t_0(\sigma_a) = \widehat{a}$ .

*Proof.* Let us first prove

$$a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_k \sigma_{t_k(a)} = a - t_k(a) + \sigma_{t_k(a)}$$

by induction on  $k \in \{0, \dots, n-1\}$ . The result is clear for  $k = 0$ . For  $k \geq 1$ , we calculate

$$\begin{aligned} a \star_0 \eta_{t_0(a)} \star_1 \cdots \star_k \sigma_{t_k(a)} &= a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_{k-1} \sigma_{t_{k-1}(a)} \\ &\quad - t_k(a \star_0 \sigma_{t_0(a)} \star_1 \cdots \star_{k-1} \sigma_{t_{k-1}(a)}) \\ &\quad + \sigma_{t_k(a)} \\ &= (a - t_{k-1}(a) + \sigma_{t_{k-1}(a)}) - t_k(a - t_{k-1}(a) + \sigma_{t_{k-1}(a)}) + \sigma_{t_k(a)} \\ &= a - t_k(a) + \sigma_{t_k(a)}, \end{aligned}$$

## 5.1. Polygraphic resolutions and contractions

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the last equality coming from the fact that  $t_k t_{k-1}(a) = t_{k-1}(a)$  and  $t_k(\sigma_{t_{k-1}(a)}) = \sigma_{t_{k-1}(a)}$ . Applying  $k = n - 1$  and (5.1.2) with  $F = id_{X^{\text{III}}}$ , we conclude that

$$s_n(\sigma_a) = a - t_{n-1}(a) + \sigma_{t_{n-1}(a)}.$$

For the second equation, we proceed similarly to show that, for all  $k \in \{0, \dots, n - 1\}$ ,

$$\sigma_{s_k(a)} \star_k \cdots \star_1 \sigma_{s_0(a)} \star_0 \widehat{a} = \widehat{a} - s_k(\widehat{a}) + \sigma_{s_k(a)} = \sigma_{s_k(a)}$$

because  $\widehat{a} = 1_{\overline{s_0(a)}}$ . Applying  $k = n - 1$  and (5.1.3) with  $G = \iota\pi$ , we conclude that

$$t_n(\sigma_a) = \sigma_{s_0(a)}.$$

□

**5.1.9. Reduced and essential monomials.** Let  $\iota$  be an unital section of  $X$ , and  $\sigma$  an  $\iota$ -contraction of an  $\omega$ -polygraph  $X$ . A 0-monomial  $u$  of  $X^{\text{III}}$  is  $\iota$ -reduced if  $\widehat{u} = u$ . A non- $\iota$ -reduced 0-monomial  $u$  of  $X^{\text{III}}$  is  $\iota$ -essential if  $u = (x \mid \vec{v})$  where  $x$  is a 0-generator of  $X$  and  $v_1, \dots, v_k$  are  $\iota$ -reduced 0-monomials of the  $\omega$ -operad  $X^{\text{III}}$ .

For  $n \geq 0$ , an  $n$ -monomial  $a$  of  $X^{\text{III}}$  is  $\sigma$ -reduced if it is an identity or in the image of  $\sigma$ . If  $\sigma$  is a right  $\iota$ -contraction of  $X$  and  $n \geq 0$ , then a non- $\sigma$ -reduced  $n$ -monomial  $a$  of  $X^{\text{III}}$  is  $\sigma$ -essential if  $a = (\alpha \mid \vec{v})$ , where  $\alpha$  is a  $n$ -generator of  $X$  and  $v_1, \dots, v_k$  are  $\iota$ -reduced 0-monomials of the  $\omega$ -operad  $X^{\text{III}}$ .

**5.1.10. Lemma.** *Let  $X$  be an  $\omega$ -polygraph and  $\iota$  a unital section of  $X$ . A right  $\iota$ -contraction  $\sigma$  of  $X$  is uniquely and entirely determined by its values on the  $\iota$ -essential 0-monomials and, for  $n \geq 1$ , on the  $\sigma$ -essential  $n$ -monomials of  $X^{\text{III}}$ .*

*Proof.* The proof follows the same arguments as in the case of associative algebras given in [24, Section 5.2], and it is divided in two steps:

- i) First, we prove that a homotopy  $\eta : F \rightarrow G$  between morphisms of  $\omega$ -operads  $F, G : X^{\text{III}} \rightarrow X^{\text{III}}$  is uniquely and entirely determined by its values on  $n$ -monomials for all  $n \geq 0$ , provided it satisfies the following relation:

$$\eta \mu_{X_n^{\text{III}}}^{\uparrow} = \eta \mu_{X_n^{\text{III}}}^{\downarrow}, \tag{5.1.11}$$

where  $\mu_{X_n^{\text{III}}}^{\uparrow}$  and  $\mu_{X_n^{\text{III}}}^{\downarrow}$  are defined considering  $X_n^{\text{III}}$  as a  $X_0^{\text{III}}$ -bimodule.

- ii) Next, we prove that the values of a right  $\iota$ -contraction on  $n$ -monomials are uniquely and entirely determined by the values on  $\iota$ -essential and  $\sigma$ -essential monomials, and that the resulting values satisfy (5.1.11).

(i) Proceed by induction on  $n \geq 0$ . For  $n = 0$ , assume that  $\eta_u : F(u) \rightarrow G(u)$  is a fixed 1-cell of  $X^{\text{III}}$  for every 0-monomial  $u$  of  $X^{\text{III}}$ . Extend  $\eta$  uniquely to every 0-cell  $a$  of  $X^{\text{III}}$  by linearity.

Now fix  $n \geq 1$  and assume that an  $(n + 1)$ -cell  $\eta_u$  of  $X^{\text{III}}$  has been chosen for every  $n$ -monomial  $u$  of  $X^{\text{III}}$ , with source and target given by the definition of homotopies, such that (5.1.11) holds for

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$n$ -monomials. By construction, the  $n$ -cells of  $X^{\text{III}}$  are linear combinations of  $n$ -monomials of  $X^{\text{III}}$  and of identities of  $(n-1)$ -cells of  $X^{\text{III}}$  up to the relation

$$\mu_{X_0^{\text{III}} \langle X_n \rangle \oplus X_{n-1}^{\text{III}}}^{\uparrow} = \mu_{X_0^{\text{III}} \langle X_n \rangle \oplus X_{n-1}^{\text{III}}}^{\downarrow}.$$

Thus we can extend  $\eta$  to all  $n$ -cells  $a$  of  $X^{\text{III}}$  by choosing a decomposition of  $a$  into a linear combination of  $n$ -monomials and an identity, and using (5.1.11) to ensure that the resulting cell does not depend on the choice of decomposition. We check that the source and target of the resulting  $(n+1)$ -cell  $\eta_a$  match the definition of homotopies by linearity of  $F, G$  and the  $\star_k$ -compositions.

(ii) First, we construct  $\sigma$  as a graded linear map by induction on  $n$ . For  $n=0$ , if  $u$  is a non- $\iota$ -essential monomial, then either  $u = \widehat{u}$ , or  $u = (x \mid \vec{v})$  where  $x$  is a 0-cell of  $X$  and some  $v_i$  is a non- $\iota$ -reduced monomial. In the former case,  $\sigma_u = 1_u$  is forced because  $u$  is  $\iota$ -reduced. In the latter case, take  $i$  maximal. Writing  $(x \mid \vec{v}) = (x \mid v_1 \cdots 1 \cdots v_k) \circ_{i,\tau} v_i$  for some shuffle permutation  $\tau$ , (5.1.6) imposes

$$\sigma_{(x \mid \vec{v})} = ((x \mid v_1 \cdots 1 \cdots v_k) \circ_{i,\tau} \sigma_{v_i}) \star_0 \sigma_{(x \mid v_1 \cdots \widehat{v}_i \cdots v_k)}.$$

Then proceed by induction on the weight of the  $v_i$  to define  $\sigma_{v_i}$  from the values of  $\sigma$  on  $\iota$ -reduced monomials.

Now let  $n \geq 1$ . For every  $n$ -monomial  $\Gamma[\alpha]$ , with  $\alpha$  a  $n$ -generator of  $X$  and  $\Gamma$  a one-hole context of  $X$ , writing

$$\Gamma[\alpha] = u \circ_{i,\tau} (\alpha \mid \vec{v})$$

and

$$(\alpha \mid \vec{v}) = \alpha \circ_{k,\tau_k} v_k \circ_{k-1,\tau_{k-1}} \cdots \circ_{1,\tau_1} v_1,$$

the equation (5.1.6) imposes that we set

$$\begin{aligned} \sigma_{\Gamma[\alpha]} := & (u \circ_{i,\tau} (s_0(\alpha) \mid \sigma_{v_1} v_2 \cdots v_k)) \star_0 \cdots \star_0 (u \circ_{i,\tau} (s_0(\alpha) \mid \widehat{v}_1 \cdots \widehat{v}_{k-1} \sigma_{v_k})) \\ & \star_0 (u \circ_{i,\tau} \sigma_{(\alpha \mid \vec{v})}) \star_0 \sigma_{u \circ_{i,\tau} (\alpha \mid \vec{v})}, \end{aligned}$$

where  $(\alpha \mid \vec{v})$  is a shortcut for  $(\alpha \mid \widehat{v}_1 \cdots \widehat{v}_k)$ . Let us check that this definition is well-founded. The  $\sigma_{v_i}$  are defined by induction on the weight of the  $v_i$ , and  $\sigma_{u \circ_{i,\tau} (\alpha \mid \vec{v})} = \sigma_{u \circ_{i,\tau} (s_0(\alpha) \mid \vec{v})}$  is defined by induction on  $n$ . It remains to check that  $\sigma_{(\alpha \mid \vec{v})}$  is defined. If  $(\alpha \mid \vec{v})$  is  $\sigma$ -essential, then it is defined by hypothesis. Otherwise,  $(\alpha \mid \vec{v})$  is  $\sigma$ -reduced, in which case  $(\alpha \mid \vec{v}) = \sigma_b$  for some  $(n-1)$ -cell  $b$  of  $X^{\text{III}}$ , which imposes  $\sigma_{(\alpha \mid \vec{v})} := 1_{\sigma_b}$ .

Now it remains only to show (5.1.11) and then apply the first point. More explicitly, we need to show

$$\sigma_{u \circ_{i,\tau} s_0(v)} + \sigma_{t_0(u) \circ_{i,\tau} v} - \sigma_{t_0(u) \circ_{i,\tau} s_0(v)} = \sigma_{u \circ_{i,\tau} t_0(v)} + \sigma_{s_0(u) \circ_{i,\tau} v} - \sigma_{s_0(u) \circ_{i,\tau} t_0(v)}$$

for all  $u, v$  two  $n$ -monomials of  $X^{\text{III}}$  and compatible elementary composition  $\circ_{i,\tau}$ . Write  $a = s_0(u)$ ,  $a' = t_0(u)$ ,  $b = s_0(v)$ ,  $b' = t_0(v)$ , and  $\cdot = \circ_{i,\tau}$ . On the one hand,

$$\begin{aligned} \sigma_{u \cdot b} + \sigma_{a' \cdot v} - \sigma_{a' \cdot b} &= (a \cdot \sigma_b) \star_0 \sigma_{u \cdot \widehat{b}} + (a' \cdot \sigma_v) \star_0 \sigma_{a' \cdot \widehat{v}} - (a' \cdot \sigma_b) \star_0 \sigma_{a' \cdot \widehat{b}} \\ &= a \cdot \sigma_b + \sigma_{u \cdot \widehat{b}} - a \cdot \widehat{b} + a' \cdot \sigma_v - a' \cdot \sigma_b, \end{aligned}$$

## 5.2. Polygraphic resolution from a convergent presentation

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and on the other hand,

$$\begin{aligned}\sigma_{u \cdot b'} + \sigma_{a \cdot v} - \sigma_a \cdot b' &= (a \cdot \sigma_{b'}) \star_0 \sigma_{u \cdot \widehat{b}'} + (a \cdot \sigma_v) \star_0 \sigma_{a \cdot \widehat{v}} - (a \cdot \sigma_{b'}) \star_0 \sigma_{a \cdot \widehat{b}'} \\ &= \sigma_{u \cdot \widehat{b}'} + a \cdot \sigma_v - a \cdot \widehat{v} \\ &= \sigma_{u \cdot \widehat{b}} + a \cdot \sigma_v - a \cdot \widehat{b}\end{aligned}$$

Therefore it remains to prove

$$a \cdot \sigma_b + a' \cdot \sigma_v = a \cdot \sigma_v + a' \cdot \sigma_b. \quad (5.1.12)$$

Since  $\star_0$ -composition in  $X^{\text{III}}$  is a morphism of  $\infty$ -operads, we have

$$u \cdot \sigma_b = u \cdot b \star_0 a' \cdot \sigma_b = a \cdot \sigma_b \star_0 u \cdot \widehat{b}.$$

Using the linear expression of  $\star_0$ -composition, we get

$$u \cdot b + a' \cdot \sigma_b - a' \cdot b = a \cdot \sigma_b + u \cdot \widehat{b} - a \cdot \widehat{b}$$

Similarly, considering  $u \cdot \sigma_v$ , we get

$$u \cdot b + a' \cdot \sigma_v - a' \cdot b = a \cdot \sigma_v + u \cdot \widehat{b} - a \cdot \widehat{b}.$$

Taking the difference of the two previous equations gives us (5.1.12).  $\square$

**5.1.13. Proposition.** *Let  $X$  be an  $\omega$ -polygraph with a fixed unital section  $\iota$ . Then  $X$  is a polygraphic resolution of the  $\omega$ -operad  $\overline{X}$  if, and only if,  $X$  admits a right  $\iota$ -contraction.*

*Proof.* Suppose that  $X$  is a polygraphic resolution of the operad  $\overline{X}$ , and define a right  $\iota$ -contraction  $\sigma$  of  $X$ . Using Lemma 5.1.10, we shall define  $\sigma$  on  $\iota$ - and  $\sigma$ -essential  $n$ -monomials of  $X^{\text{III}}$  by induction on  $n \geq 0$ . If  $(x \mid \vec{v})$  is an  $\iota$ -essential 0-monomial, then  $\pi_X(x \mid \vec{v}) = \pi_X(\overline{(x \mid \vec{v})})$  in  $\overline{X}$ , hence there exists a 1-cell  $\sigma_{(x \mid \vec{v})} : (x \mid \vec{v}) \rightarrow \overline{(x \mid \vec{v})}$  in  $X^{\text{III}}$ . Now assume that  $\sigma$  is defined on the  $n$ -cells of  $X^{\text{III}}$  for  $n \geq 0$  and let  $(\alpha \mid \vec{v})$  be a  $\sigma$ -essential  $(n+1)$ -monomial of  $X^{\text{III}}$ . The  $n$ -cells defining  $s(\sigma_{(\alpha \mid \vec{v})})$  and  $t(\sigma_{(\alpha \mid \vec{v})})$  as in (5.1.8) are parallel, so, by acyclicity of  $X$ , there exists an  $(n+2)$ -cell  $\sigma_{(\alpha \mid \vec{v})}$  with this source and target in  $X^{\text{III}}$ .

Conversely, let  $\sigma$  be a right  $\iota$ -contraction of the polygraph  $X$ , and let  $a, b$  be parallel  $n$ -cells of  $X^{\text{III}}$  for  $n \geq 1$ . We have  $t(\sigma_a) = \sigma_{s(a)} = \sigma_{s(b)} = t(\sigma_b)$  by (5.1.8), so the  $(n+1)$ -cell  $\sigma_a \star_n \sigma_b^-$  is well defined, with source  $s(\sigma_a)$  and target  $s(\sigma_b)$ . Since  $t_k(a) = t_k(b)$  for  $k \in \{0, \dots, n-1\}$ , we find that

$$(\sigma_a \star_n \sigma_b^-) \star_{n-1} \sigma_{t_{n-1}(a)}^- \star_{n-2} \cdots \star_1 \sigma_{t_0(a)}^-$$

is a well defined  $(n+1)$ -cell of  $X^{\text{III}}$  of source  $a$  and target  $b$ , thus proving that  $X_{n+1}$  is an acyclic extension of  $X_n^{\text{III}}$ . Thus  $X$  is a polygraphic resolution of  $\overline{X}$ .  $\square$

## 5.2. Polygraphic resolution from a convergent presentation

This subsection contains the main result of this article. We show how to extend a reduced left-monomial convergent shuffle 1-polygraph into a shuffle polygraphic resolution of its presented operad. The  $n$ -generators of the resolution correspond to certain overlappings of the 1-generators of the polygraph. We deduce an improved version of Squier's coherence results by showing that we can restrict the generating confluences to critical branching satisfying some minimality conditions.



## 5. Shuffle polygraphic resolutions from convergence

**5.2.1. Reduced polygraphs.** Let  $X$  be a left-monomial 1-polygraph. Recall from [3] that a 1-generator  $\alpha \in X_1$  is *right* (resp. *left*) *reduced* if  $t_0(\alpha) \in \text{Red}(X_1)$  (resp.  $s_0(\alpha) \in \text{Red}(X_1 \setminus \{\alpha\})$ ). We say that  $X$  is *reduced* when each of its 1-generators is left and right reduced. We prove that every (finite) convergent left-monomial 1-polygraph is Tietze-equivalent to a reduced (finite) convergent left-monomial 1-polygraph.

**5.2.2. Reducible divisors and crowns.** Let  $X$  be a left-monomial 1-polygraph. For  $u \in X_0^{*\text{III}}$  a 0-monomial, a *reducible divisor of  $u$*  is a minimal non-reduced submonomial of  $T(u)$ . Denote by  $D(u)$  the set of reducible divisors of  $u$ . Note that if  $v$  is a submonomial of  $u$ , then  $D(v) \subseteq D(u)$ .

Let  $u, v_1, \dots, v_k$  be 0-monomials, with  $u$  of arity  $k$ , and  $f$  a shuffle surjection such that  $(u \mid_f \vec{v})$  is defined. We say that  $(\vec{v}, f)$

- i) *creates reducible divisors* if  $D(u \mid_f \vec{v}) \setminus D(u)$  is nonempty,
- ii) *creates reducible divisors context-minimally* if, for all submonomials  $w_i$  of  $v_i$ , 0-monomials  $\vec{w}' \neq \vec{1}$ , and shuffle surjections  $g, g'$  such that  $(u \mid_f \vec{v}) = ((u \mid_g \vec{w}) \mid_{g'} \vec{w}')$ ,  $D(u \mid_g \vec{w}) = D(u)$ ,
- iii) *is a crown on  $u$*  if the  $\vec{v}$  are reduced and  $(\vec{v}, f)$  creates reducible divisors context-minimally.

We denote by  $C(u)$  the set of crowns on  $u$ . Let  $<_1$  be a total order  $X_0^{*\text{III}}$  and  $<_2$  a total order on the set of all shuffle surjections  $\coprod_{k, n_1, \dots, n_k \geq 1} S(n_1, \dots, n_k)$ . For instance, we can let  $<_1$  be a degree-lexicographic ordering on tree monomials [18, § 3.2.1], and  $<_2$  be the lexicographic ordering on shuffle surjections  $f$  when written as lists  $(f(1), \dots, f(n))$ . Then, for every 0-monomial  $u$ , the set  $C(u)$  is equipped with a total order given by the lexicographic product order of  $<_1$  and  $<_2$  on the tuples  $(w_1, \dots, w_k, f)$ .

**5.2.3. Higher-dimensional overlappings.** Let  $X$  be a left-monomial 1-polygraph and fix  $<$  a total monomial order on  $X_0^{*\text{III}}$ . Define the family of indexed sets  $\mathcal{Ov}(X) = (\mathcal{Ov}(X)_n)_{n \geq 0}$ , indexed sets  $\mathcal{Ov}(X)_n^\circ \supseteq \mathcal{Ov}(X)_n$ , and maps of indexed sets  $\psi : \mathcal{Ov}(X)_n^\circ \rightarrow X_0^{*\text{III}}$ , by induction on  $n \geq 0$ . The elements of  $\mathcal{Ov}(X)_n$  are called *n-overlappings*.

- i) Set  $\mathcal{Ov}(X)_0^\circ = \mathcal{Ov}(X)_0 := X_0$  and  $\psi$  equal to the inclusion of  $X_0$  in  $X_0^{*\text{III}}$ .
- ii)  $\mathcal{Ov}(X)_1^\circ$  is the indexed set of tuples  $(u_0, \vec{v}_1, f)$ , written  $u_0 \ll_f \vec{v}_1$ , where  $u_0 \in \mathcal{Ov}(X)_0$  is of arity  $k$ ,  $v_{1,1}, \dots, v_{1,k}$  are reduced 0-monomials, minimal such that  $D(u_0 \mid \vec{v}_1) > 0$ , and  $f$  is a shuffle surjection such that  $(u_0 \mid_f \vec{v}_1)$  is well-defined. Define the arity of  $u_0 \ll_f \vec{v}_1$  as the arity of  $(u_0 \mid_f \vec{v}_1)$  and set  $\psi(u_0 \ll_f \vec{v}_1) := (u_0 \mid_f \vec{v}_1)$ . Set  $\mathcal{Ov}(X)_1 := \mathcal{Ov}(X)_1^\circ$ .
- iii) Let  $n \geq 1$ , and suppose that we have defined  $\mathcal{Ov}(X)_n$  and  $\psi : \mathcal{Ov}(X)_n \rightarrow X_0^{*\text{III}}$ .  $\mathcal{Ov}(X)_{n+1}^\circ$  is the indexed set of tuples  $(u_n, \vec{v}_{n+1}, f)$ , written  $u_n \ll_f \vec{v}_{n+1}$ , where  $u_n \in \mathcal{Ov}(X)_n$ ,  $(u_n \mid_f \vec{v}_{n+1}) \in C(\psi(u_n))$ , and  $f$  is a shuffle surjection such that  $(\psi(u_n) \mid_f \vec{v}_{n+1})$  exists. Define the arity of  $u_n \ll_f \vec{v}_{n+1}$  as the arity of  $(\psi(u_n) \mid_f \vec{v}_{n+1})$  and set  $\psi(u_n \ll_f \vec{v}_{n+1}) := (\psi(u_n) \mid_f \vec{v}_{n+1})$ .

For  $u_{n-1} \ll_f \vec{v}_n \ll_{f'} \vec{v}_{n+1} \in \mathcal{Ov}(X)_{n+1}^\circ$ , define the indexed subset of  $C(\psi(u_{n-1} \ll_f \vec{v}_{n+1}))$

$$P(u_{n-1} \ll_f \vec{v}_n \ll_{f'} \vec{v}_{n+1}) := \left\{ (\vec{w}_n, g) \mid \begin{array}{l} u_{n-1} \ll_g \vec{w}_n \ll_{g'} \vec{w}_{n+1} \in \mathcal{Ov}(X)_{n+1}^\circ, \\ (u_{n-1} \mid_g \vec{w}_n \mid_{g'} \vec{w}_{n+1}) = (u_{n-1} \mid_f \vec{v}_n \mid_{f'} \vec{v}_{n+1}) \end{array} \right\},$$

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with lexicographic order on  $(\vec{w}_n, g)$ . Let

$$M_{n+1} := \left\{ u_{n-1} \ll_g \vec{w}_n \ll_{g'} \vec{w}_{n+1} \in \mathcal{Ov}(X)_{n+1}^\circ \left| \begin{array}{l} u_{n-1} \ll_{f'} \vec{v}_n \ll_f \vec{v}_{n+1} \in \mathcal{Ov}(X)_{n+1}^\circ, \\ (\vec{v}_n \mid_{f, f'} \vec{v}_{n+1}) \text{ reduced,} \\ (\vec{w}_n, g) = \min P(u_{n-1} \ll_{f'} \vec{v}_n \ll_f \vec{v}_{n+1}), \\ (u_{n-1} \mid_g \vec{w}_n \mid_{g'} \vec{w}_{n+1}) = (u_{n-1} \mid_f \vec{v}_n \mid_{f'} \vec{v}_{n+1}) \end{array} \right. \right\}$$

and set  $\mathcal{Ov}(X)_{n+1} := \mathcal{Ov}(X)_{n+1}^\circ \setminus M_{n+1}$ . The idea is that  $P(u_{n-1} \ll_{f'} \vec{v}_n \ll_f \vec{v}_{n+1})$  is the set of paths from  $u_{n-1}$  to  $u_{n-1} \ll_{f'} \vec{v}_n \ll_f \vec{v}_{n+1}$ , and, when  $(\vec{v}_n \mid_{f, f'} \vec{v}_{n+1})$  is reduced (which implies that the two new reducible divisors are “parallel”),  $M_{n+1}$  eliminates one of these paths following an ambient order.

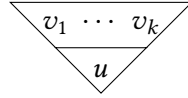
For  $u_n \in \mathcal{Ov}(X)_n$ ,  $v_{n+1,1}, \dots, v_{n+1,k}$  reduced 0-monomials, and  $f$  an appropriate shuffle surjection, define

$$C(u_n, \vec{v}_{n+1}, f) := \left\{ (\vec{w}_{n+1}, g) \left| \begin{array}{l} u_n \ll_g \vec{w}_{n+1} \in \mathcal{Ov}(X)_{n+1} \\ (\psi(u_n) \mid_g \vec{w}_{n+1}) \text{ rooted submonomial of } (\psi(u_n) \mid_f \vec{v}_{n+1}) \end{array} \right. \right\},$$

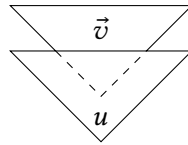
with the total order induced by the one on crowns on  $\psi(u_n)$ . Essentially,  $C(u_n, \vec{v}_{n+1})$  is the set of crowns contained in  $\vec{v}_{n+1}$ .

As for the in-line notation of shuffle operads, we will omit the shuffle surjection  $f$  from  $n$ -overlappings and crowns when possible. Thus we will write, for instance,  $u_n \ll \vec{v}_{n+1}$  for an  $(n+1)$ -overlapping and  $\vec{w}_{n+1} \in C(u_n, \vec{v}_{n+1})$  for a crown on  $u_n$  contained in  $\vec{v}_{n+1}$ .

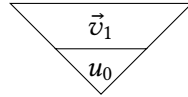
**5.2.4. Drawings of crowns and overlappings.** As a visual representation of crowns and overlappings, let us draw 0-monomials as triangles. We can draw a 0-monomial  $(u \mid_f \vec{v})$  as



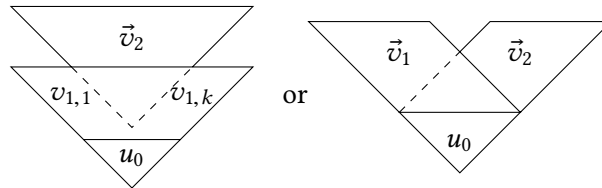
Given a 0-monomial  $u$  and a crown  $\vec{v}$ , we can identify an element of  $D(u \mid \vec{v}) \setminus D(u)$  as a triangle completing  $\vec{v}$ :



A 1-overlapping  $u_0 \ll \vec{v}_1$  looks like

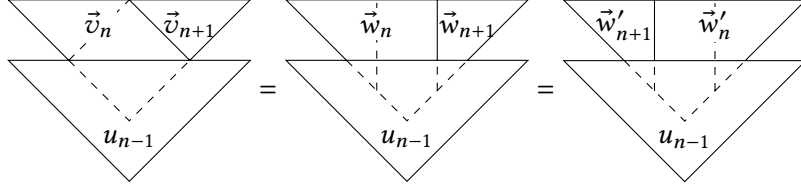


A 2-overlapping  $u_0 \ll \vec{v}_1 \ll \vec{v}_2$  can have two general forms:



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More generally, given an  $(n + 1)$ -overlapping  $u_{n-1} \ll \vec{v}_n \ll \vec{v}_{n+1}$ , the set  $P(u_{n-1} \ll \vec{v}_n \ll \vec{v}_{n+1})$  consists of elements that, for instance, look like



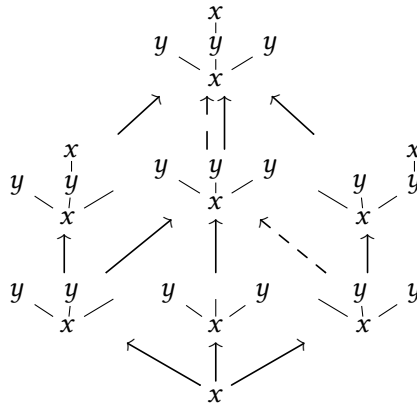
**5.2.5. Overlappings as paths of crowns.** Given a left-monomial 1-polygraph, we can interpret the  $n$ -overlappings as paths in a certain directed graph. Let  $X$  be a left-monomial 1-polygraph. For  $Y$  a set of 0-monomials, define the set of 0-monomials  $\tilde{C}(Y) := \cup_{u \in Y} \{(u \mid \vec{v}) \mid \vec{v} \in C(u)\}$ . Define  $\mathcal{G}(X) = (V, E)$  a directed graph whose vertices are  $V := X_0 \cup \cup_{n \geq 0} \tilde{C}^n(s_0(X_1))$  and whose edges are  $u \rightarrow (u \mid \vec{v})$  for all  $u \in V$  and  $\vec{v} \in C(u)$  and  $u_0 \rightarrow (u_0 \mid \vec{v}_1)$  for  $(u_0 \mid \vec{v}_1) \in s_0(X_1)$ . Then the indexed set of  $n$ -overlappings corresponds to a subset of paths of  $\mathcal{G}$ , starting in  $X_0$  and of length  $n$ , where each step of the path corresponds to the addition of a crown:

$$u_0 \ll \vec{v}_1 \ll \dots \ll \vec{v}_n \quad \leftrightarrow \quad u_0 \rightarrow (u_0 \mid \vec{v}_1) \rightarrow \dots \rightarrow (u_0 \mid \vec{v}_1 \mid \dots \mid \vec{v}_n).$$

**5.2.6. Example.** Consider the following 1-polygraph

$$X := \left\langle x \in X(3), y \in X(1) \mid y \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} y \\ x \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\alpha} 0, y \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} y \\ x \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\beta} 0, \begin{array}{c} y \\ x \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} y \\ x \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\gamma} 0, \begin{array}{c} x \\ y \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} y \\ x \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\delta} 0 \right\rangle.$$

It is left-monomial and trivially confluent. We consider the degree-lexicographic ordering on tree monomials, with  $x > y$ , and the lexicographic ordering on shuffle surjections when written as tuples. Since  $y$  is of arity 1, we do not need to mark the inputs of tree monomials, as they are always the identity permutation. Let us draw the part of the directed graph  $\mathcal{G}(X)$  consisting of all paths from  $x$  to  $(x \mid y(y \mid x)y)$ :



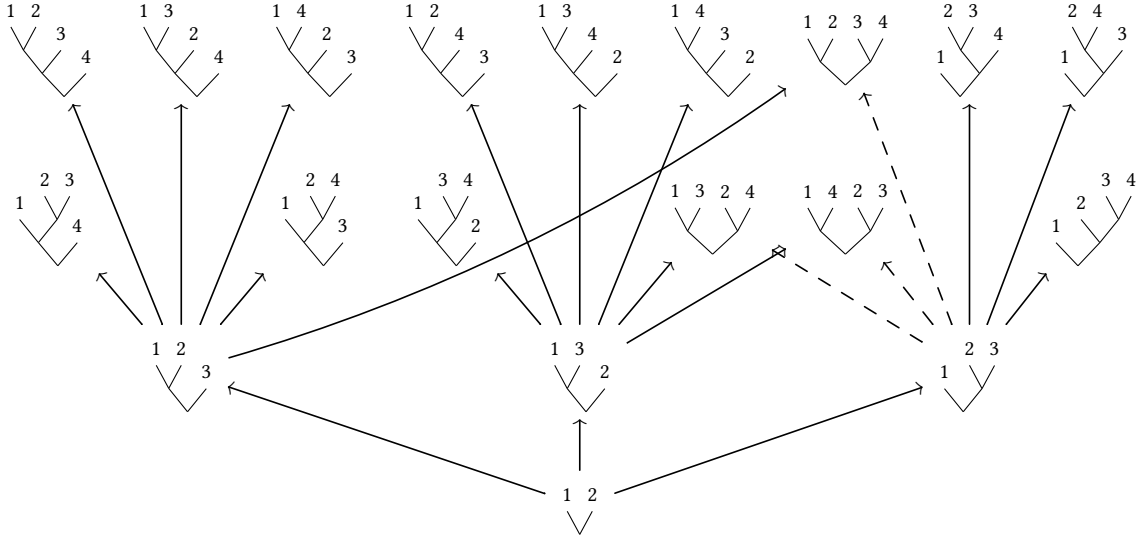
The dashed edges correspond to paths selected by  $M_2$  and  $M_3$ , that is, paths not in  $Ov(X)$ . The two edges between  $(x \mid yyy)$  and  $(x \mid y(y \mid x)y)$  correspond to the fact that  $Ov(X)$  contains the 3-overlapping  $x \ll (y, 1, y) \ll (1, y, 1) \ll (1, x, 1)$ , but not the 3-overlapping  $x \ll (y, y, 1) \ll (1, 1, y) \ll (1, x, 1)$ .

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**5.2.7. Example.** Consider the following binary quadratic 1-polygraph

$$X := \left\langle x \in X_0(2) \mid \begin{array}{l} 1 \begin{array}{c} \diagup 2 \\ x \diagdown 3 \end{array} \rightarrow 0, \quad 1 \begin{array}{c} \diagup 3 \\ x \diagdown 2 \end{array} \rightarrow 0, \quad 1 \begin{array}{c} \diagup 2 \\ x \diagdown 3 \end{array} \rightarrow 0 \end{array} \right\rangle,$$

with the degree-lexicographic ordering on tree monomials and lexicographic ordering on shuffle surjections when written as tuples. It has 15 critical branchings. Let us draw the part of the directed graph  $\mathcal{G}(X)$  corresponding to  $\mathcal{O}v(X)_2$ :



Every internal vertex of every tree monomial is  $x$ , so we omit them. The dashed edges correspond to paths not in  $\mathcal{O}v(X)$ .

**5.2.8. Description in low dimensions.** Let us look at the definition of generators of the polygraph  $\mathcal{O}v(X)$  in low dimensions. By definition, when the 1-polygraph  $X$  is reduced, the 1-overlappings in  $\mathcal{O}v(X)_1$  are the reducible monomial trees  $u_0 \ll \vec{v}_1$  where the  $v_{1,i}$  are reduced and every proper rooted submonomial of  $u_0 \ll \vec{v}_1$  is reduced. Thus every proper submonomial of  $u_0 \ll \vec{v}_1$  is reduced. We deduce that the set of 1-generators  $\mathcal{O}v(X)_1$  coincides with the sources of the 1-generators of  $X$ , and so  $\mathcal{O}v(X)_1 \simeq X_1$  because  $X$  is reduced. Note that without the reducibility condition on  $X$ , the set of 1-overlappings does not necessarily correspond to the set of sources of 1-generators of  $X$ .

When the 1-polygraph  $X$  is reduced, let  $u_0 \ll \vec{v}_1 \ll \vec{v}_2$  be a 2-overlapping in  $\mathcal{O}v(X)_2$ . As above,  $(u_0 \mid \vec{v}_1)$  corresponds to a 1-generator  $\alpha$  of  $X$ . By definition of  $\mathcal{O}v(X)_2$ , the crown  $\vec{v}_2$  makes  $(u_0 \mid \vec{v}_1 \mid \vec{v}_2)$  reducible by another 1-generator of  $X$  in context, say  $\Gamma[\beta]$ . Since  $\vec{v}_2$  is context-minimal, we find that  $((\alpha \mid \vec{v}_1), \Gamma[\beta])$  is a critical branching of  $X$ . Suppose that this critical branching is not minimal, that is, that there exists a 1-generator  $\gamma$  of  $X$  such that  $T(s_0(\beta)) \setminus T((u_0 \mid \vec{v}_1))$  not contain in  $T(s_0(\gamma))$ . Then  $\gamma$  corresponds to a crown contained in  $\vec{v}_2$  contradicting the definition of  $\vec{v}_2$ . Therefore every 2-overlapping corresponds to a minimal critical branching.

However, it should be noted that, even if  $X$  is reduced, not all minimal critical branchings of  $X$  are 2-overlappings in  $\mathcal{O}v(X)_2$ . Indeed, for the 1-polygraph  $X$  defined in Example 5.2.6, the 0-monomial  $(x \mid y y y)$  is the source of three critical branchings while the  $\mathcal{O}v(X)_2$  contains only 2-overlapping.

## 5. Shuffle polygraphic resolutions from convergence

**5.2.9. Theorem (Overlapping polygraphic resolution).** *Let  $X$  be a reduced, convergent, left-monomial 1-polygraph and  $\iota$  the unital section sending every monomial to its reduced form. Then there exist a unique  $\omega$ -polygraph structure on  $\mathcal{O}v(X)$  and a unique right  $\iota$ -contraction  $\sigma$  of  $\mathcal{O}v(X)$  such that, for all  $n$ -overlappings  $u_n$  of  $\mathcal{O}v(X)$  and reduced 0-monomials  $\vec{v}_{n+1}$  of  $X_0^{\text{III}}$ ,*

$$\sigma(u_n \mid \vec{v}_{n+1}) = \begin{cases} u_n \ll \vec{v}_{n+1} & \text{if } u_n \ll \vec{v}_{n+1} \in \mathcal{O}v(X)_{n+1}, \\ \text{an identity} & \text{if } C(u_n, \vec{v}_{n+1}) = \emptyset, \\ \sigma(u_n \mid \vec{v}_{n+1}) & \text{otherwise (tautological condition).} \end{cases} \quad (5.2.10)$$

As a consequence,  $\mathcal{O}v(X)$  is a polygraphic resolution of the operad  $\overline{X}$ .

*Proof.* By induction on  $n \geq 0$ , we simultaneously construct the source and target maps of the  $\omega$ -polygraph structure on the  $(n+1)$ -generators of  $\mathcal{O}v(X)$  and the right  $\iota$ -contraction  $\sigma : \mathcal{O}v(X)_n^{\text{III}} \rightarrow \mathcal{O}v(X)_{n+1}^{\text{III}}$ . By Lemma 5.1.10, it suffices to define  $\sigma$  on the  $\iota$ - and  $\sigma$ -essential  $n$ -monomials of the  $\omega$ -operad  $\mathcal{O}v(X)^{\text{III}}$ .

Let  $n = 0$ . The  $\iota$ -essential 0-monomials of  $\mathcal{O}v(X)^{\text{III}}$  are the  $(u_0 \mid \vec{v}_1)$  where  $u_0$  is a 0-generator of  $X$  and the  $v_{1,i}$  are reduced 0-monomials of  $X_0^{\text{III}}$  such that  $(u_0 \mid \vec{v}_1)$  is not  $(\iota)$ -reduced. By (5.1.8), it suffices to define  $\sigma(u_0 \mid \vec{v}_1)$  such that  $s_0\sigma(u_0 \mid \vec{v}_1) = (u_0 \mid \vec{v}_1)$  and  $t_0\sigma(u_0 \mid \vec{v}_1) = \widehat{(u_0 \mid \vec{v}_1)}$ .

If  $u_0 \ll v_1$  is a 1-overlapping, then we set

$$s_0(u_0 \ll \vec{v}_1) := (u_0 \mid \vec{v}_1), \quad t_0(u_0 \ll \vec{v}_1) := \widehat{(u_0 \mid \vec{v}_1)},$$

and the first case of (5.2.10) imposes  $\sigma(u_0 \mid \vec{v}_1) := u_0 \ll \vec{v}_1$ . Otherwise, since we have supposed  $(u_0 \mid \vec{v}_1)$  not reduced,  $C(u_0, \vec{v}_1)$  is nonempty. Let  $\vec{w}_1 = \min C(u_0, \vec{v}_1)$  and write  $(u_0 \mid \vec{w}_1 \mid \vec{w}_2) = (u_0 \mid \vec{v}_1)$ . Then  $u_0 \ll \vec{v}_1 \in \mathcal{O}v(X)_1$ , and since  $(\vec{w}_1 \mid \vec{w}_2) = \vec{v}_1$  is reduced,  $C(u_0 \ll \vec{w}_1, \vec{w}_2) = \emptyset$ , so by the second case of (5.2.10),  $\sigma(u_0 \ll \vec{w}_1 \mid \vec{w}_2)$  is an identity. Moreover, by (5.1.8), we know that the target of  $\sigma(u_0 \ll \vec{w}_1 \mid \vec{w}_2)$  is  $\sigma(s_0(u_0 \ll \vec{w}_1) \mid \vec{w}_2) = \sigma(u_0 \mid \vec{v}_1)$ . Thus we set

$$\sigma(u_0 \mid \vec{v}_1) := s_0\sigma(u_0 \ll \vec{w}_1 \mid \vec{w}_2) = (u_0 \ll \vec{w}_1 \mid \vec{w}_2) \star_0 \sigma(\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2).$$

Since  $X$  is terminating, we define  $\sigma(\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2) : (\widehat{(u_0 \mid \vec{w}_1)} \mid \vec{w}_2) \rightarrow (u_0 \mid \vec{w}_1 \mid \vec{w}_2)$  by well-founded induction on  $\prec_{X_1}$ , so this definition is licit.

Let  $n \geq 1$ . The essential  $n$ -cells of  $\mathcal{O}v(X)^{\text{III}}$  are the  $(u_n \mid \vec{v}_{n+1})$  where  $u_n$  is an  $n$ -overlapping and the  $v_{n+1,i}$  are reduced 0-monomials of  $X_0^{\text{III}}$  such that  $(u_n \mid \vec{v}_{n+1})$  is not  $\sigma$ -reduced. We distinguish the three cases of (5.2.10). The induction step for the  $\omega$ -polygraph structure on  $\mathcal{O}v(X)$  is entirely contained within the first case.

**First case.** First, suppose that  $u_n \ll \vec{v}_{n+1}$  is an  $(n+1)$ -overlapping. Since condition (5.2.10) imposes  $u_n \ll \vec{v}_{n+1} = \sigma(u_n \mid \vec{v}_{n+1})$ , and (5.1.8) gives us the source and target of the (not yet defined)  $(n+1)$ -cell  $\sigma(u_n \mid \vec{v}_{n+1})$ , we set

$$\begin{aligned} s_n(u_n \ll \vec{v}_{n+1}) &:= (u_n \mid \vec{v}_{n+1}) - (t_{n-1}(u_n) \mid \vec{v}_{n+1}) + \sigma(t_{n-1}(u_n) \mid \vec{v}_{n+1}), \\ t_n(u_n \ll \vec{v}_{n+1}) &:= \sigma(s_{n-1}(u_n) \mid \vec{v}_{n+1}), \end{aligned}$$

which are indeed globular, and define  $\sigma(u_n \mid \vec{v}_{n+1}) := u_n \ll \vec{v}_{n+1}$ . This gives us the polygraphic structure on the  $(n+1)$ -overlappings.

**Second case.** Next, suppose that  $C(u_n, \vec{v}_{n+1}) = \emptyset$ . Writing  $u_n = u_{n-1} \ll \vec{v}_n$ , we make the following observations:

- $u_{n-1} \ll (\vec{v}_n \mid \vec{v}_{n+1})$  does not exist.
- $C(u_{n-1}, (\vec{v}_n \mid \vec{v}_{n+1}))$  is nonempty, since it includes  $\vec{v}_n$ .
- The 0-monomials of  $(\vec{v}_n \mid \vec{v}_{n+1})$  are reduced. If not, then we could write  $(\vec{v}_n \mid \vec{v}_{n+1}) = (\vec{v}_n \mid \vec{w}_{n+1} \mid \vec{w}_{n+2})$  where  $(\vec{v}_n \mid \vec{w}_{n+1})$  is reducible and  $u_{n-1} \ll \vec{v}_n \ll \vec{w}_{n+1} \in \mathcal{O}v(X)_{n+1}^\circ$ . Since  $(\vec{v}_n \mid \vec{w}_{n+1})$  is not reduced,  $u_{n-1} \ll \vec{v}_n \ll \vec{w}_{n+1} \notin M_{n+1}$ , so  $u_{n-1} \ll \vec{v}_n \ll \vec{w}_{n+1} = u_n \ll \vec{w}_{n+1} \in \mathcal{O}v(X)_{n+1}$ . In particular,  $\vec{w}_{n+1} \in C(u_n, \vec{v}_{n+1})$ , contradicting the hypothesis that  $C(u_n, \vec{v}_{n+1})$  is empty.

In particular, the third observation says that  $(u_{n-1} \mid \vec{v}_n \mid \vec{v}_{n+1})$  is an essential  $(n-1)$ -monomial. Thus we are in the third case of the induction hypothesis. Following the calculations of the induction hypothesis in the third case below, let  $\vec{w}_n$  be the minimal element of  $C(u_{n-1}, (\vec{v}_n \mid \vec{v}_{n+1}))$  and let  $\vec{w}_{n+1}$  be 0-monomials such that  $(u_{n-1} \mid \vec{w}_n \mid \vec{w}_{n+1}) = (u_{n-1} \mid \vec{v}_n \mid \vec{v}_{n+1})$ . Then, by induction, the source and target of  $\sigma(u_{n-1} \ll \vec{w}_n \mid \vec{w}_{n+1})$  are equal.

Suppose by contradiction that  $\vec{v}_n \neq \vec{w}_n$ . Then there exists a  $(n+1)$ -overlapping  $u_{n-1} \ll \vec{v}_n \ll \vec{w}'_{n+1} \in \mathcal{O}v(X)_{n+1}$ , which corresponds to the union of  $u_{n-1} \ll \vec{v}_n$  and  $u_{n-1} \ll \vec{w}_n$ . Thus  $\vec{w}'_{n+1} \in C(u_n, \vec{v}_{n+1})$ , which contradicts the hypothesis that  $C(u_n, \vec{v}_{n+1})$  is empty. Therefore  $\vec{v}_n = \vec{w}_n$ , and we conclude that the source and target of  $\sigma(u_{n-1} \ll \vec{v}_n \mid \vec{v}_{n+1}) = \sigma(u_n \mid \vec{v}_{n+1})$  are equal, allowing us to define  $\sigma(u_n \mid \vec{v}_{n+1})$  as an identity.

**Third case.** Otherwise,  $C(u_n, \vec{v}_{n+1})$  is nonempty. Let  $\vec{w}_{n+1}$  be its minimal element, and write  $(u_n \mid \vec{v}_{n+1}) = (u_n \mid \vec{w}_{n+1} \mid \vec{w}_{n+2})$ . Suppose by contradiction that there exists  $\vec{x}_{n+2} \in C(u_n \ll \vec{w}_{n+1}, \vec{w}_{n+2})$ . Then  $\vec{w}_{n+1} = \min P(u_n \ll \vec{w}_{n+1} \ll \vec{x}_{n+2})$  because  $P(u_n \ll \vec{w}_{n+1} \ll \vec{x}_{n+2})$  is a subset of  $C(u_n, \vec{v}_{n+1})$ , and  $(\vec{w}_{n+1} \mid \vec{x}_{n+2})$  is reduced because it is a list of submonomials of  $\vec{v}_{n+1}$ , and so  $u_n \ll \vec{w}_{n+1} \ll \vec{x}_{n+2} \in M_{n+2}$ , which contradicts the hypothesis that  $\vec{x}_{n+2} \in C(u_n \ll \vec{w}_{n+1}, \vec{w}_{n+2})$ . Therefore  $C(u_n \ll \vec{w}_{n+1}, \vec{w}_{n+2}) = \emptyset$ . In addition, the 0-monomials  $\vec{w}_{n+2}$  are reduced, so this is exactly the condition of the second case, so we have the constraint that the source and target of the (not yet defined)  $(n+2)$ -cell  $\sigma(u_n \ll \vec{w}_{n+1} \mid \vec{w}_{n+2})$  are equal. To conclude this case, we prove the following lemma.

Let  $k \geq 1$  and  $\vec{v}_0, \vec{v}_1, \dots, \vec{v}_k$  0-cells of  $X^{\text{III}}$  such that  $(\vec{v}_0 \mid \vec{v}_1^i \cdots \mid \vec{v}_k^i)$  is a well-defined family of 0-cells of  $X^{\text{III}}$ , where, for  $\ell \in \{1, \dots, k\}$ ,  $\vec{v}_\ell^i$  is the subset of  $\vec{v}_\ell$  of ancestor  $v_{0,i}$ . Denote by  $(\vec{v}_0 \parallel \cdots \parallel \vec{v}_k)$  the set of  $k$ -cells

$$\underbrace{\sigma(\cdots \sigma(\sigma(v_{0,i} \mid \vec{v}_1^i) \mid \vec{v}_2^i) \mid \cdots \mid \vec{v}_{k-1}^i) \mid \vec{v}_k^i)}_k.$$

Similarly, we denote by  $\widehat{(\vec{v}_0 \mid \vec{v}_1)}$  the set of reduced 0-cells  $(\vec{v}_{0,i} \mid \vec{v}_1^i)$ . Note that, if  $u_0 \ll \vec{v}_1 \ll \cdots \ll \vec{v}_k$  is an  $k$ -overlapping, then  $u_0 \ll \vec{v}_1 \ll \cdots \ll \vec{v}_k = (u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_k)$ .

**Lemma.** For  $n \geq 2$  and  $\vec{v}_0, \dots, \vec{v}_n$  0-monomials of  $X_0^{\text{III}}$ , we have the equality of  $(n-1)$ -cells

$$\begin{aligned} \partial(\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) &= ((\vec{v}_0 \parallel \cdots \parallel \vec{v}_{n-1}) \mid \vec{v}_n) \\ &+ \sum_{k=1}^n (-1)^k (\vec{v}_0 \parallel \cdots \parallel (\widehat{\vec{v}_{n-k} \mid \vec{v}_{n-k+1}}) \parallel \cdots \parallel \vec{v}_n) \\ &+ (-1)^{n+1} (\vec{v}_0 \mid (\vec{v}_1 \parallel \cdots \parallel \vec{v}_n)) + 1_c, \end{aligned}$$

## 5. Shuffle polygraphic resolutions from convergence

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where  $\partial = s - t$  and  $c$  is some  $(n - 2)$ -cell.

*Proof.* Proceed by induction on  $n \geq 2$ . According to (5.1.8), for every  $n$ -cell  $a$ ,

$$\partial\sigma(a) = a - \sigma(\partial a) + 1_c,$$

where  $c = -t_{n-1}(a)$  is an  $(n - 1)$ -cell. For  $n = 2$ , applying this equality to  $(\vec{v}_0 \parallel \vec{v}_1 \parallel \vec{v}_2)$  gives

$$\begin{aligned} \partial(\vec{v}_0 \parallel \vec{v}_1 \parallel \vec{v}_2) &= \partial\sigma(\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) \\ &= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) - \sigma(\partial\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + 1_c \\ &= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + \sigma(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \mid \vec{v}_2) - \sigma(\vec{v}_0 \mid \vec{v}_1 \mid \vec{v}_2) + 1_c \\ &= (\sigma(\vec{v}_0 \mid \vec{v}_1) \mid \vec{v}_2) + \sigma(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \mid \vec{v}_2) - (\vec{v}_0 \mid \sigma(\vec{v}_1 \mid \vec{v}_2)) - \sigma(\vec{v}_0 \mid \widehat{(\vec{v}_1 \mid \vec{v}_2)}) + 1_{c'} + 1_c \\ &= ((\vec{v}_0 \parallel \vec{v}_1) \mid \vec{v}_2) + (-1)^1(\vec{v}_0 \parallel \widehat{(\vec{v}_1 \mid \vec{v}_2)}) + (-1)^2(\widehat{(\vec{v}_0 \mid \vec{v}_1)} \parallel \vec{v}_2) + (-1)^3(\vec{v}_0 \mid (\vec{v}_1 \parallel \vec{v}_2)) + 1_{c+c'}. \end{aligned}$$

Let  $n \geq 2$ . Recall that, for all  $(n - 1)$ -cells  $u$  and 0-cells  $\vec{v}$ ,

$$\sigma(u \mid \vec{v}) = (s_0(u) \mid \sigma(\vec{v})) \star_0 (u \mid \widehat{\vec{v}}) = \sigma(u \mid \widehat{\vec{v}}) + 1_c$$

with  $c$  an  $(n - 1)$ -cell. We calculate

$$\begin{aligned} \partial(\vec{v}_0 \parallel \cdots \parallel \vec{v}_{n+1}) &= \partial\sigma((\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) \\ &= ((\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma(\partial(\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) + 1_c \\ &= ((\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma((\vec{v}_0 \parallel \cdots \parallel \vec{v}_{n-1}) \mid \vec{v}_n \mid \vec{v}_{n+1}) \\ &\quad - \sum_{k=1}^n (-1)^k \sigma((\vec{v}_0 \parallel \cdots \parallel \widehat{(\vec{v}_{n-k} \mid \vec{v}_{n-k+1})} \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) \\ &\quad - (-1)^{n+1} \sigma(\vec{v}_0 \mid (\vec{v}_1 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) + \sigma(1_{c'}) + 1_c \\ &= ((\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) - \sigma((\vec{v}_0 \parallel \cdots \parallel \vec{v}_{n-1}) \mid \widehat{(\vec{v}_n \mid \vec{v}_{n+1})}) - 1_{c''} \\ &\quad - \sum_{k=1}^n (-1)^k (\vec{v}_0 \parallel \cdots \parallel \widehat{(\vec{v}_{n-k} \mid \vec{v}_{n-k+1})} \parallel \cdots \parallel \vec{v}_{n+1}) \\ &\quad - (-1)^{n+1} (\vec{v}_0 \mid \sigma((\vec{v}_1 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1})) + 1_{\sigma(c')} + 1_c \\ &= ((\vec{v}_0 \parallel \cdots \parallel \vec{v}_n) \mid \vec{v}_{n+1}) \\ &\quad + \sum_{k=1}^{n+1} (-1)^k (\vec{v}_0 \parallel \cdots \parallel \widehat{(\vec{v}_{n-k+1} \mid \vec{v}_{n-k+2})} \parallel \cdots \parallel \vec{v}_{n+1}) \\ &\quad + (-1)^{n+2} (\vec{v}_0 \mid (\vec{v}_1 \parallel \cdots \parallel \vec{v}_{n+1})) + 1_{c+\sigma(c')-c''}, \end{aligned}$$

which concludes the induction step, and the proof of the lemma.  $\square$

Writing  $u_n = u_0 \ll \vec{v}_1 \ll \cdots \ll \vec{v}_n$ , we apply the lemma to  $\sigma(u_n \ll \vec{w}_{n+1} \mid \vec{w}_{n+2}) = (u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2})$

to get the equation of  $(n + 1)$ -cells

$$\begin{aligned} \partial\sigma(u_n \ll \vec{w}_{n+1} \mid \vec{w}_{n+2}) = 0 &= ((u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1}) \mid \vec{w}_{n+2}) \\ &+ \sum_{k=1}^{n+2} (-1)^k (u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \widehat{(\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})} \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2}) \\ &+ (-1)^{n+3} (u_0 \mid (\vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2})) + 1_c, \end{aligned}$$

where  $c$  is an  $n$ -cell. On the righthand side, the  $(n + 1)$ -cell

$$(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \widehat{(\vec{w}_{n+1} \mid \vec{w}_{n+2})}) = \sigma(u_n \mid \vec{v}_{n+1})$$

appears. We want to define this  $(n + 1)$ -cell using the other  $(n + 1)$ -cells that appear, that is,

$$\begin{aligned} &(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1}), \quad (\vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2}), \\ &(\widehat{u_0 \parallel \vec{v}_1 \parallel \cdots \parallel (\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})} \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2}), \quad k \in \{2, \dots, n + 2\}. \end{aligned} \tag{5.2.11}$$

We define a well-founded order  $<$  on  $(n+1)$  cells of the form  $(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n)$  as follows.  $(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n) < (u'_0 \parallel \vec{v}'_1 \parallel \cdots \parallel \vec{v}'_n)$  if

- i)**  $T(u_0 \mid \vec{v}_1 \mid \cdots \mid \vec{v}_n)$  is a proper submonomial of  $T(u'_0 \mid \vec{v}'_1 \mid \cdots \mid \vec{v}'_n)$ , or
- ii)**  $T(u_0 \mid \vec{v}_1 \mid \cdots \mid \vec{v}_n) = T(u'_0 \mid \vec{v}'_1 \mid \cdots \mid \vec{v}'_n)$  and there exist  $i, j$  such that  $u_0 = u'_0$ ,  $\vec{v}_1 = \vec{v}'_1, \dots, \vec{v}_{i-1} = \vec{v}'_{i-1}$ ,  $v_{i,1} = v'_{i,1}, \dots, v_{i,j-1} = v'_{i,j-1}$ , and the weight of  $v_{i,j}$  is less than that of  $v'_{i,j}$ , or
- iii)** there exists a positive 1-cell  $f : (u'_0 \parallel \vec{v}'_1 \parallel \cdots \parallel \vec{v}'_n) \rightarrow b$  of  $X^{\text{III}}$  such that  $(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n) \in \text{Supp}(b)$ .

The relation  $<$  is an order because the 1-polygraph  $X$  is supposed reduced (so we cannot rewrite a 0-monomial into a larger 0-monomial). The relation  $<$  is well-founded because every sequence  $((u_0^i \parallel \vec{v}_1^i \parallel \cdots \parallel \vec{v}_n^i))_{i \geq 0}$  that decreases for  $<$  can be rearranged into the concatenation of a decreasing sequence for **iii)** followed by a decreasing sequence for the lexicographic order induced by **i)** and **ii)** (if we can rewrite a submonomial of a 0-monomial, then we can rewrite the 0-monomial following the same rule).

We initialize our well-founded induction on the  $(n + 1)$ -overlappings, since  $u_0 \ll \vec{v}_1 \ll \cdots \ll \vec{v}_{n+1} = (u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_{n+1})$  is already defined. We then check that all of the  $(n + 1)$ -cells of (5.2.11) are smaller than  $\sigma(u_n \mid \vec{v}_{n+1})$  for the order  $<$ :  $(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1})$  and  $(\vec{v}_1 \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2})$  satisfy **i)**, and  $(u_0 \parallel \vec{v}_1 \parallel \cdots \parallel \widehat{(\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})} \parallel \cdots \parallel \vec{v}_n \parallel \vec{w}_{n+1} \parallel \vec{w}_{n+2})$  satisfies **ii)** if  $(\vec{v}_{n-k+2} \mid \vec{v}_{n-k+3})$  is reduced, and **iii)** otherwise, by confluence of  $X$ . Thus we can define  $\sigma(u_n \mid \vec{v}_{n+1})$  by well-founded induction.

Finally, by Proposition 5.1.13, the  $\omega$ -polygraph  $\mathcal{O}v(X)$  is acyclic. Since  $X$  is reduced, by the discussion of 5.2.8,  $\mathcal{O}v(X)_{\leq 1}$  coincides with  $X$ . Therefore  $\mathcal{O}v(X)$  is a polygraphic resolution of the operad  $\overline{X}$ .  $\square$

**5.2.12. Corollary.** *Let  $X$  be a reduced, terminating, left-monomial 1-polygraph such that every minimal critical branching is confluent. Then there exists a  $\omega$ -polygraph structure on  $\mathcal{O}v(X)$  making it a polygraphic resolution of  $\overline{X}$ .*

*Proof.* Confluence (resp. critical confluence) of  $X$  is equivalent to coherent confluence (resp. coherent critical confluence) of the 2-polygraph  $(X_0, X_1, \text{Sph}(X_1))$ . Since  $X$  is critically confluent, Theorem 4.2.4 implies that  $X$  is confluent. Then, by Theorem 5.2.9,  $\mathcal{O}v(X)$  is equipped with a  $\omega$ -polygraph structure and is a polygraphic resolution of  $\overline{X}$ .  $\square$



## 5. Shuffle polygraphic resolutions from convergence

**5.2.13. Coherent presentations from convergence.** In [43], Squier introduced a method to compute a coherent presentation of a monoid from a convergent one. This construction was extended to the case of associative algebras in [24, Thm 4.3.2]. For shuffle operads, this result can be stated as follows: for a convergent left-monoidal 1-polygraph  $X$ , and a cellular extension  $Y$  of  $X_1^{\text{III}}$  that contains a 2-generator  $A_{f,g}$  of shape

$$\begin{array}{ccccc} & f & \rightarrow & b & \xrightarrow{h} \\ a & \searrow & & & \searrow \\ & & & \Downarrow A_{f,g} & \\ & g & \rightarrow & c & \xrightarrow{k} \\ & & & & \searrow \\ & & & & d \end{array}$$

with  $h$  and  $k$  positive 1-cells of  $X_1^{\text{III}}$ , for every critical branching  $(f, g)$  of  $X$ , then the 2-polygraph  $(X, Y)$  is acyclic. The 2-generator  $A_{f,g}$  is called a *generating confluence* associated to the critical branching  $(f, g)$ . Note that such a generating confluence depends on the choice of the positive cells  $h$  and  $k$  and the orientation of the 2-cell  $A_{f,g}$ . The proof of this result is done in two steps. First, we show that the 2-polygraph  $(X, Y)$  is coherently confluent: in particular, for any parallel positive 1-cells  $(f, g)$ , there exists a 2-cell  $F : f \rightarrow g$  in  $X_1^{\text{III}}[Y]$ . Next, we deduce the same property for all parallel 1-cells in  $X_1^{\text{III}}$ . The proof of this second step is entirely based on the underlying  $(2, 1)$ -categorical structure and does not depend on the generating confluences, as shown in [26, Thm. 4.3.2].

While the known proofs of the first step consider a generating confluence for every critical branching, we can restrict the family of generating confluences. Indeed, suppose that  $X$  is a convergent left-monoidal 1-polygraph. Theorem 4.2.4 proves the first part of the coherent Squier's Theorem for  $X$  with a generating confluence for each minimal critical branching. In particular, this proves that any generating confluence of a critical branching of  $X$  can be replaced by a composition of generating confluences of minimal critical branchings of  $X$ . Moreover, if  $X$  is reduced, as a consequence of Theorem 5.2.9, we can further restrict the generating confluences to those corresponding to 2-overappings in  $\mathcal{O}v(X)_2$ . Indeed,  $\mathcal{O}v(X)_2$  is an acyclic extension of the 1-operad  $\mathcal{O}v(X)_1^{\text{III}}$ , which is isomorphic to  $X_1^{\text{III}}$  when  $X$  is reduced. Hence, every generating confluence associated to a minimal critical branching of  $X$  can be replaced a composition of the following 2-generators  $u_0 \ll \vec{v}_1 \ll \vec{v}_2$  of  $\mathcal{O}v(X)_2$ , which correspond to the two shapes of critical pairs of  $X$ :

$$\begin{array}{ccc} (u_0 \ll \vec{v}_1 | \vec{v}_2) & \xrightarrow{\sigma_{((u_0 | \vec{v}_1) | \vec{v}_2)}} & ((u_0 | \vec{v}_1) | \vec{v}_2) \\ (u_0 | \vec{v}_1 | \vec{v}_2) & \Downarrow u_0 \ll \vec{v}_1 \ll \vec{v}_2 & (u_0 | \vec{v}_1 | \vec{v}_2) \\ (u_0 | \sigma(\vec{v}_1 | \vec{v}_2)) & \xrightarrow{\sigma_{(u_0 | (\vec{v}_1 | \vec{v}_2))}} & (u_0 | (\vec{v}_1 | \vec{v}_2)) \end{array} \quad \begin{array}{ccc} (u_0 \ll \vec{v}_1 | \vec{v}_2) & \xrightarrow{\sigma_{((u_0 | \vec{v}_1) | \vec{v}_2)}} & ((u_0 | \vec{v}_1) | \vec{v}_2) \\ (u_0 | \vec{v}_1 | \vec{v}_2) & \Downarrow u_0 \ll \vec{v}_1 \ll \vec{v}_2 & (u_0 | \vec{v}_1 | \vec{v}_2) \\ (u_0 \ll \vec{w}_1 | \vec{w}_2) & \xrightarrow{\sigma_{((u_0 | \vec{w}_1) | \vec{w}_2)}} & ((u_0 | \vec{w}_1) | \vec{w}_2) \end{array}$$

If  $(\vec{v}_1 | \vec{v}_2)$  is not reduced, we have the left diagram. If  $(\vec{v}_1 | \vec{v}_2)$  is reduced, we take  $\vec{w}_1 = \min C(u_0, (\vec{v}_1 | \vec{v}_2))$  and get the right diagram.

**5.2.14. The case of associative algebras.** An associative algebra can be seen as an operad concentrated in arity one. For such an operad, the  $n$ -overappings correspond to the  $n$ -cells of the polygraphic resolution defined in [24, Thm. 6.2.4], which are overappings of rules reducing monomials of a free algebra. In this case, for any  $(n + 1)$ -overlapping  $u_{n-1} \ll \vec{v}_n \ll \vec{v}_{n+1}$ , there exists a reducible 0-monomial in the list  $(\vec{v}_n | v_{n+1})$ . Thus  $M_{n+1}$  is empty and  $\mathcal{O}v(X)_n^\circ = \mathcal{O}v(X)$ .

**5.2.15. Example.** The terminating reduced 1-polygraph  $X$  defined in Example 4.1.9 does not have critical branching. As a consequence, it is convergent and can be extended into a polygraphic resolution  $\mathcal{O}v(X)$ , with  $\mathcal{O}v(X)_n$  empty for  $n \geq 2$ .

**5.2.16. Remark.** Following Proposition 3.2.5, if  $X$  is an acyclic  $\omega$ -polygraph whose underlying 1-polygraph  $X_{\leq 1}$  is left-monomial and convergent, then  $X$  is Tietze equivalent to the  $\omega$ -polygraph  $\mathcal{O}v(X_{\leq 1})$ . In particular, for every operad  $P$ , the  $\omega$ -polygraphs  $\text{Std}(P)$  and  $\mathcal{O}v(\text{Std}(P)_{\leq 1})$  are Tietze equivalent.

### 5.3. Bimodule resolutions from polygraphic resolutions

In this subsection, we show how to deduce the cohomology of a shuffle operad with coefficients in bimodules from a shuffle polygraphic resolution of the operad.

**5.3.1. Construction of a chain complex.** Let  $X$  be a shuffle  $\omega$ -polygraph, and denote  $P$  the shuffle operad presented by  $X$ . Denote by  $\bar{u}$  the image of  $u$  by the canonical projection  $\pi_X$  defined in (3.2.2). Consider the chain complex  $(P\langle X \rangle_n)_{n \geq -1}$  in the category  $\text{Bimod}(P)$  generated by  $X$ , that is, for all  $n \geq -1$ ,  $P\langle X \rangle_n := P\langle X_n \rangle$  is the free  $P$ -bimodule on  $X_n$ , where  $X_{-1}$  is the unit indexed set  $\mathbf{1}$  defined in (2.1.6). The generators of  $P\langle X \rangle_n$  are denoted  $[x]$  for  $x$  in  $X_n$ . The boundary map  $\delta_n : P\langle X \rangle_{n+1} \rightarrow P\langle X \rangle_n$  is defined as follows

i) For  $x \in X_0(k)$ , we set

$$\delta_{-1}([x]) = (\varepsilon \mid \bar{x}) - \sum_{i=1}^k (\bar{x} \mid \bar{1} \cdots \varepsilon_i \cdots \bar{1}). \quad (5.3.2)$$

ii) Consider the derivation  $[ ] : X_0^{\text{III}} \rightarrow P\langle X_0 \rangle$  defined by induction on the weight of monomials  $u$  in  $X_0^{\text{III}}$ , by setting  $[1] := 0$ ,  $[x] := x$  and

$$[u \mid \vec{v}] := ([u \mid \bar{v}_1 \cdots \bar{v}_k] + \sum_{i=1}^k (\bar{u} \mid \bar{v}_1 \cdots [v_i] \cdots \bar{v}_k).$$

We set  $\delta_0$  the *Fox differential* defined for every 1-generator  $\alpha$  in  $X_1$  by

$$\delta_0([\alpha]) := [s_0(\alpha)] - [t_0(\alpha)].$$

iii) For  $n \geq 1$ , we define the map  $[ ] : X_n^{\text{III}} \rightarrow P\langle X_n \rangle$  by setting, for

$$f = \sum_{i=1}^p \lambda_i \Gamma_i[\alpha_i] + 1_c$$

an  $n$ -cell of  $X^{\text{III}}$  with  $\alpha_i \in X_n$  and  $\Gamma_i$  is a one-hole context of  $X_0$ ,

$$[f] = \sum_{i=1}^p \lambda_i \bar{\Gamma}_i[[\alpha_i]],$$

where  $\bar{\Gamma}_i$  is the one-hole context of  $P$  induced by the context  $\Gamma_i$ . Note that  $[f]$  does not depend on the choice of decomposition, so  $[ ]$  is well defined. We set for every  $(n+1)$ -generator  $A$  in  $X_{n+1}$

$$\delta_n([A]) := [s_n(A)] - [t_n(A)].$$

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As a consequence of the globularity of the polygraph  $X$ , for all  $n \geq -1$ , we have  $\delta_{n+1}\delta_n = 0$  and thus  $P\langle X \rangle$  forms a chain complex.

**5.3.3. Lemma.** *For every 0-monomial  $u \in X_0^{*\text{III}}(k)$ , we have*

$$\delta_{-1}([u]) = (\varepsilon \mid \bar{u}) - \sum_{i=1}^k (\bar{u} \mid \bar{1} \cdots \varepsilon_i \cdots \bar{1}).$$

*Proof.* Proceed by induction on the depth of the 0-monomial  $u$ . The equality is true by definition for  $x \in X_0$ . For the induction step, consider  $(u \mid \vec{v})$  with  $u \in X_0^\ell(k)$ ,  $v_i \in X_0^{\ell_i}(\ell_i)$  for all  $1 \leq i \leq k$ :

$$\begin{aligned} \delta_{-1}([u \mid \vec{v}]) &= (\delta_{-1}([u]) \mid \bar{v}_1 \cdots \bar{v}_k) + \sum_{i=1}^k (\bar{u} \mid \bar{v}_1 \cdots \delta_{-1}([v_i]) \cdots \bar{v}_k) \\ &= (\varepsilon \mid \bar{u} \mid \bar{v}_1 \cdots \bar{v}_k) - \sum_{i=1}^k (\bar{u} \mid (\bar{1} \mid \bar{v}_1) \cdots (\varepsilon \mid v_i) \cdots (\bar{1} \mid \bar{v}_k)) \\ &\quad + \sum_{i=1}^k (\bar{u} \mid \bar{v}_1 \cdots (\varepsilon \mid \bar{v}_i) \cdots \bar{v}_k) - \sum_{i=1}^k \sum_{j=1}^{\ell_i} (\bar{u} \mid \bar{v}_1 \cdots (\bar{v}_i \mid \bar{1} \cdots \varepsilon_j \cdots \bar{1}) \cdots \bar{v}_k) \\ &= (\varepsilon \mid \overline{(u \mid \vec{v})}) - \sum_{i=1}^{\ell_1 + \cdots + \ell_k} (\overline{(u \mid \vec{v})} \mid \bar{1} \cdots \varepsilon_i \cdots \bar{1}). \end{aligned}$$

□

**5.3.4. Trivial  $P$ -bimodule.** Define the *trivial  $P$ -bimodule*, denoted by  $\Omega_P$ , as the free  $P$ -bimodule generated by the unit indexed set  $\mathbf{1}$  quotiented by the relations

$$(\varepsilon \mid \bar{u}) = \sum_{i=1}^k \bar{u} \circ_i \varepsilon \tag{5.3.5}$$

for every  $k \geq 1$  and  $\bar{u} \in P(k)$ . Every element of the  $P$ -bimodule  $\Omega_P$  can be written as a linear combination of monomials of the form  $\bar{u} \circ_i \varepsilon$  for all  $k \geq 1$ ,  $\bar{u} \in P(k)$ , and  $1 \leq i \leq k$ .

**5.3.6. Proposition.** *Let  $X$  be an acyclic shuffle  $\omega$ -polygraph, and  $P$  be the shuffle operad presented by  $X$ . Then the chain complex  $P\langle X \rangle$  is a resolution of  $\Omega_P$  in the category  $\text{Bimod}(P)$ .*

*Proof.* Note that  $\Omega_P$  is exactly the cokernel of  $\delta_{-1}$ . Thus it suffices to show that the chain complex  $P\langle X \rangle$  is exact.

Let us fix  $\iota$  a unital section of  $X$ . Following Proposition 5.1.13, the acyclicity of the polygraph  $X$  implies that it admits a right  $\iota$ -contraction. Let  $\sigma$  be such a right  $\iota$ -contraction. We define the linear map  $i_0 : P\langle \mathbf{1} \rangle \rightarrow P\langle X_0 \rangle$  by

$$i_0(u \mid v_1 \cdots (\varepsilon \mid w) \cdots v_k) := (u \mid v_1 \cdots [\widehat{w}] \cdots v_k),$$

for  $u, v_1, \dots, v_i, \dots, v_n, w \in P$ , and, for  $n \geq 1$ , the linear map  $i_n : P\langle X_{n-1} \rangle \rightarrow P\langle X_n \rangle$  by

$$i_n(u \mid v_1 \cdots ([x] \mid w_1 \cdots w_\ell) \cdots v_k) := (u \mid v_1 \cdots [\sigma_{(x|\widehat{w}_1 \cdots \widehat{w}_\ell)}] \cdots v_k),$$

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and  $u, v_1, \dots, \check{v}_i, \dots, v_k, w_1, \dots, w_\ell \in P$ . Note that the linear maps  $i_n$  are compatible with the left action of  $P$ . Hence, we prove that the maps  $i_n$  define a contracting homotopy of the complex  $P\langle X \rangle$ , by showing that the identity  $i_n \delta_{n-1} + \delta_n i_{n+1} = id_{P\langle X_n \rangle}$  holds on generators of the  $P$ -bimodule  $P\langle X_n \rangle$  as follows.

For  $n = 0$ , on the one hand, we have

$$\begin{aligned} i_0 \delta_{-1}([x] \mid w_1 \cdots w_n) &= i_0(\varepsilon \mid \bar{x} \mid \vec{w}) - \sum_{i=1}^n i_0(\bar{x} \mid (\bar{1} \mid w_1) \cdots (\varepsilon \mid w_i) \cdots (\bar{1} \mid w_n)) \\ &= [(\bar{x} \mid \vec{w})] - \sum_{i=1}^n (\bar{x} \mid w_1 \cdots [\widehat{w}_i] \cdots w_n). \end{aligned}$$

On the other, we have

$$\begin{aligned} \delta_0 i_1([x] \mid w_1 \cdots w_n) &= \delta_0[\sigma_{(x \mid \widehat{w}_1 \cdots \widehat{w}_n)}] \\ &= \delta_0[(x \mid \widehat{w}_1 \cdots \widehat{w}_n)] - \delta_0[(\bar{x} \mid \vec{w})] \\ &= ([x] \mid w_1 \cdots w_n) + \sum_{i=1}^n (\bar{x} \mid w_1 \cdots [\widehat{w}_i] \cdots w_n) - [(\bar{x} \mid \vec{w})], \end{aligned}$$

proving the equality  $\delta_0 i_1 + i_0 \delta_{-1} = id_{P\langle X_0 \rangle}$ .

For  $n \geq 1$ , by definition of the right  $\iota$ -contractions  $\sigma$ , we show that, for every  $(n-1)$ -cells  $u, w_1, \dots, w_n$  of  $X^{\text{III}}$ ,

$$i_n[(u \mid w_1 \cdots w_n)] = [\sigma_{(u \mid \widehat{w}_1 \cdots \widehat{w}_n)}].$$

Therefore, for every  $n$ -generator  $A : a \rightarrow b$  in  $X_n$ , we have

$$\begin{aligned} i_n \delta_{n-1}([A] \mid w_1 \cdots w_n) &= i_n[(a \mid \vec{w})] - i_n[(b \mid \vec{w})] = [\sigma_{(a \mid \widehat{w}_1 \cdots \widehat{w}_n)}] - [\sigma_{(b \mid \widehat{w}_1 \cdots \widehat{w}_n)}], \\ \delta_n i_{n+1}([A] \mid w_1 \cdots w_n) &= [(A \mid \widehat{w}_1 \cdots \widehat{w}_n) \star_0 \sigma_{(b \mid \widehat{w}_1 \cdots \widehat{w}_n)}] - [\sigma_{(a \mid \widehat{w}_1 \cdots \widehat{w}_n)}] \\ &= ([A] \mid w_1 \cdots w_n) + [\sigma_{(b \mid \widehat{w}_1 \cdots \widehat{w}_n)}] - [\sigma_{(a \mid \widehat{w}_1 \cdots \widehat{w}_n)}], \end{aligned}$$

proving that  $i_n \delta_{n-1} + \delta_n i_{n+1} = id_{P\langle X_n \rangle}$ . □

**5.3.7. (Co)homology of shuffle operads.** Recall that the Cartan-Eilenberg cohomology of a shuffle operad  $P$  with coefficients in a  $P$ -bimodule  $A$  is defined by

$$H_{CE}^*(P, A) := \text{Ext}_{\text{Bimod}(P)}^*(\Omega_P, A),$$

and that the Quillen cohomology of  $P$  with coefficients in an abelian group  $M$  internal to  $\text{III}Op/P$  is defined by

$$H_Q^*(P, M) := H^*(\text{Hom}_{\text{Ab}(\text{III}Op/P)}(\text{Ab}(\mathcal{X}_*), M)),$$

where  $\mathcal{X}$  is a simplicial cofibrant resolution of the operad  $P$  in the category  $\text{III}Op/P$ , and  $\text{Ab}(-) : \text{III}Op/P \rightarrow \text{Ab}(\text{III}Op/P)$  is the abelianization functor. The equivalence of categories

$$\Theta : \text{Bimod}(P) \approx \text{Ab}(\text{III}Op/P)$$

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induces a natural isomorphism

$$H_{CE}^*(P, A) \simeq \text{Ext}_{\text{Ab}(\text{III}Op/P)}^*(\Theta(\Omega_P), \Theta(A))$$

Following [5, Thm. 4.1], see also [6, Thm. 6.2.1], this induces an isomorphism

$$H_Q^*(P, \Theta(A)) \simeq H_{CE}^{*+1}(P, A).$$

**5.3.8. Finite homological type.** From Theorem 5.2.9 we deduce a generalization of Squier's homological finiteness condition [42], for finite convergence in the case of operads. We say that a shuffle operad  $P$  has *finite homological type*,  $FP_\infty$  for short, if the  $P$ -bimodule  $\Omega_P$  has a resolution in  $\text{Bimod}(P)$  by finitely generated projective bimodules. If  $P$  admits a finite convergent presentation  $X$ , then by Theorem 5.2.9, the overlapping polygraphic resolution  $Op(X)$  is finite and the complex  $P\langle Op(X) \rangle$  is a finitely generated free resolution of  $\Omega_P$ . Thus,  $P$  has homological type  $FP_\infty$ .

### 5.4. Confluence and Koszulness

In this subsection, we show that shuffle operads presented by a quadratic convergent 1-polygraph are Koszul. This result generalizes those obtained by Dotsenko and Khoroshkin in [18] for shuffle operads with quadratic Gröbner bases defined with respect to a given monomial order.

**5.4.1. Graded shuffle polygraphs.** In order to give criteria for Koszulness, we introduce the notion of a graded shuffle  $\omega$ -polygraph. Just as we defined shuffle operads as internal monoids in the presheaf category  $\text{Vect}^{\text{Ord}^\circ}$  in Section 2.1, we define *graded shuffle operads* as internal monoids in the presheaf category  $\text{grVect}^{\text{Ord}^\circ}$ . For  $n \in \mathbb{N} \cup \{\omega\}$ , a *graded shuffle  $n$ -operad* is an  $n$ -category in  $\text{grIII}Op$ , and we denote by  $\text{grIII}Op_n$  the corresponding category with internal  $n$ -functors as morphisms. In particular, the source, target and composition morphisms of graded shuffle  $n$ -operad are graded.

The category  $\text{grIII}Op_n^+$  of *graded extended  $n$ -operads* is defined similarly to  $\text{III}Op_n^+$ : its objects are pairs  $(P, X)$ , where  $P$  is a graded  $n$ -operad, and  $X$  is a graded cellular extension of  $A$ , meaning that  $X = \coprod_{i \geq 0} X^{(i)}$  and that the source and target of each  $x$  in  $X^{(i)}$  are homogeneous of degree  $i$ . In that case, the free  $(n+1)$ -operad  $P[X]$ , defined as in the nongraded case, is also graded.

A *graded  $\omega$ -polygraph* is an  $\omega$ -polygraph  $X$  such that each set  $X_n$  is graded, for  $n \geq 0$ . This notion restricts to  $n$ -polygraphs, and a 1-polygraph  $X$  is called *quadratic* if  $X_0$  is concentrated in degree 1 and  $X_1$  is concentrated in degree 2. We say that a graded  $\infty$ -polygraph  $X$  is *concentrated on the superdiagonal* if each graded set  $X_n$ , for  $n \geq 0$ , is concentrated in degree  $n+1$ . In that case, because the source and target maps are graded, for  $n \geq 1$ , the source and target of every  $n$ -cell of  $X^{\text{III}}$  are homogeneous  $(n-1)$ -cells of  $X^{\text{III}}$  of degree  $n+1$ .

**5.4.2. Koszul operads.** Let  $P$  be a (connected and graded) symmetric operad. We denote by  $\overline{B}(P)$  for the reduced bar complex on  $P$ . Recall from [20, Def. 5.2.3] that the *Koszul complex* on  $P$  is defined by

$$K(P)_{(s)} := H_s(\overline{B}(P)_{(s)}, \delta) = \ker(\delta : \overline{B}_s(P)_{(s)} \rightarrow \overline{B}_{s-1}(P)_{(s)}),$$

the second equality coming from the fact that  $\overline{B}_n(P)_{(s)} = 0$  when  $n > s$ , and where  $(s)$  denotes the degree of  $P$ . By definition, the complex  $K(P)_{(s)}$  is concentrated in degree  $s$ . The operad  $P$  is *Koszul*

if the inclusion morphism  $K(P) \hookrightarrow \bar{B}(P)$  is a quasi-isomorphism [20, Def. 5.2.8], or equivalently the homology of the reduced bar complex of  $P$  is concentrated on the diagonal [20, Thm. 5.3.3], that is,

$$H_n(\bar{B}(P)_{(s)}) = 0, \quad \text{for } n \neq s.$$

Recall that the bar-cobar construction on  $P$  is a resolution, whose abelianization is the reduced bar complex [19, § 1.1], so that the operad  $P$  is Koszul if, and only if, its Quillen homology is concentrated on the diagonal. Finally, recall from [19, Cor. 1.5] that for a symmetric operad  $P$ , there is an isomorphism

$$H_\bullet(\bar{B}(P))^u \simeq H_\bullet(\bar{B}(P^u)).$$

As a consequence, the Koszulness of a symmetric operad can be proved via its shuffle version as follows.

**5.4.3. Theorem.** *Let  $P$  be a quadratic symmetric operad. If the associated shuffle operad  $P^u$  has a quadratic convergent presentation, then  $P$  is Koszul.*

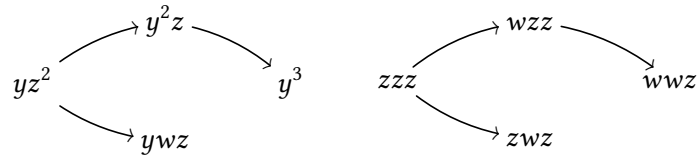
*Proof.* Let  $X$  be a quadratic convergent 1-polygraph presenting  $P^u$ . By definition, the 0-generators in  $X_0$  are concentrated in degree 1 and the 1-generators in  $X_1$  in degree 2. By construction, the polygraphic resolution  $\mathcal{O}v(X)$ , constructed in Theorem 5.2.9, is concentrated on the superdiagonal, that is for  $n \geq 2$ , the  $n$ -generators in  $\mathcal{O}v(X)_n$  are of degree  $n + 1$ ; these cells are the generators of the  $P$ -bimodules of the resolution  $P\langle \mathcal{O}v(X) \rangle$  of  $\Omega_P$  of Theorem 5.3.6. Thus, the Cartan-Eilenberg homology of  $P$  is concentrated in degree  $n + 1$ , and so the Quillen homology is concentrated on the diagonal. Following 5.4.2, we conclude that  $P$  is Koszul.  $\square$

**5.4.4. Remark.** If we consider a quadratic symmetric operad whose generators are all of arity one, using Theorem 5.4.3 we recover the similar result for quadratic associative algebras: every algebra having a quadratic convergent presentation is Koszul, as proved in [24, Prop. 7.2.2] by a polygraphic construction, see also [34, Sec. 4.3], and [8] for a such a criterion with the rewriting rules ordered with respect to a monomial order.

**5.4.5. Koszul associative algebra without monomial order.** Let  $A$  be the associative algebra presented by

$$\langle w, x, y, z \mid w^2 = wx, x^2 = yx, y^2 = yz, z^2 = wz \rangle.$$

If we orient the relations according to a monomial order, say the order generated by  $w < x < y < z$ , this gives a 1-polygraph with two critical branching that are non-confluent



Moreover, we show that any alphabetic order conduces to a similar situation of non-confluent critical branching. Instead, consider the following 1-polygraph:

$$X := \langle w, x, y, z \mid wx \rightarrow w^2, yx \rightarrow x^2, yz \rightarrow y^2, wz \rightarrow z^2 \rangle.$$

## 5. Shuffle polygraphic resolutions from convergence

This polygraph has no critical branchings, so it is confluent. To show that it terminates, we can show that the termination of  $X$  is equivalent to the termination of the following 1-polygraph

$$\langle w, x, y, z \mid wx \rightarrow w, yx \rightarrow x, yz \rightarrow y, wz \rightarrow z \rangle,$$

and this second 1-polygraph clearly terminates by considering the lengths of words. Thus  $X$  is a convergent quadratic 1-polygraph, so by [24, Prop. 7.2.2], thus the algebra  $A$  is Koszul.

**5.4.6. Koszul operad without monomial order.** Following the previous example, let  $P$  be the symmetric operad presented by

$$\left\langle w, x, y, z \in P(2) \mid \begin{array}{l} w(12) = -w, x(12) = -x, y(12) = -y, z(12) = -z, \\ J(w, w) = J(w, x), J(x, x) = J(y, x), J(y, y) = J(y, z), J(z, z) = J(w, z) \end{array} \right\rangle,$$

where  $J(\mu, \nu) := \mu(\nu(a_1, a_2), a_3) + \mu(\nu(a_2, a_3), a_1) + \mu(\nu(a_3, a_1), a_2)$  is a sort of Jacobi formula for arbitrary binary operations  $\mu$  and  $\nu$ . Note that the choice of these relations is only to avoid creating too many relations when we pass to the associated shuffle operad.

Consider  $P^u$ . If we orient the induced relations according to a monomial order, say an order where  $w < x < y < z$  and where left comb tree monomials are greater, then the fourth relation becomes the rewriting rule

$$\underline{\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array}} \rightarrow \underline{\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array}} - \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array} + \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array}.$$

Comparing the underlined tree monomials with the previous example, we find that this rule creates a non-confluent critical pair. Indeed, the other monomials cannot intervene, as they correspond to different shuffle permutations. By Proposition 4.3.2, this also means that this presentation of  $P$  does not admit a quadratic Gröbner basis.

Instead, if we orient the relations to get the shuffle 1-polygraph  $X$  with 0-generators  $w, x, y, z \in X_0(2)$  and 1-generators

$$\begin{array}{l} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ w \end{array} \rightarrow \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ w \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ w \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ w \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ w \\ \diagup \quad \diagdown \\ w \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ w \\ \diagup \quad \diagdown \\ w \end{array}, \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ y \end{array} \rightarrow \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ y \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ y \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ x \end{array} - \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ x \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ x \\ \diagup \quad \diagdown \\ x \end{array}, \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ y \end{array} \rightarrow \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ y \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ y \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ y \\ \diagup \quad \diagdown \\ y \end{array} - \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ y \\ \diagup \quad \diagdown \\ y \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ y \\ \diagup \quad \diagdown \\ y \end{array}, \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array} \rightarrow \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ w \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array} - \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ z \\ \diagup \quad \diagdown \\ z \end{array}, \end{array}$$

we get no critical branchings, so the 1-polygraph  $X$  is confluent. Moreover, following the arguments of the previous example and noting that “monomials with a non-trivial shuffle permutation cannot be rewritten into monomials with the trivial/identity shuffle permutation”, we can show that  $X$  is terminating. Thus  $X$  is a convergent quadratic 1-polygraph, so by Theorem 5.4.3,  $P$  is Koszul.

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