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Geometrization of the local Langlands correspondence

Laurent Fargues and Peter Scholze

Abstract. Following the idea of [**Far16**], we develop the foundations for the geometric Langlands program on the Fargues–Fontaine curve. In particular, we define a category of ℓ-adic sheaves on the stack Bun_G of G-bundles on the Fargues–Fontaine curve, prove a geometric Satake equivalence over the Fargues–Fontaine curve, and study the stack of L-parameters. As applications, we prove finiteness results for the cohomology of local Shimura varieties and general moduli spaces of local shtukas, and define L-parameters associated with irreducible smooth representations of $G(E)$, a map from the spectral Bernstein center to the Bernstein center, and the spectral action of the category of perfect complexes on the stack of L-parameters on the category of ℓ-adic sheaves on Bun_G.

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CHAPTER I

Introduction

I.1. The local Langlands correspondence

The local Langlands correspondence aims at a description of the irreducible smooth representations π of $G(E)$, for a reductive group G over a local field E. Until further notice, we will simplify our life by assuming that G is split; the main text treats the case of general reductive G, requiring only minor changes.

The case where E is archimedean, i.e. $E = \mathbb{R}$ or $E = \mathbb{C}$, is the subject of Langlands' classical work [**Lan89**]. Based on the work of Harish-Chandra, cf. e.g. [**HC66**], Langlands associates to each π an L-parameter, that is a continuous homomorphism

$$\varphi_\pi : W_E \to \widehat{G}(\mathbb{C})$$

where W_E is the Weil group of $E = \mathbb{R}, \mathbb{C}$ (given by $W_\mathbb{C} = \mathbb{C}^\times$ resp. a nonsplit extension $1 \to W_\mathbb{C} \to W_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$), and $\widehat{G}$ is the Langlands dual group. This is the split reductive group over $\mathbb{Z}$ whose root datum is dual to the root datum of G. The map $\pi \mapsto \varphi_\pi$ has finite fibres, and a lot of work has been done on making the fibres, the so-called L-packets, explicit. If $G = \mathrm{GL}_n$, the map $\pi \mapsto \varphi_\pi$ is essentially a bijection.

Throughout this paper, we assume that E is nonarchimedean, of residue characteristic $p > 0$, with residue field $\mathbb{F}_q$. Langlands has conjectured that one can still naturally associate an L-parameter

$$\varphi_\pi : W_E \to \widehat{G}(\mathbb{C})$$

to any irreducible smooth representation π of $G(E)$. In the nonarchimedean case, W_E is the dense subgroup of the absolute Galois group $\mathrm{Gal}(\overline{E}|E)$, given by the preimage of $\mathbb{Z} \subset \mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q)$ generated by the Frobenius $x \mapsto x^q$. This begs the question where such a parameter should come from. In particular,

(1) How does the Weil group W_E relate to the representation theory of $G(E)$?

(2) How does the Langlands dual group $\widehat{G}$ arise?

The goal of this paper is to give a natural construction of a parameter φ_π (only depending on a choice of isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$), and in the process answer questions (1) and (2).

I.2. The big picture

In algebraic geometry, to any ring A corresponds a space $\operatorname{Spec} A$. The starting point of our investigations is a careful reflection on the space $\operatorname{Spec} E$ associated with E.[1] Note that the group $G(E)$ is the automorphism group of the trivial G-torsor over $\operatorname{Spec} E$, while the Weil group of E is essentially the absolute Galois group of E, that is the (étale) fundamental group of $\operatorname{Spec} E$. Thus, $G(E)$ relates to coherent information (especially G-torsors) on $\operatorname{Spec} E$, while W_E relates to étale information on $\operatorname{Spec} E$. Moreover, the perspective of G-torsors is a good one: Namely, for general groups G there can be nontrivial G-torsors $\mathcal{E}$ on $\operatorname{Spec} E$, whose automorphism groups are then the so-called pure inner forms of Vogan [**Vog93**]. Vogan realized that from the perspective of the local Langlands correspondence, and in particular the parametrization of the fibres of $\pi \mapsto \varphi_\pi$, it is profitable to consider all pure inner forms together; in particular, he was able to formulate a precise form of the local Langlands conjecture (taking into account the fibres of $\pi \mapsto \varphi_\pi$) for pure inner forms of (quasi)split groups. All pure inner forms together arise by looking at the groupoid of all G-bundles on $\operatorname{Spec} E$: This is given by

$$[*/G](\operatorname{Spec} E) = \bigsqcup_{[\alpha] \in H^1_{\text{ét}}(\operatorname{Spec} E, G)} [*/G_\alpha(E)],$$

where $H^1_{\text{ét}}(\operatorname{Spec} E, G)$ is the set of G-torsors on $\operatorname{Spec} E$ up to isomorphism, and G_α the corresponding pure inner form of G. Also, we already note that representations of $G(E)$ are equivalent to sheaves on $[*/G(E)]$ (this is a tautology if $G(E)$ were a discrete group; in the present context of smooth representations, it is also true for the correct notion of "sheaf"), and hence sheaves on

$$[*/G](\operatorname{Spec} E) = \bigsqcup_{[\alpha] \in H^1_{\text{ét}}(\operatorname{Spec} E, G)} [*/G_\alpha(E)],$$

are equivalent to tuples $(\pi_\alpha)_{[\alpha]}$ of representations of $G_\alpha(E)$.[2]

Looking at the étale side of the correspondence, we observe that the local Langlands correspondence makes the Weil group W_E of E appear, not its absolute Galois group $\operatorname{Gal}(\overline{E}|E)$. Recall that $W_E \subset \operatorname{Gal}(\overline{E}|E)$ is the dense subgroup given as the preimage of the inclusion $\mathbb{Z} \subset \operatorname{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$, where $\operatorname{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q)$ is generated by its Frobenius morphism $x \mapsto x^q$. On the level of geometry, this change corresponds to replacing a scheme X over $\mathbb{F}_q$ with the (formal) quotient $X_{\overline{\mathbb{F}}_q}/\mathrm{Frob}$.

In the function field case $E = \mathbb{F}_q(\!(t)\!)$, we are thus led to replace $\operatorname{Spec} E$ by $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$ where $\breve{E} = \overline{\mathbb{F}}_q(\!(t)\!)$. We can actually proceed similarly in general, taking $\breve{E}$ to be the completion of the maximal unramified extension of E. For a natural definition of π_1, one then has $\pi_1(\operatorname{Spec}(\breve{E})/\varphi^{\mathbb{Z}}) = W_E$ — for example, $\operatorname{Spec} \overline{\breve{E}} \to \operatorname{Spec}(\breve{E})/\varphi^{\mathbb{Z}}$ is a W_E-torsor, where $\overline{\breve{E}}$ is a separable closure.

Let us analyze what this replacement entails on the other side of the correspondence: Looking at the coherent theory of $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$, one is led to study $\breve{E}$-vector spaces V equipped with φ-linear automorphisms σ. This is known as the category of isocrystals Isoc_E. The category of isocrystals is much richer than the category of E-vector spaces, which it contains fully faithfully. Namely, by

[1] Needless to say, the following presentation bears no relation to the historical developments of the ideas, which as usual followed a far more circuitous route. We will discuss some of our original motivation in Section I.11 below.

[2] The point of replacing $[*/G(E)]$ by $[*/G](\operatorname{Spec} E)$ was also stressed by Bernstein.

the Dieudonné–Manin classification, the category Isoc_E is semisimple, with one simple object V_λ for each rational number $\lambda \in \mathbb{Q}$. The endomorphism algebra of V_λ is given by the central simple E-algebra D_λ of Brauer invariant $\overline{\lambda} \in \mathbb{Q}/\mathbb{Z}$. Thus, there is an equivalence of categories

$$\mathrm{Isoc}_E \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Vect}_{D_\lambda} \otimes V_\lambda.$$

Here, if one writes $\lambda = \frac{s}{r}$ with coprime integers s, r, $r > 0$, then $V_\lambda \cong \breve{E}^r$ is of rank r with σ given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 0 & 1 \\ \pi^s & 0 & & & 0 \end{pmatrix}$$

where $\pi \in E$ is a uniformizer.

Above, we were considering G-torsors on $\operatorname{Spec} E$, thus we should now look at G-torsors in Isoc_E. These are known as G-isocrystals and have been extensively studied by Kottwitz [**Kot85**], [**Kot97**]. Their study has originally been motivated by the relation of isocrystals to p-divisible groups and accordingly a relation of G-isocrystals to the special fibre of Shimura varieties (parametrizing abelian varieties with G-structure, and thus p-divisible groups with G-structure). Traditionally, the set of G-isocrystals is denoted $B(E, G)$, and for $b \in B(E, G)$ we write $\mathcal{E}_b$ for the corresponding G-isocrystal. In particular, Kottwitz has isolated the class of basic G-isocrystals; for $G = \mathrm{GL}_n$, a G-isocrystal is just a rank n isocrystal, and it is basic precisely when it has only one slope λ. There is an injection $H^1_{\text{ét}}(\operatorname{Spec} E, G) \hookrightarrow B(E, G)$ as any G-torsor on $\operatorname{Spec} E$ "pulls back" to a G-torsor in Isoc_E; the image lands in $B(E, G)_{\text{basic}}$. For any $b \in B(E, G)_{\text{basic}}$, the automorphism group of $\mathcal{E}_b$ is an inner form G_b of G; the set of such inner forms of G is known as the extended pure inner forms of G. Note that for $G = \mathrm{GL}_n$, there are no nontrivial pure inner forms of G, but all inner forms of G are extended pure inner forms, precisely by the occurence of all central simple E-algebras as H_λ for some slope λ. More generally, if the center of G is connected, then all inner forms of G can be realized as extended pure inner forms. Kaletha, [**Kal14**], has extended Vogan's results on pure inner forms to extended pure inner forms, giving a precise form of the local Langlands correspondence (describing the fibres of $\pi \mapsto \varphi_\pi$) for all extended pure inner forms and thereby showing that G-isocrystals are profitable from a purely representation-theoretic point of view. We will actually argue below that it is best to include G_b for all $b \in B(E, G)$, not only the basic b; the resulting automorphism groups G_b are then inner forms of Levi subgroups of G. Thus, we are led to consider the groupoid of G-torsors in Isoc_E,

$$G\text{-Isoc} \cong \bigsqcup_{[b] \in B(E,G)} [*/G_b(E)].$$

Sheaves on this are then tuples of representations $(\pi_b)_{[b] \in B(E,G)}$ of $G_b(E)$. The local Langlands conjecture, including its expected functorial behaviour with respect to passage to inner forms and Levi subgroups, then still predicts that for any irreducible sheaf $\mathcal{F}$ — necessarily given by an irreducible representation π_b of $G_b(E)$ for some $b \in B(E, G)$ — one can associate an L-parameter $\varphi_{\mathcal{F}} : W_E \to \widehat{G}(\mathbb{C})$.

To go further, we need to bring geometry into the picture: Indeed, it will be via geometry that (sheaves on the groupoid of) G-torsors on $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$ will be related to the fundamental group W_E of $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$. The key idea is to study a moduli stack of G-torsors on $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$.

There are several ways to try to define such a moduli stack. The most naive may be the following. The category Isoc_E is an E-linear category. We may thus, for any E-algebra A, consider G-torsors in $\operatorname{Isoc}_E \otimes_E A$. The resulting moduli stack will then actually be

$$\bigsqcup_{b \in B(E,G)} [*/G_b],$$

an Artin stack over E, given by a disjoint union of classifying stacks for the algebraic groups G_b. This perspective is actually instrumental in defining the G_b as algebraic groups. However, it is not helpful for the goal of further geometrizing the situation. Namely, sheaves on $[*/G_b]$ are representations of the algebraic group G_b, while we are interested in representations of the locally profinite group $G_b(E)$.

A better perspective is to treat the choice of $\overline{\mathbb{F}}_q$ as auxiliary, and replace it by a general $\overline{\mathbb{F}}_q$-algebra R. In the equal characteristic case, we can then replace $\breve{E} = \overline{\mathbb{F}}_q(\!(t)\!)$ with $R(\!(t)\!)$. This carries a Frobenius $\varphi = \varphi_R$ acting on R. To pass to the quotient $\operatorname{Spec} R(\!(t)\!)/\varphi^{\mathbb{Z}}$, we need to assume that the Frobenius of R is an automorphism, i.e. that R is perfect. (The restriction to perfect R will become even more critical in the mixed characteristic case. For the purpose of considering ℓ-adic sheaves, the passage to perfect rings is inconsequential, as étale sheaves on a scheme X and on its perfection are naturally equivalent.) We are thus led to the moduli stack on perfect $\overline{\mathbb{F}}_q$-algebras

$$G\text{-}\mathcal{I}\mathrm{soc} : \{\text{perfect } \overline{\mathbb{F}}_q\text{-algebras}\} \to \{\text{groupoids}\} : R \mapsto \{G\text{-torsors on } \operatorname{Spec} R(\!(t)\!)/\varphi^{\mathbb{Z}}\}.$$

These are also known as families of G-isocrystals over the perfect scheme $\operatorname{Spec} R$. (Note the curly $\mathcal{I}$ in G-$\mathcal{I}$soc, to distinguish it from the groupoid G-Isoc.)

This definition can be extended to the case of mixed characteristic. Indeed, if R is a perfect $\mathbb{F}_q$-algebra, the analogue of $R[\![t]\!]$ is the unique π-adically complete flat $\mathcal{O}_E$-algebra $\widetilde{R}$ with $\widetilde{R}/\pi = R$; explicitly, $\widetilde{R} = W_{\mathcal{O}_E}(R) = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ in terms of the p-typical Witt vectors $W(R)$ or the ramified Witt vectors $W_{\mathcal{O}_E}(R)$. Thus, if E is of mixed characteristic, we define

$$G\text{-}\mathcal{I}\mathrm{soc} : \{\text{perfect } \overline{\mathbb{F}}_q\text{-algebras}\} \to \{\text{groupoids}\} : R \mapsto \{G\text{-torsors on } \operatorname{Spec}(W_{\mathcal{O}_E}(R)[\tfrac{1}{\pi}])/\varphi^{\mathbb{Z}}\}.$$

We will not use the stack G-$\mathcal{I}$soc in this paper. However, it has been highlighted recently among others implicitly by Genestier–V. Lafforgue, [**GL17**], and explicitly by Gaitsgory, [**Gai16**, Section 4.2], and Zhu, [**Zhu20**], and one can hope that the results of this paper have a parallel expression in terms of G-$\mathcal{I}$soc, so let us analyze it further in this introduction. It is often defined in the following slightly different form. Namely, v-locally on R, any G-torsor over $R(\!(t)\!)$ resp. $W_{\mathcal{O}_E}(R)[\frac{1}{\pi}]$ is trivial by a recent result of Anschütz [**Ans18**]. Choosing such a trivialization, a family of G-isocrystals is given by some element of $LG(R)$, where we define the loop group

$$LG(R) = G(R(\!(t)\!)) \text{ (resp. } LG(R) = G(W_{\mathcal{O}_E}(R)[\tfrac{1}{\pi}])).$$

Changing the trivialization of the G-torsor amounts to σ-conjugation on LG, so as v-stacks

$$G\text{-}\mathcal{I}\mathrm{soc} = LG/_{\mathrm{Ad},\sigma} LG$$

is the quotient of LG under σ-conjugation by LG.

The stack G-$\mathcal{I}$soc can be analyzed. More precisely, we have the following result.[3]

THEOREM I.2.1. *The prestack G-$\mathcal{I}$soc is a stack for the v-topology on perfect $\overline{\mathbb{F}}_q$-algebras. It admits a stratification into locally closed substacks*

$$G\text{-}\mathcal{I}\mathrm{soc}^b \subset G\text{-}\mathcal{I}\mathrm{soc}$$

for $b \in B(E,G)$, consisting of the locus where at each geometric point, the G-isocrystal is isomorphic to $\mathcal{E}_b$. Moreover, each stratum

$$G\text{-}\mathcal{I}\mathrm{soc}^b \cong [*/G_b(E)]$$

is a classifying stack for the locally profinite group $G_b(E)$.

The loop group LG is an ind-(infinite dimensional perfect scheme), so the presentation

$$G\text{-}\mathcal{I}\mathrm{soc} = LG/_{\mathrm{Ad},\sigma}LG$$

is of extremely infinite nature. We expect that this is not an issue with the presentation, but that the stack itself has no good finiteness properties; in particular note that all strata appear to be of the same dimension 0, while admitting nontrivial specialization relations. Xiao–Zhu (see [**XZ17**], [**Zhu20**]) have nonetheless been able to define a category $D(G\text{-}\mathcal{I}\mathrm{soc}, \overline{\mathbb{Q}}_\ell)$ of ℓ-adic sheaves on G-$\mathcal{I}$soc, admitting a semi-orthogonal decomposition into the various $D(G\text{-}\mathcal{I}\mathrm{soc}^b, \overline{\mathbb{Q}}_\ell)$. Each $D(G\text{-}\mathcal{I}\mathrm{soc}^b, \overline{\mathbb{Q}}_\ell) \cong D([*/G_b(E)], \overline{\mathbb{Q}}_\ell)$ is equivalent to the derived category of the category of smooth representations of $G_b(E)$ (on $\overline{\mathbb{Q}}_\ell$-vector spaces). Here, as usual, we have to fix an auxiliary prime $\ell \neq p$ and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.

At this point we have defined a stack G-$\mathcal{I}$soc, with a closed immersion

$$i : [*/G(E)] \cong G\text{-}\mathcal{I}\mathrm{soc}^1 \subset G\text{-}\mathcal{I}\mathrm{soc},$$

thus realizing a fully faithful embedding

$$i_* : D(G(E), \overline{\mathbb{Q}}_\ell) \hookrightarrow D(G\text{-}\mathcal{I}\mathrm{soc}, \overline{\mathbb{Q}}_\ell)$$

of the derived category of smooth representations of $G(E)$ into the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on G-$\mathcal{I}$soc. It is in this way that we "geometrize the representation theory of $G(E)$".

The key additional structure that we need are the Hecke operators: These will simultaneously make the Weil group W_E (i.e. $\pi_1(\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}})$) and, by a careful study, also the Langlands dual group $\widehat{G}$ appear. Recall that Hecke operators are related to modifications of G-torsors, and are parametrized by a point x of the curve where the modification happens, and the type of the modification at x (which can be combinatorially encoded in terms of a cocharacter of G — this eventually leads to the appearance of $\widehat{G}$). Often, the effect of Hecke operators is locally constant for varying x. In that case, letting x vary amounts to an action of $\pi_1(X)$, where X is the relevant curve; thus, the curve should now be $\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}}$.

Thus, if we want to define Hecke operators, we need to be able to consider modifications of G-isocrystals. These modifications ought to happen at a section of $\operatorname{Spec} R(\!(t)\!) \to \operatorname{Spec} R$ (resp. a

[3]This result seems to be well-known to experts, but we are not aware of a full reference. For the v-descent (even arc-descent), see [**Iva20**, Lemma 5.9]. The stratification is essentially constructed in [**RR96**]; the local constancy of the Kottwitz map is proved in general in Corollary III.2.8. The identification of the strata in some cases is in [**CS17**, Proposition 4.3.13], and the argument works in general, up to translating all occurences of p-divisible groups into families of isocrystals.

non-existent map $\operatorname{Spec}(W(R) \otimes_{W(\mathbb{F}_q)} E) \to \operatorname{Spec} R$). Unfortunately, the map $R \to R(\!(t)\!)$ does not admit any sections. In fact, we would certainly want to consider continuous sections; such continuous sections would then be in bijection with maps $\overline{\mathbb{F}}_q(\!(t)\!) = \breve{E} \to R$. In other words, in agreement with the motivation from the previous paragraph, the relevant curve should be $\operatorname{Spec} \breve{E}$, or really $\operatorname{Spec} \breve{E}$ modulo Frobenius — so we can naturally hope to get actions of $\pi_1(\operatorname{Spec} \breve{E}/\varphi^{\mathbb{Z}})$ by the above recipe.

However, in order for this picture to be realized we need to be in a situation where we have continuous maps $\mathbb{F}_q(\!(t)\!) \to R$. In other words, we can only hope for sections if we put ourselves into a setting where R is itself some kind of Banach ring.

This finally brings us to the setting considered in this paper. Namely, we replace the category of perfect $\overline{\mathbb{F}}_q$-schemes with the category of perfectoid spaces $\mathrm{Perf} = \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ over $\overline{\mathbb{F}}_q$. Locally any $S \in \mathrm{Perf}$ is of the form $S = \operatorname{Spa}(R, R^+)$ where R is a perfectoid Tate $\overline{\mathbb{F}}_q$-algebra: This means that R is a perfect topological algebra that admits a topologically nilpotent unit $\varpi \in R$ (called a pseudouniformizer) making it a Banach algebra over $\overline{\mathbb{F}}_q(\!(\varpi)\!)$. Moreover, $R^+ \subset R$ is an open and integrally closed subring of powerbounded elements. Often $R^+ = R^\circ$ is the subring of powerbounded elements, and we consequently use the abbreviation $\operatorname{Spa} R = \operatorname{Spa}(R, R^\circ)$. The geometric (rank 1) points of S are given by $\operatorname{Spa} C$ for complete algebraically closed nonarchimedean fields C, and as usual understanding geometric points is a key first step. We refer to [**SW20**] for an introduction to adic and perfectoid spaces.

For any $S = \operatorname{Spa}(R, R^+)$, we need to define the analogue of $\operatorname{Spec} R(\!(t)\!)/\varphi^{\mathbb{Z}}$, taking the topology of R into account. Note that for discrete R', we have

$$\operatorname{Spa} R'(\!(t)\!) = \operatorname{Spa} R' \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q(\!(t)\!),$$

and we are always free to replace $\operatorname{Spec} R'(\!(t)\!)/\varphi^{\mathbb{Z}}$ by $\operatorname{Spa} R'(\!(t)\!)/\varphi^{\mathbb{Z}}$ as they have the same category of vector bundles. This suggests that the analogue of $\operatorname{Spec} R'(\!(t)\!)$ is

$$\operatorname{Spa}(R, R^+) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q(\!(t)\!) = \mathbb{D}^*_{\operatorname{Spa}(R,R^+)},$$

a punctured open unit disc over $\operatorname{Spa}(R, R^+)$, with coordinate t. Note that

$$\operatorname{Spa}(R, R^+) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q(\!(t)\!) \subset \operatorname{Spa} R^+ \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q[\![t]\!] = \operatorname{Spa} R^+[\![t]\!]$$

is the locus where t and $\varpi \in R^+$ are invertible, where ϖ is a topologically nilpotent unit of R. The latter definition can be extended to mixed characteristic: We let

$$\operatorname{Spa}(R, R^+) \dot{\times}_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} E \subset \operatorname{Spa} R^+ \dot{\times}_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_E := \operatorname{Spa} W_{\mathcal{O}_E}(R^+)$$

be the open subset where π and $[\varpi] \in W_{\mathcal{O}_E}(R^+)$ are invertible. This space is independent of the choice of ϖ as for any other such ϖ', one has $\varpi | \varpi'^n$ and $\varpi' | \varpi^n$ for some $n \geq 1$, and then the same happens for their Teichmüller representatives. We note that the symbol $\dot{\times}$ is purely symbolic: There is of course no map of adic spaces $\operatorname{Spa} E \to \operatorname{Spa} \mathbb{F}_q$ along which a fibre product could be taken.

Definition I.2.2. *The Fargues–Fontaine curve (for the local field E, over $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}$) is the adic space over E defined by*

$$X_S = X_{S,E} = \left(\operatorname{Spa}(R, R^+) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q(\!(t)\!)\right)/\varphi^{\mathbb{Z}},$$

respectively

$$X_S = X_{S,E} = \left(\mathrm{Spa}(R, R^+) \dot{\times}_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, E\right) / \varphi^{\mathbb{Z}},$$

where the Frobenius φ acts on (R, R^+).

A novel feature, compared to the discussion of G-isocrystals, is that the action of φ is free and totally discontinuous, so the quotient by φ is well-defined in the category of adic spaces. In fact, on $Y_S = \mathrm{Spa}(R, R^+) \dot{\times}_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, E \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$ one can compare the absolute values of π and $[\varpi]$. As both are topologically nilpotent units, the ratio

$$\mathrm{rad} = \log(|[\varpi]|)/\log(|\pi|) : |Y_S| \to (0, \infty)$$

gives a well-defined continuous map. The Frobenius on $|Y_S|$ multiplies rad by q, proving that the action is free and totally discontinuous.

We note that in the function field case $E = \mathbb{F}_q(\!(t)\!)$, the space

$$Y_S = S \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q(\!(t)\!) = \mathbb{D}_S^*$$

is precisely a punctured open unit disc over S. In this picture, the radius function measures the distance to the origin: Close to the origin, the radius map is close to 0, while close to the boundary of the open unit disc it is close to ∞. The quotient by φ is however not an adic space over S anymore, as φ acts on S. Thus,

$$X_S = Y_S/\varphi^{\mathbb{Z}} = \mathbb{D}_S^*/\varphi^{\mathbb{Z}}$$

is locally an adic space of finite type over S, but not globally so. This space, for $S = \mathrm{Spa}\, C$ a geometric point, has been first studied by Hartl–Pink [**HP04**].

If $S = \mathrm{Spa}\, C$ is a geometric point but E is general, this curve (or rather a closely related schematic version) has been extensively studied by Fargues–Fontaine [**FF18**], where it was shown that it plays a central role in p-adic Hodge theory. From the perspective of adic spaces, it has been studied by Kedlaya–Liu [**KL15**]. In particular, in this case where S is a point, X_S is indeed a curve: It is a strongly noetherian adic space whose connected affinoid subsets are spectra of principal ideal domains. In particular, in this situation there is a well-behaved notion of "classical points", referring to those points that locally correspond to maximal ideals. These can be classified. In the equal characteristic case, the description of

$$Y_S = \mathbb{D}_S^* = S \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q(\!(t)\!)$$

shows that the closed points are in bijection with maps $S \to \mathrm{Spa}\, \mathbb{F}_q(\!(t)\!)$ up to Frobenius; where now one has to take the quotient under $t \mapsto t^q$. In mixed characteristic, the situation is more subtle, and brings us to the tilting construction for perfectoid spaces.

Proposition I.2.3. *If E is of mixed characteristic and $S = \mathrm{Spa}\, C$ is a geometric point, the classical points of X_C are in bijection with untilts $C^\sharp|E$ of C, up to the action of Frobenius.*

Here, we recall that for any complete algebraically closed field $C'|E$, or more generally any perfectoid Tate ring R, one can form the tilt

$$R^\flat = \varprojlim_{x \mapsto x^p} R,$$

where the addition is defined on the ring of integral elements in terms of the bijection $R^{\flat +} = \varprojlim_{x \mapsto x^p} R^+ \cong \varprojlim_{x \mapsto x^p} R^+/\pi$, where now $x \mapsto x^p$ is compatible with addition on R^+/π. Then $R^\flat$

is a perfectoid Tate algebra of characteristic p. Geometrically, sending $\mathrm{Spa}(R, R^+)$ to $\mathrm{Spa}(R^\flat, R^{\flat+})$ defines a tilting functor on perfectoid spaces $T \mapsto T^\flat$, preserving the underlying topological space and the étale site, cf. [**SW20**].

One sees that the classical points of X_S, for $S = \mathrm{Spa}\, C$ a geometric point, are in bijection with untilts $S^\sharp$ of S together with a map $S \to \mathrm{Spa}\, E$, modulo the action of Frobenius. Recall from [**SW20**] that for any adic space Z over $W(\overline{\mathbb{F}}_q)$, one defines a functor

$$Z^\diamondsuit : \mathrm{Perf} \to \mathrm{Sets} : S \mapsto \{S^\sharp, f : S^\sharp \to Z\}$$

sending a perfectoid space S over $\overline{\mathbb{F}}_q$ to pairs $S^\sharp$ of an untilt of S, and a map $S^\sharp \to Z$. If Z is an analytic adic space, then $Z^\diamondsuit$ is a diamond, that is a quotient of a perfectoid space by a pro-étale equivalence relation. Then the classical points of X_S are in bijection with the S-valued points of the diamond

$$(\mathrm{Spa}\, \breve{E})^\diamondsuit / \varphi^{\mathbb{Z}}.$$

More generally, for any $S \in \mathrm{Perf}$, maps $S \to (\mathrm{Spa}\, \breve{E})^\diamondsuit / \varphi^{\mathbb{Z}}$ are in bijection with degree 1 Cartier divisors $D_S \subset X_S$, so we define

$$\mathrm{Div}^1 = (\mathrm{Spa}\, \breve{E})^\diamondsuit / \varphi^{\mathbb{Z}}.$$

We warn the reader the action of the Frobenius here is a geometric Frobenius. In particular, it only exists on $(\mathrm{Spa}\, \breve{E})^\diamondsuit$, not on $\mathrm{Spa}\, \breve{E}$, in case E is of mixed characteristic. However, one still has $\pi_1(\mathrm{Div}^1) = W_E$.

This ends our long stream of thoughts on the geometry of $\mathrm{Spec}\, E$: We have arrived at the Fargues–Fontaine curve, in its various incarnations. To orient the reader, we recall them here:

(i) For any complete algebraically closed nonarchimedean field $C|\overline{\mathbb{F}}_q$, the curve $X_C = X_{C,E}$, a strongly noetherian adic space over E, locally the adic spectrum of a principal ideal domain. One can also construct a schematic version X_C^{alg}, with the same classical points and the same category of vector bundles. The classical points are in bijection with untilts $C^\sharp|E$ of C, up to Frobenius.

(ii) More generally, for any perfectoid space $S \in \mathrm{Perf}$, the "family of curves" X_S, again an adic space over E, but no longer strongly noetherian. If S is affinoid, there is a schematic version X_S^{alg}, with the same category of vector bundles.

(iii) The "mirror curve" $\mathrm{Div}^1 = (\mathrm{Spa}\, \breve{E})^\diamondsuit / \varphi^{\mathbb{Z}}$, which is only a diamond. For any $S \in \mathrm{Perf}$, this parametrizes "degree 1 Cartier divisors on X_S".

A peculiar phenomenon here is that there is no "absolute curve" of which all the others are the base change. Another peculiar feature is that the space of degree 1 Cartier divisors is not the curve itself.

Again, it is time to study G-torsors. This leads to the following definition.

Definition I.2.4. *Let*

$$\mathrm{Bun}_G : \mathrm{Perf} \to \{\text{groupoids}\} : S \mapsto \{G\text{-torsors on } X_S\}$$

be the moduli stack of G-torsors on the Fargues–Fontaine curve.

Remark I.2.5. Let us stress here that while "the Fargues–Fontaine curve" is not really a well-defined notion, "the moduli stack of G-torsors on the Fargues–Fontaine curve" is.

As X_S maps towards $\operatorname{Spa} \breve{E}/\varphi^{\mathbb{Z}}$, there is a natural pullback functor G-Isoc $\to \operatorname{Bun}_G(S)$. The following result is in most cases due to Fargues [**Far18b**], completed by Anschütz, [**Ans19**].

THEOREM I.2.6. *If $S = \operatorname{Spa} C$ is a geometric point, the map*

$$B(G) \to \operatorname{Bun}_G(S)/\cong$$

is a bijection. In particular, any vector bundle on X_S is a direct sum of vector bundles $\mathcal{O}_{X_S}(\lambda)$ associated to $D_{-\lambda}$, $\lambda \in \mathbb{Q}$.

Under this bijection, $b \in B(G)$ is basic if and only if the corresponding G-torsor $\mathcal{E}_b$ on X_S is semistable in the sense of Atiyah–Bott [**AB83**].

However, it is no longer true that the automorphism groups are the same. On the level of the stack, we have the following result.

THEOREM I.2.7. *The prestack Bun_G is a v-stack. It admits a stratification into locally closed substacks*

$$i^b : \operatorname{Bun}_G^b \subset \operatorname{Bun}_G$$

for $b \in B(G)$ consisting of the locus where at each geometric point, the G-torsor is isomorphic to $\mathcal{E}_b$. Moreover, each stratum

$$\operatorname{Bun}_G^b \cong [*/\widetilde{G}_b]$$

is a classifying space for a group $\widetilde{G}_b$ that is an extension of the locally profinite group $G_b(E)$ by a "unipotent group diamond".

The semistable locus $\operatorname{Bun}_G^{\mathrm{ss}} \subset \operatorname{Bun}_G$ is an open substack, and

$$\operatorname{Bun}_G^{\mathrm{ss}} = \bigsqcup_{b \in B(G)_{\mathrm{basic}}} [*/G_b(E)].$$

REMARK I.2.8. The theorem looks formally extremely similar to Theorem I.2.1. However, there is a critical difference, namely the closure relations are reversed: For Bun_G, the inclusion of Bun_G^b for $b \in B(G)$ basic is an open immersion while it was a closed immersion in Theorem I.2.1. Note that basic $b \in B(G)$ correspond to semistable G-bundles, and one would indeed expect the semistable locus to be an open substack. Generally, Bun_G behaves much like the stack of G-bundles on the projective line.

REMARK I.2.9. We define a notion of Artin stacks in this perfectoid setting, and indeed Bun_G is an Artin stack; we refer to Section I.4 for a more detailed description of our geometric results on Bun_G. This shows that Bun_G has much better finiteness properties than G-$\mathcal{I}$soc, even if it is defined on more exotic test objects.

We can define a derived category of ℓ-adic sheaves

$$D(\operatorname{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

on Bun_G. This admits a semi-orthogonal decomposition into all $D(\operatorname{Bun}_G^b, \overline{\mathbb{Q}}_\ell)$, and

$$D(\operatorname{Bun}_G^b, \overline{\mathbb{Q}}_\ell) \cong D([*/G_b(E)], \overline{\mathbb{Q}}_\ell) \cong D(G_b(E), \overline{\mathbb{Q}}_\ell)$$

is equivalent to the derived category of smooth $G_b(E)$-representations.

Remark I.2.10. It is reasonable to expect that this category is equivalent to the category $D(G\text{-}\mathcal{I}\mathrm{soc}, \overline{\mathbb{Q}}_\ell)$ defined by Xiao–Zhu. However, we do not pursue this comparison here.

Finally, we can define the Hecke stack that will bring all key players together. Consider the global Hecke stack Hck_G parametrizing pairs $(\mathcal{E}, \mathcal{E}')$ of G-bundles on X_S, together with a map $S \to \mathrm{Div}^1$ giving rise to a degree 1 Cartier divisor $D_S \subset X_S$, and an isomorphism

$$f : \mathcal{E}|_{X_S \setminus D_S} \cong \mathcal{E}'|_{X_S \setminus D_S}$$

that is meromorphic along D_S. This gives a correspondence

$$\mathrm{Bun}_G \xleftarrow{h_1} \mathrm{Hck}_G \xrightarrow{h_2} \mathrm{Bun}_G \times \mathrm{Div}^1.$$

To define the Hecke operators, we need to bound the modification, i.e. bound the poles of f along D_S. This is described by the local Hecke stack $\mathcal{H}\mathrm{ck}_G$, parametrizing pairs of G-torsors on the completion of X_S along D_S, together with an isomorphism away from D_S; thus, there is a natural map $\mathrm{Hck}_G \to \mathcal{H}\mathrm{ck}_G$ from the global to the local Hecke stack. Geometrically, $\mathcal{H}\mathrm{ck}_G$ admits a Schubert stratification in terms of the conjugacy classes of cocharacters of G; in particular, there are closed Schubert cells $\mathcal{H}\mathrm{ck}_{G,\leq\mu}$ for each conjugacy class $\mu : \mathbb{G}_m \to G$. By pullback, this defines a correspondence

$$\mathrm{Bun}_G \xleftarrow{h_{1,\leq\mu}} \mathrm{Hck}_{G,\leq\mu} \xrightarrow{h_{2,\leq\mu}} \mathrm{Bun}_G \times \mathrm{Div}^1$$

where now $h_{1,\leq\mu}$ and $h_{2,\leq\mu}$ are proper. One can then consider Hecke operators

$$Rh_{2,\leq\mu,*}h_{1,\leq\mu}^* : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G \times \mathrm{Div}^1, \Lambda).$$

The following theorem ensures that Hecke operators are necessarily locally constant as one varies the point of Div^1, and hence give rise to representations of $\pi_1(\mathrm{Div}^1) = W_E$. In the following, we are somewhat cavalier about the precise definition of $D(-, \overline{\mathbb{Q}}_\ell)$ employed, and the notion of W_E-equivariant objects: The fine print is addressed in the main text.

Theorem I.2.11. *Pullback along the map* $\mathrm{Div}^1 \to [*/W_E]$ *induces an equivalence*

$$D(\mathrm{Bun}_G \times \mathrm{Div}^1, \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Bun}_G \times [*/W_E], \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_E}.$$

Thus, Hecke operators produce W_E-equivariant objects in $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, making the Weil group appear naturally.

One also wants to understand how Hecke operators compose. This naturally leads to the study of $D(\mathcal{H}\mathrm{ck}_G, \overline{\mathbb{Q}}_\ell)$ as a monoidal category, under convolution. Here, we have the geometric Satake equivalence. In the setting of usual smooth projective curves (over $\mathbb{C}$), this was established in the papers of Lusztig [**Lus83**], Ginzburg [**Gin90**] and Mirković–Vilonen [**MV07**]. The theorem below is a first approximation; we will actually prove a more precise version with $\mathbb{Z}_\ell$-coefficients, describing all perverse sheaves on $\mathcal{H}\mathrm{ck}_G$, and applying to the Beilinson–Drinfeld Grassmannians in the spirit of Gaitsgory's paper [**Gai07**].

Theorem I.2.12. *There is a natural monoidal functor from* $\mathrm{Rep}\,\widehat{G}$ *to* $D(\mathcal{H}\mathrm{ck}_G, \overline{\mathbb{Q}}_\ell)$.

Remark I.2.13. Our proof of Theorem I.2.12 follows the strategy of Mirković–Vilonen's proof, and in particular defines a natural symmetric monoidal structure on the category of perverse sheaves by using the fusion product. This requires one to work over several copies of the base curve, and

let the points collide. It is a priori very surprising that this can be done in mixed characteristic, as it requires a space like $\operatorname{Spa}\mathbb{Q}_p \dot{\times}_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}\mathbb{Q}_p$. Spaces of this type do however exist as diamonds, and this was one of the main innovations of [**SW20**].

Remark I.2.14. Using a degeneration of the local Hecke stack, which is essentially the B_{dR}^+-affine Grassmannian of [**SW20**], to the Witt vector affine Grassmannian, Theorem I.2.12 gives a new proof of Zhu's geometric Satake equivalence for the Witt vector affine Grassmannian [**Zhu17**]. In fact, we even prove a version with $\mathbb{Z}_\ell$-coefficients, thus also recovering the result of Yu [**Yu19**].

Remark I.2.15. Regarding the formalism of ℓ-adic sheaves, we warn the reader that we are cheating slightly in the formulation of Theorem I.2.12; the definition of $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ implicit above is not the same as the one implicit in Theorem I.2.12. With torsion coefficients, the problem would disappear, and in any case the problems are essentially of technical nature.

Thus, this also makes the Langlands dual group $\widehat{G}$ appear naturally. For any representation V of $\widehat{G}$, we get a Hecke operator

$$T_V : D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_E}.$$

Moreover, the Hecke operators commute and

$$T_{V\otimes W} \cong T_V \circ T_W|_{\Delta(W_E)}$$

where we note that $T_V \circ T_W$ naturally takes values in $W_E \times W_E$-equivariant objects; the restriction on the right means the restriction to the action of the diagonal copy $\Delta(W_E) \subset W_E \times W_E$.

At this point, the representation theory of $G(E)$ (which sits fully faithfully in $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$), the Weil group W_E, and the dual group $\widehat{G}$, all interact with each other naturally. It turns out that this categorical structure is precisely what is needed to construct L-parameters for (Schur-)irreducible objects $A \in D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, and in particular for irreducible smooth representations of $G(E)$. We will discuss the construction of L-parameters below in Section I.9.

We note that the whole situation is exactly parallel to the Betti geometric Langlands situation considered by Nadler–Yun [**NY19**], and indeed the whole strategy can be described as "the geometric Langlands program on the Fargues–Fontaine curve". It is curious that our quest was to understand the local Langlands correspondence in an arithmetic setting, for potentially very ramified representations, and eventually we solved it by relating it to the global Langlands correspondence in a geometric setting, in the everywhere unramified setting.

In the rest of this introduction, we give a more detailed overview of various aspects of this picture:

(i) The Fargues–Fontaine curve (Section I.3);
(ii) The geometry of the stack Bun_G (Section I.4);
(iii) The derived category of ℓ-adic sheaves on Bun_G (Section I.5);
(iv) The geometric Satake equivalence (Section I.6);
(v) Finiteness of the cohomology of Rapoport–Zink spaces, local Shimura varieties, and more general moduli spaces of shtukas (Section I.7);
(vi) The stack of L-parameters (Section I.8);
(vii) The construction of L-parameters (Section I.9);

(viii) The spectral action (Section I.10);
(ix) The origin of the ideas fleshed out in this paper (Section I.11).

These items largely mirror the chapters of this paper, and each chapter begins with a reprise of these introductions.

I.3. The Fargues–Fontaine curve

The Fargues–Fontaine curve has been studied extensively in the book of Fargues–Fontaine [**FF18**] and further results, especially in the relative situation, have been obtained by Kedlaya–Liu [**KL15**]. In the first chapter, we reprove these foundational results, thereby also collecting and unifying certain results (proved often only for $E = \mathbb{Q}_p$).

The first results concern the Fargues–Fontaine curve $X_C = X_S$ when $S = \operatorname{Spa} C$ for some complete algebraically closed nonarchimedean field $C|\mathbb{F}_q$. We define a notion of classical points of X_C in that case; they form a subset of $|X_C|$. The basic finiteness properties of X_C are summarized in the following result.

THEOREM I.3.1. *The adic space X_C is locally the adic spectrum $\operatorname{Spa}(B, B^+)$ where B is a principal ideal domain; the classical points of $\operatorname{Spa}(B, B^+) \subset X_C$ are in bijection with the maximal ideals of B. For each classical point $x \in X_C$, the residue field of x is an untilt $C^\sharp$ of C over E, and this induces a bijection of the classical points of X_C with untilts $C^\sharp$ of C over E, taken up to the action of Frobenius.*

In the equal characteristic case, Theorem I.3.1 is an immediate consequence of the presentation $X_C = \mathbb{D}_C^*/\varphi^{\mathbb{Z}}$ and classical results in rigid-analytic geometry. In the p-adic case, we use tilting to reduce to the equal characteristic case. At one key turn, in order to understand Zariski closed subsets of X_C, we use the result that Zariski closed implies strongly Zariski closed [**BS19**]. Using these ideas, we are able to give an essentially computation-free proof.

A key result is the classification of vector bundles.

THEOREM I.3.2. *The functor from Isoc_E to vector bundles on X_C induces a bijection on isomorphism classes. In particular, there is a unique stable vector bundle $\mathcal{O}_{X_C}(\lambda)$ of any slope $\lambda \in \mathbb{Q}$, and any vector bundle $\mathcal{E}$ can be written as a direct sum of stable bundles.*

We give a new self-contained proof of Theorem I.3.2, making critical use of the v-descent results for vector bundles obtained in [**Sch17a**] and [**SW20**], and basic results on the geometry of Banach–Colmez spaces established here. The proof in the equal characteristic case by Hartl–Pink [**HP04**] and the proof of Kedlaya in the p-adic case [**Ked04**] relied on heavy computations, while the proof of Fargues–Fontaine [**FF18**] relied on the description of the Lubin–Tate and Drinfeld moduli spaces of π-divisible $\mathcal{O}$-modules. Our proof is related to the arguments of Colmez in [**Col02**].

Allowing general $S \in \operatorname{Perf}_{\overline{\mathbb{F}}_q}$, we define the moduli space of degree 1 Cartier divisors as $\operatorname{Div}^1 = \operatorname{Spd} \breve{E}/\varphi^{\mathbb{Z}}$. Given a map $S \to \operatorname{Div}^1$, one can define an associated closed Cartier divisor $D_S \subset X_S$; locally, this is given by an untilt $D_S = S^\sharp \subset X_S$ of S over E, and this embeds Div^1 into the space of closed Cartier divisors on X_S (justifying the name). Another important result is the following ampleness result, cf. [**KL15**, Proposition 6.2.4], which implies that one can define an algebraic version of the curve, admitting the same theory of vector bundles.

THEOREM I.3.3. *Assume that $S \in \mathrm{Perf}$ is affinoid. For any vector bundle $\mathcal{E}$ on X_S, the twist $\mathcal{E}(n)$ is globally generated and has no higher cohomology for all $n \gg 0$. Defining the graded ring*

$$P = \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}_{X_S}(n))$$

and the scheme $X_S^{\mathrm{alg}} = \mathrm{Proj}\, P$, there is a natural map of locally ringed spaces $X_S \to X_S^{\mathrm{alg}}$, pullback along which defines an equivalence of categories of vector bundles, preserving cohomology.

If $S = \mathrm{Spa}\, C$ for some complete algebraically closed nonarchimedean field C, then X_C^{alg} is a regular noetherian scheme of Krull dimension 1, locally the spectrum of a principal ideal domain, and its closed points are in bijection with the classical points of X_C.

We also need to understand families of vector bundles, i.e. vector bundles $\mathcal{E}$ on X_S for general S. Here, the main result is the following.

THEOREM I.3.4. *Let $S \in \mathrm{Perf}$ and let $\mathcal{E}$ be a vector bundle on X_S. Then the function taking a point $s \in S$ to the Harder–Narasimhan polygon of $\mathcal{E}|_{X_s}$ defines a semicontinuous function on S. If it is constant, then $\mathcal{E}$ admits a global Harder–Narasimhan stratification, and pro-étale locally on S one can find an isomorphism with a direct sum of $\mathcal{O}_{X_S}(\lambda)$'s.*

In particular, if $\mathcal{E}$ is everywhere semistable of slope 0, then $\mathcal{E}$ is pro-étale locally trivial, and the category of such $\mathcal{E}$ is equivalent to the category of pro-étale $\underline{E}$-local systems on S.

The key to proving Theorem I.3.4 is the construction of certain global sections of $\mathcal{E}$. To achieve this, we use v-descent techniques, and an analysis of the spaces of global sections of $\mathcal{E}$; these are known as Banach–Colmez spaces, and were first introduced (in slightly different terms) in [**Col02**].

DEFINITION I.3.5. *Let $\mathcal{E}$ be a vector bundle on X_S. The Banach–Colmez space $\mathcal{BC}(\mathcal{E})$ associated with $\mathcal{E}$ is the locally spatial diamond over S whose T-valued points, for $T \in \mathrm{Perf}_S$, are given by*

$$\mathcal{BC}(\mathcal{E})(T) = H^0(X_T, \mathcal{E}|_{X_T}).$$

Similarly, if $\mathcal{E}$ is everywhere of only negative Harder–Narasimhan slopes, the negative Banach–Colmez space $\mathcal{BC}(\mathcal{E}[1])$ is the locally spatial diamond over S whose T-valued points are

$$\mathcal{BC}(\mathcal{E}[1])(T) = H^1(X_T, \mathcal{E}|_{X_T}).$$

Implicit here is that this functor actually defines a locally spatial diamond. For this, we calculate some key examples of Banach–Colmez spaces. For example, if $\mathcal{E} = \mathcal{O}_{X_S}(\lambda)$ with $0 < \lambda \leq [E : \mathbb{Q}_p]$ (resp. all positive λ if E is of equal characteristic), then $\mathcal{BC}(\mathcal{E})$ is representable by a perfectoid open unit disc (of dimension given by the numerator of λ). A special case of this is the identification of $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ with the universal cover of a Lubin–Tate formal group law, yielding a very close relation between Lubin–Tate theory, and thus local class field theory, and the Fargues–Fontaine curve; see also [**Far18a**]. On the other hand, for larger λ, or negative λ, Banach–Colmez spaces are more exotic objects; for example, the negative Banach–Colmez space

$$\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \cong (\mathbb{A}^1_{C^\sharp})^\diamond / \underline{E}$$

is the quotient of the affine line by the translation action of $\underline{E} \subset \mathbb{A}^1_{C^\sharp}$. We remark that our proof of the classification theorem, Theorem I.3.2, ultimately relies on the negative result that $\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1])$ is not representable by a perfectoid space![4]

For the proof of Theorem I.3.4, a key result is that projectivized Banach–Colmez spaces

$$(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$$

are proper — they are the relevant analogues of "families of projective spaces over S". In particular, their image in S is a closed subset, and if the image is all of S, then we can find a nowhere vanishing section of $\mathcal{E}$ after a v-cover, as then the projectivized Banach–Colmez space is a v-cover of S. From here, Theorem I.3.4 follows easily.

I.4. The geometry of Bun_G

Let us discuss the geometry of Bun_G. Here, G can be any reductive group over a nonarchimedean local field E, with residue field $\mathbb{F}_q$ of characteristic p. Recall that Kottwitz' set $B(G) = B(E, G)$ of G-isocrystals can be described combinatorially, by two discrete invariants. The first is the Newton point

$$\nu : B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma,$$

where T is the universal Cartan of G and $\Gamma = \mathrm{Gal}(\overline{E}|E)$. More precisely, any G-isocrystal $\mathcal{E}$ defines a slope morphism $\mathbb{D} \to G_{\breve{E}}$ where $\mathbb{D}$ is the diagonalizable group with cocharacter group $\mathbb{Q}$; its definition reduces to the case of GL_n, where it amounts to the slope decomposition of isocrystals. Isomorphisms of G-isocrystals lead to conjugate slope morphisms, and this defines the map ν.

The other map is the Kottwitz invariant

$$\kappa : B(G) \to \pi_1(G_{\overline{E}})_\Gamma.$$

Its definition is indirect, starting from tori, passing to the case of G with simply connected derived group, and finally to the general case by z-extensions. Then Kottwitz shows that

$$(\nu, \kappa) : B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G_{\overline{E}})_\Gamma$$

is injective. Moreover, κ induces a bijection between $B(G)_{\mathrm{basic}}$ and $\pi_1(G_{\overline{E}})_\Gamma$. The non-basic elements can be described in terms of Levi subgroups.

Using ν and κ, one can define a partial order on $B(G)$ by declaring $b \leq b'$ if $\kappa(b) = \kappa(b')$ and $\nu_b \leq \nu_{b'}$ with respect to the dominance order.

Up to sign, one can think of ν, resp. κ, as the Harder–Narasimhan polygon, resp. first Chern class, of a G-bundle.

Theorem I.4.1. *The prestack* Bun_G *satisfies the following properties.*

(i) *The prestack* Bun_G *is a stack for the v-topology.*

(ii) *The points* $|\mathrm{Bun}_G|$ *are naturally in bijection with Kottwitz' set* $B(G)$ *of* G*-isocrystals.*

[4]Actually, we only know this for sure if E is p-adic; in the function field case, we supply a small extra argument circumventing the issue.

(iii) *The map*

$$\nu : |\,\mathrm{Bun}_G\,| \to B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma$$

is semicontinuous, and

$$\kappa : |\,\mathrm{Bun}_G\,| \to B(G) \to \pi_1(G_{\overline{E}})_\Gamma$$

is locally constant. Equivalently, the map $|\,\mathrm{Bun}_G\,| \to B(G)$ *is continuous when* $B(G)$ *is equipped with the order topology.*

(iv) *For any* $b \in B(G)$*, the corresponding subfunctor*

$$i^b : \mathrm{Bun}_G^b = \mathrm{Bun}_G \times_{|\,\mathrm{Bun}_G\,|} \{b\} \subset \mathrm{Bun}_G$$

is locally closed, and isomorphic to $[*/\widetilde{G}_b]$*, where* $\widetilde{G}_b$ *is a v-sheaf of groups such that* $\widetilde{G}_b \to *$ *is representable in locally spatial diamonds with* $\pi_0\widetilde{G}_b = G_b(E)$*. The connected component* $\widetilde{G}_b^\circ \subset \widetilde{G}_b$ *of the identity is cohomologically smooth of dimension* $\langle 2\rho, \nu_b\rangle$*.*

(v) *In particular, the semistable locus* $\mathrm{Bun}_G^{\mathrm{ss}} \subset \mathrm{Bun}_G$ *is open, and given by*

$$\mathrm{Bun}_G^{\mathrm{ss}} \cong \bigsqcup_{b \in B(G)_{\mathrm{basic}}} [*/\underline{G_b(E)}].$$

(vi) *For any* $b \in B(G)$*, there is a map*

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$$

that is representable in locally spatial diamonds, partially proper and cohomologically smooth, where $\mathcal{M}_b$ *parametrizes* G*-bundles* $\mathcal{E}$ *together with an increasing* $\mathbb{Q}$*-filtration whose associated graded is, at all geometric points, isomorphic to* $\mathcal{E}_b$ *with its slope grading. The v-stack* $\mathcal{M}_b$ *is representable in locally spatial diamonds, partially proper and cohomologically smooth over* $[*/\underline{G_b(E)}]$*.*

(vii) *The v-stack* Bun_G *is a cohomologically smooth Artin stack of dimension* 0*.*

As examples, let us analyze the case of GL_1 and GL_2. For GL_1, and general tori, everything is semistable, so

$$\mathrm{Pic} := \mathrm{Bun}_{\mathrm{GL}_1} \cong \bigsqcup_{\mathbb{Z}} [*/\underline{E^\times}].$$

For GL_2, the Kottwitz invariant gives a decomposition

$$\mathrm{Bun}_{\mathrm{GL}_2} = \bigsqcup_{\alpha \in \frac{1}{2}\mathbb{Z}} \mathrm{Bun}_{\mathrm{GL}_2}^\alpha .$$

Each connected component has a unique semistable point, given by the basic element $b \in B(\mathrm{GL}_2)_{\mathrm{basic}}$ with $\kappa(b) = \alpha$. For $b \in B(\mathrm{GL}_2)_{\mathrm{basic}} \cong \frac{1}{2}\mathbb{Z}$, the corresponding group $G_b(E)$ is given by $\mathrm{GL}_2(E)$ when $b \in \mathbb{Z}$, and by $D^\times$ when $b \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, where $D|E$ is the quaternion algebra.

The non-semistable points of $\mathrm{Bun}_{\mathrm{GL}_2}$ are given by extensions of line bundles, which are of the form $\mathcal{O}(i) \oplus \mathcal{O}(j)$ for some $i, j \in \mathbb{Z}$, with $2\alpha = i + j$. Let us understand the simplest degeneration inside $\mathrm{Bun}_{\mathrm{GL}_2}$, which is from $\mathcal{O}(\frac{1}{2})$ to $\mathcal{O} \oplus \mathcal{O}(1)$. The individual strata here are

$$[*/\underline{D^\times}],\ [*/\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1))].$$

Here

$$\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \underline{E^\times} & \mathcal{BC}(\mathcal{O}(1)) \\ 0 & \underline{E^\times} \end{pmatrix}.$$

Here $\mathcal{BC}(\mathcal{O}(1))$ is representable by a perfectoid open unit disc $\operatorname{Spd}\overline{\mathbb{F}}_q[\![t^{1/p^\infty}]\!]$.

In this case, the local chart $\mathcal{M}_b$ for $\mathrm{Bun}_{\mathrm{GL}_2}$ parametrizes rank 2 bundles $\mathcal{E}$ written as an extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

such that at all geometric points, $\mathcal{L} \cong \mathcal{O}$ and $\mathcal{L}' \cong \mathcal{O}(1)$. Fixing such isomorphisms defines a $\underline{E^\times} \times \underline{E^\times}$-torsor

$$\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$$

with $\widetilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1])$ a "negative Banach–Colmez space". This local chart shows that the local structure of Bun_G is closely related to the structure of negative Banach–Colmez spaces. It also shows that while the geometry of Bun_G is quite nonstandard, it is still fundamentally a finite-dimensional and "smooth" situation.

For general G, we still get a decomposition into connected components

$$\mathrm{Bun}_G = \bigsqcup_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^\alpha$$

and each connected component Bun_G^α admits a unique semistable point.

By a recent result of Viehmann [**Vie21b**], the map $|\mathrm{Bun}_G| \to B(G)$ is a homeomorphism. This had previously been proved for $G = \mathrm{GL}_n$ by Hansen [**Han17**] based on [**BFH+17**]; that argument was extended to some classical groups in unpublished work of Hamann.

Let us say some words about the proof of Theorem I.4.1. Part (i) has essentially been proved in [**SW20**], and part (ii) follows from the result of Fargues and Anschütz, Theorem I.2.6. In part (iii), the statement about ν reduces to GL_n by an argument of Rapoport–Richartz [**RR96**], where it is Theorem I.3.4. The statement about κ requires more work, at least in the general case: If the derived group of G is simply connected, one can reduce to tori, which are not hard to handle. In general, one approach is to use z-extensions $\widetilde{G} \to G$ to reduce to the case of simply connected derived group. For this, one needs that $\mathrm{Bun}_{\widetilde{G}} \to \mathrm{Bun}_G$ is a surjective map of v-stacks; we prove this using Beauville–Laszlo uniformization. Alternatively, one can use the abelianized Kottwitz set of Borovoi [**Bor98**], which we prove to behave well relatively over a perfectoid space S. Part (iv) is a also consequence of Theorem I.3.4. Part (v) is a consequence of parts (iii) and (iv). The key point is then part (vi), which will imply (vii) formally. The properties of $\mathcal{M}_b$ itself are easy to establish — the analysis for GL_2 above generalizes easily to show that $\widetilde{\mathcal{M}}_b$ is a successive extension of negative Banach–Colmez spaces. The key difficulty is to prove that

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$$

is cohomologically smooth. Note that as we are working with perfectoid spaces, there are no tangent spaces, and we cannot hope to prove smoothness via deformation theory. To attack this problem, we nonetheless prove a general "Jacobian criterion of cohomological smoothness". The setup here is the following.

Let S be a perfectoid space, and let $Z \to X_S$ be a smooth map of (sousperfectoid) adic spaces; this means that Z is an adic space that is locally étale over a finite-dimensional ball over X_S. In this situation, we can define a v-sheaf $\mathcal{M}_Z \to S$ parametrizing sections of $Z \to X_S$, i.e. the S'-valued points, for S'/S a perfectoid space, are given by the maps $s : X_{S'} \to Z$ lifting $X_{S'} \to X_S$. For each such section, we get the vector bundle $s^* T_{Z/X_S}$ on S', where T_{Z/X_S} is the tangent bundle.

Naively, deformations of $S' \to \mathcal{M}_Z$, i.e. of $X_{S'} \to Z$ over $X_{S'} \to X_S$, should correspond to global sections $H^0(X_{S'}, s^*T_{Z/X_S})$, and obstructions to $H^1(X_{S'}, s^*T_{Z/X_S})$. If s^*T_{Z/X_S} has everywhere only positive Harder–Narasimhan slopes, then this vanishes locally on S'. By analogy with the classical situation, we would thus expect the open subspace

$$\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z,$$

where s^*T_{Z/X_S} has positive Harder–Narasimhan slopes, to be (cohomologically) smooth over S. Our key geometric result confirms this, at least if $Z \to X_S$ is quasiprojective.

THEOREM I.4.2. *Assume that $Z \to X_S$ can, locally on S, be embedded as a Zariski closed subset of an open subset of (the adic space) $\mathbb{P}^n_{X_S}$. Then $\mathcal{M}_Z \to S$ is representable in locally spatial diamonds, compactifiable, and of locally finite* $\dim.\,\mathrm{trg}$. *Moreover, the open subset $\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z$ is cohomologically smooth over S.*

In the application, the space $Z \to X_S$ will be the flag variety parametrizing $\mathbb{Q}$-filtrations on a given G-torsor $\mathcal{E}$ on X_S. Then $\mathcal{M}_b$ will be an open subset of $\mathcal{M}_Z^{\mathrm{sm}}$.

The proof of Theorem I.4.2 requires several innovations. The first is a notion of formal smoothness, in which infinitesimal thickenings (that are not available in this perfectoid setting) are replaced by small étale neighborhoods. This leads to a notion with a close relation to the notion of absolute neighborhood retracts [**Bor67**] in classical topology. We prove that virtually all examples of cohomologically smooth maps are also formally smooth, including Banach–Colmez spaces and Bun_G. We also prove that $\mathcal{M}_Z^{\mathrm{sm}} \to S$ is formally smooth, which amounts to some delicate estimates, spreading sections $X_{T_0} \to Z$ into small neighborhoods of $T_0 \subset T$, for any Zariski closed immersion $T_0 \subset T$ of affinoid perfectoid spaces — here we crucially use the assumption that all Harder–Narasimhan slopes are positive. Coupled with the theorem that Zariski closed implies strongly Zariski closed [**BS19**] this makes it possible to write $\mathcal{M}_Z^{\mathrm{sm}}$, up to (cohomologically and formally) smooth maps, as a retract of a space that is étale over a ball over S. Certainly in classical topology, this is not enough to ensure cohomological smoothness — a coordinate cross is a retract of $\mathbb{R}^2$ — but it does imply that the constant sheaf $\mathbb{F}_\ell$ is universally locally acyclic over S. For this reason, and other applications to sheaves on Bun_G as well as geometric Satake, we thus also develop a general theory of universally locally acyclic sheaves in our setting. To finish the proof, we use a deformation to the normal cone argument to show that the dualizing complex is "the same" as the one for the Banach–Colmez space $\mathcal{BC}(s^*T_{Z/X_S})$.

I.5. ℓ-adic sheaves on Bun_G

For our results, we need to define the category of ℓ-adic sheaves on Bun_G. More precisely, we will define for each $\mathbb{Z}_\ell$-algebra Λ a category

$$D(\mathrm{Bun}_G, \Lambda)$$

of sheaves of Λ-modules on Bun_G. If Λ is killed by some power of ℓ, such a definition is the main achievement of [**Sch17a**]. Our main interest is however the case $\Lambda = \overline{\mathbb{Q}}_\ell$. In the case of schemes (of finite type over an algebraically closed field), the passage from torsion coefficients to $\overline{\mathbb{Q}}_\ell$-coefficients is largely formal: Roughly,

$$D(X, \overline{\mathbb{Q}}_\ell) = \mathrm{Ind}(\varprojlim_n D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell).$$

Behind this definition are however strong finiteness results for constructible sheaves; in particular, the morphism spaces between constructible sheaves are finite. For Bun_G, or for the category of smooth representations, there are still compact objects (given by compactly induced representations in the case of smooth representations), but their endomorphism algebras are Hecke algebras, which are infinite-dimensional. A definition along the same lines would then replace all Hecke algebras by their ℓ-adic completions, which would drastically change the category of representations.

Our definition of $D(\mathrm{Bun}_G, \Lambda)$ in general involves some new ideas, employing the idea of solid modules developed by Clausen–Scholze [**CS**] in the context of the pro-étale (or v-)site; in the end, $D(\mathrm{Bun}_G, \Lambda)$ is defined as a certain full subcategory

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \subset D_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$$

of the category $D_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$ of solid complexes of Λ-modules on the v-site of Bun_G. The formalism of solid sheaves, whose idea is due to Clausen and the second author, is developed in Chapter VII. It presents some interesting surprises; in particular, there is always a left adjoint $f_\natural$ to pullback f^*, satisfying base change and a projection formula. (In return, $Rf_!$ fails to exist in general.)

Theorem I.5.1. *Let Λ be any $\mathbb{Z}_\ell$-algebra.*

(i) *Via excision triangles, there is an infinite semiorthogonal decomposition of $D(\mathrm{Bun}_G, \Lambda)$ into the various $D(\mathrm{Bun}_G^b, \Lambda)$ for $b \in B(G)$.*

(ii) *For each $b \in B(G)$, pullback along*

$$\mathrm{Bun}_G^b \cong [*/\widetilde{G}_b] \to [*/\underline{G_b(E)}]$$

gives an equivalence

$$D([*/\underline{G_b(E)}], \Lambda) \cong D(\mathrm{Bun}_G^b, \Lambda),$$

and $D([*/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$ is equivalent to the derived category of the category of smooth representations of $G_b(E)$ on Λ-modules.

(iii) *The category $D(\mathrm{Bun}_G, \Lambda)$ is compactly generated, and a complex $A \in D(\mathrm{Bun}_G, \Lambda)$ is compact if and only if for all $b \in B(G)$, the restriction*

$$i^{b*}A \in D(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

is compact, and zero for almost all b. Here, compactness in $D(G_b(E), \Lambda)$ is equivalent to lying in the thick triangulated subcategory generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ as K runs over open pro-p-subgroups of $G_b(E)$.

(iv) *On the subcategory $D(\mathrm{Bun}_G, \Lambda)^\omega \subset D(\mathrm{Bun}_G, \Lambda)$ of compact objects, there is a Bernstein–Zelevinsky duality functor*

$$\mathbb{D}_{BZ} : (D(\mathrm{Bun}_G, \Lambda)^\omega)^{\mathrm{op}} \to D(\mathrm{Bun}_G, \Lambda)^\omega$$

with a functorial identification

$$R\,\mathrm{Hom}(A, B) \cong \pi_\natural(\mathbb{D}_{BZ}(A) \otimes_\Lambda^{\mathbb{L}} B)$$

for $B \in D(\mathrm{Bun}_G, \Lambda)$, where $\pi : \mathrm{Bun}_G \to *$ is the projection. The functor $\mathbb{D}_{BZ}$ is an equivalence, and $\mathbb{D}_{BZ}^2$ is naturally equivalent to the identity. It is compatible with usual Bernstein–Zelevinsky duality on $D(G_b(E), \Lambda)$ for basic $b \in B(G)$.

(v) *An object $A \in D(\mathrm{Bun}_G, \Lambda)$ is universally locally acyclic (with respect to $\mathrm{Bun}_G \to *$) if and only if for all $b \in B(G)$, the restriction*

$$i^{b*}A \in D(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

is admissible, i.e. for all pro-p open subgroups $K \subset G_b(E)$, the complex $(i^{b*}A)^K$ is perfect. Universally locally acyclic complexes are preserved by Verdier duality, and satisfy Verdier biduality.

This theorem extends many basic notions from representation theory — finitely presented objects, admissible representations, Bernstein–Zelevinsky duality, smooth duality — to the setting of $D(\mathrm{Bun}_G, \Lambda)$.

Parts (i) and (ii) are easy when Λ is ℓ-power torsion. In general, their proofs invoke the precise definition of $D(\mathrm{Bun}_G, \Lambda) = D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ and are somewhat subtle. Part (iii) uses that ii^{b*} admits a left adjoint, which will then automatically preserve compact objects (inducing compact generators). Using the diagram

$$[*/\underline{G_b(E)}] \xleftarrow{q_b} \mathcal{M}_b \xrightarrow{\pi_b} \mathrm{Bun}_G,$$

this left adjoint is defined as $\pi_{b\natural} q_b^*$. The verification that this is indeed a left adjoint amounts in some sense to the assertion that $\mathcal{M}_b$ is "strictly local" along the closed subspace $[*/\underline{G_b(E)}] \subset \mathcal{M}_b$ in the sense that for all $A \in D(\mathcal{M}_b, \Lambda)$, the restriction

$$R\Gamma(\mathcal{M}_b, A) \to R\Gamma([*/\underline{G_b(E)}], A)$$

is an isomorphism. This builds on a detailed analysis of the topological nature of $\mathcal{M}_b$, in particular that $\widetilde{\mathcal{M}_b} \setminus *$ is a spatial diamond, and Theorem I.5.2 below. For part (iv), the constructions in (iii) imply the existence of $\mathbb{D}_{BZ}(A)$ on a class of generators, thus in general, and similar arguments to the ones in (iii) prove the biduality. Finally, part (v) is essentially a formal consequence.

The key cohomological result for the proof is the following result, applied to $\widetilde{\mathcal{M}_b} \setminus *$ (or quotients of it). It plays on the subtle point that the point $*$ is not quasiseparated.

THEOREM I.5.2. *Let X be a spatial diamond such that $f : X \to *$ is partially proper, and of finite* dim. trg. *Then for any affinoid perfectoid space S, the base change $X_S = X \times S$ naturally admits two ends. Taking compactly supported cohomology with respect to one end (but no support condition at the other end), one has*

$$R\Gamma_{\partial\text{-}c}(X_S, A) = 0$$

for all $A \in D^+_{\blacksquare}(X, \mathbb{Z}_\ell)$ (resp. all $A \in D_{\blacksquare}(X, \mathbb{Z}_\ell)$ if f is ℓ-cohomologically smooth).

As an example, if $X = \mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t)\!)$, then $X_S = \mathbb{D}^*_S$ is an open unit disc over S, whose two ends are the origin and the boundary, and one has

$$R\Gamma_{\partial\text{-}c}(\mathbb{D}^*_S, \mathbb{Z}_\ell) = 0.$$

In particular, the cohomology of $\mathrm{Spa}\,\overline{\mathbb{F}}_q[\![t]\!]$ agrees with sections on the closed point, showing that $\mathrm{Spa}\,\overline{\mathbb{F}}_q[\![t]\!]$ is "strictly local". The same phenomenon is at work for $\mathcal{M}_b$.

I.6. The geometric Satake equivalence

In order to define the Hecke operators, we need to prove the geometric Satake equivalence, taking representations of the dual group $\widehat{G}$ to sheaves on the local Hecke stack. In order to analyze compositions of Hecke operators, it will in fact be necessary to analyze modifications at several points.

Thus, for any finite set I, we consider the moduli space $(\mathrm{Div}^1)^I$ parametrizing degree 1 Cartier divisors $D_i \subset X_S$, $i \in I$. Locally on S, each D_i defines an untilt $S_i^\sharp$ of S over E, and one can form the completion B^+ of $\mathcal{O}_{X_S}$ along the union of the D_i. Inverting the D_i defines a localization B of B^+. One can then define a positive loop group $L^+_{(\mathrm{Div}^1)^I}G$ and loop group $L_{(\mathrm{Div}^1)^I}G$, with values given by $G(B^+)$ resp. $G(B)$; for brevity, we will simply write L^+G and LG here. One can then define the local Hecke stack

$$\mathcal{H}\mathrm{ck}_G^I = [L^+G\backslash LG/L^+G] \to (\mathrm{Div}^1)^I.$$

For $d = |I|$, this is in fact already defined over the moduli space $\mathrm{Div}^d = (\mathrm{Div}^1)^d/\Sigma_d$ of degree d Cartier divisors. We will often break symmetry, and first take the quotient on the right to define the Beilinson–Drinfeld Grassmannian

$$\mathrm{Gr}_G^I = LG/L^+G \to (\mathrm{Div}^1)^I$$

so that

$$\mathcal{H}\mathrm{ck}_G^I = L^+G\backslash \mathrm{Gr}_G^I\,.$$

The Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_G^I \to (\mathrm{Div}^1)^I$ is a small v-sheaf that can be written as an increasing union of closed subsheaves that are proper and representable in spatial diamonds, by bounding the relative position; this is one main result of [**SW20**]. On the other hand, L^+G can be written as an inverse limit of truncated positive loop groups, which are representable in locally spatial diamonds and cohomologically smooth; moreover, on each bounded subset, it acts through such a finite-dimensional quotient. This essentially reduces the study of all bounded subsets of $\mathcal{H}\mathrm{ck}_G^I$ to Artin stacks.

In particular, one can write the local Hecke stack as an increasing union of closed substacks that are quasicompact over $(\mathrm{Div}^1)^I$, by bounding the relative position. In the following, we assume that the coefficients Λ are killed by some power of ℓ, so that we can use the theory from [**Sch17a**]. Let

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)^{\mathrm{bd}} \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)$$

be the full subcategory of all objects with quasicompact support over $(\mathrm{Div}^1)^I$. This is a monoidal category under convolution $\star$. Here, we use the convolution diagram

$$\mathcal{H}\mathrm{ck}_G^I \times_{(\mathrm{Div}^1)^I} \mathcal{H}\mathrm{ck}_G^I \xleftarrow{(p_1,p_2)} L^+G\backslash LG \times^{L^+G} LG/L^+G \xrightarrow{m} \mathcal{H}\mathrm{ck}_G^I$$

and define

$$A \star B = Rm_*(p_1^*A \otimes_\Lambda^{\mathbb{L}} p_2^*B).$$

The map m is ind-proper (its fibres are Gr_G^I), and in particular proper on any bounded subset; thus, proper base change ensures that this defines an associative monoidal structure.

On $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)^{\mathrm{bd}}$, one can define a relative perverse t-structure (where an object is perverse if and only if it is perverse over any geometric fibre of $(\mathrm{Div}^1)^I$). For this t-structure, the convolution

$\star$ is left t-exact (and t-exactness only fails for issues related to non-flatness over Λ). To prove that there is a well-defined t-structure, and the preservation of perversity under convolution, we adapt Braden's theorems [**Bra03**] on hyperbolic localization, and a degeneration to the Witt vector affine Grassmannian [**Zhu17**], [**BS17**]. We will discuss hyperbolic localization further below.

We remark that there is no general theory of perverse sheaves in p-adic geometry, the issue being that it is difficult to unambiguously assign a dimension to a point of an adic space (cf. [**Tem21**] for what is known about topological transcendence degrees of points, and the subtleties especially in characteristic p). In particular, we would not know how to define a notion of perverse sheaf on $(\mathrm{Div}^1)^I$ in general, which is the reason we revert to asking perversity only in the fibres. Here, we use that all geometric fibres of the stack $\mathcal{H}\mathrm{ck}_G^I \to (\mathrm{Div}^1)^I$ have only countably many points enumerated explicitly in terms of dominant cocharacters μ_i, and one can assign by hand the dimension $\sum_i \langle 2\rho, \mu_i\rangle$ of the corresponding open Schubert cells.

Remark I.6.1. It seems to have been overlooked in the literature that for any map $f : X \to S$ locally of finite type between schemes, one can define a relative perverse t-structure, with relative perversity equivalent to perversity on all geometric fibres. It is a nontrivial result that there indeed is such a t-structure; this will appear in forthcoming work of Hansen and the second author. Thus, there is a good notion of "families of perverse sheaves".

Moreover, one can restrict to the complexes $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)^{\mathrm{bd}}$ that are universally locally acyclic over $(\mathrm{Div}^1)^I$. This condition is also preserved under convolution.

Definition I.6.2. *The Satake category*

$$\mathrm{Sat}_G^I(\Lambda) \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)^{\mathrm{bd}}$$

is the category of all $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)^{\mathrm{bd}}$ that are perverse, flat over Λ (i.e., for all Λ-modules M, also $A \otimes_\Lambda^{\mathbb{L}} M$ is perverse), and universally locally acyclic over $(\mathrm{Div}^1)^I$.

Intuitively, $\mathrm{Sat}_G^I(\Lambda)$ are the "flat families of perverse sheaves on $\mathcal{H}\mathrm{ck}_G^I \to (\mathrm{Div}^1)^I$", where flatness refers both to the geometric aspect of flatness over $(\mathrm{Div}^1)^I$ (encoded in universal local acyclicity) and the algebraic aspect of flatness in the coefficients Λ. The Satake category $\mathrm{Sat}_G^I(\Lambda)$ is a monoidal category under convolution. Moreover, it is covariantly functorial in I.

In fact, the monoidal structure naturally upgrades to a symmetric monoidal structure. This relies on the fusion product, for which it is critical to allow general finite sets I. Namely, given finite sets $I_1, \ldots, I_k$, letting $I = I_1 \sqcup \ldots \sqcup I_k$, one has an isomorphism

$$\mathcal{H}\mathrm{ck}_G^I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_k} \cong \prod_{j=1}^k \mathcal{H}\mathrm{ck}_G^{I_j} \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_k}$$

where $(\mathrm{Div}^1)^{I;I_1,\ldots,I_k} \subset (\mathrm{Div}^1)^I$ is the open subset where $x_i \neq x_{i'}$ whenever $i, i' \in I$ lie in different I_j's. The exterior tensor product then defines a functor

$$\boxtimes_{j=1}^k : \prod_{j=1}^k \mathrm{Sat}_G^{I_j}(\Lambda) \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda)$$

where $\mathrm{Sat}_G^{I;I_1,\dots,I_k}(\Lambda)$ is the variant of $\mathrm{Sat}_G^I(\Lambda)$ for $\mathcal{H}\mathrm{ck}_G^I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\dots,I_k}$. However, the restriction functor

$$\mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I;I_1,\dots,I_k}(\Lambda)$$

is fully faithful, and the essential image of the exterior product lands in its essential image. Thus, we get a natural functor

$$*_{j=1}^k : \prod_{j=1}^k \mathrm{Sat}_G^{I_j}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda),$$

independent of the ordering of the I_j. In particular, for any I, we get a functor

$$\mathrm{Sat}_G^I(\Lambda) \times \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I \sqcup I}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda),$$

using functoriality of $\mathrm{Sat}_G^J(\Lambda)$ in J, which defines a symmetric monoidal structure $*$ on $\mathrm{Sat}_G^I(\Lambda)$, commuting with $\star$. This is called the fusion product. In general, for any symmetric monoidal category $(\mathcal{C}, *)$ with a commuting monoidal structure $\star$, the monoidal structure $\star$ necessarily agrees with $*$; thus, the fusion product refines the convolution product. (As usual in geometric Satake, we actually need to change $*$ slightly by introducing certain signs into the commutativity constraint, depending on the parity of the support of the perverse sheaves.)

Moreover, restricting $A \in \mathrm{Sat}_G^I(\Lambda)$ to Gr_G^I and taking the pushforward to $(\mathrm{Div}^1)^I$, all cohomology sheaves are local systems of Λ-modules on $(\mathrm{Div}^1)^I$. By a version of Drinfeld's lemma, these are equivalent to representations of W_E^I on Λ-modules. This defines a symmetric monoidal fibre functor

$$F^I : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Rep}_{W_E^I}(\Lambda),$$

where $\mathrm{Rep}_{W_E^I}(\Lambda)$ is the category of continuous representations of W_E^I on finite projective Λ-modules. Using a version of Tannaka duality, one can then build a Hopf algebra in the Ind-category of $\mathrm{Rep}_{W_E^I}(\Lambda)$ so that $\mathrm{Sat}_G^I(\Lambda)$ is given by its category of representations (internal in $\mathrm{Rep}_{W_E^I}(\Lambda)$). For any finite set I, this is given by the tensor product of I copies of the corresponding Hopf algebra for $I = \{*\}$, which in turn is given by some affine group scheme $\breve{G}$ over Λ with W_E-action.

Theorem I.6.3. *There is a canonical isomorphism $\breve{G} \cong \widehat{G}$ with the Langlands dual group, under which the action of W_E on $\breve{G}$ agrees with the usual action of W_E on $\widehat{G}$ up to an explicit cyclotomic twist. If $\sqrt{q} \in \Lambda$, the cyclotomic twist can be trivialized, and $\mathrm{Sat}_G^I(\Lambda)$ is naturally equivalent to the category of $(\widehat{G} \rtimes W_E)^I$-representations on finite projective Λ-modules.*

This theorem is thus a version of the theorem of Mirković–Vilonen [**MV07**], coupled with the refinements of Gaitsgory [**Gai07**] for general I. (We remark that we formulate a theorem valid for any Λ, not necessarily regular; such a formulation does not seem to be in the literature. Also, we give a purely local proof: Most proofs require a globalization on a (usual) curve.) Contrary to Mirković–Vilonen, we actually construct an explicit pinning of $\breve{G}$. For the proof, one can restrict to $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$; passing to a limit over n, one can actually build a group scheme over $\mathbb{Z}_\ell$. Its generic fibre is reductive, as the Satake category with $\mathbb{Q}_\ell$-coefficients is (geometrically) semisimple: For this, we again use the degeneration to the Witt vector affine Grassmannian and the decomposition theorem for schemes. To identify the reductive group, we argue first for tori, and then for rank 1 groups, where everything reduces to $G = \mathrm{PGL}_2$ which is easy to analyze by using the minuscule Schubert cell. Here, the pinning includes a cyclotomic twist as of course the cohomology of

the minuscule Schubert variety $\mathbb{P}^1$ of $\mathrm{Gr}_{\mathrm{PGL}_2}$ contains a cyclotomic twist. Afterwards, we apply hyperbolic localization in order to construct symmetric monoidal functors $\mathrm{Sat}_G \to \mathrm{Sat}_M$ for any Levi M of G, inducing dually maps $\breve{M} \to \breve{G}$. This produces many Levi subgroups of $\breve{G}_{\mathbb{Q}_\ell}$ from which it is easy to get the isomorphism with $\widehat{G}_{\mathbb{Q}_\ell}$, including a pinning. As these maps $\breve{M} \to \breve{G}$ are even defined integrally, and $\widehat{G}(\mathbb{Z}_\ell) \subset \widehat{G}(\mathbb{Q}_\ell)$ is a maximal compact open subgroup by Bruhat–Tits theory, generated by the rank 1 Levi subgroups, one can then deduce that $\breve{G} \cong \widehat{G}$ integrally, again with an explicit (cyclotomic) pinning.

We will also need the following addendum regarding a natural involution. Namely, the local Hecke stack $\mathcal{H}\mathrm{ck}_G$ has a natural involution sw given by reversing the roles of the two G-torsors; in the presentation in terms of LG, this is induced by the inversion on LG. Then sw^* induces naturally an involution of $\mathrm{Sat}_G(\Lambda)$, and this involution can be upgraded to a symmetric monoidal functor commuting with the fibre functor, thus realizing a W_E-equivariant automorphism of $\breve{G} \cong \widehat{G}$.

Proposition I.6.4. *The action of* sw^* *on* Sat_G *induces the automorphism of* $\widehat{G}$ *that is the Cartan involution of the split group* $\widehat{G}$*, conjugated by* $\widehat{\rho}(-1)$*.*

Critical to all of our arguments is the hyperbolic localization functor. In the setting of the Beilinson–Drinfeld Grassmannian, assume that $P^+, P^- \subset G$ are two opposite parabolics, with common Levi M. We get a diagram

$$\begin{tikzcd}
 & \mathrm{Gr}^I_{P^+} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\
\mathrm{Gr}^I_G & & \mathrm{Gr}^I_M . \\
 & \mathrm{Gr}^I_{P^-} \arrow[ul, "q^-"] \arrow[ur, "p^-"'] &
\end{tikzcd}$$

We get two "constant term" functors

$$\mathrm{CT}^+ = R(p^+)_!(q^+)^*, \mathrm{CT}^- = R(p^-)_* R(q^-)^! : D_{\text{ét}}(\mathrm{Gr}^I_G, \Lambda)^{\mathrm{bd}} \to D_{\text{ét}}(\mathrm{Gr}^I_M, \Lambda)^{\mathrm{bd}},$$

and one can construct a natural transformation $\mathrm{CT}^- \to \mathrm{CT}^+$. The functor CT^+ corresponds classically to the Satake transform, of integrating along orbits under the unipotent radical of U^+. Hyperbolic localization claims that the transformation $\mathrm{CT}^- \to \mathrm{CT}^+$ is an equivalence when restricted to L^+G-equivariant objects. This has many consequences; note that CT^+ is built from left adjoint functors while CT^- is built from right adjoint functors, so if they are isomorphic, hyperbolic localization has the best of both worlds. In particular, hyperbolic localization commutes with all colimits and all limits, preserves (relative) perversity, universal local acyclicity, commutes with any base change, etc. .

This is in fact a special case of the following more general assertion. Let S be any small v-stack, and $f : X \to S$ be a proper map that is representable in spatial diamonds with $\dim.\mathrm{trg}\, f < \infty$. Assume that there is an action of $\mathbb{G}_m$ on X/S, where $\mathbb{G}_m(R, R^+) = R^\times$. The fixed points $X^0 \subset X$ of the $\mathbb{G}_m$-action form a closed substack. We assume that one can define an attractor locus $X^+ \subset X$ and a repeller locus $X^- \subset X$, given by disjoint unions of locally closed subspaces, on which the

$t \in \mathbb{G}_m$-action admits a limit as $t \to 0$ (resp. $t \to \infty$). We get a diagram

$$\begin{tikzcd} & X^+ \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ X & & X^0, \\ & X^- \arrow[ul, "q^-"] \arrow[ur, "p^-"'] & \end{tikzcd}$$

generalizing (bounded parts of) the above diagram if one chooses a cocharacter $\mu : \mathbb{G}_m \to G$ whose dynamic parabolics are P^+, P^-. One can define

$$L^+ = R(p^+)_!(q^+)^*, L^- = R(p^-)_* R(q^-)^! : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X^0, \Lambda)$$

and a natural transformation $L^- \to L^+$. The following is our version of Braden's theorem [**Bra03**], cf. also [**Ric19**].

THEOREM I.6.5. *The transformation $L^- \to L^+$ is an equivalence when restricted to the essential image of $D_{\text{ét}}(X/\mathbb{G}_m, \Lambda) \to D_{\text{ét}}(X, \Lambda)$.*

The proof makes use of the following principle: If $Y \to S$ is partially proper with a $\mathbb{G}_m$-action such that the quotient stack $Y/\mathbb{G}_m$ is qcqs over S, then again Y admits two ends, and the partially compactly supported cohomology of Y with coefficients in any $A \in D_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ vanishes identically.

I.7. Cohomology of moduli spaces of shtuka

At this point, we have defined

$$D(\mathrm{Bun}_G, \Lambda),$$

and using the geometric Satake equivalence and the diagram

$$\begin{tikzcd} & \mathrm{Hck}_G^I \arrow[r, "q"] \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \mathcal{H}\mathrm{ck}_G^I \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

one can define the Hecke operator

$$T_V = Rh_{2*}(h_1^* \otimes_\Lambda^{\mathbb{L}} q^* \mathcal{S}_V) : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)$$

for any $V \in \mathrm{Sat}_G^I(\Lambda)$, where $\mathcal{S}_V$ is the corresponding sheaf on $\mathcal{H}\mathrm{ck}_G^I$. This works at least if Λ is killed by some power of ℓ. We can in fact extend this functor to all $\mathbb{Z}_\ell$-algebras Λ. Moreover, its image lies in the full subcategory of those objects that are locally constant in the direction of $(\mathrm{Div}^1)^I$, thereby giving a functor

$$T_V : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$$

to the category of W_E^I-equivariant objects in $D(\mathrm{Bun}_G, \Lambda)$. The proof is surprisingly formal: One reduces to $I = \{*\}$ by an inductive argument, and then uses that $\mathrm{Div}^1 = \mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}}$ is still just a point. More precisely, one uses that

$$D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G \times \mathrm{Spd}\, \widehat{\overline{E}}, \Lambda)$$

is an equivalence.

Remark I.7.1. To define $D(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$, we need to upgrade $D(\mathrm{Bun}_G, \Lambda)$ to a condensed ∞-category; then it is the notion of W_E^I-equivariant objects for the condensed group W_E^I.

A first consequence of our results is that T_V, forgetting the W_E^I-equivariance, preserves finiteness properties. Note that $T_V \circ T_W \cong T_{V \otimes W}$ as the geometric Satake equivalence is monoidal. This formally implies that T_V is left and right adjoint to T_{V^*}. From here, it is not hard to prove the following result.

Theorem I.7.2. *The functor $T_V : D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$ preserves compact objects and universally locally acyclic objects. Moreover, it commutes with Bernstein–Zelevinsky and Verdier duality in the sense that there are natural isomorphisms $\mathbb{D}_{\mathrm{BZ}}(T_V(A)) \cong T_{\mathrm{sw}^* V^\vee}(\mathbb{D}_{\mathrm{BZ}}(A))$ and $R\mathscr{H}\mathrm{om}(T_V(A), \Lambda) \cong T_{(\mathrm{sw}^* V)^\vee} R\mathscr{H}\mathrm{om}(A, \Lambda)$.*

Here sw^* is the involution of Sat_G^I which by Proposition I.6.4 is induced by the Cartan involution of $\widehat{G}$, conjugated by $\widehat{\rho}(-1)$.

This theorem has concrete consequences for the cohomology of moduli spaces of shtukas. For simplicity, we formulate it here with coefficients in a Λ-algebra that is killed by ℓ^n for some n; for the general formulation, we would need to discuss more precisely the foundational issues surrounding the derived categories. In [**SW20**, Lecture XXIII], for any collection $\{\mu_i\}_i$ of conjugacy classes of cocharacters with fields of definition E_i/E and $b \in B(G)$, there is defined a tower of moduli spaces of local shtukas

$$f_K : (\mathrm{Sht}_{(G,b,\mu_\bullet),K})_{K \subset G(E)} \to \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i$$

as K ranges over compact open subgroups of $G(E)$, equipped with compatible étale period maps

$$\pi_K : \mathrm{Sht}_{(G,b,\mu_\bullet),K} \to \mathrm{Gr}^{\mathrm{tw}}_{G, \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i, \leq \mu_\bullet}.$$

Here, $\mathrm{Gr}^{\mathrm{tw}}_{G, \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i} \to \prod_{i \in I} \mathrm{Spd}\, \breve{E}$ is a certain twisted form of the convolution affine Grassmannian, cf. [**SW20**, Section 23.5]. Let W be the exterior tensor product $\boxtimes_{i \in I} V_{\mu_i}$ of highest weight representations, and $\mathcal{S}_W$ the corresponding sheaf on $\mathrm{Gr}^{\mathrm{tw}}_{G, \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i}$. We continue to write $\mathcal{S}_W$ for its pullback to $\mathrm{Sht}_{(G,b,\mu_\bullet),K}$.

Corollary I.7.3. *The sheaf*

$$Rf_{K!}\mathcal{S}_W \in D([*/\underline{G_b(E)}] \times \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i, \Lambda)$$

is equipped with partial Frobenii, thus descends to an object of

$$D([*/\underline{G_b(E)}] \times \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i/\varphi_i^{\mathbb{Z}}, \Lambda).$$

This object lives in the full subcategory

$$D(G_b(E),\Lambda)^{B\prod_{i\in I} W_{E_i}} \subset D([*/\underline{G_b(E)})] \times \prod_{i\in I} \operatorname{Spd} \breve{E}_i/\varphi_i^{\mathbb{Z}}, \Lambda),$$

and its restriction to $D(G_b(E),\Lambda)$ is compact. In particular, for any admissible representation ρ of $G_b(E)$, the object

$$R\operatorname{Hom}_{G_b(E)}(Rf_{K!}\mathcal{S}_W,\rho) \in D(\Lambda)^{B\prod_{i\in I} W_{E_i}}$$

is a representation of $\prod_{i\in I} W_{E_i}$ on a perfect complex of Λ-modules. Taking the colimit over K, this gives rise to a complex of admissible $G(E)$-representations

$$\varinjlim_K R\operatorname{Hom}_{G_b(E)}(Rf_{K!}\mathcal{S}_W,\rho)$$

equipped with a $\prod_{i\in I} W_{E_i}$-action.

If ρ is compact, then so is

$$\varinjlim_K R\operatorname{Hom}_{G_b(E)}(Rf_{K!}\mathcal{S}_W,\rho)$$

as a complex of $G(E)$-representations.

Specializing to $I = \{*\}$ and μ minuscule, we get local Shimura varieties, and this proves the finiteness properties of [**RV14**, Proposition 6.1] unconditionally, as well as [**RV14**, Remark 6.2 (iii)]. We note that those properties seem inaccessible using only the definition of the moduli spaces of shtukas, i.e. without the use of Bun_G.

I.8. The stack of L-parameters

Let us discuss the other side of the Langlands correspondence, namely (the stack of) L-parameters. This has been previously done by Dat–Helm–Kurinczuk–Moss [**DHKM20**] and Zhu [**Zhu20**]. One wants to define a scheme whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the continuous 1-cocycles

$$\varphi : W_E \to \widehat{G}(\Lambda).$$

(Here, we endow $\widehat{G}$ with its usual W_E-action, that factors over a finite quotient Q of W_E. As discussed above, the difference between the two actions disappears over $\mathbb{Z}_\ell[\sqrt{q}]$, and we find it much more convenient to use the standard normalization here, so that we can sometimes make use of the algebraic group $\widehat{G} \rtimes Q$.)

There seems to be a mismatch here, in asking for an algebraic stack, but continuous cocycles. Interestingly, there is a way to phrase the continuity condition that produces a scheme. Namely, we consider Λ as a condensed $\mathbb{Z}_\ell$-algebra that is "relatively discrete over $\mathbb{Z}_\ell$". Abstract $\mathbb{Z}_\ell$-modules M embed fully faithfully into condensed $\mathbb{Z}_\ell$-modules, via sending M to $M_{\mathrm{disc}} \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$.

Theorem I.8.1. *There is a scheme $Z^1(W_E,\widehat{G})$ over $\mathbb{Z}_\ell$ whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the condensed 1-cocycles*

$$\varphi : W_E \to \widehat{G}(\Lambda),$$

where we regard Λ as a relatively discrete condensed $\mathbb{Z}_\ell$-algebra. The scheme $Z^1(W_E,\widehat{G})$ is a union of open and closed affine subschemes $Z^1(W_E/P,\widehat{G})$ as P runs through open subgroups of the wild

inertia subgroup of W_E, and each $Z^1(W_E/P, \widehat{G})$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $\dim G$.

The point here is that the inertia subgroup of W_E has a $\mathbb{Z}_\ell$-factor, and this can map in interesting ways to Λ when making this definition. To prove the theorem, following [**DHKM20**] and [**Zhu20**] we define discrete dense subgroups $W \subset W_E/P$ by discretizing the tame inertia, and the restriction $Z^1(W_E/P, \widehat{G}) \to Z^1(W, \widehat{G})$ is an isomorphism, where the latter is clearly an affine scheme.

We can also prove further results about the $\widehat{G}$-action on $Z^1(W_E, \widehat{G})$, or more precisely each $Z^1(W_E/P, \widehat{G})$. For this result, we need to exclude some small primes, but if $G = \mathrm{GL}_n$, all primes ℓ are allowed; for classical groups, all $\ell \neq 2$ are allowed. More precisely, we say that ℓ is very good for $\widehat{G}$ if the following conditions are satisfied.

(i) The (algebraic) action of W_E on $\widehat{G}$ factors over a finite quotient Q of order prime to ℓ.

(ii) The order of the fundamental group of the derived group of $\widehat{G}$ is prime to ℓ (equivalently, $\pi_0 Z(G)$ is of order prime to ℓ).

(iii) If G has factors of type B, C, or D, then $\ell \neq 2$; if it has factors of type E, F, or G, then $\ell \neq 2, 3$; and if it has factors of type E_8, then $\ell \neq 2, 3, 5$.

THEOREM I.8.2. *Assume that ℓ is a very good prime for $\widehat{G}$. Then $H^i(\widehat{G}, \mathcal{O}(Z^1(W_E/P, \widehat{G}))) = 0$ for $i > 0$ and the formation of the invariants $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ commutes with any base change. The algebra $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ admits an explicit presentation in terms of excursion operators,*

$$\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}} = \operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}}$$

where the colimit runs over all maps from a free group F_n to $W \subset W_E/P$, and $Z^1(F_n, \widehat{G}) \cong \widehat{G}^n$ with the simultaneous twisted $\widehat{G}$-conjugation.

Moreover, the ∞-category $\mathrm{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$ *is generated under cones and retracts by the image of* $\mathrm{Rep}(\widehat{G}) \to \mathrm{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$, *and* $\mathrm{Ind}\,\mathrm{Perf}(Z^1(W_E/P, \widehat{G}))$ *is equivalent to the ∞-category of modules over $\mathcal{O}(Z^1(W_E/P, \widehat{G}))$ in* $\mathrm{Ind}\,\mathrm{Perf}(B\widehat{G})$.

All of these results also hold with $\mathbb{Q}_\ell$-coefficients, without any assumption on ℓ.

With $\mathbb{Q}_\ell$-coefficients, these results are simple, as the representation theory of $\widehat{G}$ is semisimple. However, with $\mathbb{Z}_\ell$-coefficients, these results are quite subtle, and we need to dive into modular representation theory of reductive groups. Very roughly, the proof of the theorem proceeds by analyzing the closed $\widehat{G}$-orbits in the stack of L-parameters first, and then use a deformation to the normal cone to understand the behaviour near any $\widehat{G}$-orbit. We make critical use of some results of Touzé–van der Kallen [**TvdK10**]. Let us make some further remarks about the closed $\widehat{G}$-orbits.

First, the closed $\widehat{G}$-orbits in $Z^1(W_E/P, \widehat{G})_L$, for any algebraically closed field L over $\mathbb{Z}_\ell$, correspond to the *semisimple* L-parameter $\varphi : W_E \to \widehat{G}(L)$, and also biject to the geometric points of $\operatorname{Spec} \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$. Here, an L-parameter is semisimple if, whenever it factors over a parabolic $\widehat{P} \subset \widehat{G}$, it also factors over a Levi $\widehat{M} \subset \widehat{P}$. Any semisimple parameter is in fact continuous for the discrete topology on L, i.e. trivial on an open subgroup of I_E. If $L = \overline{\mathbb{F}}_\ell$ and Q is of order prime to ℓ, then φ is semisimple if and only if it factors over a finite quotient of order prime to ℓ.

Its orbit in $Z^1(W_E/P, \widehat{G})_L$ is then given by $\widehat{G}/S_\varphi$ where $S_\varphi \subset \widehat{G}$ is the centralizer of φ, which is in that case the fixed points under the action of a finite solvable group F of automorphisms of $\widehat{G}$, where the order of F is prime to ℓ. We need the following result.

THEOREM I.8.3. *Let H be a reductive group over an algebraically closed field L of characteristic ℓ. Let F be a finite group of order prime to ℓ acting on H. Then H^F is a smooth linear algebraic group whose connected component is reductive, and with $\pi_0 H^F$ of order prime to ℓ. If F is solvable, the image of* $\mathrm{Perf}(BH) \to \mathrm{Perf}(BH^F)$ *generates under cones and retracts.*

The last part of this theorem is proved by a very explicit (and exhausting) analysis of all possible cases. It would be desirable to have a more enlightening argument, possibly also removing the assumption that F is solvable. In fact, we would expect similar results to hold true in the case where W is replaced by the fundamental group of a Riemann surface. Our arguments are not general enough to handle that case.

REMARK I.8.4. While the hypotheses imposed on ℓ are surely not optimal, we are quite sure that *some* hypothesis on ℓ is required in Theorem I.8.2. For example, if $\widehat{G} = \mathrm{PGL}_2$ and $\ell = 2$, we expect problems to arise. For example, one can show that for $X = \widehat{G}$ with the conjugation action by $\widehat{G}$, the ∞-category $\mathrm{Perf}(X/\widehat{G})$ is not generated under cones and retracts by the image of $\mathrm{Rep}(\widehat{G})$. Our guess would be that the condition that ℓ does not divide the order of $\pi_1(\widehat{G}_{\mathrm{der}})$ is essential.

On the other hand, we expect that, for example by the use of z-embeddings [**Kal14**, Section 5], one can reduce all relevant questions (like the general construction of maps on Bernstein centers discussed below, or the construction of the spectral action) to the case where ℓ does not divide the order of $\pi_1(\widehat{G}_{\mathrm{der}})$. We are not taking this up here.

I.9. Construction of L-parameters

Finally, we can discuss the construction of L-parameters. Assume first for simplicity that $\Lambda = \overline{\mathbb{Q}}_\ell$ with fixed $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$, and let $A \in D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ be any Schur-irreducible object, i.e. $\mathrm{End}(A) = \overline{\mathbb{Q}}_\ell$. For example, A could correspond to an irreducible smooth representation of $G(E)$, taking the extension by zero along $[*/G(E)] \hookrightarrow \mathrm{Bun}_G$. Then, following V. Lafforgue [**Laf18**], we can define excursion operators as follows. For any representation V of $(\widehat{G} \rtimes W_E)^I$ over $\overline{\mathbb{Q}}_\ell$, together with maps

$$\alpha : \overline{\mathbb{Q}}_\ell \to V|_{\widehat{G}}, \beta : V|_{\widehat{G}} \to \overline{\mathbb{Q}}_\ell$$

when restricted to the action of the diagonal copy $\widehat{G} \subset (\widehat{G} \rtimes W_E)^I$, and elements $\gamma_i \in W_E$ for $i \in I$, we can define the endomorphism

$$A \xrightarrow{T_\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i \in I}} T_V(A) \xrightarrow{T_\beta} A$$

of A, defining an element of $\overline{\mathbb{Q}}_\ell$. With all the formalism in place, the following result is essentially due to V. Lafforgue [**Laf18**, Proposition 11.7].

PROPOSITION I.9.1. *There is a unique continuous semisimple L-parameter*

$$\varphi_A : W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

such that for all $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ *as above, the excursion operator*

$$A \xrightarrow{T_\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{T_\beta} A$$

is given by multiplication with the scalar

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i)_{i\in I})} V \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell.$$

Note that in fact, the excursion operators define elements in the Bernstein center of $G(E)$, as they define endomorphisms of the identity functor. From this perspective, let us make the following definition.

Definition I.9.2.

(i) *The Bernstein center of* $G(E)$ *is*

$$\mathcal{Z}(G(E), \Lambda) = \pi_0 \mathrm{End}(\mathrm{id}_{\mathcal{D}(G(E),\Lambda)}) = \varprojlim_{K\subset G(E)} \mathcal{Z}(\Lambda[K\backslash G(E)/K])$$

where K *runs over open pro-p subgroups of* $G(E)$*, and* $\Lambda[K\backslash G(E)/K] = \mathrm{End}_{G(E)}(c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda)$ *is the Hecke algebra of level* K*.*

(ii) *The geometric Bernstein center of* G *is*

$$\mathcal{Z}^{\mathrm{geom}}(G, \Lambda) = \pi_0 \mathrm{End}(\mathrm{id}_{\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}).$$

Inside $\mathcal{Z}^{\mathrm{geom}}(G, \Lambda)$*, we let* $\mathcal{Z}^{\mathrm{geom}}_{\mathrm{Hecke}}(G, \Lambda)$ *be the subring of all endomorphisms* $f : \mathrm{id} \to \mathrm{id}$ *commuting with Hecke operators, in the sense that for all* $V \in \mathrm{Rep}(\widehat{G}^I)$ *and* $A \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$*, one has* $T_V(f(A)) = f(T_V(A)) \in \mathrm{End}(T_V(A))$*.*

(iii) *The spectral Bernstein center of* G *is*

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda) = \mathcal{O}(Z^1(W_E, \widehat{G})_\Lambda)^{\widehat{G}},$$

the ring of global functions on the quotient stack $Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}$*.*

The inclusion $\mathcal{D}(G(E), \Lambda) \hookrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ induces a map of algebra $\mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda)$.

Now the construction of excursion operators, together with Theorem I.8.2 imply the following. Here Λ is a $\mathbb{Z}_\ell[\sqrt{q}]$-algebra.

Proposition I.9.3. *Assume that* ℓ *is invertible in* Λ*, or* ℓ *is a very good prime for* $\widehat{G}$*. Then there is a canonical map*

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}^{\mathrm{geom}}_{\mathrm{Hecke}}(G, \Lambda) \subset \mathcal{Z}^{\mathrm{geom}}(G, \Lambda),$$

and in particular a map

$$\Psi_G : \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda).$$

In fact, even for general ℓ, one gets similar maps, up to replacing $Z^1(W_E, \widehat{G})/\!\!/\widehat{G}$ by a universally homeomorphic scheme. The construction of L-parameters above is then a consequence of this map on Bernstein centers. The existence of such an integral map is due to Helm–Moss [**HM18**] in the case $G = \mathrm{GL}_n$.

Remark I.9.4. In the function-field case, a similar construction has been given by Genestier–Lafforgue [**GL17**]. We expect that the two constructions agree, and that proving this is within reach (as both are based on the cohomology of moduli spaces of shtukas), but have not thought much about it.

We make the following conjecture regarding independence of ℓ. For its formulation, we note that there is a natural $\mathbb{Q}$-algebra $\mathcal{Z}^{\mathrm{spec}}(G,\mathbb{Q})$ whose base change to $\mathbb{Q}_\ell$ is $\mathcal{Z}^{\mathrm{spec}}(G,\mathbb{Q}_\ell)$ for any $\ell \neq p$; in fact, one can take the global functions on the stack of L-parameters that are continuous for the discrete topology (i.e. trivial on an open subgroup of W_E); see also [**DHKM20**].

Conjecture I.9.5. *There is a (necessarily unique) map $\mathcal{Z}^{\mathrm{spec}}(G,\mathbb{Q}(\sqrt{q})) \to \mathcal{Z}(G(E),\mathbb{Q}(\sqrt{q}))$ that after base extension to any $\mathbb{Q}_\ell$ for $\ell \neq p$ recovers the composite*

$$\mathcal{Z}^{\mathrm{spec}}(G,\mathbb{Q}_\ell(\sqrt{q})) \to \mathcal{Z}^{\mathrm{geom}}(G,\mathbb{Q}_\ell(\sqrt{q})) \to \mathcal{Z}(G(E),\mathbb{Q}_\ell(\sqrt{q})).$$

This would ensure that the L-parameters we construct are independent of ℓ in the relevant sense. Further conjectures about this map and its relation to the stable Bernstein center have been formulated by Haines [**Hai14**] (see also [**BKV15**], [**SS13**, Section 6]). In particular, it is conjectured that for G quasisplit, the map Ψ_G is injective, and its image can be characterized as those elements of the Bernstein center of $G(E)$ whose corresponding distribution is invariant under stable conjugation.

One can also construct the map to the Bernstein center in terms of moduli spaces of local shtukas, as follows. For simplicity, we discuss this again only if Λ is a $\mathbb{Z}/\ell^n$-algebra for some n. Given I and V as above, we can consider a variant $\mathrm{Sht}_{(G,1,V),K}$ of the spaces $\mathrm{Sht}_{(G,b,\leq\mu_\bullet),K}$ considered above, where the bound is given by the support of V and we fix the element $b = 1$. They come with an étale period map

$$\pi_K : \mathrm{Sht}_{(G,1,V),K} \to \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i=1}^n \mathrm{Spd}\,\breve{E}}$$

and a perverse sheaf $\mathcal{S}_V$. When restricted to the geometric diagonal

$$\overline{x} : \mathrm{Spd}\,\widehat{\overline{E}} \to \prod_{i=1}^n \mathrm{Spd}\,\breve{E},$$

they become a corresponding moduli space of shtukas with one leg

$$f_K^\Delta : \mathrm{Sht}_{(G,1,V|_{\widehat{G}})} \to \mathrm{Spd}\,\widehat{\overline{E}}$$

with the sheaf $\mathcal{S}_{V|_{\widehat{G}}}$. The sheaf $\mathcal{S}_{V|_{\widehat{G}}}$ admits maps α (resp. β) from (resp. to) the sheaf $i_*\Lambda$, where $i : G(E)/K = \mathrm{Sht}_{(G,1,\overline{\mathbb{Q}}_\ell),K} \hookrightarrow \mathrm{Sht}_{(G,1,V|_{\widehat{G}}),K}$ is the subspace of shtukas with no legs. This produces an endomorphism

$$c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda \xrightarrow{\alpha} Rf_{K!}^\Delta \mathcal{S}_{V|_{\widehat{G}}} = (Rf_{K!}\mathcal{S}_V)_{\overline{x}} \xrightarrow{(\gamma_i)_{i\in I}} (Rf_{K!}\mathcal{S}_V)_{\overline{x}} = Rf_{K!}^\Delta \mathcal{S}_{V|_{\widehat{G}}} \xrightarrow{\beta} c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda.$$

Here, the action of $(\gamma_i)_{i\in I}$ is defined by Corollary I.7.3. It follows from the definitions that this is precisely the previous construction applied to the representation $c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda$. Note that these endomorphisms are $G(E)$-equivariant, so define elements in the Hecke algebra

$$\Lambda[K\backslash G(E)/K] = \mathrm{End}_{G(E)}(c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda);$$

in fact, these elements are central (as follows by comparison to the previous construction). Taking the inverse limit over K, one gets the elements in the Bernstein center of $G(E)$.[5]

Concerning the L-parameters we construct, we can prove the following basic results.

THEOREM I.9.6.

(i) *If $G = T$ is a torus, then $\pi \mapsto \varphi_\pi$ is the usual Langlands correspondence.*

(ii) *The correspondence $\pi \mapsto \varphi_\pi$ is compatible with twisting.*

(iii) *The correspondence $\pi \mapsto \varphi_\pi$ is compatible with central characters.*

(iv) *The correspondence $\pi \mapsto \varphi_\pi$ is compatible with passage to congradients.*

(v) *If $G' \to G$ is a map of reductive groups inducing an isomorphism of adjoint groups, π is an irreducible smooth representation of $G(E)$ and π' is an irreducible constitutent of $\pi|_{G'(E)}$, then $\varphi_{\pi'}$ is the image of φ_π under the induced map $\widehat{G} \to \widehat{G'}$.*

(vi) *If $G = G_1 \times G_2$ is a product of two groups and π is an irreducible smooth representation of $G(E)$, then $\pi = \pi_1 \boxtimes \pi_2$ for irreducible smooth representations π_i of $G_i(E)$, and $\varphi_\pi = \varphi_{\pi_1} \times \varphi_{\pi_2}$ under $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$.*

(vii) *If $G = \mathrm{Res}_{E'|E} G'$ is the Weil restriction of scalars of a reductive group G' over some finite separable extension $E'|E$, so that $G(E) = G'(E')$, then L-parameters for $G|E$ agree with L-parameters for $G'|E'$.*

(viii) *The correspondence $\pi \mapsto \varphi_\pi$ is compatible with parabolic induction.*

(ix) *For $G = \mathrm{GL}_n$ and supercuspidal π, the correspondence $\pi \mapsto \varphi_\pi$ agrees with the usual local Langlands correspondence* [**LRS93**], [**HT01**], [**Hen00**].

Note that parts (viii) and (ix) together say that for GL_n and general π, the L-parameter φ_π is what is usually called the semisimple L-parameter.

I.10. The spectral action

The categorical structure we have constructed actually produces something better. Let Λ be the ring of integers in a finite extension of $\mathbb{Q}_\ell(\sqrt{q})$. We have the stable ∞-category $\mathcal{C} = \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ of compact objects, which is linear over Λ, and functorially in the finite set I an exact monoidal functor $\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{C})^{BW_E^I}$ that is linear over $\mathrm{Rep}_\Lambda(Q^I)$; here, $\mathrm{End}_\Lambda(\mathcal{C})$ denotes the stable ∞-category of Λ-linear endofunctors of $\mathcal{C}$, and we regard it as being enriched in condensed Λ-modules via regarding $\mathcal{C}$ as enriched in relatively discrete condensed Λ-modules. A first version of the following theorem is due to Nadler–Yun [**NY19**] in the context of Betti geometric Langlands, and a more general version appeared in the work of Gaitsgory–Kazhdan–Rozenblyum–Varshavsky [**GKRV19**]. Both references, however, effectively assume that G is split, work only with characteristic 0 coefficients, and work with a discrete group in place of W_E. At least the extension to $\mathbb{Z}_\ell$-coefficients is a nontrivial matter.

[5]When the second author gave his Berkeley lectures [**SW20**], this was the construction of excursion operators that we envisaged. Note that a key step here is that the cohomology of moduli spaces of local shtukas defines a local system on $(\mathrm{Div}^1)^I$. It is however not clear how to prove this purely in terms of moduli space of shtukas. In the global function field case, this result has been obtained by Xue [**Xue20**].

Note that $Z^1(W_E, \widehat{G})$ is not quasicompact, as it has infinitely many connected components; it can be written as the increasing union of open and closed quasicompact subschemes $Z^1(W_E/P, \widehat{G})$. We say that an action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})$ on a stable ∞-category $\mathcal{C}$ is compactly supported if for all $X \in \mathcal{C}$ the functor $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G}) \to \mathcal{C}$ (induced by acting on X) factors over some $\mathrm{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$.

We stress again that for $G = \mathrm{GL}_n$, all ℓ are very good, and for classical groups, all $\ell \neq 2$ are very good.

THEOREM I.10.1. *Assume that ℓ is a very good prime for $\widehat{G}$. Let $\mathcal{C}$ be a small Λ-linear stable ∞-category. Then giving, functorially in the finite set I, an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}(\widehat{G} \rtimes Q)^I \to \mathrm{End}_{\mathbb{Z}_\ell}(\mathcal{C})^{BW_E^I}$$

is equivalent to giving a compactly supported Λ-linear action of

$$\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}).$$

Here, given a compactly supported Λ-linear action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$, one can produce such an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_{\mathbb{Z}_\ell}(\mathcal{C})^{BW_E^I}$$

functorially in I by composing the exact $\mathrm{Rep}_\Lambda(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}(\widehat{G} \rtimes Q)^I \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E^I}$$

with the action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})$.

The same result holds true with Λ a field over $\mathbb{Q}_\ell$, for any prime ℓ.

Here, the exact $\mathrm{Rep}_\Lambda(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E^I}$$

is induced by tensor products and the exact $\mathrm{Rep}_\Lambda(Q)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q) \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E}$$

corresponding to the universal $\widehat{G} \rtimes Q$-torsor, with the universal W_E-equivariance as parametrized by $Z^1(W_E, \widehat{G})/\widehat{G}$.

The key part of the proof is actually the final part of Theorem I.8.2 above, which effectively describes $\mathrm{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$ in terms of generators and relations, as does the present theorem.

In particular, we get an action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$ on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, suitably compatible with the Hecke action.

With everything in place, it is now obvious that the main conjecture is the following, cf. [**AG15**], [**Zhu20**], [**Hel20**]:[6]

[6]The formulation of the conjecture presumes a spectral action, which we have only constructed under a small assumption on ℓ; implicit is thus that a spectral action, refining the Hecke action, can be defined for all ℓ.

Conjecture I.10.2. *Assume that G is quasisplit and choose Whittaker data consisting of a Borel $B \subset G$ and generic character $\psi : U(E) \to \mathcal{O}_L^\times$ of the unipotent radical $U \subset B$, where $L/\mathbb{Q}_\ell$ is some algebraic extension; also fix $\sqrt{q} \in \mathcal{O}_L$. Then there is an equivalence*

$$\mathcal{D}(\mathrm{Bun}_G, \mathcal{O}_L)^\omega \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G})$$

of stable ∞-categories equipped with actions of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G})$. Under this correspondence, the structure sheaf of $Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G}$ maps to the Whittaker sheaf, which is the sheaf concentrated on Bun^1_G corresponding to the Whittaker representations $c\text{-}\mathrm{Ind}_{U(E)}^{G(E)}\psi$.

Here, we use the notion of complexes of coherent sheaves with nilpotent singular support, see [**AG15**]. More precisely, $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}$ is the ∞-category of bounded complexes with quasicompact support, coherent cohomology, and nilpotent singular support. With characteristic 0 coefficients, or at banal primes ℓ, the condition of nilpotent singular support is actually automatic.

If $\mathcal{W}_\psi$ is the Whittaker sheaf and we note $*$ the spectral action, the conjecture thus says that

$$\begin{aligned} \mathrm{Perf}^{\mathrm{qc}}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G}) &\longrightarrow \mathcal{D}(\mathrm{Bun}_G, \mathcal{O}_L)^\omega \\ M &\longmapsto M * \mathcal{W}_\psi \end{aligned}$$

extends to an equivalence of stable ∞-categories

$$\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G}) \cong \mathcal{D}(\mathrm{Bun}_G, \mathcal{O}_L)^\omega;$$

in particular, the first functor has to be fully faithful.

Remark I.10.3. Consider the conjecture with coefficients in $\overline{\mathbb{Q}}_\ell$. Ideally, the conjecture should also include a comparison of t-structures. Unfortunately, we did not immediately see a good candidate for matching t-structures. Ideally, this would compare the perverse t-structure on the left (which is well-defined, for abstract reasons, and appears at least implicitly in [**CS17**], [**CS19a**]; it seems to be the "correct" t-structure for questions of local-global compatibility) with some "perverse-coherent" t-structure on the right. If so, the equivalence would also yield a bijection between irreducible objects in the abelian hearts. On the left-hand side, these irreducible objects would then be enumerated by pairs (b, π_b) of an element $b \in B(G)$ and an irreducible smooth representation π_b of $G_b(E)$, by using intermediate extensions. On the right-hand side, they would likely correspond to a Frobenius-semisimple L-parameter $\varphi : W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ together with an irreducible representation of the centralizer S_φ of φ. Independently of the categorical conjecture, one can wonder whether these two sets are in fact canonically in bijection.

I.11. The origin of the ideas

Finally, let us give some account of the historical developments of these ideas. Let us first recall some of our early work in the direction of local Langlands correspondences. Fargues [**Far04**] has proved that in the cohomology of basic Rapoport–Zink spaces for GL_n (and $U(3)$) and general minuscule cocharacters, an appropriate version of the local Langlands correspondence is realized. Moreover, Fargues [**Far08**] has proved the duality isomorphism between the Lubin–Tate and Drinfeld tower. Already at this point Fargues thought of this as an attempt to geometrize the Jacquet-Langlands correspondence, see [**Far08**, Theorem 2 of the Préambule]. On the other hand, Scholze

[**Sch13**] has given a new proof of the local Langlands correspondence for GL_n. His results pointed to the idea that there ought to exist certain sheaves on the moduli stack of p-divisible groups (which, when restricted to perfect schemes, can be regarded as a "part" of the stack GL_n-$\mathcal{I}$soc considered above), giving a certain geometrization of the local Langlands correspondence, then formulated as a certain character sheaf property (inspired by the character formulas in [**Sch13**]). Related observations were also made by Boyer (cf. e.g. [**Boy09**]) and in unpublished work of Dat. However, Scholze was always uneasy with the very bad geometric properties of the stack of p-divisible groups.

At this point, both of us had essentially left behind local Langlands to study other questions. Fargues found the fundamental curve of p-adic Hodge theory in his work with Fontaine [**FF18**]; an initial critical motivation for Fargues was a development of "p-adic Hodge theory without Galois actions", i.e. for fields like $\mathbb{C}_p$. Indeed, this was required in some of his work on Rapoport–Zink spaces. On the other hand, Scholze developed perfectoid spaces [**Sch12**], motivated by the weight-monodromy conjecture. After his talk at a conference in Princeton in March 2011, Weinstein gave a talk about his results on the Lubin–Tate tower at infinite level, which made it clear that it is in fact a perfectoid space. Scholze at the time was already eager to understand the isomorphism between Lubin–Tate and Drinfeld tower, and it now became clear that it should really be an isomorphism of perfectoid spaces. This was worked out in [**SW13**]. At the time of writing of [**SW13**], the perspective of the Fargues–Fontaine curve had already become central, and we realized that the isomorphism of the towers simply amounts to two dual descriptions of the space of minuscule modifications $\mathcal{O}_X^n \to \mathcal{O}_X(\frac{1}{n})$ on the Fargues–Fontaine curve, depending on which bundle is fixed and which one is the modification. This was the first clear connection between local Langlands (as encoded in the cohomology of Lubin–Tate and Drinfeld space) and the theory of vector bundles on the Fargues–Fontaine curve, which Scholze had however not taken seriously enough. Moreover, Fargues had noted in [**FF18**], in the proof of "weakly admissible implies admissible", that modifications of vector bundles were playing an important role: the Hodge filtration of a filtered φ-module allows one to define a new vector bundle by modifying the vector bundle associated to an isocrystal i.e. by "applying a Hecke correspondence" as he said in the talk [**Far10**] at the conference in honor of Jean-Marc Fontaine.

This duality perspective also put the two dual period morphisms into the center of attention: The Hodge–de Rham period mapping, and the Hodge–Tate period mapping (which are swapped under the duality isomorphism). Thinking about the Lubin–Tate tower as part of the moduli space of elliptic curves, Scholze then realized that the Hodge–Tate period map even exists globally on the moduli space of elliptic curves with infinite level (on the level of Berkovich topological spaces, this had also been observed by Fargues before). Moreover, Scholze realized that the Hodge–Tate period map gives a substitute for the map from the moduli space of elliptic curves to the moduli space of p-divisible groups, and that the sheaves he sought for a geometric interpretation of [**Sch13**] have a better chance of existing on the target of the Hodge-Tate period map, which is simply a projective space over $\mathbb{C}_p$; he sketched these ideas in an MSRI talk [**Sch14**]. (Again, Dat has had similar ideas.) Eventually, this perspective was used in his work with Caraiani [**CS17**], [**CS19a**] to study torsion in the cohomology of Shimura varieties. The work with Caraiani required the classification of G-torsors on the Fargues–Fontaine curve, which was proved by Fargues [**Far18b**].

Increasingly taking the perspective of studying all geometric objects by mapping only perfectoid spaces in, the idea of diamonds emerged quickly, including the possibility of getting several copies of $\operatorname{Spec}\mathbb{Q}_p$ (the earliest published incarnation of this idea is [**Wei17**]), and of defining general

moduli spaces of p-adic shtukas. These ideas were laid out in Scholze's Berkeley course [**SW20**] during the MSRI trimester in Fall 2014. The eventual goal was always to adapt V. Lafforgue's work [**Laf18**] to the case of p-adic fields; the original strategy was to define the desired excursion operators via the cohomology of moduli spaces of local shtukas. At the beginning of the trimester, Scholze was still very wary about the geometric Langlands program, as it did not seem to be able to incorporate the subtle arithmetic properties of supercuspidal representations of p-adic groups. It was thus a completely unexpected conceptual leap that in fact the best perspective for the whole subject is to view the local Langlands correspondence as a geometric Langlands correspondence on the Fargues–Fontaine curve, which Fargues suggested over a coffee break at MSRI (partly inspired by having thought intensely about the space of G-bundles on the curve in relation to [**Far18b**]). Fargues was taking the perspective of Hecke eigensheaves then, seeking to construct for any (discrete) L-parameter φ an associated Hecke eigensheaf A_φ on Bun_G with eigenvalue φ. This should define a functor $\varphi \mapsto A_\varphi$, and thus carry an action of the centralizer group $S_\varphi \subset \widehat{G}$ of φ, and the corresponding S_φ-isotypic decomposition of A_φ should realize the internal structures of the L-packets. Moreover, the Hecke eigensheaf property should imply the Kottwitz conjecture [**RV14**, Conjecture 7.3] on the cohomology of local Shimura varieties. This made everything come together. In particular, it gave a compelling geometric origin for the internal structure of L-packets, and also matched the recent work of Kaletha [**Kal14**] who used basic G-isocrystals for the fine study of L-packets.

Unfortunately, the conjecture was formulated on extremely shaky grounds: It presumed that one could work with the moduli stack Bun_G as if it were an object of usual algebraic geometry. Of course, it also presumed that there is a version of geometric Satake, etc.pp. On the other hand, we realized that once we could merely *formulate* Fargues' conjecture, enough machinery is available to apply Lafforgue's ideas [**Laf18**] to get the "automorphic-to-Galois" direction and define (semisimple) L-parameters (as Genestier–Lafforgue [**GL17**] did in equal characteristic).

Since then, it has been a long and very painful process. The first step was to give a good definition of the category of geometric objects relevant to this picture, i.e. diamonds. In particular, one had to prove that the relevant affine Grassmannians have this property. This was the main result of the Berkeley course [**SW20**]. For the proof, the concept of v-sheaves was introduced, which has since taken on a life of its own also in algebraic geometry (cf. [**BM18**]). (Generally, v-descent turned out to be an extremely powerful proof technique. We use it here to reprove the basic theorems about the Fargues–Fontaine curve, recovering the main theorems of [**FF18**] and [**KL15**] with little effort.) Next, one had to develop a 6-functor formalism for the étale cohomology of diamonds, which was achieved in [**Sch17a**], at least with torsion coefficients. The passage to $\mathbb{Q}_\ell$-coefficients requires more effort than for schemes, and we will comment on it below. A central technique of [**Sch17a**] is pro-étale descent, and more generally v-descent. In fact, virtually all theorems of [**Sch17a**] are proved using such descent techniques, essentially reducing them to profinite collections of geometric points. It came as a surprise to Scholze that this process of disassembling smooth spaces into profinite sets has any power in proving geometric results, and this realization gave a big impetus to the development of condensed mathematics (which in turn fueled back into the present project).

At this point, it became possible to contemplate Fargues' conjecture. In this respect, the first result that had to be established is that $D_{\text{ét}}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z})$ is well-behaved, for example satisfies

Verdier biduality for "admissible" sheaves. We found a proof, contingent on the cohomological smoothness of a certain "chart" $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ for Bun_G near any $b \in B(G)$; this was explained in Scholze's IHÉS course [**Sch17b**]. While for $G = \mathrm{GL}_n$, the cohomological smoothness of π_b could be proved by a direct attack, in general we could only formulate it as a special case of a general "Jacobian criterion of smoothness" for spaces parametrizing sections of $Z \to X_S$ for some smooth adic space Z over the Fargues–Fontaine curve. Proving this Jacobian criterion required three further key ideas. The first is the notion of "formal smoothness", where liftings to infinitesimal thickenings (that do not exist in perfectoid geometry) are replaced by liftings to actual small open (or étale) neighborhoods. The resulting notion is closely related to the notion of absolute neighborhood retracts in classical topology [**Bor67**]. Through some actual "analysis", it is not hard to prove that the space of sections is formally smooth. Unfortunately, this does not seem to be enough to guarantee cohomological smoothness. The first issue is that formal smoothness does not imply any finite-dimensionality. Here, the second key idea comes in, which is Bhatt's realization [**BS19**] that Zariski closed immersions are strongly Zariski closed in the sense of [**Sch15**, Section II.2] (contrary to a claim made by Scholze there). At this point, it would be enough to show that spaces that are formally smooth and Zariski closed in a finite-dimensional perfectoid ball are cohomologically smooth. Unfortunately, despite many tries, we are still unable to prove that even the different notions of dimension of [**Sch17a**] (Krull dimension, dim. trg, cohomological dimension) agree for such spaces. This may well be the most important foundational open problem in the theory:

Problem I.11.1. *Let $X \subset \widetilde{\mathbb{B}}^n_C$ be Zariski closed, where $\widetilde{\mathbb{B}}^n$ is a perfectoid ball. Show that X has a well-behaved dimension.*

In fact, we find it crazy that we are able to prove all sorts of nontrivial geometric results without ever being able to unambiguously talk about dimensions!

Our attacks on this failing, a third key idea comes in: Namely, the notion of universally locally acyclic sheaves, that we also developed independently in order to prove geometric Satake. It is easy to see that formal smoothness plus finite-dimensionality implies that the constant sheaf is universally locally acyclic; it remains to see that the dualizing sheaf is invertible. This can be proved by a deformation to the normal cone (using universal local acyclicity to spread the result on the normal cone to a neighborhood). We found this argument at a conference in Luminy in July 2018; an inspiration to use a deformation to the normal cone may have been Clausen's use in the proof of the "linearization hypothesis".

These results are enough to show that $D_{\text{ét}}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z})$ is well-behaved, and are already enough to prove new finiteness results on the cohomology of Rapoport–Zink spaces (with torsion coefficients). Our next emphasis was on geometric Satake. This essentially required the theory of universally locally acyclic sheaves, and a version of Braden's hyperbolic localization theorem [**Bra03**]. We were able to find substitutes for both. Regarding universally locally acyclic sheaves, we were able to prove analogues of most basic theorems, however we failed to prove that in general they are preserved under relative Verdier duality (even while we could check it by hand in all relevant cases). Lu–Zheng [**LZ20**] then found a new characterization of universally locally acyclic sheaves, making stability under relative Verdier duality immediate. Their arguments immediately transport to our setting. Eventually we used a slightly different characterization, but in spirit the argument is still the same as theirs. Regarding hyperbolic localization, we could not follow Braden's arguments that rely on nice coordinate choices. Instead, we reduce all arguments to the following

(simple to prove) principle: If X is a (partially proper) space with a $\mathbb{G}_m$-action such that $[X/\mathbb{G}_m]$ is qcqs, and $A \in D_{\text{ét}}([X/\mathbb{G}_m], \Lambda)$, then the partially compactly supported cohomology of X with coefficients in A vanishes. The idea here is that the $\mathbb{G}_m$-action contracts X towards one of the ends. Afterwards, the proof of geometric Satake largely follows the lines of [**MV07**], although there are certain improvements in the argument; in particular, we give a simple reduction to groups of rank 1, and pin the isomorphism with the dual group.

Using these results, one has all ingredients in place, but only working with torsion coefficients. One can formally pass to ℓ-adically complete sheaves, but this leads to studying representations on Banach $\mathbb{Q}_\ell$-vector spaces, which is very unnatural. During this time, Clausen came to Bonn, and Clausen and Scholze started to develop condensed mathematics, and the theory of solid modules [**CS**]. They realized that one could also define solid $\mathbb{Z}_\ell$-sheaves on schemes or diamonds, and that this makes it possible to study representations on discrete $\mathbb{Q}_\ell$- or $\overline{\mathbb{Q}}_\ell$-vector spaces, as desired. We take this up here, and first define solid $\mathbb{Z}_\ell$-sheaves on any small v-stack, together with some 5-functor formalism (involving relative homology in place of compactly supported cohomology; its right adjoint is then pullback, so there are only 5 functors), and afterwards pass to a certain subcategory of "lisse-étale" sheaves to define the desired category $D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$, with exactly the desired properties.

In the meantime, Nadler–Yun [**NY19**] and Gaitsgory–Kazhdan–Rozenblyum–Varshavsky [**GKRV19**] showed that the categorical structures we have now constructed — $D(\text{Bun}_G, \mathbb{Q}_\ell)$ together with the action of Hecke operators — formally induce an action of the category of perfect complexes on the stack of L-parameters on $D(\text{Bun}_G, \mathbb{Q}_\ell)$, giving a categorical upgrade to the construction of L-parameters based on excursion operators. (We were aware of some weak form of this, when restricted to elliptic parameters; this was discussed in the last lecture of [**Sch17b**], based on some unpublished results of Anschütz.) Here, we make the effort of proving a result with $\mathbb{Z}_\ell$-coefficients, at least under a minor assumption on ℓ.

I.12. Acknowledgments

We apologize for the long delay in the preparation of the manuscript. Already in April 2016, an Oberwolfach Arbeitsgemeinschaft gave an introduction to these ideas. The required foundations on étale cohomology of diamonds (with torsion coefficients) were essentially worked out in the winter term 2016/17, where also the ARGOS seminar in Bonn studied the evolving manuscript [**Sch17a**], which was an enormous help. Next, we gave the CAGA course at the IHÉS in Spring 2017, [**Sch11**]. At the time we had essentially obtained enough results to achieve the construction of L-parameters; the missing ingredient was the Jacobian criterion. In particular, we understood that one has to use the local charts $\mathcal{M}_b$ to study Bun_G. In more detail, these results were discussed in the ARGOS seminar in the summer term of 2017, and the proof of geometric Satake in the ARGOS seminar in the summer term of 2018. We heartily thank all the participants of these seminars for working through these manuscripts, and their valuable feedback. In the meantime, in the spring of 2018, we revised the Berkeley notes [**SW20**] and used [**Sch17a**] to give a full construction of moduli spaces of p-adic shtukas. Then we eventually found a full proof of the Jacobian criterion in July 2018 at a conference in Luminy. Our early attempts at writing up the results ran into the issue that we wanted to first prove basic sanity results on dimensions. Later, we got sidetracked by the development of condensed mathematics with Dustin Clausen, which eventually made it

possible to use $\overline{\mathbb{Q}}_\ell$-coefficients in this paper, finally giving the most natural results. In fact, we always wanted to develop the foundations so that papers such as the paper of Kaletha–Weinstein [**KW17**] on the Kottwitz conjecture can build on a proper foundation, and prove results about general smooth $\overline{\mathbb{Q}}_\ell$-representations; we believe we have finally achieved this. In the winter term 2020/21, Scholze gave an (online) course in Bonn about these results, and wants to thank all the participants for their valuable feedback. We thank the many mathematicians that we discussed with about these results, including Johannes Anschütz, David Ben-Zvi, Joseph Bernstein, Bhargav Bhatt, Arthur-César le Bras, Ana Caraiani, Dustin Clausen, Pierre Colmez, Jean-François Dat, Vladimir Drinfeld, Jessica Fintzen, Dennis Gaitsgory, Toby Gee, David Hansen, Eugen Hellmann, Alex Ivanov, Tasho Kaletha, Kiran Kedlaya, Dmitry Kubrak, Vincent Lafforgue, Judith Ludwig, Andreas Mihatsch, Sophie Morel, Jan Nekovář, Wiesława Nizioł, Vincent Pilloni, Michael Rapoport, Timo Richarz, Will Sawin, Tony Scholl, Antoine Touzé, Roman Travkin, Wilberd van der Kallen, Yakov Varshavsky, Jared Weinstein, Zhiyou Wu, Xinwen Zhu, Konrad Zou, and undoubtedly many others. Special thanks go to David Hansen for many comments on a preliminary draft.

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I.13. Notation

Throughout most of this paper, E denotes a nonarchimedean local field with residue field $\mathbb{F}_q$ of characteristic $p > 0$, and we fix an algebraic closure $k = \overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Then $\breve{E}$ is the completed unramified extension of E with residue field k. We also fix a separable closure $\overline{E}$ of E, with absolute Galois group $\Gamma = \mathrm{Gal}(\overline{E}|E)$, containing the Weil group W_E, inertia subgroup I_E, and wild inertia P_E. The letter P usually denotes open subgroups of P_E.

The group G is usually a reductive group over E; reductive groups are always assumed to be connected.

For any topological space X, we denote by $\underline{X}$ the sheaf taking any S (in the relevant test category, usually a perfectoid space) to the continuous maps from $|S|$ to X. This is in the spirit of the passage from topological spaces to condensed sets, see [**CS**]. We make occasional use of the condensed language, but do not make use of any nontrivial results from [**CS**]. In particular, our discussion of solid ℓ-adic sheaves is self-contained.

We will occasionally use the "animated" terminology, see [**CS**], [**ČS19b**]. In particular, we use the term anima for what is variously called spaces in [**Lur09**], ∞-groupoids, or homotopy types, and for any ring A, the ∞-category of animated A-algebras is the ∞-category obtained from simplicial A-algebras by inverting weak equivalences. Thus, animated A-algebras are freely generated under sifted colimits by polynomial algebras $A[X_1, \ldots, X_n]$.

If $\mathcal{C}$ is an (∞-)category equipped with an action of a group G, we write $\mathcal{C}^{BG}$ for the (∞-)category of G-equivariant objects in $\mathcal{C}$. Note that the data here is really a functor $BG \to \mathrm{Cat}_\infty$, and $\mathcal{C}^{BG}$ is by definition the limit of this diagram. (It would be more customary to write $\mathcal{C}^G$, but this leads to inconsistent notation.)

CHAPTER II

The Fargues–Fontaine curve and vector bundles

The goal of this chapter is to define the Fargues–Fontaine curve, in its various incarnations, and the category of vector bundles on the Fargues–Fontaine curve. Throughout this chapter, we fix a nonarchimedean local field E with residue field $\mathbb{F}_q$ of characteristic p. We let $\mathcal{O}_E \subset E$ be the ring of integers, and π a uniformizing element in E.

For any perfectoid space S over $\mathbb{F}_q$, we introduce a curve $\mathcal{Y}_S$, to be thought of as the product $S \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_E$, together with an open subset $Y_S \subset \mathcal{Y}_S$ given by the locus where $\pi \neq 0$. This carries a Frobenius φ induced from the Frobenius on S, and X_S is the quotient $Y_S/\varphi^{\mathbb{Z}}$.

The first results concern the Fargues–Fontaine curve $X_C = X_S$ when $S = \operatorname{Spa} C$ for some complete algebraically closed nonarchimedean field $C|\mathbb{F}_q$. We define a notion of classical points of X_C in that case; they form a subset of $|X_C|$. The basic finiteness properties of X_C are summarized in the following result.

Theorem II.0.1 (Proposition II.1.11, Corollary II.1.12, Definition/Proposition II.1.22). *The adic space $\mathcal{Y}_C$ is locally the adic spectrum $\operatorname{Spa}(B, B^+)$ where B is a principal ideal domain; the classical points of $\operatorname{Spa}(B, B^+) \subset \mathcal{Y}_C$ are in bijection with the maximal ideals of B. For each classical point $x \in \mathcal{Y}_C$, the residue field of x is an untilt $C^\sharp$ of C over $\mathcal{O}_E$, and this induces a bijection of the classical points of $\mathcal{Y}_C$ with untilts $C^\sharp$ of C over $\mathcal{O}_E$. A similar result holds true for $Y_C \subset \mathcal{Y}_C$, and the quotient $X_C = Y_C/\varphi^{\mathbb{Z}}$.*

In the equal characteristic case, this is an immediate consequence of $\mathcal{Y}_C = \mathbb{D}_C$ and classical results in rigid-analytic geometry. In the p-adic case, we use tilting to reduce to the equal characteristic case. More precisely, if E is p-adic and E_∞ is the completion of $E(\pi^{1/p^\infty})$, then $\mathcal{Y}_C \times_{\operatorname{Spa} \mathcal{O}_E} \operatorname{Spa} \mathcal{O}_{E_\infty}$ is perfectoid, with tilt given by a perfectoid open unit disc $\widetilde{\mathbb{D}}_C$. The corresponding map $|\widetilde{\mathbb{D}}_C| \to |\mathcal{Y}_C|$ induces a surjective map on classical points, see Proposition II.1.8. At one key turn, in order to understand Zariski closed subsets of $\mathcal{Y}_C$, we use the result that Zariski closed subspaces are invariant under tilting, to reduce to $\widetilde{\mathbb{D}}_C$. More precisely, we recall the following result.

Proposition II.0.2 ([**Sch15**, Section II.2], [**BS19**], [**Sch17a**, Definition 5.7, Theorem 5.8]). *Let $S = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space with tilt $S^\flat = \operatorname{Spa}(R^\flat, R^{\flat +})$. Then a closed subspace $Z \subset |S|$ is the vanishing locus of an ideal $I \subset R$ if and only if $Z \subset |S| \cong |S^\flat|$ is the vanishing locus of an ideal $J \subset R^\flat$. In that case, there is a universal perfectoid space $S_Z \to S$ such that $|S_Z| \to |S|$ factors over Z, and $S_Z = \operatorname{Spa}(T, T^+)$ is affinoid perfectoid with $|S_Z| \to Z$ a homeomorphism, $R \to T$ surjective, $R^+ \to T^+$ almost surjective, and T^+ is the integral closure of R^+ in T.*

A key result is the classification of vector bundles.

THEOREM II.0.3 (Theorem II.2.14). *The functor from* Isoc_E *to vector bundles on* X_C *induces a bijection on isomorphism classes. In particular, there is a unique stable vector bundle* $\mathcal{O}_{X_C}(\lambda)$ *of any slope* $\lambda \in \mathbb{Q}$*, and any vector bundle* $\mathcal{E}$ *can be written as a direct sum of stable bundles.*

We give a new self-contained proof of the classification theorem, making critical use of the v-descent results for vector bundles obtained in [**Sch17a**] and [**SW20**], and basic results on the geometry of Banach–Colmez spaces established here.

Allowing general $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, we define the moduli space of degree 1 Cartier divisors as $\mathrm{Div}^1 = \mathrm{Spd}\,\breve{E}/\varphi^{\mathbb{Z}}$. Given a map $S \to \mathrm{Div}^1$, one can define an associated closed Cartier divisor $D_S \subset X_S$; locally, this is given by an untilt $D_S = S^\sharp \subset X_S$ of S over E, and this embeds Div^1 into the space of closed Cartier divisors on X_S. Another important result is the following ampleness result, cf. [**KL15**, Proposition 6.2.4], which implies that one can define an algebraic version of the curve, admitting the same theory of vector bundles.

THEOREM II.0.4 (Theorem II.2.6, Proposition II.2.7, Proposition II.2.9). *Assume that* $S \in \mathrm{Perf}$ *is affinoid. For any vector bundle* $\mathcal{E}$ *on* X_S*, the twist* $\mathcal{E}(n)$ *is globally generated and has no higher cohomology for all* $n \gg 0$*. Defining the graded ring*

$$P = \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}_{X_S}(n))$$

and the scheme $X_S^{\mathrm{alg}} = \mathrm{Proj}\, P$*, there is a natural map of locally ringed spaces* $X_S \to X_S^{\mathrm{alg}}$*, pullback along which defines an equivalence of categories of vector bundles, preserving cohomology.*

If $S = \mathrm{Spa}\, C$ *for some complete algebraically closed nonarchimedean field* C*, then* X_C^{alg} *is a regular noetherian scheme of Krull dimension* 1*, locally the spectrum of a principal ideal domain, and its closed points are in bijection with the classical points of* X_C*.*

We also need to understand families of vector bundles, i.e. vector bundles $\mathcal{E}$ on X_S for general S. Here, the main result is the following, which is originally due to Kedlaya–Liu [**KL15**].

THEOREM II.0.5 (Theorem II.2.19, Corollary II.2.20). *Let* $S \in \mathrm{Perf}$ *and let* $\mathcal{E}$ *be a vector bundle on* X_S*. Then the function taking a point* $s \in S$ *to the Harder–Narasimhan polygon of* $\mathcal{E}|_{X_s}$ *defines a semicontinuous function on* S*. If it is constant, then* $\mathcal{E}$ *admits a global Harder–Narasimhan stratification, and pro-étale locally on* S *one can find an isomorphism with a direct sum of* $\mathcal{O}_{X_S}(\lambda)$*'s.*

In particular, if $\mathcal{E}$ *is everywhere semistable of slope* 0*, then* $\mathcal{E}$ *is pro-étale locally trivial, and the category of such* $\mathcal{E}$ *is equivalent to the category of pro-étale* $\underline{E}$*-local systems on* S*.*

The key to proving this theorem is the construction of certain global sections of $\mathcal{E}$. To achieve this, we use v-descent techniques, and an analysis of the spaces of global sections of $\mathcal{E}$; these are known as Banach–Colmez spaces, and were first introduced (in slightly different terms) in [**Col02**].

DEFINITION II.0.6. *Let* $\mathcal{E}$ *be a vector bundle on* X_S*. The Banach–Colmez space* $\mathcal{BC}(\mathcal{E})$ *associated with* $\mathcal{E}$ *is the locally spatial diamond over* S *whose* T*-valued points, for* $T \in \mathrm{Perf}_S$*, are given by*

$$\mathcal{BC}(\mathcal{E})(T) = H^0(X_T, \mathcal{E}|_{X_T}).$$

Similarly, if $\mathcal{E}$ is everywhere of only negative Harder–Narasimhan slopes, the negative Banach–Colmez space $\mathcal{BC}(\mathcal{E}[1])$ is the locally spatial diamond over S whose T-valued points are

$$\mathcal{BC}(\mathcal{E}[1])(T) = H^1(X_T, \mathcal{E}|_{X_T}).$$

Implicit here is that this functor actually defines a locally spatial diamond. For this, we calculate some key examples of Banach–Colmez spaces. For example, if $\mathcal{E} = \mathcal{O}_{X_S}(\lambda)$ with $0 < \lambda \leq [E : \mathbb{Q}_p]$ (resp. all positive λ if E is of equal characteristic), then $\mathcal{BC}(\mathcal{E})$ is representable by a perfectoid open unit disc (of dimension given by the numerator of λ). A special case of this is the identification of $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ with the universal cover of a Lubin–Tate formal group law, yielding a very close relation between Lubin–Tate theory, and thus local class field theory, and the Fargues–Fontaine curve. This case actually plays a special role in getting some of the theory started, and we recall it explicitly in Section II.2.1. On the other hand, for larger λ, or negative λ, Banach–Colmez spaces are more exotic objects; for example, the negative Banach–Colmez space

$$\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \cong (\mathbb{A}^1_{C^\sharp})^\diamond / \underline{E}$$

is the quotient of the affine line by the translation action of $\underline{E} \subset \mathbb{A}^1_{C^\sharp}$.

A key result is Proposition II.2.16, stating in particular that projectivized Banach–Colmez spaces

$$(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$$

are proper — they are the relevant analogues of "families of projective spaces over S". In particular, their image in S is a closed subset, and if the image is all of S, then we can find a nowhere vanishing section of $\mathcal{E}$ after a v-cover, as then the projectivized Banach–Colmez space is a v-cover of S.

II.1. The Fargues–Fontaine curve

II.1.1. The curve $\mathcal{Y}_C$. Recall that for any perfect $\mathbb{F}_q$-algebra R, there is a unique π-adically complete flat $\mathcal{O}_E$-algebra $\widetilde{R}$ such that $\widetilde{R} = R/\pi$. There is a unique multiplicative lift $[\cdot] : R \to \widetilde{R}$ of the identity $R \to R$, called the Teichmüller lift. Explicitly, one can take

$$\widetilde{R} = W_{\mathcal{O}_E}(R)$$

in terms of the ramified Witt vectors.

The construction of the Fargues–Fontaine curve is based on this construction on the level of perfectoid spaces S over $\mathbb{F}_q$. Its construction is done in three steps. First, one constructs a curve $\mathcal{Y}_S$, an adic space over $\mathcal{O}_E$, which carries a Frobenius action φ. Passing to the locus $Y_S = \mathcal{Y}_S \setminus \{\pi = 0\}$, i.e. the base change to E, the action of φ is free and totally discontinuous, so that one can pass to the quotient $X_S = Y_S/\varphi^{\mathbb{Z}}$, which will be the Fargues–Fontaine curve.

We start by constructing $\mathcal{Y}_S$ in the affinoid case. More precisely, if $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $\mathbb{F}_q$, and $\varpi \in R^+$ is a pseudouniformizer (i.e. a topologically nilpotent unit of R), we let

$$\mathcal{Y}_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V([\varpi]).$$

This does not depend on the choice of ϖ, as for any choice of $\varpi, \varpi' \in R$, one has $\varpi | \varpi'^n$, $\varpi' | \varpi^n$ for some $n > 0$. The q-th power Frobenius of R^+ induces an automorphism φ of $\mathcal{Y}_S$. To construct the Fargues–Fontaine curve, we will eventually remove $V(\pi)$ from $\mathcal{Y}_S$ and quotient by φ, but for now we recall some properties of $\mathcal{Y}_S$.

PROPOSITION II.1.1. *The above defines an analytic adic space $\mathcal{Y}_S$ over $\mathcal{O}_E$. Letting E_∞ be the completion of $E(\pi^{1/p^\infty})$, the base change*

$$\mathcal{Y}_S \times_{\operatorname{Spa}\mathcal{O}_E} \operatorname{Spa}\mathcal{O}_{E_\infty}$$

is a perfectoid space, with tilt given by

$$S \times_{\mathbb{F}_q} \operatorname{Spa}\mathbb{F}_q[\![t^{1/p^\infty}]\!] = \mathbb{D}_{S,\mathrm{perf}},$$

a perfectoid open unit disc over S.

PROOF. One can cover $\mathcal{Y}_S$ by the subsets $\mathcal{Y}_{S,[0,n]} := \{|\pi|^n \leq |[\varpi]| \neq 0\} \subset \mathcal{Y}_S$, which are rational subsets of $\operatorname{Spa} W_{\mathcal{O}_E}(R^+)$, where $n > 0$ is some integer that we assume to be a power of p for simplicity. Then

$$\mathcal{Y}_{S,[0,n]} = \operatorname{Spa}(B_{S,[0,n]}, B^+_{S,[0,n]})$$

where

$$B_{S,[0,n]} = W_{\mathcal{O}_E}(R^+)\langle\frac{\pi^n}{[\varpi]}\rangle[\tfrac{1}{[\varpi]}]$$

and $B^+_{S,[0,n]} \subset B_{S,[0,n]}$ is the integral closure of $W_{\mathcal{O}_E}(R^+)\langle\frac{\pi^n}{[\varpi]}\rangle$. To see that $\mathcal{Y}_S$ is an adic space (i.e. the structure presheaf is a sheaf) and $\mathcal{Y}_S \times_{\operatorname{Spa}\mathcal{O}_E} \operatorname{Spa}\mathcal{O}_{E_\infty}$ is perfectoid, it is enough to prove that $B_{S,[0,n]}\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty}$ is a perfectoid Tate algebra. Indeed, the algebra $B_{S,[0,n]}$ splits off $B_{S,[0,n]}\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty}$ as a direct factor as topological $B_{S,[0,n]}$-module, and hence the sheaf property for perfectoid spaces gives the result for $\mathcal{Y}_{S,[0,n]}$ and thus all of $\mathcal{Y}_S$ (cf. the sousperfectoid property of [**HK20**], [**SW20**, Section 6.3]). Using the Frobenius automorphism of (R, R^+), one can in fact assume that $n = 1$.

Let us abbreviate

$$A = B_{S,[0,1]}\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty}$$

and $A^+ \subset A$ the integral closure of $B^+_{S,[0,1]}\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty}$. In particular

$$A_0^+ = (W_{\mathcal{O}_E}(R^+)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty})[(\tfrac{\pi}{[\varpi]})^{1/p^\infty}]^\wedge_{[\varpi]} \subset A^+,$$

and $A = A_0^+[\frac{1}{[\varpi]}]$. Note that

$$A_0^+/[\varpi] = (R^+/\varpi \otimes_{\mathbb{F}_q} \mathcal{O}_{E_\infty}/\pi)[t_1^{1/p^\infty}]/(\pi^{1/p^m} - [\varpi]^{1/p^m}t_1^{1/p^m}) \cong R^+/\varpi[t_1^{1/p^\infty}].$$

This implies already that A_0^+ is integral perfectoid by [**BMS18**, Lemma 3.10 (ii)], and thus necessarily (cf. [**BMS18**, Lemma 3.21]) $A_0^+ \to A^+$ is an almost isomorphism and $A_0^+[\frac{1}{[\varpi]}] \cong A$ is perfectoid. Moreover, one can see that the tilt of A is given by $R\langle t_1^{1/p^\infty}\rangle$, where $t_1^\sharp = \frac{\pi}{[\varpi]}$, which corresponds to the subset

$$\{|t| \leq |\varpi| \neq 0\} \subset S \times_{\mathbb{F}_q} \operatorname{Spa}\mathbb{F}_q[\![t^{1/p^\infty}]\!] = \mathbb{D}_{S,\mathrm{perf}}. \qquad \square$$

PROPOSITION II.1.2. *For any perfectoid space T over $\mathbb{F}_q$, giving an untilt $T^\sharp$ of T together with a map $T^\sharp \to \mathcal{Y}_S$ of analytic adic spaces is equivalent to giving an untilt $T^\sharp$ together with a map $T^\sharp \to \operatorname{Spa}\mathcal{O}_E$, and a map $T \to S$. In other words, there is a natural isomorphism*

$$\mathcal{Y}_S^\diamond \cong \operatorname{Spd}\mathcal{O}_E \times S.$$

Proof. Changing notation, we need to see that for any perfectoid space T over $\mathcal{O}_E$, giving a map $T \to \mathcal{Y}_S$ is equivalent to giving a map $T^\flat \to S$. Let $T = \mathrm{Spa}(A, A^+)$. Giving a map $T \to \mathcal{Y}_S$ is equivalent to giving a map $W_{\mathcal{O}_E}(R^+) \to A^+$ such that the image of $[\varpi]$ in A is invertible. By the universal property of $W_{\mathcal{O}_E}(R^+)$ in case R^+ is perfect, this is equivalent to giving a map $R^+ \to (A^+)^\flat$ such that the image of ϖ in $A^\flat$ is invertible. But this is precisely a map $T = \mathrm{Spa}(A, A^+) \to S = \mathrm{Spa}(R, R^+)$. □

In particular, there is a natural map

$$|\mathcal{Y}_S| \cong |\mathcal{Y}_S^\diamond| \cong |(\mathrm{Spa}\,\mathcal{O}_E)^\diamond \times S| \to |S|.$$

The following proposition ensures that we may glue $\mathcal{Y}_S$ for general S, i.e. for any perfectoid space S there is an analytic adic space $\mathcal{Y}_S$ equipped with an isomorphism

$$\mathcal{Y}_S^\diamond \cong \mathrm{Spd}\,\mathcal{O}_E \times S$$

(and in particular a map $|\mathcal{Y}_S| \to |S|$) such that for $U = \mathrm{Spa}(R, R^+) \subset S$ an affinoid subset, the corresponding pullback of $\mathcal{Y}_S$ is given by $\mathcal{Y}_U$.

Proposition II.1.3. *If $S' \subset S$ is an affinoid subset, then $\mathcal{Y}_{S'} \to \mathcal{Y}_S$ is an open immersion, with*

$$\begin{tikzcd} |\mathcal{Y}_{S'}| \arrow[r] \arrow[d] & |\mathcal{Y}_S| \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

cartesian.

Proof. Let $Z \subset \mathcal{Y}_S$ be the open subset corresponding to $|\mathcal{Y}_S| \times_{|S|} |S'| \subset |\mathcal{Y}_S|$. Then by functoriality of the constructions, we get a natural map $\mathcal{Y}_{S'} \to Z$. To see that it is an isomorphism, we can check after base change to $\mathcal{O}_{E_\infty}$ (as the maps on structure sheaves are naturally split injective). The base change of $\mathcal{Y}_{S'}$ and Z become perfectoid, and hence it suffices to see that one gets an isomorphism after passing to diamonds, where it follows from Proposition II.1.2. □

Next, we recall the "sections of $\mathcal{Y}_S \to S$".

Proposition II.1.4 ([**SW20**, Proposition 11.3.1]). *Let S be a perfectoid space over $\mathbb{F}_q$. The following objects are in natural bijection.*

(i) *Sections of $\mathcal{Y}_S^\diamond \to S$;*

(ii) *Morphisms $S \to \mathrm{Spd}\,\mathcal{O}_E$;*

(iii) *Untilts $S^\sharp$ over $\mathcal{O}_E$ of S.*

Moreover, given an untilt $S^\sharp$ over $\mathcal{O}_E$ of S, there is a natural closed immersion of adic spaces

$$S^\sharp \hookrightarrow \mathcal{Y}_S$$

that presents $S^\sharp$ as a closed Cartier divisor in $\mathcal{Y}_S$.

Proof. The equivalence of (i), (ii) and (iii) is a direct consequence of Proposition II.1.2. Thus, let $S^\sharp$ be an untilt of S over $\mathcal{O}_E$. We may work locally, so assume $S = \mathrm{Spa}(R, R^+)$ is affinoid. Then $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is affinoid perfectoid as well, and

$$R^{\sharp+} = W_{\mathcal{O}_E}(R^+)/\xi$$

for some nonzerodivisor $\xi \in W_{\mathcal{O}_E}(R^+)$ that can be chosen to be of the form $\pi - a[\varpi]$ for some $a \in W_{\mathcal{O}_E}(R^+)$ and suitable topologically nilpotent $\varpi \in R$ (choose $\varpi \in R^+$ a pseudouniformizing element such that $\varpi^\sharp | \pi$, and write $\pi = \varpi^\sharp \theta(a)$ for some a). To see that $S^\sharp$ defines a closed Cartier divisor in $\mathcal{Y}_S$, that is to say the sequence

$$0 \to \mathcal{O}_{\mathcal{Y}_S} \xrightarrow{\xi} \mathcal{O}_{\mathcal{Y}_S} \to i_* \mathcal{O}_{S^\sharp} \to 0$$

is exact with $i : S^\sharp \hookrightarrow \mathcal{Y}_S$, we need to see that for any open affinoid $U = \mathrm{Spa}(A, A^+) \subset \mathcal{Y}_S$ with affinoid perfectoid pullback $V = \mathrm{Spa}(B, B^+) \subset S^\sharp$, the sequence

$$0 \to A \xrightarrow{\xi} A \to B \to 0$$

is exact. To see this, we are free to localize near $S^\sharp = V(\xi) \subset \mathcal{Y}_S$. In particular, replacing S by $V^\flat$, we can assume that $V = S^\sharp$. In that case, any neighborhood of $S^\sharp = V(\xi)$ in $\mathcal{Y}_S$ contains $\{|\xi| \leq |[\varpi]|^n\}$ for some $n > 0$, so we can assume that U is of this form.

Endow A with the spectral norm, where we normalize the norm on each completed residue field of $\mathcal{Y}_S$ by $|[\varpi]| = \frac{1}{q}$. We claim that with this choice of norm, one has

$$|\xi a| \geq q^{-n} |a|$$

for all $a \in A$. In particular, this implies that $\xi : A \to A$ is injective, and has closed image (as the preimage of any Cauchy sequence in the image is a Cauchy image). On the other hand, $R^\sharp$ is the separated completion of A/ξ, so $B = A/\xi$.

To verify the claimed inequality, it is enough to see that the norm of $|a|$ is equal to the supremum norm over $\{|\xi| = |[\varpi|^n\}$. In fact, it is enough to consider the points in the Shilov boundary, i.e. those points $\mathrm{Spa}(C, \mathcal{O}_C) \to U$ that admit a specialization $\mathrm{Spa}(C, C^+) \to \mathcal{Y}_S$ whose image is not contained in U; any such is necessarily contained in $\{|\xi = [\varpi]^n\}$. This will in fact hold for all functions on $U \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}\,\mathcal{O}_{E_\infty}$, for which the claim reduces to the tilt, which is an affinoid subset of $\mathbb{D}_{S,\mathrm{perf}}$. By approximation, it then reduces to the case of affinoid subsets of $\mathbb{D}_S$, where it is well-known that the maximum is taken on the Shilov boundary. (Note that this question immediately reduces to the case that S is a geometric point.) $\square$

Remark II.1.5. The preceding Cartier divisor satisfies the stronger property of being a "relative Cartier divisor" in the sense that for all $s \in S$ its pullback to $\mathcal{Y}_{\mathrm{Spa}(K(s), K(s)^+)}$ is a Cartier divisor.

Now let us analyze the case $S = \mathrm{Spa}\, C$ for some complete algebraically closed nonarchimedean field over $\mathbb{F}_q$.

Example II.1.6. Assume that $E = \mathbb{F}_q(\!(t)\!)$ is of equal characteristic. Then $\mathcal{Y}_C = \mathbb{D}_C$ is an open unit disc over C, with coordinate t. In particular, inside $|\mathcal{Y}_C|$, we have the subset of classical points $|\mathcal{Y}_C|^{\mathrm{cl}} \subset |\mathcal{Y}_C|$, which can be identified as

$$|\mathcal{Y}_C|^{\mathrm{cl}} = \{x \in C \mid |x| < 1\}.$$

Note that these classical points are in bijection with maps $\mathcal{O}_E \to C$ (over $\mathbb{F}_q$), i.e. with "untilts of C over $\mathcal{O}_E$".

With suitable modifications, the same picture exists also when E is of mixed characteristic.

Definition/Proposition II.1.7. *Any untilt $C^\sharp$ of C over $\mathcal{O}_E$ defines a closed Cartier divisor $\operatorname{Spa} C^\sharp \hookrightarrow \mathcal{Y}_S$, and in particular a closed point of $|\mathcal{Y}_C|$. This induces an injection from the set of such untilts to $|\mathcal{Y}_C|$.*

The set of classical points $|\mathcal{Y}_C|^{\mathrm{cl}} \subset |\mathcal{Y}_C|$ is defined to be the set of such points.

Proof. We have seen that any untilt $C^\sharp$ defines such a map $\operatorname{Spa} C^\sharp \hookrightarrow \mathcal{Y}_S$. As it is a closed Cartier divisor, the corresponding point is closed in $|\mathcal{Y}_C|$. One can recover $C^\sharp$ as the completed residue field at the point, together with the map $W_{\mathcal{O}_E}(\mathcal{O}_C) \to \mathcal{O}_{C^\sharp}$, which induces the isomorphism $\mathcal{O}_C \cong \mathcal{O}_{C^\sharp}^\flat$ and thus $C \cong (C^\sharp)^\flat$, giving the untilt structure on $C^\sharp$; this shows that the map is injective. $\square$

Recall that $\mathcal{Y}_C$ is preperfectoid. In fact, if one picks a uniformizer $\pi \in E$ and lets E_∞ be the completion of $E(\pi^{1/p^\infty})$, then $\mathcal{Y}_C \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ is perfectoid, and its tilt is given by

$$\operatorname{Spa} C \times \operatorname{Spa} \mathcal{O}_{E_\infty}^\flat \cong \operatorname{Spa} C \times \operatorname{Spa} \mathbb{F}_q[\![t^{1/p^\infty}]\!].$$

Thus, we get a map

$$|\mathbb{D}_C| = |\operatorname{Spa} C \times \operatorname{Spa} \mathbb{F}_q[\![t]\!]| \cong |\operatorname{Spa} C \times \operatorname{Spa} \mathbb{F}_q[\![t^{1/p^\infty}]\!]| \cong |\mathcal{Y}_C \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}| \to |\mathcal{Y}_C|.$$

Proposition II.1.8. *Under this map, the classical points $|\mathbb{D}_C|^{\mathrm{cl}} = \{x \in C \mid |x| < 1\} \subset |\mathbb{D}_C|$ are exactly the preimage of the classical points $|\mathcal{Y}_C|^{\mathrm{cl}} \subset |\mathcal{Y}_C|$.*

Unraveling the definitions, one sees that the map

$$\{x \in C \mid |x| < 1\} = |\mathbb{D}_C|^{\mathrm{cl}} \to |\mathcal{Y}_C|^{\mathrm{cl}}$$

sends any $x \in C$ with $|x| < 1$ to the closed point defined by the ideal $(\pi - [x])$. In particular, the proposition shows that any classical point of $\mathcal{Y}_C$ can be written in this form.

Proof. This is clear as classical points are defined in terms of maps of diamonds, which are compatible with this tilting construction on topological spaces. $\square$

The formation of classical points is also compatible with changing C in the following sense.

Proposition II.1.9. *Let $C'|C$ be an extension of complete algebraically closed nonarchimedean fields over $\mathbb{F}_q$, inducing the map $\mathcal{Y}_{C'} \to \mathcal{Y}_C$. A point $x \in |\mathcal{Y}_C|$ is classical if and only if its preimage in $|\mathcal{Y}_{C'}|$ is a classical point. Moreover, if $x \in |\mathcal{Y}_C|$ is a rank-1-point that is not classical, then there is some $C'|C$ such that the preimage of x contains a nonempty open subset of $|\mathcal{Y}_{C'}|$.*

In other words, one can recognize classical points as those points that actually stay points after any base change; all other rank 1 points actually contain whole open subsets after some base change.

Proof. It is clear that if x is classical, then its preimage is a classical point. Conversely, if $x \in |\mathcal{Y}_C|$ is a rank 1 point, and $S = \operatorname{Spd} K(x)$, the point x is given by a morphism $S \to \operatorname{Spa} C \times \operatorname{Spd} \mathcal{O}_E$. If the preimage of x is a classical point, the induced morphism $S \to \operatorname{Spa} C$ becomes an isomorphism after pullback via $\operatorname{Spa} C' \to \operatorname{Spa} C$. Since S is a v-sheaf ([**Sch17a**, Proposition 11.9]) and $\operatorname{Spa} C' \to \operatorname{Spa} C$ a v-cover, the morphism $S \to \operatorname{Spa} C$ is an isomorphism, and thus x is a classical point.

Now assume that x is nonclassical rank-1-point; we want to find $C'|C$ such that the preimage of x contains an open subset of $|\mathcal{Y}_{C'}|$. By Proposition II.1.8, it is enough to prove the similar result for $\mathbb{D}_C$, using that $|\mathbb{D}_{C'}| \to |\mathcal{Y}_{C'}|$ is open.[1] Thus, assume $x \in |\mathbb{D}_C|$ is a non-classical point. Let C' be a completed algebraic closure of the corresponding residue field. Then the preimage of x in $|\mathbb{D}_{C'}|$ has a tautological section $\widetilde{x} \in \mathbb{D}_{C'}(C')$ which is a classical point, and the preimage of x contains a small disc $\mathbb{B}(\widetilde{x}, r) \subset \mathbb{D}_{C'}$ for some $r > 0$. Indeed, this follows from the description of the rank 1 points of $\mathbb{D}_C$ as being either the Gauss norm for some disc $\mathbb{B}(x, r_0) \subset \mathbb{D}_C$ of radius $r_0 > 0$, or the infimum of such over a decreasing sequence of balls (but with radii not converging to zero). See Lemma II.1.10. □

Lemma II.1.10. *Let $x \in \mathbb{D}_C(C)$, $\rho \in (0, 1]$, and $x_\rho \in |\mathbb{D}_C|$ be the Gauss norm with radius ρ centered at x. The preimage of x_ρ in $|\mathbb{D}_{C(x_\rho)}|$ contains the open disk with radius ρ centered at $x_\rho \in \mathbb{D}_{C(x_\rho)}(C(x_\rho))$.*

Proof. We can suppose $x = 0$. The point x_ρ is given by the morphism $C\langle T\rangle \to C(x_\rho)$ that sends T to t. Let $y \in |\mathbb{D}_{C(x_\rho)}|$. This corresponds to a morphism $C(x_\rho)\langle T\rangle \to C(x_\rho)(y)$. Let us note $u \in C(x_\rho)(y)$ the image of T via the preceding map. Suppose y lies in the open disk with radius ρ centered at x_ρ. This means $|u - t| < \rho = |t|$. Let us remark that this implies that for any $n \geq 1$, $|u^n - t^n| < \rho^n$. For $f = \sum_{n\geq 0} a_n T \in C\langle T\rangle$, one then has

$$|\sum_{n\geq 0} a_n(u^n - t^n)| < \sup_{n\geq 0} |a_n|\rho^n = |\sum_{n\geq 0} a_n t^n|.$$

We deduce that

$$|\sum_{n\geq 0} a_n u^n| = |\sum_{n\geq 0} a_n t^n| = |f(x_\rho)|.$$ □

There is in fact another characterization of the classical points in terms of maximal ideals.

Proposition II.1.11. *Let $U = \operatorname{Sp}(B, B^+) \subset \mathcal{Y}_C$ be an affinoid subset. Then for any maximal ideal $\mathfrak{m} \subset B$, the quotient $B/\mathfrak{m}$ is a nonarchimedean field, inducing an injection $\operatorname{Spm}(B) \hookrightarrow |U|$. This gives a bijection between $\operatorname{Spm}(B)$ and $|U|^{\mathrm{cl}} := |U| \cap |\mathcal{Y}_C|^{\mathrm{cl}} \subset |\mathcal{Y}_C|$.*

Proof. First, if $x \in |U|^{\mathrm{cl}}$, then it corresponds to a closed Cartier divisor $\operatorname{Spa} C^\sharp \hookrightarrow U \subset \mathcal{Y}_C$, and thus defines a maximal ideal of B, yielding an injection $|U|^{\mathrm{cl}} \hookrightarrow \operatorname{Spm}(B)$. We need to see that this is a bijection.

Note that using the tilting map $|\mathbb{D}_C| \to |\mathcal{Y}_C|$, one sees that the preimage of U in $|\mathbb{D}_C|$ has only finitely many connected components (any quasicompact open subset of $|\mathbb{D}_C|$ has finitely many

[1] Any quasicompact open of $|\mathbb{D}_{C'}|$ is the base change of a quasicompact open of $|\mathcal{Y}_{C'} \times_{\operatorname{Spa}\mathcal{O}_E} \operatorname{Spa}\mathcal{O}_{E'}|$ for a finite extension $E'|E$. Passing to the Galois hull of E' and taking the orbit of the open subset under the Galois group, it then follows from the map being a quotient map, as any map of analytic adic spaces.

connected components); we can thus assume that U is connected. In that case, we claim that any nonzero element $f \in B$ vanishes only at classical points of $|U|$. By Proposition II.1.9, it suffices to see that for any nonempty open subset $U' \subset U$, the map $\mathcal{O}(U) \to \mathcal{O}(U')$ is injective. In fact, if $V(f)$ contains a nonclassical point, it also contains a nonclassical rank 1 point as $V(f)$ is generalizing, then after base changing to some $C'|C$, $V(f)$ contains an open subset U', and this is impossible if $\mathcal{O}(U) \hookrightarrow \mathcal{O}(U')$. For this it suffices to prove that $\mathcal{O}(V) \hookrightarrow \mathcal{O}(V')$ where V is a connected component of $U \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$, and V' the intersection of $U' \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ with this connected component. Now for any $g \in \mathcal{O}(V) \setminus \{0\}$, $V(g) \neq V$, as perfectoid spaces are uniform (and hence vanishing at all points implies vanishing). We thus have to prove that for any Zariski closed subset $Z \subsetneq V$, $V' \not\subset Z$.

By Proposition II.0.2, it suffices to prove the similar property for open subsets $V' \subset V \subset \mathbb{D}_{C,\mathrm{perf}}$, with V connected. But then $V' = W'_{\mathrm{perf}}$ and $V = W_{\mathrm{perf}}$ for $W' \subset W \subset \mathbb{D}_C$, and $\mathcal{O}(V) \to \mathcal{O}(V')$ is topologically free (with basis t^i, $i \in [0,1) \cap \mathbb{Z}[p^{-1}]$) over the corresponding map $\mathcal{O}(W) \to \mathcal{O}(W')$ of classical Tate algebras over C, for which injectivity is classical. □

The previous proposition implies that, once U is connected, the rings B are principal ideal domains (cf. [**Ked16**]).

Corollary II.1.12. *Let $U = \mathrm{Spa}(B, B^+) \subset \mathcal{Y}_C$ be an affinoid subset. Then U has finitely many connected components. Assuming that U is connected, the ring B is a principal ideal domain.*

Proof. We have already seen in the preceding proof that U has finitely many connected components. Moreover, each maximal ideal of B is principal, as it comes from a closed Cartier divisor on U. Now take any nonzero $f \in B$. We have seen that the vanishing locus of f is contained in $|U|^{\mathrm{cl}}$, and it is also closed in $|U|$. It it thus a spectral space with no nontrivial specializations, and therefore a profinite set. We claim that it is in fact discrete. For this, let $x \in V(f)$ be any point. We get a generator $\xi_x \in B$ for the corresponding maximal ideal. We claim that there is some $n \geq 1$ such that $f = \xi_x^n g$ where g does not vanish at x. Assume otherwise. Note that the spectral norm on U is given by the supremum over finitely many points, the Shilov boundary of U (cf. proof of Proposition II.1.4). We may normalize ξ_x so that its norm at all of these finitely many points is ≥ 1. Then for any n, if $f = \xi_x^n g_n$, one has $||g_n|| \leq ||f||$. But inside the open neighborhood $U_x = \{|\xi_x| \leq |[\varpi]|\}$ of x, this implies that $||f||_{U_x} \leq |[\varpi]|^n ||f||$ for all n, and thus $||f||_{U_x} = 0$ as $n \to \infty$. Thus, f vanishes on all of U_x, which is a contradiction.

By the above, we can write $f = \xi_x^n g$ where g does not vanish at x. But then g does not vanish in a neighborhood of x, and therefore $x \in V(f)$ is an isolated point, and hence $V(f)$ is profinite and discrete, and thus finite. Enumerating these points $x_1, \ldots, x_m$, we can thus write $f = \xi_{x_1}^{n_1} \cdots \xi_{x_m}^{n_m} g$ where g does not vanish at $x_1, \ldots, x_m$, and thus vanishes nowhere, and hence is a unit. This finishes the proof. □

Remark II.1.13. The main new ingredient compared to [**Ked16**] or [**FF18**, Theorem 2.5.1] that allows us to shorten the proof is Proposition II.0.2, i.e. the use of the fact (proved in [**BS19**]) that "Zariski closed implies strongly Zariski closed" in the terminology of [**Sch15**, Section II.2].

Later (cf. Proposition IV.7.3), we will also need the following lemma about non-classical points of $Y_C = \mathcal{Y}_C \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}\,E$.

LEMMA II.1.14. *There is a point $x \in |Y_C|$, with completed residue field $K(x)$, such that the induced map $\mathrm{Gal}(\overline{K(x)}|K(x)) \to I_E$ is surjective, where I_E is the inertia subgroup of the absolute Galois group of E.*

Note that a priori we have a map $\mathrm{Gal}(\overline{K(x)}|K(x)) \to \mathrm{Gal}(\overline{E}|E)$, but it is clear that its image is contained in I_E, as $K(x)$ contains $\breve{E}$.

PROOF. In fact, we can be explicit: Looking at the surjection

$$|\mathbb{D}_C^*| \to |Y_C|$$

from the tilting construction, the image of any Gaußpoint (corresponding to a disc of radius r, $0 < r < 1$, around the origin) will have the desired property. This follows from the observation that this locus of Gaußpoints lifts uniquely to $|Y_C \times_{\mathrm{Spa}\,\breve{E}} \mathrm{Spa}\,E'|$ for any finite extension $E'|\breve{E}$. In fact, this cover admits a similar surjection from a punctured open unit disc over C, and there is again one Gaußpoint for each radius (i.e. the set of Gaußpoints maps isomorphically to $(0,\infty)$ via $\mathrm{rad} : |Y_C| \to (0,\infty)$). □

II.1.2. The Fargues–Fontaine curve. Now we can define the Fargues–Fontaine curve.

DEFINITION II.1.15. *For any perfectoid space S over $\mathbb{F}_q$, the relative Fargues–Fontaine curve is*

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

where

$$Y_S = \mathcal{Y}_S \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}\,E = \mathcal{Y}_S \setminus V(\pi),$$

which for affinoid $S = \mathrm{Spa}(R, R^+)$ with pseudouniformizer ϖ is given by

$$Y_S = \mathrm{Spa}\,W_{\mathcal{O}_E}(R^+) \setminus V(\pi[\varpi]).$$

To see that this is well-formed, we note the following proposition.

PROPOSITION II.1.16. *The action of φ on Y_S is free and totally discontinuous. In fact, if $S = \mathrm{Spa}(R, R^+)$ is affinoid and $\varpi \in R$ is a pseudouniformizer, one can define a map*

$$\mathrm{rad} : |Y_S| \longrightarrow (0,\infty)$$

taking any point $x \in Y_S$ with rank-1-generalization $\widetilde{x}$ to $\log|[\varpi](\widetilde{x})|/\log|\pi(\widetilde{x})|$. This factorizes through the Berkovich space quotient of $|Y_S|$ and satisfies $\mathrm{rad} \circ \varphi = q \cdot \mathrm{rad}$.

For any interval $I = [a,b] \subset (0,\infty)$ with rational ends (possibly with $a = b$), there is the open subset

$$Y_{S,I} = \{|\pi|^b \leq |[\varpi]| \leq |\pi|^a\} \subset \mathrm{rad}^{-1}(I) \subset Y_S$$

which is in fact a rational open subset of $\mathrm{Spa}\,W_{\mathcal{O}_E}(R^+)$ and thus affinoid,

$$Y_{S,I} = \mathrm{Spa}(B_{S,I}, B_{S,I}^+),$$

and one can form X_S as the quotient of $Y_{S,[1,q]}$ via the identification $\varphi : Y_{S,[1,1]} \cong Y_{S,[q,q]}$. In particular, X_S is qcqs in case S is affinoid.

PROOF. This follows directly from the definitions. □

In terms of the preceding radius function, the end 0 corresponds to the boundary divisor (π), and ∞ to the boundary divisor $([\varpi])$.

For each $s \in S$ corresponding to a map $\mathrm{Spa}(K(s), K(s)^+) \to S$, functoriality defines a morphism $X_{K(s),K(s)^+} \to X_S$. We way think of X_S as the collection of curves $(X_{K(s),K(s)^+})_{s\in S}$, the one defined and studied in [**FF18**], merged in a "family of curves". Although X_S does not sit over S, the absolute Frobenius $\varphi \times \varphi$ of $S \times \mathrm{Spd}(E)$ acts trivially on the topological space and one has

$$|X_S| \cong |X_S^\diamond| \cong |S \times \mathrm{Spd}(E)/\varphi^{\mathbb{Z}} \times \mathrm{id}| \cong |S \times \mathrm{Spd}(E)/\mathrm{id} \times \varphi^{\mathbb{Z}}| \longrightarrow |S|.$$

Thus the topological space $|X_S|$ sits over $|S|$, and for all S the map $|X_S| \to |S|$ is qcqs. Here, we used the following identification of the diamond.

Proposition II.1.17. *There is a natural isomorphism*

$$Y_S^\diamond \cong S \times \mathrm{Spd}(E),$$

descending to an isomorphism

$$X_S^\diamond \cong (S \times \mathrm{Spd}(E))/\varphi^{\mathbb{Z}} \times \mathrm{id}.$$

Proof. This is immediate from Proposition II.1.2. □

Moreover, we have the following version of Proposition II.1.4.

Proposition II.1.18. *The following objects are naturally in bijection.*

(i) *Sections of* $Y_S^\diamond \to S$*;*
(ii) *Maps* $S \to \mathrm{Spd}(E)$*;*
(iii) *Untilts* $S^\sharp$ *over* E *of* S*.*

Given such a datum, in particular an untilt $S^\sharp$ *over* E *of* S*, there is a natural closed immersion* $S^\sharp \hookrightarrow Y_S$ *presenting* $S^\sharp$ *as a closed Cartier divisor in* Y_S*. The composite map* $S^\sharp \to Y_S \to X_S$ *is still a closed Cartier divisor, and depends only on the composite* $S \to \mathrm{Spd}(E) \to \mathrm{Spd}(E)/\varphi^{\mathbb{Z}}$*. In this way, any map* $S \to \mathrm{Spd}(E)/\varphi^{\mathbb{Z}}$ *defines a closed Cartier divisor* $D \subset X_S$*; this gives an injection of* $\mathrm{Spd}(E)/\varphi^{\mathbb{Z}}$ *into the space of closed Cartier divisors on* X_S*.*

Proof. This is immediate from Proposition II.1.4. □

Definition II.1.19. *A closed Cartier divisor of degree* 1 *on* X_S *is a closed Cartier divisor* $D \subset X_S$ *that arises from a map* $S \to \mathrm{Spd}(E)/\varphi^{\mathbb{Z}}$*. Equivalently, it arises locally on* S *from an untilt* $S^\sharp$ *over* E *of* S*.*

In particular, we see that the moduli space Div^1 of degree 1 closed Cartier divisors is given by

$$\mathrm{Div}^1 = \mathrm{Spd}(E)/\varphi^{\mathbb{Z}}.$$

Note that something strange is happening in the formalism here: Usually the curve itself would represent the moduli space of degree 1 Cartier divisors!

Remark II.1.20. In [**Far18a**, Définition 2.6] Fargues gives a definition of a Cartier divisor of degree 1 on X_S equivalent to the preceding one, similar to the definition of a relative Cartier divisor in classical algebraic geometry.

Proposition II.1.21. *The map* $\mathrm{Div}^1 \to *$ *is proper, representable in spatial diamonds, and cohomologically smooth.*

Proof. First, $\mathrm{Spd}(E) \to *$ is representable in locally spatial diamonds and cohomologically smooth by [**Sch17a**, Proposition 24.5] (for $E = \mathbb{Q}_p$, which formally implies the case of E finite over $\mathbb{Q}_p$, and the equal characteristic case is handled in the proof). As $|\mathrm{Spd}(E) \times S| \cong |Y_S| \to |S|$, we see that $\varphi^{\mathbb{Z}}$ acts totally discontinuously with quotient $|\mathrm{Spd}(E)/\varphi^{\mathbb{Z}} \times S| \cong |X_S| \to |S|$ being qcqs in case $|S|$ is qcqs; thus, $\mathrm{Spd}(E)/\varphi^{\mathbb{Z}} \to *$ is representable in spatial diamonds, in particular qcqs. Then being proper follows from the valuative criterion [**Sch17a**, Proposition 18.3]. □

In particular, the map

$$|X_S| = |\mathrm{Div}^1 \times S| \longrightarrow |S|$$

is open and closed. *We can thus picture* X_S *as being "a proper and smooth family over* S*".*

Further motivation for Definition II.1.19 is given by the following.

Definition/Proposition II.1.22. *The classical points of* X_C *are* $|X_C|^{\mathrm{cl}} := |Y_C|^{\mathrm{cl}}/\varphi^{\mathbb{Z}} \subset |X_C| = |Y_C|/\varphi^{\mathbb{Z}}$. *They are in bijection with* $(\mathrm{Spd}(E)/\varphi^{\mathbb{Z}})(C) = \mathrm{Div}^1(C)$, *i.e. are given untilts of* C *over* E *up to Frobenius, or by degree* 1 *closed Cartier divisors on* X_C. *For any affinoid open subset* $U = \mathrm{Spa}(B, B^+) \subset X_C$, *the maximal ideals of* B *are in bijection with* $|U|^{\mathrm{cl}} = |U| \cap |X_C|^{\mathrm{cl}}$. *Any such* U *has only finitely many connected components, and if* U *is connected, then* B *is a principal ideal domain.*

Proof. This follows immediately from Proposition II.1.11 and Corollary II.1.12 if U lifts to Y_C. In general, $Y_C \to X_C$ is locally split, so the result is true locally on U; and then it easily follows by gluing in general. □

II.2. Vector bundles on the Fargues–Fontaine curve

Let us recall a few basic facts about the cohomology of vector bundles. Suppose $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid. Then Y_S is "Stein", one has $Y_S = \bigcup_{I \subset (0,\infty)} Y_{(R,R^+),I}$ where

(i) as before I is a compact interval with rational ends
(ii) $Y_{(R,R^+),I}$ is affinoid sous-perfectoid
(iii) for $I_1 \subset I_2$, the restriction morphism $\mathcal{O}(Y_{(R,R^+),I_2}) \to \mathcal{O}(Y_{(R,R^+),I_1})$ has dense image.

Let $\mathcal{F}$ be a vector bundle on Y_S. Point (2) implies that $H^i(Y_{S,I}, \mathcal{F}_{|Y_{(R,R^+),I}}) = 0$ when $i > 0$. Point (3) implies that $R^1 \varprojlim_I \Gamma(Y_{(R,R^+),I}, \mathcal{F}) = 0$ ([**Gro61**, 0.13.2.4]). We thus have $H^i(Y_S, \mathcal{F}) = 0$ when $i > 0$.

Thus, if $\mathcal{E}$ is a vector bundle on X_S, one has

$$R\Gamma(X_S, \mathcal{E}) = \left[H^0(Y_S, \mathcal{E}_{|Y_S}) \xrightarrow{Id-\varphi} H^0(Y_S, \mathcal{E}_{|Y_S})\right].$$

In particular, this vanishes in degree > 1.

Moreover, one has the following important (cohomological) descent result.

Proposition II.2.1. *Let S be a perfectoid space over $\mathbb{F}_q$ and $\mathcal{E}$ a vector bundle on X_S. The functor taking any $T \in \mathrm{Perf}_S$ to*

$$R\Gamma(X_T, \mathcal{E}|_{X_T})$$

is a v-sheaf of complexes. In fact, the functor taking any $T \in \mathrm{Perf}_S$ to $H^0(Y_T, \mathcal{E}|_{Y_T})$ is a v-sheaf, whose cohomology vanishes in case T is affinoid.

Moreover, sending S to the groupoid of vector bundles on X_S defines a v-stack.

Proof. By the displayed formula for $R\Gamma(X_S, \mathcal{E})$ as Frobenius fixed points, it suffices to prove the result about Y_T. We can assume that $S = \mathrm{Spa}(R, R^+)$ is affinoid, pick a pseudouniformizer $\varpi \in R$, and one can further reduce to the similar claim for $Y_{T,I}$ for any compact interval I with rational ends. We need to see that for any v-cover $T = \mathrm{Spa}(R', R'^+)$, the corresponding Čech complex

$$0 \to \mathcal{O}(Y_{S,I}) \to \mathcal{O}(Y_{T,I}) \to \mathcal{O}(Y_{T\times_S T,I}) \to \dots$$

of E-Banach spaces is exact. This can be checked after taking a completed tensor product with $E_\infty = E(\pi^{1/p^\infty})^\wedge$. In that case, all algebras become perfectoid, and $Y_{T,I} \times_E E_\infty \to Y_{S,I} \times_E E_\infty$ is a v-cover of affinoid perfectoid spaces, so the result follows from [**Sch17a**, Theorem 8.7, Proposition 8.8].

Similarly, one proves v-descent for the groupoid of vector bundles, cf. [**SW20**, Lemma 17.1.8, Proposition 19.5.3]. □

If $[\mathcal{E}_1 \to \mathcal{E}_0]$ is a complex of vector bundles on X_S sitting in homological degrees $[0, 1]$, such that $H^0(X_T, \mathcal{E}_1|_{X_T}) = 0$ for all $T \in \mathrm{Perf}_S$, we let

$$\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) : T \mapsto \mathbb{H}^0(X_T, [\mathcal{E}_1 \to \mathcal{E}_0]|_{X_T})$$

be the corresponding v-sheaf on Perf_S. We refer to this as *the Banach–Colmez space associated with* $[\mathcal{E}_1 \to \mathcal{E}_0]$. We will usually apply this only when either of $\mathcal{E}_1$ and $\mathcal{E}_0$ is zero.

Let us also recall the basic examples of vector bundles. Already here it is useful to fix an algebraically closed field $k|\mathbb{F}_q$, e.g. $k = \overline{\mathbb{F}}_q$. Let $\breve{E} = W_{\mathcal{O}_E}(k)[\frac{1}{\pi}]$, the complete unramified extension of E with residue field k. Recall that, functorially in $S \in \mathrm{Perf}_k$, there is a natural exact $\otimes$-functor

$$\begin{aligned} \mathrm{Isoc}_k &\longrightarrow \mathrm{Bun}(X_S) \\ (D, \varphi) &\longmapsto \mathcal{E}(D, \varphi) \end{aligned}$$

from the category of isocrystals (of a finite-dimensional $\breve{E}$-vector space D equipped with a σ-linear automorphism $\varphi : D \xrightarrow{\sim} D$) to the category of vector bundles on X_S, defined via descending $D \otimes_{\breve{E}} \mathcal{O}_{Y_S}$ to X_S via $\varphi \otimes \varphi$. We denote by $\mathcal{O}_{X_S}(n)$ the image of $(\breve{E}, \pi^{-n}\sigma)$ (note the change of sign — the functor $\mathcal{E}$ reverses slopes); more generally, if $(D_\lambda, \varphi_\lambda)$ is the simple isocrystal of slope $\lambda \in \mathbb{Q}$ in the Dieudonné–Manin classification, we let $\mathcal{O}_{X_S}(-\lambda) = \mathcal{E}(D_\lambda, \varphi_\lambda)$.

II.2.1. Lubin–Tate formal groups. The claim of this paper is that the Fargues–Fontaine curve enables a geometrization of the local Langlands correspondence. As a warm-up, let us recall the relation between $\mathcal{O}_{X_S}(1)$ and local class field theory in the form of Lubin–Tate theory.

Up to isomorphism, there is a unique 1-dimensional formal group G over $\mathcal{O}_{\breve{E}}$ with action by $\mathcal{O}_E$, such that the two induced actions on Lie G coincide; this is "the" Lubin–Tate formal group $G = G_{\mathrm{LT}}$ of E. Fixing a uniformizer $\pi \in E$, we normalize this as follows. First, any Lubin–Tate formal group law over $\mathcal{O}_E$ is the unique (up to unique isomorphism) lift of a 1-dimensional formal group over k whose Lie algebra has the correct $\mathcal{O}_E$-action. Now, if E is p-adic then G_k is classified by Dieudonné theory by a finite projective $W_{\mathcal{O}_E}(k)$-module M equipped with a σ-linear isomorphism $F : M[\frac{1}{\pi}] \cong M[\frac{1}{\pi}]$ such that $M \subset F(M) \subset \frac{1}{\pi}M$.[2] Here, we take $M = W_{\mathcal{O}_E}(k)$ with $F = \frac{1}{\pi}\sigma$. One can similarly define G in equal characteristic, but actually we will explain a different way to pin down the choice just below; under our normalization, G is already defined over $\mathcal{O}_E$.

After passing to the generic fibre, G_E is isomorphic to the additive group $\mathbb{G}_a$, compatibly with the $\mathcal{O}_E$-action, and one can choose a coordinate on $G \cong \operatorname{Spf} \mathcal{O}_E[\![X]\!]$ so that explicitly, the logarithm map is given by

$$\log_G : G_E \to \mathbb{G}_{a,E} : X \mapsto X + \tfrac{1}{\pi}X^q + \tfrac{1}{\pi^2}X^{q^2} + \ldots + \tfrac{1}{\pi^n}X^{q^n} + \ldots.$$

Regarding the convergence of $\log_G$, we note that in fact it defines a map of rigid-analytic varieties (i.e. adic spaces locally of finite type over E)

$$\log_G : G_E^{\mathrm{ad}} \cong \mathbb{D}_E \to \mathbb{G}_{a,E}^{\mathrm{ad}}$$

from the open unit disc

$$G_E^{\mathrm{ad}} \cong \operatorname{Spa} \mathcal{O}_E[\![X]\!] \times_{\operatorname{Spa} \mathcal{O}_E} \operatorname{Spa} E$$

to the adic space corresponding to $\mathbb{G}_a$. From the formula, one sees that in small enough discs it defines an isomorphism, and via rescaling by powers of π (which on the level of G_E^{ad} defines finite étale covers of degree q, while it is an isomorphism on $\mathbb{G}_{a,E}^{\mathrm{ad}}$), one sees that one has an exact sequence

$$0 \to G_E^{\mathrm{ad}}[\pi^\infty] \to G_E^{\mathrm{ad}} \to \mathbb{G}_{a,E}^{\mathrm{ad}} \to 0$$

on the big étale site of adic spaces over $\operatorname{Spa} E$, where $G_E^{\mathrm{ad}}[\pi^\infty] \subset G_E^{\mathrm{ad}}$ is the torsion subgroup. This is, in fact, the generic fibre of $G[\pi^\infty] = \bigcup_n G[\pi^n]$ over $\operatorname{Spa} \mathcal{O}_E$, and each $G[\pi^n] = \operatorname{Spa} A_n$ is represented by some finite $\mathcal{O}_E$-algebra A_n of degree q^n. Inductively, $G[\pi^{n-1}] \subset G[\pi^n]$ giving a map $A_n \to A_{n-1}$; after inverting π, this is split, and the other factor is a totally ramified extension $E_n|E$. Then

$$G_E^{\mathrm{ad}} = \bigcup_n \operatorname{Spa} A_n[\tfrac{1}{\pi}] = \bigsqcup_n \operatorname{Spa} E_n.$$

We also need the "universal cover" of G, defined as

$$\widetilde{G} = \varprojlim_{\times\pi} G \cong \operatorname{Spf} \mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!],$$

where the inverse limit is over the multiplication by π maps. The isomorphism with $\operatorname{Spf} \mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!]$ is evident modulo π, but as this gives a perfect algebra, we see that in fact the isomorphism lifts uniquely to $\mathcal{O}_E$. Explicitly, the coordinate $\widetilde{X}$ is given by

$$\widetilde{X} = \lim_{n\to\infty} X_n^{q^n}$$

[2] As in [**SW20**, p. 99], we renormalize usual covariant Dieudonné theory for p-divisible groups by dividing F by p; and then in the case of π-divisible $\mathcal{O}_E$-modules as here, we base change along $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E \to W_{\mathcal{O}_E}(k)$.

where X_n is the coordinate on the n-th copy of G in the formula $\widetilde{G} = \varprojlim_{\times\pi} G$; in fact, $\widetilde{X} \equiv X_n^{q^n}$ modulo π^n. In particular, the logarithm map

$$\log_G : \widetilde{G}_E \to G_E \to \mathbb{G}_{a,E}$$

is given by the series

$$\sum_{i\in\mathbb{Z}} \pi^i \widetilde{X}^{q^{-i}}.$$

Note that for any π-adically complete $\mathcal{O}_E$-algebra A, one has

$$\widetilde{G}(A) \cong \widetilde{G}(A/\pi) = \mathrm{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(A/\pi))[\tfrac{1}{\pi}].$$

Indeed, the first equality follows from $\mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!]$ being relatively perfect over $\mathcal{O}_E$, and the second equality by noting that any element of $G(A/\pi)$ is π^n-torsion for some n. A different description based on $\widetilde{G} = \mathrm{Spf}\,\mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!]$ is

$$\widetilde{G}(A) = \varprojlim_{x\mapsto x^p} A^{\circ\circ} = A^{\flat,\circ\circ} \subset A^\flat,$$

the subset of topologically nilpotent elements of the tilt.

This is related to the line bundle $\mathcal{O}_{X_S}(1)$ as follows.

Proposition II.2.2. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$ and let $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ be an untilt of S over E, giving rise to the closed immersion $S^\sharp \hookrightarrow X_S$. Let $\mathcal{O}_{X_S}(1)$ be the line bundle on X_S corresponding to the isocrystal (E, π^{-1}). Then the map*

$$\widetilde{G}(R^{\sharp+}) \cong R^{\circ\circ} \to H^0(Y_S, \mathcal{O}_{Y_S}) : X \mapsto \sum_{i\in\mathbb{Z}} \pi^i [X^{q^{-i}}]$$

defines a natural isomorphism

$$\widetilde{G}(R^{\sharp+}) \cong H^0(X_S, \mathcal{O}_{X_S}(1)) = H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi=\pi}.$$

Under this isomorphism, the map

$$H^0(X_S, \mathcal{O}_{X_S}(1)) \to H^0(S^\sharp, \mathcal{O}_{S^\sharp}) = R^\sharp$$

of evaluation at $S^\sharp$ is given by the logarithm map

$$\log_G : \widetilde{G}(R^{\sharp+}) \to G(R^{\sharp+}) \to R^\sharp.$$

Proof. The compatibility with the logarithm map is clear from the explicit formulas. Assume first that E is of characteristic p. Then $H^0(Y_S, \mathcal{O}_{Y_S})$, where $Y = \mathbb{D}_S^*$ is a punctured open unit disc over S, can be explicitly understood as certain power series $\sum_{i\in\mathbb{Z}} r_i \pi^i$ with coefficients $r_i \in R$ (subject to convergence conditions as $i \to \pm\infty$). Then

$$H^0(X_S, \mathcal{O}_{X_S}(1)) = H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi=\pi}$$

amounts to those series such that $r_i = r_{i+1}^q$ for all $i \in \mathbb{Z}$. Thus, all r_i are determined by r_0, which in turn can be any topologically nilpotent element of R. This gives the desired isomorphism

$$H^0(X_S, \mathcal{O}_{X_S}(1)) \cong R^{\circ\circ} = \widetilde{G}(R^+) = \widetilde{G}(R^{\sharp+})$$

(as $\widetilde{G} = \mathrm{Spa}\mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!]$ and $R^\sharp = R$).

If E is p-adic, then we argue as follows. First, as in the proof of Proposition II.2.5 below, one can rewrite $H^0(X_S, \mathcal{O}_{X_S}(1))$ as $B_{R,[1,\infty]}^{\varphi=\pi}$ where

$$B_{R,[1,\infty]} = \mathcal{O}(Y_{[1,\infty]}), \text{ for } Y_{[1,\infty]} = \{|[\varpi]| \leq |\pi| \neq 0\} \subset \operatorname{Spa} W_{\mathcal{O}_E}(R^+).$$

By the contracting property of Frobenius, one can also replace $B_{R,[1,\infty]}$ with the crystalline period ring B_{crys}^+ of $R^{\sharp+}/\pi$ here, and then [**SW13**, Theorem A] gives the desired

$$B_{R,[1,\infty]}^{\varphi=\pi} = \operatorname{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(R^{\sharp+}/\pi))[\tfrac{1}{\pi}] = \widetilde{G}(R^{\sharp+}/\pi) = \widetilde{G}(R^{\sharp+}).$$

That this agrees with the explicit formula follows from [**SW13**, Lemma 3.5.1]. □

Recall also that the field E_∞ obtained as the completion of the union of all E_n is perfectoid — in fact, one has a closed immersion $\operatorname{Spf} \mathcal{O}_{E_\infty} \hookrightarrow \widetilde{G} = \operatorname{Spf} \mathcal{O}_E[\![\widetilde{X}^{1/p^\infty}]\!]$, which induces an isomorphism $\operatorname{Spf} \mathcal{O}_{E^\infty}^\flat \cong \operatorname{Spf} \mathbb{F}_q[\![X^{1/p^\infty}]\!]$. Over E_∞, we have an isomorphism $\mathcal{O}_E \cong (T_\pi G)(\mathcal{O}_{E_\infty}) \subset \widetilde{G}(\mathcal{O}_{E_\infty})$. By the proposition, if $S^\sharp$ lives over E_∞, we get a nonzero section of $\mathcal{O}_{X_S}(1)$, vanishing at $S^\sharp \subset X_S$.

Proposition II.2.3. *For any perfectoid space S with untilt $S^\sharp$ over E_∞, the above construction construction defines an exact sequence*

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\sharp} \to 0$$

of $\mathcal{O}_{X_S}$-modules.

Proof. The above constructions show that one has a map $\mathcal{O}_{X_S} \to \mathcal{I}(1)$ where $\mathcal{I} \subset \mathcal{O}_{X_S}$ is the ideal sheaf of $S^\sharp$, which by Proposition II.1.18 is a line bundle. To see that this map is an isomorphism, it suffices to check on geometric points, so we can assume that $S = \operatorname{Spa} C$ for some complete algebraically closed extension C of $\mathbb{F}_q$. We have now fixed some nonzero global section of $\mathcal{O}_{X_S}(1)$, which by the above corresponds to some nonzero topologically nilpotent $X = \varpi \in C$, explicitly given by

$$f = \sum_{i \in \mathbb{Z}} \pi^i [X^{q^{-i}}] \in H^0(Y_C, \mathcal{O}_{Y_C})^{\varphi=\pi}.$$

This is the base change of the function

$$\sum_{i \in \mathbb{Z}} \pi^i X^{q^{-i}} \in \mathcal{O}((\operatorname{Spa} \mathcal{O}_E[\![X^{1/p^\infty}]\!])_E \setminus V(X)),$$

so it is enough to determine the vanishing locus of this function. But note that under the identification $\widetilde{G} = \operatorname{Spf} \mathcal{O}_E[\![X^{1/p^\infty}]\!]$, this is precisely the logarithm function

$$\log_G : \widetilde{G}_E^{\mathrm{ad}} \setminus \{0\} \to \mathbb{G}_{a,E}^{\mathrm{ad}};$$

thus, it is enough to determine the vanishing locus of the logarithm function. But this is precisely

$$\bigsqcup_n \operatorname{Spa} E_\infty \subset \widetilde{G}_E^{\mathrm{ad}} \setminus \{0\},$$

with a simple zero at each of these points. This gives exactly the claimed statement (the $\mathbb{Z}$ many copies are translates under π, which are also translates under φ as $\varphi(f) = \pi f$). □

Corollary II.2.4 ([**Far18a**, Proposition 2.12]). *There is a well-defined map* $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \to \mathrm{Div}^1$ *sending a nonzero section* $f \in H^0(X_S, \mathcal{O}_{X_S}(1))$ *to the closed Cartier divisor given by* $V(f)$. *This descends to an isomorphism*

$$(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})/\underline{E}^\times \cong \mathrm{Div}^1.$$

Proof. Note that $\mathcal{BC}(\mathcal{O}(1)) \cong \mathrm{Spd}\, \mathbb{F}_q[\![X^{1/p^\infty}]\!]$ by Proposition II.2.2, and hence $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \cong \mathrm{Spa}\, \mathbb{F}_q(\!(X^{1/p^\infty})\!)$ is representable by a perfectoid space. In fact, it is naturally isomorphic to $\mathrm{Spd}\, E_\infty = \mathrm{Spa}\, E_\infty^\flat$, and the previous proposition ensures that the map to Div^1 is well-defined and corresponds to the projection $\mathrm{Spd}\, E_\infty \to \mathrm{Spd}\, E \to \mathrm{Spd}\, E/\varphi^{\mathbb{Z}} = \mathrm{Div}^1$. Here, the first map $\mathrm{Spd}\, E_\infty \to \mathrm{Spd}\, E$ is a quotient under $\underline{\mathcal{O}_E^\times}$, and the second map $\mathrm{Spd}\, E \to \mathrm{Spd}\, /\varphi^{\mathbb{Z}}$ then corresponds to the quotient by $\pi^{\mathbb{Z}}$, as $\varphi = \pi$ on $\mathcal{BC}(\mathcal{O}(1))$. □

In particular, if one works on Perf_k, then $\mathrm{Div}^1 = \mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}}$, whose $\pi_1^{\text{ét}}$ is given by the absolute Galois group of E. On the other hand, the preceding gives a canonical $\underline{E}^\times$-torsor, giving a natural map from the absolute Galois group of E to the profinite completion of $E^\times$. By comparison with Lubin–Tate theory, this is the usual *Artin reciprocity map*, see [**Far18a**, Section 2.3] for more details.

II.2.2. Absolute Banach–Colmez spaces. In this section, we analyze the Banach–Colmez spaces in the case $\mathcal{E} = \mathcal{E}(D)$ for some isocrystal $D = (D, \varphi)$. We then sometimes write $\mathcal{BC}(D)$ and $\mathcal{BC}(D[1])$ for the corresponding functors on Perf_k; or also $\mathcal{BC}(\mathcal{O}(\lambda))$, $\mathcal{BC}(\mathcal{O}(\lambda)[1])$ for $\lambda \in \mathbb{Q}$ when $D = D_{-\lambda}$. These are in fact already defined for all $S \in \mathrm{Perf}_{\mathbb{F}_q}$.

Proposition II.2.5. *Let* $\lambda \in \mathbb{Q}$.

(i) *If* $\lambda < 0$, *then* $H^0(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$ *for all* $S \in \mathrm{Perf}_{\mathbb{F}_q}$. *Moreover, the projection from*

$$\mathcal{BC}(\mathcal{O}(\lambda)[1]) : S \mapsto H^1(X_S, \mathcal{O}_{X_S}(\lambda))$$

to the point $*$ *is relatively representable in locally spatial diamonds, partially proper, and cohomologically smooth.*

(ii) *For* $\lambda = 0$, *the map*

$$\underline{E} \to \mathcal{BC}(\mathcal{O})$$

is an isomorphism of pro-étale sheaves, and the pro-étale sheafification of $S \mapsto H^1(X_S, \mathcal{O}_{X_S})$ *vanishes. In particular, for all* S *one gets an isomorphism*

$$R\Gamma_{\text{proét}}(S, \underline{E}) \to R\Gamma(X_S, \mathcal{O}_{X_S}).$$

(iii) *For* $\lambda > 0$, *one has* $H^1(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$ *for all affinoid* $S \in \mathrm{Perf}_{\mathbb{F}_q}$, *and the projection from*

$$\mathcal{BC}(\mathcal{O}(\lambda)) : S \mapsto H^0(X_S, \mathcal{O}_{X_S}(\lambda))$$

to the point $*$ *is relatively representable in locally spatial diamonds, partially proper, and cohomologically smooth.*

(iv) *If* $0 < \lambda \leq [E : \mathbb{Q}_p]$ *(resp. for all positive* λ *if* E *is of equal characteristic), there is an isomorphism*

$$\mathcal{BC}(\mathcal{O}(\lambda)) \cong \mathrm{Spd}\, k[\![x_1^{1/p^\infty}, \ldots, x_r^{1/p^\infty}]\!]$$

where $\lambda = r/s$ *with coprime integers* $r, s > 0$.

Proof. For all statements, we can reduce to the case $\lambda = n \in \mathbb{Z}$ by replacing E by its unramified extension of degree s. Regarding the vanishing of $H^1(X_S, \mathcal{O}_{X_S}(n))$ for $n > 0$ and $S = \mathrm{Spa}(R, R^+)$ affinoid, pick a pseudouniformizer $\varpi \in R$. In terms of the presentation of X_S as gluing $Y_{S,[1,q]}$ along $\varphi : Y_{S,[1,1]} \cong Y_{S,[q,q]}$, it suffices to see that

$$\varphi - \pi^n : B_{R,[1,q]} \to B_{R,[1,1]}$$

is surjective. Any element of $B_{R,[1,1]}$ can be written as the sum of an element of $B_{R,[0,1]}[\frac{1}{\pi}]$ and an element of $[\varpi]B_{R,[1,\infty]}$, corresponding to the affinoid subsets

$$Y_{S,[0,1]} = \{|\pi| \leq |[\varpi]| \neq 0\} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$$

resp.

$$Y_{S,[1,\infty]} = \{|[\varpi]| \leq |\pi| \neq 0\} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+).$$

If $f \in B_{R,[0,1]}$, then the series

$$g = \varphi^{-1}(f) + \pi^n \varphi^{-2}(f) + \pi^{2n}\varphi^{-3}(f) + \dots$$

converges in $B_{R,[0,q]}$ and thus in $B_{R,[1,q]}$, and $f = \varphi(g) - \pi^n g$. The same then applies to elements of $B_{R,[0,1]}[\frac{1}{\pi}]$. On the other hand, if $f \in [\varpi]B_{R,[1,\infty]}$, then the series

$$g = -\pi^{-n} f - \varphi(f) - \pi^n \varphi^2(f) - \dots$$

converges in $B_{R,[1,q]}$, and $f = \varphi(g) - \pi^n g$.

In fact, the same arguments prove that the map

$$[B_{R,[1,\infty]} \xrightarrow{\varphi - \pi^n} B_{R,[1,\infty]}] \to [B_{R,[1,q]} \xrightarrow{\varphi = \pi^n} B_{R,[1,1]}]$$

is a quasi-isomorphism. Indeed, we have a short exact sequence

$$0 \to W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}] \to B_{R,[1,\infty]} \oplus B_{R,[0,q]}[\tfrac{1}{\pi}] \to B_{R,[1,q]} \to 0$$

(obtained from sheafyness of $W_{\mathcal{O}_E}(R^+)[\frac{1}{\pi}]$ when endowed with the π-adic topology on $W_{\mathcal{O}_E}(R^+)$), and similarly

$$0 \to W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}] \to B_{R,[1,\infty]} \oplus B_{R,[0,1]}[\tfrac{1}{\pi}] \to B_{R,[1,1]} \to 0.$$

Therefore, it suffices to see that the maps

$$B_{R,[0,q]}[\tfrac{1}{\pi}] \xrightarrow{\varphi - \pi^n} B_{R,[0,1]}[\tfrac{1}{\pi}]$$

and

$$W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}] \xrightarrow{\varphi - \pi^n} W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}]$$

are isomorphisms. In both cases, this follows from convergence of $\varphi^{-1} + \pi^n\varphi^{-2} + \pi^{2n}\varphi^{-3} + \dots$ on these algebras, giving an explicit inverse.

For part (iv), note that in equal characteristic one can describe $\mathcal{O}(Y_{S,I}) = B_{R,I}$, for $S = \mathrm{Spa}(R, R^+)$ affinoid, explicitly as power series $\sum_{i \in \mathbb{Z}} r_i \pi^i$ with $r_i \in R$, satisfying some convergence conditions as $i \to \pm\infty$. Taking the part where $\varphi = \pi^n$, we require $\varphi(r_i) = r_{i+n}$, and we see that we can freely choose $r_1, \dots, r_n$. The required convergence holds precisely when all r_i are topologically nilpotent, giving the isomorphism in that case. If E is p-adic, we can reduce to $E = \mathbb{Q}_p$ (but now λ rational, $0 < \lambda \leq 1$), taking a pushforward of the sheaf along $X_{S,E} = X_{S,\mathbb{Q}_p} \times_{\mathbb{Q}_p} E \to X_{S,\mathbb{Q}_p}$. In that case, the result follows from the equality $H^0(X_S, \mathcal{O}_{X_S}(\lambda)) = B_{R,[1,\infty]}^{\varphi^r = p^s}$ proved above, and [**SW13**, Theorem A, Proposition 3.1.3 (iii)].

In particular, for affinoid S we can choose a fibrewise nonzero map $\mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1)$, by taking a map $S \to \mathcal{BC}(\mathcal{O}(1)) \cong \operatorname{Spd} \mathbb{F}_q[\![x^{1/p^\infty}]\!]$ sending x to a pseudouniformizer. By Proposition II.2.3, for any $n \in \mathbb{Z}$, we get an exact sequence

$$0 \to \mathcal{O}_{X_S}(n) \to \mathcal{O}_{X_S}(n+1) \to \mathcal{O}_{S^\sharp} \to 0.$$

Applying this for $n > 0$, we get inductively an exact sequence

$$0 \to \mathcal{BC}(\mathcal{O}(n))|_S \to \mathcal{BC}(\mathcal{O}(n+1))|_S \to (\mathbb{A}^1_{S^\sharp})^\diamond \to 0.$$

Starting with the base case $n = 1$ already handled, this allows one to prove part (iii) by induction, using [**Sch17a**, Proposition 23.13].

Now for part (ii), we use the sequence for $n = 0$. In that case, for $S = \operatorname{Spa}(R, R^+)$, we get an exact sequence

$$0 \to H^0(X_S, \mathcal{O}_{X_S}) \to H^0(X_S, \mathcal{O}_{X_S}(1)) \to R^\sharp \to H^1(X_S, \mathcal{O}_{X_S}) \to 0$$

where the map in the middle can be identified with the logarithm map of the universal cover of the Lubin–Tate formal group. This is pro-étale locally surjective, with kernel given by $\underline{E}$, proving (ii).

Finally, for part (i), we first treat the case $n = -1$, where we get an exact sequence

$$0 \to \underline{E} \to (\mathbb{A}^1_{S^\sharp})^\diamond \to \mathcal{BC}(\mathcal{O}(-1)[1])|_S \to 0$$

showing in particular the vanishing of $H^0(X_S, \mathcal{O}_{X_S}(-1)) = 0$. As $\underline{E} \to (\mathbb{A}^1_{S^\sharp})^\diamond$ is a closed immersion, the result follows from [**Sch17a**, Proposition 24.2]. Now for $n < -1$, the result follows by induction from the sequence

$$0 \to (\mathbb{A}^1_{S^\sharp})^\diamond \to \mathcal{BC}(\mathcal{O}(-n)[1])|_S \to \mathcal{BC}(\mathcal{O}(-n+1)[1])|_S \to 0$$

and [**Sch17a**, Proposition 23.13]. □

II.2.3. The algebraic curve. We recall the following important ampleness result.

THEOREM II.2.6 ([**KL15**, Proposition 6.2.4]). *Let $S = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$ and let $\mathcal{E}$ be any vector bundle on X_S. Then there is an integer n_0 such that for all $n \geq n_0$, the vector bundle $\mathcal{E}(n)$ is globally generated, i.e. there is a surjective map*

$$\mathcal{O}_{X_S}^m \to \mathcal{E}(n)$$

for some $m \geq 0$, and moreover $H^1(X_S, \mathcal{E}(n)) = 0$.

PROOF. Pick a pseudouniformizer $\varpi \in R$, thus defining a radius function on Y_S. Write X_S as the quotient of $Y_{S,[1,q]}$ along the isomorphism $\varphi : Y_{S,[1,1]} \cong Y_{S,[q,q]}$. Correspondingly, $\mathcal{E}$ is given by some finite projective $B_{R,[1,q]}$-module $M_{[1,q]}$, with base changes $M_{[1,1]}$ and $M_{[q,q]}$ to $B_{R,[1,1]}$ and $B_{R,[q,q]}$, and an isomorphism $\varphi_M : M_{[q,q]} \cong M_{[1,1]}$, linear over $\varphi : B_{R,[q,q]} \cong B_{R,[1,1]}$.

For convenience, we first reduce to the case that $M_{[1,q]}$ is free (cf. [**KL15**, Corollary 1.5.3]). Indeed, pick a surjection $\psi : F_{[1,q]} := B^m_{R,[1,q]} \to M_{[1,q]}$. We want to endow the source with a similar

φ-module structure $\varphi_F : F_{[q,q]} \cong F_{[1,1]}$ (with obvious notation), making ψ equivariant. For this, we would like to find a lift

$$\begin{tikzcd} F_{[q,q]} \arrow[r, dashed, "\varphi_F"] \arrow[d, "\psi"] & F_{[1,1]} \arrow[d, "\psi"] \\ M_{[q,q]} \arrow[r, "\varphi_M"] & M_{[1,1]} \end{tikzcd}$$

such that φ_F is an isomorphism. Let $N_{[1,q]} = \ker(\psi)$, with base change $N_{[q,q]}$, $N_{[1,1]}$. Choosing a splitting $F_{[1,q]} \cong M_{[1,q]} \oplus N_{[1,q]}$, we see that we could find φ_F if there is an isomorphism $\varphi^* N_{[q,q]} \cong N_{[1,1]}$ of $B_{R,[1,1]}$-modules. But in the Grothendieck group of $B_{R,[1,1]}$-modules, both are given by $[B^m_{R,[1,1]}] - [M_{[1,1]}]$. Thus, after possibly adding a free module (i.e. increasing m), they are isomorphic, giving the claim.

Thus, we can assume that $M_{[1,q]} \cong B^m_{R,[1,q]}$ is a free $B_{R,[1,q]}$-module, and then

$$\varphi_M = A^{-1}\varphi$$

for some matrix $A \in \mathrm{GL}_m(B_{R,[1,1]})$. Actually, repeating the above argument starting with the presentation of X_S as the quotient of $Y_{S,[q^{-1},q]}$ via identifying $Y_{S,[q^{-1},1]}$ with $Y_{S,[1,q]}$, one can ensure that

$$A \in \mathrm{GL}_m(B_{R,[q^{-1},1]}).$$

Twisting by $\mathcal{O}_{X_S}(n)$ amounts to replacing A by $A\pi^n$. Let us choose integers N and N' such that

- the matrix A has entries in $\pi^N W_{\mathcal{O}_E}(R^+)\langle(\frac{[\varpi]}{\pi})^{\pm 1}\rangle$
- the matrix A^{-1} has entries in $\pi^{-N'} W_{\mathcal{O}_E}(R^+)\langle\frac{\pi}{[\varpi]}, \frac{[\varpi]^{1/q}}{\pi}\rangle$.

By twisting, we can replace N and N' by $N+n$ and $N'+n$; we can thus arrange that $qN > N'$, $N > 0$.

Fix some rational r such that $1 < r \leq q$. We will now show that there are m elements

$$v_1, \ldots, v_m \in (B^m_{R,[1,q]})^{\varphi = A} = H^0(X_S, \mathcal{E})$$

that form a basis of $B^m_{R,[r,q]}$. Repeating the above analysis for different strips (and different choices of pseudouniformizers $\varpi \in R$ to get overlapping strips), we can then get global generation of $\mathcal{E}$.

In fact, we will choose v_i to be of the form $[\varpi]^M e_i - v_i'$, for some positive integer M chosen later, where $e_i \in B^m_{R,[1,q]}$ is the i-th basis vector and v_i' is such that

$$||v_i'||_{B_{R,[r,q]}} \leq ||[\varpi]^{M+1}||_{B_{R,[r,q]}} = q^{-M-1}.$$

Here, we endow all $B_{R,I}$ with the spectral norm, normalizing the norms on all completed residue fields via $||[\varpi]|| = \frac{1}{q}$. These $v_1, \ldots, v_m$ restrict to a basis of $B^m_{R,[r,q]}$ since the base change matrix from the canonical basis is given by an element of

$$[\varpi^M](\mathrm{Id} + [\varpi] M_m(B^\circ_{R,[r,q]})) \subset \mathrm{GL}_m(B_{R,[r,q]}).$$

In order to find the v_i', it suffices to prove that the map

$$\varphi - A : B^m_{R,[1,q]} \to B^m_{R,[1,1]}$$

is surjective (yielding $H^1(X_S, \mathcal{E}) = 0$), in the following quantitative way: If, for some positive integer M chosen later,

$$w \in \pi^M W_{\mathcal{O}_E}(R^+)\langle(\tfrac{[\varpi]}{\pi})^{\pm 1}\rangle^m \subset B^m_{R,[1,1]},$$

then there is some

$$v \in B^m_{R,[1,q]}$$

such that

$$(\varphi - A)v = w \text{ and } ||v||_{B_{R,[r,q]}} \leq q^{-M-1}. \tag{II.2.1}$$

Indeed, we can then apply this to $w_i = (\varphi - A)([\varpi]^M e_i)$ (since $N > 0$ and thus A has entries in $W_{\mathcal{O}_E}(R^+)\langle(\frac{[\varpi]}{\pi})^{\pm 1}\rangle$), getting some v_i' with $w_i = (\varphi - A)(v_i')$ and

$$||v_i'||_{B_{R,[r,q]}} \leq q^{-M-1},$$

as desired.

Thus, take any

$$w \in \pi^M W_{\mathcal{O}_E}(R^+)\langle(\tfrac{[\varpi]}{\pi})^{\pm 1}\rangle^m.$$

We can write

$$w = w_1 + w_2 \text{ where } w_1 \in [\varpi]^{N-1}\pi^{M-N+1}W_{\mathcal{O}_E}(R^+)\langle\tfrac{\pi}{[\varpi]}\rangle^m,\ w_2 \in [\varpi]^N\pi^{M-N}W_{\mathcal{O}_E}(R^+)\langle\tfrac{[\varpi]}{\pi}\rangle^m.$$

Let

$$v = \varphi^{-1}(w_1) - A^{-1}w_2 \in B^m_{R,[1,q]} \text{ so that } w' := w - \varphi(v) + Av = \varphi(A^{-1}w_2) + A\varphi^{-1}(w_1).$$

Note that (as $N > 0$)

$$A\varphi^{-1}(w_1) \in \pi^N[\varpi]^{(N-1)/q}\pi^{M-N+1}W_{\mathcal{O}_E}(R^+)\langle\tfrac{\pi}{[\varpi]^{1/q}}\rangle^m \subset \pi^{M+1}W_{\mathcal{O}_E}(R^+)\langle(\tfrac{[\varpi]}{\pi})^{\pm 1}\rangle^m$$

and also (as $qN > N'$)

$$\varphi(A^{-1}w_2) \in \pi^{-N'}[\varpi]^{Nq}\pi^M W_{\mathcal{O}_E}(R^+)\langle\tfrac{[\varpi]^q}{\pi}\rangle^m \subset \pi^{M+1}W_{\mathcal{O}_E}(R^+)\langle(\tfrac{[\varpi]}{\pi})^{\pm 1}\rangle^m$$

so that

$$w' \in \pi^{M+1}W_{\mathcal{O}_E}(R^+)\langle(\frac{[\varpi]}{\pi})^{\pm 1}\rangle^m.$$

If one can thus prove the required bounds on v, this process will converge and prove the desired statement. It remains to estimate v. On the one hand, its norm is clearly bounded in terms of the norm of w (as both w_1 and w_2 are, and φ^{-1} and A^{-1} are bounded operators), and thus, since when one iterates w goes to zero, v goes to zero, and the process converges by summing to obtain some v such that $(\varphi - A)v = w$. But we need an improved estimate over $B_{R,[r,q]}$ to obtain (II.2.1). Note that the norm of $\varphi^{-1}(w_1)$ is bounded above by the norm of $[\varpi]^{(N-1)/q}\pi^{M-N+1}$, which in $B_{R,[r,q]}$ is given by $q^{-(N-1)/q-rM+rN-r}$. This is at most q^{-M-1} once M is large enough. On the other hand, $w_2 \in \pi^M W_{\mathcal{O}_E}(R^+)\langle\frac{[\varpi]}{\pi}\rangle^m$ and so the norm of $A^{-1}w_2$ is bounded by the norm of $\pi^{-N'}\pi^M$, which in $B_{R,[r,q]}$ is given by $q^{rN'-rM}$. Again, this is at most q^{-M-1} once M is large enough. Thus, taking M large enough (depending only on N, N' and $r > 1$), the process above converges, giving the desired result. □

We have the following general *GAGA theorem*. Its proof is an axiomatization of [**KL15**, Theorem 6.3.9].

Proposition II.2.7 (GAGA). *Let $(X, \mathcal{O}_X)$ be a locally ringed spectral space equipped with a line bundle $\mathcal{O}_X(1)$ such that for any vector bundle $\mathcal{E}$ on X, there is some n_0 such that for all $n \geq n_0$, the bundle $\mathcal{E}(n)$ is globally generated. Moreover, assume that for $i > 0$, the cohomology group $H^i(X, \mathcal{E}(n)) = 0$ vanishes for all sufficiently large n.*

Let $P = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n))$ be the graded ring and $X^{\mathrm{alg}} = \mathrm{Proj}(P)$. There is a natural map $(X, \mathcal{O}_X) \to X^{\mathrm{alg}}$ of locally ringed topological spaces, and pullback along this map induces an equivalence of categories between vector bundles on X^{alg} and vector bundles on $(X, \mathcal{O}_X)$. Moreover, for any vector bundle $\mathcal{E}^{\mathrm{alg}}$ on X^{alg} with pullback $\mathcal{E}$ to X, the map

$$H^i(X^{\mathrm{alg}}, \mathcal{E}^{\mathrm{alg}}) \to H^i(X, \mathcal{E})$$

is an isomorphism for all $i \geq 0$.

Recall that for any graded ring $P = \bigoplus_{n \geq 0} P_n$, one can define a separated scheme $\mathrm{Proj}(P)$ by gluing $\mathrm{Spec}\, P[f^{-1}]_0$ for all $f \in P_n$, $n > 0$, where $P[f^{-1}]_0 = \varinjlim_i f^{-i} P_{in}$ is the degree 0 part of $P[f^{-1}]$. In our situation, if n is large enough so that $\mathcal{O}_X(n)$ is globally generated, then it is enough to consider only $f \in P_n$ for this given n, and in fact only a finite set of them (as X is quasicompact); in particular, $\mathrm{Proj}(P)$ is quasicompact. Moreover, one sees that there is a tautological line bundle $\mathcal{O}_{\mathrm{Proj}(P)}(n)$ for all sufficiently large n, compatible with tensor products; thus, there is also a tautological line bundle $\mathcal{O}_{\mathrm{Proj}(P)}(1)$, which is an ample line bundle on $\mathrm{Proj}(P)$. The pullback of $\mathcal{O}_{\mathrm{Proj}(P)}(1)$ is then given by $\mathcal{O}_X(1)$.

Proof. The construction of the map $f : (X, \mathcal{O}_X) \to X^{\mathrm{alg}}$ is formal (and does not rely on any assumptions): if $g \in P_n$, then on the non-vanishing locus $U = D(g) \subset X$, there is an isomorphism $g_{|U} : \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_U(n)$. Now, for $x = \frac{a}{g^k} \in P[g^{-1}]_0$, $g_{|U}^{-k} \circ a \in \mathcal{O}(U)$, and this defines a morphism of rings $P[g^{-1}]_0 \to \mathcal{O}(U)$. One deduces a morphism of locally ringed spaces $U \to D^+(g)$, and those glue when g varies to a morphism of locally ringed spaces $(X, \mathcal{O}_X) \to X^{\mathrm{alg}}$.

We consider the functor taking any vector bundle $\mathcal{E}$ on X to the quasicoherent $\mathcal{O}_{X^{\mathrm{alg}}}$-module $\overline{\mathcal{E}}$ associated to the graded P-module $\bigoplus_{n \geq 0} H^0(X, \mathcal{E}(n))$. This functor is exact as $H^1(X, \mathcal{E}(n)) = 0$ for all sufficiently large n, and it commutes with twisting by $\mathcal{O}(1)$. We claim that it takes values in vector bundles on X^{alg}. To see this, take a surjection $\mathcal{O}_X^m \to \mathcal{E}(n)$ with kernel $\mathcal{F}$, again a vector bundle. The map $\mathcal{O}_X^m \to \mathcal{E}(n)$ splits after twisting, i.e. for any $f \in P_{n'}$ with n' large enough, there is a map $\mathcal{E}(n - n') \to \mathcal{O}_X^m$ such that $\mathcal{E}(n - n') \to \mathcal{O}_X^m \to \mathcal{E}(n)$ is multiplication by f. Indeed, the obstruction to such a splitting is a class in $H^1(X, \mathscr{H}\mathrm{om}(\mathcal{E}, \mathcal{F})(n'))$ which vanishes for n' large enough. This implies that $\overline{\mathcal{E}}$ is a vector bundle on $\mathrm{Spec}\, P[f^{-1}]_0$ for any such f, and these cover X^{alg}.

There is a natural map $f^* \overline{\mathcal{E}} \to \mathcal{E}$, and the preceding arguments show that this is an isomorphism (on the preimage of any $\mathrm{Spec}\, P[f^{-1}]_0$, and thus globally). It now remains to show that if $\mathcal{E}^{\mathrm{alg}}$ is any vector bundle on X^{alg}, the map

$$H^i(X^{\mathrm{alg}}, \mathcal{E}^{\mathrm{alg}}) \to H^i(X, \mathcal{E})$$

is an isomorphism for all $i \geq 0$. By ampleness of $\mathcal{O}_{X^{\mathrm{alg}}}(1)$, there is some surjection $\mathcal{O}_{X^{\mathrm{alg}}}(-n)^m \to (E^{\mathrm{alg}})^\vee$, with kernel a vector bundle $\mathcal{F}$. Dualizing, we get an injection $\mathcal{E} \to \mathcal{O}_{X^{\mathrm{alg}}}(n)^m$ with cokernel a vector bundle. This already gives injectivity on H^0 by reduction to $\mathcal{O}_{X^{\mathrm{alg}}}(n)$ where it is clear.

Applying this injectivity also for $\mathcal{F}$, we then get bijectivity on H^0. This already implies that we get an equivalence of categories (exact in both directions). Finally, picking $f_1, \ldots, f_m \in P_n$ so that the $\operatorname{Spec} P[f_i^{-1}]_0$ cover X^{alg}, we can look at the corresponding Čech complex. Each term is a filtered colimit of global sections of vector bundles $\mathcal{E}(n)$ along multiplication by products of powers of f_i's. This reduces the assertion to the case of H^0 and the vanishing of $H^i(X, \mathcal{E}(n))$ for n large enough. □

Remark II.2.8. One can check that X^{alg} is up to canonical isomorphism independent of the choice of a line bundle $\mathcal{O}_X(1)$ satisfying the preceding properties.

In particular, for any affinoid perfectoid space S over $\mathbb{F}_q$, we can define the algebraic curve

$$X_S^{\mathrm{alg}} = \operatorname{Proj} \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}_{X_S}(n)).$$

There is a well-defined map $X_S \to X_S^{\mathrm{alg}}$ of locally ringed spectral spaces, pullback along which defines an equivalence of categories of vector bundles, and is compatible with cohomology.

Notably, this connects the present discussion to the original definition of the Fargues–Fontaine curve as given in [**FF18**], where the case $S = \operatorname{Spa}(F, \mathcal{O}_F)$ is considered, for a perfectoid field F of characteristic p. We will restrict ourselves, as above, to the case that $F = C$ is algebraically closed.

Proposition II.2.9. *Let C be a complete algebraically closed nonarchimedean field over $\mathbb{F}_q$. Then X_C^{alg} is a connected regular noetherian scheme of Krull dimension 1, and the map $|X_C| \to |X_C^{\mathrm{alg}}|$ induces a bijection between $|X_C|^{\mathrm{cl}}$ and the closed points of $|X_C^{\mathrm{alg}}|$. Moreover, for any classical point $x \in |X_C|$, the complement $X_C^{\mathrm{alg}} \setminus \{x\}$ is the spectrum of a principal ideal domain.*

Proof. Let $x \in |X_C|^{\mathrm{cl}}$ be any classical point, corresponding to some untilt $C^\sharp$ over E of C. Using Lubin–Tate formal groups, we see that there is an exact sequence

$$0 \to \mathcal{O}_{X_C} \to \mathcal{O}_{X_C}(1) \to \mathcal{O}_{C^\sharp} \to 0$$

on X_C. The corresponding section $f \in H^0(X_C, \mathcal{O}_{X_C}(1))$ defines its vanishing locus in X_C^{alg}, which is then also given by $\operatorname{Spec} C^\sharp$. In particular, x defines a closed point of $|X_C^{\mathrm{alg}}|$. Now we want to show that $P[f^{-1}]_0$ is a principal ideal domain. Thus, take any nonzero $g \in H^0(X_C, \mathcal{O}_{X_C}(n))$. This has finitely many zeroes on X_C, all at classical points $x_1, \ldots, x_m$. For each x_i, we have a section $f_{x_i} \in H^0(X_C, \mathcal{O}_{X_C}(1))$ as before, and then $g = f_1^{n_1} \cdots f_m^{n_m} h$ for some $n_i \geq 1$, and some $h \in H^0(X_C, \mathcal{O}_{X_C}(n'))$ that is everywhere nonzero. In particular, h defines an isomorphism $\mathcal{O}_{X_C} \to \mathcal{O}_{X_C}(n')$, whence $n' = 0$, and $h \in E^\times$. This decomposition implies easily that $P[f^{-1}]_0$ is indeed a principal ideal domain, and it shows that all maximal ideals arise from classical points of $|X_C|$, finishing the proof. □

II.2.4. Classification of vector bundles. At this point, we can recall the classification of vector bundles over X_C; so here we take $S = \operatorname{Spa} C$ for a complete algebraically closed nonarchimedean field C over $\mathbb{F}_q$. First, one classifies line bundles.

Proposition II.2.10. *The map $\mathbb{Z} \to \operatorname{Pic}(X_C)$, $n \mapsto \mathcal{O}_{X_C}(n)$, is an isomorphism.*

Proof. By Proposition II.2.9, any line bundle becomes trivial after removing one closed point $x \in X_C^{\mathrm{alg}}$. As the local rings of X_C^{alg} are discrete valuation rings, this implies that any line bundle is of the form $\mathcal{O}_{X_C}(n[x])$ for some $n \in \mathbb{Z}$. But $\mathcal{O}_{X_C}([x]) \cong \mathcal{O}_{X_C}(1)$ by Proposition II.2.2, so the result follows. □

In particular, one can define the degree of any vector bundle $\mathcal{E}$ on X_C via

$$\deg(\mathcal{E}) = \deg(\det(\mathcal{E})) \in \mathbb{Z}$$

where $\det(\mathcal{E})$ is the determinant line bundle, and $\deg : \mathrm{Pic}(X_C) \cong \mathbb{Z}$ is the isomorphism from the proposition. Of course, one can also define the rank $\mathrm{rk}(\mathcal{E})$ of any vector bundle, and thus for any nonzero vector bundle its slope

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \in \mathbb{Q}.$$

It is easy to see that this satisfies the Harder–Narasimhan axiomatics (for example, rank and degree are additive in short exact sequences). In particular, one can define semistable vector bundles as those vector bundles $\mathcal{E}$ such that for all proper nonzero $\mathcal{F} \subset \mathcal{E}$, one has $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. One says that $\mathcal{E}$ is stable if in fact $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for all such $\mathcal{F}$.

Example II.2.11. For any $\lambda \in \mathbb{Q}$, the bundle $\mathcal{O}_{X_C}(\lambda)$ is stable of slope λ. Indeed, assume that $0 \neq \mathcal{F} \subsetneq \mathcal{O}_{X_C}(\lambda)$ is a proper nonzero subbundle, and let $r = \mathrm{rk}(\mathcal{F})$, $s = \deg(\mathcal{F})$. Passing to r-th wedge powers, we get an injection

$$\det(\mathcal{F}) \cong \mathcal{O}_{X_C}(s) \hookrightarrow \mathcal{O}_{X_C}(r\lambda)^m,$$

using that $\bigwedge^r \mathcal{O}_{X_C}(\lambda)$ is a direct sum of copies of $\mathcal{O}_{X_C}(r\lambda)$. This implies that $s \leq r\lambda$. Moreover, if we have equality, then r is at least the denominator of λ, which is the rank of $\mathcal{O}_{X_C}(\lambda)$, i.e. $\mathcal{F}$ has the same rank as $\mathcal{O}_{X_C}(\lambda)$. Thus, $\mathcal{O}_{X_C}(\lambda)$ is stable.

Proposition II.2.12. *Any vector bundle $\mathcal{E}$ on X_C admits a unique exhaustive separating $\mathbb{Q}$-indexed filtration by saturated subbundles $\mathcal{E}^{\geq\lambda} \subset \mathcal{E}$, called the Harder–Narasimhan filtration, such that*

$$\mathcal{E}^\lambda := \mathcal{E}^{\geq\lambda}/\mathcal{E}^{>\lambda}, \text{ where } \mathcal{E}^{>\lambda} = \bigcup_{\lambda'>\lambda} \mathcal{E}^{\geq\lambda'},$$

is semistable of slope λ. The formation of the Harder–Narasimhan filtration is functorial in $\mathcal{E}$. □

As a preparation for the next theorem, we note that the Harder–Narasimhan filtration is also compatible with change of C.

Proposition II.2.13. *Let $\mathcal{E}$ be a vector bundle on X_C, and let $C'|C$ be an extension of complete algebraically closed nonarchimedean fields, with pullback $\mathcal{E}'$ of $\mathcal{E}$ to $X_{C'}$. Then $(\mathcal{E}')^{\geq\lambda}$ is the pullback of $\mathcal{E}^{\geq\lambda}$.*

Similarly, if $E'|E$ is a finite separable extension of degree r, and $\mathcal{E}'$ is the pullback of $\mathcal{E}$ along $X_{C,E'} = X_{C,E} \otimes_E E' \to X_{C,E} = X_C$, then $(\mathcal{E}')^{\geq\lambda}$ is the pullback of $\mathcal{E}^{\geq\lambda/r}$.

Proof. Consider the case of $C'|C$. By uniqueness of the Harder–Narasimhan filtration, it suffices to see that pullbacks of semistable vector bundles remain semistable. Thus, assume that $\mathcal{E}$ is semistable, and assume by way of contradiction that $\mathcal{E}'$ is not semistable. By induction on the

rank, we can assume that the formation of the Harder–Narasimhan filtration of $\mathcal{E}'$ is compatible with any base change. Consider the first nontrivial piece of the Harder–Narasimhan filtration $0 \neq \mathcal{F} \subsetneq \mathcal{E}$. This is a vector bundle on $X_{C'}$ with $\mu(\mathcal{F}) > \mu(\mathcal{E})$. We claim that $\mathcal{F}$ descends to X_C. By Proposition II.2.1, it suffices to see that the two pullbacks of $\mathcal{F}$ to $X_{C'\widehat{\otimes}_C C'}$ agree. This is true as there are no nonzero maps from $\mathcal{F}$ to $\mathcal{E}'/\mathcal{F}$ after base change to X_R for any perfectoid C'-algebra R: If there were such a nonzero map, there would also be a nonzero map for some choice of $R = C''$ a complete algebraically closed nonarchimedean field. But then $\mathcal{F}$ is still semistable and all pieces of the Harder–Narasimhan filtration of $\mathcal{E}'/\mathcal{F}$ are of smaller slope, so such maps do not exist.

For an extension $E'|E$, the similar arguments work, using Galois descent instead (noting that one may assume that $E'|E$ is Galois by passing to Galois hulls). Note that the pullback of $\mathcal{O}_{X_{C,E}}(1)$ is $\mathcal{O}_{X_{C,E'}}(r)$, causing the mismatch in slopes. $\square$

The main theorem on the classification of vector bundles is the following. Our proof follows the arguments of Hartl-Pink, [**HP04**], to reduce to Lemma II.2.15 below. However, we give a new and direct proof of this key lemma, which avoids any hard computations by using the geometry of diamonds and v-descent. We thus get a new proof of the classification theorem.[3]

THEOREM II.2.14. *Any vector bundle $\mathcal{E}$ on X_C is isomorphic to a direct sum of vector bundles of the form $\mathcal{O}_{X_C}(\lambda)$ with $\lambda \in \mathbb{Q}$. If $\mathcal{E}$ is semistable of slope λ, then $\mathcal{E} \cong \mathcal{O}_{X_C}(\lambda)^m$ for some $m \geq 0$.*

PROOF. We argue by induction on the rank n of $\mathcal{E}$, so assume the theorem in rank $\leq n-1$ (and for all choices of E); the case $n = 1$ has been handled already. By the vanishing of $H^1(X_C, \mathcal{O}_{X_C}(\lambda)) = 0$ for $\lambda > 0$, the theorem follows for $\mathcal{E}$ if $\mathcal{E}$ is not semistable. Thus assume $\mathcal{E}$ is semistable of slope $\lambda = \frac{s}{r}$ with $s \in \mathbb{Z}$ and $r > 0$ coprime. It suffices to find a nonzero map $\mathcal{O}_{X_C}(\lambda) \to \mathcal{E}$: Indeed, by stability of $\mathcal{O}_{X_C}(\lambda)$, the map is necessarily injective (the category of semistable vector bundles of slope λ is abelian with simple objects the stable vector bundles of slope λ), and the quotient will then again be semistable of slope λ, and thus by induction isomorphic to $\mathcal{O}_{X_C}(\lambda)^{m-1}$. One finishes by observing that $\mathrm{Ext}^1_{X_C}(\mathcal{O}_{X_C}(\lambda), \mathcal{O}_{X_C}(\lambda)) = 0$ by Proposition II.2.5 (ii).

Thus, it suffices to find a nonzero map $\mathcal{O}_{X_C}(\lambda) \to \mathcal{E}$. Let $E'|E$ be the unramified extension of degree r, and consider the covering $f : X_{C,E'} = X_{C,E} \otimes_E E' \to X_{C,E} = X_C$. Then $\mathcal{O}_{X_C}(\lambda) = f_*\mathcal{O}_{X_{C,E'}}(s)$, and so it suffices to find a nonzero map $\mathcal{O}_{X_{C,E'}}(s) \to f^*\mathcal{E}$. In other words, up to changing E, we can assume that $\lambda \in \mathbb{Z}$. Then by twisting, we can assume $\lambda = 0$.

Next, we observe that we are free to replace C by an extension. Indeed, consider the v-sheaf sending $S \in \mathrm{Perf}_C$ to the isomorphisms $\mathcal{E} \cong \mathcal{O}^n_{X_C}$. This is a v-quasitorsor under $\underline{\mathrm{GL}_n(E)}$ (using Proposition II.2.5 (ii)). If there is some extension of C where we can find a nonzero section of $\mathcal{E}$ (and thus also trivialize $\mathcal{E}$), then it is a v-torsor under $\underline{\mathrm{GL}_n(E)}$. By v-descent of $\underline{\mathrm{GL}_n(E)}$-torsors, cf. [**Sch17a**, Lemma 10.13], it is then representable by a space pro-étale over $\overline{\mathrm{Spa}\, C}$, and thus admits a section.

[3]First, it has been proven for E of equal characteristic in [**HP04**] and for p-adic E by Kedlaya in [**Ked04**]; both of these proofs used heavy computations to prove Lemma II.2.15. A more elegant proof was given by Fargues–Fontaine [**FF18**] (for all E) by reducing to the description of the Lubin–Tate and Drinfeld moduli spaces of π-divisible $\mathcal{O}_E$-modules, and their Grothendieck–Messing period morphisms (which arguably also involve some nontrivial computations). Finally, for p-adic E a proof is implicit in Colmez' work [**Col02**] on Banach–Colmez spaces.

Let $d \geq 0$ be minimal such that there is an injection $\mathcal{O}_{X_C}(-d) \hookrightarrow \mathcal{E}$, possibly after base enlarging C; by Theorem II.2.6 some such d exists. We want to see that $d = 0$, so assume $d > 0$ by way of contradiction. By minimality of d, the quotient $\mathcal{F} = \mathcal{E}/\mathcal{O}_{X_C}(-d)$ is a vector bundle, and by induction the classification theorem holds true for $\mathcal{F}$.

If $d \geq 2$, then we can by induction find an injection $\mathcal{O}_{X_C}(-d+2) \hookrightarrow \mathcal{F}$; taking the pullback defines an extension

$$\mathcal{O}_{X_C}(-d) \to \mathcal{G} \to \mathcal{O}_{X_C}(-d+2).$$

so by twisting

$$\mathcal{O}_{X_C}(-1) \to \mathcal{G}(d-1) \to \mathcal{O}_{X_C}(1).$$

By the key lemma, Lemma II.2.15 below, we would, possibly after enlarging C, get an injection $\mathcal{O}_{X_C} \hookrightarrow \mathcal{G}(d-1)$, and hence an injection $\mathcal{O}_{X_C}(-d+1) \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{E}$, contradicting our choice of d.

Thus, we may assume that $d = 1$. If $\mathcal{F}$ is not semistable, then it admits a subbundle $\mathcal{F}' \subset \mathcal{F}$ of degree ≥ 1 and rank $\leq n-2$. Applying the classification theorem to the pullback

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E}' \to \mathcal{F}' \to 0$$

of $\mathcal{F}'$, which is of slope ≥ 0, we then get that $\mathcal{E}' \subset \mathcal{E}$ has a global section.

It remains the case that $d = -1$ and that $\mathcal{F}$ is semistable, thus necessarily isomorphic to $\mathcal{O}_{X_C}(\frac{1}{n-1})$. This is the content of the next lemma. □

Lemma II.2.15. *Let*

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \mathcal{O}_{X_C}(\tfrac{1}{n}) \to 0$$

be an extension of vector bundles on X_C, for some $n \geq 1$. Then there is some extension $C'|C$ of complete algebraically closed nonarchimedean fields such that $H^0(X_{C'}, \mathcal{E}|_{X_{C'}}) \neq 0$.

Proof. Assume the contrary. Passing to Banach–Colmez spaces, we find an injection of v-sheaves

$$f : \mathcal{BC}(\mathcal{O}_{X_C}(\tfrac{1}{n})) \hookrightarrow \mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]).$$

The image cannot be contained in the classical points (as these form a totally disconnected subset while the source is connected), so the image contains some non-classical point. After base change to some $C'|C$, we thus find that the image contains some nonempty open subset of $\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1])$, as follows from the presentation

$$\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) = (\mathbb{A}^1_{C^\sharp})^\diamond/\underline{E}$$

and the similar behaviour of non-classical points of $\mathbb{A}^1_{C^\sharp}$. Translating this nonempty open subset to the origin, we find that the image of f contains an open neighborhood of 0, and then by rescaling by the contracting of $E^\times$, we find that the map f must be surjective, and thus an isomorphism.

In particular, this would mean that $\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1])$ is a perfectoid space. This is patently absurd if E is p-adic, as then the given presentation shows that $(\mathbb{A}^1_{C^\sharp})^\diamond$ is pro-étale over a perfectoid space and thus itself a perfectoid space, but $\mathbb{A}^1_{C^\sharp}$ is clearly not a perfectoid space.[4]

In general, we can argue as follows. There is a nonzero map

$$\mathcal{BC}(\mathcal{O}_{X_C}(\tfrac{1}{n})) \to (\mathbb{A}^1_{C^\sharp})^\diamond$$

[4]We believe that also when E is of equal characteristic, $\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1])$ is not a perfectoid space, but we were not able to settle this easily.

as $H^0(\mathcal{O}_{X_C}(\frac{1}{n}))$ maps nontrivially to its fibre at the chosen untilt $\operatorname{Spa} C^\sharp \hookrightarrow X_C$. If f is an isomorphism, we would then get a nonzero map

$$(\mathbb{A}^1_{C^\sharp})^\diamond/\underline{E} \cong \mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \xrightarrow{f^{-1}} \mathcal{BC}(\mathcal{O}_{X_C}(\tfrac{1}{n})) \to (\mathbb{A}^1_{C^\sharp})^\diamond.$$

On the other hand, one can classify all E-linear maps $(\mathbb{A}^1_{C^\sharp})^\diamond \to (\mathbb{A}^1_{C^\sharp})^\diamond$. The latter are the same as maps $\mathbb{A}^1_{C^\sharp} \to \mathbb{A}^1_{C^\sharp}$ if E is p-adic (by [**SW20**, Proposition 10.2.3]), respectively maps $\mathbb{A}^1_{C,\mathrm{perf}} \to \mathbb{A}^1_{C,\mathrm{perf}}$ if E is of equal characteristic. Thus, they are given by some convergent power series $g(X)$ that is additive, i.e. $g(X+Y) = g(X) + g(Y)$, and satisfies $g(aX) = ag(X)$ for all $a \in E$. (If E is of characteristic p, then g may a priori involve fractional powers X^{1/p^i}.) The equation $g(\pi X) = \pi g(X)$ alone in fact shows that only the linear coefficient of g may be nonzero, so $g(X) = cX$ for some $c \in C^\sharp$, and thus g is either an isomorphism or zero. But our given map is nonzero with nontrivial kernel, giving a contradiction. □

II.2.5. Families of vector bundles. Using the ampleness of $\mathcal{O}(1)$, we can now prove the following result on relative Banach–Colmez spaces.

PROPOSITION II.2.16. *Let S be a perfectoid space over $\mathbb{F}_q$. Let $\mathcal{E}$ be a vector bundle on X_S. Then the Banach–Colmez space*

$$\mathcal{BC}(\mathcal{E}) : T \mapsto H^0(X_T, \mathcal{E}|_{X_T})$$

is a locally spatial diamond, partially proper over S. Moreover, the projectivized Banach–Colmez space

$$(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$$

is a locally spatial diamond, proper over S.

PROOF. Note that $\mathcal{BC}(\mathcal{E}) \to S$ is separated: As it is a group, it suffices to see that the zero section is closed, but being zero can be checked through the vanishing at all untilts $T^\sharp$ of T, where it defines Zariski closed conditions. Using Theorem II.2.6, choose a surjection $\mathcal{O}_{X_S}(-n)^m \to \mathcal{E}^\vee$. Dualizing, we get an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{X_S}(n)^m \to \mathcal{F} \to 0$$

for some vector bundle $\mathcal{F}$. As $\mathcal{BC}(\mathcal{F})$ is separated, we see that $\mathcal{BC}(\mathcal{E}) \subset \mathcal{BC}(\mathcal{O}_{X_S}(n)^m)$ is closed, so the first part follows from Proposition II.2.5 (iii). For the second part, we may assume that S is qcqs. It is also enough to prove the similar result for $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\pi^{\mathbb{Z}}$ as the $\underline{\mathcal{O}_E^\times}$-action is free (so one can apply the last part of [**Sch17a**, Proposition 11.24]). This follows from the following general lemma about contracting group actions on locally spectral spaces, noting that checking the conditions formally reduces to the case of $\mathcal{BC}(\mathcal{O}_{X_S}(n)^m)$ and from there to $\mathbb{A}^1_{S^\sharp}$ by evaluating sections at some collection of untilts. □

LEMMA II.2.17. *Let X be a taut locally spectral space such that for any $x \in X$, the set $X_x \subset X$ of generalizations of x is a totally ordered chain under specialization. Let $\gamma : X \xrightarrow{\sim} X$ be an automorphism of X such that the subset $X_0 \subset X$ of fixed points is a spectral space. Moreover, assume that*

(i) *for all $x \in X$, the sequence $\gamma^n(x)$ for $n \to \infty$ converges towards X_0, i.e. for all open neighborhoods U of X_0, one has $\gamma^n(x) \in U$ for all sufficiently positive n;*

(ii) *for all $x \in X \setminus X_0$, the sequence $\gamma^n(x)$ for $n \to -\infty$ diverges, i.e. for all quasicompact open subspaces $U \subset X$, one has $\gamma^n(x) \notin U$ for all sufficiently negative n.*

Then $X_0 \subset X$ is a closed subspace, the action of γ on $X \setminus X_0$ is free and totally discontinuous (i.e. the action map $(X \setminus X_0) \times \mathbb{Z} \to (X \setminus X_0) \times (X \setminus X_0)$ is a closed immersion), and the quotient $(X \setminus X_0)/\gamma^{\mathbb{Z}}$ is a spectral space.

Remark II.2.18. For applications of this lemma, we recall the following facts:

(i) If X is any locally spatial diamond, then $|X|$ is a locally spectral space such that all for all $x \in |X|$, the set of generalizations of x in $|X|$ is a totally ordered chain under specialization. Indeed, this follows from [**Sch17a**, Proposition 11.19] and the similar property for analytic adic spaces.

(ii) If X is in addition partially proper over a spatial diamond, then $|X|$ is taut by [**Sch17a**, Proposition 18.10].

This means that the first sentence of the lemma is practically always satisfied.

Proof. Let $U \subset X$ be some quasicompact open neighborhood of X_0. First, we claim that one can arrange that $\gamma(U) \subset U$. Indeed, one has

$$U \subset \gamma^{-1}(U) \cup \gamma^{-2}(U) \cup \ldots \cup \gamma^{-n}(U) \cup \ldots,$$

as for any $x \in U \subset X$, also $\gamma^n(x) \in U$ for all sufficiently large n by assumption, and so $x \in \gamma^{-n}(U)$ for some $n > 0$. By quasicompacity of U, this implies that $U \subset \gamma^{-1}(U) \cup \ldots \cup \gamma^{-n}(U)$ for some n, and then $U' = U \cup \gamma^{-1}(U) \cup \ldots \cup \gamma^{-n+1}(U)$ is a quasicompact open neighborhood of X_0 with $\gamma(U') \subset U'$.

Now fix a quasicompact open neighborhood U of X_0 with $\gamma(U) \subset U$. We claim that

$$X_0 = \bigcap_{n \geq 0} \gamma^n(U).$$

Indeed, if $x \in X \setminus X_0$, then by assumption there is some positive n such that $\gamma^{-n}(x) \notin U$, giving the result.

In particular, for any other quasicompact open neighborhood V of X_0, there is some n such that $\gamma^n(U) \subset V$. Indeed, the sequence of spaces $\gamma^n(U) \setminus V$ is a decreasing sequence of spectral spaces with empty inverse limit, and so one of the terms is empty.

Consider the closure $\overline{U} \subset X$ of U in X. As X is taut, this is still quasicompact. Repeating the above argument, we see that for some $n > 0$, one has

$$\overline{U} \subset \gamma^{-1}(U) \cup \ldots \cup \gamma^{-n}(U) = \gamma^{-n}(U).$$

This implies that the sequences $\{\gamma^n(U)\}_{n \geq 0}$ and $\{\gamma^n(\overline{U})\}_{n \geq 0}$ are cofinal. In particular,

$$X_0 = \bigcap_{n \geq 0} \gamma^n(U) = \bigcap_{n \geq 0} \gamma^n(\overline{U})$$

is a closed subset of X.

Next, we check that any point $x \in X \setminus X_0$ has an open neighborhood V such that $\{\gamma^n(V)\}_{n \in \mathbb{Z}}$ are pairwise disjoint; for this it suffices to arrange that $V \cap \gamma^i(V) = \emptyset$ for all $i > 0$. For this,

note that if n is chosen such that $\gamma^n(\overline{U}) \subset U$, then up to rescaling by a power of γ, we can assume that $x \in U \setminus \gamma^{n+1}(\overline{U})$. Let $V \subset U \setminus \gamma^{n+1}(\overline{U})$ be a quasicompact open neighborhood of x. Then $\gamma^i(V) \cap V = \emptyset$ as soon as $i \geq n+1$. For the finitely many $i = 1, \ldots, n$, we can use a quasicompacity argument, and reduce to proving that if X_x is the localization of X at x (i.e., the set of all generalizations of x), then $X_x \cap \gamma^i(X_x) = \emptyset$ for $i = 1, \ldots, n$. By our assumption on X, the space X_x has a unique generic point $\eta \in X_x$ (X_x is pro-constructible in a spectral space thus spectral and by our hypothesis X_x is irreducible), which must then also be the unique generic point of $\gamma^i(X_x)$ if $X_x \cap \gamma^i(X_x) \neq \emptyset$. Thus, if $X_x \cap \gamma^i(X_x) \neq \emptyset$, then $\gamma^i(\eta) = \eta$, so $\eta \in X_0$. But X_0 is closed, so that $x \in X_0$, which is a contradiction.

In particular, the action of γ on $X \setminus X_0$ is free and totally discontinuous, and the quotient $\overline{X} = (X \setminus X_0)/\gamma^{\mathbb{Z}}$ is a locally spectral space which is locally isomorphic to $X \setminus X_0$. A basis of open neighborhoods of $\overline{X}$ is given by the image of quasicompact open subsets $V \subset X \setminus X_0$ for which $\{\gamma^n(V)\}_{n \in \mathbb{Z}}$ are pairwise disjoint; it follows that these are quasicompact open subsets of $\overline{X}$. Also, the intersection of two such subsets is of the same form, so the quotient $\overline{X}$ is quasiseparated. Finally, note that $U \setminus \gamma(U) \to \overline{X}$ is a bijective continuous map, and the source is a spectral space (as $\gamma(U) \subset U$ is a quasicompact open subspace of the spectral space U), and in particular quasicompact, and so $\overline{X}$ is quasicompact. □

The result on properness of the projectivized Banach–Colmez space enables us to give quick proofs of the main results of [**KL15**] (including an extension to the case of general E, in particular of equal characteristic).

THEOREM II.2.19 ([**KL15**, Theorem 7.4.5, Theorem 7.4.9, Theorem 7.3.7, Proposition 7.3.6]). *Let S be a perfectoid space over $\mathbb{F}_q$ and let $\mathcal{E}$ be a vector bundle over X_S of constant rank n.*

(i) *The function taking a geometric point $\operatorname{Spa} C \to S$ of S to the Harder–Narasimhan polygon of $\mathcal{E}|_{X_C}$ is upper semicontinuous.*

(ii) *Assume that the Harder–Narasimhan polygon of $\mathcal{E}$ is constant. Then there exists a global (separated exhaustive decreasing) Harder–Narasimhan filtration*

$$\mathcal{E}^{\geq \lambda} \subset \mathcal{E}$$

specializing to the Harder–Narasimhan filtration at each point. Moreover, after replacing S by a pro-étale cover, the Harder–Narasimhan filtration can be split, and there are isomorphisms

$$\mathcal{E}^{\lambda} \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$$

for some integers $n_\lambda \geq 0$.

PROOF. Note that the Harder–Narasimhan polygon can be described as the convex hull of the points (i, d_i) for $i = 0, \ldots, n$, where d_i is the maximal integer such that $H^0(X_C, (\wedge^i \mathcal{E})(-d_i)|_{X_C}) \neq 0$. To prove part (i), it therefore suffices to show that for any vector bundle $\mathcal{F}$ on X_S, the locus of all geometric points $\operatorname{Spa} C \to S$ for which $H^0(X_C, \mathcal{F}|_{X_C}) \neq 0$ is closed in S. But note that this is precisely the image of

$$(\mathcal{BC}(\mathcal{F}) \setminus \{0\})/\underline{E^\times} \to S.$$

As this map is proper by Proposition II.2.16 (ii), its image is closed. To see that the endpoint of the Harder–Narasimhan polygon is locally constant, apply the preceding also to the dual of the determinant of $\mathcal{E}$.

For part (ii), it is enough to prove that v-locally on S, there exists an isomorphism $\mathcal{E} \cong \bigoplus_\lambda \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$. Indeed, the desired global Harder–Narasimhan filtration will then exist v-locally, and it necessarily descends. The trivialization of each $\mathcal{E}^\lambda$ amounts to a torsor under some locally profinite group, and can thus be done after a pro-étale cover by [**Sch17a**, Lemma 10.13]. Then the ability to split the filtration follows from Proposition II.2.5 (i).

We argue by induction on the rank of $\mathcal{E}$. Let λ be the maximal slope of $\mathcal{E}$. We claim that v-locally on S, there is a map $\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$ that is nonzero in each fibre. Indeed, finding such a map is equivalent to finding a fibrewise nonzero map $\mathcal{O}_{X_S} \to \mathcal{F} = \mathscr{H}\mathrm{om}(\mathcal{O}_{X_S}(\lambda), \mathcal{E})$. But then

$$\mathcal{BC}(\mathcal{F}) \setminus \{0\} \to (\mathcal{BC}(\mathcal{F}) \setminus \{0\})/\underline{E^\times} \to S$$

is a v-cover over which such a map exists: The first map is an $\underline{E^\times}$-torsor and thus a v-cover, while the second map is proper and surjective on geometric points, thus surjective by [**Sch17a**, Lemma 12.11]. The dual map $\mathcal{E}^\vee \to \mathcal{O}_{X_S}(-\lambda)$ is surjective as can be checked over geometric points (using that $\mathcal{O}_{X_C}(-\lambda)$ is stable), thus the cokernel of $\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$ is a vector bundle $\mathcal{E}'$, that again has constant Harder–Narasimhan polygon. By induction, one can find an isomorphism

$$\mathcal{E}' \cong \bigoplus_{\lambda' \leq \lambda} \mathcal{O}_{X_S}(\lambda')^{n'_{\lambda'}}.$$

By Proposition II.2.5 (i)–(ii), the extension

$$0 \to \mathcal{O}_{X_C}(\lambda) \to \mathcal{E} \to \bigoplus_{\lambda' \leq \lambda} \mathcal{O}_{X_S}(\lambda')^{n'_{\lambda'}} \to 0$$

can be split after a further pro-étale cover, finishing the proof. □

Let us explicitly note the following corollary.

Corollary II.2.20 ([**KL15**, Theorem 8.5.12]). *Let S be a perfectoid space. The category of pro-étale $\underline{E}$-local systems $\mathbb{L}$ is equivalent to the category of vector bundles on X_S whose Harder–Narasimhan polygon is constant* 0*, via $\mathbb{L} \mapsto \mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S}$.*

Proof. First, the functor is fully faithful, as we can see by pro-étale descent (to assume $\mathbb{L}$ is trivial) and Proposition II.2.5. Now essential surjectivity follows from Theorem II.2.19. □

II.3. Further results on Banach–Colmez spaces

We include some further results on Banach–Colmez spaces.

II.3.1. Cohomology of families of vector bundles. First, we generalize the vanishing results of Proposition II.2.5 to families of vector bundles. A key tool is given by the following result, which is a small strengthening of [**KL15**, Lemma 8.8.13].

Proposition II.3.1. *Let S be a perfectoid space over $\mathbb{F}_q$, and let $\mathcal{E}$ be a vector bundle on X_S such that all Harder–Narasimhan slopes of $\mathcal{E}$ at all geometric points are nonnegative. Then locally (in the analytic topology) on S, there is an exact sequence*

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{F} \to \mathcal{E} \to 0$$

where $\mathcal{F}$ is semistable of degree 0 *at all geometric points.*

Proof. We may assume that the degree of $\mathcal{E}$ is constant, given by some $d \geq 0$. We can assume $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid and pick d untilts $S_i^\sharp = \mathrm{Spa}(R_i^\sharp, R_i^{\sharp +})$ over E, $i = 1, \dots, d$, such that $S_1^\sharp, \dots, S_d^\sharp \subset X_S$ are pairwise disjoint; more precisely, choose d maps $S \to \mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}$. Let W_i be the fibre of $\mathcal{E}$ over $R_i^\sharp$, which is a finite projective $R_i^\sharp$-module. For any rank 1 quotients $W_i \to R_i^\sharp$, we can pull back the sequence

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{O}_{X_S}^d \to \bigoplus_{i=1}^d \mathcal{O}_{S_i^\sharp} \to 0,$$

obtained from Proposition II.2.3, along

$$\mathcal{E} \to \bigoplus_{i=1}^d \mathcal{E} \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{S_i^\sharp} = \bigoplus_{i=1}^d W_i \otimes_{R_i^\sharp} \mathcal{O}_{S_i^\sharp} \to \bigoplus_{i=1}^d \mathcal{O}_{S_i^\sharp}^d$$

to get an extension

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{E}' \to \mathcal{E} \to 0.$$

We claim that one can choose, locally on S, the rank 1 quotients so that $\mathcal{E}'$ is semistable of degree 0. For this, we argue by induction on $i = 1, \dots, d$ that one can choose (locally on S) the quotients of $W_1, \dots, W_i$ so that the modification

$$0 \to \mathcal{O}_{X_S}(-1)^i \to \mathcal{E}_i \to \mathcal{E} \to 0,$$

invoking only the quotients of $W_1, \dots, W_i$, has the property that the Harder–Narasimhan slopes of $\mathcal{E}_i$ at all geometric points are nonnegative. It is in fact enough to handle the case $i = 1$, as the inductive step then reduces to the similar assertion for the bundle $\mathcal{E}_1$ which is of smaller degree.

First, we handle the case that $S = \mathrm{Spa}(K, K^+)$ for a perfectoid field K (not necessarily algebraically closed). Then $\mathcal{E}$ has a Harder–Narasimhan filtration, and look at the subbundle $\mathcal{E}^{\geq \lambda} \subset \mathcal{E}$ of maximal slope, where necessarily $\lambda > 0$. Also recall that any nonsplit extension

$$0 \to \mathcal{O}_{X_S}(-1) \to \mathcal{G} \to \mathcal{E}^{\geq \lambda} \to 0$$

necessarily has nonnegative Harder–Narasimhan slopes (by looking at the Harder–Narasimhan polygon, which has to lie strictly above the -1-line). Thus, it is enough to ensure that the pullback of the extension to $\mathcal{E}^{\geq \lambda}$ is nonsplit. But if it splits, then the given map $\mathcal{E}^{\geq \lambda} \to \mathcal{O}_{S_1^\sharp}$ lifts to a map $\mathcal{E}^{\geq \lambda} \to \mathcal{O}_{X_S}$; by consideration of slopes, this map is necessarily trivial. Thus, if we let $W_1' \subset W_1$ be the fibre of $\mathcal{E}^{\geq \lambda} \subset \mathcal{E}$ at $S_1^\sharp$, it suffices to pick a quotient $W_1 \to K_1^\sharp$ whose restriction to W_1' is nonzero.

Going back to general affinoid S, pick any point $s \in S$. By the preceding argument, we can locally on S find a quotient $W_1 \to R_1^\sharp$ such that the corresponding extension $\mathcal{E}_1$ has the property that the Harder–Narasimhan slopes at s are still nonnegative. By Theorem II.2.19, the same is true in an open neighborhood, finishing the proof. □

In applications, it is often more useful to have the following variant, switching which of the two bundles is trivialized, at the expense of assuming strictly positive slopes (and allowing étale localizations in place of analytic localizations — this is probably unnecessary).

Proposition II.3.2. *Let S be a perfectoid space over $\mathbb{F}_q$ and let $\mathcal{E}$ be a vector bundle on X_S such that at all geometric points of S, all Harder–Narasimhan slopes of $\mathcal{E}$ are positive. Then étale locally on S, there is a short exact sequence*

$$0 \to \mathcal{G} \to \mathcal{O}_{X_S}^m \to \mathcal{E} \to 0$$

where $\mathcal{G}$ is semistable of slope -1 at all geometric points.

Proof. We can assume that $\mathcal{E}$ has constant degree d and rank r; we set $m = d + r$. Inside $\mathcal{BC}(\mathcal{E})^m$, we can look at the locus $U \subset \mathcal{BC}(\mathcal{E})^m$ of those maps $\mathcal{O}_{X_S}^m \to \mathcal{E}$ that are surjective and whose kernel is semistable of slope -1. This is an open subdiamond of $\mathcal{BC}(\mathcal{E})^m$: This is clear for the condition of surjectivity (say, as the cokernel of the universal map $\mathcal{O}_{X_T}^n \to \mathcal{E}|_{X_T}$ is supported on a closed subset of X_T, whose image is then closed in T), and then the locus where the kernel is semistable of slope -1 is open by Theorem II.2.19. By Proposition II.3.1, we see moreover that all geometric fibres of $U \to S$ are nonempty. It thus suffices to prove that for any geometric point $T = \mathrm{Spa}(C, C^+) \to S$, given as a cofiltered inverse limit of étale maps $S_i = \mathrm{Spa}(R_i, R_i^+) \to S$, and any section $s \in \mathcal{BC}(\mathcal{E})(T)$, one can find a sequence of i's and sections $s_i \in \mathcal{BC}(\mathcal{E})(S_i)$ such that $s_i|_T \to s$ as $i \to \infty$. Indeed, applying this to $\mathcal{E}^m$ in place of $\mathcal{E}$ and some section of U over T, one of the s_i will then lie in $U(S_i)$, giving the desired short exact sequence.

To prove that one can approximate s, we argue in a way similar to the proof of Theorem II.2.6. To facilitate the estimates, it is useful to assume that all Harder–Narasimhan slopes of $\mathcal{E}$ at T are integral; this can always be achieved through pulling back $\mathcal{E}$ to a cover $f : X_{S,E'} = X_S \otimes_E E' \to X_S$ for some unramified extension $E'|E$ (as this pullback multiplies slopes by $[E' : E]$), noting that $\mathcal{E}$ is a direct factor of $f_* f^* \mathcal{E} = \mathcal{E} \otimes_E E'$. We analyze $\mathcal{E}$ in terms of its pullback to $Y_{S,[1,q]}$ and the isomorphism over $Y_{S,[q,q]} \cong Y_{S,[1,1]}$. Note that as $B_{C,[1,q]}$ is a principal ideal domain, the pullback of $\mathcal{E}$ to $Y_{C,[1,q]}$ is necessarily free, and by approximation we can already find a basis over some $Y_{S_i,[1,q]}$; replacing S by S_i we can then assume that the pullback of $\mathcal{E}$ to $Y_{S,[1,q]}$ is free. The descent datum is then given by $A^{-1}\varphi$ for some matrix $A \in \mathrm{GL}_n(B_{R,[1,1]})$. After pullback to T, by Theorem II.2.14 and the assumption of integral slopes, one can in fact choose a basis so that A is a diagonal matrix D with positive powers of π along the diagonal. Approximating this basis, we can assume that $A - D \in \pi^N B^+_{(R,R^+),[1,1]}$ for any chosen $N > 0$.

Now the map

$$\varphi - D : B^r_{R,[1,q]} \to B^r_{R,[1,1]}$$

is surjective by Proposition II.2.5, and in fact there is some M (depending only on D, not on R) such that for any $x \in (B^+_{(R,R^+),[1,1]})^r$, there is some $y \in \pi^{-M}(B^+_{(R,R^+),[1,q]})^r$ with $x = (\varphi - D)(y)$. There are two ways to see the existence of M: Either by an explicit reading of the proof of Proposition II.2.5, or as follows. Assume no such M exists; then we can find perfectoid algebras $R_0, R_1, \ldots$ with integral elements $R_i^+ \subset R_i$ and pseudouniformizers $\varpi_i \in R_i$, and sections $x_i \in (B^+_{(R_i,R_i^+),[1,1]})^r$ such that there is no $y_i \in \pi^{-2i}(B^+_{(R_i,R_i^+),[1,q]})^r$ with $x_i = (\varphi - D)(y_i)$. Let R^+ be the product of all R_i^+, and $R = R^+[\frac{1}{\varpi}]$ where $\varpi = (\varpi_i)_i \in R^+ = \prod_i R_i^+$. Then all x_i define elements of $(B^+_{(R,R^+),[1,1]})^r$, and $x = x_0 + \pi x_1 + \pi^2 x_2 + \ldots$ another element. By surjectivity of $\varphi - D$, there is some $y \in (B_{(R,R^+),[1,q]})^r$ with $x = (\varphi - D)(y)$. But then $y \in \pi^{-i}(B^+_{(R,R^+),[1,q]})^r$ for some i, and then projecting along $(R, R^+) \to (R_i, R_i^+)$ contradicts the choice of x_i.

Taking $N > M$ above, one sees that also

$$\varphi - A = \varphi - D + (D - A) : B^r_{R,[1,q]} \to B^r_{R,[1,1]}$$

is surjective, with the same bound (in particular independent of R). But now the section s of $\mathcal{E}$ over X_T can be approximated by a section s'_i of $B_{R_i,[1,q]}$, so that its image under $\varphi - A$ will be small. By the preceding surjectivity, we can then replace s'_i by $s_i = s'_i + \epsilon_i$ for some still small ϵ_i such that

$$s_i \in B^{\varphi=A}_{R_i,[1,q]} = H^0(X_{S_i}, \mathcal{E}|_{X_{S_i}}).$$

This gives the desired conclusion. □

One can prove the following variants.

Corollary II.3.3. *Let S be a perfectoid space over $\mathbb{F}_q$ and let $\mathcal{E}$ be a vector bundle on X_S.*

(i) *Assume that all Harder–Narasimhan slopes of $\mathcal{E}$ are $\geq \frac{1}{r}$. Then locally on S, for some $m \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}^m_{X_S} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{F}$ is fibrewise semistable of slope $\frac{1}{r}$.

(ii) *Assume that all Harder–Narasimhan slopes of $\mathcal{E}$ are $\geq \frac{1}{r}$. Then locally on S, for some $m \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_{X_S}(\tfrac{1}{2r})^m \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{F}$ is fibrewise semistable of slope $\frac{1}{r}$.

(iii) *Assume that all Harder–Narasimhan slopes of $\mathcal{E}$ are $> \frac{1}{r}$. Then étale locally on S, for some $m \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{X_S}(\tfrac{1}{r})^m \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{G}$ is fibrewise semistable of slope 0.

(iv) *Assume that all Harder–Narasimhan slopes of $\mathcal{E}$ are $> \frac{1}{r}$. Then étale locally on S, for some $m \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{X_S}(\tfrac{1}{r})^m \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{G}$ is fibrewise semistable of slope $\frac{1}{2r}$.

Proof. We can suppose S is affinoid. We start with (i). Let $\pi_r : X_{S,r} = Y_S/\varphi^{r\mathbb{Z}} \to X_S$ be the finite étale cover $X_{S,E_r} = X_{S,E} \otimes_E E_r \to X_{S,E} = X_S$, where E_r is the unramified extension of degree r of E. We apply Proposition II.3.1 to $(\pi_r^*\mathcal{E})(-1)$. We get, locally on S, a short exact sequence

$$0 \to \mathcal{O}^{m'}_{X_{S,E_r}} \to \mathcal{F}' \to \pi_r^*\mathcal{E} \to 0$$

where $\mathcal{F}'$ is fiberwise semistable of slope 1. Thus, applying π_{r*}, we get a short exact sequence

$$0 \to \mathcal{O}^{m'r}_{X_S} \to \pi_{r*}\mathcal{F}' \to \pi_{r*}\pi_r^*\mathcal{E} \to 0.$$

Here $\pi_{r*}\mathcal{F}'$ is fiberwise semistable of slope $\frac{1}{r}$. As $\mathcal{E}$ is a direct summand of $\pi_{r*}\pi_r^*\mathcal{E} = \mathcal{E} \otimes_E E_r$, we get via pullback a similar exact sequence.

Arguing similarly with $(\pi_{2r}^*\mathcal{E})(-2)$, we get part (ii) of the corollary. Invoking Proposition II.3.2 instead, we get parts (iii) and (iv). $\square$

PROPOSITION II.3.4. *Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, and let $\mathcal{E}$ be a vector bundle on X_S.*

(i) *If at all geometric points of S, all slopes of $\mathcal{E}$ are negative, then $H^0(X_S, \mathcal{E}) = 0$.*

(ii) *If at all geometric points of S, all slopes of $\mathcal{E}$ are nonnegative, then there is a pro-étale cover $\widetilde{S} \to S$ such that*

$$H^1(X_{\widetilde{S}}, \mathcal{E}|_{X_{\widetilde{S}}}) = 0.$$

(iii) *If at all geometric points of S, all slopes of $\mathcal{E}$ are positive, then there is an étale cover $S' \to S$ such that for any affinoid perfectoid T over S', one has $H^1(X_T, \mathcal{E}|_{X_T}) = 0$.*

PROOF. Part (i) can be checked on geometric points, where it follows from Theorem II.2.14 and Proposition II.2.5 (i). For part (ii), we use Proposition II.3.1 to produce locally on S an exact sequence

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{E}' \to \mathcal{E} \to 0$$

where $\mathcal{E}'$ is everywhere semistable of degree 0. By Theorem II.2.19 we can find a pro-étale cover of S over which $\mathcal{E}' \cong \mathcal{O}_{X_S}^r$. By the vanishing of H^2, this induces a surjection from $H^1(X_S, \mathcal{O}_{X_S})^r$ onto $H^1(X, \mathcal{E})$. Since $H^1(X_S, \mathcal{O}_{X_S}) = H^1_{\text{proét}}(S, \underline{E})$ by Proposition II.2.5 (ii), this vanishes pro-étale locally on S.

For part (iii), we use Corollary II.3.3 (iii) to produce an étale cover of S over which there is an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^m \to \mathcal{E} \to 0.$$

For any affinoid $T|S$, this induces a surjection from $H^1(X_T, \mathcal{O}_{X_T}(\frac{1}{r})^m)$ onto $H^1(X_T, \mathcal{E}|_{X_T})$, so we conclude by Proposition II.2.5 (iii). $\square$

II.3.2. Families of Banach–Colmez spaces. We can now prove the following strengthening of Proposition II.2.16.

PROPOSITION II.3.5. *Let S be a perfectoid space over $\mathbb{F}_q$. Let $[\mathcal{E}_1 \to \mathcal{E}_0]$ be a map of vector bundles on X_S such that at all geometric points of S, the bundle $\mathcal{E}_1$ has only negative Harder–Narasimhan slopes.*

(i) *The Banach–Colmez space*

$$\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) : T \mapsto \mathbb{H}^0(X_T, [\mathcal{E}_1 \to \mathcal{E}_0]|_{X_T})$$

is a locally spatial diamond, partially proper over S.

(ii) *The projectivized Banach–Colmez space*

$$(\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \setminus \{0\})/\underline{E^\times}$$

is a locally spatial diamond, proper over S.

(iii) *Assume that all Harder–Narasimhan slopes of $\mathcal{E}_0$ at all geometric points are positive. Then*

$$\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \to S$$

is cohomologically smooth.

Proof. All assertions are étale local (in fact v-local) on S. For parts (i) and (ii), let us first simplify the form of the complex $[\mathcal{E}_1 \to \mathcal{E}_0]$. By Theorem II.2.6, we can find (for S affinoid) some $d > 0$ and a surjection

$$\mathcal{O}_{X_S}(-d)^m \to \mathcal{E}_0.$$

Let $\mathcal{E}_1'$ be the kernel of $\mathcal{E}_1 \oplus \mathcal{O}_{X_S}(-d)^m \to \mathcal{E}_0$. Then we find a quasi-isomorphism

$$[\mathcal{E}_1' \to \mathcal{O}_{X_S}(-d)^m] \to [\mathcal{E}_1 \to \mathcal{E}_0].$$

Note also that $\mathcal{E}_1'$ still satisfies the assumption on negative slopes. We get an exact sequence

$$0 \to \mathcal{BC}([\mathcal{E}_1' \to \mathcal{O}_{X_S}(-d)^m]) \to \mathcal{BC}(\mathcal{E}_1'[1]) \to \mathcal{BC}(\mathcal{O}_{X_S}(-d)^m[1]).$$

As $\mathcal{BC}(\mathcal{O}_{X_S}(-d)^m[1])$ is separated by Proposition II.2.5 (i), we see that parts (i) and (ii) reduce to the case of $\mathcal{BC}(\mathcal{E}_1'[1])$. Now applying Corollary II.3.3 (iv) to the dual of $\mathcal{E}_1'$, we get (étale locally on S) an exact sequence

$$0 \to \mathcal{BC}(\mathcal{E}_1'[1]) \to \mathcal{BC}(\mathcal{O}_{X_S}(-\tfrac{1}{r})^m[1]) \to \mathcal{BC}(\mathcal{G}[1])$$

where $\mathcal{G}$ is semistable of slope $\frac{1}{2r}$ everywhere. In particular, $\mathcal{BC}(\mathcal{G}[1])$ is separated over S by pro-étale descent and Proposition II.2.5 (i). Thus, $\mathcal{BC}(\mathcal{E}_1'[1]) \subset \mathcal{BC}(\mathcal{O}_{X_S}(-\frac{1}{r})^m[1])$ is a closed subfunctor, finishing the proof of part (i) by applying Proposition II.2.5 (i) again. Part (ii) is then reduced to the similar assertion for $\mathcal{BC}(\mathcal{O}_{X_S}(-\frac{1}{r})^m[1])$. Replacing E by its unramified extension of degree r, this reduces to $\mathcal{BC}(\mathcal{O}_{X_S}(-1)^m[1])$. Now, as in the proof of Proposition II.2.16, this follows from Lemma II.2.17, where one checks the required contracting property of multiplication by π by using the presentation

$$\mathcal{BC}(\mathcal{O}_{X_S}(-1)[1]) = (\mathbb{A}^1_{S^\sharp})^\diamond/\underline{E}$$

for an untilt $S^\sharp$ of S over E.

It remains to prove part (iii). Note that one has a short exact sequence

$$0 \to \mathcal{BC}(\mathcal{E}_0) \to \mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \to \mathcal{BC}(\mathcal{E}_1[1]) \to 0;$$

by [**Sch17a**, Proposition 23.13], we can thus handle $\mathcal{BC}(\mathcal{E}_1[1])$ and $\mathcal{BC}(\mathcal{E}_0)$ individually. For the case of $\mathcal{BC}(\mathcal{E}_0)$, we use Corollary II.3.3 (iv) to get, pro-étale locally on S, an exact sequence

$$0 \to \mathcal{O}_{X_S}(\tfrac{1}{2r})^{m'} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^m \to \mathcal{E}_0 \to 0,$$

inducing a similar sequence of Banach–Colmez spaces. Then the result follows from [**Sch17a**, Proposition 23.13]. For the case of $\mathcal{BC}(\mathcal{E}_1[1])$, choose a surjection $\mathcal{O}_{X_S}(-d)^m \to \mathcal{E}_1^\vee$ for some $d > 0$; we get an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{O}_{X_S}(d)^m \to \mathcal{F} \to 0$$

where necessarily all Harder–Narasimhan slopes of $\mathcal{F}$ are positive everywhere. This gives an exact sequence

$$0 \to \mathcal{BC}(\mathcal{O}_{X_S}(d)^m) \to \mathcal{BC}(\mathcal{F}) \to \mathcal{BC}(\mathcal{E}_1[1]) \to 0,$$

so the result follows from [**Sch17a**, Proposition 23.13] and the case of positive slopes already established. □

II.3.3. Punctured absolute Banach–Colmez spaces. Finally, we analyze punctured absolute Banach–Colmez spaces. Recall that, in the situation of Proposition II.2.5 (iv), one has

$$\mathcal{BC}(\mathcal{O}(d)) \cong \mathrm{Spd}(k[\![x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]\!]),$$

so the v-sheaf $\mathcal{BC}(\mathcal{O}(d))$ fails to be a perfectoid space, or even a diamond, as it contains the non-analytic point $\mathrm{Spd}\, k$. However, passing to the punctured Banach–Colmez space

$$\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\} \cong \mathrm{Spa}(k[\![x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]\!])_{\mathrm{an}}$$

identifies with the analytic points, which form a perfectoid space; in fact, a qcqs perfectoid space. These objects first showed up in [**Far18a**] in the case of positive slopes. It was remarked in [**Far18a**] that the punctured version $\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}$ is a diamond for all $d \geq 1$, that is moreover simply connected when $d > 2$. This plays a key role in [**Far18a**] since after base changing from $\mathrm{Spd}\, k$ to $\mathrm{Spa}\, C$ this is not simply connected anymore. In the above example,

$$\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\} = \mathrm{Spa}(k[\![x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]\!]) \setminus V(x_1, \cdots, x_d)$$

is a qcqs perfectoid space that is simply connected when $d > 1$. After base changing to $\mathrm{Spa}(C)$ this is a punctured n-dimensional open ball over $\mathrm{Spa}(C)$ that is not quasicompact anymore, and not simply connected. Thus, some new interesting phenomena appear when we consider absolute Banach–Colmez spaces.

Let us first continue the discussion with the case of $\mathcal{O}(d)$ for $d \geq 1$. In that case, there is a relation to Cartier divisors. Recall that any closed Cartier divisor $D \subset X_S$ is given by a line bundle $\mathcal{I}$ on X_S together with an injection $\mathcal{I} \hookrightarrow \mathcal{O}_{X_S}$ with closed image. We will only consider the case of relative Cartier divisors, so that this map stays injective after base change to any geometric point. Now Theorem II.2.19 implies that after replacing S by an open and closed cover, $\mathcal{I}$ is of degree $-d$ for some integer $d \geq 0$, and that there is an $\underline{E^\times}$-torsor of isomorphisms $\mathcal{I} \cong \mathcal{O}_{X_S}(-d)$. This shows that the v-sheaf Div sending any S to the closed relative Cartier divisors is given by

$$\mathrm{Div} = \bigsqcup_{d \geq 0} \mathrm{Div}^d, \ \mathrm{Div}^d \cong (\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\})/\underline{E^\times}.$$

Note that we are implicitly using a different definition of Div^1 here, but Corollary II.2.4 shows that they agree.

In particular, the moduli space Div^d of degree d Cartier divisors is given by the projectivized Banach–Colmez space for $\mathcal{O}(d)$. On the other hand, in terms of divisors we can see the following proposition. Recall that one can take sums of Cartier divisors (by tensoring their ideal sheaves).

Proposition II.3.6. *For any $d \geq 1$, the sum map*

$$(\mathrm{Div}^1)^d \to \mathrm{Div}^d : (D_1, \ldots, D_d) \mapsto D_1 + \ldots + D_d$$

is a quasi-pro-étale cover, identifying

$$\mathrm{Div}^d = (\mathrm{Div}^1)^d / \Sigma_d,$$

where Σ_d is the symmetric group. In particular, Div^d is a diamond.

Proof. By Proposition II.2.16 (ii), all occuring spaces are proper over $*$. In particular, the sum map is proper. To check surjectivity as v-sheaves, we can then check on geometric points, where it

follows from Proposition II.2.9 (in whose proof we checked that any element of P_d is a product of elements of P_1). In fact, we even get bijectivity up to the Σ_d-action, and thus the isomorphism

$$\mathrm{Div}^d = (\mathrm{Div}^1)^d/\Sigma_d$$

as v-sheaves. But the projection $(\mathrm{Div}^1)^d \to (\mathrm{Div}^1)^d/\Sigma_d$ is quasi-pro-étale by [**Sch17a**, Lemma 7.19, Definition 10.1 (i)]. As $\mathrm{Div}^1 = \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$ is a diamond, it follows that Div^d is a diamond by [**Sch17a**, Proposition 11.4, Proposition 11.6]. □

Now we can analyze the case of general absolute Banach–Colmez spaces.

Proposition II.3.7. *Let D be an isocrystal with only negative slopes (resp. with only positive slopes), and work on* Perf_k.

(i) *The punctured Banach–Colmez space $\mathcal{BC}(D) \setminus \{0\}$ (resp. $\mathcal{BC}(D[1]) \setminus \{0\}$) is a spatial diamond.*

(ii) *The quotient*

$$\big(\mathcal{BC}(D) \setminus \{0\}\big)/\underline{E^\times} \longrightarrow * \ \big(\text{resp. } \big(\mathcal{BC}(D[1]) \setminus \{0\}\big)/\underline{E^\times} \longrightarrow *\big)$$

is proper, representable in spatial diamonds, and cohomologically smooth.

Proof. Part (ii) follows from Proposition II.3.5 and (for the cohomological smoothness after taking the quotient by $\underline{E^\times}$) [**Sch17a**, Proposition 24.2].

For part (i), we are going to apply Lemma II.3.8, so we first want to see that $\mathcal{BC}(D) \setminus \{0\}$ is a spatial v-sheaf. By the Dieudonné–Manin classification, we can find a basis for D so that φ is E-rational and $U := \varphi^N$ is a diagonal matrix with entries powers of π for some $N > 0$; this essentially means that V is decent in the sense of [**RZ96**, Definition 1.8]. Then $\mathcal{BC}(D)$ (resp. $\mathcal{BC}(D[1])$) is already defined on $\mathrm{Perf}_{\mathbb{F}_q}$, and the action of U agrees with the action of Frob^N. Moreover, the action of U^{-1} (resp. U) on $|\,\mathcal{BC}(D) \times_{\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q(\!(t^{1/p^\infty})\!)|$ (resp. $|\,\mathcal{BC}(D[1]) \times_{\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q(\!(t^{1/p^\infty})\!)|$) still satisfies the hypotheses of Lemma II.2.17. This implies that

$$(\mathcal{BC}(D) \setminus \{0\})/\varphi^N \times_{\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q(\!(t^{1/p^\infty})\!)$$

is a spatial diamond, which can be translated into

$$(\mathcal{BC}(D) \setminus \{0\}) \times_{\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q(\!(t^{1/p^\infty})\!)/\varphi^N$$

being a spatial diamond, as the absolute Frobenius acts trivially on the topological space. But $\mathrm{Spa}\,\mathbb{F}_q(\!(t^{1/p^\infty})\!)/\varphi^N \to *$ is qcqs, even proper, and cohomologically smooth. We can thus apply point (i) of Lemma II.3.8 to conclude that $\mathcal{BC}(D) \setminus \{0\}$ (resp. $\mathcal{BC}(D[1]) \setminus \{0\}$, for which the same argument applies) is spatial.

It remains to see that it is a diamond. One easily reduces to the case that D is simple, and allowing ourselves to replace E by a finite unramified extension, to D of rank 1. The case of positive Banach–Colmez spaces now follows from Proposition II.3.6, as it is an $\underline{E^\times}$-torsor over a diamond (so [**Sch17a**, Proposition 11.7] applies). It remains to prove that this is a diamond in the case of a negative absolute Banach–Colmez spaces, i.e. for $D = (E, \pi^n\varphi)$ with $n > 0$. Then $\mathcal{D} := \mathcal{BC}(D[1]) \setminus \{0\}$ classifies extensions

$$0 \longrightarrow \mathcal{O}_{X_S}(-n) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X_S} \longrightarrow 0$$

that are geometrically fiberwise non split on S (remark that those extensions are rigid). We now apply point (ii) of Lemma II.3.8 using the Harder–Narasimhan stratification of $\mathcal{D}$ defined by $\mathcal{E}$. We can pass to the subsheaf of $\mathcal{D}$ where $\mathcal{E}$ is, at each geometric point, isomorphic to a given rank 2 bundle, necessarily of the form $\mathcal{O}_{X_S}(-n+i) \oplus \mathcal{O}_{X_S}(-i)$ for some $0 < i \leq \frac{n}{2}$ or to $\mathcal{O}_{X_S}(-\frac{n}{2})$.

On such a stratum $\mathcal{D}_\alpha \subset \mathcal{D}$ there is a global Harder–Narasimhan filtration by Theorem II.2.19, and trivializing the graded piece of lowest slope defines a pro-étale morphism $\widetilde{\mathcal{D}}_\alpha \to \mathcal{D}_\alpha$. For $S \to \widetilde{\mathcal{D}}_\alpha$ there is a morphism from $\mathcal{O}_{X_S}(-n)$ to this quotient of $\mathcal{E}$ by composing with the inclusion $\mathcal{O}_{X_S}(-n) \hookrightarrow \mathcal{E}$. Since the extension is non-split geometrically fiberwise on S, this morphism is non-zero geometrically fiberwise. This defines a morphism from $\widetilde{\mathcal{D}}_\alpha \to X$ to a punctured positive absolute Banach–Colmez space X, which is a diamond. Thus

$$\widetilde{\mathcal{D}}_\alpha \subset \widetilde{\mathcal{D}}_\alpha \times X$$

where the latter is a diamond as $\widetilde{\mathcal{D}}_\alpha \to *$ is representable in diamonds, so [**Sch17a**, Proposition 11.10] shows that $\widetilde{\mathcal{D}}_\alpha$ is a diamond. □

Lemma II.3.8. *Let $\mathcal{F}$ be a small v-sheaf.*

(i) *Suppose there exists a surjective qcqs cohomologically smooth morphism $D \to \mathcal{F}$ where D is a spatial diamond. Then $\mathcal{F}$ is a spatial v-sheaf.*

(ii) *Suppose moreover there is a family of locally closed generalizing subsets $(X_\alpha)_\alpha$, $X_\alpha \subset |\mathcal{F}|$, such that for each α the associated subsheaf of $\mathcal{F}$ is a diamond. Then $\mathcal{F}$ is a spatial diamond.*

Proof. For point (i), since D is qcqs and $D \to \mathcal{F}$ qcqs surjective, $\mathcal{F}$ is qcqs. Since cohomologically smooth implies universally open we can apply [**Sch17a**, Lemma 2.10] to conclude it is spatial. For point (ii) we apply [**Sch17a**, Theorem 12.18]. Let $\mathcal{G}_\alpha \subset \mathcal{F}$ be associated to X_α. From [**Sch17a**, Lemma 7.6] we deduce that $\mathcal{G}_\alpha \hookrightarrow \mathcal{F}$ is quasi-pro-étale. This implies the result. □

Remark II.3.9. The proof of Proposition II.3.7 for negative absolute Banach–Colmez space goes the same way as the proof of the fact that $\mathrm{Gr}_{\leq \mu}$ is a spatial diamond, [**SW20**, Theorem 19.2.4]. One first proves this is a spatial v-sheaf and then one stratifies it by locally closed generalizing subsets that are diamonds.

Remark II.3.10. One has to be careful that although the absolute $\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}$ is a spatial diamond, $(\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\})/\pi^{\mathbb{Z}}$ is not spatial anymore since not quasiseparated, [**Far18a**, Remarque 2.15]. In this context the good object is the morphism $(\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\})/\pi^{\mathbb{Z}} \to *$ that is representable in spatial diamonds.

Remark II.3.11. In the equal characteristic case, $E = \mathbb{F}_q(\!(\pi)\!)$, the structure of punctured positive absolute Banach–Colmez spaces is much simpler since they are perfectoid spaces. Nevertheless the structure of the punctured negative one is not, they are only spatial diamonds. We will see below that $\mathcal{BC}(\mathcal{O}(-1)[1])$ is stratified into the open part, which can be written as the quotient of $\mathrm{Spd}\, k(\!(t)\!)$ by the action of a profinite group, and a point $\mathrm{Spd}\, k$. However, the degeneration to the point happens at the boundary of the open unit disc as $|t| \to 1$, not as $|t| \to 0$ as in $\mathrm{Spa}\, k[\![t]\!]$. Thus $\mathcal{BC}(\mathcal{O}(-1)[1])$ is a rather strange geometric object.[5]

[5]This example was critical in convincing us to not try to develop a version of the theory of diamonds that would allow non-analytic test objects like $\mathrm{Spa}\, k[\![t^{1/p^\infty}]\!]$ and would thus make $\mathcal{BC}(\mathcal{O}(1))$ itself representable: After all, in the

Example II.3.12. The absolute $\mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\}$ classifies extensions

$$0 \longrightarrow \mathcal{O}_{X_S}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X_S} \to 0$$

that are non-split fiberwise on S. Any such extension is, at each geometric point, isomorphic to $\mathcal{O}_{X_S}(-\frac{1}{2})$. Parametrizing isomorphisms $\mathcal{E} \cong \mathcal{O}_{X_S}(-\frac{1}{2})$ defines a $\underline{D^\times}$-torsor, where D is the quaternion algebra over E; here we use Theorem II.2.19. Remark that if $0 \to \mathcal{O}_{X_S}(-1) \to \mathcal{O}_{X_S}(-\frac{1}{2}) \to \mathcal{L} \to 0$ is an extension, then taking the determinant automatically fixes an isomorphism $\mathcal{L} \cong \mathcal{O}_{X_S}$, and thus the $\underline{E^\times}$-torsor of isomorphisms between $\mathcal{O}_{X_S}$ and $\mathcal{L}$ is trivial. From this we deduce that

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\} \simeq (\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\})/\underline{\mathrm{SL}_1(D)}$$

with

$$\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\} \simeq \mathrm{Spa}(k(\!(x^{1/p^\infty})\!))$$

the punctured universal cover of a 1-dimensional formal π-divisible $\mathcal{O}_E$-module of height 2.

Let us compare this with our previous description of $\mathcal{BC}(\mathcal{O}(-1)[1])$ after pullback to $\mathrm{Spa}(C)$, fixing an untilt $C^\sharp$ over E and $t \in H^0(X_C, \mathcal{O}_{X_C}(1)) \setminus \{0\}$: the exact sequence

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{O}_{X_C} \to \mathcal{O}_{C^\sharp} \to 0$$

induces an isomorphism

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \times_k \mathrm{Spa}(C) \cong (\mathbb{A}^1_{C^\sharp})^\diamond/\underline{E}.$$

We thus have

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\} \times_k \mathrm{Spa}(C) \cong (\Omega_{C^\sharp})^\diamond/\underline{E}$$

where $\Omega = \mathbb{A}^1_E \setminus \underline{E}$ is Drinfeld's upper half plane over E.

We deduce an isomorphism

$$\big((\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\}) \times_k \mathrm{Spa}(C)\big)/\underline{\mathrm{SL}_1(D)} \cong (\Omega_{C^\sharp})^\diamond/\underline{E}.$$

This isomorphism is in fact deduced from the isomorphism between Lubin–Tate and Drinfeld towers. In fact, [**SW13**], the Lubin–Tate tower in infinite level, LT_∞ over $\mathrm{Spa}(C^\sharp)$, is the moduli of modifications $\mathcal{O}^2_{X_S} \hookrightarrow \mathcal{O}_{X_S}(\frac{1}{2})$ at the point of the curve defined by the untilt $C^\sharp$. From this one deduces a $\underline{D^\times}$-equivariant isomorphism

$$\mathrm{LT}^\flat_\infty \Big/ \begin{pmatrix} 1 & \underline{E} \\ 0 & \underline{E^\times} \end{pmatrix} \cong (\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\}) \times_k \mathrm{Spa}(C).$$

Dividing this isomorphism by $\underline{\mathrm{SL}_1(D)}$ one obtains the preceding isomorphism.

Example II.3.13. The absolute $\mathcal{BC}(\mathcal{O}(-2)[1]) \setminus \{0\}$ classifies extensions

$$0 \longrightarrow \mathcal{O}_{X_S}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X_S}(1) \longrightarrow 0$$

that are non-split fiberwise on S. There is only one Harder–Narasimhan stratum and geometrically fiberwise on S, $\mathcal{E}$ is a trivial vector bundle. The moduli of surjections $\mathcal{O}^2_{X_S} \twoheadrightarrow \mathcal{O}_{X_S}(1)$ is the open subset

$$U \subset (\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}) \times_k (\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})$$

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context of absolute Banach–Colmez spaces, $\mathcal{BC}(\mathcal{O}(1)) = \mathrm{Spd}\, k[\![t^{1/p^\infty}]\!]$ and $\mathcal{BC}(\mathcal{O}(-1)[1])$ play very similar roles, so the formalism should also treat them similarly.

equal to

$$U = (\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})^2 \setminus (\underline{E}^\times \times 1).\Delta$$

where Δ is the diagonal of $(\mathcal{BC}(\mathcal{O}(1))\setminus\{0\})^2$, that is to say couples (x, y) of sections of $H^0(X_S, \mathcal{O}_{X_S}(1))$ that are fiberwise/S non-zero and linearly independent over E. Here $(\underline{E}^\times \times 1).\Delta$ is a locally profinite union of copies of Δ.

Again, by consideration of determinants, $\ker(\mathcal{O}_{X_S}^2 \twoheadrightarrow \mathcal{O}_{X_S}(1))$ is canonically identified with $\mathcal{O}_{X_S}(-1)$. This implies that

$$\mathcal{BC}(\mathcal{O}(-2)[1]) \setminus \{0\} = U/\underline{\mathrm{SL}_2(E)}.$$

CHAPTER III

Bun_G

Throughout this chapter, we fix a reductive group G over the nonarchimedean local field E. As it will be important to study Bun_G over a geometric base point, we fix from now on a complete algebraically closed field k over $\mathbb{F}_q$ and work with perfectoid spaces S over $\mathrm{Spd}\, k$; write Perf_k for the category.

Definition III.0.1. *Let* Bun_G *be the prestack taking a perfectoid space* $S \in \mathrm{Perf}_k$ *to the groupoid of* G*-bundles on* X_S*.*

The main results of this chapter are summarized in the following theorem.

Theorem III.0.2 (Proposition III.1.3; Theorem III.2.2; Theorem III.2.3 and Theorem III.2.7; Theorem III.4.5; Proposition III.5.3). *The prestack* Bun_G *satisfies the following properties.*

(i) *The prestack* Bun_G *is a small v-stack.*

(ii) *The points* $|\mathrm{Bun}_G|$ *are naturally in bijection with Kottwitz' set* $B(G)$ *of* G*-isocrystals.*

(iii) *The map*

$$\nu : |\mathrm{Bun}_G| \to B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma$$

is semicontinuous, and

$$\kappa : |\mathrm{Bun}_G| \to B(G) \to \pi_1(G_{\overline{E}})_\Gamma$$

is locally constant. Equivalently, the map $|\mathrm{Bun}_G| \to B(G)$ *is continuous when* $B(G)$ *is equipped with the order topology.*

(iv) *The semistable locus* $\mathrm{Bun}_G^{\mathrm{ss}} \subset \mathrm{Bun}_G$ *is open, and given by*

$$\mathrm{Bun}_G^{\mathrm{ss}} \cong \bigsqcup_{b \in B(G)_{\mathrm{basic}}} [*/\underline{G_b(E)}].$$

(v) *For any* $b \in B(G)$*, the corresponding subfunctor*

$$i^b : \mathrm{Bun}_G^b = \mathrm{Bun}_G \times_{|\mathrm{Bun}_G|} \{b\} \subset \mathrm{Bun}_G$$

is locally closed, and isomorphic to $[*/\widetilde{G}_b]$*, where* $\widetilde{G}_b$ *is a v-sheaf of groups such that* $\widetilde{G}_b \to *$ *is representable in locally spatial diamonds with* $\pi_0 \widetilde{G}_b = G_b(E)$*. The connected component* $\widetilde{G}_b^\circ \subset \widetilde{G}_b$ *of the identity is cohomologically smooth of dimension* $\langle 2\rho, \nu_b \rangle$*.*

The hardest part of this theorem is that κ is locally constant. We give two proofs of this fact. If the derived group of G is simply connected, one can reduce to tori, which are not hard to handle. In general, one approach is to use z-extensions $\widetilde{G} \to G$ to reduce to the case of simply connected derived group. For this, one needs that $\mathrm{Bun}_{\widetilde{G}} \to \mathrm{Bun}_G$ is a surjective map of v-stacks;

we prove this using Beauville–Laszlo uniformization. Alternatively, at least for p-adic E, one can use the abelianized Kottwitz set of Borovoi [**Bor98**], which we prove to behave well relatively over a perfectoid space S.

III.1. Generalities

There is a good notion of G-torsors in p-adic geometry:

Definition/Proposition III.1.1 ([**SW20**, Proposition 19.5.1][1]). *Let X be a sousperfectoid space over E. The following categories are naturally equivalent.*

(i) *The category of adic spaces $T \to X$ with a G-action such that étale locally on X, there is a G-equivariant isomorphism $T \cong G \times X$.*

(ii) *The category of étale sheaves $\mathcal{Q}$ on X equipped with an action of G such that étale locally, $\mathcal{Q} \cong G$.*

(iii) *The category of exact $\otimes$-functors*

$$\mathrm{Rep}_E G \to \mathrm{Bun}(X)$$

to the category of vector bundles on X.

A G-bundle on X is an exact $\otimes$-functor

$$\mathrm{Rep}_E G \to \mathrm{Bun}(X);$$

by the preceding, it can equivalently be considered in a geometric or cohomological manner.

In particular, G-torsors up to isomorphism are classified by $H^1_{\text{ét}}(X, G)$. By Proposition II.2.1, the following defines a v-stack.

Definition III.1.2. *Let Bun_G be the v-stack taking a perfectoid space $S \in \mathrm{Perf}_k$ to the groupoid of G-bundles on X_S.*

Our goal in this chapter is to analyze this v-stack. Before going on, let us quickly observe that it is small, i.e there are perfectoid spaces S, R with a v-surjection $S \to \mathrm{Bun}_G$ and a v-surjection $R \to S \times_{\mathrm{Bun}_G} S$.

Proposition III.1.3. *The v-stack Bun_G is small.*

Proof. It is enough to prove that if $S_i = \mathrm{Spa}(R_i, R_i^+)$, $i \in I$, is an ω_1-cofiltered inverse system of affinoid perfectoid spaces with inverse limit $S = \mathrm{Spa}(R, R^+)$, then

$$\mathrm{Bun}_G(S) = \varinjlim \mathrm{Bun}_G(S_i).$$

Indeed, then any section of Bun_G over an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ factors over $S' = \mathrm{Spa}(R', R'^+)$ for some topologically countably generated perfectoid algebra R'. But there is only a set worth of such R' up to isomorphism, and then taking the disjoint union $T = \bigsqcup_{S', \alpha \in \mathrm{Bun}_G(S')} S'$ gives a perfectoid space that surjects onto Bun_G. Moreover, the equivalence relation $T \times_{\mathrm{Bun}_G} T$ satisfies the same limit property, and hence also admits a similar surjection.

[1] The reference applies in the case of $\mathbb{Z}_p$, but it extends verbatim to $\mathcal{O}_E$.

To see the claim, note first that $R = \varinjlim R_i$ as any Cauchy sequence already lies in some R_i. The same applies to $B_{R,I}$ for any interval I, and hence one sees that

$$\mathrm{Bun}(X_S) = \varinjlim \mathrm{Bun}(X_{S_i}).$$

Now the definition of G-torsors gives the claim. □

This proof uses virtually no knowledge about Bun_G and shows that any reasonable v-stack is small.

III.2. The topological space $|\mathrm{Bun}_G|$

III.2.1. Points. As a first step, we recall the classification of G-bundles on the Fargues–Fontaine curve over geometric points. This is based on the following definition of Kottwitz, [**Kot85**].

Definition III.2.1. *A G-isocrystal is an exact $\otimes$-functor*

$$\mathrm{Rep}_E\, G \to \mathrm{Isoc}_E\,.$$

The set of isomorphism classes of G-isocrystals is denoted $B(G)$.

By Steinberg's theorem, the underlying fibre functor to $\breve{E}$-vector spaces is isomorphic to the standard fibre functor; this shows one can identify $B(G)$ with the quotient of $G(\breve{E})$ under σ-conjugation.

Composing with the exact $\otimes$-functor

$$\mathrm{Isoc}_E \to \mathrm{Bun}(X_S) : D \mapsto \mathcal{E}(D)$$

any G-isocrystal defines a G-bundle on X_S, for any $S \in \mathrm{Perf}_k$.

Theorem III.2.2 ([**Far18b**], [**Ans19**]). *For any complete algebraically closed nonarchimedean field C over k, the construction above defines a bijection*

$$B(G) \to \mathrm{Bun}_G(C)/\cong\,.$$

Proof. For the convenience of the reader, and as some of the constructions will resurface later, we give a sketch of the proof in [**Ans19**]. Any G-bundle on X_C has its Harder–Narasimhan filtration, and the formation of the Harder–Narasimhan filtration is compatible with tensor products. This implies that any exact $\otimes$-functor $\mathrm{Rep}_E\, G \to \mathrm{Bun}(X_C)$ lifts canonically to an exact $\otimes$-functor $\mathrm{Rep}_E\, G \to \mathrm{FilBun}(X_C)$ to $\mathbb{Q}$-filtered vector bundles. To check exactness, note that if E is p-adic, the category $\mathrm{Rep}_E\, G$ is semisimple and thus exactness reduces to additivity, which is clear. If E is of equal characteristic, one needs to argue more carefully, and we refer to the proof of [**Ans19**, Theorem 3.11].

We can now project $\mathrm{Rep}_E\, G \to \mathrm{FilBun}(X_C)$ to the category $\mathrm{GrBun}(X_C)$ of $\mathbb{Q}$-graded vector bundles, and note that the essential image of this functor is landing in the category of bundles $\bigoplus_\lambda \mathcal{E}^\lambda$ such that each $\mathcal{E}^\lambda$ is semistable of slope λ. This category is in fact equivalent to Isoc_E by Theorem II.2.14 and Proposition II.2.5 (ii). Thus, it suffices to see that the filtration on the exact $\otimes$-functor $\mathrm{Rep}_E\, G \to \mathrm{FilBun}(X_C)$ can be split.

Looking at splittings locally on X_C, they exist, and form a torsor under a unipotent group scheme U over X_C, where U is parametrizing automorphisms of the filtered fibre functor that are

trivial on the graded pieces. One can then filter U by vector bundles of positive slopes, and using the vanishing of their H^1, we get the desired splitting. $\square$

In particular, using [**Sch17a**, Proposition 12.7] it follows that the map

$$B(G) \longrightarrow |\mathrm{Bun}_G|$$

is a bijection.

III.2.2. Harder–Narasimhan stratification. Now we need to recall Kottwitz's description of $B(G)$. This relies on two invariants, *the Newton point and the Kottwitz point.* Let $\overline{E}$ be a separable closure of E and fix a maximal torus inside a Borel subgroup $T \subset B \subset G_{\overline{E}}$; the set of dominant cocharacters $X_*(T)^+$ is naturally independent of the choice of T and B, and acquires an action of $\Gamma = \mathrm{Gal}(\overline{E}|E)$ via its identification with

$$\mathrm{Hom}(\mathbb{G}_{m\overline{E}}, G_{\overline{E}})/G(\overline{E})\text{−conjugacy}.$$

The Newton point is a map

$$\begin{aligned} \nu : B(G) &\longrightarrow (X_*(T)^+_{\mathbb{Q}})^\Gamma \\ b &\longmapsto \nu_b. \end{aligned}$$

When $G = \mathrm{GL}_n$, then $X_*(T) \cong \mathbb{Z}^n$ and the target is the set of nonincreasing sequences of rational numbers, which are the slopes of the Newton polygon of the corresponding isocrystal. The Kottwitz point is a map

$$\begin{aligned} \kappa : B(G) &\longrightarrow \pi_1(G)_\Gamma \\ b &\longmapsto \kappa(b), \end{aligned}$$

where $\pi_1(G) := \pi_1(G_{\overline{\mathbb{Q}}_p}) = X_*(T)/(\text{coroot lattice})$ is the Borovoi fundamental group. For $G = \mathrm{GL}_n$, this is naturally isomorphic to $\mathbb{Z}$, and in this case $\kappa(b)$ is the endpoint of the Newton polygon. In general, this compatibility is expressed by saying that the images of $\kappa(b)$ and ν_b in

$$\pi_1(G)^\Gamma_{\mathbb{Q}}$$

agree (using an averaging operation for $\kappa(b)$). However, this means that in general $\kappa(b)$ is not determined by ν_b, as $\pi_1(G_E)_\Gamma$ may contain torsion.

The definition of κ is done in steps. First, one defines it for tori, where it is actually a bijection. Then one defines it for G whose derived group is simply connected; in that case, it is simply done via passage to the torus G/G_{der} which does not change π_1. In general, one uses a z-extension $\widetilde{G} \to G$ such that $\widetilde{G}$ has simply connected derived group, observing that $B(\widetilde{G}) \to B(G)$ is surjective.

Borovoi, [**Bor98**], gave a more canonical construction of κ as an abelianization map that does not use the choice of a z-extension, at least in the case of p-adic E. We will recall the construction in Section III.2.4.2.

Finally, recall that $(\nu, \kappa) : B(G) \to (X^*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma$ is injective.

Let $v \mapsto v^* = w_0(-v)$ be the involution of the positive Weyl chamber $X_*(T)^+_{\mathbb{Q}}$ where w_0 is the longest element of the Weyl group. The Harder–Narasimhan polygon of $\mathcal{E}_b$ is ν_b^*. Its first Chern class is $-\kappa(b)$.

We need to understand how ν and κ vary on $B(G)$. The following result follows from Theorem II.2.19 and [**RR96**, Lemma 2.2].

THEOREM III.2.3 ([**SW20**, Corollary 22.5.1]). *The map*

$$\nu^* : |\mathrm{Bun}_G| \cong B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma$$

is upper semicontinuous.

We will later prove in Theorem III.2.7 that κ is locally constant on Bun_G.

III.2.3. Geometrically fiberwise trivial G-bundles. Let

$$[*/\underline{G(E)}]$$

be the classifying stack of pro-étale $\underline{G(E)}$-torsors, and

$$\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$$

be the substack of geometrically fiberwise trivial G-bundles. One has $H^0(X_S, \mathcal{O}_{X_S}) = \underline{E}(S)$ and thus $\underline{G(E)}$ acts on the trivial G-bundle. From this we deduce a morphism

$$[*/\underline{G(E)}] \longrightarrow \mathrm{Bun}_G^1 .$$

We are going to prove that this is an isomorphism. Let us note that, although this is an isomorphism at the level of geometric points, we can not apply [**SW20**, Lemma 12.5] since it is not clear that it is qcqs.

Theorems III.2.3 and III.2.7 (to follow) taken together imply that the locus

$$\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$$

is an open substack. One of our proofs of Theorem III.2.7 will however require this statement as an input. Of course, when $\pi_1(G)_\Gamma$ is torsion free, that is to say $H^1(E, G) = \{1\}$, Theorem III.2.3 is enough to obtain the openness.

THEOREM III.2.4. *The substack* $\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$ *is open, and the map*

$$[*/\underline{G(E)}] \xrightarrow{\sim} \mathrm{Bun}_G^1$$

defined above is an isomorphism.

PROOF. Let $S \in \mathrm{Perf}_k$ be qcqs with a map to Bun_G. We need to see that the subset of $|S|$ over which this map is trivial at any geometric point is open; and that if this is all of S, then the data is equivalent to a pro-étale $\underline{G(E)}$-torsor.

Let us check the openness assertion. If $T \to S$ is surjective with T qcqs then $|T| \to |S|$ is a quotient map. We can thus assume that S is strictly totally disconnected. The locus where the Newton point is identically zero is an open subset of S by Theorem III.2.3, so passing to this open subset, we can assume that the Newton point is zero. In that case, for any algebraic representation $\rho : G \to \mathrm{GL}_n$, the corresponding rank n vector bundle on X_S is trivial. Now, geometrically fiberwise on S trivial vector bundles on X_S are equivalent to $\underline{E}$-local systems on S by Corollary II.2.20. On the other hand, as S is strictly totally disconnected, all $\underline{E}$-local systems on S

are trivial (Lemma III.2.6 for $G = \mathrm{GL}_n$), and their category is equivalent to the category of finite free modules over $\mathcal{C}^0(|S|, E)$. Thus, the preceding discussion defines a fibre functor on $\mathrm{Rep}_E(G)$ with values in $\mathcal{C}^0(|S|, E) = \mathcal{C}^0(\pi_0(S), E)$, i.e. a G-torsor over $\mathrm{Spec}(\mathcal{C}^0(\pi_0(S), E))$. Note that for all $s \in \pi_0(S)$, the local ring $\varinjlim_{U \ni s} \mathcal{C}^0(|U|, E)$ is henselian as the local ring of the analytic adic space $\underline{\pi_0(S)} \times \mathrm{Spa}(E)$ at s. This implies that if this G-torsor is trivial at some point of S, then it is trivial in a neighborhood. This concludes the openness assertion.

Moreover, the preceding argument shows that the map $* \to \mathrm{Bun}_G^1$ is a pro-étale cover. As $* \times_{\mathrm{Bun}_G^1} * = \underline{G(E)}$, as automorphisms of the trivial G-torsor are given by $\underline{G(E)}$, we thus get the desired isomorphism. □

Remark III.2.5 ([**Sch17a**, Lemma 10.13]). If S is a perfectoid space, and $T \to S$ is a pro-étale $\underline{G(E)}$-torsor then T is representable by a perfectoid space. In fact, $T = \varprojlim_K \underline{K}\backslash T$ where K goes through the set of compact open subgroups of $G(E)$. By descent of étale separated morphisms ([**Sch17a**, Proposition 9.7]), for each such K, $\underline{K}\backslash T$ is represented by a separated étale perfectoid space over S. The transition morphisms in the preceding limit are finite étale.

Lemma III.2.6. *Let S be a strictly totally disconnected perfectoid space. Then any pro-étale $\underline{G(E)}$-torsor on S is trivial.*

Proof. Let $T \to S$ be such a torsor. Fix a compact open subgroup $K \subset G(E)$. Since $\underline{K}\backslash T \to S$ is an étale cover of perfectoid spaces it has a section and we can assume $T \to S$ is in fact a $\underline{K}$-torsor. Now, $T = \varprojlim_U \underline{U}\backslash T$ where U goes through the set of distinguished open subgroups of K. Each $\underline{U}\backslash T \to S$ is an étale $\underline{K/U}$-torsor and is trivial. One concludes using that if $U' \subset U$ then $\underline{K/U'}(S) \to \underline{K/U}(S)$ is surjective. □

III.2.4. Local constancy of the Kottwitz invariant. A central result is the following.

Theorem III.2.7. *The map*

$$\kappa : |\mathrm{Bun}_G| \cong B(G) \to \pi_1(G)_\Gamma$$

is locally constant.

Let us note the following corollary. We give a new proof (and slight strengthening) of a result of Rapoport–Richartz (when $p \mid |\pi_1(G)|$ the original proof used p-adic nearby cycles and relied on a finite type hypothesis).

Corollary III.2.8 ([**RR96**, Corollary 3.11]). *Let S be an $\mathbb{F}_q$-scheme and $\mathcal{E}$ an G-isocrystal on S. The map $|S| \to \pi_1(G)_\Gamma$ that sends a geometric point $\bar{s} \to S$ to $\kappa(\mathcal{E}_{\bar{s}})$ is locally constant.*

Proof. We can suppose $S = \mathrm{Spec}(R)$ is affine and defined over k. We get a small v-sheaf $\mathrm{Spd}(R, R)$, and $\mathcal{E}$ defines a morphism $\mathrm{Spd}(R, R) \to \mathrm{Bun}_G$. The induced map $\kappa : |\mathrm{Spd}(R, R)| \to |\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$ is locally constant by Theorem III.2.7. As open and closed subsets of $\mathrm{Spd}(R, R)$ are in bijection with open and closed subschemes of $\mathrm{Spec}(R)$ (by [**SW20**, Proposition 18.3.1] applied to morphisms to $* \sqcup *$), we can thus assume that $\kappa : |\mathrm{Spd}(R, R)| \to |\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$ is constant. But now for any geometric point $\bar{s} \to S$, the element $\kappa(\mathcal{E}_{\bar{s}}) \in \pi_1(G)_\Gamma$ agrees with the image of $* = |\mathrm{Spd}(\bar{s}, \bar{s})| \to |\mathrm{Spd}(R, R)| \to |\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$, giving the desired result. □

Remark III.2.9. There is a natural map $|\operatorname{Spd}(R,R)| \to |\operatorname{Spa}(R,R)|$, the latter of which admits two natural maps to $|\operatorname{Spec}(R)|$ (given by the support of the valuation, or the prime ideal of all elements of norm < 1). However, the maps κ defined on $|\operatorname{Spd}(R,R)|$ and $|\operatorname{Spec}(R)|$ do not make this diagram commute. Still, there is also the map $|\operatorname{Spec}(R)| \to |\operatorname{Spd}(R,R)|$, used in the proof, and this is continuous, and commutes with the κ maps.

We give two different proofs of Theorem III.2.7.

III.2.4.1. *First proof.* For the first proof of Theorem III.2.7, we also need the following lemma that we will prove in the next section.

Lemma III.2.10. *Let $\widetilde{G} \to G$ be a central extension with kernel a torus. Then*

$$\mathrm{Bun}_{\widetilde{G}} \to \mathrm{Bun}_G$$

is a surjective map of v-stacks.

In fact, up to correctly interpreting all the relevant structure, if $Z \subset \widetilde{G}$ is the kernel, then Bun_Z is a Picard stack (as for commutative Z one can tensor Z-bundles) which acts on $\mathrm{Bun}_{\widetilde{G}}$, and Bun_G is the quotient stack. It is in fact clear that it is a quasitorsor, and the lemma ensures surjectivity.

First Proof of Theorem III.2.7. Picking a z-extension, we can by Lemma III.2.10 reduce to the case that G has simply connected derived group. Then we may replace G by G/G_{der}, and so reduce to the case that G is a torus. By a further application of Lemma III.2.10, we can reduce to the case that G is an induced torus. In that case $\pi_1(G)_\Gamma$ is torsion-free, and so the Kottwitz map is determined by the Newton map, so the result follows from Theorem III.2.3, noting that in the case of tori there are no nontrivial order relations so semicontinuity means local constancy. $\square$

It remains to prove Lemma III.2.10. This will be done in the next section, using Beauville–Laszlo uniformization.

III.2.4.2. *Second proof.* For this proof, we assume that E is p-adic (otherwise certain non-étale finite flat group schemes may appear). We define

$$B_{ab}(G) = H^1(W_E, [G_{sc}(\overline{\breve{E}}) \to G(\overline{\breve{E}})]),$$

the abelianized Kottwitz set (cohomology with coefficient in a crossed module, see [**Bor98**] and [**Lab99**, Appendix B]). There is an abelianization map

$$B(G) \longrightarrow B_{ab}(G)$$

deduced from the morphism $[1 \to G] \to [G_{sc} \to G]$. If T is a maximal torus in G with reciprocal image T_{sc} in G_{sc} then

$$[T_{sc} \to T] \longrightarrow [G_{sc} \to G]$$

is a homotopy equivalence. If Z, resp. Z_{sc}, is the center of G, resp. G_{sc}, there is a homotopy equivalence

$$[Z_{sc} \to Z] \longrightarrow [G_{sc} \to G].$$

Lemma III.2.11. *There is an identification*

$$B_{ab}(G) = \pi_1(G)_\Gamma$$

through which Kottwitz map κ is identified with the abelianization map $B(G) \to B_{ab}(G)$.

Proof. Choose a maximal torus T in G. One has

$$\begin{aligned} B_{ab}(G) &= H^1(W_E, [T_{sc}(\overline{\breve{E}}) \to T(\overline{\breve{E}})]) \\ &= \mathrm{coker}\big(B(T_{sc}) \to B(T)\big) \end{aligned}$$

since $H^2(W_E, T_{sc}(\overline{\breve{E}})) = 0$ (use [**Ser94**, Chapter II.3.3 example (c)] and [**Ser94**, Chapter III.2.3 Theorem 1']). The result is deduced using Kottwitz description of $B(T_{sc})$ and $B(T) = X_*(T)_\Gamma$. $\square$

For $S \in \mathrm{Perf}_k$ there is a morphism of sites

$$\tau : (X_S)_{\text{ét}} \longrightarrow S_{\text{ét}}$$

deduced from the identifications

$$(X_S)_{\text{ét}} = (X_S^\diamond)_{\text{ét}} = (\mathrm{Div}^1_S)_{\text{ét}}$$

and the projection $\mathrm{Div}^1_S \to S$. Equivalently, τ^* takes any étale $T \to S$ to $X_T \to X_S$, which is again étale.

We now interpret some étale cohomology groups of the curve as Galois cohomology groups, as in [**Far18b**] where this type of computation was done for the schematical curve attached to an algebraically closed perfectoid field.

Proposition III.2.12. *Let $S \in \mathrm{Perf}_k$.*

(i) *Let $\mathcal{F}$ be a locally constant sheaf of finite abelian groups on $\mathrm{Spa}(E)_{\text{ét}}$. One has*

$$R\tau_*\mathcal{F}_{|X_S} = \underline{R\Gamma_{\text{ét}}(\mathrm{Spa}(E), \mathcal{F})}$$

as a constant complex on $S_{\text{ét}}$.

(ii) *If D is a diagonalizable algebraic group over E, the pro-étale sheaf associated to*

$$T/S \longmapsto H^1_{\text{ét}}(X_T, D)$$

is the constant sheaf with value $H^1(W_E, D(\overline{\breve{E}}))$.

Proof. Let us note $\mathcal{G} = \mathcal{F}|_{X_S}$. There is a natural morphism $\underline{R\Gamma_{\text{ét}}(\mathrm{Spa}(E), \mathcal{F})} \to R\tau_*\mathcal{G}$. The morphism $\mathrm{Div}^1_S \to S$ is proper and applying [**Sch17a**, Corollary 16.10 (ii)], we are reduced to prove that

$$H^\bullet_{\text{ét}}(\mathrm{Spa}(E), \mathcal{F}) \xrightarrow{\sim} H^\bullet_{\text{ét}}(X_{C,C^+}, \mathcal{G})$$

when C is an algebraically closed field. Since X_{C,C^+} is quasicompact quasiseparated

$$H^\bullet_{\text{ét}}(X_{C,C^+} \widehat{\otimes}_E \widehat{\overline{E}}, \mathcal{G}) = \varinjlim_{E'|E \text{ finite}} H^\bullet(X_{C,C^+} \otimes_E E', \mathcal{G}),$$

and, using Galois descent, it thus suffices to prove that the left member vanishes in positive degrees, and equals $\mathcal{F}|_{\overline{E}}$ in degree 0.

Let $K = \mathbb{F}_q(\!(T)\!)$ and $X_{(C,C^+),K}$ the equal characteristic Fargues–Fontaine curve over $\mathrm{Spa}(K)$. Identifying $\widehat{\overline{E}}^\flat$ with $\widehat{K^{\mathrm{sep}}}$, one has

$$(X_{(C,C^+),E}\widehat{\otimes}_E\widehat{\overline{E}})^\flat = X_{(C,C^+),K}\widehat{\otimes}\widehat{K^{\mathrm{sep}}}.$$

Using this we are reduced to prove that for any prime number n, for $i > 0$

$$H^i_{\text{ét}}(X_{(C,C^+),K}\widehat{\otimes}_K\widehat{K^{\mathrm{sep}}}, \mathbb{Z}/n\mathbb{Z}) = 0.$$

This is reduced, as above, to prove that any class in $H^i_{\text{ét}}(X_{(C,C^+),K}, \mathbb{Z}/n\mathbb{Z})$ is killed by pullback to a finite separable extension of K.

When $n \neq p$ one has

$$R\Gamma_{\text{ét}}(X_{(C,C^+),K}, \mathbb{Z}/n\mathbb{Z}) = R\Gamma(\varphi^{\mathbb{Z}}, R\Gamma_{\text{ét}}(\mathbb{D}^*_{C,C^+}, \mathbb{Z}/n\mathbb{Z}))$$

where T is the coordinate on the open punctured disk

$$\mathbb{D}^*_{C,C^+} = \mathrm{Spa}(C, C^+) \times \mathrm{Spa}(K).$$

One has $H^k(\mathbb{D}^*_{C,C^+}, \mathbb{Z}/n\mathbb{Z})) = 0$ for $k > 1$, and this is equal, via Kummer theory, to $\mathbb{Z}/n\mathbb{Z}(-1)$ for $k = 1$. The Kummer covering of $\mathbb{D}^*_{C,C^+}$ induced by $T \mapsto T^n$ kills any class in $H^1(\mathbb{D}^*_{C,C^+}, \mathbb{Z}/n\mathbb{Z})$. Also $H^0(\mathbb{D}^*_{C,C^+}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and the class in $H^1(\varphi^{\mathbb{Z}}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ is killed by passing up along an unramified extension of K of degree n.

When $n = p$ we use Artin-Schreier theory. Since C is an algebraically closed field we have $H^i(X_{(C,C^+),K}, \mathcal{O}) = 0$ when $i > 0$. Since the adic space $X_{(C,C^+),K}$ is noetherian we deduce that $H^i_{\text{ét}}(X_{(C,C^+),K}, \mathcal{O}) = 0$ for $i > 0$. Thus, $H^i_{\text{ét}}(X_{(C,C^+),K}, \mathbb{Z}/n\mathbb{Z})$ is 0 for $i > 1$ and $\mathrm{coker}(K \xrightarrow{F-Id} K)$ when $i = 1$, which is killed by pullback to an Artin-Schreier extension of K. This finishes the proof of point (1).

For point (2). There is a natural morphism

$$H^1(W_E, D(\overline{\breve{E}})) \to H^1(X_S, D)$$

(see just after the proof of this proposition). Suppose first that D is a torus. Then point (2) is the computation of the coarse moduli space of Bun_D as a pro-étale stack. This itself is a consequence of Theorem III.2.4 using a translation argument from 1 to any $[b] \in B(D)$ (use the Picard stack structure on Bun_D).

For any D we use the exact sequence

$$1 \longrightarrow D^0 \longrightarrow D \longrightarrow \underline{\pi}_0(D) \longrightarrow 1.$$

For T/S there is a diagram

$$\begin{array}{ccccccc}
H^0(E, \underline{\pi}_0(D)) & \longrightarrow & B(D^0) & \longrightarrow & H^1(W_E, D(\overline{\breve{E}})) & \longrightarrow & H^1(E, \underline{\pi}_0(D)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^0(X_T, \underline{\pi}_0(D)) & \longrightarrow & H^1_{\text{ét}}(X_T, D^0) & \longrightarrow & H^1_{\text{ét}}(X_T, D) & \longrightarrow & H^1_{\text{ét}}(X_T, \underline{\pi}_0(D)) & &
\end{array}$$

since $H^2(W_E, D^0(\overline{\breve{E}})) = 0$ and $H^\bullet(W_E, \underline{\pi}_0(D)(\overline{\breve{E}})) = H^\bullet(W_E, \underline{\pi}_0(D)(\overline{E}))$. The result is then deduced from part (1) and the torus case. $\square$

For $S \in \mathrm{Perf}_k$ there is a natural morphism of groups

$$B_{ab}(G) \longrightarrow H^1_{\text{ét}}(X_S, [G_{sc} \to G]).$$

This is deduced from the natural continuous morphism of sites

$$(X_S)_{\text{ét}} \longrightarrow \{\text{discrete } W_E\text{-sets}\}.$$

Proposition III.2.13. *For $S \in \mathrm{Perf}_k$, the pro-étale sheaf on S associated with*

$$T/S \longmapsto H^1_{\text{ét}}(X_T, [G_{sc} \to G])$$

is the constant sheaf with value $B_{ab}(G)$.

Proof. We use the homotopy equivalence $[Z_{sc} \to Z] \to [G_{sc} \to G]$. There is a diagram

$$\begin{tikzcd}
H^1(W_E, Z_{sc}(\overline{\breve{E}})) \arrow[r] \arrow[d, "\simeq"] & H^1(W_E, Z(\overline{\breve{E}})) \arrow[r] \arrow[d, "\simeq"] & H^1(W_E, [G_{sc}(\overline{\breve{E}}) \to G(\overline{\breve{E}})]) \arrow[r] \arrow[d] & H^2(W_E, Z_{sc}(\overline{\breve{E}})) \arrow[d, "\simeq"] \\
H^1_{\text{ét}}(X_T, Z_{sc}) \arrow[r] & H^1_{\text{ét}}(X_T, Z) \arrow[r] & H^1_{\text{ét}}(X_T, [Z_{sc} \to Z]) \arrow[r] & H^2_{\text{ét}}(X_T, Z_{sc})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & H^2(W_E, Z(\overline{\breve{E}})) \arrow[r, "\simeq"] \arrow[d, hook] & H^2(E, \underline{\pi}_0(Z)) \arrow[d, "\simeq"] \\
\arrow[r] & H^2_{\text{ét}}(X_T, Z) \arrow[r] & H^2_{\text{ét}}(X_T, \underline{\pi}_0(Z)).
\end{tikzcd}$$

Using Proposition III.2.12 and some diagram chasing we conclude. □

Second Proof of Theorem III.2.7. The theorem is now deduced from the preceding Proposition III.2.13 and the abelianization map $H^1_{\text{ét}}(X_S, G) \to H^1_{\text{ét}}(X_S, [G_{sc} \to G])$. □

Remark III.2.14. Let X_C^{alg} be the schematical curve associated to $C|\mathbb{F}_q$ algebraically closed. The results of [**Far18b**] for the étale cohomology of torsion local systems, [**Far18b**, Theorem 3.7] and the vanishing of the $H^2(X_C, T)$ for a torus T, [**Far18b**, Theorem 2.7], can be stated in a more uniform way; if D is a diagonalizable group over E then $H^i(W_E, D(\overline{\breve{E}})) \xrightarrow{\sim} H^i_{\text{ét}}(X_C, D)$ for $0 \leq i \leq 2$. *Weil cohomology of E is the natural cohomology theory that corresponds to étale cohomology of the curve.* For example Theorem III.2.2 can be restated as $H^1(W_E, G(\overline{\breve{E}})) \xrightarrow{\sim} H^1_{\text{ét}}(X_C, G)$ for a reductive group G.

III.2.5. The explicit description of $|\mathrm{Bun}_G|$. Theorem III.2.3 and Theorem III.2.7 imply that the map

$$|\mathrm{Bun}_G| \to B(G)$$

is continuous when the target is endowed with the topology induced by the order on $(X_*(T)^+_{\mathbb{Q}})^\Gamma$ and the discrete topology on $\pi_1(G)_\Gamma$.

Conjecture III.2.15. *The map $|\mathrm{Bun}_G| \to B(G)$ is a homeomorphism.*

In other words, whenever $b, b' \in B(G)$ such that $b > b'$, there should be a specialization from b to b' in $|\operatorname{Bun}_G|$.

The conjecture is known for $G = \mathrm{GL}_n$ by work of Hansen, [**Han17**], based on [**BFH$^+$17**]. The argument has been extended to some other classical groups in unpublished work of Hamann. While finishing our manuscript, a proof for general G has been given by Viehmann [**Vie21b**].

We will later prove some weak form of the conjecture in Corollary IV.1.23, determining the connected components of Bun_G using simple geometric considerations.

III.3. Beauville–Laszlo uniformization

Recall from [**SW20**, Lecture XIX] the B_{dR}^+-affine Grassmannian

$$\mathrm{Gr}_G$$

of G over $\operatorname{Spd} E$, sending an affinoid perfectoid $S = \operatorname{Spa}(R, R^+)$ over $\operatorname{Spd} E$ to the G-torsors over $\operatorname{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ with a trivialization over $B_{\mathrm{dR}}(R^\sharp)$; here $R^\sharp/E$ is the untilt of R given by $S \to \operatorname{Spd}(E)$. In [**SW20**] this was considered over $\operatorname{Spa}(C^\flat)$ for some $C|E$ algebraically closed but we want now to consider it in a more "absolute" way over $\operatorname{Spd}(E)$.

Since any G-torsor over $\operatorname{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ is trivial locally on $\operatorname{Spa}(R, R^+)_{\text{ét}}$, this coincides with the étale sheaf associated to the presheaf $(R, R^+) \mapsto G(B_{\mathrm{dR}}(R^\sharp))/G(B_{\mathrm{dR}}^+(R^\sharp))$.

This has an interpretation as a Beilinson–Drinfeld type affine Grassmannian. If $\mathcal{E}, \mathcal{E}' \in \operatorname{Bun}_G(S)$ and $D \in \operatorname{Div}^1(S)$, a modification between $\mathcal{E}$ and $\mathcal{E}'$ at D is an isomorphism

$$\mathcal{E}|_{X_S \setminus D} \xrightarrow{\sim} \mathcal{E}'|_{X_S \setminus D}$$

that is meromorphic along D. The latter means that for any representation in $\operatorname{Rep}_E(G)$, the associated isomorphism between vector bundles, $\mathcal{F}|_{X_S \setminus D} \xrightarrow{\sim} \mathcal{F}'|_{X_S \setminus D}$ extends to a morphism $\mathcal{F} \to \mathcal{F}'(kD)$ for $k \gg 0$ via $\mathcal{F}' \hookrightarrow \mathcal{F}'(kD)$. Beauville–Laszlo gluing then identifies

$$\mathrm{Gr}_G / \varphi^{\mathbb{Z}} \longrightarrow \operatorname{Div}^1$$

with the moduli of $D \in \operatorname{Div}^1(S)$, $\mathcal{E} \in \operatorname{Bun}_G(X_S)$, and a modification between the trivial G-bundle and $\mathcal{E}$ at D, cf. [**SW20**, Proposition 19.1.2]. This defines a morphism of v-stacks

$$\mathrm{Gr}_G \longrightarrow \operatorname{Bun}_G .$$

Proposition III.3.1. *The Beauville–Laszlo morphism*

$$\mathrm{Gr}_G \longrightarrow \operatorname{Bun}_G$$

is a surjective map of v-stacks; in fact, of pro-étale stacks.

Proof. Pick any $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}_k$ affinoid perfectoid with a map to Bun_G, given by some G-bundle $\mathcal{E}$ on X_S. Fix an untilt $S^\sharp$ of S over $\operatorname{Spa}(E)$. To prove surjectivity as pro-étale stacks, we can assume that S is strictly totally disconnected. By [**Far18b**, Théorème 7.1] (in case G quasisplit) and [**Ans19**, Theorem 6.5] (for general G), for any connected component $\operatorname{Spa}(C, C^+) \subset S$ of S, the map $\mathrm{Gr}_G(C) \to \operatorname{Bun}_G(C)$ is surjective, so in particular, we can pick a modification $\mathcal{E}'_C$ of $\mathcal{E}|_{X_{C,C^+}}$ at $\operatorname{Spa}(C^\sharp, C^{\sharp,+}) \hookrightarrow X_{C,C^+}$ such that $\mathcal{E}'_C$ is trivial.

Now, since S is strictly totally disconnected, we can trivialize $\mathcal{E}$ at the completion at $S^\sharp$; as $\mathrm{Gr}_G(S) \to \mathrm{Gr}_G(C)$ is surjective (Lemma III.3.2), we can lift $\mathcal{E}'_C$ to a modification $\mathcal{E}'$ of $\mathcal{E}$. Now Theorem III.2.4 implies that $\mathcal{E}'$ is trivial in a neighborhood of the given point, and as S is strictly totally disconnected, the corresponding $\underline{G(E)}$-torsor is trivial (Lemma III.2.6), so we can trivialize $\mathcal{E}'$ in a neighborhood of the given point. This shows that locally on S, the bundle $\mathcal{E}$ is in the image of $\mathrm{Gr}_G \to \mathrm{Bun}_G$, as desired. □

Lemma III.3.2. *For $S = \mathrm{Spa}(R, R^+)$ a strictly totally disconnected perfectoid space over $\mathrm{Spa}(E)$, and $s \in S$, the map $\mathrm{Gr}_G(R) \to \mathrm{Gr}_G(K(s))$ is surjective.*

Proof. Set $C = K(s)$. First note that $R \to C$ is surjective, as any connected component of S is an intersection of open and closed subsets, and thus Zariski closed.

Since C is algebraically closed, $\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}}(C))/G(B^+_{\mathrm{dR}}(C))$. For any finite degree extension $E'|E$ in C, since $S \otimes_E E' \to S$ is étale and S strictly totally disconnected, E' can be embedded in R. Fix such an $E' \subset R$ that splits G and a pair $T \subset B$ inside $G_{E'}$. Let $\xi \in W_{\mathcal{O}_E}(R^{\flat,+})$ be a generator of the kernel of θ. The Cartan decomposition

$$G(B_{\mathrm{dR}}(C)) = \coprod_{\mu \in X^*(T)^+} G(B^+_{\mathrm{dR}}(C))\mu(\xi)G(B^+_{\mathrm{dR}}(C))$$

shows then that we only need to prove that $G(B^+_{\mathrm{dR}}(R)) \to G(B^+_{\mathrm{dR}}(C))$ is surjective.

Let us first remark that $G(R) \to G(C)$ is surjective. In fact, since S is totally disconnected it suffices to check that $G(\mathcal{O}_{S,s}) \to G(C)$ is surjective. But this is a consequence of the smoothness of G and the fact that $\mathcal{O}_{S,s}$ is Henselian (with residue field C).

Moreover $G(B^+_{\mathrm{dR}}(R)) = \varprojlim_{n \geq 1} G(B^+_{\mathrm{dR}}(R)/\operatorname{Fil}^n)$. Using the surjectivity of $\mathrm{Lie}(G) \otimes R \to \mathrm{Lie}(G) \otimes C$ the result is then deduced by an approximation argument. □

Remark III.3.3. In the "classical case" of the moduli of G-bundles over a proper smooth algebraic curve over a field k, G/k, Proposition III.3.1 is true only when G is semi-simple ([**DS95**]). Typically this is false for GL_n in general. The main reason why it is true in our situation is that $\mathrm{Pic}^0(X_{C,C^+})$ is trivial, equivalently that $X_C^{\mathrm{alg}} \setminus \{x\}$ is the spectrum of a principal ideal domain in Proposition II.2.9.

Lemma III.3.4. *If G is split then*

$$\mathrm{Gr}_G = \varinjlim_{\mu \in X_*(T)^+} \mathrm{Gr}_{G,\leq\mu}$$

as a v-sheaf, where the index set is a partially ordered set according to the dominance order ($\mu \leq \mu'$ if $\mu' - \mu$ is a nonnegative integral sum of positive coroots).

Proof. Consider a morphism $S \to \mathrm{Gr}_G$ with S quasicompact quasiseparated. Fix an embedding $G \hookrightarrow \mathrm{GL}_n$ such that the image of T lies in the standard maximal torus of GL_n. This induces an embedding $X_*(T)^+ \hookrightarrow \mathbb{Z}^n$. One checks easily that the image of $|S| \to |\mathrm{Gr}_{\mathrm{GL}_n}|$ lies in a finite union of affine Schubert cells. Since the fibers of $X_*(T)^+ \to \mathbb{Z}^n/\mathfrak{S}_n$ are finite we deduce that there is a finite collection $(\mu_i)_i$, $\mu_i \in X_*(T)^+$, such that the image of $|S| \to |\mathrm{Gr}_G|$ lies in $\cup_i |\mathrm{Gr}_{G,\leq\mu_i}|$. By [**SW20**, Proposition 19.2.3], $S_i := S \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leq\mu_i}$ is closed in S and thus quasicompact. Since the

morphism $\coprod_i S_i \to S$ is surjective at the level of points with quasicompact source it is quasicompact and thus a v-cover. This allows us to conclude. □

LEMMA III.3.5. *Suppose $\widetilde{G} \to G$ is a central extension with kernel a torus. Then*

$$\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$$

is a surjective map of v-sheaves.

PROOF. Up to replacing E by a finite degree extension we can suppose G and $\widetilde{G}$ are split. Fix $\widetilde{T} \to T$ inside $\widetilde{G} \to G$. According to Lemma III.3.4 it is enough to prove that for any $\widetilde{\mu} \in X_*(\widetilde{T})^+$, if $\mu \in X_*(T)^+$ is its image in G, then $\mathrm{Gr}_{\widetilde{G},\leq\widetilde{\mu}} \to \mathrm{Gr}_{G,\leq\mu}$ is surjective. This is clearly surjective at the level of points since, if D is the kernel of $\widetilde{G} \to G$, then $H^1_{\text{ét}}(\mathrm{Spec}(B_{\mathrm{dR}}(C)), D) = 0$, and thus $\widetilde{G}(B_{\mathrm{dR}}(C)) \to G(B_{\mathrm{dR}}(C))$ is surjective. Since $\mathrm{Gr}_{\widetilde{G},\leq\widetilde{\mu}}$ is quasicompact over $\mathrm{Spd}(E)$ and $\mathrm{Gr}_{G,\leq\mu}$ quasiseparated over $\mathrm{Spd}(E)$ (both are proper according to [**SW20**, Proposition 19.2.3]), $\mathrm{Gr}_{\widetilde{G},\leq\widetilde{\mu}} \to \mathrm{Gr}_{G,\leq\mu}$ is quasicompact and thus a v-cover by [**Sch17a**, Lemma 12.11]. □

Using Proposition III.3.1, we thus have now a proof of Lemma III.2.10.

Let us record a few facts we can deduce from the preceding results.

PROPOSITION III.3.6. *Suppose G is split.*

(i) *There is a locally constant map $|\mathrm{Gr}_G| \to \pi_1(G)$ inducing a decomposition in open/closed subsheaves*

$$\mathrm{Gr}_G = \coprod_{\alpha\in\pi_1(G)} \mathrm{Gr}_G^\alpha$$

characterized by $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_G^{\mu^\sharp}$.

(ii) *The composite*

$$|\mathrm{Gr}_G| \xrightarrow{\textit{Beauville–Laszlo}} |\,\mathrm{Bun}_G\,| \xrightarrow{\kappa} \pi_1(G)$$

is the opposite of the preceding map.

(iii) *For each $\alpha \in \pi_1(G)$,*

$$\mathrm{Gr}_G^\alpha = \varinjlim_{\mu\in X_*(T)^+,\mu^\sharp=\alpha} \mathrm{Gr}_{G,\leq\mu}^\alpha$$

as a filtered colimit of v-sheaves.

PROOF. Point (1) is reduced to the case when G_{der} is simply connected using Lemma III.3.5 and a z-extension. Now, if G_{der} is simply connected, for $\mu_1, \mu_2 \in X_*(T)^+$, $\mu_1 \leq \mu_2$ implies $\mu_1^\sharp = \mu_2^\sharp$. The result is then deduced from the fact that $\mathrm{Gr}_{G,\leq\mu} \subset \mathrm{Gr}_G$ is closed for any μ.

Point (2) is can be similarly reduced first to the case that G_{der} is simply connected by passage to a z-extension; then to the case of a torus by taking the quotient by G_{der}; then to the case of an induced torus by another z-extension; and then to $\mathbb{G}_m$ by changing E. In that case, it follows from Proposition II.2.3.

Point (3) is deduced from Lemma III.3.4 and the fact that for $\alpha \in \pi_1(G)$, $\{\mu \in X_*(T)^+ \mid \mu^\sharp = \alpha\}$ is a filtered ordered set. □

When G is not split, choosing $E'|E$ Galois of finite degree splitting G, using the formula $\mathrm{Gr}_G \times_{\mathrm{Spd}(E)} \mathrm{Spd}(E') = \mathrm{Gr}_{G_{E'}}$, one deduces:

(i) There is a decomposition $\mathrm{Gr}_G = \coprod_{\bar{\alpha}\in\Gamma\backslash\pi_1(G)} \mathrm{Gr}_G^{\bar{\alpha}}$ and a formula $\mathrm{Gr}_G = \varinjlim_{\bar{\mu}\in\Gamma\backslash X^*(T)^+} \mathrm{Gr}_{G,\leq\bar{\mu}}$ such that $\mathrm{Gr}_{G,\leq\bar{\mu}} \subset \mathrm{Gr}_G^{\bar{\mu}^\sharp}$.

(ii) The composite $|\mathrm{Gr}_G| \xrightarrow{\mathrm{BL}} |\,\mathrm{Bun}_G\,| \xrightarrow{\kappa} \pi_1(G)_\Gamma$ is induced by the opposite of $\Gamma\backslash\pi_1(G) \to \pi_1(G)_\Gamma$ and the preceding decomposition.

This description of the Kottwitz map, together with Proposition III.3.1, in fact gives another proof of Theorem III.2.7.

III.4. The semistable locus

III.4.1. Pure inner twisting. Recall the following. In the particular case of non abelian group cohomology this is called "torsion au moyen d'un cocycle" in [**Ser94**, I.5.3].

Proposition III.4.1. *Let X be a topos, H a group in X and T an H-torsor. Let $H^T = \underline{\mathrm{Aut}}(T)$ as a group in X. Then:*

(i) *H^T is the "pure inner twisting" of H by T, $H^T = H \overset{H}{\wedge} T$ where H acts by conjugation on H. In particular $[H^T] \in H^1(X, H_{\mathrm{ad}})$ is the image of $[T]$ via $H^1(X,H) \to H^1(X, H_{\mathrm{ad}})$.*

(ii) *The morphism of stacks in X, $[*/H] \to [*/H^T]$, that sends an H-torsor S to $\underline{\mathrm{Isom}}(S,T)$, is an equivalence.*

In the following we use the cohomological description of G-bundles on the curve as G-torsors on the étale site of the sous-perfectoid space X_S (here G is seen as an E-adic group, for (R,R^+) a sous-perfectoid E-algebra its $\mathrm{Spa}(R,R^+)$-points being $G(R)$).

Proposition III.4.2. *Let $S \in \mathrm{Perf}_k$, $b \in B(G)$ basic and $\mathcal{E}_b \to X_S$ the associated étale G-torsor. Then the étale sheaf of groups $G_b \times_{\mathrm{Spa}(E)} X_S$ over X_S is the pure inner twisting of $G \times_{\mathrm{Spa}(E)} X_S$ by $\mathcal{E}_b$.*

Proof. One has

$$\mathcal{E}_b = (G_{\breve{E}} \times_{\mathrm{Spa}(\breve{E})} Y_S)/((b\sigma)\times\varphi)^{\mathbb{Z}} \longrightarrow X_S$$

where $b\sigma$ acts on $G_{\breve{E}}$ by translation on the right. The $G \times X_S = (G_{\breve{E}} \times_{\mathrm{Spa}(\breve{E})} Y_S)/(Id\times\varphi)^{\mathbb{Z}}$-torsor structure is given by multiplication on the left on $G_{\breve{E}}$. The group $G_b \times X_S = (G_b \times Y_S)/(Id\times\varphi)^{\mathbb{Z}}$ acts on this torsor on the right via the morphism $G_b \to G_{\breve{E}}$, which gives a morphism

$$G_b \times X_S \to \underline{\mathrm{Aut}}(\mathcal{E}_b).$$

After pullback via the étale cover $Y_S \to X_S$ and evaluation on $T = \mathrm{Spa}(R,R^+) \to Y_S$ affinoid sous-perfectoid, this is identified with the map

$$G_b(R) = \{g \in G(\breve{E}\otimes_E R) \mid g\cdot(b\sigma\otimes 1) = (b\sigma\otimes 1)\cdot g\} \longrightarrow G(R)$$

deduced from the $\breve{E}$-algebra structure of R. But as b is basic, the natural map

$$G_b \times_E \breve{E} \to G \times_E \breve{E}$$

is an isomorphism, cf. [**RZ96**, Corollary 1.14]. □

Thus, *extended pure inner forms, as defined by Kottwitz, become pure inner forms, as defined by Vogan, when pulled back to the curve.*

COROLLARY III.4.3. *For b basic there is an isomorphism of v-stacks*

$$\mathrm{Bun}_G \simeq \mathrm{Bun}_{G_b}$$

that induces an isomorphism $\mathrm{Bun}_G^b \simeq \mathrm{Bun}_{G_b}^1$.

EXAMPLE III.4.4. Take $G = \mathrm{GL}_n$ and (D, φ) an isoclinic isocrystal of height n. Let $B = \mathrm{End}(D, \varphi)$ be the associated simple algebra over E. Since (D, φ) is isoclinic the action of B on $\mathcal{E}(D, \varphi)$ induces an isomorphism $B \otimes_E \mathcal{O}_{X_S} \xrightarrow{\sim} \underline{\mathrm{End}}(\mathcal{E}(D, \varphi))$ for any $S \in \mathrm{Perf}_k$. The stack $\mathrm{Bun}_{B^\times}$ is identified with the stack $S \mapsto \{\text{rank 1 locally free } B \otimes_E \mathcal{O}_{X_S}\text{-modules}\}$. There is then an isomorphism (Morita equivalence)

$$\begin{aligned} \mathrm{Bun}_{\mathrm{GL}_n} &\xrightarrow{\sim} \mathrm{Bun}_{B^\times} \\ \mathcal{E} &\longmapsto \underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(\mathcal{E}, \mathcal{E}(D, \varphi)). \end{aligned}$$

III.4.2. Description of the semi-stable locus. Recall from [**Far18b**] that a G-bundle over X_{C,C^+} is semistable if and only if it corresponds to some basic element of $B(G)$. Also recall that for any $b \in B(G)$, the automorphism group of the corresponding G-isocrystal defines an algebraic group G_b over G. If G is quasisplit, then G_b is an inner form of a Levi subgroup of G, and it is an inner form of G precisely when b is basic. More generally G_b is an inner form of a Levi subgroup of the quasisplit inner form of G.

THEOREM III.4.5. *The semistable locus*

$$\mathrm{Bun}_G^{\mathrm{ss}} \subset \mathrm{Bun}_G$$

is open, and there is a canonical decomposition as open/closed substacks

$$\mathrm{Bun}_G^{\mathrm{ss}} = \coprod_{b \in B(G)_{\mathrm{basic}}} \mathrm{Bun}_G^b.$$

For b basic there is an isomorphism

$$[*/\underline{G_b(E)}] \xrightarrow{\sim} \mathrm{Bun}_G^b.$$

PROOF. Theorem III.2.3 implies that $\mathrm{Bun}_G^{\mathrm{ss}}$ is open. Recall that the basic elements of $B(G)$ map isomorphically to $\pi_1(G)_\Gamma$ via the Kottwitz map. Thus Theorem III.2.7 gives a disjoint decomposition

$$\mathrm{Bun}_G^{\mathrm{ss}} = \coprod_{b \in B(G)_{\mathrm{basic}}} \mathrm{Bun}_G^b.$$

The result is then a consequence of Proposition III.4.2 and Theorem III.2.4. □

EXAMPLE III.4.6. For a torus T, $\mathrm{Bun}_T = \mathrm{Bun}_T^{ss}$ and there is an exact sequence of Picard stacks

$$0 \longrightarrow [*/\underline{T(E)}] \longrightarrow \mathrm{Bun}_T \longrightarrow \underline{X_*(T)_\Gamma} \longrightarrow 0,$$

where we recall that $X_*(T)_\Gamma = B(T)$. The fiber of $\mathrm{Bun}_T \to \underline{B(T)}$ over β is a gerbe banded by $\underline{T(E)}$ over $*$ via the action Bun_T^1. This gerbe is neutralized after choosing some b such that $[b] = \beta$. In case there is a section of $T(\breve{E}) \twoheadrightarrow B(T)$, for example if $B(T)$ is torsion free, then $\mathrm{Bun}_T \simeq [*/\underline{T(E)}] \times \underline{X_*(T)_\Gamma}$ as a Picard stack.

III.4.3. Splittings of the Harder–Narasimhan filtration. We can also consider the following moduli problem, parametrizing G-bundles with a splitting of their Harder–Narasimhan filtration.

Proposition III.4.7. *Consider the functor* $\mathrm{Bun}_G^{\text{HN-split}}$ *taking each* $S \in \mathrm{Perf}_k$ *to the groupoid of exact* $\otimes$*-functors from* $\mathrm{Rep}_E G$ *to the category of* $\mathbb{Q}$*-graded vector bundles* $\mathcal{E} = \bigoplus_\lambda \mathcal{E}^\lambda$ *on* X_S *such that* $\mathcal{E}^\lambda$ *is everywhere semistable of slope* λ *for all* $\lambda \in \mathbb{Q}$*. For any* $b \in B(G)$*, the bundle* $\mathcal{E}_b$ *naturally refines to a* $\mathbb{Q}$*-graded bundle* $\mathcal{E}_b^{\mathrm{gr}}$*, using the* $\mathbb{Q}$*-grading on isocrystals, and for* S *affinoid the natural map*

$$G_b \times_E X_S^{\mathrm{alg}} \to \underline{\mathrm{Aut}}(\mathcal{E}_b^{\mathrm{gr}})$$

of group schemes over X_S^{alg} *is an isomorphism. In particular, we get a natural map*

$$\bigsqcup_{b \in B(G)} [*/\underline{G_b(E)}] \to \mathrm{Bun}_G^{\text{HN-split}},$$

and this is an isomorphism.

Proof. Recall that the natural map $G_b \times_E \breve{E} \to G \times_E \breve{E}$, recording the map of underlying $\breve{E}$-vector spaces, is a closed immersion identifying $G_b \times_E \breve{E}$ with the centralizer of the slope homomorphism $\nu_b : \mathbb{D} \to G \times_E \breve{E}$, cf. [**RZ96**, Corollary 1.14]. This implies that the natural map

$$G_b \times_E X_S^{\mathrm{alg}} \to \underline{\mathrm{Aut}}(\mathcal{E}_b^{\mathrm{gr}})$$

is an isomorphism.

We get the evident functor from $\bigsqcup_{b \in B(G)} [*/\underline{G_b(E)}]$ to this moduli problem, and it is clearly fully faithful. To see that it is surjective, take any strictly totally disconnected S and an exact $\otimes$-functor $\mathcal{E}^{\mathrm{gr}}$ from $\mathrm{Rep}_E G$ to such $\mathbb{Q}$-graded vector bundles. For any point $s \in S$, note that $\mathbb{Q}$-graded vector bundles of the given form on $X_{K(s)}$ are equivalent to Isoc_E, so at $s \in S$ there is an isomorphism with some $\mathcal{E}_b^{\mathrm{gr}}$. The type of the $\mathbb{Q}$-filtration is locally constant, so after replacing S by an open neighborhood of s, we can assume that

$$\underline{\mathrm{Isom}}(\mathcal{E}_b^{\mathrm{gr}}, \mathcal{E}^{\mathrm{gr}})$$

defines an $\underline{\mathrm{Aut}}(\mathcal{E}_b^{\mathrm{gr}})$-torsor over X_S^{alg}, i.e. a G_b-torsor over S. This defines a map $S \to \mathrm{Bun}_{G_b}$, taking s into $\mathrm{Bun}_{G_b}^1$, and by Theorem III.2.4 and Lemma III.2.6 it follows that after replacing S by an open neighborhood, we can assume that the torsor is trivial. This concludes the proof of surjectivity. $\square$

III.5. Non-semistable points

III.5.1. Structure of $\underline{\mathrm{Aut}}(\mathcal{E}_b)$. Next, we aim to describe the non-semi-stable strata Bun_G^b. Thus, fix any $b \in B(G)$ and consider the associated G-bundle $\mathcal{E}_b$ on X_S. For any algebraic representation $\rho : G \to \mathrm{GL}_n$, the corresponding vector bundle $\rho_*\mathcal{E}_b$ has its Harder–Narasimhan filtration $(\rho_*\mathcal{E}_b)^{\geq\lambda} \subset \rho_*\mathcal{E}_b$, $\lambda \in \mathbb{Q}$. If G is quasisplit and we fix a Borel $B \subset G$, then this defines a reduction of $\mathcal{E}_b$ to a parabolic $P \subset G$ containing B.

Now inside the automorphism v-sheaf

$$\widetilde{G}_b = \underline{\mathrm{Aut}}(\mathcal{E}_b)$$

(which necessarily preserves the Harder–Narasimhan filtration of $\rho_*\mathcal{E}_b$ for any $\rho \in \mathrm{Rep}_E(G)$) one can consider for any $\lambda > 0$ the subgroup

$$\widetilde{G}_b^{\geq\lambda} \subset \widetilde{G}_b$$

of all automorphisms $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ such that

$$(\gamma - 1)(\rho_*\mathcal{E}_b)^{\geq\lambda'} \subset (\rho_*\mathcal{E}_b)^{\geq\lambda'+\lambda}$$

for all λ' and all representations ρ of G. We also set $\widetilde{G}_b^{>\lambda} = \bigcup_{\lambda'>\lambda} \widetilde{G}_b^{\geq\lambda'}$, noting that this union is eventually constant.

As $G_b(E)$ is the automorphism group of the isocrystal corresponding to b, and $H^0(X_S, \mathcal{O}_{X_S}) = \underline{E}(S)$, we have a natural injection

$$\underline{G_b(E)} \hookrightarrow \widetilde{G}_b.$$

Now, for any automorphism γ of $\mathcal{E}_b$ and any representation ρ, γ induces an automorphism of the $\mathbb{Q}$-graded vector bundle

$$\bigoplus_{\lambda\in\mathbb{Q}} \mathrm{Gr}^\lambda(\rho_*\mathcal{E}_b).$$

Using Proposition III.4.7, we deduce that the preceding injection has a section and

$$\widetilde{G}_b = \widetilde{G}_b^{>0} \rtimes \underline{G_b(E)}.$$

For a G-bundle $\mathcal{E}$ on X_S we note $\mathrm{ad}\,\mathcal{E}$ for its adjoint bundle deduced by pushforward by the adjoint representation $G \to \mathrm{GL}(\mathrm{Lie}(G))$. This is in fact a Lie algebra bundle.

Proposition III.5.1. *One has*

$$\widetilde{G}_b = \widetilde{G}_b^{>0} \rtimes \underline{G_b(E)},$$

and for any $\lambda > 0$, there is a natural isomorphism

$$\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda} \xrightarrow{\sim} \mathcal{BC}((\mathrm{ad}\,\mathcal{E}_b)^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda}),$$

the Banach–Colmez space associated to the slope $-\lambda$ isoclinic part of $(\mathrm{Lie}(G) \otimes_E \breve{E}, \mathrm{Ad}(b)\sigma)$.

In particular, $\widetilde{G}_b$ is an extension of $\underline{G_b(E)}$ by a successive extension of positive Banach–Colmez spaces, and thus $\widetilde{G}_b \to *$ is representable in locally spatial diamonds, of dimension $\langle 2\rho, \nu_b\rangle$ (where as usual ρ is the half-sum of the positive roots).

We refer to [**SR72**, Section IV] and [**Zie15**] for some general discussion of filtered and graded fibre functors.

Proof. We already saw the first part. For the second part, suppose $S = \mathrm{Spa}(R, R^+)$ is affinoid. Let X_R^{alg} be the schematical curve. We use the GAGA correspondence, Proposition II.2.7. Now, we apply Proposition III.5.2 to X_R^{alg} and the G-bundle $\mathcal{E}_b$ associated to b on X_R^{alg}. Let H be the inner twisting of $G \times X_R^{\mathrm{alg}}$ by $\mathcal{E}_b$ as a reductive group scheme over X_R^{alg}. It is equipped with a filtration $(H^{\geq\lambda})_{\lambda\geq 0}$ satisfying

- $H^{\geq 0}/H^{>0} \cong G_b \times_E X_R^{\mathrm{alg}}$,
- for $\lambda > 0$, $H^{\geq\lambda}/H^{>\lambda} = (\mathrm{ad}\,\mathcal{E}_b)^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda}$,
- $\widetilde{G}_b^{\geq\lambda}(S) = H^{\geq\lambda}(X_R^{\mathrm{alg}})$,

functorially in (R, R^+); the first part uses Proposition III.4.7. Since $H^1(X_R^{\mathrm{alg}}, \mathcal{O}(\mu)) = 0$ as soon as $\mu > 0$, we deduce by induction on $\mu > 0$, starting with $\mu \gg 0$ and using the computation of $H^{\geq\mu}/H^{>\mu}$, that $H^1_{\text{ét}}(X_R^{\mathrm{alg}}, H^{\geq\mu}) = 0$ for $\mu > 0$. From this we deduce that

$$H^{\geq\lambda}(X_R^{\mathrm{alg}})/H^{>\lambda}(X_R^{\mathrm{alg}}) = (H^{\geq\lambda}/H^{>\lambda})(X_R^{\mathrm{alg}}).$$

Finally, it remains to compute the dimension. This is given by

$$\sum_{\lambda>0} \lambda \cdot \dim(\mathrm{ad}\,\mathcal{E}_b)^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda},$$

which is given by $\langle 2\rho, \nu_b\rangle$. □

Proposition III.5.2. *Let G be a reductive group over a field K, and let X be a scheme over K. Let $\mathcal{E}$ be a G-bundle on X with automorphism group scheme H/X (an inner form of $G \times_K X$, cf. Proposition III.4.1). Consider a $\mathbb{Q}$-filtration on the fibre functor $\mathrm{Rep}_K(G) \to \{$Vector bundles on $X\}$ associated with $\mathcal{E}$. Defining groups $H^{\geq\lambda} \subset H$ for $\lambda \geq 0$ as before, they are smooth group schemes, $H^{\geq 0}$ is a parabolic subgroup with unipotent radical $H^{>0}$, the Lie algebra of $H^{\geq\lambda}$ is given by $(\mathrm{ad}\,E)^{\geq\lambda} \subset \mathrm{Lie}\,\mathrm{ad}\,E = \mathrm{Lie}\,\widetilde{G}$, and the quotient $H^{\geq\lambda}/H^{>\lambda}$ is a vector group, thus*

$$H^{\geq\lambda}/H^{>\lambda} \cong (\mathrm{ad}\,\mathcal{E})^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E})^{>\lambda}$$

Proof. The Lie algebra of H is $\mathrm{ad}\,\mathcal{E}$. All statements can be checked étale locally on X. According to [**Zie15**, Theorem 1.3] the $\mathbb{Q}$-filtration on the fiber functor is split locally on X. Moreover $\mathcal{E}$ is split étale locally on X. We can thus suppose that $\mathcal{E}$ is the trivial G-bundle and the filtration given by some $\nu : \mathbb{D}_{/X} \to G \times_K X$. Then the statement is easily checked, see [**SR72**]. □

III.5.1.1. *The quasi-split case.* Suppose now that G is moreover quasi-split. Fix $A \subset T \subset B$ with A a maximal split torus and T a maximal torus of G inside a Borel subgroup B. Up to σ-conjugating b one can suppose that $\nu_b : \mathbb{D} \to A$ and $\nu_b \in X_*(A)^+_{\mathbb{Q}}$. Let M_b be the centralizer of ν_b and P_b^+ the parabolic subgroup associated to ν_b, the weights of ν_b in $\mathrm{Lie}(P_b^+)$ are ≥ 0. One has $B \subset P_b^+$, P_b^+ is a standard parabolic subgroup with standard Levi subgroup M_b. Let P_b^- be the opposite parabolic subgroup, the weights of ν_b in $\mathrm{Lie}(P_b^-)$ are ≤ 0. One has $b \in M_b(\breve{E})$ and we denote it b_M as an element of $M_b(\breve{E})$.

Then, if

$$Q = \mathcal{E}_{b_M} \overset{M_b}{\times} P_b^-,$$
$$R_u Q = \mathcal{E}_{b_M} \overset{M_b}{\times} R_u P_b^-$$

as X_R^{alg}-group-schemes, one has

$$\widetilde{G}_b(R, R^+) = Q(X_R^{\mathrm{alg}})$$
$$\widetilde{G}_b^{>0}(R, R^+) = R_u Q(X_R^{\mathrm{alg}}).$$

III.5.2. Description of Harder–Narasimhan strata. Now we can describe the structure of the stratum Bun_G^b.

Proposition III.5.3. *Let $b \in B(G)$ be any element given by some G-isocrystal. The induced map $x_b : * \to \mathrm{Bun}_G^b$ is a surjective map of v-stacks, and $* \times_{\mathrm{Bun}_G^b} * \cong \widetilde{G}_b$, so that*

$$\mathrm{Bun}_G^b \cong [*/\widetilde{G}_b]$$

is the classifying stack of $\widetilde{G}_b$-torsors. In particular, the map $\widetilde{G}_b \to \pi_0 \widetilde{G}_b \cong \underline{G_b(E)}$ induces a map

$$\mathrm{Bun}_G^b \to [*/\underline{G_b(E)}]$$

that admits a splitting.

Proof. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_k$ be strictly totally disconnected and let $\mathcal{E}$ be a G-bundle on X_R^{alg}, the schematical curve, all of whose geometric fibers are isomorphic to $\mathcal{E}_b$. In particular, the Harder–Narasimhan polygon of $\rho_* \mathcal{E}$ is constant for all representations $\rho : G \to \mathrm{GL}_n$, and thus by Theorem II.2.19, the vector bundle $\rho_* \mathcal{E}$ admits a relative Harder–Narasimhan filtration. This defines a $\mathbb{Q}$-filtration on the fiber functor $\mathrm{Rep}_E(G) \to \{\text{vector bundles on } X_R^{\mathrm{alg}}\}$ defined by $\mathcal{E}$, and exactness can be checked on geometric points where it holds by the classification of G-bundles. Since for any ρ, the Harder–Narasimhan polygon of $\rho_* \mathcal{E}_b$ and the one of $\rho_* \mathcal{E}$ are equal, the two filtered fiber functors on $\mathrm{Rep}_E(G)$ defined by $\mathcal{E}$ and $\mathcal{E}_b$ are of the same type. Thus, étale locally on X_R^{alg} those two filtered fiber functors are isomorphic. Let $H = \underline{\mathrm{Aut}}(\mathcal{E}_b)$ and $H^{\geq 0} = \underline{\mathrm{Aut}}_{\mathrm{filtered}}(\mathcal{E}_b)$ as group schemes over X_R^{alg}, cf. Proposition III.5.1. Now, look at

$$T = \underline{\mathrm{Isom}}_{\mathrm{filtered}}(\mathcal{E}_b, \mathcal{E}).$$

This is an $H^{\geq 0}$-torsor over X_R^{alg} that is a reduction to $H^{\geq 0}$ of the H-torsor $\underline{\mathrm{Isom}}(\mathcal{E}_b, \mathcal{E})$. Let us look at the image of $[T] \in H^1_{\text{ét}}(X_R^{\mathrm{alg}}, H^{\geq 0})$ in $H^1_{\text{ét}}(X_R^{\mathrm{alg}}, H^{\geq 0}/H^{>0})$, that is to say the $H^{\geq 0}/H^{>0}$-torsor $T/H^{>0}$. This parametrizes isomorphisms of graded fiber functors between the two obtained by semi-simplifying the filtered fiber functors attached to $\mathcal{E}_b$ and $\mathcal{E}$. By Proposition III.4.7, this torsor is locally trivial. Now the triviality of T follows from the vanishing of $H^1(X_R^{\mathrm{alg}}, H^{>0})$. In fact, for $\lambda > 0$, $H^1(X_R^{\mathrm{alg}}, H^{\geq \lambda}/H^{>\lambda}) = 0$ since $H^1(X_R^{\mathrm{alg}}, \mathcal{O}(\lambda)) = 0$.

It is clear that $* \times_{\mathrm{Bun}_G^b} *$ is given by $\widetilde{G}_b$, so the rest follows formally. $\square$

Remark III.5.4 (Followup to Remark III.2.5). From the vanishing of $H^1_v(S, \widetilde{G}_b^{>0})$ for S affinoid perfectoid one deduces that for such S, any $\widetilde{G}_b$-torsor is of the form $T \times \widetilde{G}_b^{>0}$ where $T \to S$ is a $\underline{G_b(E)}$-torsor. Here the action of $g_1 \rtimes g_2 \in \underline{G_b(E)} \rtimes \widetilde{G}_b^{>0}$ on $T \times \widetilde{G}_b^{>0}$ is given by $(x, y) \mapsto (g_1 \cdot x, g_1 g_2 y g_1^{-1})$. In particular *any $\widetilde{G}_b$-torsor is representable in locally spatial diamonds.*

CHAPTER IV

Geometry of diamonds

In this chapter, we extend various results on schemes to the setting of diamonds, showing that many advanced results in étale cohomology of schemes have analogues for diamonds.

In Section IV.1, we introduce a notion of Artin v-stacks, and discuss some basic properties; in particular, we show that Bun_G is a cohomologically smooth Artin v-stack. Moreover, we can define a notion of dimension for Artin v-stacks, which we use to determine the connected components of Bun_G. In Section IV.2, we develop the theory of universally locally acyclic sheaves. In Section IV.3, we introduce a notion of formal smoothness for maps of v-stacks. In Section IV.4, we use the previous sections to prove the Jacobian criterion for cohomological smoothness, by establishing first formal smoothness, and universal local acyclicity. In Section IV.5, we prove a result on the vanishing of certain partially compactly supported cohomology groups, ensuring that for example $\mathrm{Spd}\, k[\![x_1, \ldots, x_d]\!]$ behaves like a strictly local scheme for $D_{\text{ét}}$. In Section IV.6, we establish Braden's theorem on hyperbolic localization in the world of diamonds. Finally, in Section IV.7, we establish several version of Drinfeld's lemma in the present setup. The theme here is the idea $\pi_1((\mathrm{Div}^1)^I) = W_E^I$. Unfortunately, we know no definition of π_1 making this true, but for example it becomes true when considering Λ-local systems for any Λ.

IV.1. Artin stacks

IV.1.1. Generalities.

IV.1.1.1. *Definition and basic properties.* In this paper, we consider many small v-stacks like Bun_G as above. However, they are stacky in some controlled way, in that they are Artin v-stacks in the sense of the following definition.

Definition IV.1.1. *An Artin v-stack is a small v-stack X such that the diagonal $\Delta_X : X \to X \times X$ is representable in locally spatial diamonds, and there is some surjective map $f : U \to X$ from a locally spatial diamond U such that f is separated and cohomologically smooth.*

Remark IV.1.2. We are making the assumption that f is separated, because only in this case we have defined cohomological smoothness. This means that we are imposing some (probably unwanted) very mild separatedness conditions on Artin v-stacks. In particular, it implies that Δ_X *is quasiseparated*: Let $f : U \to X$ be as in the definition, and assume without loss of generality that U is a disjoint union of spatial diamonds (replacing it by an open cover if necessary), so in particular U is quasiseparated. As f is separated, the map $U \times_X U \to U$ is separated, and in particular $U \times_X U$ is again quasiseparated. This is the pullback of $\Delta_X : X \to X \times X$ along the surjection $U \times U \to X \times X$, so Δ_X is quasiseparated.

Remark IV.1.3. *The stack* Bun_G *is not quasiseparated.* In fact, $[*/\underline{G(E)}]$ is already not quasiseparated since the sheaf of automorphisms of the trivial G-bundle, $\underline{G(E)}$, is not quasicompact. This is different from the "classical situation" of the stack of G-bundles on a proper smooth curve, this one being quasiseparated (although not separated). In the "classical schematical case" of Artin stacks it is a very mild assumption to suppose that Artin stacks are quasiseparated. In our situation this would be a much too strong assumption, but it is still a very mild assumption to suppose that the diagonal is quasiseparated.

Remark IV.1.4. By Remark IV.1.2, for any Artin v-stack X, the diagonal Δ_X is quasiseparated. Conversely, let X be any small v-stack, and assume that there is some surjective map $U \to X$ from a small v-sheaf. Then:

(i) If U is quasiseparated, then Δ_X is quasiseparated, by the argument of Remark IV.1.2.

(ii) If U is a locally spatial diamond and $U \to X$ is representable in locally spatial diamonds, then Δ_X is quasiseparated (as we may without loss of generality assume that U is quasiseparated, so that (i) applies), and to check that Δ_X is representable in locally spatial diamonds, it suffices to see that Δ_X is representable in diamonds. Indeed, [**Sch17a**, Proposition 13.4 (v)] shows that if Δ_X is quasiseparated and representable in diamonds, then representability in locally spatial diamonds can be checked v-locally on the target. But the pullback of Δ_X along $U \times U \to X \times X$ is $U \times_X U$, which is a locally spatial diamond as we assumed that $U \to X$ is representable in locally spatial diamonds.

(iii) Finally, in the situation of (ii), checking whether Δ_X is representable in diamonds can be done after pullback along a map $V \to X \times X$ that is surjective as a map of pro-étale stacks, by [**Sch17a**, Proposition 13.2 (iii)].

In particular, if there is a map $f : U \to X$ from a locally spatial diamond U such that f is separated, cohomologically smooth, representable in locally spatial diamonds, and surjective as a map of pro-étale stacks, then X is an Artin v-stack. If one only has a map $f : U \to X$ from a locally spatial diamond such that f is separated, cohomologically smooth, representable in locally spatial diamonds, and surjective as a map of v-stacks, then it remains to prove that Δ_X is representable in diamonds, which can be done after pullback along a map $V \to X \times X$ that is surjective as a map of pro-étale stacks.

Remark IV.1.5. Since cohomologically smooth morphisms are open, to prove that a separated, representable in locally spatial diamonds, cohomologically smooth morphism $U \to X$ is surjective, it suffices to verify it on geometric points.

Remark IV.1.6. If X is a small v-stack with a map $g : X \to S$ to some "base" small v-stack S, one might introduce a notion of an "Artin v-stack over S", asking instead that $\Delta_{X/S} : X \to X \times_S X$ is representable in locally spatial diamonds; note that the condition on the chart $f : U \to X$ will evidently remain the same as in the absolute case. We note that as long as the diagonal of S is representable in locally spatial diamonds (for example, S is an Artin v-stack itself), this agrees with the absolute notion. Indeed, if $\Delta_{X/S}$ and Δ_S are representable in locally spatial diamonds, then also Δ_X is representable in locally spatial diamonds, as $X \times_S X \to X \times X$ is a pullback of Δ_S and thus representable in locally spatial diamonds, so Δ_X is the composite of the two maps $X \to X \times_S X \to X \times X$ both of which are representable in locally spatial diamonds. Conversely, assume that X and S are such that their diagonals are representable in locally spatial diamonds.

Then both X and $X \times_S X$ are representable in locally spatial diamonds over $X \times X$, thus any map between them is.

Example IV.1.7. Any locally spatial diamond is an Artin v-stack.

Before giving other examples let us state a few properties.

Proposition IV.1.8.

(i) *Any fibre product of Artin v-stacks is an Artin v-stack.*

(ii) *Let $S \to *$ be a pro-étale surjective, representable in locally spatial diamonds, separated and cohomologically smooth morphism of v-sheaves. The v-stack X is an Artin v-stack if and only if $X \times S$ is an Artin v-stack.*

(iii) *If X is an Artin v-stack and $f : Y \to X$ is representable in locally spatial diamonds, then Y is an Artin v-stack.*

Proof. For point (1), if $X = X_2 \times_{X_1} X_3$ is such a fibre product and $f_i : U_i \to X_i$ are separated, representable in locally spatial diamonds, and cohomologically smooth surjective maps from locally spatial diamonds U_i, then $U = (U_1 \times_{X_2} U_2) \times_{U_2} (U_2 \times_{X_2} U_3)$ is itself a locally spatial diamond (using that Δ_{X_2} is representable in locally spatial diamonds), and the projection $f : U \to X$ is a separated, representable in locally spatial diamonds, and cohomologically smooth surjection. For the diagonal, since Δ_{X_2} and Δ_{X_3} are representable in locally spatial diamonds, $\Delta_{X_2} \times \Delta_{X_3} : X_2 \times X_3 \to (X_2 \times X_3) \times (X_2 \times X_3)$ is representable in locally spatial diamonds. Since Δ_{X_1} is representable in locally spatial diamonds, its pullback by $X_2 \times X_3 \to X_1 \times X_1$, that is to say $u : X_2 \times_{X_1} X_3 \to X_2 \times X_3$, is representable in locally spatial diamonds. Thus, $\Delta_{X_2 \times_{X_1} X_3}$ is a map between stacks that are representable in locally spatial diamonds over $(X_2 \times X_3) \times (X_2 \times X_3)$, and thus is representable in locally spatial diamonds.

For point (2), suppose $X \times S$ is an Artin v-stack. If U is a locally spatial diamond and $U \to X \times S$ is separated, representable in locally spatial diamonds, cohomologically smooth, and surjective, then the composite $U \to X \times S \to X$ is too. It remains to see that Δ_X is representable in locally spatial diamonds. By Remark IV.1.4 it suffices to prove that the pullback of Δ_X by $X \times X \times S \to X \times X$ is representable in locally spatial diamonds. But this pullback is the composite of $\Delta_{X \times S}$ with $X \times X \times S \times S \to X \times X \times S$, and we conclude since the projection $S \times S \to S$ is representable in locally spatial diamonds for evident reasons.

For point (3), if U is a locally spatial diamond and $U \to X$ is surjective, separated, representable in locally spatial diamonds, and cohomologically smooth, then $V = U \times_X Y$ is a locally spatial diamond, and $V \to Y$ is surjective, separated, representable in locally spatial diamonds, and cohomologically smooth. It remains to see that Δ_Y is representable in locally spatial diamonds. By Remark IV.1.4, it suffices to see that Δ_Y is representable in diamonds. But we can write Δ_Y as the composite $Y \to Y \times_X Y \to Y \times_k Y$. The first map is 0-truncated and injective and thus representable in diamonds by [**Sch17a**, Proposition 11.10], while the second map is a pullback of Δ_X. □

We can now give more examples.

Example IV.1.9.

(i) According to point (2) of Proposition IV.1.8, the v-stack X is an Artin v-stack if and only if $X \times \operatorname{Spd} E$, resp. $X \times \operatorname{Spa}(\mathbb{F}_q(\!(t^{1/p^\infty})\!))$, is an Artin v-stack. *To check that X is an Artin v-stack we can thus replace the base point $*$ by* $\operatorname{Spd} E$, *resp.* $\operatorname{Spa} \mathbb{F}_q(\!(t^{1/p^\infty})\!)$.

(ii) For example, *any small v-sheaf X such that $X \to *$ is representable in locally spatial diamonds is an Artin v-stack*; e.g. $X = *$.

(iii) Using point (3) of Proposition IV.1.8 and [**Sch17a**, Proposition 11.20] we deduce that *any locally closed substack of an Artin v-stack is an Artin v-stack.*

(iv) Let G be a locally profinite group that admits a closed embedding into $\mathrm{GL}_n(E)$ for some n. Then *the classifying stack $[*/\underline{G}]$ is an Artin v-stack.* For this it suffices to see that $[\operatorname{Spd} E/\underline{G}] = \operatorname{Spd} E \times [*/\underline{G}]$ is an Artin v-stack. Now let $H = \mathrm{GL}_{n,E}^{\diamondsuit}$; then there is a closed immersion $\underline{G} \times \operatorname{Spd} E \hookrightarrow H$. The map $H \to \operatorname{Spd} E$ is representable in locally spatial diamonds, separated, and cohomologically smooth; hence so is $H/\underline{G} \to [\operatorname{Spd} E/\underline{G}]$ (by [**Sch17a**, Proposition 13.4 (iv), Proposition 23.15]), and $H/\underline{G}$ is a locally spatial diamond (itself cohomologically smooth over $\operatorname{Spd} E$ by [**Sch17a**, Proposition 24.2] since this becomes cohomologically smooth over the separated étale cover $H/\underline{K} \to H/\underline{G}$ for some compact open pro-p subgroup K of G). It is clear that the diagonal of $[*/\underline{G}]$ is representable in locally spatial diamonds.

Remark IV.1.10. If G is a smooth algebraic group over the field k then $\operatorname{Spec}(k) \to [\operatorname{Spec}(k)/G]$ is a smooth presentation of the Artin stack $[\operatorname{Spec}(k)/G]$. However, in the situation of point (4) of Example IV.1.9 the map $f : * \to [*/\underline{G}]$ is not cohomologically smooth since for its pullback $\widetilde{f} : \underline{G} \to *$, the sheaf $\widetilde{f}^! \Lambda$ is the sheaf of distributions on G with values in Λ.

IV.1.1.2. *Smooth morphisms of Artin v-stacks.* Notions that can be checked locally with respect to cohomologically smooth maps can be extended to Artin v-stacks (except possibly for subtleties regarding separatedness). In particular:

Definition IV.1.11. *Let $f : Y \to X$ be a map of Artin v-stacks. Assume that there is some separated, representable in locally spatial diamonds, and cohomologically smooth surjection $g : V \to Y$ from a locally spatial diamond V such that $f \circ g : V \to X$ is separated. Then f is cohomologically smooth if for any (equivalently, one) such g, the map $f \circ g : V \to X$ (which is separated by assumption, and automatically representable in locally spatial diamonds) is cohomologically smooth.*

In the preceding definition the "equivalently, one" assertion is deduced from [**Sch17a**, Proposition 23.13] that says that cohomological smoothness is "cohomologically smooth local on the source". More precisely, if checked for one then for all $g : V \to X$ separated cohomologically smooth (not necessarily surjective) from a locally spatial diamond V, $f \circ g$ is separated cohomologically smooth.

Convention IV.1.12. In the following, whenever we say that a map $f : Y \to X$ of Artin v-stacks is cohomologically smooth, we demand that there is some separated, representable in locally spatial diamonds, and cohomologically smooth surjection $g : V \to Y$ from a locally spatial diamond V such that $f \circ g : V \to X$ is separated. Note that this condition can be tested after taking covers $U \to X$ by separated, representable in locally spatial diamonds, and cohomologically smooth surjections; i.e. after replacing X by $X \times_Y U$ and Y by U. If X and Y have the property that one can find a cover $U \to X$, $V \to Y$, as above with U and V perfectoid spaces, and Δ_X is representable in perfectoid spaces, then the condition is automatic, as all maps of perfectoid spaces

are locally separated. That being said there is no reason that this is true in general since there are morphisms of spatial diamonds that are not locally separated.

We will not try to give a completely general 6-functor formalism that includes functors $Rf_!$ and $Rf^!$ for stacky maps f (this would require some ∞-categorical setting). However, we can extend the functor $Rf^!$ to cohomologically smooth maps of Artin v-stacks. Let Λ be a ring killed by some integer n prime to p, or an adic ring as in [**Sch17a**, Section 26].

Definition IV.1.13. *Let $f : Y \to X$ be a cohomologically smooth map of Artin v-stacks. The dualizing complex $Rf^!\Lambda \in D_{\text{ét}}(Y, \Lambda)$ is the invertible object equipped with isomorphisms*

$$Rg^!(Rf^!\Lambda) \cong R(f \circ g)^!\Lambda$$

for all separated, representable in locally spatial diamonds, and cohomologically smooth maps $g : V \to Y$ from a locally spatial diamond V, such that for all cohomologically smooth maps $h : V' \to V$ between such $g' : V' \to Y$ and $g : V \to Y$, the composite isomorphism

$$R(g')^!(Rf^!\Lambda) \cong R(f \circ g')^!\Lambda \cong R(f \circ g \circ h)^!\Lambda \cong Rh^!(R(f \circ g)^!\Lambda) \cong Rh^!(Rg^!(Rf^!\Lambda)) \cong R(g')^!(Rf^!\Lambda)$$

is the identity.

As $Rf^!\Lambda$ is locally concentrated in one degree, it is easy to see that $Rf^!\Lambda$ is unique up to unique isomorphism. Let us be more precise. Let $\mathcal{C}$ be the category whose objects are separated cohomologically smooth morphisms $V \to Y$ with V a locally spatial diamond, and morphisms $(V' \xrightarrow{g'} Y) \to (V \xrightarrow{g} Y)$ are couples (h, α) where $h : V' \to V$ is separated cohomologically smooth and $\alpha : g \circ h \Rightarrow g'$ is a 2-morphism. Then the rule

$$(V \xrightarrow{g} Y) \longmapsto R\mathscr{H}\mathrm{om}_\Lambda(Rg^!\Lambda, R(f \circ g)^!\Lambda)$$

defines an element of

$$2\text{-}\varprojlim_{(V \to Y) \in \mathcal{C}} \{\text{invertible objects in } D_{\text{ét}}(V, \Lambda)\} \cong \{\text{invertible objects in } D_{\text{ét}}(Y, \Lambda)\}.$$

Remark IV.1.14. If $g : V \to Y$ is a compactifiable representable in locally spatial diamonds morphism of small v-stacks with $\dim.\operatorname{trg} g < \infty$ such that $f \circ g$ satisfies the same hypothesis, it is not clear that $Rg^!(Rf^!\Lambda) \cong R(f \circ g)^!\Lambda$. This is a priori true only when V is a locally spatial diamond and g is separated cohomologically smooth, the only case we will need.

Definition IV.1.15. *Let $f : Y \to X$ be a cohomologically smooth map of Artin v-stacks. The functor*

$$Rf^! : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$$

is given by $Rf^! = Rf^!\Lambda \otimes^{\mathbb{L}}_{\Lambda} f^*$.

Remark IV.1.16. Checking after a cohomologically smooth cover, one sees that $Rf^!$ preserves all limits (and colimits) and hence admits a left adjoint $Rf_!$.

Definition IV.1.17. *Let $f : Y \to X$ be a cohomologically smooth map of Artin v-stacks and let $\ell \neq p$ be a prime. Then f is pure of ℓ-dimension $d \in \frac{1}{2}\mathbb{Z}$ if $Rf^!\mathbb{F}_\ell$ sits locally in homological degree $2d$.*

As $Rf^!\mathbb{F}_\ell$ is locally constant, any cohomologically smooth map $f : Y \to X$ of Artin v-stacks decomposes uniquely into a disjoint union of $f_d : Y_d \to X$ that are pure of ℓ-dimension d. A priori this decomposition may depend on ℓ and include half-integers d, but this will not happen in any examples that we study.

IV.1.2. The case of Bun_G.

IV.1.2.1. *Smooth charts on* Bun_G. One important example is the following. We use Beauville–Laszlo uniformization to construct cohomologically smooth charts on Bun_G. More refined charts will be constructed in Theorem V.3.7. For $\bar{\mu} \in X_*(T)^+/\Gamma$ we note $\mathrm{Gr}_{G,\bar{\mu}}$ for the subsheaf of Gr_G such that $\mathrm{Gr}_{G,\bar{\mu}} \times_{\mathrm{Spd}(E)} \mathrm{Spd}(E') = \coprod_{\mu' \equiv \mu} \mathrm{Gr}_{G,\mu'}$ where $E'|E$ is a finite degree Galois extension splitting G. We will use the following simple proposition.

Proposition IV.1.18. *For any $\mu \in X_*(T)^+$, the open Schubert cell $\mathrm{Gr}_{G,\mu} / \mathrm{Spd}\, E'$ is cohomologically smooth of ℓ-dimension $\langle 2\rho, \mu \rangle$.*

We defer the proof to Proposition VI.2.4 as we do not want to make a digression on Gr_G here.

Theorem IV.1.19. *The stack Bun_G is a cohomologically smooth Artin v-stack of ℓ-dimension 0. The Beauville–Laszlo map defines a separated cohomologically smooth cover*

$$\coprod_{\bar{\mu} \in X_*(T)^+/\Gamma} [\underline{G(E)} \backslash \mathrm{Gr}_{G,\bar{\mu}}] \longrightarrow \mathrm{Bun}_G\,.$$

Proof. We check first that Δ_{Bun_G} is representable in locally spatial diamonds. For this, it suffices to see that for a perfectoid space S with two G-bundles $\mathcal{E}_1, \mathcal{E}_2$ on X_S, the functor of isomorphisms between $\mathcal{E}_1$ and $\mathcal{E}_2$ is representable by a locally spatial diamond over S. By the Tannakian formalism, one can reduce to vector bundles. For example, according to Chevalley, one can find a faithful linear representation $\rho : G \to \mathrm{GL}_n$, a representation $\rho' : \mathrm{GL}_n \to \mathrm{GL}(W)$, and a line $D \subset W$ such that G is the stabilizer of D inside $\mathrm{GL}(V)$. Then G-bundles on X_S are the same as a rank n vector bundle $\mathcal{E}$ together with a sub-line bundle $\mathcal{L}$ inside $\rho'_*\mathcal{E}$. In terms of those data, isomorphisms between $(\mathcal{E}_1, \mathcal{L}_1)$ and $(\mathcal{E}_2, \mathcal{L}_2)$ are given by a couple (α, β) where $\alpha : \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$, and $\beta : \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ satisfy $(\rho'_*\alpha)_{|\mathcal{L}_1} = \beta$. Since the category of locally spatial diamonds is stable under finite projective limits we are reduced to the case of the linear group. Now the result is given by Lemma IV.1.20.

It remains to construct cohomologically smooth charts for Bun_G. We first prove that the morphism

$$\pi : \coprod_{\bar{\mu} \in X_*(T)^+/\Gamma} [\underline{G(E)} \backslash \mathrm{Gr}_{G,\bar{\mu}}] \longrightarrow \mathrm{Bun}_G \times_k \mathrm{Spd}\, \breve{E}$$

is separated cohomologically smooth. Since this is surjective at the level of geometric points we deduce that it is a v-cover, cf. Remark IV.1.5.

To verify this, note that for a perfectoid space S mapping to $\mathrm{Bun}_G \times_k \mathrm{Spd}\, \breve{E}$ corresponding to a G-bundle $\mathcal{E}$ on X_S as well as a map $S \to \mathrm{Spd}\, E$ inducing an untilt $S^\sharp/E$ and a closed immersion $i : S^\sharp \to X_S$, the fibre of π over S parametrizes modifications of $\mathcal{E}$ of locally constant type that are trivial at each geometric point. This is open in the space of all modifications of $\mathcal{E}$ of locally constant

type, which is v-locally isomorphic to $\bigsqcup_{\bar{\mu}} \mathrm{Gr}_{G,\bar{\mu},E} \times_{\mathrm{Spd}\, E} S \to S$. Thus, Proposition IV.1.18 gives the desired cohomological smoothness.

Moreover, the preceding argument shows that when restricted to $[\underline{G(E)}\backslash \mathrm{Gr}_{G,\bar{\mu}}]$, the map π has ℓ-dimension equal to $\langle 2\rho, \mu\rangle$. Thus, it now suffices to see that $[\underline{G(E)}\backslash \mathrm{Gr}_{G,\bar{\mu}}]$ is an ℓ-cohomologically smooth Artin v-stack of ℓ-dimension equal to $\langle 2\rho, \mu\rangle$. But the map

$$[\underline{G(E)}\backslash \mathrm{Gr}_{G,\bar{\mu}}] \to [\mathrm{Spd}\, E/\underline{G(E)}]$$

is representable in locally spatial diamonds and ℓ-cohomologically smooth of ℓ-dimension equal to $\langle 2\rho, \mu\rangle$, as $\mathrm{Gr}_{G,\bar{\mu}} \to *$ is by Proposition IV.1.18. We conclude by using that $[*/\underline{G(E)}] \to *$ is an Artin v-stack, cohomologically smooth of ℓ-dimension 0, by Example IV.1.9 (4). □

Lemma IV.1.20. *For $\mathcal{E}_1, \mathcal{E}_2$ vector bundles on X_S, the sheaf $T/S \mapsto \{$surjections $\mathcal{E}_1|_{X_T} \twoheadrightarrow \mathcal{E}_2|_{X_T}\}$, resp. $T/S \mapsto \mathrm{Isom}(\mathcal{E}_1|_{X_T}, \mathcal{E}_2|_{X_T})$, is representable by an open subdiamond of $\mathcal{BC}(\mathcal{E}_1^\vee \otimes \mathcal{E}_2)$. In particular, those are locally spatial diamonds.*

Proof. The case of isomorphisms is reduced to the case of surjections since a morphism u of vector bundles is an isomorphisms if and only if u and $u^\vee$ are surjective. For any morphism $g : \mathcal{E}_1 \to \mathcal{E}_2$, the support of its cokernel is a closed subset of $|X_S|$, whose image in $|S|$ is thus closed; this implies the result. □

Remark IV.1.21. It would be tempting to study $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ using the preceding charts. But, contrary to the sheaves coming from the geometric Satake correspondence, the sheaves on Gr_G obtained via pullback from Bun_G are not locally constant on open Schubert strata. We will prefer other smooth charts to study $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$, see Theorem V.3.7.

Moreover, each HN stratum Bun_G^b gives another example.

Proposition IV.1.22. *For every $b \in B(G)$, the stratum Bun_G^b is a cohomologically smooth Artin v-stack of ℓ-dimension $-\langle 2\rho, \nu_b\rangle$.*

Proof. Under the identification $\mathrm{Bun}_G^b \cong [*/\widetilde{G}_b]$, note that we have a map $[*/\widetilde{G}_b] \to [*/\underline{G_b(E)}]$ where the target is a cohomologically smooth Artin v-stack of dimension 0, while the fibre admits a cohomologically smooth surjection from $*$ (as positive Banach–Colmez spaces are cohomologically smooth) of ℓ-dimension $\langle 2\rho, \nu_b\rangle$. This gives the result. □

IV.1.2.2. *Connected components of* Bun_G. A consequence is that we can classify the connected components of Bun_G.

Corollary IV.1.23. *The Kottwitz map induces a bijection*

$$\kappa : \pi_0(\mathrm{Bun}_G) \to \pi_1(G)_\Gamma.$$

Proof. The Kottwitz map is well-defined and surjective. It remains to see that it is injective. To see this, recall that the basic elements of $B(G)$ biject via κ to $\pi_1(G)_\Gamma$. Thus, it suffices to see that any nonempty open subsheaf U of Bun_G contains a basic point. Note that the topological space $(X_*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma$ equipped with the product topology given by the order on $(X_*(T)^+_{\mathbb{Q}})^\Gamma$ and the discrete topology on $\pi_1(G)_\Gamma$, is (T0), and an increasing union of finite open subspaces; and $|\,\mathrm{Bun}_G\,|$ maps continuously to it. Pick some finite open $V \subset (X_*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma$ such that its

preimage in U is a nonempty open $U' \subset U$. Then U' is a nonempty finite (T0) space, and thus has an open point by Lemma IV.1.24.

Thus, there is some $b \in B(G)$ such that $\mathrm{Bun}_G^b \subset U \subset \mathrm{Bun}_G$ is open. Combining Theorem IV.1.19 and Proposition IV.1.22, this forces $-\langle 2\rho, \nu_b \rangle = 0$, i.e. ν_b is central, which means that b is basic. □

LEMMA IV.1.24. *If X is a nonempty finite spectral space, that is to say a finite (T0) topological space, there exists an open point $x \in X$.*

PROOF. Take x maximal for the specialization relation, i.e. x is a maximal point. Then, since X is (T0), $X \setminus \{x\} = \cup_{y \neq x} \overline{\{y\}}$, a finite union of closed spaces thus closed. □

IV.2. Universally locally acyclic sheaves

IV.2.1. Definition and basic properties. In many of our results, and in particular in the (formulation and) proof of the geometric Satake equivalence, a critical role is played by the notion of universally locally acyclic (ULA) sheaves. Roughly speaking, for a morphism $f : X \to S$ of schemes, these are constructible complexes of étale sheaves A on X whose relative cohomology is constant in all fibres of S, even locally. Technically, one requires that for all geometric points $\overline{x}$ of X mapping to a geometric point $\overline{s}$ of S and a generization $\overline{t}$ of $\overline{s}$ in S, the natural map

$$R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$

is an isomorphism, where $X_{\overline{x}}$ is the strict henselization of X at $\overline{x}$ (and $S_{\overline{s}}$ is defined similarly). Moreover, the same property should hold universally after any base change along $S' \to S$.[1] By [**Ill06**, Corollary 3.5], universal local acyclicity is equivalent to asking that, again after any base change, the map

$$R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A)$$

is an isomorphism; we prefer the latter formulation as strict henselizations admit analogues for adic spaces, while the actual fibre over a point is only a pseudo-adic space in Huber's sense [**Hub96**].

In the world of adic spaces, there are not enough specializations to make this an interesting definition; for example, there are no specializations from $\mathrm{Gr}_{G,\mu}$ into $\mathrm{Gr}_{G,\leq\mu} \setminus \mathrm{Gr}_{G,\mu}$. Thus, we need to adapt the definition by adding a condition on preservation of constructibility that is automatic in the scheme case under standard finiteness hypothesis, but becomes highly nontrivial in the case of adic spaces.

DEFINITION IV.2.1. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$ *and let $A \in D_{\text{ét}}(X, \Lambda)$ for some Λ with $n\Lambda = 0$ with n prime to p.*

(i) *The sheaf of complexes A is f-locally acyclic if*
(a) *For all geometric points $\overline{x}$ of X with image $\overline{s}$ in S and a generization $\overline{t}$ of $\overline{s}$, the map*

$$R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A)$$

is an isomorphism.

[1] Recently, Gabber proved that this is automatic when S is noetherian and f is of finite type, cf. [**LZ19**, Corollary 6.6].

(b) *For all separated étale maps $j : U \to X$ such that $f \circ j$ is quasicompact, the complex $R(f \circ j)_!(A|_U) \in D_{\text{ét}}(S, \Lambda)$ is perfect-constructible.*

(ii) *The sheaf of complexes A is f-universally locally acyclic if for any map $S' \to S$ of locally spatial diamonds with base change $f' : X' = X \times_S S' \to S'$ and $A' \in D_{\text{ét}}(X', \Lambda)$ the pullback of A, the sheaf of complexes A' is f'-locally acyclic.*

Recall that if (K, K^+) is an affinoid Huber field, $S = \mathrm{Spa}(K, K^+)$, then $|S| = |\operatorname{Spec}(K^+/K^{00})|$ as a topological spectral space, that is identified with the totally ordered set of open prime ideals in K^+. For any $s \in S$, $S_s \subset S$ is pro-constructible generalizing. For example, the maximal generalization is $\mathrm{Spa}(K, \mathcal{O}_K) = \cap_{a \in \mathcal{O}_K} \{|a| \leq 1\}$.

Remark IV.2.2. In the setup of condition (a), note that $X_{\overline{x}}$ is a strictly local space, i.e. of the form $\mathrm{Spa}(C, C^+)$ where C is algebraically closed and $C^+ \subset C$ is an open and bounded valuation subring; thus, $R\Gamma(X_{\overline{x}}, A) = A_{\overline{x}}$ is just the stalk of A. Moreover, $S_{\overline{t}} \subset S_{\overline{s}}$ is a quasicompact pro-constructible generalizing subspace, and thus $X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}} \subset X_{\overline{x}}$ is itself a quasicompact pro-constructible generalizing subset that is strictly local. Its closed point $\overline{y}$ is the minimal generization of $\overline{x}$ mapping to $\overline{t}$, and $R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A) = A_{\overline{y}}$ is the stalk at $\overline{y}$. Thus, condition (a) means that A is "overconvergent" along the horizontal lifts of generizations of S.

Remark IV.2.3.

(i) Another way to phrase the "relative overconvergence condition" (a), is to say that if $\bar{s}$ is a geometric point of S, $\bar{t}$ a generization of $\bar{s}$, $j : X \times_S S_{\bar{t}} \hookrightarrow X \times_S S_{\bar{s}}$, a pro-constructible generalizing immersion, and $B = A|_{X \times_S S_{\bar{s}}}$, then $B = Rj_* j^* B$ (use quasicompact base change).

(ii) Still another way to phrase it is to say that for any $\mathrm{Spa}(C, C^+) \to S$, if $B = A|_{X \times_S \mathrm{Spa}(C, C^+)}$, and $j : X \times_S \mathrm{Spa}(C, \mathcal{O}_C) \hookrightarrow X \times_S \mathrm{Spa}(C, C^+)$, then $B = Rj_* j^* B$.

(iii) Still another way is to say that if $\bar{x} \mapsto \bar{s}$ and $f_{\bar{x}} : X_{\bar{x}} \to S_{\bar{s}}$ then $Rf_{\bar{x}*} A|_{X_{\bar{x}}}$ is overconvergent i.e. constant.

In fact, asking for condition (a) universally, i.e. after any base change, amounts to asking that A is overconvergent.

Proposition IV.2.4. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$ *and let $A \in D_{\text{ét}}(X, \Lambda)$ for some Λ with $n\Lambda = 0$ with n prime to p. The condition (a) of Definition IV.2.1 holds after any base change $S' \to S$ if and only if A is overconvergent, i.e. for any specialization $\overline{y} \rightsquigarrow \overline{x}$ of geometric points of X, the map $A_{\overline{x}} \to A_{\overline{y}}$ is an isomorphism.*

Proof. The condition is clearly sufficient. For necessity, take the base change along $X_{\overline{x}} \to S$. Then $\overline{x}$ lifts to a section $\overline{x}' : X_{\overline{x}} \to X \times_S X_{\overline{x}}$, and applying the relative overconvergence condition to $\overline{x}' \mapsto \overline{x}$ and the generization $\overline{y}$ of $\overline{x}$, we see that $A_{\overline{x}} \to A_{\overline{y}}$ is an isomorphism. □

Proposition IV.2.5. *Local acyclicity descends along v-covers of the target. More precisely, in the setup of Definition IV.2.1, if $S' \to S$ is a v-cover and A' is f'-locally acyclic, then automatically A is f-locally acyclic.*

Proof. Condition (a) follows by lifting geometric points, and condition (b) descends by [**Sch17a**, Proposition 20.13]. □

PROPOSITION IV.2.6. *Let Y be a spatial diamond.*

(i) *If $\mathcal{F}$ is a constructible étale sheaf of Λ-modules on Y, then $\mathcal{F}$ is locally constant if and only if $\mathcal{F}$ is overconvergent.*

(ii) *If $A \in D_{\text{ét},pc}(Y, \Lambda)$, then A is overconvergent if and only if it is locally a constant perfect complex of Λ-modules.*

PROOF. For a geometric point $\overline{y}$ of Y, writing $Y_{\overline{y}} = \mathrm{Spa}(C, C^+) = \varprojlim_{\overline{y} \to U} U$ as a limit of the étale neighborhoods, according to [**Sch17a**, Proposition 20.7],

$$2\text{-}\varinjlim_{\overline{y} \to U} \mathrm{Cons}(U, \Lambda) = \mathrm{Cons}(Y_{\overline{y}}, \Lambda).$$

An étale sheaf on $Y_{\overline{y}} = \mathrm{Spa}(C, C^+)$ is locally constant if and only if it is constant if and only if it is overconvergent. This gives point (1). Point (2) goes the same way using [**Sch17a**, Proposition 20.15]. □

REMARK IV.2.7. The preceding argument shows that if $\mathcal{F}$ is constructible on Y then $\mathcal{F}$ is locally constant in a neighborhood of any maximal point of Y. For example, if $Y = X^\diamond$ with X a K-rigid space, then any constructible sheaf on Y is locally constant in a neighborhood of all classical Tate points of X. Thus, the difference between constructible and locally constant sheaves shows up at rank > 1 valuations.

EXAMPLE IV.2.8. Let $j : \mathbb{B}^1_K \setminus \{0\} \hookrightarrow \mathbb{B}^1_K$ be the inclusion of the punctured disk inside the disk. Then $j_!\Lambda$ is not constructible since not locally constant around $\{0\}$. Nevertheless, if $R \in |K^\times|$ and x is the coordinate on $\mathbb{B}^1_K$, $j_R : \{R \leq |x| \leq 1\} \hookrightarrow \mathbb{B}^1_K$, $j_{R!}\Lambda$ is constructible and $j_!\Lambda = \varinjlim_{R \to 0} j_{R!}\Lambda$. The category of étale sheaves of Λ-modules on a spatial diamond is the Ind-category of constructible étale sheaves, cf. [**Sch17a**, Proposition 20.6].

PROPOSITION IV.2.9. *Assume that $f : S \to S$ is the identity. Then $A \in D_{\text{ét}}(S, \Lambda)$ is f-locally acyclic if and only if it is locally constant with perfect fibres.*

PROOF. Applying part (b) of the definition, we see that A is perfect-constructible. On the other hand, part (a) says that A is overconvergent. This implies that A is locally constant by Proposition IV.2.6. □

Let us finish with a basic example of universally locally acyclic sheaves relevant to the smooth base change theorem. A more general result will be given in Proposition IV.2.13.

PROPOSITION IV.2.10. *Assume that $f : X \to S$ is a separated map of locally spatial diamonds that is ℓ-cohomological smooth for all divisors ℓ of n, where $n\Lambda = 0$. If $A \in D_{\text{ét}}(X, \Lambda)$ is locally constant with perfect fibres, then A is f-universally locally acyclic.*

PROOF. It is enough to show that A is f-locally acyclic, as the hypotheses are stable under base change. Condition (a) follows directly from A being locally constant. Condition (b) follows from the preservation of constructible sheaves of complexes under $Rf_!$ if f is quasicompact, separated and cohomologically smooth, see [**Sch17a**, Proposition 23.12 (ii)]. □

IV.2.2. Proper push-forward, smooth pull-back. In the "classical algebraic case", if $Y \xrightarrow{g} X \xrightarrow{f} S$ are morphisms of finite type between noetherian schemes, using proper and smooth base change:

(i) if g is proper and $A \in D^b_c(Y, \Lambda)$ is $f \circ g$ -locally acyclic then Rg_*A is f-locally acyclic;
(ii) if g is smooth and $A \in D^b_c(X, \Lambda)$ is f-locally acyclic then g^*A is $f \circ g$-locally acyclic. Moreover if g is surjective then $A \in D^b_c(X, \Lambda)$ is f-locally acyclic if and only if g^*A is $f \circ g$-locally acyclic.

We are going to see that the same phenomenon happens in our context. The fact that local acyclicity is smooth local on the source is essential to define local acyclicity for morphisms of Artin v-stacks, cf. Definition IV.2.31.

Proposition IV.2.11. *Let $g : Y \to X$, $f : X \to S$ be maps of locally spatial diamonds where g is proper and f is compactifiable and locally* $\dim.\operatorname{trg} g, \dim.\operatorname{trg} f < \infty$. *Assume that $A \in D_{\text{ét}}(Y, \Lambda)$ is $f \circ g$-locally acyclic (resp. $f \circ g$-universally locally acylic). Then $Rg_*A \in D_{\text{ét}}(X, \Lambda)$ is f-locally acyclic (resp. f-universally locally acyclic).*

Proof. It is enough to consider the locally acyclic case, as the hypotheses are stable under base change. For condition (a), we use Remark IV.2.3 (2). Let $\bar{s}$ be a geometric point of S, with $S_{\bar{s}} = \operatorname{Spa}(C, C^+)$. Let us look at the cartesian diagram

$$\begin{array}{ccc} Y \times_{\operatorname{Spa}(C,C^+)} \operatorname{Spa}(C, \mathcal{O}_C) & \xhookrightarrow{k} & Y \\ \downarrow & & \downarrow g \\ X \times_{\operatorname{Spa}(C,C^+)} \operatorname{Spa}(C, \mathcal{O}_C) & \xhookrightarrow{j} & X \end{array}$$

one has by local acyclicity of $f \circ g$, $A = Rk_*k^*A$. Applying Rg_*, this gives the desired

$$Rg_*A = Rj_*j^*(Rg_*A).$$

For condition (b), take any separated étale map $j : U \to X$ such that $f \circ j$ is quasicompact, and set $j' : V = U \times_X Y \to Y$, which is an étale map such that $f \circ g \circ j'$ is quasicompact. Let $g' : V \to U$ denote the pullback of g. Using proper base change and $Rg_* = Rg_!$, we see that

$$R(f \circ j)_! j^* Rg_*A = R(f \circ j)_! Rg'_! j'^* A = R(f \circ g \circ j')_! j'^* A,$$

which is perfect-constructible by the assumption that A is $f \circ g$-locally acyclic. □

In particular we have the following that generalizes the "proper and smooth case".

Corollary IV.2.12. *Let $f : X \to S$ be a proper map of locally spatial diamonds with* $\dim.\operatorname{trg} f < \infty$ *and $A \in D_{\text{ét}}(X, \Lambda)$ that is f-locally acyclic. Then Rf_*A is locally a constant perfect complex of Λ-modules.*

Next proposition says that *local acyclicity is "cohomologically smooth local" on the source.*

Proposition IV.2.13. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$. *For the statements in the locally acyclic case below, assume that S is spatial and that the cohomological dimension of $U_{\text{ét}}$ for all quasicompact separated étale $U \to S$ is $\leq N$ for some fixed integer N.*

Let $A \in D_{\text{ét}}(X, \Lambda)$ where $n\Lambda = 0$ for some n prime to p and let $g : Y \to X$ be a separated map of locally spatial diamonds that is ℓ-cohomologically smooth for all ℓ dividing n.

(i) *If A is f-locally acyclic (resp. f-universally locally acyclic), then g^*A is $f \circ g$-locally acyclic (resp. $f \circ g$-universally locally acyclic).*

(ii) *Conversely, if g^*A is $f \circ g$-locally acyclic (resp. $f \circ g$-universally locally acyclic) and g is surjective, then A is f-locally acyclic (resp. f-universally locally acyclic).*

Proof. It is enough to handle the locally acyclic case with the assumption on S; then the universally locally acyclic case follows by testing after pullbacks to strictly totally disconnected spaces. Let us treat point (i). We can assume that X and Y are qcqs, i.e. spatial. In fact, this is clear for condition (a). For condition (b), if $j : V \to Y$ is separated étale such that $f \circ g \circ j$ is quasicompact, up to replacing S by an open cover we can suppose that S is spatial and thus V is spatial (since f, g, and j are separated, $f \circ g \circ j$ is separated quasicompact, and thus S spatial implies X spatial). Then one can replace Y, resp. X, by the quasicompact open subsets $j(V)$, resp. $(g \circ j)(V)$, that are separated over S and thus spatial too.

Condition (a) follows as pullbacks preserve stalks. For condition (b), let $j : V \to Y$ be any quasicompact separated étale map. Then by the projection formula for $g \circ j$, one has

$$R(f \circ g \circ j)_! j^* g^* A = Rf_!(A \otimes^{\mathbb{L}}_{\Lambda} R(g \circ j)_! \Lambda).$$

As $g \circ j : V \to X$ is a quasicompact separated ℓ-cohomologically smooth map, it follows that $R(g \circ j)_! \Lambda \in D_{\text{ét}}(X, \Lambda)$ is perfect-constructible by [**Sch17a**, Proposition 23.12 (ii)]. Thus, the desired result follows from Lemma IV.2.14.

In the converse direction, i.e. for part (ii), condition (a) of A being f-locally acyclic follows by lifting geometric points from X to Y and noting that stalks do not change. For condition (b), we may replace X by U to reduce to the assertion that $Rf_! A \in D_{\text{ét}}(S, \Lambda)$ is perfect-constructible. Consider the thick triangulated subcategory $\mathcal{C}$ of $D_{\text{ét}}(X, \Lambda)$ of all $B \in D_{\text{ét}}(X, \Lambda)$ such that $Rf_!(A \otimes^{\mathbb{L}}_{\Lambda} B) \in D_{\text{ét}}(S, \Lambda)$ is perfect-constructible. We have to see that $\Lambda \in \mathcal{C}$. We know that for all perfect-constructible $C \in D_{\text{ét}}(Y, \Lambda)$, the perfect-constructible complex $Rg_! C$ lies in $\mathcal{C}$. Indeed, using [**Sch17a**, Proposition 20.17], this reduces to the case $C = j_! \Lambda$ where $j : U \to Y$ is a quasicompact separated étale map, and then

$$Rf_!(A \otimes^{\mathbb{L}}_{\Lambda} R(g \circ j)_! \Lambda) = R(f \circ g)_!(g^* A \otimes^{\mathbb{L}}_{\Lambda} Rj_! \Lambda),$$

which is perfect-constructible as g^*A is $f \circ g$-locally acyclic. Thus, it is enough to show that the set of $Rg_! C \in D_{\text{ét}}(X, \Lambda)$ with $C \in D_{\text{ét}}(Y, \Lambda)$ perfect constructible form a set of compact generators of $D_{\text{ét}}(X, \Lambda)$. Equivalently, for any complex $B \in D_{\text{ét}}(X, \Lambda)$ with $R\operatorname{Hom}_{D_{\text{ét}}(X,\Lambda)}(Rg_! C, B) = 0$ for all perfect-constructible $C \in D_{\text{ét}}(Y, \Lambda)$, then $B = 0$. The hypothesis is equivalent to $R\operatorname{Hom}_{D_{\text{ét}}(Y,\Lambda)}(C, Rg^! B) = 0$ for all such C. By [**Sch17a**, Proposition 20.17] and the standing assumptions on finite cohomological dimension (on S, f and g), this implies that $Rg^! B = 0$. As g is ℓ-cohomologically smooth, this is equivalent to $g^* B = 0$, which implies $B = 0$ as g is surjective. □

Lemma IV.2.14. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$. *Suppose there exists an integer N such that the cohomological dimension of $U_{\text{ét}}$ is bounded by N for all $U \to X$ separated étale. Let $A \in D_{\text{ét}}(X, \Lambda)$ be f-locally acyclic and $B \in D_{\text{ét}}(X, \Lambda)$ be perfect-constructible. Then $A \otimes^{\mathbb{L}}_{\Lambda} B$ satisfies condition (ii) of Definition*

IV.2.1: for any $j : U \to X$ separated étale such that $f \circ j$ is quasicompact, $R(f \circ j)_! j^*(A \otimes^{\mathbb{L}}_\Lambda B)$ is perfect-constructible.

Proof. We can suppose X is spatial. According to [**Sch17a**, Proposition 20.17], B lies in the triangulated subcategory generated by $j'_!\Lambda$ where $j' : U' \to X$ is separated quasicompact étale. For such a B, using the projection formula, $A \otimes^{\mathbb{L}}_\Lambda Rj'_!\Lambda = Rj'_! j'^* A$. Thus, if $V = U \times_X U'$ with projection $k : U \times_X U' \to U$,

$$j^* Rj'_! j'^* A = Rk_! k^* j^* A.$$

We thus have

$$R(f \circ j)_! j^*(A \otimes^{\mathbb{L}}_\Lambda Rj'_! B) = R(f \circ j \circ k)_!(j \circ k)^*(A)$$

and we can conclude. □

IV.2.3. Local acyclicity and duality. In this section, we prove that universal local acyclicity behaves well with respect to Verdier duality.

IV.2.3.1. *Compatibility with base change.* We note that for f-ULA sheaves, the formation of the (relative) Verdier dual

$$\mathbb{D}_{X/S}(A) := R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!\Lambda)$$

commutes with base change in S.

Proposition IV.2.15. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\mathrm{trg}\, f < \infty$ *and let $A \in D_{\text{ét}}(X, \Lambda)$ be f-universally locally acyclic. Let $g : S' \to S$ be a map of locally spatial diamonds with pullback $f' : X' = X \times_S S' \to S'$, $\widetilde{g} : X' \to X$. Then the composite*

$$\widetilde{g}^* \mathbb{D}_{X/S}(A) \to \mathbb{D}_{X'/S'}(\widetilde{g}^* A)$$

of the natural maps

$$\widetilde{g}^* R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!\Lambda) \to R\mathscr{H}\mathrm{om}_\Lambda(\widetilde{g}^* A, \widetilde{g}^* Rf^!\Lambda) \to R\mathscr{H}\mathrm{om}_\Lambda(\widetilde{g}^* A, Rf'^!\Lambda)$$

is an isomorphism.

More generally, for any $B \in D_{\text{ét}}(S, \Lambda)$, the map

$$\widetilde{g}^* R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! B) \to R\mathscr{H}\mathrm{om}_\Lambda(\widetilde{g}^* A, Rf'^! B)$$

is an isomorphism.

Proof. The assertion is local, so we may assume that X, S and S' are spatial. By choosing a strictly totally disconnected cover S'' of S', one reduces the result for $S' \to S$ to the cases of $S'' \to S'$ and $S'' \to S$, so we may assume that S' is strictly totally disconnected. In that case, by [**Sch17a**, Proposition 20.17], whose hypothesis apply as $X' \to S'$ is of finite dim. trg and S' is strictly totally disconnected, it suffices to check on global sections over all quasicompact separated étale maps $V' \to X'$. According to Lemma IV.2.16 we can write S' as a cofiltered limit of quasicompact open subsets S'_i of finite-dimensional balls over S. Then V' comes via pullback from a quasicompact

separated étale map $V_i \to X \times_S S'_i$ for i large enough by [**Sch17a**, Proposition 11.23]. We thus have a diagram with cartesian squares

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V' \arrow[r] \arrow[d] \arrow[dr, "h_i"] & {} \\
X' \arrow[r] \arrow[d] & X \times_S S'_i \arrow[r] \arrow[d] & X \arrow[d] \\
S' \arrow[r] & S'_i \arrow[r] & S.
\end{tikzcd}$$

The result we want to prove is immediate when $S' \to S$ is cohomologically smooth. Up to replacing $X \to S$ by $V' \to S'_i$ we are thus reduced to prove that $R\Gamma(X', \widetilde{g}^* \mathbb{D}_{X/S}(A)) \xrightarrow{\sim} R\Gamma(X', \mathbb{D}_{X'/S'}(\widetilde{g}^* A))$.

Thus, it suffices to check the result after applying Rf'_*. In that case,

$$\begin{aligned} Rf'_* \widetilde{g}^* R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! B) &= g^* Rf_* R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! B) \\ &= g^* R\mathscr{H}\mathrm{om}_\Lambda(Rf_! A, B), \end{aligned}$$

using [**Sch17a**, Proposition 17.6, Theorem 1.8 (iv)]. On the other hand,

$$\begin{aligned} Rf'_* R\mathscr{H}\mathrm{om}_\Lambda(\widetilde{g}^* A, Rf'^! B) &= R\mathscr{H}\mathrm{om}_\Lambda(Rf'_! \widetilde{g}^* A, B) \\ &= R\mathscr{H}\mathrm{om}_\Lambda(g^* Rf_! A, B) \end{aligned}$$

using [**Sch17a**, Theorem 1.8 (iv), Theorem 1.9 (ii)]. But by condition (b) of being f-locally acyclic, the complex $Rf_! A \in D_{\text{ét}}(S, \Lambda)$ is perfect-constructible, and thus the formation of $R\mathscr{H}\mathrm{om}_\Lambda(Rf_! A, B)$ commutes with any base change by Lemma IV.2.17. □

Lemma IV.2.16. *Let S be a spatial diamond and $X \to S$ be a morphism from an affinoid perfectoid space to S. Then one can write $X = \varprojlim_i U_i$ where U_i is a quasicompact open subset inside a finite dimensional ball over S, and the projective limit is cofiltered.*

Proof. If $I = \mathcal{O}(X)^+$, one has a closed immersion over S defined by elements of I, $X \hookrightarrow \mathbb{B}^I_S$ where $\mathbb{B}^I_S$ is the spatial diamond over S that represents the functor $T/S \mapsto (\mathcal{O}(T)^+)^I$ (an "infinite dimensional perfectoid ball over S" when S is perfectoid). Now,

$$\mathbb{B}^I_S = \varprojlim_{J \subset I} \mathbb{B}^J_S$$

where J goes through the set of finite subsets of I and $\mathbb{B}^I_S \to \mathbb{B}^J_S$ is the corresponding projection. For each such J the composite $X \hookrightarrow \mathbb{B}^I_S \twoheadrightarrow \mathbb{B}^J_S$ is a spatial morphism of spatial diamonds. Its image is a pro-constructible generalizing subset of $\mathbb{B}^J_S$ and can thus be written as $\bigcap_{\alpha \in A_J} U_\alpha$ where U_α is a quasicompact open subset of $\mathbb{B}^J_S$. Then one has

$$X = \varprojlim_{J \subset I} \varprojlim_{\alpha \in A_J} U_\alpha. \qquad \square$$

Lemma IV.2.17. *Let X be a spatial diamond and $A \in D_{\text{ét}}(X, \Lambda)$ perfect-constructible, and let $B \in D_{\text{ét}}(X, \Lambda)$ be arbitrary. Then the formation of $R\mathscr{H}\mathrm{om}_\Lambda(A, B)$ commutes with any base change.*

Proof. Using [**Sch17a**, Proposition 20.16 (iii)] this is reduced to the case when $A = j_!(\mathcal{L}|_Z)$ where $j : U \to X$ is separated quasicompact étale, $Z \subset U$ closed constructible, and $\mathcal{L} \in D_{\text{ét}}(U, \Lambda)$ locally constant with perfect fibres. If $j' : U \setminus Z \to X$, that is again quasicompact (since Z is constructible inside U) separated étale, using the exact sequence $0 \to j'_!\mathcal{L} \to j_!\mathcal{L} \to j_!(\mathcal{L}_{|Z}) \to 0$, this is reduced to the case of A of the form $j_!\mathcal{L}$. In this case $R\mathscr{H}\mathrm{om}_\Lambda(A, -)$ is given by $Rj_*(\mathcal{L}^\vee \otimes^{\mathbb{L}}_\Lambda -)$, and this commutes with any base change by quasicompact base change, [**Sch17a**, Proposition 17.6]. □

One has to be careful that, in general, *the naive dual of a perfect constructible complex is not constructible.* The following lemma says that in fact it is overconvergent, so never constructible unless locally constant.

Lemma IV.2.18. *For X a spatial diamond and $A \in D_{\text{ét}}(X, \Lambda)$ perfect constructible, $R\mathscr{H}\mathrm{om}_\Lambda(A, \Lambda)$ is overconvergent.*

Proof. Using Lemma IV.2.17 this is reduced to the case when $X = \mathrm{Spa}(C, C^+)$. Moreover, one can assume that $A = j_!\Lambda$ for some quasicompact open immersion $j : U \to X$. Then $R\mathscr{H}\mathrm{om}_\Lambda(A, \Lambda) = Rj_*\Lambda = \Lambda$, which is overconvergent. □

IV.2.3.2. *Twisted inverse images.* Also, if A is f-ULA, then one can relate appropriately A-twisted versions of f^* and $Rf^!$.

Proposition IV.2.19. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\mathrm{trg}\, f < \infty$ *and let $A \in D_{\text{ét}}(X, \Lambda)$ be f-universally locally acyclic. Then for all $B \in D_{\text{ét}}(S, \Lambda)$, the natural map*

$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda f^*B \to R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!B)$$

given as the composite

$$R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!\Lambda) \otimes^{\mathbb{L}}_\Lambda f^*B \to R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!\Lambda \otimes^{\mathbb{L}}_\Lambda f^*B) \to R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!B)$$

is an isomorphism.

Proof. First, we note that both sides commute with any base change, by Proposition IV.2.15.

It suffices to check that we get an isomorphism on stalks at all geometric points $\mathrm{Spa}(C, C^+)$ of X. For this, we may base change along the associated map $\mathrm{Spa}(C, C^+) \to S$ to reduce to the case that $S = \mathrm{Spa}(C, C^+)$ is strictly local, and we need to check that we get an isomorphism at the stalk of a section $s : S \to X$. We may also assume that X is spatial, in which case X is of bounded cohomological dimension, so [**Sch17a**, Proposition 20.17] applies, and perfect-constructible complexes are the same thing as compact objects in $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$.

Next, we note that the functor $B \mapsto R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!B)$ is right adjoint to $A' \mapsto Rf_!(A \otimes^{\mathbb{L}}_\Lambda A')$. The latter functor preserves perfect-constructible complexes, i.e. compact objects, by condition (b), see Lemma IV.2.14. Thus, $B \mapsto R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!B)$ commutes with arbitrary direct sums, see Lemma IV.2.20. Obviously, the functor $B \mapsto \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda f^*B$ also commutes with arbitrary direct sums, so it follows that it suffices to check the assertion for $B = j_!\Lambda$ for some quasicompact open immersion $j : S' = \mathrm{Spa}(C, C'^+) \to S$ (the shifts of those compact objects generate $D_{\text{ét}}(S, \Lambda)$). If $S' = S$, then $B = \Lambda$ and the result is clear. Otherwise, the stalk of $j_!\Lambda$ at the closed point is zero,

and thus the stalk of the left-hand side $\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B$ at our fixed section is zero. It remains to see that the stalk of

$$R\mathscr{H}\mathrm{om}_{\Lambda}(A, Rf^! j_! \Lambda)$$

at the (closed point of) the section $s : S \to X$ is zero. This stalk is given by the filtered colimit over all quasicompact open neighborhoods $U \subset X$ of $s(S)$ of

$$R\,\mathrm{Hom}_{D_{\text{ét}}(U,\Lambda)}(A|_U, Rf^! j_! \Lambda|_U) = R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(Rf_{U!}A|_U, j_!\Lambda),$$

where $f_U : U \to S$ denotes the restriction of f (the possibility to use only open embeddings in place of general étale maps results from the observation that the intersection of all these open subsets is the strictly local space S already; the set of open neighborhoods of $s(S)$ is cofinal among étale neighborhoods of $s(S)$).

Now we claim that the inverse systems of all such U and of the compactifications $\overline{U}^{/S}$ are cofinal. Note that the intersection of all $\overline{U}^{/S}$ (taken inside $\overline{X}^{/S}$) is simply $s(S)$: Indeed, given any point $x \in \overline{X}^{/S} \setminus s(S)$, there are disjoint open neighborhoods $x \in V$ and $s(S) \subset U$. In fact, the maximal Hausdorff quotient $|\overline{X}^{/S}|^B$ is compact Hausdorff by [**Sch17a**, Proposition 13.11] and its points can be identified with rank 1 points of $\overline{X}^{/S}$, which are the same as rank 1 points of X. But as $s(S) \subset \overline{X}^{/S}$ is closed, no point outside $s(S)$ admits the same rank 1 generalization, so x and $s(S)$ define distinct points of the Hausdorff spaces $|\overline{X}^{/S}|$, so that the desired disjoint open neighborhoods exist $x \in V$ and $s(S) \subset U$ exist. Then $x \notin \overline{U}^{/S}$. Thus,

$$s(S) = \bigcap_{U \supset s(S)} \overline{U}^{/X}.$$

Now given any open neighborhood U of $s(S)$, the complement $|\overline{X}^{/S}| \setminus U$ is quasicompact, which then implies that there is some U' such that $\overline{U'}^{/S} \subset U$. It follows that the direct systems of

$$Rf_{U!}A|_U$$

and

$$Rf_{\overline{U}^{/S}*} Ri^!_{\overline{U}^{/S}} A$$

are equivalent, where $i_{\overline{U}^{/S}} : \overline{U}^{/S} \to X$ and $f_{\overline{U}^{/S}} : \overline{U}^{/S} \to S$ are the evident maps. Now observe that if $j'_X : X_\eta = X \times_S \mathrm{Spa}(C, \mathcal{O}_C) \hookrightarrow X$ denotes the proconstructible generalizing immersion, then condition (a) in being f-locally acyclic implies that $A = Rj'_{X*}A|_{X_\eta}$ (see Remark IV.2.3), and then

$$\begin{aligned} Rf_{\overline{U}^{/S}*} Ri^!_{\overline{U}^{/S}} A &= Rf_{\overline{U}^{/S}*} Ri^!_{\overline{U}^{/S}} Rj'_{X*} A|_{X_\eta} \\ &= Rf_{\overline{U}^{/S}*} Rj'_{\overline{U}^{/S}*} Ri^!_{\overline{U}^{/S}_\eta} A|_{X_\eta} \\ &= Rj'_{S*} Rf_{\overline{U}^{/S}_\eta *} Ri^!_{\overline{U}^{/S}_\eta} A|_{X_\eta} \end{aligned}$$

with hopefully evident notation; in particular, $j'_S : \mathrm{Spa}(C, \mathcal{O}_C) \to S = \mathrm{Spa}(C, C^+)$ denotes the pro-open immersion of the generic point on the base.

In summary, we can rewrite the stalk of $R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! j_!\Lambda)$ at $s(S)$ as the filtered colimit of

$$R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(Rj'_{S*}Rf_{\overline{U}_\eta^{/S}*}Ri^!_{\overline{U}_\eta^{/S}}A, j_!\Lambda),$$

and we need to prove that this vanishes. This follows from the observation that for all $M \in D_{\text{ét}}(\mathrm{Spa}(C, \mathcal{O}_C), \Lambda) = D(\Lambda)$, one has

$$R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(Rj'_{S*}M, j_!\Lambda) = 0.$$

For this, note that $Rj'_{S*}M = M$ is just the constant sheaf given by the complex of Λ-modules M, and one has a triangle

$$R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(M, j_!\Lambda) \to R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(M, \Lambda) \to R\,\mathrm{Hom}_{D_{\text{ét}}(S,\Lambda)}(M, i_*\Lambda),$$

where i denotes the complementary closed immersion. Both the second and last term are given by $R\,\mathrm{Hom}_\Lambda(M, \Lambda)$, finishing the proof. □

We used the following classical lemma, cf. [**Nee96**, Theorem 5.1].

Lemma IV.2.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories such that $\mathcal{C}$ is compactly generated. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be such that G is right adjoint to F. If F sends compact objects to compact objects then G commutes with arbitrary direct sums.*

Proof. Since $\mathcal{C}$ is compactly generated it suffices to prove that for any compact object A in $\mathcal{C}$ and any collection $(B_i)_i$ of objects of $\mathcal{D}$, $\mathrm{Hom}(A, \oplus_i G(B_i)) \xrightarrow{\sim} \mathrm{Hom}(A, G(\oplus_i B_i))$. By compactness of A, $\mathrm{Hom}(A, \oplus_i G(B_i)) = \oplus_i \mathrm{Hom}(A, G(B_i))$, by adjunction this is equal to $\oplus_i \mathrm{Hom}(F(A), B_i)$, since $F(A)$ is compact this is equal to $\mathrm{Hom}(F(A), \oplus_i B_i)$, and by adjunction this is $\mathrm{Hom}(A, G(\oplus_i B_i))$. □

Remark IV.2.21.

(i) In fact, outside of the overconvergence condition (a) in Definition IV.2.1, *the property of Proposition IV.2.19 characterizes locally acyclic complexes* under the assumption that locally on X the exists an integer N such that for any $U \to X$ quasicompact separated étale the cohomological dimension of $X_{\text{ét}}$ is bounded by N. More precisely, if $j : U \to X$ is separated étale with $f \circ j$ quasicompact then $R\,\mathrm{Hom}_\Lambda(R(f \circ j)_! A, B) = R\Gamma(U, R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! B))$. Thus, if for all B one has $\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda f^* B \xrightarrow{\sim} R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! B)$ then $R(f \circ j)_! A$ is compact since $R\Gamma(U, -)$ commutes with arbitrary direct sums.

(ii) Outside of the overconvergence condition (a) in Definition IV.2.1, the property of Proposition IV.2.19 universally on S characterize universally locally acyclic objects. In fact, using [**Sch17a**, Proposition 20.13], the constructibility property is reduced to the case when the base is strictly totally disconnected, in which case we can apply point (1).

Finally, let us note that all the previous results extend to the setting where the base S is a general small v-stack, taking the following definition.

Definition IV.2.22. *Let $f : X \to S$ be a map of small v-stacks that is compactifiable and representable in locally spatial diamonds with locally* $\dim.\mathrm{trg}\, f < \infty$. *Let $A \in D_{\text{ét}}(X, \Lambda)$. Then A is f-universally locally acyclic if for any map $S' \to S$ from a locally spatial diamond S' with pullback $f' : X' = X \times_S S' \to S'$, the complex $A|_{X'} \in D_{\text{ét}}(X', \Lambda)$ is f'-locally acyclic.*

IV.2.3.3. *Dualizability.* From the previous two propositions, we can deduce an analogue of a recent result of Lu-Zheng, [**LZ20**], characterizing universal local acyclicity in terms of dualizability in a certain monoidal category. We actually propose a different such characterization closer to how dualizability will appear later in the discussion of geometric Satake. In terms of applications to abstract properties of universal local acyclicity, such as its preservation by Verdier duality, it leads to the same results.

Fix a base small v-stack S, and a coefficient ring Λ (killed by some integer prime to p). We define a 2-category $\mathcal{C}_S$ as follows. The objects of $\mathcal{C}_S$ are maps $f : X \to S$ of small v-stacks that are compactifiable, representable in locally spatial diamonds, with locally $\dim.\operatorname{trg} f < \infty$. For any $X, Y \in \mathcal{C}_S$, the category of maps $\mathrm{Fun}_{\mathcal{C}_S}(X, Y)$ is the category $D_{\text{ét}}(X \times_S Y, \Lambda)$. Note that any such $A \in D_{\text{ét}}(X \times_S Y, \Lambda)$ defines in particular a functor

$$D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda) : B \mapsto R\pi_{2!}(A \otimes^{\mathbb{L}}_{\Lambda} \pi_1^* B)$$

with kernel A, where $\pi_1 : X \times_S Y \to X$, $\pi_2 : X \times_S Y \to Y$ are the two projections. The composition in $\mathcal{C}_S$ is now defined to be compatible with this association. More precisely, the composition

$$\mathrm{Fun}_{\mathcal{C}_S}(X, Y) \times \mathrm{Fun}_{\mathcal{C}_S}(Y, Z) \to \mathrm{Fun}_{\mathcal{C}_S}(X, Z)$$

is defined to be the functor

$$D_{\text{ét}}(X \times_S Y, \Lambda) \times D_{\text{ét}}(Y \times_S Z, \Lambda) \to D_{\text{ét}}(X \times_S Z, \Lambda) : (A, B) \mapsto A \star B = R\pi_{13!}(\pi_{12}^* A \otimes^{\mathbb{L}}_{\Lambda} \pi_{23}^* B)$$

where π_{ij} denotes the various projections on $X \times_S Y \times_S Z$. It follows from the projection formula that this indeed defines a 2-category $\mathcal{C}_S$. The identity morphism is given by $R\Delta_!\Lambda = R\Delta_*\Lambda \in D_{\text{ét}}(X \times_S X, \Lambda)$, where $\Delta : X \hookrightarrow X \times_S X$ is the diagonal. We note that $\mathcal{C}_S$ is naturally equivalent to $\mathcal{C}_S^{\mathrm{op}}$. Indeed, $D_{\text{ét}}(X \times_S Y, \Lambda)$ is invariant under switching X and Y, and the definition of composition (and coherences) is compatible with this switch.

Recall that in any 2-category $\mathcal{C}$, there is a notion of adjoints. Namely, a morphism $f : X \to Y$ is a left adjoint of $g : Y \to X$ if there are maps $\alpha : \mathrm{id}_X \to gf$ and $\beta : fg \to \mathrm{id}_Y$ such that the composites

$$f \xrightarrow{f\alpha} fgf \xrightarrow{\beta f} f, \ g \xrightarrow{\alpha g} gfg \xrightarrow{g\beta} g$$

are the identity. If a right adjoint g of f exists, it is (together with the accompanying data) moreover unique up to unique isomorphism. As is clear from the definition, any functor of 2-categories preserves adjunctions. In particular, this applies to pullback functors $\mathcal{C}_S \to \mathcal{C}_{S'}$ for maps $S' \to S$ of small v-stacks, or to the functor from $\mathcal{C}_S$ to triangulated categories taking X to $D_{\text{ét}}(X, \Lambda)$ and $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, Y)$ to the functor $R\pi_{2!}(A \otimes^{\mathbb{L}}_{\Lambda} \pi_1^* -)$ with kernel A.

THEOREM IV.2.23. *Let S be a small v-stack and $X \in \mathcal{C}_S$, and $A \in D_{\text{ét}}(X, \Lambda)$. The following conditions are equivalent.*

(i) *The complex A is f-universally locally acyclic.*

(ii) *The natural map*

$$p_1^* \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} p_2^* A \to R\mathscr{H}\mathrm{om}_{\Lambda}(p_1^* A, Rp_2^! A)$$

is an isomorphism, where $p_1, p_2 : X \times_S X \to X$ are the two projections.

(iii) *The object $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, S)$ is a left adjoint in $\mathcal{C}_S$. In that case, its right adjoint is given by*

$$\mathbb{D}_{X/S}(A) \in D_{\text{ét}}(X, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(S, X).$$

Proof. That (i) implies (ii) follows from Proposition IV.2.15 and Proposition IV.2.19. For (ii) implies (iii), we claim that $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, S)$ is indeed a left adjoint of $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(A)$. The composites are given by

$$A \star \mathbb{D}_{X/S}(A) = Rf_!(\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} A) \in D_{\text{ét}}(S, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(S, S)$$

and

$$\mathbb{D}_{X/S}(A) \star A = p_1^* A \otimes^{\mathbb{L}}_{\Lambda} p_2^* \mathbb{D}_{X/S}(A) \in D_{\text{ét}}(X \times_S X, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(X, X).$$

Then we take $\beta : A \star \mathbb{D}_{X/S}(A) \to \mathrm{id}_S$ to be given by the map $Rf_!(\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} A) \to \Lambda$ adjoint to the map $\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} A \to Rf^! \Lambda$ which is just the tautological pairing. On the other hand, for $\alpha : \mathrm{id}_X \to \mathbb{D}_{X/S}(A) \star A$, we have to produce a map

$$R\Delta_! \Lambda \to p_1^* A \otimes^{\mathbb{L}}_{\Lambda} p_2^* \mathbb{D}_{X/S}(A).$$

Using (ii), the right-hand side is naturally isomorphic to $R\mathscr{H}\mathrm{om}_{\Lambda}(p_1^* A, Rp_2^! A)$. Now maps from $R\Delta_! \Lambda$ are adjoint to sections of

$$R\Delta^! R\mathscr{H}\mathrm{om}_{\Lambda}(p_1^* A, Rp_2^! A) \cong R\mathscr{H}\mathrm{om}_{\Lambda}(A, A)$$

(using [**Sch17a**, Theorem 1.8 (v)]), which has the natural identity section. It remains to prove that certain composites are the identity. This follows from a straightforward diagram chase.

Finally, it remains to prove that (iii) implies (i). We can assume that S is strictly totally disconnected. It follows that the functor $Rf_!(A \otimes^{\mathbb{L}}_{\Lambda} -)$ admits a right adjoint that commutes with all colimits. This implies that condition (b) in Definition IV.2.1 is satisfied. In fact, more precisely we see that the right adjoint $R\mathscr{H}\mathrm{om}_{\Lambda}(A, Rf^! -)$ is given by $A' \otimes^{\mathbb{L}}_{\Lambda} f^* -$ for some $A' \in D_{\text{ét}}(X, \Lambda)$, and by using the self-duality of $\mathcal{C}_S^{\mathrm{op}}$, we also see that $R\mathscr{H}\mathrm{om}_{\Lambda}(A', Rf^! -)$ is given by $A \otimes^{\mathbb{L}}_{\Lambda} f^* -$. Applied to the constant sheaf, this shows in particular that $A \cong R\mathscr{H}\mathrm{om}_{\Lambda}(A', Rf^! \Lambda)$ is a Verdier dual. For condition (a), we can assume that $S = \mathrm{Spa}(C, C^+)$ and reduce to checking overconvergence along sections $s : S \to X$. In fact, using part (2) of Remark IV.2.3, let $j : \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(C, C^+)$ be the pro-open immersion, with pullback $j_X : X \times_{\mathrm{Spa}(C,C^+)} \mathrm{Spa}(C, \mathcal{O}_C) \to X$, and $f_\eta : X \times_{\mathrm{Spa}(C,C^+)} \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(C, \mathcal{O}_C)$ the restriction of f. To see the overconvergence, it is enough to see that $A \cong Rj_{X*} A_0$ for some A_0. But

$$\begin{aligned} A &\cong R\mathscr{H}\mathrm{om}_{\Lambda}(A', Rf^! \Lambda) \cong R\mathscr{H}\mathrm{om}_{\Lambda}(A', Rf^! Rj_* \Lambda) \\ &\cong R\mathscr{H}\mathrm{om}_{\Lambda}(A', Rj_{X*} Rf_\eta^! \Lambda) \cong Rj_{X*} R\mathscr{H}\mathrm{om}_{\Lambda}(j_X^* A', Rf_\eta^! \Lambda), \end{aligned}$$

giving the desired overconvergence. □

Before moving on, let us observe the following relative variant.

Proposition IV.2.24. *Let S be a small v-stack and $X, Y \in \mathcal{C}_S$. If Y/S is proper and $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, Y) = D_{\text{ét}}(X \times_S Y, \Lambda)$ is p_2-universally locally acyclic, then it is a left adjoint and the right adjoint is given by*

$$\mathbb{D}_{X \times_S Y/Y}(A) \in D_{\text{ét}}(X \times_S Y, \Lambda) \cong D_{\text{ét}}(Y \times_S X, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(Y, X).$$

The assumption that Y/S is proper is important here. Already if $X = S$ and $A = \Lambda \in D_{\text{ét}}(Y, \Lambda)$, which is always id_Y-universally locally acyclic, being a left adjoint in $\mathcal{C}_S$ implies that there is some $B \in D_{\text{ét}}(Y, \Lambda)$ for which $Rf_* \cong Rf_!(B \otimes^{\mathbb{L}}_{\Lambda} -)$.

Proof. We need to produce the maps α and β again. Let us give the construction of α, which is the harder part. First, using the various projections π_{ij} on $X \times_S Y \times_S X$, we have

$$\mathbb{D}_{X\times_S Y/Y}(A) \star A \cong R\pi_{13!}(\pi_{12}^*\mathbb{D}_{X\times_S Y/Y}(A) \otimes_\Lambda^{\mathbb{L}} \pi_{23}^* A) \cong R\pi_{13*}R\mathscr{H}\mathrm{om}_\Lambda(\pi_{12}^*A, R\pi_{23}^!A)$$

using that A is p_2-universally locally acyclic, and properness of π_{13} (which is a base change of $Y \to S$). Now giving a map $R\Delta_!\Lambda \to \mathbb{D}_{X\times_S Y/Y}(A)\star A$, for $\Delta = \Delta_{X/S}$, amounts to finding a section of

$$R\Delta^! R\pi_{13*}R\mathscr{H}\mathrm{om}_\Lambda(\pi_{12}^*A, R\pi_{23}^!A) \cong Rp_{1*}R\Delta^!_{X\times_S Y/Y}R\mathscr{H}\mathrm{om}_\Lambda(\pi_{12}^*A, R\pi_{23}^!A) \cong Rp_{1*}R\mathscr{H}\mathrm{om}_\Lambda(A, A),$$

where we can take the identity. □

Theorem IV.2.23 has the following notable consequences.

Corollary IV.2.25. *Let $f : X \to S$ be a compactifiable map of locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$ *and let $A \in D_{\text{ét}}(X, \Lambda)$ be f-universally locally acyclic. Then $\mathbb{D}_{X/S}(A)$ is again f-universally locally acyclic, and the biduality map*

$$A \to \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$$

is an isomorphism.

If $f_i : X_i \to S$ for $i = 1, 2$ are compactifiable maps of small v-stacks that are representable in locally spatial diamonds with locally $\dim.\operatorname{trg} f_i < \infty$ *and $A_i \in D_{\text{ét}}(X_i, \Lambda)$ are f_i-universally locally acyclic, then also $A_1 \boxtimes A_2 \in D_{\text{ét}}(X_1 \times_S X_2, \Lambda)$ is $f_1 \times_S f_2$-universally locally acyclic, and the natural map*

$$\mathbb{D}_{X_1/S}(A_1) \boxtimes \mathbb{D}_{X_2/S}(A_2) \to \mathbb{D}_{X_1\times_S X_2/S}(A_1 \boxtimes A_2)$$

is an isomorphism.

Proof. By Theorem IV.2.23, the object $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S, X)$ is a right adjoint of $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, S)$. But $\mathcal{C}_S \cong \mathcal{C}_S^{\mathrm{op}}$; under this equivalence, this means that $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S, X)$ is a left adjoint of $A \in \mathrm{Fun}_{\mathcal{C}_S}(S, X)$. Thus, applying Theorem IV.2.23 again, the result follows.

For the second statement, note that $A_i \in D_{\text{ét}}(X_i, \Lambda)$ define left adjoints, hence so does

$$A_1 \star A_2 = A_1 \boxtimes A_2 \in D_{\text{ét}}(X_1 \times_S X_2, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(X_1 \times_S X_2, S).$$

Its right adjoint is the similar composition, giving the claim. □

The final statement admits the following generalization concerning "compositions" of universally locally acyclic sheaves.

Proposition IV.2.26. *Let $g : Y \to X$ and $f : X \to S$ be compactifiable maps of small v-stacks representable in locally spatial diamonds with locally* $\dim.\operatorname{trg} f, \dim.\operatorname{trg} g < \infty$, *and let $A \in D_{\text{ét}}(X, \Lambda)$ be f-universally acyclic and $B \in D_{\text{ét}}(Y, \Lambda)$ be g-universally locally acyclic. Then $g^*A \otimes_\Lambda^{\mathbb{L}} B$ is $f \circ g$-universally locally acyclic, and there is natural isomorphism*

$$\mathbb{D}_{Y/S}(g^*A \otimes_\Lambda^{\mathbb{L}} B) \cong g^*\mathbb{D}_{X/S}(A) \otimes_\Lambda^{\mathbb{L}} \mathbb{D}_{Y/X}(B).$$

Proof. It is easy to see that condition (a) of being locally acyclic holds universally, so it suffices to identify the functor $R\mathscr{H}\mathrm{om}_\Lambda(g^*A \otimes^{\mathbb{L}}_\Lambda B, R(f \circ g)^!-)$. We compute:

$$\begin{aligned} R\mathscr{H}\mathrm{om}_\Lambda(g^*A \otimes^{\mathbb{L}}_\Lambda B, R(f \circ g)^!-) &= R\mathscr{H}\mathrm{om}_\Lambda(B, R\mathscr{H}\mathrm{om}_\Lambda(g^*A, Rg^!Rf^!-)) \\ &= R\mathscr{H}\mathrm{om}_\Lambda(B, Rg^!R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^!-)) \\ &= \mathbb{D}_{Y/X}(B) \otimes^{\mathbb{L}}_\Lambda g^*R\mathscr{H}\mathrm{om}(A, Rf^!-) \\ &= \mathbb{D}_{Y/X}(B) \otimes^{\mathbb{L}}_\Lambda g^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda g^*f^*-, \end{aligned}$$

implying that it commutes with colimits, hence its left adjoint preserves perfect-constructible complexes (after reduction to strictly totally disconnected S and X and Y spatial). Moreover, evaluating this functor at Λ gives the identification of the Verdier dual. □

Let us also note another corollary of Theorem IV.2.23 concerning retracts.

Corollary IV.2.27. *Let $f : X \to S$ and $g : Y \to S$ be maps of small v-stacks that are compactifiable and representable in locally spatial diamonds with locally* $\dim.\operatorname{trg} f, \dim.\operatorname{trg} g < \infty$. *Assume that f is a retract of g over S, i.e. there are maps $i : X \to Y$, $r : Y \to X$ over S such that $ri = \mathrm{id}_X$. If Λ is g-universally locally acyclic, then Λ is f-universally locally acyclic.*

Proof. One can check this directly from the definitions, or note that the map in $\mathcal{C}_S$ given by $\Lambda \in D_{\text{ét}}(X, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(X, S)$ is a retract of the map given by $\Lambda \in D_{\text{ét}}(Y, \Lambda) = \mathrm{Fun}_{\mathcal{C}_S}(Y, S)$, from which one can easily obtain adjointness. □

Moreover, in some cases the converse to Proposition IV.2.11 holds.

Proposition IV.2.28. *Let $g : Y \to X$, $f : X \to S$ be maps of locally spatial diamonds where g is proper and quasi-pro-étale and f is compactifiable and locally* $\dim.\operatorname{trg} f < \infty$. *Then $A \in D_{\text{ét}}(Y, \Lambda)$ is $f \circ g$-universally locally acyclic if and only if $Rg_*A \in D_{\text{ét}}(X, \Lambda)$ is f-universally locally acyclic.*

Proof. One direction is given by Proposition IV.2.11. For the converse, assume that Rg_*A is f-universally locally acylic. To see that A is $h = f \circ g$-universally locally acyclic, it suffices by Theorem IV.2.23 that the map

$$p_{1,Y}^* R\mathscr{H}\mathrm{om}(A, Rh^!\Lambda) \otimes^{\mathbb{L}}_\Lambda p_{2,Y}^* A \to R\mathscr{H}\mathrm{om}(p_{1,Y}^* A, Rp_{2,Y}^! A)$$

in $D_{\text{ét}}(Y \times_S Y, \Lambda)$ is an isomorphism, where $p_{1,Y}, p_{2,Y} : Y \times_S Y \to Y$ are the two projections. As $g \times_S g : Y \times_S Y \to X \times_S X$ is proper and quasi-pro-étale, pushforward along $g \times_S g$ is conservative: By testing on stalks, this follows from the observation that for a profinite set T, the global sections functor $R\Gamma(T, -)$ is conservative on $D(T, \Lambda)$ (as one can write any stalk as a filtered colimit of functors that are direct summands of the global sections functor). Applying $R(g \times_S g)_* = R(g \times_S g)_!$ to the displayed map, we get the map

$$p_{1,X}^* R\mathscr{H}\mathrm{om}(Rg_*A, Rf^!\Lambda) \otimes^{\mathbb{L}}_\Lambda p_{1,Y}^* Rg_*A \to R\mathscr{H}\mathrm{om}(p_{1,X}^* Rg_*A, Rp_{2,X}^! Rg_*A)$$

where $p_{1,X}, p_{2,X} : X \times_S X \to X$ are the two projections. This is an isomorphism precisely when Rg_*A is f-universally locally acyclic. □

The following corollary shows that smooth base change generalizes to universally locally acyclic maps. The general version of this corollary was suggested by David Hansen.

Corollary IV.2.29 (ULA base change). *Consider a cartesian diagram of small v-stacks*

$$\begin{tikzcd} X' \arrow[r, "\widetilde{g}"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with f representable in locally spatial diamonds, compactifiable, locally $\dim.\operatorname{trg} f < \infty$. *Assume that Λ is f-universally locally acyclic. Then the base change map*

$$f^* Rg_* A \to R\widetilde{g}_* f'^* A$$

is an isomorphism. More generally, if $B \in D_{\text{ét}}(X, \Lambda)$ is f-universally locally acyclic and $A \in D_{\text{ét}}(S', \Lambda)$, then

$$(f^* Rg_* A) \otimes^{\mathbb{L}}_{\Lambda} B \xrightarrow{\sim} R\widetilde{g}_*(f'^* A \otimes^{\mathbb{L}}_{\Lambda} \widetilde{g}^* B).$$

Proof. We apply Proposition IV.2.19 to the universally locally acyclic $\mathbb{D}_{X/S}(B)$, so that by Corollary IV.2.25, we get

$$f^* Rg_* A \otimes^{\mathbb{L}}_{\Lambda} B \cong R\mathscr{H}\mathrm{om}_{\Lambda}(\mathbb{D}_{X/S}(B), Rf^! Rg_* A).$$

By [**Sch17a**, Theorem 1.9 (iii)], $Rf^! Rg_* A \cong R\widetilde{g}_* Rf'^! A$, and then one can rewrite further as

$$R\mathscr{H}\mathrm{om}_{\Lambda}(\mathbb{D}_{X/S}(B), R\widetilde{g}_* Rf'^! A) \cong R\widetilde{g}_* R\mathscr{H}\mathrm{om}_{\Lambda}(\widetilde{g}^* \mathbb{D}_{X/S}(B), Rf'^! A).$$

Now another application of Proposition IV.2.19 gives the result. □

Finally let us note the following consequence of Theorem IV.2.23 and [**LZ20**].

Proposition IV.2.30. *Let K be a complete non-archimedean field with residue characteristic p, $f : X \to S$ a separated morphism of K-schemes locally of finite type, and $A \in D^b_c(X, \Lambda)$. Then A is f-universally acyclic if and only if its analytification A^{ad} is $f^{ad,\diamond}$-universally locally acyclic, where $f^{ad,\diamond} : X^{ad,\diamond} \to S^{ad,\diamond}$.*

Proof. The criterion of Theorem IV.2.23 (2) applies similarly in the algebraic case by [**LZ20**], and all operations are compatible with passing to analytic adic spaces (and diamonds). □

For example, if $S = \operatorname{Spec} K$ then any A is f-universally locally acyclic and thus A^{ad} is $f^{ad,\diamond}$-universally acyclic. This gives plenty of examples of ULA sheaves.

IV.2.4. Local acyclicity for morphisms of Artin v-stacks. Using the descent results Remark IV.2.2 and Proposition IV.2.13, one can extend the previous definition and results to the case of maps of Artin v-stacks as follows.

Definition IV.2.31. *Let $f : X \to S$ be a map of Artin v-stacks and assume that there is some separated, representable in locally spatial diamonds, and cohomologically smooth surjection $g : U \to X$ from a locally spatial diamond U such that $f \circ g : U \to S$ is compactifiable with locally* $\dim.\operatorname{trg}(f \circ g) < \infty$. *Then $A \in D_{\text{ét}}(X, \Lambda)$ is f-universally locally acyclic if $g^* A$ is $f \circ g$-universally locally acyclic.*

All previous results concerning universally locally acyclic complexes also hold in this setting (assuming that the relevant operations are defined in the case of interest – we did not define $Rf_!$ and $Rf^!$ for general stacky maps), and follow by the reduction to the case when S and X are locally spatial diamonds. In particular, the characterization in terms of dualizability gives the following.

PROPOSITION IV.2.32. *Let $f : X \to S$ be a cohomologically smooth map of Artin v-stacks, and let $A \in D_{\text{ét}}(X, \Lambda)$. Consider $X \times_S X$ with its two projections $p_1, p_2 : X \times_S X \to X$. Then A is f-universally locally acyclic if and only if the natural map*

$$p_1^* R\mathscr{H}\mathrm{om}_\Lambda(A, \Lambda) \otimes_\Lambda^{\mathbb{L}} p_2^* A \to R\mathscr{H}\mathrm{om}_\Lambda(p_1^* A, p_2^* A)$$

is an isomorphism.

PROOF. Taking a chart for S, we can assume that S is a locally spatial diamond, and then taking a presentation for X we can assume that also X is a locally spatial diamond, noting that the condition commutes with smooth base change. In that case, replacing some occurences of p_2^* by $Rp_2^!$ using cohomological smoothness, the result follows from Theorem IV.2.23. □

There is a simple characterization of ℓ-cohomological smoothness in terms of universal local acyclicity.

PROPOSITION IV.2.33. *Let $f : X \to S$ be a compactifiable map of v-stacks that is representable in locally spatial diamonds with locally* $\dim.\operatorname{trg} f < \infty$. *Then f is ℓ-cohomologically smooth if and only if $\mathbb{F}_\ell$ is f-universally locally acyclic and its Verdier dual $Rf^!\mathbb{F}_\ell$ is invertible.*

Note that in checking whether $\mathbb{F}_\ell$ is f-universally locally acyclic, condition (a) of Definition IV.2.1 is automatic. Also, by Theorem IV.2.23, the condition that $\mathbb{F}_\ell$ is f-universally locally acyclic is equivalent to the condition that the natural map

$$p_1^* Rf^! \mathbb{F}_\ell \to Rp_2^! \mathbb{F}_\ell$$

is an isomorphism, where $p_1, p_2 : X \times_S X \to X$ are the two projections. Thus, *f is ℓ-cohomologically smooth if and only if $Rf^!\mathbb{F}_\ell$ is invertible and its formation commutes with any base change.*

PROOF. The conditions are clearly necessary. For the converse, we may assume that S is strictly totally disconnected. By Proposition IV.2.19, the natural transformation of functors

$$Rf^!\Lambda \otimes_\Lambda^{\mathbb{L}} f^* \to Rf^!$$

is an equivalence. As $Rf^!\Lambda$ is assumed to be invertible (it commutes with base change by Proposition IV.2.15), this shows that the condition of [**Sch17a**, Definition 23.8] is satisfied. □

In particular, we can resolve a question from [**Sch17a**].

COROLLARY IV.2.34. *The map $f : \operatorname{Spd} \mathcal{O}_E \to \operatorname{Spd} \mathbb{F}_q$ is ℓ-cohomologically smooth for all $\ell \neq p$.*

PROOF. This is clear if E is of equal characteristic, so assume that E is p-adic. First, we prove that $\mathbb{F}_\ell$ is f-ULA. This follows from $f' : \operatorname{Spd} \mathcal{O}_{\widetilde{E}} \cong \operatorname{Spd} \mathbb{F}_q[\![t^{1/p^\infty}]\!] \to \operatorname{Spd} \mathbb{F}_q$ being ℓ-cohomologically smooth, where $\widetilde{E}/E$ is some totally ramified $\mathbb{Z}_p$-extension, by the argument of the proof of [**Sch17a**, Proposition 24.3] (in essence, the compactly supported pushforward for any base change of f are

the $\mathbb{Z}_p$-invariants inside the compactly supported pushforward for the corresponding base change of f', so constructibility of the latter implies constructibility of the former).

It remains to show that $Rf^!\mathbb{F}_\ell$ is invertible. If $j : \operatorname{Spd} E \to \operatorname{Spd} \mathcal{O}_E$ is the open immersion with complement $i : \operatorname{Spd} \mathbb{F}_q \to \operatorname{Spd} \mathcal{O}_E$, we have $Ri^!Rf^!\mathbb{F}_\ell = \mathbb{F}_\ell$ by transitivity, and $j^*Rf^!\mathbb{F}_\ell \cong \mathbb{F}_\ell(1)[2]$ by [**Sch17a**, Proposition 24.5], so we get a triangle

$$i_*\mathbb{F}_\ell \to Rf^!\mathbb{F}_\ell \to Rj_*\mathbb{F}_\ell(1)[2].$$

Using this, one computes $Rf^!\mathbb{F}_\ell \cong \mathbb{F}_\ell(1)[2]$, as desired. □

IV.3. Formal smoothness

IV.3.1. Definition. A key step in the proof of Theorem IV.4.2, the Jacobian criterion of cohomological smoothness, is the following notion of formal smoothness.

Definition IV.3.1. *Let $f : Y \to X$ be a map of v-stacks. Then f is formally smooth if for any affinoid perfectoid space S of characteristic p with a Zariski closed subspace $S_0 \subset S$, and any commutative diagram*

$$\begin{tikzcd} S_0 \arrow[r, "g_0"] \arrow[d] & Y \arrow[d, "f"] \\ S \arrow[r, "h"] & X, \end{tikzcd}$$

there is some étale map $S' \to S$ containing S_0 in its image and a map $g : S' \to Y$ fitting in a commutative diagram

$$\begin{tikzcd} S' \times_S S_0 \arrow[r] \arrow[d] & S_0 \arrow[r, "g_0"] \arrow[d] & Y \arrow[d, "f"] \\ S' \arrow[r] \arrow[urr, "g"] & S \arrow[r, "h"] & X. \end{tikzcd}$$

This kind of formal smoothness is closely related to the notion of *absolute neighborhood retracts* ([**Bor67**], [**Dol80**]). In fact, suppose $Y \to X$ is formally smooth with Y and X affinoid perfectoid. Choose a Zariski closed embedding $Y \hookrightarrow \mathbb{B}^I_X$ for some set I. Then there exists an étale neighborhood $U \to \mathbb{B}^I_X$ of Y such that the closed embedding

$$i : Y \times_{\mathbb{B}^I_X} U \hookrightarrow U$$

admits a retraction $r : U \to Y \times_{\mathbb{B}^I_X} U$, $r \circ i = Id$. Thus, Y/X is a retract of an (étale) neighborhood inside a ball$/X$.

IV.3.2. Examples and basic properties. We will see that formally smooth morphisms share analogous properties to cohomologically smooth morphisms. Let's begin with the following observations:

(i) The composite of two formally smooth morphisms is formally smooth,

(ii) The formally smooth property is stable under pullback: if $Y \to X$ is formally smooth and $X' \to X$ is any map then $Y \times_X X' \to X'$ is formally smooth.

(iii) Separated étale maps are formally smooth.

(iv) For morphisms of locally spatial diamonds, formal smoothness is étale local on the source and the target.

Let us observe that, like cohomologically smooth morphisms, formally smooth morphisms are universally open.

PROPOSITION IV.3.2. *Formally smooth maps are universally open.*

PROOF. Let $Y \to X$ be formally smooth. We can suppose X is affinoid perfectoid. Since any open subsheaf of Y is formally smooth over X we are reduced to prove that the image of $Y \to X$ is open. Let $S \to Y$ be a morphism with S affinoid perfectoid. Choose a Zariski closed embedding $S \hookrightarrow \mathbb{B}^I_X$ for some set I. The formal smoothness assumption implies that there exists an étale neighborhood $U \to \mathbb{B}^I_X$ of $S \subset \mathbb{B}^I_X$ such that $S \times_{\mathbb{B}^I_X} U \to S \to Y$ extends to a map $U \to Y$; in particular, the image of $S \to Y \to X$ is contained in the image of $U \to Y \to X$, and it suffices to prove that the latter is open. We can suppose U is quasicompact and separated over $\mathbb{B}^I_X$. Writing $\mathbb{B}^I_X = \varprojlim_J \mathbb{B}^J_X$ where J goes through the set of finite subsets of I, there exists a some $J \subset I$ finite and $V \to \mathbb{B}^J_X$ such that $U \to \mathbb{B}^I_X$ is the pullback of $V \to \mathbb{B}^J_X$ via the projection $\mathbb{B}^I_X \to \mathbb{B}^J_X$, cf. [**Sch17a**, Proposition 6.4]. Since $\mathbb{B}^I_X \to \mathbb{B}^J_X$ is a v-cover,

$$\mathrm{Im}(U \to X) = \mathrm{Im}(V \to X).$$

Now, using that $V \to \mathbb{B}^J_X \to X$ is open, since cohomologically smooth for example, this is an open subset of X. □

Let us begin with some concrete examples. In the following, $\mathbb{B} \to *$ is the v-sheaf $\mathcal{O}^+$ on Perf_k and $\mathbb{A}^1 \to *$ is the v-sheaf $\mathcal{O}$.

PROPOSITION IV.3.3. *The morphisms $\mathbb{B} \to *$, $\mathbb{A}^1 \to *$ and $\mathrm{Spd}\,\mathcal{O}_E \to *$ are formally smooth.*

PROOF. Let $S_0 = \mathrm{Spa}(R_0, R_0^+) \hookrightarrow S = \mathrm{Spa}(R, R^+)$ be a Zariski closed embedding of affinoid perfectoid spaces. Then $R \to R_0$ is surjective, which immediately shows that $\mathbb{A}^1 \to *$ is formally smooth. The case of $\mathbb{B} \to *$ follows as $\mathbb{B} \subset \mathbb{A}^1$ is open. For $\mathrm{Spd}\,\mathcal{O}_E$, note that any untilt of S_0 can be given by some element $\xi \in W_{\mathcal{O}_E}(R_0^+)$ of the form $\xi_0 = \pi + \sum_{n=0}^{\infty} \pi^i [r_{i,0}]$ where all $r_i \in R_0^{\circ\circ}$. But $R^{\circ\circ} \to R_0^{\circ\circ}$ is surjective (cf. the discussion after [**Sch17a**, Definition 5.7]), so one can lift all $r_{i,0} \in R_0^{\circ\circ}$ to $r_i \in R^{\circ\circ}$, and then $\xi = \pi + \sum_{n=0}^{\infty} \pi^i [r_i]$ defines an untilt of S over $\mathcal{O}_E$ lifting the given one on S_0. □

COROLLARY IV.3.4. *Is $f : Y \to X$ is a smooth morphism of analytic adic spaces over $\mathbb{Z}_p$ then $f^\diamond : Y^\diamond \to X^\diamond$ is formally smooth.*

PROOF. Any smooth morphism is locally étale over a finite-dimensional ball. □

Let us remark the following.

PROPOSITION IV.3.5. *If $f : Y \to X$ is a formally smooth and surjective map of v-stacks, then f is surjective as a map of étale stacks. Equivalently, in case X is a perfectoid space, the map f splits over an étale cover of X.*

Proof. We can suppose X is affinoid. There exists a surjective morphism $X' \to X$ with X' affinoid perfectoid and a section $s : X' \to Y$ of $Y \to X$ over X'. Let us choose a Zariski closed embedding $X' \hookrightarrow \mathbb{B}^I_X$. Applying the formal smoothness property we deduce there is an étale neighborhood $U \to \mathbb{B}^I_X$ of $X' \subset \mathbb{B}^I_X$ and a section over U of $Y \to X$. It thus suffices to see that $U \to X$ admits a section over an étale cover of X. As in the proof of Proposition IV.3.2 there exists a finite subset $J \subset I$, and quasicompact étale map $V \to \mathbb{B}^J_X$ such that $U \to \mathbb{B}^I_X$ is the pullback of $V \to \mathbb{B}^J_X$ via the projection $\mathbb{B}^I_X \to \mathbb{B}^J_X$. This reduces us to the case I is finite. We may also replace V by its image in $\mathbb{B}^J_X$. At geometric points, the splitting follows from [**Sch17a**, Lemma 9.5]. Approximating a section over a geometric point over an étale neighborhood then gives the desired splitting on an étale cover. □

According to [**Sch17a**, Proposition 23.13] cohomological smoothness is cohomologically smooth local on the source. The same holds for formally smooth morphisms.

Corollary IV.3.6. *Let $f : Y \to X$ be a morphism of v-stacks. Suppose there exists a v-surjective formally smooth morphism of v-stacks $g : Y' \to Y$ such that $f \circ g$ is formally smooth. Then f is formally smooth.*

Proof. Given a test diagram $g_0 : S_0 \to Y$, $h : S \to X$ as in Definition IV.3.1, we can first lift $S_0 \to Y$ étale locally to Y' by Proposition IV.3.5, and the required étale neighborhoods lift to S by [**Sch17a**, Proposition 6.4] applied to S_0 as the intersection of all open neighborhoods in S. Thus, the diagram can be lifted to a similar test diagram for $Y' \to X$, which admits a solution by assumption. □

Let us remark the following.

Proposition IV.3.7. *The stack $\mathrm{Bun}_G \to *$ is formally smooth.*

Proof. Let $S_0 = \mathrm{Spa}(R_0, R_0^+) \subset S = \mathrm{Spa}(R, R^+)$ be a Zariski closed immersion of affinoid perfectoid spaces over $\mathrm{Spd}\, k$, and fix a pseudouniformizer $\varpi \in R$. Let $\mathcal{E}_0$ be a G-bundle on X_{S_0}. Pick any geometric point $\mathrm{Spa}(C, C^+) \to S_0$; we intend to find an étale neighborhood $U \to S$ of $\mathrm{Spa}(C, C^+)$ in S such that the G-bundle over $U \times_S S_0$ extends to U.

Note that the pullback of $\mathcal{E}_0$ to $Y_{C,[1,q]}$ is a trivial G-bundle, by Theorem III.2.2. From [**GR03**, Proposition 5.4.21] (applied with $R = \varinjlim_V \mathcal{O}^+(Y_{V,[1,q]})$, $t = \pi$ and $I = 0$, where $V \to S_0$ runs over étale neighborhoods of $\mathrm{Spa}(C, C^+)$ in S_0; all of these lift to S) it follows that after passing to an étale neighborhood as above, we can assume that the pullback of $\mathcal{E}_0$ to $Y_{S_0,[1,q]}$ is a trivial G-bundle. In that case, $\mathcal{E}_0$ is given by some matrix $A \in G(B_{R_0,[1,1]})$ encoding the descent. Applying [**GR03**, Proposition 5.4.21] again, with

$$R = \varinjlim_{S_0 \subset U \subset S} \mathcal{O}^+(Y_{U,[1,1]}),\ t = \pi,\ I = \ker(R \to \varinjlim_U \mathcal{O}^+(Y_{S_0,[1,1]})),$$

then shows that we may lift A into a neighborhood, as desired. □

The following is the analog of Proposition II.3.5 (iii).

Proposition IV.3.8. *Let S be a perfectoid space and let $[\mathcal{E}_1 \to \mathcal{E}_0]$ be a map of vector bundles on X_S such that $\mathcal{E}_0$ is everywhere of positive Harder–Narasimhan slopes, and $\mathcal{E}_1$ is everywhere of negative Harder–Narasimhan slopes. Then $\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \to S$ is formally smooth.*

Proof. Using the exact sequence

$$0 \to \mathcal{BC}(\mathcal{E}_0) \to \mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \to \mathcal{BC}(\mathcal{E}_1[1]) \to 0$$

and Proposition II.3.4 (iii), one reduces to the individual cases of $\mathcal{BC}(\mathcal{E}_0)$ and $\mathcal{BC}(\mathcal{E}_1[1])$. For $\mathcal{E} = \mathcal{E}_0$, we can use Corollary II.3.3 to choose étale locally on S a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^m \to \mathcal{E} \to 0$$

where $\mathcal{G}$ is fiberwise on S semistable of positive slope. Moreover, by Proposition II.3.4 (iii), one can also ensure that $H^1(X_{S'}, \mathcal{G}|_{X_{S'}}) = 0$ for all affinoid perfectoid spaces $S' \to S$. In particular, if $S_0 \subset S$ is a Zariski closed immersion of affinoid perfectoid spaces, the map $\mathcal{O}_{X_S}(\frac{1}{r})^m(S_0) \to \mathcal{E}(S_0)$ is surjective, and we can replace $\mathcal{E}$ by $\mathcal{O}_{X_S}(\frac{1}{r})^m$. But then Proposition II.2.5 (iv) shows that this Banach–Colmez space is representable by a perfectoid open unit disc, which is formally smooth.

For $\mathcal{E} = \mathcal{E}_1$, we can use Theorem II.2.6 to find a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{X_S}(d)^m \to \mathcal{G} \to 0$$

for some $d, m > 0$, and this induces an exact sequence

$$0 \to \mathcal{BC}(\mathcal{G}) \to \mathcal{BC}(\mathcal{O}_{X_S}(d)^m) \to \mathcal{BC}(\mathcal{E}[1]) \to 0$$

where the middle term is formally smooth by the preceding, and the map $\mathcal{BC}(\mathcal{O}_{X_S}(d)^m) \to \mathcal{BC}(\mathcal{E}[1])$ is formally smooth (as étale locally surjective and its fibre $\mathcal{BC}(\mathcal{G})$ is formally smooth). We conclude by Corollary IV.3.6. □

IV.4. A Jacobian criterion

The goal of this section is to prove that certain geometrically defined diamonds are cohomologically smooth when one expects them to be. We regard this result as the most profound in the theory of diamonds so far: While we cannot control much of the geometry of these diamonds, in particular we have no way to relate them to (perfectoid) balls in any reasonable way, we can still prove relative Poincaré duality for them. The spaces considered below also appear quite naturally in a variety of contexts, so we expect the result to have many applications.

The setup is the following. Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid adic spaces — defining this concept of smoothness will be done in a first subsection, but it is essentially just a family of smooth rigid spaces over X_S, in the usual sense. One can then consider the v-sheaf $\mathcal{M}_Z$ of sections of $Z \to X_S$, sending any perfectoid space $S' \to S$ to the set of maps $X_{S'} \to Z$ lifting $X_{S'} \to X_S$. In general, we cannot prove that $\mathcal{M}_Z$ is a locally spatial diamond, but this turns out to be true when Z is quasiprojective in the sense that it is a Zariski closed subspace of an open subset of (the adic space) $\mathbb{P}^n_{X_S}$ for some $n \geq 0$.

In general, the space $\mathcal{M}_Z \to S$ is not (cohomologically) smooth: If tangent spaces of $\mathcal{M}_Z \to S$ would exist, one would expect their fibre over $S' \to \mathcal{M}_Z$, given by some section $s : X_{S'} \to Z$, to be given by $H^0(X_{S'}, s^*T_{Z/X_S})$, where T_{Z/X_S} is the tangent bundle of $Z \to X_S$; and then an obstruction space would be given by $H^1(X_{S'}, s^*T_{Z/X_S})$. Thus, one can expect smoothness to hold

only when $H^1(X_{S'}, s^*T_{Z/X_S})$ vanishes. This holds true, locally on S', if all slopes of s^*T_{Z/X_S} are positive (by Proposition II.3.4 (iii)), suggesting the following definition.

Definition IV.4.1. *Let $\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z$ be the open subfunctor of all sections $s : X_{S'} \to Z$ such that s^*T_{Z/X_S} has everywhere positive Harder–Narasimhan slopes.*

Roughly speaking, one expects $\mathcal{M}_Z^{\mathrm{sm}}$ to look infinitesimally like the Banach–Colmez space $\mathcal{BC}(s^*T_{Z/X_S})$; these indeed are cohomologically smooth when all slopes are positive, by Proposition II.3.5 (iii). Unfortunately, we are unable to prove a direct relation of this sort; however, we will be able to relate these spaces via a "deformation to the normal cone".

Our goal is to prove the following theorem.

Theorem IV.4.2. *Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid spaces such that Z admits a Zariski closed immersion into an open subset of (the adic space) $\mathbb{P}^n_{X_S}$ for some $n \geq 0$. Then $\mathcal{M}_Z$ is a locally spatial diamond, the map $\mathcal{M}_Z \to S$ is compactifiable, and $\mathcal{M}_Z^{\mathrm{sm}} \to S$ is cohomologically smooth.*

Moreover, for a geometric point $x : \operatorname{Spa} C \to \mathcal{M}_Z^{\mathrm{sm}}$ given by a map $\operatorname{Spa} C \to S$ and a section $s : X_C \to Z$, the map $\mathcal{M}_Z^{\mathrm{sm}} \to S$ is at x of ℓ-dimension equal to the degree of s^*T_{Z/X_S}.

Remark IV.4.3. The map $\mathcal{M}_Z^{\mathrm{sm}} \to S$ is a natural example of a map that is only locally of finite dimension, but not globally so (as there are many connected components of increasing dimension).

Remark IV.4.4. In the "classical context" of algebraic curves the preceding theorem is the following (easy) result. Let X/k be a proper smooth curve and $Z \to X$ be quasi-projective smooth. Consider $\mathcal{M}_Z$ the functor on k-schemes that sends S to morphisms $s : X \times_k S \to Z$ over X. This is representable by a quasi-projective scheme over $\operatorname{Spec}(k)$. Let $\mathcal{M}_Z^{sm}$ be the open sub-scheme defined by the condition that if $s : X \times_k S \to Z$ is an S-point of $\mathcal{M}_Z$ then the vector bundle $s^*T_{Z/X}$ has no H^1 fiberwise on S. Then $\mathcal{M}_Z \to \operatorname{Spec}(k)$ is smooth.

Remark IV.4.5. Suppose that W is a smooth quasi-projective E-scheme. The moduli space $\mathcal{M}_Z$ with $Z = W \times_E X_S$ classifies morphisms $X_S \to W$ i.e. $\mathcal{M}_Z$ is a moduli of morphisms from families of Fargues–Fontaine curves to W. This is some kind of "Gromov–Witten" situation.

Remark IV.4.6. We could have made the a priori weaker assumption that Z admits a Zariski closed immersion inside an open subset of $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle on X_S. Nevertheless, since the result is local on S and we can suppose it is affinoid perfectoid, and since when S is affinoid perfectoid $\mathcal{O}_{X_S}(1)$ "is ample" i.e. there is a surjection $\mathcal{O}_{X_S}(-N)^n \twoheadrightarrow \mathcal{E}$ for $N, n \gg 0$, this assumption is equivalent to the one we made i.e. we can suppose $\mathcal{E}$ is free.

Example IV.4.7 (The Quot diamond). Let $\mathcal{E}$ be a vector bundle on X_S. We denote by

$$\mathrm{Quot}_{\mathcal{E}} \longrightarrow S$$

the moduli space over S of locally free quotients of $\mathcal{E}$. Fixing the rank of such a quotient, one sees that $\mathrm{Quot}_{\mathcal{E}}$ is a finite disjoint union of spaces $\mathcal{M}_Z$ with $Z \to X_S$ a Grassmannian of quotients of $\mathcal{E}$. This is thus representable in locally spatial diamonds, compactifiable, of locally finite dim. trg.

Let $\mathrm{Quot}_{\mathcal{E}}^{\mathrm{sm}} \subset \mathrm{Quot}_{\mathcal{E}}$ be the open subset parametrizing quotients $u : \mathcal{E} \to \mathcal{F}$ such that fiberwise, the greatest slope of $\ker u$ is strictly less than the smallest slope of $\mathcal{F}$. According to Theorem IV.4.2 this is cohomologically smooth over S.

Fix an integer $n \geq 1$. For some $N \in \mathbb{Z}$ and $r \in \mathbb{N}_{\geq 1}$, let $\mathrm{Quot}^{n,\mathrm{sm},\circ}_{\mathcal{O}(N)^r}$ be the open subset of $\mathrm{Quot}^{\mathrm{sm}}_{\mathcal{O}(N)^r}$ where the quotient has rank n and its slopes are greater than N. When N and r vary one constructs, as in the "classical case", cohomologically smooth charts on $\mathrm{Bun}_{\mathrm{GL}_n}$ using $\mathrm{Quot}^{n,\mathrm{sm},\circ}_{\mathcal{O}(N)^r}$. In fact, the morphism

$$\mathrm{Quot}^{n,\mathrm{sm},\circ}_{\mathcal{O}(N)^r} \longrightarrow \mathrm{Bun}_{\mathrm{GL}_n}$$

given by the quotient of $\mathcal{O}(N)^r$ is separated cohomologically smooth. When pulled back by a morphism $S \to \mathrm{Bun}_{\mathrm{GL}_n}$ with S perfectoid, this is an open subset of a positive Banach–Colmez space.

We will not use the Quot diamond in the following. In section V.3, using the Jacobian criterion, we will construct charts on Bun_G for any G that are better suited to our needs.

IV.4.1. Smooth maps of sous-perfectoid adic spaces. We need some background about smooth morphisms of adic spaces in non-noetherian settings. We choose the setting of sous-perfectoid adic spaces as defined by Hansen-Kedlaya, [**HK20**], cf. [**SW20**, Section 6.3]. Recall that an adic space X is sous-perfectoid if it is analytic and admits an open cover by $U = \mathrm{Spa}(R, R^+)$ where each R is a sous-perfectoid Tate algebra, meaning that there is some perfectoid R-algebra $\widetilde{R}$ such that $R \to \widetilde{R}$ is a split injection in the category of topological R-modules.

The class of sous-perfectoid rings R is stable under passage to rational localizations, finite étale maps, and $R\langle T_1, \ldots, T_n\rangle$. As smooth maps should be built from these basic examples, we can hope for a good theory of smooth maps of sous-perfectoid spaces.

Recall that a map $f : Y \to X$ of sous-perfectoid adic spaces is étale if locally on the source and target it can be written as an open immersion followed by a finite étale map.

Definition IV.4.8. *Let $f : Y \to X$ be a map of sous-perfectoid adic spaces. Then f is smooth if one can cover Y by open subsets $V \subset Y$ such that there are étale maps $V \to \mathbb{B}^d_X$ for some integer $d \geq 0$.*

It can immediately be checked that analytifications of smooth schemes satisfy this condition.

Proposition IV.4.9. *Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid sous-perfectoid adic space, and let $f_0 : Y_0 \to \mathrm{Spec}\, A$ be a smooth map of schemes. Let $f : Y \to X$ be the analytification of $f_0 : Y_0 \to \mathrm{Spec}\, A$, representing the functor taking $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ to the $\mathrm{Spec}\, B$-valued points of $Y_0 \to \mathrm{Spec}\, A$. Then $f : Y \to X$ is smooth.*

Proof. Locally, f_0 is the composite of an étale and the projection from affine space. This means that its analytification is locally étale over the projection from the analytification of affine space, which is a union of balls, giving the result. □

Let us analyze some basic properties of smooth maps of sous-perfectoid adic spaces.

Proposition IV.4.10. *Let $f : Y \to X$ and $g : Z \to Y$ be maps of sous-perfectoid adic spaces.*

(i) *The property of f being smooth is local on Y.*

(ii) *If f and g are smooth, then so is $f \circ g : Z \to X$.*

(iii) *If $h : X' \to X$ is any map of sous-perfectoid adic spaces and f is smooth, then the fibre product $Y' = Y \times_X X'$ in adic spaces exists, is sous-perfectoid, and $f' : Y' \to X'$ is smooth.*

(iv) *The map f is universally open.*

(v) *If f is surjective, then there is some étale cover $X' \to X$ with a lift $X' \to Y$.*

Regarding part (1), we note that we will see in Proposition IV.4.18 that the property of f being smooth is in fact étale local on Y (and thus smooth local on X and Y, using (5)).

Proof. Part (1) is clear from the definition. For part (2), the composite is locally a composite of an étale map, a projection from a ball, an étale map, and another projection from a ball; but we can swap the two middle maps, and use that composites of étale maps are étale. Part (3) is again clear, by the stability properties of sous-perfectoid rings mentioned above. For part (4), it is now enough to see that f is open, and we can assume that f is a composite of an étale map and the projection from a ball, both of which are open. For part (5), using that f is open, we can work locally on Y and thus assume again that it is a composite of an étale map and the projection from a ball; we can then replace Y by its open image in the ball. By [**Sch17a**, Lemma 9.5], for any geometric point $\mathrm{Spa}(C, C^+) \to X$ of X, we can find a lift to Y. Writing the geometric point as the limit of affinoid étale neighborhoods, the map to $Y \subset \mathbb{B}^d_X$ can be approximated at some finite stage, and then openness of Y ensures that it will still lie in Y. This gives the desired étale cover of X over which f splits. □

Of course, the most important structure of a smooth morphism is its module of Kähler differentials. Recall that if Y is sous-perfectoid, then one can define a stack (for the étale topology) of vector bundles on Y, such that for $Y = \mathrm{Spa}(R, R^+)$ affinoid with R sous-perfectoid, the category of vector bundles is equivalent to the category of finite projective R-modules; see [**KL15**], [**SW20**, Theorem 5.2.8, Proposition 6.3.4]. By definition, a vector bundle on Y is an $\mathcal{O}_Y$-module that is locally free of finite rank.

Definition IV.4.11. *Let $f : Y \to X$ be a smooth map of sous-perfectoid adic spaces, with diagonal $\Delta_f : Y \to Y \times_X Y$. Let $\mathcal{I}_{Y/X} \subset \mathcal{O}_{Y \times_X Y}$ be the ideal sheaf. Then*

$$\Omega^1_{Y/X} := \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$$

considered as $\mathcal{O}_{Y \times_X Y}/\mathcal{I}_{Y/X} = \mathcal{O}_Y$-module.

It follows from the definition that there is a canonical $\mathcal{O}_X$-linear derivation $d : \mathcal{O}_Y \to \Omega^1_{Y/X}$, given by $g \mapsto g \otimes 1 - 1 \otimes g$.

Proposition IV.4.12. *Let $f : Y \to X$ be a smooth map of sous-perfectoid adic spaces. Then $\Omega^1_{Y/X}$ is a vector bundle on Y. There is a unique open and closed decomposition $Y = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_n$ such that $\Omega^1_{Y/X}|_{Y_d}$ is of rank d for all $d = 0, \ldots, n$. In that case, for any open subset $V \subset Y_d$ with an étale map $V \to \mathbb{B}^{d'}_X$, necessarily $d' = d$.*

We will say that f is smooth of dimension d if $\Omega^1_{Y/X}$ is of rank d. By the proposition, this is equivalent to asking that Y can be covered by open subsets V that admit étale maps $V \to \mathbb{B}^d_X$. In particular, f is smooth of dimension 0 if and only if it is étale.

Proof. It is enough to show that if f is a composite of an étale map $Y \to \mathbb{B}^d_X$ with the projection to X, then $\Omega^1_{Y/X}$ is isomorphic to $\mathcal{O}^d_Y$. Indeed, this implies that $\Omega^1_{Y/X}$ is a vector bundle in general, of the expected rank; and the decomposition into open and closed pieces is then a general property of vector bundles.

Let $Y' = \mathbb{B}^d_X$. Then $Y \times_X Y \to Y' \times_X Y'$ is étale, and the map $Y \to Y' \times_{Y' \times_X Y'} (Y \times_X Y)$ is an open immersion (as the diagonal of the étale map $Y \to Y'$). It follows that $\mathcal{I}_{Y/X}$ is the pullback of $\mathcal{I}_{Y'/X'}$. But $Y' \hookrightarrow Y' \times_X Y'$ is of the form

$$\mathrm{Spa}(R\langle T_1, \dots, T_n\rangle, R^+\langle T_1, \dots, T_n\rangle) \hookrightarrow \mathrm{Spa}(R\langle T_1^{(1)}, \dots, T_n^{(1)}, T_1^{(2)}, \dots, T_n^{(2)}\rangle, R^+\langle T_1^{(1)}, \dots, T_n^{(2)}\rangle)$$

if $X = \mathrm{Spa}(R, R^+)$, and the ideal sheaf is given by $(T_1^{(1)} - T_1^{(2)}, \dots, T_n^{(1)} - T_n^{(2)})$. This defines a regular sequence after any étale localization, by the lemma below. This gives the claim. □

Lemma IV.4.13. *Let $X = \mathrm{Spa}(R\langle T_1, \dots, T_n\rangle, R^+\langle T_1, \dots, T_n\rangle)$ where R is a sous-perfectoid Tate ring, let $Y = \mathrm{Spa}(S, S^+)$ where S is a sous-perfectoid Tate ring, and let $f : Y \to X$ be a smooth map. Then $T_1, \dots, T_n$ define a regular sequence on S and $(T_1, \dots, T_n)S \subset S$ is a closed ideal.*

Proof. By induction, one can reduce to the case $n = 1$. The claim can be checked locally, so we can assume that Y is étale over $\mathbb{B}^d_X$ for some d; replacing X by $\mathbb{B}^d_X$, we can then assume that f is étale. Let $Y_0 \subset Y$ be the base change to $X_0 = \mathrm{Spa}(R, R^+) = V(T) \subset X$; then Y and $Y_0 \times_{X_0} X$ are both étale over X and become isomorphic over $X_0 \subset X$. By spreading of étale maps, this implies that they are isomorphic after base change to $X' = \mathrm{Spa}(R\langle T'\rangle, R^+\langle T'\rangle)$ where $T' = \varpi^n T$ for some n (and ϖ is a pseudouniformizer of R). This easily implies the result. □

Locally around a section, any smooth space is a ball:

Lemma IV.4.14. *Let $f : Y \to X$ be a smooth map of sous-perfectoid spaces with a section $s : \mathrm{Spa}(K, K^+) \to Y$ for some point $\mathrm{Spa}(K, K^+) \to X$. Then there are open neighborhoods $U \subset X$ of $\mathrm{Spa}(K, K^+)$ and $V \subset Y$ of $s(\mathrm{Spa}(K, K^+))$ such that $U \cong \mathbb{B}^d_V$.*

Proof. We can assume that X and Y are affinoid. If f is étale, then any section extends to a small neighborhood (e.g. by [**Sch17a**, Lemma 15.6, Lemma 12.17]), and any section is necessarily étale and thus open, giving the result in that case. In general, we may work locally around the given section, so we can assume that f is the composite of an étale map $Y \to \mathbb{B}^d_X$ and the projection to X. Using the étale case already handled, we can assume that Y is an open subset of $\mathbb{B}^d_X$. Any section $\mathrm{Spa}(K, K^+) \to \mathbb{B}^d_X$ has a cofinal system of neighborhoods that are small balls over open subsets of X: The section is given by d elements $T_1, \dots, T_d \in K^+$, and after picking a pseudouniformizer ϖ and shrinking X, one can find global sections $T'_1, \dots, T'_d$ of $\mathcal{O}^+_X(X)$ such that $T_i \equiv T'_i \mod \varpi^n$. Then $\{|T'_1|, \dots, |T'_d| \leq |\varpi|^n\}$ is a small ball over X, and the intersection of all these is $\mathrm{Spa}(K, K^+)$. Thus, one of these neighborhoods is contained in Y, as desired. □

Proposition IV.4.15. *Let $f_i : Y_i \to X$, $i = 1, 2$, be smooth maps of sous-perfectoid adic spaces, and let $g : Y_1 \to Y_2$ be a map over X.*

(i) *If g is smooth, then the sequence*

$$0 \to g^*\Omega^1_{Y_2/X} \to \Omega^1_{Y_1/X} \to \Omega^1_{Y_1/Y_2} \to 0$$

is exact.

(ii) *Conversely, if* $g^*\Omega^1_{Y_2/X} \to \Omega^1_{Y_1/X}$ *is a locally split injection, then* g *is smooth.*

In particular, if $g^*\Omega^1_{Y_2/X} \to \Omega^1_{Y_1/X}$ *is an isomorphism, then* g *is étale.*

Proof. Part (1) follows from a routine reduction to the case of projections from balls, where it is clear. For part (2), we may assume that $Y_1 \to \mathbb{B}^{d_1}_X$ and $Y_2 \to \mathbb{B}^{d_2}_X$ are étale. It suffices to see that the composite $Y_1 \to Y_2 \to \mathbb{B}^{d_2}_X$ is smooth, as g is the composite of its base change $Y_1 \times_{\mathbb{B}^{d_2}_X} Y_2 \to Y_2$ with the section $Y_1 \to Y_1 \times_{\mathbb{B}^{d_2}_X} Y_2$ of the étale map $Y_1 \times_{\mathbb{B}^{d_2}_X} Y_2 \to Y_1$; any such section is automatically itself étale. Thus, we may assume that $Y_2 = \mathbb{B}^{d_2}_X$. Locally on Y_1, we may find a projection $g' : \mathbb{B}^{d_1}_X \to \mathbb{B}^{d_1-d_2}_X$ so that

$$(g'^*\Omega^1_{\mathbb{B}^{d_1-d_2}_X/X})|_{Y_1}$$

is an orthogonal complement of $g^*\Omega^1_{Y_2/X}$. Thus, we can assume that $d_1 = d_2 =: d$, and $g^*\Omega^1_{Y_2/X} \to \Omega^1_{Y_1/X}$ is an isomorphism.

Our aim is now to prove that $g : Y_1 \to Y_2 = \mathbb{B}^d_X$ is étale. We may assume that all of X, Y_1 and Y_2 are affinoid. Passing to the fibre over a point $S = \mathrm{Spa}(K, K^+) \to X$, this follows from a result of Huber, [**Hub96**, Proposition 1.6.9 (iii)]. The resulting étale map $Y_{1,S} \to Y_{2,S}$ deforms uniquely to a quasicompact separated étale map $Y'_{1,U} \to Y_{2,U}$ for a small enough neighborhood $U \subset X$ of S, by [**Sch17a**, Lemma 12.17]. Moreover, the map $Y_{1,U} \to Y_{2,U}$ lifts uniquely to $Y_{1,U} \to Y'_{1,U}$ for U small enough, by the same result. Replacing X by U, Y_1 by $Y_{1,U}$ and Y_2 by $Y'_{1,U}$, we can now assume that $g : Y_1 \to Y_2$ is a map between sous-perfectoid spaces smooth over X that is an isomorphism on one fibre. It is enough to see that it is then an isomorphism in a neighborhood. To see this, we may in fact work locally on Y_2.

For this, we study $Y_1 \subset Y_1 \times_X Y_2 \to Y_2$: Here $Y_1 \times_X Y_2 \to Y_2$ is smooth, and $Y_1 \subset Y_1 \times_X Y_2$ is locally the vanishing locus of d functions (as $Y_2 \subset Y_2 \times_X Y_2$ is). Moreover, over fibres lying over the given point of X, the map $Y_1 \to Y_2$ becomes an isomorphism, and in particular gives a section of $Y_1 \times_X Y_2 \to Y_2$. By Lemma IV.4.14, after shrinking Y_2, we can assume that there is an open neighborhood $V \subset Y_1 \times_X Y_2$ such that $V \cong \mathbb{B}^d_{Y_2}$. Inside there, Y_1 is (locally) given by the vanishing of d functions, and is only a point in one fibre. Now the result follows from the next lemma, using Y_2 in place of X. □

Lemma IV.4.16. *Let* $X = \mathrm{Spa}(A, A^+)$ *be a sous-perfectoid affinoid adic space with a point* $X' = \mathrm{Spa}(K, K^+) \to X$. *Let* $f_1, \ldots, f_n \in A^+\langle T_1, \ldots, T_n\rangle$ *be functions such that*

$$K \to K\langle T_1, \ldots, T_n\rangle/(f_1, \ldots, f_n)$$

is an isomorphism. Then, after replacing X *by an open neighborhood of* X', *the map*

$$A \to A\langle T_1, \ldots, T_n\rangle/(f_1, \ldots, f_n)$$

is an isomorphism.

Proof. For any ring B with elements $g_1, \ldots, g_n \in B$, consider the homological Koszul complex

$$\mathrm{Kos}(B, (g_i)_{i=1}^n) = [B \to B^n \to \ldots \to B^n \xrightarrow{(g_1,\ldots,g_n)} B].$$

We claim that, after shrinking X, we can in fact arrange that

$$A \to \mathrm{Kos}(A\langle T_1, \dots, T_n\rangle, (f_i)_{i=1}^n)$$

is a quasi-isomorphism.

Note that all terms of these complexes are free Banach-A-modules, and thus the formation of this complex commutes with all base changes; and one can use descent to establish the statement. In particular, we can reduce first to the case that X is perfectoid, and then to the case that X is strictly totally disconnected. In that case, the map $A \to K$ is automatically surjective, and so we can arrange that under the isomorphism $K \cong K\langle T_1, \dots, T_n\rangle/(f_1, \dots, f_n)$, all T_i are mapped to 0. Moreover, applying another change of basis, we can arrange that the image of f_i in $K\langle T_1, \dots, T_n\rangle/(T_1, \dots, T_n)^2$ is given by $a_i T_i$ for some nonzero scalar $a_i \in K^+$. Note that we are in fact allowed to also localize on $\mathbb{B}^n_X$ around the origin, as away from the origin the functions $f_1, \dots, f_n$ locally generate the unit ideal (in the fibre, but thus in a small neighborhood). Doing such a localization, we can now arrange that $f_i \equiv T_i \mod \varpi$ for some pseudouniformizer $\varpi \in A^+$. But now in fact

$$A^+ \to \mathrm{Kos}(A^+\langle T_1, \dots, T_n\rangle, (f_i)_{i=1}^n)$$

is a quasi-isomorphism, as can be checked modulo ϖ, where it is the quasi-isomorphism

$$A^+/\varpi \to \mathrm{Kos}(A^+/\varpi[T_1, \dots, T_n], (T_i)_{i=1}^n). \qquad \square$$

Let us draw some consequences. First, we have the following form of the Jacobian criterion in this setting.

Proposition IV.4.17. *Let $f : Y \to X$ be a smooth map of sous-perfectoid adic spaces, and let $f_1, \dots, f_r \in \mathcal{O}_Y(Y)$ be global functions such that $df_1, \dots, df_r \in \Omega^1_{Y/X}(Y)$ can locally be extended to a basis of $\Omega^1_{Y/X}$. Then $Z = V(f_1, \dots, f_r) \subset Y$ is a sous-perfectoid space smooth over X.*

Proof. We can assume that all $f_i \in \mathcal{O}^+_Y(Y)$ by rescaling, and we can locally find $f_{r+1}, \dots, f_n \in \mathcal{O}^+_Y(Y)$ such that $df_1, \dots, df_n$ is a basis of $\Omega^1_{Y/X}$. This induces an étale map $Y \to \mathbb{B}^n_X$, and then $V(f_1, \dots, f_r) \subset Y$ is the pullback of $\mathbb{B}^r_X \subset \mathbb{B}^n_X$, giving the desired result. $\square$

Moreover, we can prove that being smooth is étale local on the source.

Proposition IV.4.18. *Let $f : Y \to X$ be a map of sous-perfectoid adic spaces. Assume that there is some étale cover $j : V \to Y$ such that $f \circ j$ is smooth. Then f is smooth.*

Proof. By étale descent of vector bundles on sous-perfectoid adic spaces, $\Omega^1_{Y/X} := \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is a vector bundle, together with an $\mathcal{O}_X$-linear derivation $d : \mathcal{O}_Y \to \Omega^1_{Y/X}$. We claim that locally we can find functions $f_1, \dots, f_n \in \mathcal{O}_Y$ such that $df_1, \dots, df_n \in \Omega^1_{Y/X}$ is a basis. To do this, it suffices to find such functions over all fibres $\mathrm{Spa}(K, K^+) \to X$, as any approximation will then still be a basis (small perturbations of a basis are still a basis). But over fibres, the equivalence of the constructions in [**Hub96**, 1.6.2] shows that the df for $f \in \mathcal{O}_X$ form generators of $\Omega^1_{Y/X}$.

Thus, assume that there are global sections $f_1, \dots, f_n$ such that $df_1, \dots, df_n \in \Omega^1_{Y/X}$ are a basis. Rescaling the f_i if necessary, they define a map $g : Y \to \mathbb{B}^d_X$ that induces an isomorphism

$g^*\Omega^1_{\mathbb{B}^d_X/X} \to \Omega^1_{Y/X}$. By Proposition IV.4.15, the map $Y \to \mathbb{B}^d_X$ is étale locally on Y étale. We may assume that Y and X are affinoid; in particular, all maps are separated. Then by [**Sch17a**, Lemma 15.6, Proposition 11.30] also $Y \to \mathbb{B}^d_X$ is étale. □

Finally, we note that if Y and Y' are both smooth over a sous-perfectoid space X, then the concept of Zariski closed immersions $Y \hookrightarrow Y'$ over X is well-behaved.

Proposition IV.4.19. *Let $f : Y \to X$, $f' : Y' \to X$ be smooth maps of sous-perfectoid adic spaces, and let $g : Y \to Y'$ be a map over X. The following conditions are equivalent.*

(i) *There is a cover of Y' by open affinoid $V' = \mathrm{Spa}(S', S'^+)$ such that $V = Y \times_{Y'} V' = \mathrm{Spa}(S, S^+)$ is affinoid and $S' \to S$ is surjective, with $S^+ \subset S$ the integral closure of the image of S'^+.*

(ii) *For any open affinoid $V' = \mathrm{Spa}(S', S'^+) \subset Y'$, the preimage $V = Y \times_{Y'} V' = \mathrm{Spa}(S, S^+)$ is affinoid and $S' \to S$ is surjective, with $S^+ \subset S$ the integral closure of the image of S'^+.*

Moreover, in this case the ideal sheaf $\mathcal{I}_{Y \subset Y'} \subset \mathcal{O}_{Y'}$ is pseudocoherent in the sense of [**KL16**], *and locally generated by sections $f_1, \ldots, f_r \in \mathcal{O}_{Y'}$ such that $df_1, \ldots, df_r \in \Omega^1_{Y'/X}$ can locally be extended to a basis.*

Proof. We first analyze the local structure under condition (1), so assume that $Y' = \mathrm{Spa}(S', S'^+)$ and $Y = \mathrm{Spa}(S, S^+)$ are affinoid, with $S' \to S$ surjective and $S^+ \subset S$ the integral closure of the image of S'^+. It follows that $g^*\Omega^1_{Y'/X} \to \Omega^1_{Y/X}$ is surjective, and letting d' and d be the respective dimensions of Y' and Y (which we may assume to be constant), we see that $r = d' - d \geq 0$ and that locally we can find $f_1, \ldots, f_r \in \mathcal{I}_{Y \subset Y'}$ so that $df_1, \ldots, df_r$ generate the kernel of $g^*\Omega^1_{Y'/X} \to \Omega^1_{Y/X}$ (as the kernel is generated by the closure of the image of $\mathcal{I}_{Y \subset Y'}$). By Proposition IV.4.17, the vanishing locus of the f_i defines a sous-perfectoid space $Z \subset Y'$ that is smooth over X. The induced map $Y \to Z$ induces an isomorphism on differentials, hence is étale by Proposition IV.4.15; but it is also a closed immersion, hence locally an isomorphism.

We see that the ideal sheaf $\mathcal{I}_{Y \subset Y'}$ is locally generated by sections $f_1, \ldots, f_r$ as in the statement of the proposition. By the proof of Proposition IV.4.17 and Lemma IV.4.13, it follows that the ideal sheaf $\mathcal{I}_{Y \subset Y'}$ is pseudocoherent in the sense of [**KL16**].

To finish the proof, it suffices to show that (1) implies (2). By the gluing result for pseudocoherent modules of [**KL16**], the pseudocoherent sheaf $\mathcal{I}_{Y \subset Y'}$ over V' corresponds to a pseudocoherent module $I \subset S'$, and then necessarily $V = \mathrm{Spa}(S, S^+)$ where $S = S'/I$ with $S^+ \subset S$ the integral closure of the image of S'^+. □

Definition IV.4.20. *In the setup of Proposition IV.4.19, the map g is a Zariski closed immersion if the equivalent conditions are satisfied.*

IV.4.2. Maps from X_S into $\mathbb{P}^n$. Our arguments make critical use of the assumption that in Theorem IV.4.2, the space $Z \to X_S$ is locally closed in $\mathbb{P}^n_{X_S}$. For this reason, we analyze the special case of $\mathbb{P}^n$ in this section.

Proposition IV.4.21. *Let $n \geq 0$ and consider the small v-sheaf $\mathcal{M}_{\mathbb{P}^n}$ taking any perfectoid space S to the set of maps $X_S \to \mathbb{P}^n_E$. Then $\mathcal{M}_{\mathbb{P}^n} \to *$ is partially proper and representable in*

locally spatial diamonds, and admits a decomposition into open and closed subspaces

$$\mathcal{M}_{\mathbb{P}^n} = \bigsqcup_{m \geq 0} \mathcal{M}_{\mathbb{P}^n}^m$$

such that each $\mathcal{M}_{\mathbb{P}^n}^m \to *$ *has finite* $\dim.\operatorname{trg}$*, and the degree of the pullback of* $\mathcal{O}_{\mathbb{P}^n}(1)$ *to* $X_{\mathcal{M}_{\mathbb{P}^n}^m}$ *is* m*. In fact, there is a canonical open immersion*

$$\mathcal{M}_{\mathbb{P}^n}^m \hookrightarrow (\mathcal{BC}(\mathcal{O}_{X_S}(m)^{n+1}) \setminus \{0\})/\underline{E^\times}.$$

Proof. The degree of the pullback $\mathcal{L}/X_S$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ to X_S defines an open and closed decomposition according to all $m \in \mathbb{Z}$. Fix some m. Then over the corresponding subspace $\mathcal{M}_{\mathbb{P}^n}^m$, we can fix a trivialization $\mathcal{L} \cong \mathcal{O}_{X_S}(m)$, which amounts to an $\underline{E^\times}$-torsor. After this trivialization, one parametrizes $n+1$ sections of $\mathcal{L} \cong \mathcal{O}_{X_S}(m)$ without common zeroes. The condition of no common zeroes is an open condition on S: Indeed, the common zeroes form a closed subspace of $|X_S|$, and the map $|X_S| \to |S|$ is closed (see the proof of Lemma IV.1.20). This implies the desired description. □

Proposition IV.4.22. *Let* S *be a perfectoid space and let* $Z \to X_S$ *be a smooth map of sous-perfectoid adic spaces such that* Z *admits a Zariski closed embedding into an open subspace of* $\mathbb{P}^n_{X_S}$*. Then the induced functor*

$$\mathcal{M}_Z \to \mathcal{M}_{\mathbb{P}^n_{X_S}}$$

is locally closed. More precisely, for any perfectoid space $T \to \mathcal{M}_{\mathbb{P}^n_{X_S}}$*, the preimage of* $\mathcal{M}_Z$ *is representable by some perfectoid space* $T_Z \subset T$ *that is étale locally Zariski closed in* T*, i.e. there is some étale cover of* T *by affinoid perfectoid* $T' = \operatorname{Spa}(R, R^+) \to T$ *such that* $T_Z \times_T T' = \operatorname{Spa}(R_Z, R_Z^+)$ *is affinoid perfectoid, with* $R \to R_Z$ *surjective and* $R_Z^+ \subset R_Z$ *the integral closure of the image of* R^+*.*

In particular, the map $\mathcal{M}_Z \to S$ *is representable in locally spatial diamonds and compactifiable, of locally finite* $\dim.\operatorname{trg}$*.*

Proof. Choose an open subspace $W \subset \mathbb{P}^n_{X_S}$ such that Z is Zariski closed in W. For any perfectoid space T with a map $T \to \mathcal{M}_{\mathbb{P}^n_{X_S}}$ corresponding to a map $X_T \to \mathbb{P}^n_{X_S}$ over X_S, the locus $T_W \subset T$ where the section factors over W is open. Indeed, this locus is the complement of the image in $|T|$ of the preimage of $|\mathbb{P}^n_{X_S} \setminus W|$ under $|X_T| \to |\mathbb{P}^n_{X_S}|$, and $|X_T| \to |T|$ is closed.

Replacing T by T_W, we can assume that the section $X_T \to \mathbb{P}^n_{X_S}$ factors over W. We may also assume that $T = \operatorname{Spa}(R, R^+)$ is affinoid perfectoid and that $S = T$. Pick a pseudouniformizer $\varpi \in R$, in particular defining the cover

$$Y_{S,[1,q]} = \{|\pi|^q \leq |[\varpi]| \leq |\pi|\} \subset \operatorname{Spa} W_{\mathcal{O}_E}(R^+)$$

of X_S. The pullback of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ to X_S along this section, and then to $Y_{S,[1,q]}$, is étale locally trivial, as when S is a geometric point, $Y_{S,[1,q]}$ is affinoid with ring of functions a principal ideal domain by Corollary II.1.12. Replacing W by a small étale neighborhood of this section and correspondingly shrinking S, we can assume that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ to $W_{[1,q]} = W \times_{X_S} Y_{S,[1,q]}$ is trivial. In that case the pullback $Z_{[1,q]} \to Y_{S,[1,q]}$ of $Z \to X_S$ is Zariski closed in an open subset of

$$\mathbb{A}^{n+1}_{Y_{S,[1,q]}}.$$

Inside $\mathbb{A}^{n+1}_{Y_{S,[1,q]}}$, the image of $Y_{\mathrm{Spa}(K,K^+),[1,q]}$ (via the given section) for a point $\mathrm{Spa}(K,K^+) \to S$ is an intersection of small balls over $Y_{S',[1,q]}$ for small neighborhoods $S' \subset S$ of $\mathrm{Spa}(K,K^+)$. Thus, one of these balls is contained in the open subset of which $Z_{[1,q]}$ is a Zariski closed subset. Thus, after this further localization, we can assume that there is a Zariski closed immersion

$$Z_{[1,q]} \hookrightarrow \mathbb{B}^{n+1}_{Y_{S,[1,q]}},$$

and in particular $Z_{[1,q]}$ is affinoid and cut out by global functions on $\mathbb{B}^{n+1}_{Y_{S,[1,q]}}$ by Proposition IV.4.19. Pulling back these functions along the given section $Y_{S,[1,q]} \to \mathbb{B}^{n+1}_{Y_{S,[1,q]}}$, it suffices to see that if $S = \mathrm{Spa}(R,R^+)$ is an affinoid perfectoid space of characteristic p with a choice of pseudouniformizer $\varpi \in R$ and

$$f \in B_{R,[1,q]} = \mathcal{O}(Y_{S,[1,q]})$$

is a function, then there is a universal perfectoid space $S' \subset S$ for which the pullback of f is zero, and $S' \subset S$ is Zariski closed. This is given by Lemma IV.4.23. □

Lemma IV.4.23. *Let $S = \mathrm{Spa}(R,R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ be affinoid perfectoid with a fixed pseudo-uniformizer ϖ, $I \subset (0,\infty)$ a compact interval with rational ends, and $Z \subset |Y_{S,I}|$ a closed subset defined by the vanishing locus of an ideal $J \subset \mathcal{O}(Y_{S,I})$. Then, via the open projection $\upsilon : |Y_{S,I}| \to |S|$, the closed subset $|S| \setminus \upsilon(|Y_{S,I}| \setminus Z)$ is Zariski closed. The corresponding Zariski closed perfectoid subspace of S is universal for perfectoid spaces $T \to S$ such that $J \mapsto 0$ via $\mathcal{O}(Y_{S,I}) \to \mathcal{O}(Y_{T,I})$.*

Proof. Since $Y^{\diamond}_{S,I} \to S$ is cohomologically smooth, υ is open. We can suppose $J = (f)$ with $f \in \mathcal{O}(Y_{S,I})$. For any untilt of $\mathbb{F}_q((\varpi^{1/p^\infty}))$ over E such that $|\pi|^b \leq |[\varpi^\sharp]| \leq |\pi|^a$ if $I = [a,b]$, we get a corresponding untilt $R^\sharp$ of R over E, with a map $B_{R,I} \to R^\sharp$. The locus where the image of f in $R^\sharp$ vanishes is Zariski closed by Proposition II.0.2. Intersecting these Zariski closed subsets over varying such untilts gives the vanishing locus of f, as in any fibre, f vanishes as soon at it vanishes at infinitely many untilts (e.g., by Corollary II.1.12), and all rings are sous-perfectoid, in particular uniform, so vanishing at all points implies vanishing. □

IV.4.3. Formal smoothness of $\mathcal{M}^{sm}_Z$. The key result we need is the following.

Proposition IV.4.24. *Let $S = \mathrm{Spa}(R,R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$ and let $Z \to X_S$ be a smooth map of sous-perfectoid adic spaces that is Zariski closed in an open subspace of $\mathbb{P}^n_{X_S}$. Then $\mathcal{M}^{\mathrm{sm}}_Z \to S$ is formally smooth.*

Proof. Pick a test diagram as in Definition IV.3.1; we can and do assume that the S from there is the given S, replacing the S in this proposition if necessary. This means we have a diagram

$$\begin{tikzcd} & Z \arrow[d, dotted] \\ X_{S_0} \arrow[ur, "s_0"] \arrow[r, hook] & X_S \end{tikzcd}$$

and, up to replacing S by an étale neighborhood of S_0 we try to extend the section s_0 to a section over X_S (the dotted line in the diagram). Fix a geometric point $\mathrm{Spa}(C,C^+) \to S_0$; we will always

allow ourselves to pass to étale neighborhoods of this point. Fix a pseudouniformizer $\varpi \in R$ and consider the affinoid cover $Y_{S,[1,q]} \to X_S$; recall that

$$Y_{S,[1,q]} = \{|\pi|^q \leq |[\varpi]| \leq |\pi|\} \subset \operatorname{Spa} W_{\mathcal{O}_E}(R^+)$$

and we also consider its boundary annuli

$$Y_{S,[1,1]} = \{|[\varpi]| = |\pi|\}, Y_{S,[q,q]} = \{|\pi|^q = |[\varpi]|\} \subset Y_{S,[1,q]}.$$

Let $Z_{[1,q]} \to Z$ be its pullback; with pullback $Z_{[1,1]}, Z_{[q,q]} \subset Z_{[1,q]}$ of $Y_{S,[1,1]}, Y_{S,[q,q]} \subset Y_{S,[1,q]}$. In particular, Z is obtained from $Z_{[1,q]}$ via identification of its open subsets $Z_{[1,1]}, Z_{[q,q]}$ along the isomorphism $\varphi : Z_{[1,1]} \to Z_{[q,q]}$.

Arguing as in the proof of Proposition IV.4.22, we can after étale localization on S embed

$$Z_{[1,q]} \hookrightarrow \mathbb{B}^{n+1}_{Y_{S,[1,q]}}$$

as a Zariski closed subset. We thus have a diagram

$$\begin{tikzcd} Z_{[1,q]} \arrow[r, hook, "\text{Zariski closed}"] & \mathbb{B}^{n+1}_{Y_{S,[1,q]}} \arrow[d] \\ Y_{S_0,[1,q]} \arrow[u] \arrow[r, hook] & Y_{S,[1,q]}. \end{tikzcd}$$

In particular, $Z_{[1,q]}$ is affinoid.

Next, consider the Kähler differentials $\Omega^1_{Z_{[1,q]}/Y_{S,[1,q]}}$. Again, as $B_{C,[1,q]}$ is a principal ideal domain, its restriction to the section $Y_{\operatorname{Spa}(C,C^+),[1,q]} \subset Z_{[1,q]}$ is trivial, and thus it is trivial in a small neighborhood. It follows that after a further étale localization we can assume that $\Omega^1_{Z_{[1,q]}/Y_{S,[1,q]}} \cong \mathcal{O}^r_{Z_{[1,q]}}$ is trivial. On the Zariski closed subset $Z_{0,[1,q]} \subset Z_{[1,q]}$, this implies that we may find functions $f_1, \ldots, f_r \in \mathcal{O}(Z_{0,[1,q]})$ vanishing on the section $Y_{S_0,[1,q]} \to Z_{0,[1,q]}$ and locally generating the ideal of this closed immersion (use Proposition IV.4.19). In particular,

$$df_1, \ldots, df_r \in \Omega^1_{Z_{0,[1,q]}/Y_{S_0,[1,q]}}$$

are generators at the image of the section $Y_{S_0,[1,q]} \to Z_{0,[1,q]}$, and thus in an open neighborhood. Picking lifts of the f_i to $\mathcal{O}(Z_{[1,q]})$ and shrinking $Z_{[1,q]}$, Proposition IV.4.15 implies that they define an étale map

$$Z_{[1,q]} \to \mathbb{B}^r_{Y_{S,[1,q]}}.$$

Moreover, over $\{0\}_{Y_{S_0,[1,q]}} \subset \mathbb{B}^r_{Y_{S,[1,q]}}$, this map admits a section. Shrinking further around this section, we can thus arrange that there are open immersions

$$(\pi^N \mathbb{B})^r_{Y_{S,[1,q]}} \subset Z_{[1,q]} \subset \mathbb{B}^r_{Y_{S,[1,q]}},$$

and that the section over $Y_{S_0,[1,q]}$ is given by the zero section.

The isomorphism $\varphi : Z_{[1,1]} \to Z_{[q,q]}$ induces a map

$$\varphi' : (\pi^N \mathbb{B})^r_{Y_{S,[1,1]}} \to \mathbb{B}^r_{Y_{S,[q,q]}}.$$

Recall that for any compact interval $I \subset (0, \infty)$, the space

$$Y_{S,I} = \operatorname{Spa}(B_{R,I}, B^+_{(R,R^+),I})$$

is affinoid. The map φ' is then given by a map

$$\alpha : B_{R,[q,q]}\langle T_1, \ldots, T_r\rangle \to B_{R,[1,1]}\langle \pi^{-N}T_1, \ldots, \pi^{-N}T_r\rangle$$

linear over the isomorphism $\varphi : B_{R,[q,q]} \to B_{R,[1,1]}$. The map α is determined by the images of $T_1, \ldots, T_n$ which are elements

$$\alpha_i \in B^+_{(R,R^+),[1,1]}\langle \pi^{-N}T_1, \ldots, \pi^{-N}T_r\rangle.$$

These have the property that on the quotient $B_{R_0,[1,1]}$ they vanish at $T_1 = \ldots = T_r = 0$ (as over S_0, the zero section is φ-invariant). Moreover, over the geometric point $\mathrm{Spa}(C, C^+) \to S_0$ fixed at the beginning, we can apply a linear change of coordinates in order to ensure that the derivative at the origin is given by a standard matrix for an isocrystal of negative slopes; i.e., there are cycles $1, \ldots, r_1$; $r_1 + 1, \ldots, r_2$; $\ldots$; $r_{a-1} + 1, \ldots, r_a = r$ and positive integers $d_1, \ldots, d_a$ such that

$$\alpha_i \equiv T_{i+1} \text{ in } B_{C,[1,1]}[T_1, \ldots, T_r]/(T_1, \ldots, T_r)^2$$

if $i \neq r_j$ for some $j = 1, \ldots, a$, and

$$\alpha_{r_j} \equiv \pi^{-d_j}T_{r_{j-1}+1} \text{ in } B_{C,[1,1]}[T_1, \ldots, T_r]/(T_1, \ldots, T_r)^2.$$

(Here, we set $r_0 = 0$.) Approximating this linear change of basis over an étale neighborhood, we respect the condition that the α_i's vanish at $T_1 = \ldots = T_r = 0$ over S_0, while we can for any large M arrange

$$\alpha_i \equiv T_{i+1} \text{ in } B^+_{(R,R^+),[1,1]}/\pi^M[\pi^{-N}T_1, \ldots, \pi^{-N}T_r]/(\pi^{-N}T_1, \ldots, \pi^{-N}T_r)^2$$

if $i \neq r_j$ and

$$\alpha_{r_j} \equiv \pi^{-d_j}T_{r_{j-1}+1} \text{ in } B^+_{(R,R^+),[1,1]}/\pi^M[\pi^{-N}T_1, \ldots, \pi^{-N}T_r]/(\pi^{-N}T_1, \ldots, \pi^{-N}T_r)^2.$$

Moreover, rescaling all T_i by powers of π, and passing to a smaller neighborhood around S_0, we can then even ensure that

$$\alpha_i \in T_{i+1} + \pi^M B^+_{(R,R^+),[1,1]}\langle T_1, \ldots, T_r\rangle$$

for $i \neq r_j$ and

$$\alpha_{r_j} \in \pi^{-d_j}T_{r_{j-1}+1} + \pi^M B^+_{(R,R^+),[1,1]}\langle T_1, \ldots, T_r\rangle.$$

At this point, the integers $d_1, \ldots, d_a$ are fixed, while we allow ourselves to choose M later, depending only on these.

From this point on, we will no longer change S and S_0, and instead will merely change coordinates in the balls (by automorphisms). More precisely, we study the effect of replacing T_i by $T_i + \epsilon_i$ for some

$$\epsilon_i \in \pi^d \ker(B^+_{(R,R^+),[1,q]} \to B_{R_0,[1,q]})$$

where we take d to be at least the maximum of all d_j. This replaces α_i by a new power series α'_i, given by

$$\alpha'_i(T_1, \ldots, T_r) = \alpha_i(T_1, \ldots, T_i + \epsilon_i, \ldots, T_r) - \varphi(\epsilon_i)$$

and the α'_i's still vanish at $T_1 = \ldots = T_r = 0$ over S_0. Their nonconstant coefficients will still have the same properties as for α_i (the linear coefficients are unchanged, while all other coefficients are divisible by π^M), and the constant coefficient satisfies

$$\alpha'_i(0, \ldots, 0) \equiv \alpha_i(0, \ldots, 0) + \epsilon_{i+1} - \varphi(\epsilon_i) \text{ in } B^+_{(R,R^+),[1,1]}/\pi^{M+d}$$

if $i \neq r_j$ and

$$\alpha'_{r_j}(0,\dots,0) \equiv \alpha_{r_j}(0,\dots,0) + \pi^{-d_j}\epsilon_{r_{j-1}+1} - \varphi(\epsilon_{r_j}) \text{ in } B^+_{(R,R^+),[1,1]}/\pi^{M+d}.$$

Assume that by some inductive procedure we already achieved $\alpha_i(0,\dots,0) \in \pi^{N'}B^+_{(R,R^+),[1,1]}$ for some $N' \geq M$. By Lemma IV.4.25 below, there is some constant c depending only on $d_1,\dots,d_a$ such that we can then find $\epsilon_i \in \pi^{N'-c}B^+_{(R,R^+),[1,1]}$, vanishing over R_0, with

$$\alpha_i(0,\dots,0) = \varphi(\epsilon_i) - \epsilon_{i+1}$$

for $i \neq r_j$ and

$$\alpha_{r_j}(0,\dots,0) = \varphi(\epsilon_{r_j}) - \pi^{-d_j}\epsilon_{r_{j-1}+1}.$$

This means that $\alpha'_i(0,\dots,0) \in \pi^{M+N'-c}B^+_{(R,R^+),[1,1]}$, so if we choose $M > c$ in the beginning (which we can), then this inductive procedure converges, and in the limit we get a change of basis after which the zero section defines a φ-invariant section of $Z_{[1,q]}$, thus a section $s : X_S \to Z$, as desired. Note that we arranged that this section agrees with s_0 over S_0, as all coordinate changes did not affect the situation over S_0. □

We used the following quantitative version of vanishing of $H^1(X_S,\mathcal{E})$ for $\mathcal{E}$ of positive slopes.

Lemma IV.4.25. *Fix a standard Dieudonné module of negative slopes, given explicitly on a basis $e_1,\dots,e_r$ by fixing cycles $1,\dots,r_1$; $r_1+1\dots,r_2$; $\dots$; $r_{a-1}+1,\dots,r_a = r$ and positive integers $d_1,\dots,d_a > 0$, via*

$$\varphi(e_i) = e_{i+1} \text{ for } i \neq r_j,\ \varphi(e_{r_j}) = \pi^{-d_j}e_{r_{j-1}+1}.$$

Then there is an integer $c \geq 0$ with the following property.

Let $S = \mathrm{Spa}(R,R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$ with Zariski closed subspace $S_0 = \mathrm{Spa}(R_0,R_0^+)$, and a pseudouniformizer $\varpi \in R$. Let

$$I^+_{[1,q]} = \ker(B^+_{(R,R^+),[1,q]} \to B^+_{(R_0,R_0^+),[1,q]}),\ I^+_{[1,1]} = \ker(B^+_{(R,R^+),[1,1]} \to B^+_{(R_0,R_0^+),[1,1]}).$$

Then for all $f_1,\dots,f_r \in I^+_{[1,1]}$ one can find $g_1,\dots,g_r \in \pi^{-c}I^+_{[1,q]}$ such that

$$f_i = \varphi(g_i) - g_{i+1} \text{ for } i \neq r_j,\ f_{r_j} = \varphi(g_{r_j}) - \pi^{-d_j}g_{r_{j-1}+1}.$$

Proof. We may evidently assume that $a = 1$; set $d = d_1$. By linearity, we can assume that all but one of the f_i's is equal to zero. Thus, it suffices to see that for all positive integers r and N there is $c \geq 0$ such that for all $f \in I^+_{[1,1]}$ one can find some $g \in \pi^{-c}I^+_{[1,q^r]}$ (for the evident definition of $I^+_{[1,q^r]}$) such that

$$f = \varphi^r(g) - \pi^{-d}g.$$

Replacing E by its unramified extension of degree r, we can then assume that $r = 1$. At this point, we want to reduce to the qualitative version given by Lemma IV.4.26 below, saying that the map

$$\varphi - \pi^{-d} : I_{[1,q]} \to I_{[1,1]}$$

is surjective. Indeed, assume a constant c as desired would not exist. Then for any integer $i \geq 0$ we can find some Zariski closed immersion $S_{0,i} = \mathrm{Spa}(R_{0,i},R_{0,i}^+) \subset S_i = \mathrm{Spa}(R_i,R_i^+)$, with choices of pseudouniformizers $\varpi_i \in R_i$, as well as elements $f_i \in I^+_{[1,1],i}$ such that there is no $g \in \pi^{-2i}I^+_{[1,q],i}$ with

$f_i = \varphi(g_i) - \pi^{-d} g_i$. Then we can define $R^+ = \prod_i R_i^+$ with $\varpi = (\varpi_i)_i \in R^+$, and $R = R^+[\frac{1}{\varpi}]$, which defines an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$, containing a Zariski closed subspace $S_0 \subset S$ defined similarly. Moreover, the sequence $(\pi^i f_i)_i$ defines an element of $f \in I_{[1,1]}^+$. As $\varphi - \pi^{-d} : I_{[1,q]} \to I_{[1,1]}$ is surjective by Lemma IV.4.26, we can find some $g \in I_{[1,q]}$ with $f = \varphi(g) - \pi^{-d} g$. Then $\pi^c g \in I_{[1,q]}^+$ for some c, and restricting g to $S_{0,i} \subset S_i$ with $i > c$ gives the desired contradiction. □

We reduced to the following qualitative version.

Lemma IV.4.26. *Let d be a positive integer, let $S_0 = \mathrm{Spa}(R_0, R_0^+) \subset S = \mathrm{Spa}(R, R^+)$ be a Zariski closed immersion of affinoid perfectoid spaces over $\mathbb{F}_q$, and let $\varpi \in R$ be a pseudouniformizer. Let*

$$I_{[1,q]} = \ker(B_{R,[1,q]} \to B_{R_0,[1,q]}) \ , \ I_{[1,1]} = \ker(B_{R,[1,1]} \to B_{R_0,[1,1]}).$$

Then the map

$$\varphi - \pi^{-d} : I_{[1,q]} \to I_{[1,1]}$$

is surjective.

Proof. By the snake lemma and the vanishing $H^1(X_S, \mathcal{O}_{X_S}(d)) = 0$ (Proposition II.2.5 (iii)), the lemma is equivalent to the surjectivity of

$$H^0(X_S, \mathcal{O}_{X_S}(d)) \to H^0(X_{S_0}, \mathcal{O}_{X_{S_0}}(d)).$$

For $d \leq [E : \mathbb{Q}_p]$ (or if E is of equal characteristic), this follows directly from Proposition II.2.5 (iv) and the surjectivity of $R^{\circ\circ} \to R_0^{\circ\circ}$. In general, we can either note that the proof of Proposition II.2.5 (iii) also proves the lemma, or argue by induction by choosing an exact sequence

$$0 \to \mathcal{O}_{X_S}(d-2) \to \mathcal{O}_{X_S}(d-1)^2 \to \mathcal{O}_{X_S}(d) \to 0$$

(the Koszul complex for two linearly independent sections of $H^0(X_S, \mathcal{O}_{X_S}(1))$), and use the vanishing of $H^1(X_{S_0}, \mathcal{O}_{X_{S_0}}(d-2)) = 0$ for $d > 2$, Proposition II.2.5 (iii). This induction gets started as long as $E \neq \mathbb{Q}_p$. For $E = \mathbb{Q}_p$, we can write $\mathcal{O}_{X_S}(d)$ as a direct summand of $\pi_* \pi^* \mathcal{O}_{X_S}(d)$ for any extension $\pi : X_{S,E} \to X_S$ with $E \neq \mathbb{Q}_p$. □

IV.4.4. Universal local acyclicity of $\mathcal{M}_Z^{sm} \to S$. The next step in the proof of Theorem IV.4.2 is to show that $\mathbb{F}_\ell$ is universally locally acyclic.

Proposition IV.4.27. *Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid spaces such that Z is Zariski closed inside an open subset of $\mathbb{P}^n_{X_S}$ for some $n \geq 0$. Then, for any $\ell \neq p$, the sheaf $\mathbb{F}_\ell$ is universally locally acyclic for the map*

$$\mathcal{M}_Z^{\mathrm{sm}} \to S.$$

Proof. Recall from Proposition IV.4.22 that $\mathcal{M}_Z \to \mathcal{M}_{\mathbb{P}^n_{X_S}}$ is a locally closed immersion, and the open embedding

$$\mathcal{M}_{\mathbb{P}^n_{X_S}} \hookrightarrow \bigsqcup_{m \geq 0} (\mathcal{BC}(\mathcal{O}_{X_S}(m)^{n+1}) \setminus \{0\})/\underline{E^\times}$$

from Proposition IV.4.21. In the following, we fix some m and work on the preimage of

$$(\mathcal{BC}(\mathcal{O}_{X_S}(m)^{n+1}) \setminus \{0\})/\underline{E^\times}.$$

We choose a surjection $T \to \mathcal{BC}(\mathcal{O}_{X_S}(m)^{n+1} \setminus \{0\})/\underline{E^\times}$ from a perfectoid space T as in Lemma IV.4.28; in particular, g is separated, representable in locally spatial diamonds, cohomologically smooth, and formally smooth. Moreover, locally T admits a Zariski closed immersion into the perfectoid ball $\widetilde{\mathbb{B}}^n_S$ over S. Taking the pullback of T to $\mathcal{M}_Z$, we get a surjection $T_Z \to \mathcal{M}_Z$ for some perfectoid space T_Z such that étale locally T_Z admits a Zariski closed immersion into a space étale over $\widetilde{\mathbb{B}}^n_S$.

It follows that one can cover $\mathcal{M}_Z$ via maps $h_0 : T_0 \to \mathcal{M}_Z$ that are separated, representable in locally spatial diamonds, cohomologically smooth, and formally smooth, and such that T_0 admits a Zariski closed immersion into some space étale over $\widetilde{\mathbb{B}}^n_S$. By Proposition IV.4.24, we can, up to further replacement of T_0 by an étale cover, assume that the map h_0 extends to a map $h : T \to \mathcal{M}_Z$ for some perfectoid space T étale over $\widetilde{\mathbb{B}}^n_S$. Moreover, as $T_0 \to \mathcal{M}_Z$ is formally smooth, we can, after a further étale localization, lift the map $T \to \mathcal{M}_Z$ to a retraction $T \to T_0$; thus, T_0 is a retract of a space that is étale over a perfectoid ball. Now the result follows from Corollary IV.2.27. $\square$

We used the following presentation of certain projectivized Banach–Colmez spaces.

Lemma IV.4.28. *Let S be a perfectoid space over $\mathbb{F}_q$ and let $\mathcal{E}$ be a vector bundle on X_S that is everywhere of nonnegative Harder–Narasimhan slopes. There is a perfectoid space $T \to S$ that is locally Zariski closed in a perfectoid ball $\widetilde{\mathbb{B}}^n_S$ over S and that admits a surjective map*

$$T \to (\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$$

over S that is separated, representable in locally spatial diamonds, cohomologically smooth, and formally smooth.

Proof. The target parametrizes line bundles $\mathcal{L}$ on X_S of slope zero together with a section of $\mathcal{E} \otimes \mathcal{L}$ that is nonzero fibrewise on S. Parametrizing in addition an injection $\mathcal{L} \hookrightarrow \mathcal{O}_{X_S}(1)$ defines a map that is separated, representable in locally spatial diamonds, cohomologically smooth, and formally smooth (by Proposition II.3.5 and Proposition IV.3.8). Over this cover, one has locally on S an untilt $S^\sharp$ over E corresponding to the support of the cokernel of $\mathcal{L} \to \mathcal{O}_{X_S}(1)$, and one parametrizes nonzero sections of $\mathcal{E}(1)$ that vanish at $S^\sharp \hookrightarrow X_S$. This is Zariski closed (by [**BS19**, Theorem 7.4, Remark 7.5]) inside the space of all sections of $\mathcal{E}(1)$. We see that it suffices to prove the similar result with $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$ replaced by $\mathcal{BC}(\mathcal{E}(1)) \times \operatorname{Spd} E$, and this reduces to the individual factors. For $\mathcal{BC}(\mathcal{E}(1))$, the result follows from the argument in Proposition IV.3.8. For $\operatorname{Spd} E$, there is nothing to do in equal characteristic, so assume that E is p-adic. Then we reduce to $[*/\underline{\mathcal{O}_E^\times}]$ as the fibres of $\operatorname{Spd} E \to [*/\underline{\mathcal{O}_E^\times}]$ over perfectoid spaces are given by $\mathcal{BC}(\mathcal{L}) \setminus \{0\}$ for some line bundle $\mathcal{L}$ of slope 1, and this in turn admits covers of the desired form. Finally, for $[*/\underline{\mathcal{O}_E^\times}]$, we can pass to the étale cover $[*/\underline{1 + p^2\mathcal{O}_E}] \cong [*/\underline{\mathcal{O}_E}]$, or to $[*/\underline{E}]$. This, finally, admits a surjection from a perfectoid open unit disc $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ with the desired properties by passing to Banach–Colmez spaces in an exact sequence

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(\tfrac{1}{2}) \to \mathcal{O}_{X_S}(1) \to 0$$

and using Proposition II.3.4. $\square$

IV.4.5. Deformation to the normal cone. The final step in the proof of Theorem IV.4.2 is a deformation to the normal cone.

By Proposition IV.4.22 and Proposition IV.4.27 (and Proposition IV.2.33), in order to prove Theorem IV.4.2 it only remains to prove that $Rf^!\mathbb{F}_\ell$ is invertible and sitting in the expected cohomological degree. Picking a v-cover $T \to \mathcal{M}_Z^{\mathrm{sm}}$ by some perfectoid space T and using that the formation of $Rf^!\mathbb{F}_\ell$ commutes with any base change by Proposition IV.2.15, it suffices to prove the following result.

Proposition IV.4.29. *Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid spaces such that Z admits a Zariski closed immersion into an open subset of (the adic space) $\mathbb{P}^n_{X_S}$ for some $n \geq 0$. Let $f : \mathcal{M}_Z \to S$ be the moduli space of sections of $Z \to X_S$. Moreover, let $s : X_S \to Z$ be a section such that s^*T_{Z/X_S} is everywhere of positive Harder–Narasimhan slopes, and of degree d.*

Let $t : S \to \mathcal{M}_Z$ be the section of f corresponding to s. Then $t^*Rf^!\mathbb{F}_\ell$ is étale locally on S isomorphic to $\mathbb{F}_\ell[2d]$.

Proof. We will prove this by deformation to the normal cone. In order to avoid a general discussion of blow-ups etc., we will instead take an approach based on the local structure of Z near a section as exhibited in the proof of Proposition IV.4.24.

We are free to make v-localizations on S (as being cohomologically smooth can be checked after a v-cover), and replace Z by an open neighborhood of $s(X_S)$. With this freedom, we can follow the proof of Proposition IV.4.24 and ensure that $S = \mathrm{Spa}(R, R^+)$ is strictly totally disconnected with pseudouniformizer ϖ, the pullback $Z_{[1,q]} \to Y_{S,[1,q]}$ of $Z \to X_S$ to

$$Y_{S,[1,q]} = \{|\pi|^q \leq |[\varpi]| \leq |\pi|\} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$$

satisfies

$$\pi^N \mathbb{B}^r_{Y_{S,[1,q]}} \subset Z_{[1,q]} \subset \mathbb{B}^r_{Y_{S,[1,q]}}$$

and the gluing isomorphism is given by power series

$$\alpha_i \in T_{i+1} + \pi^M B^+_{(R,R^+),[1,1]}\langle T_1, \ldots, T_r\rangle$$

resp.

$$\alpha_{r_j} \in \pi^{-d_j} T^{r_j-1+1} + \pi^M B^+_{(R,R^+),[1,1]}\langle T_1, \ldots, T_r\rangle$$

with notation following the proof of Proposition IV.4.24. Moreover, the constant coefficients of all α_i vanish. These in fact define a map

$$\varphi : \pi^d \mathbb{B}^r_{Y_{S,[1,1]}} \to \mathbb{B}^r_{Y_{S,[q,q]}}$$

preserving the origin, where d is the maximum of the d_j.

For any $n \geq N, d$, we can look at the subset

$$Z^{(n)}_{[1,q]} = \pi^n \mathbb{B}^r_{Y_{S,[1,q]}} \cup \varphi(\pi^n \mathbb{B}^r_{Y_{S,[1,1]}}) \subset Z_{[1,q]},$$

which descends to an open subset $Z^{(n)} \subset Z$. Letting $T_i^{(n)} = \pi^{-n} T_i$, the gluing is then given by power series $\alpha_i^{(n)}$ given by

$$\alpha_i^{(n)} = \pi^{-n} \alpha_i(\pi^n T_1, \ldots, \pi^n T_n)$$

which satisfy the same conditions, but the nonlinear coefficients of $\alpha_i^{(n)}$ become more divisible by π. The limit

$$\alpha_i^{(\infty)} = \varinjlim_{n\to\infty} \alpha_i^{(n)} \in B_{R,[1,1]}\langle T_1, \dots, T_r\rangle$$

exists, and is linear in the T_i.

Let $S' = S \times \underline{\mathbb{N}_{\geq N} \cup \{\infty\}}$, using the profinite set $\mathbb{N}_{\geq N} \cup \{\infty\}$. Let $Z' \to X_{S'}$ be the smooth map of sous-perfectoid spaces obtained by descending

$$Z'_{[1,q]} = \mathbb{B}^r_{Y_{S',[1,q]}} \cup \varphi'(\mathbb{B}^r_{Y_{S',[q,q]}})$$

along the isomorphism φ' given by the power series

$$\alpha'_i = (\alpha_i^{(N)}, \alpha_i^{(N+1)}, \dots, \alpha_i^{(\infty)}) \in B_{R',[1,1]}\langle T_1, \dots, T_r\rangle.$$

Then the fibre of $Z' \to X_{S'}$ over $S \times \{n\}$ is given by $Z^{(n)}$, while its fibre over $S \times \{\infty\}$ is given by an open subset $Z^{(\infty)}$ of the Banach–Colmez space $\mathcal{BC}(s^*T_{Z/X_S})$. Moreover, letting $S'^{(>N)} \subset S'$ be the complement of $S \times \{N\}$, there is natural isomorphism $\gamma : S'^{(>N)} \to S'$ given by the shift $S \times \{n+1\} \cong S \times \{n\}$, and this lifts to an open immersion

$$\gamma : Z'^{(>N)} = Z' \times_{X_{S'}} X_{S'^{(>N)}} \hookrightarrow Z'.$$

We need to check that $Z' \to X_{S'}$ still satisfies the relevant quasiprojectivity assumption.

Lemma IV.4.30. *The space $Z' \to X_{S'}$ admits a Zariski closed immersion into an open subset of $\mathbb{P}^m_{X_{S'}}$ for some $m \geq 0$.*

Proof. One may perform a parallel construction with Z replaced by an open subset of $\mathbb{P}^m_{X_S}$, reducing us to the case that Z is open in $\mathbb{P}^m_{X_S}$. In that case, the key observation is that the blow-up of $\mathbb{P}^m_{X_S}$ at the section $s : X_S \to \mathbb{P}^m_{X_S}$ is still projective, which is an easy consequence of X_S admitting enough line bundles. □

Let $f' : \mathcal{M}_{Z'} \to S'$ be the projection, with fibres $f^{(n)}$ and $f^{(\infty)}$. By Proposition IV.4.27, both $\mathbb{F}_\ell$ and $Rf'^!\mathbb{F}_\ell$ are f'-universally locally acyclic. In particular, the formation of $Rf'^!\mathbb{F}_\ell$ commutes with base change, and we see that the restriction of $Rf'^!\mathbb{F}_\ell$ to the fibre over ∞ is étale locally isomorphic to $\mathbb{F}_\ell[2d]$, as an open subset of $\mathcal{BC}(s^*T_{Z/X_S})$. As S is strictly totally disonnected, one can choose a global isomorphism with $\mathbb{F}_\ell[2d]$.

The map from $\mathbb{F}_\ell[2d]$ to the fibre of $Rf'^!\mathbb{F}_\ell$ over ∞ extends to a small neighborhood; passing to this small neighborhood, we can assume that there is a map

$$\beta : \mathbb{F}_\ell[2d] \to Rf'^!\mathbb{F}_\ell$$

that is an isomorphism in the fibre over ∞. We can assume that this map is γ-equivariant (passing to a smaller neighborhood). Let Q be the cone of β. Then Q is still f'-universally locally acyclic, as is its Verdier dual

$$\mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q) = R\mathscr{H}\mathrm{om}_{\mathcal{M}_{Z'}}(Q, Rf'^!\mathbb{F}_\ell).$$

In particular, $Rf'_!\mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q) \in D_{\text{ét}}(S', \mathbb{F}_\ell)$ is constructible, and its restriction to $S \times \{\infty\}$ is trivial. This implies (e.g. by [**Sch17a**, Proposition 20.7]) that its restriction to $S \times \{n, n+1, \dots, \infty\}$ is

trivial for some $n \gg 0$. Passing to this subset, we can assume that $Rf'_! \mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q) = 0$. Taking Verdier duals and using Corollary IV.2.25, this implies that $Rf'_* Q = 0$.

In particular, for all $n \geq n_0$, we have $Rf_*^{(n)} Q|_{\mathcal{M}_{Z^{(n)}}} = 0$. Using the γ-equivariance, this is equivalent to

$$Rf_*^{(n)}(Q|_{\mathcal{M}_Z^{(n_0)}})|_{\mathcal{M}_{Z^{(n)}}} = 0,$$

regarding $\mathcal{M}_{Z^{(n)}} \subset \mathcal{M}_{Z^{(n_0)}}$ as an open subset. Taking the colimit over all n and using that the system $\mathcal{M}_{Z^{(n)}} \subset \mathcal{M}_{Z^{(n_0)}}$ has intersection $s(S) \subset \mathcal{M}_Z$ and is cofinal with a system of spatial diamonds of finite cohomological dimension (as can be checked in the case of projective space), [**Sch17a**, Proposition 14.9] implies that

$$s^* Q|_{\mathcal{M}_{Z^{(n_0)}}} = \varinjlim_n Rf_*^{(n)}(Q|_{\mathcal{M}_{Z^{(n_0)}}})|_{\mathcal{M}_{Z^{(n)}}} = 0$$

(by applying it to the global sections on any quasicompact separated étale $\widetilde{S} \to S$), and thus the map

$$s^* \beta|_{\mathcal{M}_Z} : \mathbb{F}_\ell[2d] \to s^* Rf^! \mathbb{F}_\ell$$

is an isomorphism, as desired. This finishes the proof of Proposition IV.4.29 and thus of Theorem IV.4.2. □

The idea of the preceding proof is the following. Let $C \to X_S \times \mathbb{A}^1$ be the open subset of the deformation to the normal cone of $s : X_S \hookrightarrow Z$ (we did not develop the necessary formalism to give a precise meaning to this in the context of smooth sous-perfectoid spaces, but it could be done) whose fiber at $0 \in \mathbb{A}^1$ is the normal cone of the immersion s (the divisor over $0 \in \mathbb{A}^1$ of the deformation to the normal cone is the union of two divisors: the projective completion of the normal cone and the blow-up of Z along X_S, both meeting at infinity inside the projective completion). One has a diagram

$$\begin{tikzcd} X_S \times \mathbb{A}^1 \arrow[r, hook] \arrow[dr] & C \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

where outside $t = 0 \in \mathbb{A}^1$ this is given by the section $s : X_S \hookrightarrow Z$, i.e. the pullback over $\mathbb{G}_m$ of the preceding diagram gives the inclusion $X_S \times \mathbb{G}_m \hookrightarrow Z \times \mathbb{G}_m$, and at $t = 0$ this is the inclusion of X_S inside the normal cone of the section s. Let us note moreover that C is equipped with a $\mathbb{G}_m$-action compatible with the one on $\mathbb{A}^1$.

This gives rise to an $\underline{E^\times}$-equivariant morphism with an equivariant section

$$\begin{tikzcd} \mathcal{M}_C \arrow[d, shift left, "g"] \\ S \times \underline{E} \arrow[u, bend left, "s"] \end{tikzcd}$$

whose fiber at $0 \in \underline{E}$ is the zero section of $\mathcal{BC}(s^* T_{Z/X_S}) \to S$, and is isomorphic to $\mathcal{M}_Z \times \underline{E^\times}$ equipped with the section s outside of 0. Now, the complex $s'^* Rg^! \mathbb{F}_\ell$ is $\underline{E^\times}$-equivariant on $S \times \underline{E}$. Its fiber outside $0 \in \underline{E}$, i.e. its restriction to $S \times \underline{E^\times}$, is $s^* Rf^! \mathbb{F}_\ell$, and its fiber at 0 is $\mathbb{F}_\ell(d)[2d]$,

$d = \deg(T_{Z/X_S})$ (since g is universally locally acyclic the dualizing complexe commutes with base change).

Now one checks that one can replace the preceding diagram by a quasicompact $\underline{\mathcal{O}_E \setminus \{0\}}$-invariant open subset $U \subset \mathcal{M}_C$ together with an equivariant diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, shift left, "h"] & \mathcal{M}_{\mathcal{C}} \arrow[d, shift left, "g"] \\ S \times \underline{\mathcal{O}_E} \arrow[r, hook] \arrow[u, bend left, "t"] & S \times \underline{E}. \arrow[u, bend left, "s"] \end{tikzcd}$$

In the preceding proof one replaces $\underline{\mathcal{O}_E}$ by $\underline{\pi^{\mathbb{N}\cup\{\infty\}}} \subset \underline{\mathcal{O}_E}$, which does not change anything for the argument. One concludes using that $\times\pi$ "contracts everything to 0" and some constructibility argument using the fact that U is spatial and some complexes are h-universally locally acyclic (see the argument "$Rf'_!\mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q) \in D_{\text{ét}}(S', \mathbb{F}_\ell)$ is constructible" at the end of the proof of Proposition IV.4.29).

IV.5. Partial compactly supported cohomology

Let us start by recalling the following basic vanishing result. Let C be a complete algebraically closed nonarchimedean field C with pseudouniformizer $\varpi \in C$. Let $\operatorname{Spa}\mathbb{Z}(\!(t)\!) \times_{\mathbb{Z}} \operatorname{Spa} C = \mathbb{D}_C^*$ be the punctured open unit disc over C, and consider the subsets

$$j : U = \{|t| \leq |\varpi|\} \hookrightarrow \mathbb{D}_C^*, j' : U' = \{|t| \geq |\varpi|\} \hookrightarrow \mathbb{D}_C^*$$

Note that the punctured open unit disc has two ends: Towards the origin, and towards the boundary. The open subsets U and U' contain one end each.

Lemma IV.5.1. *The partially compactly supported cohomology groups*

$$R\Gamma(\mathbb{D}_C^*, j_!\Lambda) = 0 = R\Gamma(\mathbb{D}_C^*, j'_!\Lambda)$$

vanish.

As usual Λ is any coefficient ring killed by an integer n prime to p.

Proof. We treat the vanishing $R\Gamma(\mathbb{D}_C^*, j_!\Lambda) = 0$, the other one being similar. Let $k : \mathbb{D}_C^* \hookrightarrow \mathbb{D}_C$ be the inclusion. One has an exact triangle

$$(kj)_!\Lambda \longrightarrow Rk_*j_!\Lambda \longrightarrow i_*A \xrightarrow{+1}$$

where $i : \{0\} \hookrightarrow \mathbb{D}_C$. One has $H^0(A) = \Lambda$, $H^1(A) = \Lambda(1)$, $H^i(A) = 0$ for $i \neq 0, 1$, since $A = \varinjlim_n R\Gamma(U_n, \Lambda)$ with $U_n = \{|t| \leq |\varpi^n|\} \subset \mathbb{D}_C^*$being a punctured disc. We thus have to prove that the preceding triangle induces an isomorphism $A \xrightarrow{\sim} R\Gamma_c(U, \Lambda)[1]$. Let $\widetilde{j} : \mathbb{P}^1 \setminus \{0, \infty\} \hookrightarrow \mathbb{P}^1 \setminus \{0\}$. There is a commutative diagram

$$\begin{tikzcd} R\Gamma_c(U, \Lambda) \arrow[r] \arrow[d, "\simeq"] & R\Gamma(\mathbb{D}_C^*, j_!\Lambda) \arrow[r] \arrow[d] & A \arrow[r, "+1"] \arrow[d, equal] & {} \\ R\Gamma_c(\mathbb{P}^1 \setminus \{0, \infty\}, \Lambda) \arrow[r] & R\Gamma(\mathbb{P}^1 \setminus \{0\}, \widetilde{j}_!\Lambda) \arrow[r] & A \arrow[r, "+1"] & . \end{tikzcd}$$

It thus suffices to check that

$$R\Gamma(\mathbb{P}^1 \setminus \{0\}, \widetilde{j}_!\Lambda) = 0,$$

for example in the algebraic setting using comparison theorems, which is an easy exercise. □

Our goal now is to prove a very general version of such a result. Fix an algebraically closed field $k|\mathbb{F}_q$ and work on Perf_k. Let X be a spatial diamond such that $f : X \to * = \mathrm{Spd}\, k$ is partially proper with $\dim.\,\mathrm{trg}\, f < \infty$. Then the base change $X \times_k S$ of X to any spatial diamond S is not itself quasicompact. Rather, it has two ends, and we will in this section study the cohomology with compact support towards one of the ends.

To analyze the situation, pick quasi-pro-étale and universally open surjections $\widetilde{X} \to X$ and $\widetilde{S} \to S$ from affinoid perfectoid spaces (using [**Sch17a**, Proposition 11.24]), and pick maps $\widetilde{X} \to \mathrm{Spa}\, k(\!(t)\!)$ and $\widetilde{S} \to \mathrm{Spa}\, k(\!(u)\!)$ by choosing pseudouniformizers. We get a correspondence

$$\begin{tikzcd} & \widetilde{X} \times \widetilde{S} \arrow[dl] \arrow[dr] & \\ X \times S & & \mathrm{Spa}\, k(\!(t)\!) \times \mathrm{Spa}\, k(\!(u)\!) = \mathbb{D}^*_{k(\!(u)\!)} \end{tikzcd}$$

where all maps are qcqs, and the left map is (universally) open. Now $\mathrm{Spa}\, k(\!(t)\!) \times \mathrm{Spa}\, k(\!(u)\!)$ is a punctured open unit disc over $\mathrm{Spa}\, k(\!(u)\!)$, and one can write it as the increasing union of the affinoid subspaces

$$\{|u|^b \leq |t| \leq |u|^a\} \subset \mathrm{Spa}\, k(\!(t)\!) \times \mathrm{Spa}\, k(\!(u)\!)$$

for varying rational $0 < a \leq b < \infty$. For any two choices of pseudouniformizers, a power of one divides the other, so it follows that if $\widetilde{U}_{a,b} \subset \widetilde{X} \times \widetilde{S}$ denotes the preimage of $\{|t|^b \leq |u| \leq |t|^a\}$, then the doubly-indexed ind-system $\{\widetilde{U}_{a,b}\}_{0<a\leq b<\infty}$ is independent of the choice of the maps $\widetilde{X} \to \mathrm{Spa}\, k(\!(t)\!)$, $\widetilde{S} \to \mathrm{Spa}\, k(\!(u)\!)$.

Let $U_{a,b} \subset X \times S$ be the image of $\widetilde{U}_{a,b}$. As $\widetilde{X} \times \widetilde{S} \to X \times S$ is open, this is a qcqs open subset of $X \times S$. Moreover, the doubly indexed ind-system $\{U_{a,b}\}_{0<a\leq b<\infty}$ is independent of all choices made.

We let $U_a = \bigcup_{b<\infty} U_{a,b}$ and $U_b = \bigcup_{a>0} U_{a,b}$ and let

$$j_{a,b} : U_{a,b} \to X \times S \ , \ j_a : U_a \to X \times S \ , \ j_b : U_b \to X \times S$$

be the open immersions. We can now define the cohomology groups of interest, or rather the version of pushforward along $\beta : X \times S \to S$. As usual, Λ is a coefficient ring killed by some integer n prime to p.

Definition IV.5.2. *The functors*

$$R\beta_{!+}, R\beta_{!-} : D_{\text{ét}}(X \times S, \Lambda) \to D_{\text{ét}}(S, \Lambda)$$

are defined by

$$R\beta_{!+}C := \varinjlim_a R\beta_*(j_{a!}C|_{U_a}),$$
$$R\beta_{!-}C := \varinjlim_b R\beta_*(j_{b!}C|_{U_b})$$

for $C \in D_{\text{ét}}(X \times S, \Lambda)$.

As the ind-systems of U_a and U_b are independent of all choices, these functors are canonical.

The main result is the following. Here $\alpha : X \times S \to X$ and $\beta : X \times S \to S$ are the two projections.

THEOREM IV.5.3. *Assume that $C = \alpha^* A \otimes_\Lambda^{\mathbb{L}} \beta^* B$ for $A \in D_{\text{ét}}(X, \Lambda)$ and $B \in D_{\text{ét}}(S, \Lambda)$. Then*

$$R\beta_{!+} C = 0 = R\beta_{!-} C.$$

REMARK IV.5.4. The essential case for applications is $C = \alpha^* A$, i.e. $B = \Lambda$, and $S = \operatorname{Spa} k(\!(t)\!)$. In other words, we take any coefficient system A on X, pull it back to $X \times \operatorname{Spa} k(\!(t)\!)$, and then take the partially compactly supported cohomology (relative to S). However, it is sometimes useful to know the result in the relative case, i.e. for general S, and then it is also natural to allow twists by $B \in D_{\text{ét}}(S, \Lambda)$.

PROOF. We write the proof for $R\beta_{!+}$; the other case is exactly the same. Let $X_\bullet \to X$ be a simplicial hypercover by affinoid perfectoid spaces $X_i = \operatorname{Spa}(R_i, R_i^+)$ which are partially proper over $\operatorname{Spa} k$ (i.e., R_i^+ is minimal, i.e. the integral closure of $k + R^{\circ\circ}$). As X is a spatial diamond, we can arrange that the X_i are the compactifications of quasi-pro-étale maps to X (since X is spatial it admits an hypercover $X_\bullet \to X$ with X_i affinoid perfectoid and $X_i \to X$ quasi-pro-étale, since $X \to \operatorname{Spd} k$ is partially proper this extends to a hypercover $X_\bullet^c \to X$ where X_i^c is Huber's canonical compactification over $\operatorname{Spa}(k)$); in particular, $g_i : X_i \to X$ satisfies $\dim. \operatorname{trg} g_i = 0 < \infty$. Let $\beta_\bullet : X_\bullet \times S \to S$ be the corresponding projection. We claim that

$$R\beta_{!+} C$$

is the limit of $R\beta_{\bullet,!+}(C|_{X_\bullet \times S})$, for any $C \in D_{\text{ét}}(X \times S, \Lambda)$. Writing C as a limit of its Postnikov truncations ([**Sch17a**, Proposition 14.15]), we can assume $C \in D^+_{\text{ét}}(X \times S, \Lambda)$. Now $g_i : X_i \to X$ is a qcqs map between spaces partially proper over $*$, so g_i is proper, and hence so its base change $h_i : X_i \times S \to X \times S$. This implies that

$$R\beta_{i,!+}(h_i^* C) = R\beta_{!+}(Rh_{i*} h_i^* C),$$

as $j_{U!}$ commutes with Rh_{i*} by [**Sch17a**, Theorem 19.2]. Now by [**Sch17a**, Proposition 17.3], one sees that C is the limit of $Rh_{i*} h_i^* C$. But $R\beta_{!+}$ commutes with this limit, using that the filtered colimit does as everything lies in D^+ (with a uniform bound).

By the preceding reduction, we may assume that $X = \operatorname{Spa}(R, R^+)$ is an affinoid perfectoid space. We can even assume that X has no nonsplit finite étale covers (by taking the X_i above to be compactifications of strictly totally disconnected spaces). In that case, there is a map $g : X \to Y = \operatorname{Spa} K$, where K is the completed algebraic closure of $k(\!(t)\!)$, which is necessarily proper (as X and Y are partially proper over $*$), and as above one has

$$R\beta_{!+} C = R\beta_{Y,!+}(Rh_* C)$$

where $\beta_Y : Y \times S \to S$ is the projection and $h : X \times S \to Y \times S$ is the base change of g. Let $\alpha_Y : Y \times S \to Y$ be the other projection. Then the projection formula (and properness of h) [**Sch17a**, Proposition 22.11] show that

$$Rh_* C = Rh_*(\alpha^* A \otimes_\Lambda^{\mathbb{L}} \beta^* B) \cong Rh_* \alpha^* A \otimes_\Lambda^{\mathbb{L}} \beta_Y^* B$$

and $Rh_* \alpha^* A \cong \alpha_Y^* Rg_* A$ by proper base change.

In other words, we can reduce to the case $X = \operatorname{Spa} K$; in particular $A \in D_{\text{ét}}(\operatorname{Spa} K, \Lambda) = D(\Lambda)$ is just a complex of Λ-modules. In that case, define $U_{a,b}$ and U_a as above but taking $X = \widetilde{X} \to \operatorname{Spa} k(\!(t)\!)$ the natural map. We claim that in this case for all $a > 0$

$$R\beta_*(j_{a!}C|_{U_a}) = 0.$$

To prove this, it suffices to see that for all $a' > a > 0$, the cone of

$$R\beta_*(j_{a'!}C|_{U_{a'}}) \to R\beta_*(j_{a!}C|_{U_a})$$

vanishes, as $R\beta_*(j_{a!}C|_{U_a})$ is the limit of these cones as $a' \to \infty$. Now these cones depend on only a quasicompact part of $X \times S$, and hence their formation commutes with any base change in S, cf. [**Sch17a**, Proposition 17.6]. Therefore, we can reduce to the case $S = \operatorname{Spa}(L, L^+)$ for some complete algebraically closed nonarchimedean field L with open and bounded valuation subring $L^+ \subset L$, and check on global sections $R\Gamma(S, -)$. Moreover, the cone commutes with all direct sums in C, so one can assume that $A \in D_{\text{ét}}(\operatorname{Spa} K, \Lambda) = D_{\text{ét}}(\Lambda)$ is simply given by $A = \Lambda$.

It remains to prove the following statement: For all $B \in D_{\text{ét}}(\operatorname{Spa}(L, L^+), \Lambda)$ one has

$$R\Gamma(\operatorname{Spa} K \times \operatorname{Spa}(L, L^+), j_{a!}B|_{U_a}) = 0.$$

If the stalk of B at the closed point vanishes, this follows from proper base change (writing $\operatorname{Spa} K \times \operatorname{Spa}(L, L^+)$ as the union of its subspaces proper over $\operatorname{Spa}(L, L^+)$), [**Sch17a**, Theorem 19.2]. Thus we may assume that B is concentrated at the closed point of S. Analyzing the structure of U_a, one checks that in fact there is some map $k(\!(u)\!) \to L$ such that the fibres over the closed point of $\operatorname{Spa}(L, L^+)$ of U_a and $\{|t| \leq |u|^a\}$ agree. (A priori, it is a union of such subsets for a profinite set of maps $k(\!(u)\!) \to L$, but for any two such choices one is contained in the other, by comparing valuations of the pseudouniformizers.) Thus, we can now assume that $U_a = \{|t| \leq |u|^a\}$, and we can also reduce to the case that B is constant. Now using as above that the cones for $a' > a > 0$ commute with any base change in S and commute with direct sums in B, we can reduce to $B = \Lambda$ and the rank-1-geometric point $S = \operatorname{Spa} L$ where L is the completed algebraic closure of $k(\!(u)\!)$.

At this point, we can further replace $\operatorname{Spa} K$ by $\operatorname{Spa} k(\!(t)\!)$: One can write $\operatorname{Spa} K$ as the inverse limit over finite extensions of $\operatorname{Spa} k(\!(t)\!)$, each of which is isomorphic to $\operatorname{Spa} k(\!(t')\!)$, and although a priori $R\Gamma(\operatorname{Spa} K \times \operatorname{Spa} L, j_{a!}\Lambda)$ does not take this inverse limit to a colimit, this does happen after passing to cones for maps for $a' > a > 0$, which suffices as above. Finally, we have reduced to Lemma IV.5.1. $\square$

IV.6. Hyperbolic localization

In this section we extend some results of Braden, [**Bra03**], to the world of diamonds. Our presentation is also inspired by the work of Richarz, [**Ric19**]. We will use these results throughout our discussion of geometric Satake, starting in Section VI.3.

Let S be a small v-stack, and let $f : X \to S$ be proper and representable in spatial diamonds with $\dim.\operatorname{trg} f < \infty$, and assume that there is a $\mathbb{G}_m$-action on X/S, where $\mathbb{G}_m$ is the v-sheaf sending $\operatorname{Spa}(R, R^+)$ to $R^\times$. The fixed point space $X^0 := X^{\mathbb{G}_m} \subset X$ defines a closed subfunctor.

We make the following assumption about the $\mathbb{G}_m$-action. Here, $(\mathbb{A}^1)^+$ (resp. $(\mathbb{A}^1)^-$) denotes the affine line $\operatorname{Spa}(R, R^+) \mapsto R$ with the natural $\mathbb{G}_m$-action (resp. its inverse).

Hypothesis IV.6.1. *There is a decomposition of X^0 into open and closed subsets $X_1^0, \dots, X_n^0$ such that for each $i = 1, \dots, n$, there are locally closed $\mathbb{G}_m$-stable subfunctors $X_i^+, X_i^- \subset X$ with $X^0 \cap X_i^+ = X_i^0$ (resp. $X^0 \cap X_i^- = X_i^0$) such that the $\mathbb{G}_m$-action on X_i^+ (resp. X_i^-) extends to a $\mathbb{G}_m$-equivariant map $(\mathbb{A}^1)^+ \times X_i^+ \to X_i^+$ (resp. $(\mathbb{A}^1)^- \times X_i^- \to X_i^-$), and such that*

$$X = \bigcup_{i=1}^n X_i^+ = \bigcup_{i=1}^n X_i^-.$$

We let

$$X^+ = \bigsqcup_{i=1}^n X_i^+$$
$$X^- = \bigsqcup_{i=1}^n X_i^-,$$

so that there are natural maps

$$q^+ : X^+ \to X$$
$$q^- : X^- \to X,$$

as well as closed immersions

$$i^+ : X^0 \to X^+$$
$$i^- : X^0 \to X^-$$

and projections

$$p^+ : X^+ \to X^0$$
$$p^- : X^- \to X^0;$$

here p^+ is given by the restriction of $(\mathbb{A}^1)^+ \times X_i^+ \to X_i^+$ to $\{0\} \times X_i^+$, and p^- is defined analogously.

Although the decomposition of X^0 into X_i^0 for $i = 1, \dots, n$ is a choice, ultimately the functors X^+ and X^- are independent of any choice. Indeed, we have the following functorial description.

Proposition IV.6.2. *Consider the functor $(X^+)'$ sending any perfectoid space T over S to the set of $\mathbb{G}_m$-equivariant maps from $(\mathbb{A}^1)^+$ to X. There is a natural map $X^+ \to (X^+)'$, as there is a natural $\mathbb{G}_m$-equivariant map $(\mathbb{A}^1)^+ \times X^+ \to X^+ \to X$. The map $X^+ \to (X^+)'$ is an isomorphism.*

Analogously, X^- classifies the set of $\mathbb{G}_m$-equivariant maps from $(\mathbb{A}^1)^-$ to X.

Proof. It is enough to handle the case of X^+. There is a natural map $(X^+)' \to X^0$ given by evaluating the $\mathbb{G}_m$-equivariant map on $(\mathbb{A}^1)^+ \times (X^+)' \to X$ on $\{0\} \times X^+$. Let $(X_i^+)' = (X^+)' \times_{X^0} X_i^0$; it is enough to prove that $X_i^+ \to (X_i^+)'$ is an isomorphism. For this, it is enough to prove that the map $(X_i^+)' \to X$ given by evaluation at 1 is an injection whose image is contained in the locally closed subspace $X_i^+ \subset X$. This can be checked after pullback to an affinoid perfectoid base space $S = \mathrm{Spa}(R, R^+)$. As X/S is proper (in particular, separated) and $\mathbb{G}_m \times (X_i^+)' \subset (\mathbb{A}^1)^+ \times (X_i^+)'$ is dense, it follows that the map $(X_i^+)' \to X$ is an injection. To bound its image, we can argue on geometric points. If $x \in |X|$ is any point in the image of $|(X_i^+)'|$, and $\varpi \in R$ is a pseudouniformizer

with induced action γ on X, then the sequence $\gamma^n(x)$ converges to a point of $|X_i^0|$ for $n \to \infty$. On the other hand, if $x \notin |X_i^+|$, then $x \in |X_j^+|$ for some $j \neq i$, which implies that $\gamma^n(x)$ converges to a point of $|X_j^0|$ for $n \to \infty$; this is a contradiction.

Thus, $(X_i^+)'$ embeds into $X_i^+ \subset X$, but it also contains X_i^+, so indeed $X_i^+ = (X_i^+)'$. □

Lemma IV.6.3. *The map $j : X^0 \to X^+ \times_X X^-$ is an open and closed immersion. More precisely, for any $i = 1, \dots, n$, the map $j_i : X_i^0 \to X_i^+ \times_X X_i^-$ is an isomorphism.*

Proof. It is enough to prove that for any $i = 1, \dots, n$, the map $j_i : X_i^0 \to X_i^+ \times_X X_i^-$ is an isomorphism. As it is a closed immersion (as $X_i^0 \to X$ is a closed immersion and the target embeds into X), it is enough to prove that it is bijective on geometric rank 1 points. Thus, we can assume $S = \operatorname{Spa} C$, and let $x : \operatorname{Spa} C = S \to X$ be a section that factors over $X_i^+ \times_X X_i^-$. Then the $\mathbb{G}_m$-action on x extends to a $\mathbb{G}_m$-equivariant map $g : \mathbb{P}^1_C \to X$. Consider the preimage of X_i^+ under g; this is a locally closed subfunctor, and it contains all geometric points. Indeed, on $(\mathbb{A}^1)^+_C$, the map g factors over X_i^+ by hypothesis, and at ∞, it maps into $X_i^0 \subset X_i^+$. This implies that the preimage of X_i^+ under g is all of $\mathbb{P}^1_C$. In particular, we get a map

$$(\mathbb{A}^1)^+ \times \mathbb{P}^1_C \to (\mathbb{A}^1)^+ \times X_i^+ \to X_i^+$$

which, when restricted to the copy of $\mathbb{G}_m$ embedded via $t \mapsto (t, t^{-1})$, is constant with value x. By continuity (and separatedness of X_i^+), this implies that it is also constant with value x when restricted to $\mathbb{A}^1$ embedded via $t \mapsto (t, t^{-1})$, i.e. the point $(0, \infty) \in (\mathbb{A}^1)^+ \times \mathbb{P}^1_C$ maps to x. On the other hand, when restricted to $\mathbb{G}_m \times \{\infty\}$, the map is constant with values in X_i^0, and thus by continuity also on $(\mathbb{A}^1)^+ \times \{\infty\}$. This implies that $x \in X_i^0$, as desired. □

In this setup, we can define two functors $D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X^0, \Lambda)$. We use the diagrams

$$\begin{tikzcd} X^\pm \arrow[r, "q^\pm"] \arrow[d, "p^\pm"] & X \\ X^0. \end{tikzcd}$$

Definition IV.6.4. *Define the functors*

$$L^+_{X/S} = R(p^+)_!(q^+)^* : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X^0, \Lambda),$$
$$L^-_{X/S} = R(p^-)_* R(q^-)^! : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X^0, \Lambda),$$

and a natural transformation $L^-_{X/S} \to L^+_{X/S}$ as follows. First, there are natural transformations

$$R(i^+)^! = R(p^+)_! R(i^+)_! R(i^+)^! \to R(p^+)_! \ , \ R(p^-)_* = (i^-)^*(p^-)^* R(p^-)_* \to (i^-)^*,$$

and the desired transformation $L^-_{X/S} \to L^+_{X/S}$ arises as a composite

$$L^-_{X/S} = R(p^-)_* R(q^-)^! \to (i^-)^* R(q^-)^! \to (Ri^+)^!(q^+)^* \to R(p^+)_!(q^+)^* = L^+_{X/S},$$

where the middle map $(i^-)^*R(q^-)^! \to (Ri^+)^!(q^+)^*$ *of functors* $D_{\text{ét}}(X,\Lambda) \to D_{\text{ét}}(X^0,\Lambda)$ *is defined as the following composite*

$$\begin{aligned}(i^-)^*R(q^-)^! &\to (i^-)^*R(q^-)^!R(q^+)_*(q^+)^* \\ &= (i^-)^*R(\widetilde{q}^-)_*R(\widetilde{q}^+)^!(q^+)^* \\ &\to (i^-)^*R(\widetilde{q}^-)_*j_*j^*R(\widetilde{q}^+)^!(q^+)^* \\ &= (i^-)^*(i^-)_*Rj^!R(\widetilde{q}^+)^!(q^+)^* \\ &= R(i^+)^!(q^+)^*,\end{aligned}$$

using base change in the cartesian diagram

$$\begin{tikzcd} X^0 \arrow[r, hook, "j"] & X^+ \times_X X^- \arrow[r, "\widetilde{q}^-"] \arrow[d, "\widetilde{q}^+"] & X^+ \arrow[d, "q^+"] \\ & X^- \arrow[r, "q^-"] & X. \end{tikzcd}$$

Equivalently, it is enough to define for each $i = 1, \ldots, n$ *a natural transformation* $(i_i^-)^*R(q_i^-)^! \to (Ri_i^+)^!(q_i^+)^*$ *of functors* $D_{\text{ét}}(X,\Lambda) \to D_{\text{ét}}(X_i^0,\Lambda)$. *As*

$$\begin{tikzcd} X_i^0 \arrow[r, "i_i^+"] \arrow[d, "i_i^-"] & X_i^+ \arrow[d, "q_i^+"] \\ X_i^- \arrow[r, "q_i^-"] & X \end{tikzcd}$$

is cartesian, this arises as the composite

$$(i_i^-)^*R(q_i^-)^! \to (i_i^-)^*R(q_i^-)^!R(q_i^+)_*(q_i^+)^* = (i_i^-)^*(i_i^-)_*R(i^+)^!(q_i^+)^* = R(i^+)^!(q_i^+)^*.$$

The following is our version of Braden's theorem, [**Bra03**], cf. [**Ric19**, Theorem B].

THEOREM IV.6.5. *For any* $A \in D_{\text{ét}}(X/\mathbb{G}_m,\Lambda)$ *whose restriction to* X *we continue to denote by* A, *the map*

$$L^-_{X/S}A \to L^+_{X/S}A$$

is an isomorphism. In fact, moreover for any $A^+ \in D_{\text{ét}}(X^+/\mathbb{G}_m,\Lambda)$, *the map*

$$R(i^+)^!A^+ \to R(p^+)_!A^+$$

is an isomorphism, and for any $A^- \in D_{\text{ét}}(X^-/\mathbb{G}_m,\Lambda)$, *the map*

$$R(p^-)_*A^- \to (i^-)^*A^-$$

is an isomorphism, so that

$$L^-_{X/S}A = R(p^-)_*R(q^-)^!A \cong (i^-)^*R(q^-)^!A \cong R(i^+)^!(q^+)^*A \cong R(p^+)_!(q^+)^*A = L^+_{X/S}A$$

is a series of isomorphisms.

Before we start with the proof, we prove a certain general result about cohomology groups on spaces with "two ends", a flow connecting the two ends, and cohomology of sheaves, equivariant for the flow, that are compactly supported at only one end.

PROPOSITION IV.6.6. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, $\varpi \in R$ a pseudouniformizer, let $f : Y \to S$ be a partially proper map of locally spatial diamonds, and assume that Y is equipped with a $\mathbb{G}_m$-action over S. Assume that the quotient v-stack $Y/\mathbb{G}_m$ is qcqs. In that case, we can find a quasicompact open subset $V \subset Y$ such that $\mathbb{G}_m \times V \to Y$ is surjective and quasicompact. Write*

$$\mathbb{G}_{m,S} = \varinjlim_{n \geq 0} U_n \ , \ U_n = \{x \in \mathbb{G}_{m,S} \mid |x| \leq |\varpi|^{-n}\},$$

and let $j_n : V_n \subset Y$ be the open image of the cohomologically smooth map $U_n \times_S V \subset \mathbb{G}_{m,S} \times_S Y \to Y$. In this situation, we define for any $A \in D_{\text{ét}}(Y, \Lambda)$ the relative cohomology with partial supports

$$\varinjlim_n Rf_*(j_{n!}A|_{V_n}) \in D_{\text{ét}}(S, \Lambda) \ ;$$

this functor is canonically independent of the choices made in its definition.

For any $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ (resp. any $A \in D_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ if $\dim.\operatorname{trg} f < \infty$),

$$\varinjlim_n Rf_*(j_{n!}A|_{V_n}) = 0.$$

REMARK IV.6.7. Assume that $S = \mathrm{Spa}\, C$ is a geometric rank 1 point. Then we set

$$R\Gamma_{c+}(Y, A) = \varinjlim_n R\Gamma(Y, j_{n!}A_{V_n}),$$

which is exactly the above functor $\varinjlim_n Rf_*(j_{n!}A|_{V_n})$ under the identification $D_{\text{ét}}(S, \Lambda) = D(\Lambda)$. Roughly speaking, the space Y has two ends, one given by $\bigcup_{n<0} \gamma^n(V)$ for V large enough, where γ is the automorphism of Y induced by $\varpi \in \mathbb{G}_m(S)$, and the other given by $\bigcup_{n>0} \gamma^n(V)$. We are considering the cohomology groups of Y that have compact support in one of these directions, but not in the other. If one replaces the $\mathbb{G}_m$-action by its inverse, this implies a similar result for the direction of compact support interchanged.

PROOF. The proof is analogous to the proof of Theorem IV.5.3, reducing to the case $Y = \mathbb{G}_m \times S$, where it is a simple computation. Indeed, one can assume that $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ by a Postnikov limit argument (in case $\dim.\operatorname{trg} f < \infty$). Finding a v-hypercover of Y by spaces with $\mathbb{G}_m$-action of the form $\mathbb{G}_m \times X_i$, where each X_i is a proper spatial diamond over S, and using v-hyperdescent, one reduces to the case $Y = \mathbb{G}_m \times X_i$. Then there is a projection $\mathbb{G}_m \times X_i \to \mathbb{G}_{m,S}$, and one reduces to $Y = \mathbb{G}_{m,S}$. In that case, the same arguments as in the previous section apply. $\square$

PROOF OF THEOREM IV.6.5. We can assume that $A \in D^+_{\text{ét}}(X/\mathbb{G}_m, \Lambda)$ by pulling through the Postnikov limit $\varprojlim_n \tau^{\geq -n} A$, noting that $L^+_{X/S}$ commutes with limits while $(q^+)^*$ commutes with Postnikov limits and $R(p^+)_!$ as well by finite cohomological dimension.

By choosing a v-hypercover of S by disjoint unions of strictly totally disconnected spaces $S_\bullet$, and using v-hyperdescent, we can assume that S is a strictly totally disconnected space; indeed, $L^+_{X/S}$ commutes with all limits, while $(q^+)^*$ and $R(p^+)_!$ commute with any base change and so preserve cartesian objects, and thus also commute with the hyperdescent.

We start by proving that for any $A^+ \in D_{\text{ét}}(X^+/\mathbb{G}_m, \Lambda)$, the map

$$R(i^+)^! A^+ \to R(p^+)_! A^+$$

is an isomorphism, and similarly for any $A^- \in D_{\text{ét}}(X^-/\mathbb{G}_m, \Lambda)$, the map

$$R(p^-)_* A^- \to (i^-)^* A^-$$

is an isomorphism. Let $j^+ : X^+ \setminus X^0 \hookrightarrow X^+$, $j^- : X^- \setminus X^0 \hookrightarrow X^-$ denote the open embeddings. Then there are exact triangles

$$(i^+)_* R(i^+)^! A^+ \to A^+ \to R(j^+)_*(j^+)^* A^+ \ , \ (j^-)_!(j^-)^* A^- \to A^- \to (i^-)_*(i^-)^* A^-.$$

Using these triangles, we see that it is enough to see that for any $B^+ \in D_{\text{ét}}((X^+ \setminus X^0)/\mathbb{G}_m, \Lambda)$, $B^- \in D_{\text{ét}}((X^- \setminus X^0)/\mathbb{G}_m, \Lambda)$, one has

$$R(p^+)_! R(j^+)_* B^+ = 0 \ , \ R(p^-)_* R(j^-)_! B^- = 0$$

as objects in $D_{\text{ét}}(X^0, \Lambda)$. This follows from Proposition IV.6.6 applied to $S = X^0$ and $Y = X^+ \setminus X^0$ (resp. $Y = X^- \setminus X^0$), and the following lemma.

Lemma IV.6.8. *The $\mathbb{G}_m$-action on $X^+ \setminus X^0$ (resp. $X^- \setminus X^0$) has the property that the quotient v-stack $(X^+ \setminus X^0)/\mathbb{G}_m$ (resp. $(X^- \setminus X^0)/\mathbb{G}_m$) is qcqs over S (thus, over X^0).*

Proof. It is enough to do the case of $X^+ \setminus X^0$, and we may restrict to $X_i^+ \setminus X_i^0$. We can assume that $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space, and fix a pseudouniformizer $\varpi \in R$. As $\mathbb{G}_{m,S}/\varpi^{\mathbb{Z}}$ is qcqs (in fact proper — a Tate elliptic curve), it is equivalent to prove that $(X^+ \setminus X^0)/\gamma^{\mathbb{Z}}$ is qcqs, where γ is the automorphism given by the action of $\varpi \in \mathbb{G}_m(S)$.

Now we use the criterion of Lemma II.2.17 for the action of γ on $|X_i^+|$. As a locally closed partially proper subspace of the proper spatial diamond X over S, the locally spectral space $|X_i^+|$ is taut, and the condition on generizations is always fulfilled for locally spatial diamonds. The spectral closed subspace $|X_i^0| \subset |X_i^+|$ is fixed by γ, and by assumption for all $x \in |X_i^+|$, the sequence $\gamma^n(x)$ for $n \to \infty$ converges to a point of $|X_i^0|$ (as the $\mathbb{G}_m$-action extends to $(\mathbb{A}^1)^+$). It remains to see that for all $x \in |X_i^+ \setminus X_i^0|$, the sequence $\gamma^n(x)$ for $n \to -\infty$ diverges in $|X_i^+|$. But $x \in |X_j^-|$ for some j, and $j \neq i$ by Lemma IV.6.3. Thus, $\gamma^n(x)$, for $n \to -\infty$, converges to a point of $|X_j^0|$, which is outside of $|X_i^+|$, so the sequence diverges in $|X_i^+|$. □

Now it remains to see that for any $A \in D^+_{\text{ét}}(X/\mathbb{G}_m, \Lambda)$, the map

$$(i^-)^* R(q^-)^! A \to R(i^+)^! (q^+)^* A$$

in $D^+_{\text{ét}}(X^0, \Lambda) = D^+((X^0)_{\text{ét}}, \Lambda)$ is an isomorphism. This can be done locally on X^0, so fix some $i \in \{1, \ldots, n\}$, and choose a quasicompact open neighborhood $U_0 \subset X$ of X_i^0 that does not meet any X_j^0 for $j \neq i$ and such that $X_i^+ \cap U_0, X_i^- \cap U_0 \subset U_0$ are closed. The $\mathbb{G}_m$-orbit $Y = \mathbb{G}_m \cdot U_0 \subset X$ is still open, and contains X_i^+ and X_i^-, necessarily as closed subsets.

We are now in the situation of the next proposition. To check conditions (ii) and (iii) of that proposition, note that we may find a quasicompact open subspace $V \subset Y$ such that $Y = \gamma^{\mathbb{Z}} \cdot V$ by averaging U_0 over $\{|\varpi| \leq |t| \leq 1\} \subset \mathbb{G}_m$. Let W be the closure of $\bigcup_{n \geq 0} \gamma^n(V) \subset X$. To check (iii), it suffices (by symmetry) to see that

$$\bigcap_{m \geq 0} \gamma^m(W) = \overline{X_i^-}$$

in X. Note that $X_i^- \subset \bigcup_{n\geq 0}\gamma^n(V)$ (as for all $x \in X_i^-$, the sequence $\gamma^{-n}(x)$ converges into $X_i^0 \subset V$), so $\overline{X_i^-}$ is contained in W, and thus in $\bigcap_{m\geq 0}\gamma^m(W)$. To prove the converse inclusion, let $W' = \bigcap_{m\geq 0}\gamma^m(W)$. If $\overline{X_i^-} \subsetneq W'$, then there is some $j \neq i$ such that X_j^- contains a quasicompact open subset $A \subset W'$. Then $\widetilde{A} = \bigcup_{n\geq 0}\gamma^{-n}(A)$ is a γ^{-1}-invariant open subset of W' whose closure is $\gamma^{-(\mathbb{N}\cup\{\infty\})}\cdot\overline{A}$; in particular, replacing A by $\gamma^{-n}(A)$ if necessary, we can arrange that this closure is contained in any given small neighborhood of X_j^0, and in particular intersects V trivially. Then $\gamma^n(V)\cap A = \gamma^n(V\cap\gamma^{-n}(A)) = \emptyset$ for all $n \geq 0$, and hence A intersects $\bigcup_{n\geq 0}\gamma^n(V)$ trivially, and then also its closure W. But we assumed that $A \subset W' \subset W$, giving a contradiction. $\square$

Proposition IV.6.9. *Let $S = \mathrm{Spa}(R, R^+)$ be a strictly totally disconnected perfectoid space, let $f : Y \to S$ be a compactifiable map of locally spatial diamonds, and assume that Y/S is equipped with a $\mathbb{G}_m$-action, with fixed points $Y^0 \subset Y$, and the following properties.*

(i) *There are $\mathbb{G}_m$-invariant closed subspaces $q^+ : Y^+ \subset Y$, $q^- : Y^- \subset Y$, containing Y^0 (via $i^+ : Y^0 \to Y^+$, $i^- : Y^0 \to Y^-$) such that the action maps extend to maps $(\mathbb{A}^1)^+ \times Y^+ \to Y^+$ resp. $(\mathbb{A}^1)^- \times Y^- \to Y^-$.*

(ii) *The quotient v-stack $Y/\mathbb{G}_m$ is quasicompact. In particular, picking a pseudouniformizer $\varpi \in R$ with induced action γ on Y, we can find some quasicompact open $V \subset Y$ such that $Y = \gamma^{\mathbb{Z}}\cdot V$.*

(iii) *With V as in (ii), let W^- be the closure of $\gamma^{\mathbb{N}}\cdot V$ and W^+ the closure of $\gamma^{-\mathbb{N}}\cdot V$. Then $\bigcap_{n\geq 0}\gamma^n(W^-) = Y^-$ and $\bigcap_{n\geq 0}\gamma^{-n}(W^+) = Y^+$.*

Then Y^0 is a spatial diamond, the diagram

$$\begin{tikzcd} Y^0 \arrow[r, "i^+"] \arrow[d, "i^-"] & Y^+ \arrow[d, "q^+"] \\ Y^- \arrow[r, "q^-"] & Y \end{tikzcd}$$

is cartesian, the quotient v-stacks $(Y\setminus Y^+)/\mathbb{G}_m$ and $(Y\setminus Y^-)/\mathbb{G}_m$ are qcqs, and for all $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ (resp. all $A \in D_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ if $\dim.\operatorname{trg} f < \infty$) whose pullback to Y we continue to denote by A, the map

$$(i^-)^*R(q^-)^!A \to (i^-)^*R(q^-)^!(q^+)_*(q^+)^*A = (i^-)^*(i^-)_*R(i^+)^!(q^+)^*A = R(i^+)^!(q^+)^*A$$

in $D_{\text{ét}}(Y^0, \Lambda)$ is an isomorphism.

Proof. Note that $Y^0/\mathbb{G}_m \subset Y/\mathbb{G}_m$ is closed and thus $Y^0/\mathbb{G}_m$ is quasicompact. As the $\mathbb{G}_m$-action is trivial on Y^0, this implies that Y^0 is quasicompact. As $Y^0 \subset Y$ is closed and $Y \to S$ is compacifiable and in particular quasiseparated, we see that $Y^0 \to S$ is qcqs. That the diagram is cartesian follows from the proof of Lemma IV.6.3.

Next, we check that $(Y\setminus Y^+)/\mathbb{G}_m$ and $(Y\setminus Y^-)/\mathbb{G}_m$ are qcqs. By symmetry and as $\mathbb{G}_{m,S}/\gamma^{\mathbb{Z}}$ is qcqs, it suffices to see that $(Y\setminus Y^-)/\gamma^{\mathbb{Z}}$ is qcqs. First, we check that it is quasiseparated. Take any quasicompact open subspace $V^- \subset Y\setminus Y_i^-$; we need to see there are only finitely many n with $V^- \cap \gamma^n(V^-) \neq \emptyset$. We can assume that $V^- \subset \bigcup_{n\geq 0}(V)$ (translating by a power of γ if necessary), and then V^- is covered by the open subsets $V^- \setminus \gamma^m(W^-) \subset V^-$ by the claim above.

By quasicompacity, the intersection of V^- with $\gamma^m(W^-)$ is empty for some large enough m, but then also the intersection of V^- with $\gamma^{m'}(V^-) \subset \gamma^m(W^-)$ for $m' \geq m$ is empty.

To see that $(Y \setminus Y^-)/\gamma^{\mathbb{Z}}$ is quasicompact, note that $V \setminus \bigcup_{n>0} \gamma^n(V)$ is a spectral space (as it is closed in V) that maps bijectively to $(Y \setminus Y^-)/\gamma^{\mathbb{Z}}$.

Now, for the cohomological statement, we can as usual assume that $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$ by a Postnikov limit argument. Then we are interested in checking that a map in $D^+_{\text{ét}}(Y_0, \Lambda) = D^+(Y_{0,\text{ét}}, \Lambda)$ (cf. [**Sch17a**, Remark 14.14]) is an isomorphism, so we need to check that the sections over all quasicompact separated étale $Y_0' \to Y_0$ agree. Now we claim that any such quasicompact separated étale $Y_0' \to Y_0$ lifts to a $\mathbb{G}_m$-equivariant quasicompact separated étale map $Y' \to Y$; this will then allow us to assume $Y_0' = Y_0$ via passing to the pullback of everything to Y'.

To see that one may lift $Y_0' \to Y_0$ to $Y' \to Y$, consider the open subspace $V^{(n)} \subset Y$ given as the intersection of $\bigcup_{m \geq n} \gamma^m(V)$ with $\bigcup_{m \leq -n} \gamma^m(V)$. It follows from the topological situation that this is still quasicompact, and that the intersection of all $V^{(n)}$ is equal to Y_0 (using condition (iii)). Let $Y^{(n)} = \gamma^{\mathbb{Z}} \cdot V^{(n)} \subset Y$. Then γ-equivariant quasicompact separated étale maps to $Y^{(n)}$ are equivalent to quasicompact separated étale maps to $V^{(n)}$ together with isomorphisms between the two pullbacks to $V^{(n)} \cap \gamma(V^{(n)})$. The latter data extends uniquely from Y_0 to $V^{(n)}$ for small enough n by [**Sch17a**, Proposition 11.23]. Repeating a similar argument after taking a product with $\mathbb{G}_{m,S}/\gamma^{\mathbb{Z}}$ (which is qcqs), and observing that the $Y^{(n)}$ are cofinal with their $\mathbb{G}_m$-orbits, one can then attain $\mathbb{G}_m$-equivariance.

We have now reduced to checking the statement on global sections. Now consider the compactification $j : Y \hookrightarrow \overline{Y} = \overline{Y}^{/S} \to S$. Note that $\overline{Y}$ satisfies all the same conditions of the proposition. Restricted to Y_0, this gives a quasicompact open immersion $j_0 : Y_0 \hookrightarrow \overline{Y_0}$. By the above argument, this quasicompact open immersion spreads to a quasicompact open immersion into $\overline{Y}$, and by taking it small enough in the argument above, we can assume that it is contained in Y. This allows us to assume that j is quasicompact. In that case the functor Rj_* commutes with all operations in question by [**Sch17a**, Proposition 17.6, Proposition 23.16 (i)]. Thus, we can now moreover assume that Y is partially proper.

Our goal now is to prove that when Y is partially proper and $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$, the map

$$(i^-)^* R(q^-)^! A \to R(i^+)^!(q^+)^* A$$

becomes an isomorphism after applying Rf_{0*} where $f_0 : Y_0 \to S$ is the proper map. For this, we define another functor $D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(S, \Lambda)$, as follows. Let $j_n : V_n = \bigcup_{m \geq -n} \gamma^n(V) \hookrightarrow Y$ for $n \geq 0$. Then we consider

$$A \mapsto F(A) = \varinjlim_n Rf_*(j_{n!} A|_{V_n}) : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(S, \Lambda).$$

Lemma IV.6.10. *Let $j^- : Y \setminus Y^- \to Y$, $j^+ : Y \setminus Y^+ \to Y$ denote the open immersions.*

(i) *If $A = Rj^-_* A^-$ for $A^- \in D^+_{\text{ét}}((Y \setminus Y^-)/\mathbb{G}_m, \Lambda)$, then $F(A) = 0$.*
(ii) *If $A = j^+_! A^+$ for $A^+ \in D^+_{\text{ét}}((Y \setminus Y^+)/\mathbb{G}_m, \Lambda)$, then $F(A) = 0$.*

Proof. This follows Proposition IV.6.6 and condition (iii). □

There are natural transformations $Rf_{0*}R(p^-)_*R(q^-)^! \to F \to Rf_{0*}R(p^+)_!(q^+)^*$, and the lemma implies that these are equivalences when evaluated on $A \in D^+_{\text{ét}}(Y/\mathbb{G}_m, \Lambda)$. Using that also $(Y^+ \setminus Y^0)/\mathbb{G}_m$ and $(Y^- \setminus Y^0)/\mathbb{G}_m$ are qcqs (as closed subspaces of $(Y \setminus Y^-)/\mathbb{G}_m$ resp. $(Y \setminus Y^+)/\mathbb{G}_m$) so that we can apply Proposition IV.6.6 again as in the beginning of the proof of Theorem IV.6.5, we get an isomorphism

$$Rf_{0*}(i^-)^*R(q^-)^!A \cong Rf_{0*}R(p^-)_*R(q^-)^!A \cong F(A) \cong Rf_{0*}R(p^+)_!(q^+)^*A \cong Rf_{0*}R(i^+)^!(q^+)^*A.$$

We need to see that this implies that also the map

$$Rf_{0*}(i^-)^*R(q^-)^!A \to Rf_{0*}R(i^+)^!(q^+)^*A$$

defined in the statement of the proposition is an isomorphism. For this, observe that this map is an isomorphism if and only if for $A = j^+_! A^+$ with $A^+ \in D_{\text{ét}}((Y \setminus Y^+)/\mathbb{G}_m, \Lambda)$, one has

$$Rf_{0*}(i^-)^*R(q^-)^!A = 0.$$

But this follows from the existence of some isomorphism

$$Rf_{0*}(i^-)^*R(q^-)^!A \cong Rf_{0*}R(i^+)^!(q^+)^*A = 0,$$

using $(q^+)^*A = (q^+)^*j^+_! A^+ = 0$. □

Using Theorem IV.6.5, we give the following definition.

Definition IV.6.11. *Let $f : X \to S$ with $\mathbb{G}_m$-action be as above, satisfying Hypothesis IV.6.1. Let $D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-}mon} \subset D_{\text{ét}}(X, \Lambda)$ be the full subcategory generated under finite colimits and retracts by the image of $D_{\text{ét}}(X/\mathbb{G}_m, \Lambda) \to D_{\text{ét}}(X, \Lambda)$. The hyperbolic localization functor is the functor*

$$L_{X/S} : D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-}mon} \to D_{\text{ét}}(X^0, \Lambda)$$

given by $L^-_{X/S} \cong L^+_{X/S}$.

We observe that Theorem IV.6.5 implies the following further results.

Proposition IV.6.12. *In the situation of Definition IV.6.11, let $g : S' \to S$ be a map of small v-stacks, with pullback $f' : X' = X \times_S S' \to S'$, $g_X : X' \to X$, $g^0 : X^{0\prime} \to X^0$. Then there are natural equivalences*

$$g^{0*}L_{X/S} \cong L_{X'/S'}g^*_X\ ,\ L_{X/S}Rg_{X*} \cong Rg^0_*L_{X'/S'}\ ,\ L_{X/S}Rg_{X!} \cong Rg_{0!}L_{X'/S'}\ ,\ Rg^{0!}L_{X/S} \cong L_{X'/S'}Rg^!_X,$$

the latter two in case g is compactifiable and representable in locally spatial diamonds with $\dim.\operatorname{trg} g < \infty$ *(so that the relevant functors are defined).*

Proof. The first and third assertions are clear for $L^+_{X/S}$, while the second and fourth assertions are clear for $L^-_{X/S}$. □

Proposition IV.6.13. *In the situation of Definition IV.6.11, let $A \in D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-}mon}$ and $B \in D_{\text{ét}}(S, \Lambda)$. Let $L'_{X/S}$ denote the hyperbolic localization functor for the inverse $\mathbb{G}_m$-action. Then there is natural isomorphism*

$$R\mathscr{H}\mathrm{om}(L_{X/S}(A), Rf^{0!}B) \cong L'_{X/S}R\mathscr{H}\mathrm{om}(A, Rf^!B).$$

In particular, taking $B = \Lambda$, hyperbolic localization commutes with Verdier duality, up to changing the $\mathbb{G}_m$-action.

Proof. More generally, for all $A \in D_{\text{ét}}(X, \Lambda)$ and $B \in D_{\text{ét}}(S, \Lambda)$, we have a natural isomorphism $R\mathscr{H}\mathrm{om}(L^+_{X/S}(A), Rf^{0!}B) \cong L'^-_{X/S} R\mathscr{H}\mathrm{om}(A, Rf^! B)$. Indeed,

$$\begin{aligned} R\mathscr{H}\mathrm{om}(L^+_{X/S}(A), Rf^{0!}B) &= R\mathscr{H}\mathrm{om}(R(p^+)_!(q^+)^*A, Rf^{0!}B) \cong R(p^+)_* R\mathscr{H}\mathrm{om}((q^+)^*A, R(p^+)^! Rf^{0!}B) \\ &\cong R(p^+)_* R\mathscr{H}\mathrm{om}((q^+)^*A, R(q^+)^! Rf^! B) \cong R(p^+)_* R(q^+)^! R\mathscr{H}\mathrm{om}(A, Rf^! B). \end{aligned}$$

□

Proposition IV.6.14. *In the situation of Definition IV.6.11, assume that $A \in D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-}mon}$ is f-universally locally acyclic. Then $L_{X/S}(A) \in D_{\text{ét}}(X^0, \Lambda)$ is universally locally acyclic with respect to $f^0 : X^0 \subset X \to S$.*

Proof. As the assumption is stable under base change, we may assume that S is strictly totally disconnected, and it suffices to see that $L_{X/S}(A)$ is f_0-locally acyclic. For condition (a), we can in fact assume that $S = \mathrm{Spa}(C, C^+)$ is strictly local; let $j : S_0 = \mathrm{Spa}(C, \mathcal{O}_C) \subset S$ be the generic open point. Then we have to see that $L_{X/S}(A) = Rj^0_*(L_{X/S}(A)|_{X^0 \times_S S_0})$, where $j^0 : X^0 \times_S S_0 \to X^0$ is the pullback of j. But this follows from Proposition IV.6.12 and the corresponding property of A.

For condition (b), it suffices to see that the functor $R\mathscr{H}\mathrm{om}_\Lambda(L_{X/S}(A), Rf^{0!}-)$ commutes with all direct sums, as then its left adjoint $Rf_{0!}(A \otimes^{\mathbb{L}}_\Lambda -)$ preserves perfect-constructible complexes. For this, we compute this functor:

$$\begin{aligned} R\mathscr{H}\mathrm{om}_\Lambda(L_{X/S}(A), Rf^{0!}-) &\cong L'_{X/S} R\mathscr{H}\mathrm{om}_\Lambda(A, Rf^! -) \\ &\cong L'_{X/S}(\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda f^* -). \end{aligned}$$

Here, we used Proposition IV.6.13 and Proposition IV.2.19. The final functor clearly commutes with all direct sums, giving the desired result. □

IV.7. Drinfeld's lemma

As a final topic of this chapter, we prove the version of Drinfeld's lemma that we will need in this paper. Contrary to the classical formulation [**Dri80**, Theorem 2.1], cf. also [**Lau04**, Theorem 8.1.4], this version actually makes the Weil group of E, not the absolute Galois group of E, appear. (Also, it is worth remarking that usually, a global Galois group appears, not a local Galois group.)

In this section, we work on Perf_k where $k = \overline{\mathbb{F}}_q$. In that case, we can write the moduli space of degree 1 Cartier divisors on the Fargues–Fontaine curve as $\mathrm{Div}^1 = \mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}}$. This admits a natural map

$$\psi : \mathrm{Div}^1 \to [*/\underline{W_E}]$$

to the classifying space of the Weil group of E. Indeed, if $C = \widehat{\overline{E}}$ is a completed algebraic closure of E, then there is an action of W_E on $\mathrm{Spd}\, C$, with the inertia subgroup $I_E \subset W_E$ acting via its usual action, while Frobenius elements act via the composite of the usual action and the Frobenius of $\mathrm{Spd}\, C$. More precisely, $\tau \in W_E$ acts as $\tau \circ \mathrm{Frob}^{-\deg \tau}$ where $\deg : W_E \to \mathbb{Z}$ is the projection; note that this as a map over $\mathrm{Spd}\, k$ as on $\mathrm{Spd}\, k$ the two Frobenii cancel. The natural map

$$[\mathrm{Spd}\, C/\underline{W_E}] \to [\mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}}]$$

is an isomorphism, thus yielding the natural map

$$\psi : [\operatorname{Spd} \breve{E}/\varphi^{\mathbb{Z}}] \cong [\operatorname{Spd} C/\underline{W_E}] \to [*/\underline{W_E}].$$

One could equivalently compute

$$\underline{W_E} \times \operatorname{Spd} C \cong \operatorname{Spd} C \times_{\operatorname{Div}^1} \operatorname{Spd} C$$

for the natural map $\operatorname{Spd} C \to \operatorname{Div}^1$ to arrive at the result.

In particular, for any small v-stack X, we get a natural map

$$\psi_X : X \times \operatorname{Div}^1 \to X \times [*/\underline{W_E}].$$

Proposition IV.7.1. *The functor*

$$\psi_X^* : D_{\text{ét}}(X \times [*/\underline{W_E}], \Lambda) \to D_{\text{ét}}(X \times \operatorname{Div}^1, \Lambda)$$

is fully faithful. If the natural pullback functor

$$D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X \times \operatorname{Spd} C, \Lambda)$$

is an equivalence, then ψ_X^* is also an equivalence.

Proof. We apply descent along $* \to [*/\underline{W_E}]$. This describes $D_{\text{ét}}(X \times [*/\underline{W_E}], \Lambda)$ in terms of cartesian objects in $D_{\text{ét}}(X \times \underline{W_E}^\bullet, \Lambda)$, and $D_{\text{ét}}(X \times \operatorname{Div}^1, \Lambda)$ in terms of cartesian objects in $D_{\text{ét}}(X \times \operatorname{Spd} C \times \underline{W_E}^\bullet, \Lambda)$. By [**Sch17a**, Theorem 1.13], all functors $D_{\text{ét}}(X \times \underline{W_E}^\bullet, \Lambda) \to D_{\text{ét}}(X \times \operatorname{Spd} C \times \underline{W_E}^\bullet, \Lambda)$ are fully faithful; this implies the fully faithfulness. Moreover, for essential surjectivity on cartesian objects it is enough to know essential surjectivity on the degree 0 part of the simplicial resolution, i.e. for $D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X \times \operatorname{Spd} C, \Lambda)$, giving the desired result. □

We note the following immediate corollary.

Corollary IV.7.2. *For any finite set I, pullback along $X \times (\operatorname{Div}^1)^I \to X \times [*/\underline{W_E^I}]$ induces a fully faithful functor*

$$D_{\text{ét}}(X \times [*/\underline{W_E^I}], \Lambda) \to D_{\text{ét}}(X \times (\operatorname{Div}^1)^I, \Lambda).$$

Proof. This follows inductively from Proposition IV.7.1. □

We need the following refinement, see Proposition VI.9.2. For any small v-stack Y, let

$$D_{\text{lc}}(Y, \Lambda) \subset D_{\text{ét}}(Y, \Lambda)$$

be the full subcategory of all objects that are locally constant with perfect fibres.

Proposition IV.7.3. *For any finite set I and any small v-stack X, the functor*

$$D_{\text{lc}}(X \times [*/\underline{W_E^I}], \Lambda) \to D_{\text{lc}}(X \times (\operatorname{Div}^1)^I, \Lambda)$$

is an equivalence of categories.

We will mostly be using this in case X is a point.

Proof. By Corollary IV.7.2, the functor is fully faithful. By induction, we can reduce to the case that I has one element. By descent, we can assume that X is strictly totally disconnected. Note that $X \times \mathrm{Div}^1$ is a spatial diamond, and using [**Sch17a**, Proposition 20.15] we can reduce to the case that $X = \mathrm{Spa}(C, C^+)$ is strictly local (by writing any connected component as a cofiltered inverse limit of its open and closed neighborhoods to see that then any object is locally in the image of the functor). Moreover, the category D_{lc} is unchanged if we replace $\mathrm{Spa}(C, C^+)$ by $\mathrm{Spa}(C, \mathcal{O}_C)$, so we can assume that X is even a geometric rank 1 point.

At this point, we need to simplify the coefficient ring Λ. The algebra Λ is a $\mathbb{Z}/n\mathbb{Z}$-algebra for some n prime to p; we can then assume n is a power of some prime $\ell \neq p$, and in fact even $n = \ell$ by an induction argument. By [**Sch17a**, Proposition 20.15], we can also assume that Λ is a finitely generated $\mathbb{F}_\ell$-algebra. Taking a surjection from a polynomial algebra, one can then assume that $\Lambda = \mathbb{F}_\ell[T_1, \ldots, T_d]$. Applying [**Sch17a**, Proposition 20.15] again, we can assume that Λ is the localization of $\mathbb{F}_\ell[T_1, \ldots, T_d]$ at a closed point, or applying faithfully flat descent in the coefficients, that Λ is the completion of $\mathbb{F}_\ell[T_1, \ldots, T_d]$ at a closed point, but equipped with the discrete topology. Also note that this ring is regular, so all truncations of perfect complexes are perfect, and we can assume that the complex is concentrated in degree 0.

We are now in the following situation. We have an étale sheaf A of $\Lambda \cong \mathbb{F}_{\ell^r}[\![T_1, \ldots, T_d]\!]$-modules on $S = \mathrm{Spa}\, C \times \mathrm{Div}^1$, such that for some finitely generated Λ-module M, there are étale local isomorphisms between A and the constant Λ-module associated to M. Our goal is to see that after pullback along the $\underline{W_E}$-torsor

$$\widetilde{S} = \mathrm{Spa}\, C \times \mathrm{Spd}\, \widehat{\overline{E}} \to S = \mathrm{Spa}\, C \times \mathrm{Div}^1,$$

there is an isomorphism between A and M. To see this, we will also need to analyze the behaviour at a carefully chosen geometric point. In fact, by Lemma II.1.14 we can find a point $\mathrm{Spa}\, K \to Y_C$ of the curve Y_C associated with C such that the induced map $\mathrm{Gal}(\overline{K}|K) \to I_E$ is surjective. This induces a point $y : \mathrm{Spd}\, K \to S$, and we can lift it to a geometric point $\widetilde{y} : \mathrm{Spd}\, \widehat{\overline{K}} \to \widetilde{S}$. Define M as the stalk of A at $\widetilde{y}$; our goal is then to prove the existence of a unique isomorphism between $A|_{\widetilde{Y}}$ and M that is the identity at $\widetilde{y}$.

To prove this, we first reduce modulo $(T_1, \ldots, T_d)^n$. Then $\Lambda_n := \Lambda/(T_1, \ldots, T_d)^n$ is a finite ring, and the space of isomorphisms between A and M is parametrized by a scheme finite étale over Y. By [**SW20**, Lemma 16.3.2], all such finite étale covers come via pullback from finite étale covers of Div^1, and are thus trivialized after pullback to $\widetilde{Y}$; this implies that there is a unique isomorphism $A/(T_1, \ldots, T_d)^n \cong M/(T_1, \ldots, T_d)^n$ reducing to the identity at $\widetilde{y}$.

Taking the limit over n, we get an isomorphism $\widehat{A}|_{\widetilde{Y}} \cong \widehat{M}|_{\widetilde{Y}}$ between the pro-étale sheaves $\widehat{A} = \varprojlim_n A/(T_1, \ldots, T_d)^n$ and $\widehat{M} = \varprojlim_n M/(T_1, \ldots, T_d)^n$ after pullback to $\widetilde{Y}$. This gives in particular an automorphism of $\widehat{M}$ over

$$\widetilde{Y} \times_Y \widetilde{Y} \cong \underline{W_E} \times \widetilde{Y},$$

and thus by connectedness of $\widetilde{Y}$ a continuous map $W_E \to \mathrm{Aut}_\Lambda(\widehat{M})$ (in fact, it extends continuously to the absolute Galois group of E). We claim that this map is trivial on an open subgroup of I_E (but not necessarily on an open subgroup of the absolute Galois group of E — here it is necessary to pass to the Weil group). Indeed, restricting the map $W_E \to \mathrm{Aut}_\Lambda(\widehat{M})$ to $\mathrm{Gal}(\overline{K}|K)$ gives a

map $\mathrm{Gal}(\overline{K}|K) \to \mathrm{Aut}_\Lambda(\widehat{M})$ that is in fact continuous for the discrete topology on the target, as a local system of A-modules on $\mathrm{Spd}\, K$ is given a continuous representation of $\mathrm{Gal}(\overline{K}|K)$. As $\mathrm{Gal}(\overline{K}|K) \to I_E$ is surjective, we get the claim.

By equivariance under an open subgroup of I_E, we find that the isomorphism $\widehat{A}|_{\widetilde{Y}} \cong \widehat{M}|_{\widetilde{Y}}$ descends, necessarily uniquely, to an isomorphism over

$$\mathrm{Spa}\, C \times \mathrm{Spd}\, E'$$

for some finite extension $E'|\breve{E}$. Now we take the pushforward of the isomorphism $\widehat{A}|_{\mathrm{Spa}\, C \times \mathrm{Spd}\, E'} \cong \widehat{M}|_{\mathrm{Spa}\, C \times \mathrm{Spd}\, E'}$ to the étale site of $\mathrm{Spa}\, C \times \mathrm{Spd}\, E'$. As any étale $U \to \mathrm{Spa}\, C \times \mathrm{Spd}\, E'$ is locally connected, we have $H^0(U, M) = H^0(U, \widehat{M})$ and then also $H^0(U, A) = H^0(U, \widehat{A})$ (as A is étale locally isomorphic to M) for all such U, so we get the desired isomorphism $A|_{\mathrm{Spa}\, C \times \mathrm{Spd}\, E'} \cong M|_{\mathrm{Spa}\, C \times \mathrm{Spd}\, E'}$. $\square$

CHAPTER V

$D_{\text{ét}}(\mathrm{Bun}_G)$

In this chapter, we want to understand the basic structure of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$, building it up from all $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda)$, where we continue to work in the setting where Λ is killed by some integer n prime to p.

Throughout this chapter, we fix an algebraically closed field $k|\mathbb{F}_q$ and work on Perf_k. Our goal is to prove the following theorem.

Theorem V.0.1 (Theorem V.3.7, Proposition V.3.6; Proposition V.2.2, Theorem V.1.1; Theorem V.4.1; Theorem V.5.1; Theorem V.7.1). *Let Λ be any ring killed by some integer n prime to p.*

(o) *For any $b \in B(G)$, there is a map*

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$$

that is representable in locally spatial diamonds, partially proper and cohomologically smooth, where $\mathcal{M}_b$ parametrizes G-bundles $\mathcal{E}$ together with an increasing $\mathbb{Q}$-filtration whose associated graded is, at all geometric points, isomorphic to $\mathcal{E}_b$ with its slope grading. The v-stack $\mathcal{M}_b$ is representable in locally spatial diamonds, partially proper and cohomologically smooth over $[*/\underline{G_b(E)}]$.

(i) *Via excision triangles, there is an infinite semiorthogonal decomposition of $D(\mathrm{Bun}_G, \Lambda)$ into the various $D(\mathrm{Bun}_G^b, \Lambda)$ for $b \in B(G)$.*

(ii) *For each $b \in B(G)$, pullback along*

$$\mathrm{Bun}_G^b \cong [*/\widetilde{G}_b] \to [*/\underline{G_b(E)}]$$

gives an equivalence

$$D([*/\underline{G_b(E)}], \Lambda) \cong D(\mathrm{Bun}_G^b, \Lambda),$$

and $D([*/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$ is equivalent to the derived category of the category of smooth representations of $G_b(E)$ on Λ-modules.

(iii) *The category $D(\mathrm{Bun}_G, \Lambda)$ is compactly generated, and a complex $A \in D(\mathrm{Bun}_G, \Lambda)$ is compact if and only if for all $b \in B(G)$, the restriction*

$$i^{b*} A \in D(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

is compact, and zero for almost all b. Here, compactness in $D(G_b(E), \Lambda)$ is equivalent to lying in the thick triangulated subcategory generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ as K runs over open pro-p-subgroups of $G_b(E)$.

(iv) *On the subcategory* $D(\mathrm{Bun}_G, \Lambda)^\omega \subset D(\mathrm{Bun}_G, \Lambda)$ *of compact objects, there is a Bernstein–Zelevinsky duality functor*

$$\mathbb{D}_{BZ} : (D(\mathrm{Bun}_G, \Lambda)^\omega)^{\mathrm{op}} \to D(\mathrm{Bun}_G, \Lambda)^\omega$$

with a functorial identification

$$R\,\mathrm{Hom}(A, B) \cong \pi_\natural(\mathbb{D}_{BZ}(A) \otimes^{\mathbb{L}}_\Lambda B)$$

for $B \in D(\mathrm{Bun}_G, \Lambda)$*, where* $\pi : \mathrm{Bun}_G \to *$ *is the projection. The functor* $\mathbb{D}_{BZ}$ *is an equivalence, and* $\mathbb{D}^2_{BZ}$ *is naturally equivalent to the identity. It is compatible with usual Bernstein–Zelevinsky duality on* $D(G_b(E), \Lambda)$ *for basic* $b \in B(G)$*.*

(v) *An object* $A \in D(\mathrm{Bun}_G, \Lambda)$ *is universally locally acyclic (with respect to* $\mathrm{Bun}_G \to *$*) if and only if for all* $b \in B(G)$*, the restriction*

$$i^{b*}A \in D(\mathrm{Bun}^b_G, \Lambda) \cong D(G_b(E), \Lambda)$$

is admissible, i.e. for all pro-p open subgroups $K \subset G_b(E)$*, the complex* $(i^{b*}A)^K$ *is perfect. Universally locally acyclic complexes are preserved by Verdier duality, and satisfy Verdier biduality.*

V.1. Classifying stacks

First, we want to understand $D_{\text{ét}}([*/\underline{G}], \Lambda)$ for a locally pro-p-group G. Fix a coefficient ring Λ such that $n\Lambda = 0$ for some n prime to p, and assume that G is locally pro-p. Our aim is to prove the following theorem.

Theorem V.1.1. *Let* $D(G, \Lambda)$ *be the derived category of the category of smooth representations of* G *on* Λ*-modules. There is a natural symmetric monoidal equivalence*

$$D(G, \Lambda) \simeq D_{\text{ét}}([*/\underline{G}], \Lambda)$$

under which the functor $D(G, \Lambda) \to D(\Lambda)$ *forgetting the* G*-action gets identified with the pullback functor* $D_{\text{ét}}([*/\underline{G}], \Lambda) \to D_{\text{ét}}(*, \Lambda) = D(\Lambda)$ *under the projection* $* \to [*/\underline{G}]$*.*

The same result holds true for the base change $[\mathrm{Spa}\, C/\underline{G}] = [*/\underline{G}] \times \mathrm{Spa}\, C$ *for any complete algebraically closed nonarchimedean field* C/k*; more precisely, the base change functor*

$$D_{\text{ét}}([*/\underline{G}], \Lambda) \to D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G}], \Lambda)$$

is an equivalence.

Note that indeed

$$D_{\text{ét}}(*, \Lambda) = D(\Lambda).$$

This follows from applying [**Sch17a**, Theorem 1.13 (ii)] to the small v-stack $X = *$. In fact, for any complete algebraically closed field C, one has $D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda) = D(\Lambda)$ and there is a sequence

$$D(\Lambda) \longrightarrow D_{\text{ét}}(*, \Lambda) \xrightarrow{\text{fully faithful}} D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda) = D(\Lambda)$$

and $D(\Lambda) \to D_{\text{ét}}(*, \Lambda)$ is thus an equivalence.

Proof. We start by constructing a functor

$$D(G,\Lambda) \to D_{\text{ét}}([*/\underline{G}],\Lambda)$$

compatible with the derived tensor product and the forgetful functors. For this, one first constructs a functor from the category of smooth representations of G on Λ-modules to the heart of $D_{\text{ét}}([*/\underline{G}],\Lambda)$; note that this heart is a full subcategory of the heart of $D([*/\underline{G}]_v,\Lambda)$, which is the category of v-sheaves on $[*/\underline{G}]$. Now one can send a smooth G-representation V to the v-sheaf $\mathcal{F}_V$ on $[*/\underline{G}]$ that takes a perfectoid space X with a $\underline{G}$-torsor $\widetilde{X} \to X$ to the set of all continuous G-equivariant maps from $|\widetilde{X}|$ to V. In fact one checks that for any perfectoid space S and any locally profinite set A,

$$|S \times \underline{A}| = |S| \times A,$$

and thus $|\widetilde{X}|$ has a continuous G-action. As v-covers induce quotient maps by [**Sch17a**, Proposition 12.9], this is indeed a v-sheaf. Moreover, after pullback along $* \to [*/\underline{G}]$, it is given by the functor which sends X to the set of continuous G-equivariant maps from $|X| \times G = |X \times \underline{G}|$ to V. These are canonically the same (via restriction to $X \times \{1\}$) as continuous maps $|X| \to V$, so that $\mathcal{F}_V|_* = \underline{V}$ is the v-sheaf corresponding to V. As V is discrete, this is a disjoint union of points, and in particular (after pullback to any $\operatorname{Spa} C$) an étale sheaf. According to [**Sch17a**, Definition 14.13], this implies that $\mathcal{F}_V \in D_{\text{ét}}([*/\underline{G}],\Lambda)$, as desired.

From now on, we will simply write $\underline{V}$ for $\mathcal{F}_V$. Given any complex of smooth G-representations $V^\bullet$, one can form the corresponding complex $\underline{V}^\bullet$ of v-sheaves on $[*/\underline{G}]$, which defines an object of $D_{\text{ét}}([*/\underline{G}],\Lambda) \subset D([*/\underline{G}]_v,\Lambda)$ (using [**Sch17a**, Proposition 14.16]), giving the desired functor $D(G,\Lambda) \to D_{\text{ét}}([*/\underline{G}],\Lambda)$ compatible with the forgetful functors. One checks that this functor is compatible with derived tensor products by unraveling the definitions.

To check whether the functor is an equivalence, we may by [**Sch17a**, Theorem 1.13 (ii)] replace $[*/\underline{G}]$ by its base change $[\operatorname{Spa} C/\underline{G}] = [*/\underline{G}] \times \operatorname{Spa} C$, where C is some complete algebraically closed nonarchimedean field.

For the v-stack $X = [\operatorname{Spa} C/\underline{G}]$, we can also consider its étale site $X_{\text{ét}} \subset X_v$ consisting of all $Y \in X_v$ which are étale (and locally separated) over X. This recovers a classical site.

Lemma V.1.2. *The étale site $X_{\text{ét}}$ is equivalent to the category G-Set of discrete G-sets, via sending a discrete set S with continuous G-action to $[\underline{S} \times \operatorname{Spa} C/\underline{G}]$.*

Proof. It is clear that the functor $S \mapsto [\underline{S} \times \operatorname{Spa} C/\underline{G}]$ maps to $X_{\text{ét}} \subset X_v$ (as the pullback to $\operatorname{Spa} C$ is given by $\underline{S} \times \operatorname{Spa} C$), and is fully faithful. Conversely, if $Y \to X = [\operatorname{Spa} C/\underline{G}]$ is étale, then the pullback of Y to $\operatorname{Spa} C$ is a discrete set, on which $\underline{G}$ acts continuously, giving the descent datum defining Y. □

Lemma V.1.3. *There is a natural equivalence $D(G,\Lambda) \simeq D(G\text{-}\operatorname{Set},\Lambda)$, such that the following diagram commutes*

$$\begin{tikzcd}
D(G,\Lambda) \arrow[r, "\cong"] \arrow[d] & D(G\text{-}\operatorname{Set},\Lambda) \arrow[r, "\cong"] & D([\operatorname{Spa} C/\underline{G}]_{\text{ét}},\Lambda) \arrow[d] \\
D_{\text{ét}}([*/\underline{G}],\Lambda) \arrow[rr, hook] & & D_{\text{ét}}([\operatorname{Spa} C/\underline{G}],\Lambda).
\end{tikzcd}$$

Proof. It is enough to give an equivalence of abelian categories between smooth G-representations on Λ-modules, and sheaves of Λ-modules on discrete G-sets. The construction of the functor is as before, and it is clearly fully faithful. But any sheaf of Λ-modules $\mathcal{F}$ on G-Set comes from the smooth G-representation $V = \varinjlim_{H \subset G} \mathcal{F}(G/H)$, where H runs over all open subgroups of G. One directly verifies that the diagram commutes. $\square$

It remains to see that the natural functor

$$D([\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}, \Lambda) \to D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G}], \Lambda)$$

is an equivalence. We claim that this reduces to the case that G is pro-p: We first reduce fully faithfulness to this case. For this, we have to see that, if $\lambda : X_v \to X_{\text{ét}}$ denotes the map of sites, then for any $A \in D([\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}, \Lambda)$, the natural map

$$A \to R\lambda_* \lambda^* A$$

is an equivalence. This can be checked locally on $[\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}$, meaning that we can replace G by an open pro-p-subgroup. Similarly, for essential surjectivity, one needs to see that for all $B \in D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G}], \Lambda)$, the map $\lambda^* R\lambda_* B \to B$ is an equivalence, which can again be checked locally.

Thus, we can assume that G is pro-p. Note that $([\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}, \Lambda)$ is locally of cohomological dimension 0, as there is no continuous group cohomology of pro-p-groups on Λ-modules if $n\Lambda = 0$ for n prime to p. This implies (cf. [**Sta**, Tag 0719]) that $D([\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}, \Lambda)$ is left-complete. As $D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G}], \Lambda)$ is also left-complete by [**Sch17a**, Proposition 14.11], it is enough to see that the functor

$$D^+([\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}, \Lambda) \to D^+_{\text{ét}}([\mathrm{Spa}\, C/\underline{G}], \Lambda)$$

is an equivalence. First, we check fully faithfulness, i.e. that the unit $\mathrm{id} \to R\lambda_* \lambda^*$ of the adjunction is an equivalence. For this, it is enough to see that for any étale sheaf of Λ-modules, i.e. any smooth G-representation V, one has

$$R\Gamma([\mathrm{Spa}\, C/\underline{G}]_v, \underline{V}) = V^G,$$

i.e. its H^0 is V^G and there are no higher H^i. However, one can compute v-cohomology using the Cech nerve for the cover $\mathrm{Spa}\, C \to [\mathrm{Spa}\, C/\underline{G}]$, which produces the complex of continuous cochains, giving the desired result.

Finally, for essential surjectivity, it is now enough to check on the heart. But if $\mathcal{F}$ is a v-sheaf on $[\mathrm{Spa}\, C/\underline{G}]$ whose pullback to $\mathrm{Spa}\, C$ is an étale sheaf, then this pullback is a disjoint union of points, thus separated and étale, and therefore $\mathcal{F}$ is itself a v-stack which is étale over $[\mathrm{Spa}\, C/\underline{G}]$, and so defines an object in the topos $[\mathrm{Spa}\, C/\underline{G}]_{\text{ét}}$. $\square$

Corollary V.1.4. *The operation*

$$R\mathscr{H}\mathrm{om}_\Lambda(-, \Lambda) : D_{\text{ét}}([*/\underline{G}], \Lambda)^{\mathrm{op}} \to D_{\text{ét}}([*/\underline{G}], \Lambda)$$

corresponds to the derived smooth duality functor

$$A \mapsto (A^*)^{\mathrm{sm}} : D(G, \Lambda)^{\mathrm{op}} \to D(G, \Lambda)$$

induced on derived categories by the left-exact smooth duality functor

$$V \mapsto (V^*)^{\mathrm{sm}} = \{f : V \to \Lambda \mid \exists H \subset G \text{ open } \forall h \in H, v \in V : f(hv) = f(v)\}.$$

Proof. The operation $A \mapsto (A^*)^{\mathrm{sm}}$ on $D(G,\Lambda)$ satisfies the adjunction

$$\mathrm{Hom}_{D(G,\Lambda)}(B,(A^*)^{\mathrm{sm}}) = \mathrm{Hom}_{D(G,\Lambda)}(B \otimes^{\mathbb{L}}_{\Lambda} A, \Lambda)$$

for all $B \in D(G,\Lambda)$. As $R\mathscr{H}\mathrm{om}_\Lambda(-,\Lambda)$ is characterized by the similar adjunction in $D_{\text{ét}}([*/\underline{G}],\Lambda)$, we get the result. □

V.2. Étale sheaves on strata

We want to describe $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ via its strata Bun_G^b. For this, we need the following result saying roughly that connected Banach–Colmez spaces are "contractible".

Proposition V.2.1. *Let $f : S' \to S$ be a map of small v-stacks that is a torsor under $\mathcal{BC}(\mathcal{E})$ resp. $\mathcal{BC}(\mathcal{E}[1])$, where $\mathcal{E}$ is a vector bundle on X_S that is everywhere of positive (resp. negative) slopes. Then the pullback functor*

$$f^* : D_{\text{ét}}(S,\Lambda) \to D_{\text{ét}}(S',\Lambda)$$

is fully faithful.

Proof. By descent [**Sch17a**, Proposition 17.3, Remark 17.4], the problem is v-local on S, and in particular one can assume that the torsor is split. In the positive case, we can use Corollary II.3.3 (iv) to find pro-étale locally on S a short exact sequence

$$0 \to \mathcal{O}_{X_S}(\tfrac{1}{2r})^{m'} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^m \to \mathcal{E} \to 0,$$

inducing a similar sequence on Banach–Colmez spaces. This reduces us to the case $\mathcal{E} = \mathcal{O}_{X_S}(\frac{1}{n})$ for some n (as then pullback under $\mathcal{BC}(\mathcal{O}_{X_S}(\frac{1}{r})^m) \to S$ is fully faithful, as is pullback under $\mathcal{BC}(\mathcal{O}_{X_S}(\frac{1}{r})^m) \to S' = \mathcal{BC}(\mathcal{E})$). In that case, $\mathcal{BC}(\mathcal{E})$ is a 1-dimensional perfectoid open unit ball over S by Proposition II.2.5 (iv), in particular cohomologically smooth. It suffices to see that $Rf^!$ is fully faithful, for which it suffices that for all $A \in D_{\text{ét}}(S,\Lambda)$, the adjunction map

$$Rf_! Rf^! A \to A$$

is an equivalence. But note that both $Rf_!$ and $Rf^!$ commute with any base change by [**Sch17a**, Proposition 22.19, Proposition 23.12]. Thus, we may by passage to stalks reduce to the case $S = \mathrm{Spa}(C, C^+)$ where C is a complete algebraically closed nonarchimedean field and $C^+ \subset C$ an open and bounded valuation subring, and in fact we only need to check the statement on global sections. If the stalk of A at the closed point $s \in S$ is zero, then the same holds true for $Rf_! Rf^! A$ as $Rf^! A$ agrees with $f^* A$ up to twist, so this follows from proper base change, [**Sch17a**, Theorem 19.2]. This allows us to reduce to the case that A is constant, and then as both $Rf^!$ and $Rf_!$ commute with all direct sums, even to the case $A = \Lambda$. Thus, we are reduced to the computation of the cohomology of the perfectoid open unit disc.

The case of negative Banach–Colmez spaces follows by taking an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{X_S}(d)^m \to \mathcal{G} \to 0$$

and using the exact sequence

$$0 \to \mathcal{BC}(\mathcal{O}_{X_S}(d)^m) \to \mathcal{BC}(\mathcal{G}) \to \mathcal{BC}(\mathcal{E}[1]) \to 0.$$ □

Now we can formulate the desired result.

Proposition V.2.2. *For any $b \in B(G)$, the map*

$$\mathrm{Bun}_G^b = [*/\widetilde{G}_b] \to [*/\underline{G_b(E)}]$$

induces via pullback an equivalence

$$D_{\text{ét}}(G_b(E), \Lambda) \cong D_{\text{ét}}([*/\underline{G_b(E)}, \Lambda]) \cong D_{\text{ét}}([*/\widetilde{G}_b], \Lambda).$$

Moreover, for any complete algebraically closed nonarchimedean field C/k, the map

$$D_{\text{ét}}([*/\widetilde{G}_b], \Lambda) \to D_{\text{ét}}([\mathrm{Spa}\, C/\widetilde{G}_b], \Lambda)$$

is an equivalence.

Proof. Using [**Sch17a**, Theorem 1.13] and Theorem V.1.1, it is enough to prove that the functor

$$D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G_b(E)}], \Lambda) \to D_{\text{ét}}([\mathrm{Spa}\, C/\widetilde{G}_b], \Lambda)$$

is an equivalence. For this, it is enough to prove that pullback under the section $[\mathrm{Spa}\, C/\underline{G_b(E)}] \to [\mathrm{Spa}\, C/\widetilde{G}_b]$ induces a fully faithful functor

$$D_{\text{ét}}([\mathrm{Spa}\, C/\widetilde{G}_b], \Lambda) \to D_{\text{ét}}([\mathrm{Spa}\, C/\underline{G_b(E)}], \Lambda),$$

which follows from Proposition V.2.1. □

We see that $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is glued from the categories $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$, which are entirely representation-theoretic.[1] In particular, this implies that the base field plays no role:

Corollary V.2.3. *For any complete algebraically closed nonarchimedean field C and any locally closed substack $U \subset \mathrm{Bun}_G$, the functor*

$$D_{\text{ét}}(U, \Lambda) \to D_{\text{ét}}(U \times \mathrm{Spa}\, C, \Lambda)$$

is an equivalence of categories.

Although this seems like a purely technical result, it will actually play a key role when we study Hecke operators.

Proof. Fully faithfulness holds true by [**Sch17a**, Theorem 1.13 (ii)]. To see that it is an equivalence of categories, it is enough to check on all quasicompact locally closed substacks $U \subset \mathrm{Bun}_G$. These are stratified into finitely many locally closed substacks $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$, and we have the corresponding excision exact sequences, so to get essential surjectivity, it is enough to check on each stratum Bun_G^b. Now it follows from Proposition V.2.2. □

[1] It would be very interesting to understand the gluing of these categories in terms of pure representation theory.

V.3. Local charts

For any $b \in B(G)$ we wish to construct a chart

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$$

whose image contains Bun_G^b, such that π_b *is separated, representable in locally spatial diamonds and cohomologically smooth*, and whose geometry can be understood explicitly.

Example V.3.1. Before we discuss the general case, let us briefly discuss the first interesting case, namely $G = \mathrm{GL}_2$ and the non-basic element b corresponding to $\mathcal{O}(1) \oplus \mathcal{O}$. In that case, we let $\mathcal{M}_b$ be the moduli space of extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

where $\mathcal{L}$ is of degree 0 and $\mathcal{L}'$ is of degree 1. Mapping such an extension to $\mathcal{E}$ defines the map $\mathcal{M}_b \to \mathrm{Bun}_G$.

Note that there is a natural $\underline{E^\times} \times \underline{E^\times}$-torsor $\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$, parametrizing isomorphisms $\mathcal{L} \cong \mathcal{O}$ and $\mathcal{L}' \cong \mathcal{O}(1)$. On the other hand, it is clear that $\widetilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1])$ is a negative absolute Banach–Colmez space; thus, $\mathcal{M}_b$ is very explicit.

Any extension $\mathcal{E}$ parametrized by $\mathcal{M}_b$ is either isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$, or to $\mathcal{O}(\frac{1}{2})$. The fibres of $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_{\mathrm{GL}_2}$ over a rank 2 bundle $\mathcal{E}$ are given by an open subset of the projectivized Banach–Colmez space $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$. Thus, these fibres interpolate between $(\mathcal{BC}(\mathcal{O}(\frac{1}{2})) \setminus \{0\})/\underline{E^\times}$, which is cohomologically smooth by Proposition II.3.7, and

$$(\mathcal{BC}(\mathcal{O} \oplus \mathcal{O}(1)) \setminus \{0\})/\underline{E^\times} = (\underline{E} \times \mathbb{D} \setminus \{0\})/\underline{E^\times}.$$

The latter is still cohomologically smooth, although $\underline{E} \times \mathbb{D}$ is not — the quotient by $\underline{E^\times}$ gets rid of the disconnected nature of the space. In this case, and in fact in complete generality for all $b \in B(\mathrm{GL}_n)$, one can actually check cohomological smoothness of π_b by hand. To handle the general case, we had to prove the Jacobian criterion, Theorem IV.4.2.

Coming back to the general case, we can in fact construct all $\mathcal{M}_b$ together, as follows.

Definition V.3.2. *The v-stack $\mathcal{M}$ is the moduli stack taking $S \in \mathrm{Perf}_k$ to the groupoid of G-bundles $\mathcal{E}$ on X_S together with an increasing separated and exhaustive $\mathbb{Q}$-filtration $(\rho_*\mathcal{E})^{\leq\lambda} \subset \rho_*\mathcal{E}$ (ranging over algebraic representations $\rho : G \to \mathrm{GL}_n$, and compatible with exact sequences and tensor products) on the corresponding fibre functor such that (letting $(\rho_*\mathcal{E})^{<\lambda} = \bigcup_{\lambda'<\lambda}(\rho_*\mathcal{E})^{\leq\lambda'}$) the quotient*

$$(\rho_*\mathcal{E})^\lambda = (\rho_*\mathcal{E})^{\leq\lambda}/(\rho_*\mathcal{E})^{<\lambda}$$

is a semistable vector bundle of slope λ, for all $\lambda \in \mathbb{Q}$ and representations $\rho : G \to \mathrm{GL}_n$.

Note that by passing to the associated graded, $\mathcal{M}$ maps to the moduli stack of G-bundles in the category of $\mathbb{Q}$-graded vector bundles on X_S where the weight λ piece is semistable of slope λ. By Proposition III.4.7, this is isomorphic to $\bigsqcup_{b \in B(G)}[*/\underline{G_b(E)}]$. In particular $\mathcal{M}$ decomposes naturally into a disjoint union

$$\mathcal{M} = \bigsqcup_{b \in B(G)} \mathcal{M}_b,$$

and for each $b \in B(G)$, we have natural maps

$$q_b : \mathcal{M}_b \to [*/\underline{G_b(E)}].$$

Example V.3.3. When $G = \mathrm{GL}_n$, $\mathcal{M}$ sends S to the groupoid of filtered vector bundles $0 = \mathrm{Fil}_0 \mathcal{E} \subsetneq \mathrm{Fil}_1 \mathcal{E} \subsetneq \cdots \subsetneq \mathrm{Fil}_r \mathcal{E} = \mathcal{E}$ for some r, such that $\mathrm{Fil}_{i+1} \mathcal{E}/\mathrm{Fil}_i \mathcal{E}$ is semistable and the slopes $(\mu(\mathrm{Fil}_{i+1} \mathcal{E}/\mathrm{Fil}_i \mathcal{E}))_{0 \le i < r}$ form an increasing sequence (the opposite condition to the one defining the Harder–Narasimhan filtration of a vector bundle). The maps q_b send such a vector bundle to the graded vector bundle $\bigoplus_{i=0}^{r-1} \mathrm{Fil}_{i+1} \mathcal{E}/\mathrm{Fil}_i \mathcal{E}$.

Example V.3.4. Suppose G is quasisplit. Let M_b be the centralizer of the slope morphism, as a Levi subgroup. Let P_b be the parabolic subgroup with Levi M_b such that the weight of ν_b on Lie P_b are positive. There is a diagram

$$\begin{tikzcd} \mathrm{Bun}_{P_b} \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ \mathrm{Bun}_{M_b} & \end{tikzcd}$$

induced by the inclusion $P_b \subset G$ and the quotient map $P_b \to M_b$. There is a cartesian diagram

$$\begin{tikzcd} \mathcal{M}_b \arrow[r] \arrow[d] & \mathrm{Bun}_{P_b} \arrow[d] \\ \mathrm{Bun}^b_{M_b} \arrow[r, hook] & \mathrm{Bun}_{M_b}. \end{tikzcd}$$

Proposition V.3.5. *For any $b \in B(G)$, the map*

$$q_b : \mathcal{M}_b \to [*/\underline{G_b(E)}].$$

is partially proper, representable in locally spatial diamonds, and cohomologically smooth, of dimension $\langle 2\rho, \nu_b \rangle$. In fact, after pullback along $* \to [*/\underline{G_b(E)}]$, it is a successive torsor under negative Banach–Colmez spaces.

In particular, $\mathcal{M}_b$ is a cohomologically smooth Artin v-stack, of dimension $\langle 2\rho, \nu_b \rangle$.

Proof. It suffices to check everything after pullback after pullback by the v-cover $* \to [*/\underline{G_b(E)}]$, $\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$. Let $\widetilde{G} \to X_S$ be the automorphism group of $\mathcal{E}_b \to X_S$, see Proposition III.5.2, the pure inner twisting of $G \times X_S$ by $\mathcal{E}_b$. This is equipped with a parabolic subgroup $\widetilde{G}^{\le 0}$, and moreover a filtration $(\widetilde{G}^{\le \lambda})_{\lambda \le 0}$ with unipotent radical $\widetilde{G}^{<0}$. This is the opposite parabolic subgroup to the one used in the proof of Proposition III.5.1. Then $\widetilde{\mathcal{M}}_b(S)$ is identified with the set of $\widetilde{G}^{<0}$-torsors. The result is deduced using the description of the graded pieces of $(\widetilde{G}^{\le \lambda})_{\lambda < 0}$ as vector bundles of negative slopes. □

We first prove some structural results about $\mathcal{M}_b$ and its universal $\underline{G_b(E)}$-torsor $\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$. *A general theme here is the subtle distinction between the absolute property of being a (locally spatial) diamond (which $\widetilde{\mathcal{M}}_b$ is not, but it has a large open part $\widetilde{\mathcal{M}}_b^\circ \subset \widetilde{\mathcal{M}}_b$ that is) and the relative notion of $\widetilde{\mathcal{M}}_b \to *$ being representable in (locally spatial) diamonds (which $\widetilde{\mathcal{M}}_b$ is), and some related subtle distinctions on absolute and relative quasicompactness.*

Proposition V.3.6. *The map $\mathcal{M}_b \to [*/\underline{G_b(E)}]$ has a section $[*/\underline{G_b(E)}] \to \mathcal{M}_b$ given by the closed substack where $\mathcal{E}$ is (at every geometric point) isomorphic to $\mathcal{E}_b$, in which case $(\rho_*\mathcal{E})^{\leq\lambda} \subset \rho_*\mathcal{E}$ is a splitting of the Harder–Narasimhan filtration of $\rho_*\mathcal{E}$ for all representations $\rho : G \to \mathrm{GL}_n$.*

Consider the open complement $\mathcal{M}_b^\circ = \mathcal{M}_b \setminus [*/\underline{G_b(E)}]$, with preimage $\widetilde{\mathcal{M}}_b^\circ = \widetilde{\mathcal{M}}_b \setminus \{*\}$. Then $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond.

Moreover, if $U_\pi = \nu_b^N(\pi) \in G_b(E)$ for any large enough N (so that $\nu_b^N : \mathbb{G}_m \to G_b$ is a well-defined cocharacter), then $\widetilde{\mathcal{M}}_b^\circ/U_\pi^{\mathbb{Z}} \to *$ is proper.

Proof. To check that the substack where $\mathcal{E}$ is at every geometric point isomorphic to $\mathcal{E}_b$ is closed, note that by semicontinuity it suffices to see that everywhere on $\mathcal{M}_b$, the Newton point of $\mathcal{E}$ is bounded by b. By [**RR96**, Lemma 2.2 (iv)], this reduces to the case of $G = \mathrm{GL}_n$, where it is a simple consequence of the Harder–Narasimhan formalism (the Newton polygon of an extension is always bounded by the Newton polygon of the split extension). On this closed substack, $\mathcal{E}$ has two transverse filtrations, given by $(\rho_*\mathcal{E})^{\leq\lambda}$ and the Harder–Narasimhan filtration; it follows that $\mathcal{E}$ upgrades to a G-bundle in $\mathbb{Q}$-graded vector bundles on the Fargues–Fontaine curve, with the weight λ-piece semistable of slope λ. We see that this gives the desired section of $\mathcal{M}_b \to [*/\underline{G_b(E)}]$, using again Proposition III.4.7.

We claim that the action of U_π on $|\widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!)|$ satisfies the hypotheses of Lemma II.2.17, with fixed point locus given by the closed subspace $*$ considered in the previous paragraph. Writing $\widetilde{\mathcal{M}}_b$ as a successive extension of Banach–Colmez spaces, it is clear that for all $x \in |\widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!)|$ which are not in the closed substack, the sequence $U_\pi^{-n}(x)$ leaves any quasicompact open subspace for large n: Look at the first step in the successive extensions where x does not project to the origin. Then x gives an element in the fiber over the origin, which is a negative Banach–Colmez space, on which $U_\pi = \nu_b^N(\pi)$ acts via a positive power of π; thus $U_\pi^{-n}(x)$ leaves any quasicompact open subspace of this Banach–Colmez space. In particular, it follows that the fixed points locus of U_π is precisely the origin. To apply Lemma II.2.17, it remains to see that for all $x \in |\widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!)|$ and quasicompact open neighborhoods U of the origin, one has $\gamma^m(x) \in U$ for all sufficiently large m. This can be reduced to the case of GL_n by the Tannakian formalism, so assume $G = \mathrm{GL}_n$ for this argument. Now fix a map $f : \mathrm{Spa}(C, C^+) \to \widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!)$ having x in its image; it suffices to construct a map

$$\mathrm{Spa}(C, C^+) \times \underline{\mathbb{N} \cup \{\infty\}} \to \widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!)$$

whose restriction to $\mathrm{Spa}(C, C^+) \times \{0\}$ is f and which is equivariant for the γ-action, with γ acting on the left via shift on the profinite set $\mathbb{N} \cup \{\infty\}$. The map f classifies some $\mathbb{Q}$-filtered vector bundle $\mathcal{E}^{\leq\lambda} \subset \mathcal{E}$ of rank n on X_C with $\bigoplus_\lambda \mathcal{E}^\lambda \cong \mathcal{E}_b$ as $\mathbb{Q}$-graded vector bundles. After pullback to $Y_{C,[1,q]}$, the filtration is split, so we can find an isomorphism

$$\alpha : \mathcal{E}|_{Y_{C,[1,q]}} \cong \mathcal{E}_b|_{Y_{C,[1,q]}}$$

of $\mathbb{Q}$-filtered vector bundles on $Y_{C,[1,q]}$, such that α reduces on graded pieces to the given identification. The descent datum is now given by some isomorphism of $\mathbb{Q}$-filtered vector bundles

$$\beta : \varphi^*(\mathcal{E}_b|_{Y_{C,[q,q]}}) \cong \mathcal{E}_b|_{Y_{C,[1,1]}}$$

that reduces to the standard Frobenius on graded pieces. In other words, β is the standard Frobenius on $\mathcal{E}_b$ multiplied by some

$$\beta' : \mathcal{E}_b|_{Y_{C,[1,1]}} \cong \mathcal{E}_b|_{Y_{C,[1,1]}}$$

and with respect to the $\mathbb{Q}$-grading on $\mathcal{E}_b$, the map β' is the identity plus a lower triangular matrix.

The action of γ replaces β' by its U_π-conjugate. This multiplies all lower triangular entries by powers of π, so

$$(\beta', \gamma(\beta'), \gamma^2(\beta'), \dots, 1)$$

composed with the standard Frobenius defines an isomorphism

$$\varphi^*(\mathcal{E}_b|_{Y_{S,[q,q]}}) \cong \mathcal{E}_b|_{Y_{S,[1,1]}}$$

where $S = \mathrm{Spa}(C, C^+) \times \underline{\mathbb{N} \cup \{\infty\}}$. Using this as a descent datum defines a $\mathbb{Q}$-filtered vector bundle on X_S defining the required map

$$S = \mathrm{Spa}(C, C^+) \times \underline{\mathbb{N} \cup \{\infty\}} \to \widetilde{\mathcal{M}}_b \times_k \mathrm{Spa}\, k(\!(t)\!).$$

This finishes the verification of the hypotheses of Lemma II.2.17.

It is clear from the definition that $\widetilde{\mathcal{M}}_b^\circ \to *$ is partially proper (as the theory of vector bundles on the Fargues–Fontaine curve does not depend on R^+). Thus, to show that $\widetilde{\mathcal{M}}_b^\circ / U_\pi^{\mathbb{Z}} \to *$ is proper, it suffices to see that the map is quasicompact, which can be checked after base change to $\mathrm{Spa}\, k(\!(t)\!)$; then it follows from the previous discussion and Lemma II.2.17.

It remains to see that $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond. For this, we pick a representative $b \in G(\breve{E})$ of the σ-conjugacy class that is decent in the sense of [**RZ96**, Definition 1.8]. In particular, $b \in G(E_s)$ for some unramified extension $E_s|E$ of degree s, and $\widetilde{\mathcal{M}}_b^\circ$ is already defined over $\mathrm{Perf}_{\mathbb{F}_{q^s}}$. Let Frob_s be the Frobenius $x \mapsto x^{q^s}$. As b is decent, the action of $U_\pi = \nu_b^N(\pi)$ on $\widetilde{\mathcal{M}}_b^\circ$ agrees with the action of a power of Frob_s for N large enough. We know that

$$\widetilde{\mathcal{M}}_b^\circ / U_\pi^{\mathbb{Z}} \times_k \mathrm{Spa}\, k(\!(t)\!)$$

is a spatial diamond (as it is proper over $\mathrm{Spa}\, k(\!(t)\!)$). Replacing U_π by Frob_s, and moving the quotient by Frobenius to the other factor (which is allowed as the absolute Frobenius acts trivially on topological spaces) one sees that also

$$\widetilde{\mathcal{M}}_b^\circ \times_k \mathrm{Spa}\, k(\!(t)\!) / \mathrm{Frob}_s^{\mathbb{Z}}$$

is a spatial diamond. But $\mathrm{Spa}\, k(\!(t)\!) / \mathrm{Frob}_s^{\mathbb{Z}} \to *$ is proper and cohomologically smooth. Thus, Lemma II.3.8 shows that it is a spatial v-sheaf. By [**Sch17a**, Theorem 12.18], to see that $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond, it suffices to check on points. Writing $\widetilde{\mathcal{M}}_b$ as a successive extension of Banach–Colmez spaces, any point in $\widetilde{\mathcal{M}}_b^\circ$ has a minimal step where it does not map to the origin. Then its image is a nontrivial point of an absolute Banach–Colmez space, and a punctured absolute Banach–Colmez space is a diamond by Proposition II.3.7; the result follows. □

The following theorem gives the desired local charts for Bun_G; its proof is based on the Jacobian criterion for (cohomological) smoothness, Theorem IV.4.2.

THEOREM V.3.7. *The map $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ forgetting the filtration is partially proper, representable in locally spatial diamonds, and cohomologically smooth of ℓ-dimension $\langle 2\rho, \nu_b \rangle$.*

Proof. Let $S \to \mathrm{Bun}_G$ be some map for a perfectoid space S, given by some G-bundle $\mathcal{E}$ on X_S. Then $\mathcal{M}_b \times_{\mathrm{Bun}_G} S$ parametrizes $\mathbb{Q}$-filtrations on $\mathcal{E}$ whose associated graded $\mathbb{Q}$-bundle corresponds to b. Such $\mathbb{Q}$-filtrations are parametrized by sections of a smooth projective flag variety $Z = \mathcal{E}/\mathcal{P} \to X_S$ for some parabolic $\mathcal{P} \subset G \times_E X$. Moreover, the condition on the associated graded bundle and Proposition III.5.2 imply that $\mathcal{M}_b$ is an open subspace of $\mathcal{M}_Z^{\mathrm{sm}}$, so the result follows from Theorem IV.4.2. $\square$

V.4. Compact generation

We are now revisiting the notion of finite type smooth representation in terms of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.

The goal of this section is to prove the following theorem. As above, we fix some coefficient ring Λ such that $n\Lambda = 0$ for some n prime to p.

Theorem V.4.1. *For any locally closed substack $U \subset \mathrm{Bun}_G$, the triangulated category $D_{\text{ét}}(U, \Lambda)$ is compactly generated. An object $A \in D_{\text{ét}}(U, \Lambda)$ is compact if and only if for all $b \in B(G)$ contained in U, the restriction*

$$i^{b*} A \in D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

along $i^b : \mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ is compact, and zero for almost all b. Here, compactness in $D(G_b(E), \Lambda)$ is equivalent to lying in the thick triangulated subcategory generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ as K runs over open pro-p-subgroups of $G_b(E)$.

To prove the theorem, we exhibit a class of compact projective generators. The key result is that $\widetilde{\mathcal{M}}_b$ behaves like a strictly local scheme; in some vague sense, $\widetilde{\mathcal{M}}_b$ *is the strict henselization of* Bun_G *at* b.

Proposition V.4.2. *Let $b \in B(G)$. For any $A \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b, \Lambda)$ with stalk $A_0 = i^* A \in D_{\text{ét}}(*, \Lambda) \cong D(\Lambda)$ at the closed point $i : * \subset \widetilde{\mathcal{M}}_b$, the map*

$$R\Gamma(\widetilde{\mathcal{M}}_b, A) \to A_0$$

is an isomorphism. In particular, $R\Gamma(\widetilde{\mathcal{M}}_b, -)$ commutes with all direct sums.

Proof. Replacing A by the cone of $A \to i_* A_0$, we can assume that $A = j_! A'$ for some $A' \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b^\circ, \Lambda)$. We have to see that

$$R\Gamma(\widetilde{\mathcal{M}}_b, j_! A') = 0.$$

But this follows from Theorem IV.5.3 (applied with $X = \widetilde{\mathcal{M}}_b^\circ$ and $S = \mathrm{Spa}\, k(\!(t)\!)$, noting that base change along $S \to *$ follows from smooth base change), using that the partial compactification $\widetilde{\mathcal{M}}_b^\circ \subset \widetilde{\mathcal{M}}_b$ is precisely a compactification towards one of the two ends of $\widetilde{\mathcal{M}}_b^\circ$, as follows from the behaviour of the Frobenius action exhibited in the proof of Proposition V.3.6. $\square$

Remark V.4.3.

(i) Consider the v-sheaf $X = \mathrm{Spd}(k[\![x_1, \dots, x_d]\!])$ and the quasicompact open subset $U = \mathrm{Spd}(k[\![x_1, \dots, x_d]\!]) \setminus V(x_1, \dots, x_d)$ that is representable by a perfectoid space. When base changed to $S = \mathrm{Spa}(k(\!(t)\!))$, X becomes isomorphic to an open unit disk, U becomes the punctured unit disk that has two ends: the origin and the exterior of the disk. The picture is thus analogous to the preceding one with $\widetilde{\mathcal{M}}_b$, and for any $A \in D_{\text{ét}}(X, \Lambda)$ one has $R\Gamma(X, A) \cong i^* A$ where $i : \mathrm{Spd}(k) \hookrightarrow X$ is $V(x_1, \dots, x_d)$.

(ii) This applies more generally to the v-sheaf associated to any $W(k)$-affine formal scheme $\mathrm{Spf}(R)$ with R an I-adic ring. Then, for any $A \in D_{\text{ét}}(\mathrm{Spd}(R), \Lambda)$ one has $R\Gamma(\mathrm{Spd}(R), A) = R\Gamma(\mathrm{Spd}(R/I), i^*A)$ with $i : \mathrm{Spd}(R/I) \hookrightarrow \mathrm{Spd}(R)$. In particular, for $A \in D_{\text{ét}}(\mathrm{Spd}(R) \setminus V(I), \Lambda)$, where here $\mathrm{Spd}(R) \setminus V(I)$ is representable by a spatial diamond and even a perfectoid space if R is a k-algebra, one has

$$R\Gamma(\mathrm{Spd}(R) \setminus V(I), A) \cong R\Gamma(\mathrm{Spd}(R/I), i^*Rj_*A).$$

Thus, Proposition V.4.2 can be seen as a result about "nearby cycles on the strict henselianization of Bun_G at b".

Corollary V.4.4. *Let $b \in B(G)$ and let $K \subset G_b(E)$ be an open pro-p-subgroup. Then for any $A \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b/\underline{K}, \Lambda)$ with pullback $A_0 = i^*A \in D_{\text{ét}}([*/\underline{K}], \Lambda) \cong D(K, \Lambda)$ corresponding to a complex V of smooth K-representations, the map*

$$R\Gamma(\widetilde{\mathcal{M}}_b/\underline{K}, A) \to R\Gamma([*/\underline{K}], A_0) \cong V^K$$

is an isomorphism. In particular, $R\Gamma(\widetilde{\mathcal{M}}_b/\underline{K}, -)$ commutes with all direct sums.

Proof. This follows formally from Proposition V.4.2 by descent along $\psi : \widetilde{\mathcal{M}}_b \to \widetilde{\mathcal{M}}_b/\underline{K}$; more precisely, by writing any A as a direct summand of $\psi_*\psi^*A$. $\square$

Remark V.4.5. Let

$$i : [*/\underline{G_b(E)}] \hookrightarrow \mathcal{M}_b \hookleftarrow \mathcal{M}_b^\circ : j$$

be the usual diagram. From the corollary, one deduces that if one regards

$$i^*Rj_*A \in D_{\text{ét}}([*/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$$

as a complex of $G_b(E)$-representations, then this is given by

$$R\Gamma(\widetilde{\mathcal{M}}_b^\circ, A).$$

As $\widetilde{\mathcal{M}}_b^\circ$ is qcqs (and of finite cohomological dimension) by Proposition V.3.6, this commutes with all direct sums in A.

For example, in the case of GL_2 and $\mathcal{E}_b = \mathcal{O}(1) \oplus \mathcal{O}$, one has

$$\widetilde{\mathcal{M}}_b^\circ = \mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\} = \mathrm{Spa}\, k(\!(t^{1/p^\infty})\!)/\underline{SL_1(D)}$$

by Example II.3.12 and Example V.3.1. Thus, in this case one compute

$$i^*Rj_*A = R\Gamma(\mathrm{Spa}\, k(\!(t^{1/p^\infty})\!)/\underline{SL_1(D)}, A)$$

which is a very explicit formula. If one would use the presentation

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \times \mathrm{Spa}\, C = (\mathbb{A}^1_{C^\sharp})^\diamond/\underline{E}$$

instead, it would be considerably more difficult to compute the answer. In particular, we critically used quasicompacity of the absolute $\widetilde{\mathcal{M}}_b^\circ$, its base change $\widetilde{\mathcal{M}}_b^\circ \times_k \mathrm{Spa}\, C$ is no longer quasicompact. *This highlights the importance of working with absolute objects, and of using the right local charts.*

In fact, Theorem V.3.7, smooth base change, and this formula for i^*Rj_* show that the gluing of the representation-theoretic strata $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ into $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is encoded in the spaces $\widetilde{\mathcal{M}}_b^\circ$, showing that the local charts $\mathcal{M}_b$ are of fundamental and not just technical importance.

Now we can prove Theorem V.4.1.

Proof of Theorem V.4.1. As for closed immersions i, the functor i^* preserves compact objects, it is enough to handle the case that U is an open substack. Let $b \in |U| \subset \mathrm{Bun}_G \cong B(G)$ be any point of U, and let $K \subset G_b(E)$ be an open pro-p-subgroup, giving rise to the map

$$f_K : \widetilde{\mathcal{M}}_b/\underline{K} \to \mathrm{Bun}_G\,.$$

By Corollary V.4.4 and Theorem V.3.7, we see that $A_K^b := Rf_{K!}Rf_K^!\Lambda \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is compact; in fact,

$$\begin{aligned} R\,\mathrm{Hom}(A_K^b, B) &\cong R\,\mathrm{Hom}(Rf_K^!\Lambda, Rf_K^! B) \cong R\,\mathrm{Hom}(Rf_K^!\Lambda, f_K^* B \otimes_\Lambda^{\mathbb{L}} Rf_K^!\Lambda) \\ &\cong R\Gamma(\widetilde{\mathcal{M}}_b/\underline{K}, f_K^* B) \cong (i^{b*}B)^K. \end{aligned}$$

From this computation, we see that the collection of objects A_K^b for varying $b \in |U|$ and $K \subset G_b(E)$ open pro-p form a class of compact generators: Indeed, if B is nonzero, then $(i^{b*}B)^K$ must be nonzero for some b and K.

To prove the characterization of compact objects, we argue by induction on the number of points of $|U|$, noting that any compact object must be concentrated on a quasicompact substack, and thus on finitely many points. So assume that $|U|$ is finite, $b \in |U|$ is a closed point and $j : V = U \setminus \{b\} \subset U$ is the open complement, so we know the result for V. It suffices to prove that j^* preserves compact objects. Indeed, then $A \in D_{\text{ét}}(U, \Lambda)$ is compact if and only if j^*A and $i^{b*}A$ are compact, and this gives by induction the desired characterization.

To see that j^* preserves compact objects, we can check on the given generators. For generators $A_K^{b'}$ corresponding to $b' \neq b$ we get $j^*A_K^{b'} = A_K^{b'}$, so there is nothing to prove. On the other hand, $j^*A_K^b = Rf_{K!}^\circ f_K^{\circ!}\Lambda$ for

$$f_K^\circ : \widetilde{\mathcal{M}}_b^\circ/\underline{K} \to V \subset \mathrm{Bun}_G\,.$$

The compactness of $j^*A_K^b$ then follows from $R\Gamma(\widetilde{\mathcal{M}}_b^\circ/\underline{K}, -)$ commuting with all direct sums. But this is true as $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond of finite dim. trg, by Proposition V.3.6 and Proposition V.3.5, and taking cohomology under K is exact. □

V.5. Bernstein–Zelevinsky duality

We note that one can define a Bernstein–Zelevinsky involution on (the compact objects of) $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. More precisely, we have the following result. Here, in anticipation of some functor introduced later, we write

$$\pi_\natural : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \to D(*, \Lambda) = D(\Lambda) : A \mapsto R\pi_!(A \otimes_\Lambda^{\mathbb{L}} R\pi^!\Lambda)$$

for the left adjoint of π^*, $\pi : \mathrm{Bun}_G \to *$.

Theorem V.5.1. *For any compact object $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$, there is a unique compact object $\mathbb{D}_{\mathrm{BZ}}(A) \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ with a functorial identification*

$$R\,\mathrm{Hom}(\mathbb{D}_{\mathrm{BZ}}(A), B) \cong \pi_\natural(A \otimes_\Lambda^{\mathbb{L}} B)$$

for $B \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. Moreover, the functor $\mathbb{D}_{\mathrm{BZ}}$ is a contravariant autoequivalence of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega$, and $\mathbb{D}_{\mathrm{BZ}}^2$ is naturally isomorphic to the identity.

If $U \subset \mathrm{Bun}_G$ is an open substack and A is concentrated on U, then so is $\mathbb{D}_{\mathrm{BZ}}(A)$. In particular, $\mathbb{D}_{\mathrm{BZ}}$ restricts to an autoequivalence of the compact objects in $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ for $b \in B(G)$ basic, and in that setting it is the usual Bernstein–Zelevinsky involution.

Proof. By the Yoneda lemma, the uniqueness of $\mathbb{D}_{\mathrm{BZ}}(A)$ is clear. For the existence, it suffices to check on a system of generators, like $A = i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ for varying $b \in B(G)$ and $K \subset G_b(E)$ open pro-p. In that case, $\mathbb{D}_{\mathrm{BZ}}(A) = A_K^b$ by Corollary V.4.4. This also shows that if A is concentrated on U, then so is $\mathbb{D}_{\mathrm{BZ}}(A)$.

Now note that

$$R\,\mathrm{Hom}(\mathbb{D}_{\mathrm{BZ}}(A), B) \cong \pi_\natural(A \otimes_\Lambda^{\mathbb{L}} B) \cong \pi_\natural(B \otimes_\Lambda^{\mathbb{L}} A) \cong R\,\mathrm{Hom}(\mathbb{D}_{\mathrm{BZ}}(B), A).$$

In particular, taking $B = \mathbb{D}_{\mathrm{BZ}}(A)$, we see that there is a natural functorial map $\mathbb{D}_{\mathrm{BZ}}^2(A) \to A$. We claim that this is an equivalence. It suffices to check on generators. We have seen that the Bernstein–Zelevinsky dual of $i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ is A_K^b. Its restriction to Bun_G^b is again $i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$, so one easily checks that the map $\mathbb{D}_{\mathrm{BZ}}^2(A) \to A$ is an isomorphism over Bun_G^b. To see that it is an isomorphism everywhere, one needs to see that if $B = Rj_*B'$, $B' \in D_{\text{ét}}(U, \Lambda)$ for some open substack $j : U \subset \mathrm{Bun}_G$ not containing Bun_G^b, then

$$\pi_\natural(A_K^b \otimes_\Lambda^{\mathbb{L}} B) = 0.$$

Twisting a few things away and using the definition of $A_K^b = Rf_{K!}Rf_K^!\Lambda$, this follows from the assertion that for all $A' \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b^\circ/\underline{K}, \Lambda)$, with $j_K : \widetilde{\mathcal{M}}_b^\circ/\underline{K} \hookrightarrow \widetilde{\mathcal{M}}_b/\underline{K}$ the open immersion, one has

$$R\Gamma_c(\widetilde{\mathcal{M}}_b/\underline{K}, Rj_{K*}A') = 0.$$

Using the trace map for $\widetilde{\mathcal{M}}_b \to \widetilde{\mathcal{M}}_b/\underline{K}$, this follows from Theorem IV.5.3, applied as before with $X = \widetilde{\mathcal{M}}_b^\circ$ and $S = \mathrm{Spa}\, k(\!(t)\!)$, noting that base change along $S \to *$ holds by smooth base change and is conservative.

The comparison to Bernstein–Zelevinsky duality follows formally by taking B corresponding to the regular representation of $G_b(E)$, in which case $\pi_\natural(A \otimes_\Lambda^{\mathbb{L}} B)$ is isomorphic to the underlying chain complex of A. Moreover, as the regular representation has two commuting $G_b(E)$-actions, there is a residual $G_b(E)$-action, which is the usual action on A. This gives the usual definition of the Bernstein–Zelevinsky involution as $R\,\mathrm{Hom}$ into the regular representation. □

V.6. Verdier duality

It turns out that one can also understand how Verdier duality acts on $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. The key result is the following.

Theorem V.6.1. *Let $j : V \hookrightarrow U$ be an open immersion of open substacks of Bun_G. For any $A \in D_{\text{ét}}(V, \Lambda)$, the natural map*

$$j_! R\mathscr{H}\mathrm{om}_\Lambda(A, \Lambda) \to R\mathscr{H}\mathrm{om}_\Lambda(Rj_*A, \Lambda)$$

is an isomorphism in $D_{\text{ét}}(U, \Lambda)$.

Note that one always has

$$Rj_*R\mathscr{H}\mathrm{om}_\Lambda(A,\Lambda) = R\mathscr{H}\mathrm{om}_\Lambda(j_!A,\Lambda)\ ;$$

the theorem asserts that this is also true with $j_!$ and Rj_* exchanged, which is related to a local biduality statement: If $A \in D_{\text{ét}}(U,\Lambda)$ is reflexive, the theorem implies formally that $j_!A \in D_{\text{ét}}(U,\Lambda)$ is reflexive.

Proof. We can assume that U and V are quasicompact, and then by induction, we can assume that $V = U \setminus \{b\}$ for some closed $b \in |U|$. The map is clearly an isomorphism over V, so it suffices to see that for the compact objects

$$A_K^b = Rf_{K!}f_K^!\Lambda,\ \ f_K : \widetilde{\mathcal{M}}_b/\underline{K} \to U \subset \mathrm{Bun}_G,$$

one gets an isomorphism after applying $R\,\mathrm{Hom}(A_K^b,-)$. As $R\,\mathrm{Hom}(A_K^b,B) = (i^{b*}B)^K$, we see that the left-hand side

$$R\,\mathrm{Hom}(A_K^b, j_!R\mathscr{H}\mathrm{om}_\Lambda(A,\Lambda)) = 0$$

vanishes. On the other hand, using the left adjoint $\pi_\natural : D_{\text{ét}}(\mathrm{Bun}_G,\Lambda) \to D(*,\Lambda) \cong D(\Lambda)$ of pullback, we have

$$R\,\mathrm{Hom}(A_K^b, R\mathscr{H}\mathrm{om}_\Lambda(Rj_*A,\Lambda)) \cong R\,\mathrm{Hom}(A_K^b \otimes_\Lambda^{\mathbb{L}} Rj_*A,\Lambda) \cong R\,\mathrm{Hom}(\pi_\natural(A_K^b \otimes_\Lambda^{\mathbb{L}} Rj_*A),\Lambda).$$

But by Theorem V.5.1, and the identification of the Bernstein–Zelevinsky dual of A_K^b as $i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$, one has

$$\pi_\natural(A_K^b \otimes_\Lambda^{\mathbb{L}} Rj_*A) \cong R\,\mathrm{Hom}(i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda, Rj_*A) \cong R\,\mathrm{Hom}(c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda, Ri^{b!}Rj_*A) = 0. \qquad \square$$

Recall that as a cohomologically smooth Artin stack of dimension 0, Bun_G admits a dualizing complex $D_{\mathrm{Bun}_G} \in D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ that is locally isomorphic to $\Lambda[0]$.

Theorem V.6.2. *For any open substack $U \subset \mathrm{Bun}_G$, an object $A \in D_{\text{ét}}(U,\Lambda)$ is reflexive, i.e. the natural map*

$$A \to R\mathscr{H}\mathrm{om}_\Lambda(R\mathscr{H}\mathrm{om}_\Lambda(A,D_U),D_U)$$

is an equivalence, if and only if for all $b \in B(G)$ lying in U with corresponding locally closed stratum $i^b : \mathrm{Bun}_G^b \to U$, the restriction

$$i^{b*}A \in D_{\text{ét}}(\mathrm{Bun}_G^b,\Lambda) = D_{\text{ét}}([*/\widetilde{G}_b],\Lambda) \cong D_{\text{ét}}([*/\underline{G_b(E)}],\Lambda) = D(G_b(E),\Lambda)$$

is reflexive as a complex of admissible $G_b(E)$-representations; this means that the complex of K-invariants is reflexive in $D(\Lambda)$ for all open pro-p-subgroups $K \subset G_b(E)$.

In the definition of reflexivity, we can replace D_U by Λ (as this changes the dual by a twist, and then the bidual stays the same). The theorem follows immediately from the following result.

Lemma V.6.3. *Let $U \subset \mathrm{Bun}_G$ be an open substack and $A \in D_{\text{ét}}(U,\Lambda)$. For any $b \in B(G)$ lying in U, there is a natural isomorphism*

$$i^{b*}R\mathscr{H}\mathrm{om}_\Lambda(R\mathscr{H}\mathrm{om}_\Lambda(A,\Lambda),\Lambda) \cong R\mathscr{H}\mathrm{om}_\Lambda(R\mathscr{H}\mathrm{om}_\Lambda(i^{b*}A,\Lambda),\Lambda).$$

Proof. We may assume that $U \subset \mathrm{Bun}_G$ is the set of generalizations of b, and let $j : V = U \setminus \{b\} \hookrightarrow U$. Let $B = j^*A$. Using the exact triangle

$$j_!B \to A \to i^b_* i^{b*} A \to,$$

and the invertibility of $i^{b!}\Lambda$ (as Bun^b_G is also cohomologically smooth), it is enough to prove that

$$i^{b*} R\mathscr{H}\mathrm{om}_\Lambda(R\mathscr{H}\mathrm{om}_\Lambda(j_!B, \Lambda), \Lambda) = 0.$$

But $R\mathscr{H}\mathrm{om}_\Lambda(j_!B, \Lambda) = Rj_* R\mathscr{H}\mathrm{om}_\Lambda(B, \Lambda)$, and by Theorem V.6.1,

$$R\mathscr{H}\mathrm{om}_\Lambda(Rj_* R\mathscr{H}\mathrm{om}_\Lambda(B, \Lambda), \Lambda) = j_! R\mathscr{H}\mathrm{om}_\Lambda(R\mathscr{H}\mathrm{om}_\Lambda(B, \Lambda), \Lambda). \qquad \square$$

V.7. ULA sheaves

Finally, we want to classify the objects $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ that are universally locally acyclic with respect to $\mathrm{Bun}_G \to *$. Our goal is to prove the following theorem. *This gives a geometric interpretation of the classical notion of admissible representation in terms of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.*

Theorem V.7.1. *Let $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. Then A is universally locally acyclic with respect to $\mathrm{Bun}_G \to *$ if and only if for all $b \in B(G)$, the pullback $i^{b*}A$ to $i^b : \mathrm{Bun}^b_G \hookrightarrow \mathrm{Bun}_G$ corresponds under $D_{\text{ét}}(\mathrm{Bun}^b_G, \Lambda) \cong D(G_b(E), \Lambda)$ to a complex M_b of smooth $G_b(E)$-representations for which M_b^K is a perfect complex of Λ-modules for all open pro-p subgroups $K \subset G_b(E)$.*

We want to use Proposition IV.2.32. As preparation, we need to understand $D_{\text{ét}}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$. More generally, we have the following result.

Proposition V.7.2. *Let G_1 and G_2 be two reductive groups over E, and let $G = G_1 \times G_2$. Then $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_1} \times \mathrm{Bun}_{G_2}$, giving rise to an exterior tensor product*

$$- \boxtimes - : D_{\text{ét}}(\mathrm{Bun}_{G_1}, \Lambda) \times D_{\text{ét}}(\mathrm{Bun}_{G_2}, \Lambda) \to D_{\text{ét}}(\mathrm{Bun}_G, \Lambda).$$

For all compact objects $A_i \in D_{\text{ét}}(\mathrm{Bun}_{G_i}, \Lambda)$, $i = 1, 2$, the exterior tensor product $A_1 \boxtimes A_2 \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is compact, these objects form a class of compact generators, and for all further objects $B_i \in D_{\text{ét}}(\mathrm{Bun}_{G_i}, \Lambda)$, $i = 1, 2$, the natural map

$$R\,\mathrm{Hom}(A_1, B_1) \otimes^{\mathbb{L}}_\Lambda R\,\mathrm{Hom}(A_2, B_2) \to R\,\mathrm{Hom}(A_1 \boxtimes A_2, B_1 \boxtimes B_2)$$

is an isomorphism.

Remark V.7.3. The proposition says that as Λ-linear presentable stable ∞-categories, the exterior tensor product functor

$$\mathcal{D}_{\text{ét}}(\mathrm{Bun}_{G_1}, \Lambda) \otimes_{\mathcal{D}(\Lambda)} \mathcal{D}_{\text{ét}}(\mathrm{Bun}_{G_2}, \Lambda) \to \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$$

is an equivalence.

Proof. We use the compact generators $A_i = A^{b_i}_{K_i}$ for certain $b_i \in B(G_i)$, $K_i \subset G_{i,b_i}(E)$ open pro-p. These give rise to $b = (b_1, b_2) \in B(G)$ and $K = K_1 \times K_2 \subset G_b(E) = G_{1,b_1}(E) \times G_{2,b_2}(E)$, and using

$$\mathcal{M}_b \cong \mathcal{M}_{b_1} \times \mathcal{M}_{b_2}$$

and the Künneth formula, one concludes that $A_1 \boxtimes A_2 \cong A_K^b$, which is again compact. As $B(G) = B(G_1) \times B(G_2)$ and open pro-p subgroups of the form $K_1 \times K_2 \subset G_b(E)$ are cofinal, we see that these objects form a set of compact generators.

Moreover, as $R\operatorname{Hom}(A_K^b, B) = (i^{b*}B)^K$ for all $B \in D_{\text{ét}}(\operatorname{Bun}_G, \Lambda)$ and similarly for $A_{K_i}^{b_i}$, we also see that the map

$$R\operatorname{Hom}(A_{K_1}^{b_1}, B_1) \otimes_\Lambda^{\mathbb{L}} R\operatorname{Hom}(A_{K_2}^{b_2}, B_2) \to R\operatorname{Hom}(A_K^b, B_1 \boxtimes B_2)$$

is an isomorphism. As these objects generate, the same follows for all compact A_1, A_2. □

Now we can prove Theorem V.7.1.

Proof of Theorem V.7.1. By Proposition IV.2.32, we see that A being universally locally acyclic is equivalent to the map

$$p_1^* R\mathscr{H}\mathrm{om}(A, \Lambda) \otimes_\Lambda^{\mathbb{L}} p_2^* A \to R\mathscr{H}\mathrm{om}(p_1^* A, p_2^* A)$$

in $D_{\text{ét}}(\operatorname{Bun}_G \times \operatorname{Bun}_G, \Lambda) \cong D_{\text{ét}}(\operatorname{Bun}_{G\times G}, \Lambda)$ being an isomorphism.

By Proposition V.7.2, this is equivalent to being an isomorphism after applying $R\operatorname{Hom}(A_1 \boxtimes A_2, -)$ for varying compact $A_i \in D_{\text{ét}}(\operatorname{Bun}_{G_i}, \Lambda)$. Using Proposition V.7.2, the left-hand side is given by

$$R\operatorname{Hom}(A_1, R\mathscr{H}\mathrm{om}(A, \Lambda)) \otimes_\Lambda^{\mathbb{L}} R\operatorname{Hom}(A_2, A) \cong R\operatorname{Hom}(\pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A), \Lambda) \otimes_\Lambda^{\mathbb{L}} R\operatorname{Hom}(A_2, A).$$

The right-hand side is, using $\pi : \operatorname{Bun}_G \to *$ for the projection,

$$\begin{aligned} R\operatorname{Hom}(A_1 \boxtimes A_2, R\mathscr{H}\mathrm{om}(p_1^* A, p_2^* A)) &\cong R\operatorname{Hom}((A_1 \otimes_\Lambda^{\mathbb{L}} A) \boxtimes A_2, p_2^* A) \\ &\cong R\operatorname{Hom}(p_1^*(A_1 \otimes_\Lambda^{\mathbb{L}} A), p_2^* R\mathscr{H}\mathrm{om}_\Lambda(A_2, A)) \\ &\cong R\operatorname{Hom}(A_1 \otimes_\Lambda^{\mathbb{L}} A, Rp_{1*} p_2^* R\mathscr{H}\mathrm{om}_\Lambda(A_2, A)) \\ &\cong R\operatorname{Hom}(A_1 \otimes_\Lambda^{\mathbb{L}} A, \pi^* R\operatorname{Hom}(A_2, A)) \\ &\cong R\operatorname{Hom}(\pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A), R\operatorname{Hom}(A_2, A)), \end{aligned}$$

using usual adjunctions and smooth base change for p_2 and π several times. Under these isomorphisms, the map

$$R\operatorname{Hom}(\pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A), \Lambda) \otimes_\Lambda^{\mathbb{L}} R\operatorname{Hom}(A_2, A) \to R\operatorname{Hom}(\pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A), R\operatorname{Hom}(A_2, A))$$

is the natural map. This is an isomorphism as soon as $\pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A) \in D(\Lambda)$ is perfect for all compact A_1. In fact, the converse is also true: If one takes $A_2 = \mathbb{D}_{\mathrm{BZ}}(A_1)$, then $R\operatorname{Hom}(A_2, A) = \pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A)$, and hence it follows that for $M = \pi_\natural(A_1 \otimes_\Lambda^{\mathbb{L}} A) \in D(\Lambda)$, the map

$$R\operatorname{Hom}(M, \Lambda) \otimes_\Lambda^{\mathbb{L}} M \to R\operatorname{Hom}(M, M)$$

is an isomorphism, which means that M is dualizable in $D(\Lambda)$, i.e. perfect.

Now we use the system of compact generators given by $i_!^b c\text{-}\operatorname{Ind}_K^{G_b(E)} \Lambda$ for varying $b \in B(G)$, with locally closed immersion $i^b : \operatorname{Bun}_G^b \to \operatorname{Bun}_G$, and $K \subset G_b(E)$ open pro-p. This translates the condition on perfectness of $R\pi_!(A_1 \otimes_\Lambda^{\mathbb{L}} R\pi^! \Lambda \otimes_\Lambda^{\mathbb{L}} A)$ into the desired condition on stalks. □

CHAPTER VI

Geometric Satake

As before, we fix a nonarchimedean local field E with residue field $\mathbb{F}_q$ of characteristic p and a uniformizer $\pi \in E$. Recall that for any perfectoid space S over $\mathbb{F}_q$, we defined the "curve" $\mathcal{Y}_S$ over $\mathcal{O}_E$, as well as $Y_S = \mathcal{Y}_S \setminus V(\pi)$ and the quotient $X_S = Y_S/\varphi^{\mathbb{Z}}$.

In this chapter, we are interested in studying modifications of G-torsors on these spaces, and perverse sheaves on such. Our discussion will mirror this three-step procedure of the construction of X_S: If one has understood the basic theory over $\mathcal{Y}_S$, the basic results carry over easily to Y_S and then to X_S. While as in previous chapters our main focus is on X_S, in this chapter we will actually make critical use of $\mathcal{Y}_S$ in order to degenerate to the Witt vector affine Grassmannian, and hence to apply some results from the setting of schemes (notably the decomposition theorem). As the discussion here is very much of a local sort, one can usually reduce easily to the case that G is split, and hence admits a (split) reductive model over $\mathcal{O}_E$, and we will often fix such a split model of G.

For any $d \geq 0$, we consider the moduli space $\mathrm{Div}^d_{\mathcal{Y}}$ parametrizing degree d Cartier divisors $D \subset \mathcal{Y}_S$. For affinoid S, one can form the completion B^+ of $\mathcal{O}_{X_S}$ along D. Inverting D defines a localization B of B^+. One can then define a positive loop group $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G$ and loop group $L_{\mathrm{Div}^d_{\mathcal{Y}}} G$, with values given by $G(B^+)$ resp. $G(B)$; for brevity, we will simply write L^+G and LG here. One can then define the local Hecke stack

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} = [L^+G \backslash LG / L^+G] \to \mathrm{Div}^d_{\mathcal{Y}}$$

We will often break symmetry, and first take the quotient on the right to define the Beilinson–Drinfeld Grassmannian

$$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} = LG/L^+G \to \mathrm{Div}^d_{\mathcal{Y}}$$

so that

$$\mathcal{H}\mathrm{ck}^I_G = L^+G \backslash \mathrm{Gr}^I_G\,.$$

The Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Div}^d_{\mathcal{Y}}$ is a small v-sheaf that can be written as an increasing union of closed subsheaves that are proper and representable in spatial diamonds, by bounding the relative position; this is one main result of [**SW20**]. On the other hand, L^+G can be written as an inverse limit of truncated positive loop groups, which are representable in locally spatial diamonds and cohomologically smooth; moreover, on each bounded subset, it acts through such a finite-dimensional quotient. This essentially reduces the study of all bounded subsets of $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ to Artin stacks.

For any small v-stack $S \to \mathrm{Div}^d_{\mathcal{Y}}$, we let

$$\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

be the pullback. Let

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}} \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

be the full subcategory of all objects with quasicompact support over $\mathrm{Div}^d_{\mathcal{Y}}$. This is a monoidal category under convolution $\star$. Here, we use the convolution diagram

$$\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \times_S \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \xleftarrow{(p_1,p_2)} L^+G\backslash LG \times^{L^+G} LG/L^+G \xrightarrow{m} \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

and define

$$A \star B = Rm_*(p_1^* A \otimes^{\mathbb{L}}_{\Lambda} p_2^* B).$$

On $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$, one can define a relative perverse t-structure (where an object is perverse if and only if it is perverse over any geometric fibre of S), see Section VI.7. In particular, this t-structure is compatible with any base change in S. For this t-structure, the convolution $\star$ is left t-exact (and t-exactness only fails for issues related to non-flatness over Λ). To prove this, we reinterpret convolution as fusion, and use some results on hyperbolic localization.

Moreover, one can restrict to the complexes $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ that are universally locally acyclic over S. This condition is also preserved under convolution. For $d = 1$, or in general when S maps to the locus of distinct untilts $(\mathrm{Div}^d_{\mathcal{Y}})_{\neq} \subset \mathrm{Div}^d_{\mathcal{Y}}$, one can describe the category of universally locally acyclic by the condition that the restriction to any Schubert cell is locally constant with perfect fibres. To prove that all such sheaves are universally locally acyclic, we also introduce (for $d = 1$) the affine flag variety, in Section VI.5, and use their Demazure resolutions.

Definition VI.0.1. *The Satake category*

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

is the category of all $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ that are perverse, flat over Λ (i.e., for all Λ-modules M, also $A \otimes^{\mathbb{L}}_{\Lambda} M$ is perverse), and universally locally acyclic over $(\mathrm{Div}^1)^I$.

Intuitively, $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ are the "flat families of perverse sheaves on $\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \to S$", where flatness refers both to the geometric aspect of flatness over S (encoded in universal local acyclicity) and the algebraic aspect of flatness in the coefficients Λ. The Satake category $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ is a monoidal category under convolution. The forgetful functor

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is fully faithful. If $d = 1$ and $S = \mathrm{Spd}\,\overline{\mathbb{F}}_q$, then one can compare it to the category considered by Zhu [**Zhu17**] and Yu [**Yu19**], defined in terms of the Witt vector affine Grassmannian. Moreover, the categories for $S = \mathrm{Spd}\,\mathcal{O}_C$ and $S = \mathrm{Spd}\,C$ are naturally equivalent to the category for $S = \mathrm{Spd}\,\overline{\mathbb{F}}_q$, via the base change functors; here $C = \widehat{\overline{E}}$. Thus, the Satake category is, after picking a reductive model of G, naturally the same for the Witt vector affine Grassmannian and the B^+_{dR}-affine Grassmannian. At this point, we could in principle use Zhu's results [**Zhu17**] (refined integrally by Yu [**Yu19**]) to identify the Satake category with the category of representations of $\widehat{G}$, at least when

G is unramified. However, for the applications we actually need finer knowledge of the functoriality of the Satake equivalence including the case for $d > 1$; we thus prove everything we need directly.

More precisely, we now switch to $(\mathrm{Div}^1_X)^d$ in place of $\mathrm{Div}^d_{\mathcal{Y}}$, replacing also the use of $\mathcal{Y}_S$ by X_S; the two local Hecke stacks are locally isomorphic, so this poses no problems. For any finite set I, let

$$\mathcal{H}\mathrm{ck}^I_G = \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_X)^I}$$

and consider the monoidal category

$$\mathrm{Sat}^I_G(\Lambda) = \mathrm{Sat}(\mathcal{H}\mathrm{ck}^I_G, \Lambda).$$

In fact, the monoidal structure naturally upgrades to a symmetric monoidal structure. This relies on the fusion product, for which it is critical to allow general finite sets I. Namely, given finite sets $I_1, \ldots, I_k$, letting $I = I_1 \sqcup \ldots \sqcup I_k$, one has an isomorphism

$$\mathcal{H}\mathrm{ck}^I_G \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_k} \cong \prod_{j=1}^k \mathcal{H}\mathrm{ck}^{I_j}_G \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_k}$$

where $(\mathrm{Div}^1)^{I;I_1,\ldots,I_k} \subset (\mathrm{Div}^1)^I$ is the open subset where $x_i \neq x_{i'}$ whenever $i, i' \in I$ lie in different I_j's. The exterior tensor product then defines a functor

$$\boxtimes^k_{j=1} : \prod_{j=1}^k \mathrm{Sat}^{I_j}_G(\Lambda) \to \mathrm{Sat}^{I;I_1,\ldots,I_k}_G(\Lambda)$$

where $\mathrm{Sat}^{I;I_1,\ldots,I_k}_G(\Lambda)$ is the variant of $\mathrm{Sat}^I_G(\Lambda)$ for $\mathcal{H}\mathrm{ck}^I_G \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_k}$. However, the restriction functor

$$\mathrm{Sat}^I_G(\Lambda) \to \mathrm{Sat}^{I;I_1,\ldots,I_k}_G(\Lambda)$$

is fully faithful, and the essential image of the exterior product lands in its essential image. Thus, we get a natural functor

$$*^k_{j=1} : \prod_{j=1}^k \mathrm{Sat}^{I_j}_G(\Lambda) \to \mathrm{Sat}^I_G(\Lambda),$$

independent of the ordering of the I_j. In particular, for any I, we get a functor

$$\mathrm{Sat}^I_G(\Lambda) \times \mathrm{Sat}^I_G(\Lambda) \to \mathrm{Sat}^{I \sqcup I}_G(\Lambda) \to \mathrm{Sat}^I_G(\Lambda),$$

using functoriality of $\mathrm{Sat}^J_G(\Lambda)$ in J, which defines a symmetric monoidal structure $*$ on $\mathrm{Sat}^I_G(\Lambda)$, commuting with $\star$. This is called the fusion product. In general, for any symmetric monoidal category $(\mathcal{C}, *)$ with a commuting monoidal structure $\star$, the monoidal structure $\star$ necessarily agrees with $*$; thus, the fusion product refines the convolution product. (As usual in geometric Satake, we actually need to change $*$ slightly by introducing certain signs into the commutativity constraint, depending on the parity of the support of the perverse sheaves.)

Moreover, restricting $A \in \mathrm{Sat}^I_G(\Lambda)$ to Gr^I_G and taking the pushforward to $(\mathrm{Div}^1)^I$, all cohomology sheaves are local systems of Λ-modules on $(\mathrm{Div}^1)^I$. By a version of Drinfeld's lemma, these are equivalent to representations of W^I_E on Λ-modules. This defines a symmetric monoidal fibre functor

$$F^I : \mathrm{Sat}^I_G(\Lambda) \to \mathrm{Rep}_{W^I_E}(\Lambda),$$

where $\mathrm{Rep}_{W_E^I}(\Lambda)$ is the category of continuous representations of W_E^I on finite projective Λ-modules. Using a version of Tannaka duality, one can then build a Hopf algebra in the Ind-category of $\mathrm{Rep}_{W_E^I}(\Lambda)$ so that $\mathrm{Sat}_G^I(\Lambda)$ is given by its category of representations (internal in $\mathrm{Rep}_{W_E^I}(\Lambda)$). For any finite set I, this is given by the tensor product of I copies of the corresponding Hopf algebra for $I = \{*\}$, which in turn is given by some affine group scheme $\breve{G}$ over Λ with W_E-action.

Theorem VI.0.2 (Theorem VI.11.1). *There is a canonical isomorphism $\breve{G} \cong \widehat{G}$ with the Langlands dual group, under which the action of W_E on $\breve{G}$ agrees with the usual action of W_E on $\widehat{G}$ up to an explicit cyclotomic twist. If $\sqrt{q} \in \Lambda$, the cyclotomic twist can be trivialized, and $\mathrm{Sat}_G^I(\Lambda)$ is naturally equivalent to the category of $(\widehat{G} \rtimes W_E)^I$-representations on finite projective Λ-modules.*

For the proof, one can restrict to $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$; passing to a limit over n, one can actually build a group scheme over $\mathbb{Z}_\ell$. Its generic fibre is reductive, as the Satake category with $\mathbb{Q}_\ell$-coefficients is (geometrically) semisimple: For this, we use the degeneration to the Witt vector affine Grassmannian and the decomposition theorem for schemes. To identify the reductive group, we argue first for tori, and then for rank 1 groups, where everything reduces to $G = \mathrm{PGL}_2$ which is easy to analyze by using the minuscule Schubert cell. Here, the pinning includes a cyclotomic twist as of course the cohomology of the minuscule Schubert variety $\mathbb{P}^1$ of $\mathrm{Gr}_{\mathrm{PGL}_2}$ contains a cyclotomic twist. Afterwards, we apply hyperbolic localization in order to construct symmetric monoidal functors $\mathrm{Sat}_G \to \mathrm{Sat}_M$ for any Levi M of G, inducing dually maps $\breve{M} \to \breve{G}$. This produces many Levi subgroups of $\breve{G}_{\mathbb{Q}_\ell}$ from which it is easy to get the isomorphism with $\widehat{G}_{\mathbb{Q}_\ell}$, including a pinning. As these maps $\breve{M} \to \breve{G}$ are even defined integrally, and $\widehat{G}(\mathbb{Z}_\ell) \subset \widehat{G}(\mathbb{Q}_\ell)$ is a maximal compact open subgroup by Bruhat–Tits theory, generated by the rank 1 Levi subgroups, one can then deduce that $\breve{G} \cong \widehat{G}$ integrally, again with an explicit (cyclotomic) pinning.

We will also need the following addendum regarding a natural involution. Namely, the local Hecke stack $\mathcal{H}\mathrm{ck}_G^I$ has a natural involution sw given by reversing the roles of the two G-torsors; in the presentation in terms of LG, this is induced by the inversion on LG. Then sw^* induces naturally an involution of $\mathrm{Sat}_G^I(\Lambda)$, and thus involution can be upgraded to a symmetric monoidal functor commuting with the fibre functor, thus realizing a W_E-equivariant automorphism of $\check{G} \cong \widehat{G}$.

Proposition VI.0.3 (Proposition VI.12.1). *The action of* sw^* *on* Sat_G^I *induces the automorphism of $\widehat{G}$ that is the Cartan involution of the split group $\widehat{G}$, conjugated by $\widehat{\rho}(-1)$.*

VI.1. The Beilinson–Drinfeld Grassmannian

First, we define the base space of the Beilinson–Drinfeld Grassmannian for any $d \geq 0$.

Definition VI.1.1. *For any $d \geq 0$, consider the small v-sheaves on* $\mathrm{Perf}_{\mathbb{F}_q}$ *given by*

$$\mathrm{Div}^d_{\mathcal{Y}} = (\mathrm{Spd}\,\mathcal{O}_E)^d/\Sigma_d,\ \mathrm{Div}^d_Y = (\mathrm{Spd}\,E)^d/\Sigma_d,\ \mathrm{Div}^d_X = \mathrm{Div}^d = (\mathrm{Spd}\,E/\varphi^{\mathbb{Z}})^d/\Sigma_d,$$

where Σ_d is the symmetric group.

Proposition VI.1.2. *For any $d \geq 0$, there is a functorial injection*

(i) *from* $\mathrm{Div}^d_{\mathcal{Y}}(S)$ *into the set of closed Cartier divisors on $\mathcal{Y}_S$,*

(ii) *from* $\mathrm{Div}^d_Y(S)$ *into the set of closed Cartier divisors on Y_S,*

(iii) *from* $\mathrm{Div}^d_X(S)$ *into the set of closed Cartier divisors on* X_S.

Moreover, in case (i) and (ii), if $S = \mathrm{Spa}(R, R^+)$ *is affinoid perfectoid, then for any closed Cartier divisor* $D \subset \mathcal{Y}_S$ *resp.* $D \subset Y_S$ *in the image of this embedding, the adic space* $D = \mathrm{Spa}(Q, Q^+)$ *is affinoid. In case (iii), the same happens locally in the analytic topology on* S.

Proof. We handle case (i) first. Over $(\mathrm{Spd}\,\mathcal{O}_E)^d$ and for $S = \mathrm{Spa}(R, R^+)$ affinoid, we get d untilts $R_i^\sharp$, $i = 1, \dots, d$ of R, and there are elements $\xi_i \in W_{\mathcal{O}_E}(R^+)$ generating the kernel of $\theta_i : W_{\mathcal{O}_E}(R^+) \to R_i^{\sharp +}$. Each of the ξ_i defines a closed Cartier divisor by Proposition II.1.4. Then $\xi = \prod_i \xi_i$ defines another closed Cartier divisor, given by $\mathrm{Spa}(A, A^+)$ for $A = W_{\mathcal{O}_E}(R^+)[\frac{1}{[\varpi]}]/\xi$, and A^+ the integral closure of $W_{\mathcal{O}_E}(R^+)/\xi$, where $\varpi \in R$ is a pseudouniformizer.

Now the ideal sheaf of this closed Cartier divisor is a line bundle, and by [**SW20**, Proposition 19.5.3], line bundles on $\mathcal{Y}_S$ satisfy v-descent. Thus, even if we are only given a map $S \to \mathrm{Div}^d_{\mathcal{Y}} = (\mathrm{Spd}\,\mathcal{O}_E)^d/\Sigma_d$, we can still define a line bundle $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}_S}$, and it still defines a closed Cartier divisor as this can be checked v-locally. Also, $V(\mathcal{I}) \subset \mathcal{Y}_S$ is quasicompact over S, as this can again be checked v-locally. This implies that it is contained in some affinoid $\mathcal{Y}_{S,[0,n]}$, and hence $D = \mathrm{Spa}(A, A^+)$ is affinoid in general.

The case (ii) now follows formally by passing to an open subset, and case (iii) by passing to the quotient under Frobenius. □

Remark VI.1.3. As in [**Far18a**] one checks that $\mathrm{Div}^d_{\mathcal{Y}}(S)$ is the set of "relative Cartier divisors" of degree d, that is to say Cartier divisors that give degree d Cartier divisors when pulled back via any geometric point $\mathrm{Spa}(C, C^+) \to S$. The same holds for Div^d_Y and Div^d_X.

In the following we will consider a perfectoid space S equipped with a map $f : S \to \mathrm{Div}^d_{\mathcal{Y}}$ (resp. $f : S \to \mathrm{Div}^d_Y$, resp. $f : S \to \mathrm{Div}^d_X$). We denote by $D_S \subset \mathcal{Y}_S$ (resp. $D_S \subset Y_S$, $D_S \subset X_S$) the corresponding closed Cartier divisor. Let $\mathcal{I}_S \subset \mathcal{O}_{\mathcal{Y}_S}$ (resp. $\mathcal{I}_S \subset \mathcal{O}_{Y_S}$, $\mathcal{I}_S \subset \mathcal{O}_{X_S}$) be the corresponding invertible ideal sheaf.

Let us note the following descent result for vector bundles.

Proposition VI.1.4. *Sending* S *as above to the category of vector bundles on* D_S *defines a v-stack.*

Proof. Any vector bundle on D_S defines a v-sheaf on Perf_S: This reduces formally to the structure sheaf of D_S, which then further reduces to the structure of $\mathcal{O}_{\mathcal{Y}_S}$ (resp. $\mathcal{O}_{Y_S}$, resp. $\mathcal{O}_{X_S}$). It remains to prove that v-descent of vector bundles is effective. The case of X_S reduces to Y_S as locally on S, the relevant D_S is isomorphic; and clearly Y_S reduces to $\mathcal{Y}_S$.

Now assume first that $S = \mathrm{Spa}(C, C^+)$ for some complete algebraically closed C. Then D_S is given by a finite sum of degree 1 Cartier divisors on $\mathcal{Y}_S$, and one can reduce by induction to the case of degree 1 Cartier divisors, where the result is [**SW20**, Lemma 17.1.8] applied to the corresponding untilt of S.

On the other hand, assume that $T \to S$ is an étale cover with a vector bundle $\mathcal{E}_T$ on D_T equipped with a descent datum to D_S; we want to prove descent to D_S. By the argument of de Jong–van der Put [**dJvdP96**, Proposition 3.2.2], cf. [**KL15**, Proposition 8.2.20], one can reduce to

the case that $T \to S$ is a finite étale cover. Then $D_T \to D_S$ is also finite étale (as $\mathcal{Y}_T \to \mathcal{Y}_S$ is), and the result follows from usual finite étale descent.

Now let S be general and $T \to S$ a v-cover with a vector bundle $\mathcal{E}_T$ on D_T equipped with a descent datum to D_S. For any geometric point $\mathrm{Spa}(C, C^+) \to S$, one can descent $\mathcal{E}_{T \times_S \mathrm{Spa}(C,C^+)}$ to a vector bundle $\mathcal{E}_{\mathrm{Spa}(C,C^+)}$ on $D_{\mathrm{Spa}(C,C^+)}$. Now we follow the proof of [**SW20**, Lemma 17.1.8] to see that one can in fact descend $\mathcal{E}_T$ in an étale neighborhood of $\mathrm{Spa}(C, C^+)$, which is enough by the previous paragraph. We can assume that S and T are affinoid. As $\mathcal{E}_{\mathrm{Spa}(C,C^+)}$ is necessarily free, also $\mathcal{E}_{T \times_S \mathrm{Spa}(C,C^+)}$ is free, and by an étale localization we can assume that $\mathcal{E}_T$ is free. Then the descent datum is given by some matrix with coefficients in $\mathcal{O}_{D_{T \times_S T}}$. Moreover, by approximating the basis coming via pullback from $\mathcal{E}_{\mathrm{Spa}(C,C^+)}$, we can ensure that this matrix has coefficients in $1 + [\varpi]\mathcal{O}^+_{\mathcal{Y}_{T \times_S T}}(\mathcal{Y}_{T \times_S T,[0,n]})/\xi$ for some n so that $D_S \subset \mathcal{Y}_{S,[0,n]}$; here ξ is a generator of $\mathcal{I}_S$. Now one uses that the v-cohomology group $H^1_v(S, \mathcal{O}(\mathcal{Y}_{[0,n]})^+)$ is almost zero, as follows from almost vanishing in the perfectoid case, and writing it as a direct summand of the positive structure sheaf of the base change to $\mathcal{O}_E[\pi^{1/p^\infty}]^\wedge$. Then the argument from [**SW20**, Lemma 17.1.8] applies, showing that one can successively improve the basis to produce a basis invariant under the descent datum in the limit. $\square$

Assuming that D_S is affinoid, as is the case locally on S, we let

$$B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S) \text{ (resp. } B^+_{\mathrm{Div}^d_Y}(S), \text{ resp. } B^+_{\mathrm{Div}^d_X}(S))$$

be (the global sections of) the completion of $\mathcal{O}_{\mathcal{Y}_S}$ along $\mathcal{I}_S$ (resp. of $\mathcal{O}_{Y_S}$ along $\mathcal{I}_S$, resp. of $\mathcal{O}_{X_S}$ along $\mathcal{I}_S$), and

$$B_{\mathrm{Div}^d_{(-)}}(S) = B^+_{\mathrm{Div}^d_{(-)}}(S)[\tfrac{1}{\mathcal{I}_S}].$$

This defines v-sheaves $B^+_{\mathrm{Div}^d_{(-)}} \subset B_{\mathrm{Div}^d_{(-)}}$ over $\mathrm{Div}^d_{(-)}$ in all three cases. In the case of $d = 1$, those rings are the ones that are usually denoted B^+_{dR}, resp. B_{dR}.

Definition VI.1.5. *Let Z be an affine scheme over $\mathcal{O}_E$. The positive loop space $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} Z$ (resp. loop space $L_{\mathrm{Div}^d_{\mathcal{Y}}} Z$) of Z is the v-sheaf over $\mathrm{Div}^d_{\mathcal{Y}}$ given by*

$$S \mapsto L^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S) = Z(B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)) \text{ (resp. } S \mapsto L_{\mathrm{Div}^d_{\mathcal{Y}}}(S) = Z(B_{\mathrm{Div}^d_{\mathcal{Y}}}(S))).$$

Similarly, if Z is an affine scheme over E, one defines the positive loop space $L^+_{\mathrm{Div}^d_Y} Z$ and $L^+_{\mathrm{Div}^d_X} Z$ (resp. loop space $L_{\mathrm{Div}^d_Y} Z$ and $L_{\mathrm{Div}^d_X} Z$).

We note that we use affinity of Z to see that these are actually v-sheaves. Now we can define the local Hecke stacks.

Definition VI.1.6. *Let G be a reductive group over $\mathcal{O}_E$ (resp. over E, resp. over E). The local Hecke stack $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ (resp. $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_Y}$, resp. $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_X}$) is the functor sending an affinoid perfectoid $S \to \mathrm{Div}^d_{\mathcal{Y}}$ (resp. $S \to \mathrm{Div}^d_Y$, resp. $S \to \mathrm{Div}^d_X$, assuming that D_S is affinoid) to the groupoid of pairs of G-bundles $\mathcal{E}_1, \mathcal{E}_2$ over $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ (resp. over $B^+_{\mathrm{Div}^d_Y}(S)$, resp. over $B^+_{\mathrm{Div}^d_X}(S)$) together with an isomorphism $\mathcal{E}_1 \cong \mathcal{E}_2$ over $B_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ (resp. over $B_{\mathrm{Div}^d_Y}(S)$, resp. over $B_{\mathrm{Div}^d_X}(S)$).*

PROPOSITION VI.1.7. *The local Hecke stack* $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ *(resp.* $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_Y}$*, resp.* $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_X}$*) is a small v-stack. There is a natural isomorphism of étale stacks over* $\mathrm{Div}^d_{\mathcal{Y}}$ *(resp. over* Div^d_Y*, resp. over* Div^d_X*)*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \cong (L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)\backslash(L_{\mathrm{Div}^d_{\mathcal{Y}}}G)/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)$$

(resp.

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_Y} \cong (L^+_{\mathrm{Div}^d_Y}G)\backslash(L_{\mathrm{Div}^d_Y}G)/(L^+_{\mathrm{Div}^d_Y}G),$$

resp.

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_X} \cong (L^+_{\mathrm{Div}^d_X}G)\backslash(L_{\mathrm{Div}^d_X}G)/(L^+_{\mathrm{Div}^d_X}G).)$$

PROOF. The category of vector bundles over $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}$ (resp. $B^+_{\mathrm{Div}^d_Y}$, $B^+_{\mathrm{Div}^d_X}$) satisfies v-descent: It is enough to check this modulo powers of the ideal $\mathcal{I}_S$, where the result is Proposition VI.1.4. By the Tannakian formalism, it follows that the category of G-bundles also satisfies v-descent, so one can descend $\mathcal{E}_1, \mathcal{E}_2$. The isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ over $B_{\mathrm{Div}^d_{\mathcal{Y}_S}}$ (resp. over $B_{\mathrm{Div}^d_Y}(S)$, resp. over $B_{\mathrm{Div}^d_X}(S)$) is then given by a section of an affine scheme over the respective ring, which again satisfies v-descent. Smallness follows from the argument of Proposition III.1.3.

Any G-bundle over $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ is étale locally on S trivial. Indeed, if S is a geometric point then $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ is a product of complete discrete valuation rings with algebraically closed residue field, so that all G-torsors are trivial. In general, note that triviality of the G-torsor over $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ is implied by triviality modulo $\mathcal{I}_S$ (as one can always lift sections over nilpotent thickenings). Then the result follows from [**GR03**, Proposition 5.4.21]. Trivializing $\mathcal{E}_1$ and $\mathcal{E}_2$ étale locally then directly produces the given presentations. □

Similarly, one can define the Beilinson–Drinfeld Grassmannians.

DEFINITION VI.1.8. *Let G be a reductive group over $\mathcal{O}_E$ (resp. over E, resp. over E). The Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ *(resp.* $\mathrm{Gr}_{G,\mathrm{Div}^d_Y}$*, resp.* $\mathrm{Gr}_{G,\mathrm{Div}^d_X}$*) is the functor sending an affinoid perfectoid* $S \to \mathrm{Div}^d_{\mathcal{Y}}$ *(resp.* $S \to \mathrm{Div}^d_Y$*, resp.* $S \to \mathrm{Div}^d_X$*, assuming again that D_S is affinoid) to the groupoid of G-bundles $\mathcal{E}$ over* $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ *(resp. over* $B^+_{\mathrm{Div}^d_Y}(S)$*, resp. over* $B^+_{\mathrm{Div}^d_X}(S)$*) together with a trivialization of $\mathcal{E}$ over* $B_{\mathrm{Div}^d_{\mathcal{Y}}}(S)$ *(resp. over* $B_{\mathrm{Div}^d_Y}(S)$*, resp. over* $B_{\mathrm{Div}^d_X}(S)$*).*

PROPOSITION VI.1.9. *The Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ *(resp.* $\mathrm{Gr}_{G,\mathrm{Div}^d_Y}$*, resp.* $\mathrm{Gr}_{G,\mathrm{Div}^d_X}$*) is a small v-sheaf. There is a natural isomorphism of étale sheaves over* $\mathrm{Div}^d_{\mathcal{Y}}$ *(resp. over* Div^d_Y*, resp. over* Div^d_X*)*

$$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \cong (L_{\mathrm{Div}^d_{\mathcal{Y}}}G)/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)$$

(resp.

$$\mathrm{Gr}_{G,\mathrm{Div}^d_Y} \cong (L_{\mathrm{Div}^d_Y}G)/(L^+_{\mathrm{Div}^d_Y}G),$$

resp.

$$\mathrm{Gr}_{G,\mathrm{Div}^d_X} \cong (L_{\mathrm{Div}^d_X}G)/(L^+_{\mathrm{Div}^d_X}G).)$$

PROOF. The proof is identical to the proof of Proposition VI.1.7. □

The positive loop group $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G$ admits the natural filtration by closed subgroups

$$(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m} \subset L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G$$

defined for all $m \geq 1$ by the kernel of

$$G(B^+_{\mathrm{Div}^d_{\mathcal{Y}}}) \to G(B^+_{\mathrm{Div}^1_{\mathcal{Y}}}/\mathcal{I}_S^m);$$

we refer to these as the principal congruence subgroups of $L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G$. Similar definitions of course apply also over Div^d_Y and Div^d_X. For $d = 1$, one can easily describe the graded pieces of this filtration (and again the result also holds for Div^1_Y and Div^1_X).

Proposition VI.1.10. *There are natural isomorphisms*

$$L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G/(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq 1} \cong G^\diamond$$

and

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m}/(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m+1} \cong (\mathrm{Lie}\, G)^\diamond\{m\}$$

where $\{m\}$ signifies a "Breuil-Kisin twist" by $\mathcal{I}_S^m/\mathcal{I}_S^{m+1}$.

Proof. The first equality follows directly from the definitions, while the second comes from the exponential. □

For general d, we still have the following result.

Proposition VI.1.11. *For any d and $m \geq 1$, the quotient*

$$(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m}/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m+1}$$

sends a perfectoid space $S \to \mathrm{Div}^d_{\mathcal{Y}}$ with corresponding Cartier divisor $D_S \subset Y_S$ with ideal sheaf $\mathcal{I}_S$ to

$$(\mathrm{Lie}\, G \otimes_{\mathcal{O}_E} \mathcal{I}_S^m/\mathcal{I}_S^{m+1})(S)$$

where $\mathcal{I}_S^m/\mathcal{I}_S^{m+1}$ is a line bundle on D_S. This is representable in locally spatial diamonds, partially proper, and cohomologically smooth of ℓ-dimension equal to d times the dimension of G.

Moreover, one can filter

$$(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m}/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m+1} \times_{\mathrm{Div}^d_{\mathcal{Y}}} (\mathrm{Div}^1_{\mathcal{Y}})^d$$

with subquotients given by twists of

$$(\mathrm{Lie}\, G)^\diamond \times_{\mathrm{Div}^1_{\mathcal{Y}}, \pi_i} (\mathrm{Div}^1_{\mathcal{Y}})^d$$

where $\pi_i : (\mathrm{Div}^1_{\mathcal{Y}})^d \to \mathrm{Div}^1_{\mathcal{Y}}$ is the projection to the i-th factor.

Proof. The description of the subquotient follows from the exponential sequence again. To check that it is representable in locally spatial diamonds, partially proper, and cohomologically smooth, we can pull back to $(\mathrm{Div}^1_{\mathcal{Y}})^d$, and then these properties follow from the existence of the given filtration. For this in turn, note that over $(\mathrm{Div}^1_{\mathcal{Y}})^d$, we have d ideal sheaves $\mathcal{I}_1, \ldots, \mathcal{I}_d$, and one can filter $\mathcal{O}_{D_S}$ by $\mathcal{O}_{D_S}/\mathcal{I}_1$, $\mathcal{I}_1/\mathcal{I}_1\mathcal{I}_2$, …, $\mathcal{I}_1 \cdots \mathcal{I}_{d-1}/\mathcal{I}_1 \cdots \mathcal{I}_d$, each of which is, after pullback to an affinoid perfectoid space S, isomorphic to $\mathcal{O}_{S_i^\sharp}$. □

One can also show that the first quotient is cohomologically smooth, but this is slightly more subtle.

PROPOSITION VI.1.12. *For any d, the quotient*

$$L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq 1} \to \mathrm{Div}^d_{\mathcal{Y}}$$

parametrizes over a perfectoid space $S \to \mathrm{Div}^d_{\mathcal{Y}}$ maps $D_S \to G$. For any quasiprojective smooth scheme Z over $\mathcal{O}_E$, the sheaf

$$T_Z \to \mathrm{Div}^d_{\mathcal{Y}}$$

taking a perfectoid S over $\mathrm{Div}^d_{\mathcal{Y}}$ to maps $D_S \to Z$ is representable in locally spatial diamonds, partially proper, and cohomologically smooth over $\mathrm{Div}^d_{\mathcal{Y}}$ of ℓ-dimension equal to d times the dimension of Z; in particular, this applies to this quotient group.

PROOF. The description of the quotient group is clear. To analyze T_Z, we first note that if Z is an affine space, then the result holds true, as was proved in the previous proposition. In fact, after pullback via the quasi-pro-étale surjective morphism $(\mathrm{Div}^1_{\mathcal{Y}})^d \to \mathrm{Div}^d_{\mathcal{Y}}$, there is a sequence of morphisms

$$T_{\mathbb{A}^n} = W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{d+1} = (\mathrm{Div}^1_{\mathcal{Y}})^d$$

where, for S affinoid perfectoid with $S \to W_{i+1}$ giving rise to the untilts $(S_1^\sharp, \ldots, S_d^\sharp) \in (\mathrm{Div}^1_{\mathcal{Y}})^d(S)$, $W_i \times_{W_{i+1}} S \to S$ is locally on S isomorphic to $\mathbb{A}^{n,\diamond}_{S_i^\sharp}$.

If $Z' \to Z$ is any separated étale map between schemes over $\mathcal{O}_E$, we claim that $T_{Z'} \to T_Z$ is also separated étale. For this, we analyze the pullback along any $S \to T_Z$ given by some perfectoid space S and a map $D_S \to Z$. Then $D' = D_S \times_Z Z' \to D_S$ is separated étale, and the fibre product $T_{Z'} \times_{T_Z} S$ parametrizes $S' \to S$ with a lift $D_{S'} \to D'$ over D_S. By Lemma VI.1.13 below, this is representable by a perfectoid space separated étale over S. In case $Z' \to Z$ is an open immersion, it follows that $T_{Z'} \to T_Z$ is injective and étale, thus an open immersion.

Now note that for any geometric point of T_Z, the corresponding map $D_S \to Z$ has finite image, and is thus contained in some open affine subscheme. It follows that T_Z admits an open cover by $T_{Z'}$ for affine Z'. If Z is affine, then one sees directly that T_Z is representable in locally spatial diamonds and partially proper by taking a closed immersion into $\mathbb{A}^n_{\mathcal{O}_E}$ for some n. For cohomological smoothness, we observe that we can in fact choose these affines so that they admit étale maps to $\mathbb{A}^d_{\mathcal{O}_E}$, as again we only need to arrange this at finitely many points at a time. Now the result follows from the discussion of $\mathbb{A}^d_{\mathcal{O}_E}$ and of separated étale maps. □

LEMMA VI.1.13. *Let S be a perfectoid space with a map $S \to \mathrm{Div}^d_{\mathcal{Y}}$ giving rise to the Cartier divisor $D_S \subset \mathcal{Y}_S$. Let $D' \to D_S$ be a separated étale map. Then there is a separated étale map $S' \to S$ such that for $T \to S$, maps $T \to S'$ over S are equivalent to lifts $D_T \to D'$ over D_S.*

PROOF. By descent of separated étale maps [**Sch17a**, Proposition 9.7], we can assume that S is strictly totally disconnected. Exhausting D' by a rising union of quasicompact subspaces, we can assume that D' is quasicompact. In any geometric fibre, D' is then a disjoint union of open subsets (as any geometric fibre is, up to nilpotents, a finite disjoint union of untilts $\mathrm{Spa}(C_i^\sharp, C_i^{\sharp+})$), and this description spreads into a small neighborhood by [**Sch17a**, Proposition 11.23, Lemma 15.6]. We

can thus reduce to the case that $D' \to D_S$ is an open immersion. Now the lemma follows from the observation that the map $|D_S| \to |S|$ is closed. $\square$

VI.2. Schubert varieties

Now we recall the Schubert varieties. Assume in this section that G is a split reductive group over $\mathcal{O}_E$ (or over E, but in that case we can always choose a model over $\mathcal{O}_E$). Fix a split torus and Borel $T \subset B \subset G$. Note that we can always pass to the situation of split G by making a finite étale extension of $\mathcal{O}_E$ resp. E; this way, the results of this section are useful in the general case. Similarly, the cases of X_S and Y_S reduce easily to the case of $\mathcal{Y}_S$, so we only do the latter case explicitly here.

Assume first that $d = 1$. In that case, for every geometric point $S = \mathrm{Spa}(C, C^+) \to \mathrm{Div}^1_{\mathcal{Y}} = \mathrm{Spd}\,\mathcal{O}_E$ given by an untilt $S^\sharp = \mathrm{Spa}(C^\sharp, C^{\sharp +})$ of S, one has $B^+_{\mathrm{Div}^1_{\mathcal{Y}}} = B^+_{\mathrm{dR}}(C^\sharp)$ and $B_{\mathrm{Div}^1_{\mathcal{Y}}} = B_{\mathrm{dR}}(C^\sharp)$ for the usual definition of B^+_{dR} and B_{dR} (relative to $\mathcal{O}_E$). Recall that $B^+_{\mathrm{dR}}(C^\sharp)$ is a complete discrete valuation ring with residue field $C^\sharp$, fraction field $B_{\mathrm{dR}}(C^\sharp)$, and uniformizer ξ. It follows that by the Cartan decomposition

$$G(B_{\mathrm{dR}}(C^\sharp)) = \bigsqcup_{\mu \in X_*(T)^+} G(B^+_{\mathrm{dR}}(C^\sharp))\mu(\xi)G(B^+_{\mathrm{dR}}(C^\sharp)),$$

so as a set

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}(S)/\cong\ = X_*(T)^+,$$

the dominant cocharacters of T. Recall that on $X_*(T)^+$, we have the dominance order, where $\mu \geq \mu'$ if $\mu - \mu'$ is a sum of positive coroots with $\mathbb{Z}_{\geq 0}$-coefficients.

Remark VI.2.1. Since we work over $\mathcal{Y}$ and do not restrict ourselves to Y, we include the case of the Cartier divisor $\pi = 0$. For this divisor, $C^\sharp = C$ and $B^+_{\mathrm{dR}}(C^\sharp) = W_{\mathcal{O}_E}(C)$.

Definition VI.2.2. *For any $\mu \in X_*(T)^+$, let*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

be the subfunctor of all those maps $S \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$ such that at all geometric points $S' = \mathrm{Spa}(C, C^+) \to S$, the corresponding S'-valued point is given by some $\mu' \in X_*(T)^+$ with $\mu' \leq \mu$. Moreover,

$$\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \subset \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

is the preimage of $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$.

Recall the following result.

Proposition VI.2.3 ([**SW20**, Proposition 20.3.6]). *The inclusion*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

is a closed subfunctor and

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}} = \varinjlim_{\mu} \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu};$$

thus, similar properties hold for $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$. *Here, the index category is the partially ordered set of* μ*'s under the dominance order, which is a disjoint union (over* $\pi_1(G)$*) of filtered partially ordered sets.*

The map $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \to \mathrm{Div}^1_{\mathcal{Y}}$ *is proper and representable in spatial diamonds.*

Proof. It is enough to prove the assertions over $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$ as this is a v-cover of $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$. Then [**SW20**, Proposition 20.3.6] gives the results, except for the assertion that

$$\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} = \varinjlim_{\mu} \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu}.$$

For this, note that the map from right to left is clearly an injection. For surjectivity, note that for any quasicompact S with a map $S \to \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$, only finitely many strata can be met, as the meromorphic isomorphism of G-bundles necessarily has bounded poles. This, coupled with the fact $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^1_{\mathcal{Y}}$ is separated while $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \to \mathrm{Div}^1_{\mathcal{Y}}$ is proper, implies that the map $\bigsqcup_\mu \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \to \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$ is a v-cover, whence we get the desired surjectivity. □

In particular,

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu} \setminus \bigcup_{\mu'<\mu} \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu'} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu}$$

is an open subfunctor, and similarly its preimage $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \subset \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\leq\mu}$. By the Cartan decomposition, the space $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu}$ has only one point in every geometric fibre over $\mathrm{Div}^1_{\mathcal{Y}}$. This point can in fact be defined as a global section

$$[\mu] : \mathrm{Div}^1_{\mathcal{Y}} \to \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu}$$

given by $\mu(\xi) \in (L_{\mathrm{Div}^d_{\mathcal{Y}}} G)(S)$ whenever ξ is a local generator of $\mathcal{I}_S$; up to the action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G$, this is independent of the choice of ξ.

Proposition VI.2.4. *The map*

$$[\mu] : \mathrm{Div}^1_{\mathcal{Y}} \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu}$$

given by μ *is a v-cover. This gives an isomorphism*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \cong [\mathrm{Div}^1_{\mathcal{Y}}/(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu]$$

where $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu \subset L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G$ *is the closed subgroup stabilizing* $[\mu] \in \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}(S)$. *Recalling the principal congruence subgroups*

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m} \subset L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G,$$

we let

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m}_\mu = (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu \cap (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m} \subset (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu.$$

Then

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu/(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq 1}_\mu \cong (P^-_\mu)^\diamond \subset L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G/(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq 1} \cong G^\diamond$$

and

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m}_{\mu} / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m+1}_{\mu} \cong (\mathrm{Lie}\, G)^{\diamond}_{\mu \leq m}\{m\} \subset (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m} / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m+1} \cong (\mathrm{Lie}\, G)^{\diamond}\{m\},$$

where $P^-_\mu \subset G$ is the parabolic with Lie algebra $(\mathrm{Lie}\, G)_{\mu \leq 0}$, and $(\mathrm{Lie}\, G)_{\mu \leq m} \subset \mathrm{Lie}\, G$ is the subspace on which μ acts via weights $\leq m$ via the adjoint action.

In particular,

$$\mathrm{Gr}_{G, \mathrm{Div}^1_{\mathcal{Y}}, \mu} \cong L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu$$

is cohomologically smooth of ℓ-dimension $\langle 2\rho, \mu \rangle$ over $\mathrm{Div}^1_{\mathcal{Y}}$.

Proof. We first handle the case $G = \mathrm{GL}_n$ with its standard upper-triangular Borel and diagonal torus. In that case, μ is given by some sequence $k_1 \geq \ldots \geq k_n$ of integers, and $\mathrm{Gr}_{G, \mathrm{Div}^1_{\mathcal{Y}}, \mu}$ parametrizes $B^+_{\mathrm{Div}^1_{\mathcal{Y}}}$-lattices

$$\Xi \subset B^n_{\mathrm{Div}^1_{\mathcal{Y}}}$$

that are of relative position μ at all points. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space with a map $S \to \mathrm{Div}^1_{\mathcal{Y}} = \mathrm{Spd}\, \mathcal{O}_E$ given by an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$ over $\mathcal{O}_E$ of S. By the proof of [**SW20**, Proposition 19.4.2], the $R^\sharp$-modules

$$\mathrm{Fil}^i_\Xi (R^\sharp)^n = (\xi^i \Xi \cap B^+_{\mathrm{dR}}(R^\sharp)^n) / (\xi^i \Xi \cap \xi B^+_{\mathrm{dR}}(R^\sharp)^n)$$

are finite projective of rank equal to the number of occurrences of $-i$ among $k_1, \ldots, k_n$. Localizing, we may assume that they are finite free. We may then pick a basis $e_1, \ldots, e_n$ of $(R^\sharp)^n$ so that any $\mathrm{Fil}^i_\Xi (R^\sharp)^n$ is freely generated by a subset $e_1, \ldots, e_{n_i}$ of $e_1, \ldots, e_n$. Lifting $e_{n_{i-1}+1}, \ldots, e_{n_i}$ to elements of $f_{n_{i-1}+1}, \ldots, f_{n_i} \in \xi^i \Xi \cap B^+_{\mathrm{dR}}(R^\sharp)^n$, and setting $g_{n_{i-1}+1} = \xi^{-i} f_{n_{i-1}+1}, \ldots, g_{n_i} = \xi^{-i} f_{n_i}$, or equivalently $g_j = \xi^{k_j} f_j$ for $j = 1, \ldots, n$, one sees that $f_1, \ldots, f_n$ form a $B^+_{\mathrm{dR}}(R^\sharp)$-basis of $B^+_{\mathrm{dR}}(R^\sharp)^n$, and $g_1, \ldots, g_n$ will form a $B^+_{\mathrm{dR}}(R^\sharp)$-basis of Ξ. Thus, changing basis to the f_i's, one has moved Ξ to the lattice

$$\xi^{k_1} B^+_{\mathrm{dR}}(R^\sharp) \oplus \ldots \oplus \xi^{k_n} B^+_{\mathrm{dR}}(R^\sharp).$$

This is the lattice corresponding to $[\mu] \in \mathrm{Gr}_{\mathrm{GL}_n, \mathrm{Div}^1_{\mathcal{Y}}}$, showing that the map

$$\mathrm{Div}^1_{\mathcal{Y}} \to \mathcal{H}\mathrm{ck}_{\mathrm{GL}_n, \mathrm{Div}^1_{\mathcal{Y}}, \mu}$$

is indeed surjective.

Moreover, the stabilizer $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)_\mu$ of $\xi^{k_1} B^+_{\mathrm{dR}}(R^\sharp) \oplus \ldots \oplus \xi^{k_n} B^+_{\mathrm{dR}}(R^\sharp)$ in $L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n$ is the set of all matrices $A = (A_{ij}) \in \mathrm{GL}_n(B^+_{\mathrm{Div}^1_{\mathcal{Y}}})$ such that for $i < j$, $A_{ij} \in \xi^{k_i - k_j} B^+_{\mathrm{Div}^1_{\mathcal{Y}}}$. This easily implies the description of

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)_\mu / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)^{\geq 1}_\mu \cong (P^-_\mu)^\diamond \subset \mathrm{GL}_n^\diamond$$

and

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)^{\geq m}_\mu / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)^{\geq m+1}_\mu \cong (\mathrm{Lie}\, \mathrm{GL}_n)^\diamond_{\mu \leq m}\{m\} \subset (\mathrm{Lie}\, \mathrm{GL}_n)^\diamond\{m\}.$$

The description also implies that $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)_\mu$ contains $L^+_{\mathrm{Div}^1_{\mathcal{Y}}} P^-_\mu$ and $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} U_a)^{\geq \mu(a)}$ for any positive root a.

In general, picking a closed immersion of G into GL_n (compatible with the torus and the Borel), one sees that

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq 1}_\mu \subset (P^-_\mu)^\diamond \subset G^\diamond$$

and

$$(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m}_\mu / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq m+1}_\mu \subset (\mathrm{Lie}\, G)^\diamond_{\mu \leq m}\{m\} \subset (\mathrm{Lie}\, G)^\diamond\{m\}$$

as these subquotients embed into the similar subquotient for GL_n. Moreover, one sees that $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu$ contains $L^+_{\mathrm{Div}^1_{\mathcal{Y}}} P^-_\mu$ and $(L^+_{\mathrm{Div}^1_{\mathcal{Y}}} U_a)^{\geq \mu(a)}$ for any positive root a. These imply that the two displayed inclusions are actually equalities.

A consequence of these considerations is that the map

$$L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu \to L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{GL}_n)_\mu$$

is a closed immersion (as this happens on all subquotients for the principal congruence filtration). The target is isomorphic to $\mathrm{Gr}_{\mathrm{GL}_n, \mathrm{Div}^1_{\mathcal{Y}}, \mu}$, which contains $\mathrm{Gr}_{G, \mathrm{Div}^1_{\mathcal{Y}}, \mu}$ as a closed subspace (by [**SW20**, Proposition 20.3.7]). We see that we get an inclusion

$$L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu \hookrightarrow \mathrm{Gr}_{G, \mathrm{Div}^1_{\mathcal{Y}}, \mu} \subset \mathrm{Gr}_{\mathrm{GL}_n, \mathrm{Div}^1_{\mathcal{Y}}, \mu}$$

of closed subspaces, with the same geometric points: This implies that it is an isomorphism (e.g., as the map is then necessarily a closed immersion, thus qcqs, so one can apply [**Sch17a**, Lemma 12.5]). From here, all statements follow. □

Remark VI.2.5. The map

$$\mathrm{Gr}_{G, \mathrm{Div}^1_{\mathcal{Y}}, \mu} = L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)_\mu \to L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G)^{\geq 1}_\mu \cong (G/P_{-\mu})^\diamond$$

is the Białynicki-Birula map, see [**CS17**].

Passing to general d, we first note that any geometric fibre of

$$\mathcal{H}\mathrm{ck}_{G, \mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Div}^d_{\mathcal{Y}}$$

is isomorphic to a product of geometric fibres of $\mathcal{H}\mathrm{ck}_{G, \mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^1_{\mathcal{Y}}$. More precisely, if $f : \mathrm{Spa}(C, C^+) \to \mathrm{Div}^d_{\mathcal{Y}}$ is a geometric point, it is given by an unordered tuple $\mathrm{Spa}(C_i^\sharp, C_i^{\sharp +})$, $i \in I$ with $|I| = d$, of untilts over $\mathcal{O}_E$. Some of these may be equal, so one can partition I into sets $I_1, \ldots, I_r$ of equal untilts. Then we really have r untilts, given by maps $f_1, \ldots, f_r : \mathrm{Spa}(C, C^+) \to \mathrm{Div}^1_{\mathcal{Y}}$, and one has an isomorphism

$$\mathcal{H}\mathrm{ck}_{G, \mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}, f} \mathrm{Spa}(C, C^+) \cong \prod_{i=1}^r \mathcal{H}\mathrm{ck}_{G, \mathrm{Div}^1_{\mathcal{Y}}} \times_{\mathrm{Div}^1_{\mathcal{Y}}, f_i} \mathrm{Spa}(C, C^+),$$

and similarly

$$L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G \times_{\mathrm{Div}^d_{\mathcal{Y}},f} \mathrm{Spa}(C, C^+) \cong \prod_{i=1}^r L^+_{\mathrm{Div}^1_{\mathcal{Y}}} G \times_{\mathrm{Div}^1_{\mathcal{Y}},f_i} \mathrm{Spa}(C, C^+),$$

$$L_{\mathrm{Div}^d_{\mathcal{Y}}} G \times_{\mathrm{Div}^d_{\mathcal{Y}},f} \mathrm{Spa}(C, C^+) \cong \prod_{i=1}^r L_{\mathrm{Div}^1_{\mathcal{Y}}} G \times_{\mathrm{Div}^1_{\mathcal{Y}},f_i} \mathrm{Spa}(C, C^+),$$

$$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}},f} \mathrm{Spa}(C, C^+) \cong \prod_{i=1}^r \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \times_{\mathrm{Div}^1_{\mathcal{Y}},f_i} \mathrm{Spa}(C, C^+).$$

Indeed, it suffices to prove this on the level of the positive loop and loop group, where in turn it follows from a similar decomposition of $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}$ after pullback, which is clear.

In particular, we can define the following version of Schubert varieties.

Definition VI.2.6. *For any unordered collection $\mu_\bullet = (\mu_j)_{j \in J}$ of elements $\mu_j \in X_*(T)^+$ with $|J| = d$, let*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$$

be the subfunctor of all those $S \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ such that at all geometric points $\mathrm{Spa}(C, C^+) \to S$, then equipped with an (unordered) tuple of d untilts $\mathrm{Spa}(C_i^\sharp, C_i^{\sharp +})$, $i \in I$ with $|I| = d$, there is some bijection between $\psi : I \cong J$ such that the relative position of $\mathcal{E}_1$ and $\mathcal{E}_2$ at $\mathrm{Spa}(C_i^\sharp, C_i^{\sharp +})$ is bounded by

$$\sum_{j \in J, C^\sharp_{\psi(j)} \cong C_i^\sharp} \mu_j.$$

Let

$$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \subset \mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$$

be the preimage of $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$.

Proposition VI.2.7. *The inclusion*

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$$

is a closed subfunctor. The map $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \to \mathrm{Div}^d_{\mathcal{Y}}$ is proper, representable in spatial diamonds, and of finite $\dim.\mathrm{trg}$.

Proof. This can be checked after pullback to $(\mathrm{Div}^1_{\mathcal{Y}})^d$. Then it follows from [**SW20**, Proposition 20.5.4]. □

Moreover, we have the following result. Here, we let

$$(L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{<m} = L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G / (L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G)^{\geq m}$$

be the quotient by the principal congruence subgroup.

Proposition VI.2.8. *For any $\mu_\bullet = (\mu_j)_{j\in J}$ as above, the action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G$ on $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet}$ factors over $(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)^{<m}$ where m is chosen so that for $\mu = \sum_{j\in J}\mu_j$, all weights of μ on $\mathrm{Lie}\,G$ are $\leq m$.*

Proof. We need to see that the action of $(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)^{\geq m}$ is trivial. As everything is separated, this can be checked on geometric points, where one reduces to $d = 1$ by a decomposition into products. Then it follows from Proposition VI.2.4. □

VI.3. Semi-infinite orbits

For this section, we continue to assume that G is split, and again we only spell out the case of $\mathrm{Div}^d_{\mathcal{Y}}$; analogous results hold for Div^d_Y and Div^d_X, and follow easily from the case presented.

Previously, we stratified the affine Grassmanian using the Cartan decomposition, the strata being affine Schubert cells. We now use the Iwasawa decomposition to obtain another stratification by semi-infinite orbits.

Fix a cocharacter $\lambda : \mathbb{G}_m \to T \subset G$, inducing a Levi M_λ with Lie algebra $(\mathrm{Lie}\,G)_{\lambda=0}$, a parabolic $P_\lambda = P^+_\lambda$ with Lie algebra $(\mathrm{Lie}\,G)_{\lambda\geq 0}$ and its unipotent radical U_λ with Lie algebra $(\mathrm{Lie}\,G)_{\lambda>0}$. We get an action of the v-sheaf $\mathbb{G}_m$ (taking an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ of characteristic p to $R^\times$) on $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ via the composition of the Teichmüller map

$$[\cdot] : \mathbb{G}_m \to L^+_{\mathrm{Div}^d_{\mathcal{Y}}}\mathbb{G}_m,$$

the map

$$L^+_{\mathrm{Div}^d_{\mathcal{Y}}}\lambda : L^+_{\mathrm{Div}^d_{\mathcal{Y}}}\mathbb{G}_m \to L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G$$

and the action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G$ on $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$. We wish to apply Braden's theorem in this setup. For this purpose, we need to verify Hypothesis IV.6.1. To construct the required stratification, we use the affine Grassmannian

$$\mathrm{Gr}_{P_\lambda,\mathrm{Div}^d_{\mathcal{Y}}}$$

associated to the parabolic P_λ. Note that this admits a map

$$\mathrm{Gr}_{P_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Gr}_{M_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^d_{\mathcal{Y}}}$$

where M_λ is the Levi quotient of P_λ and $\overline{M}_\lambda$ is the maximal torus quotient of M_λ (the cocenter). Then $\mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^d_{\mathcal{Y}}}$ admits a surjection from a disjoint union of copies of $(\mathrm{Div}^1_{\mathcal{Y}})^d$ parametrized by $X_*(\overline{M}_\lambda)^d$. While there are many identifications between these copies, the sum $\overline{\mu} := \sum_{i=1}^d \overline{\mu}_i \in X_*(\overline{M}_\lambda)$ defines a well-defined locally constant function

$$\mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \to X_*(\overline{M}_\lambda). \tag{VI.3.1}$$

More precisely, for $\mathrm{Spa}(C, C^+) \to \mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^d_{\mathcal{Y}}}$ a geometric point, let $C_1^\sharp, \ldots, C_r^\sharp$ be the corresponding distinct untilts with $1 \leq r \leq d$. Then

$$\mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} \mathrm{Spa}(C, C^+) \cong \prod_{i=1}^r \mathrm{Gr}_{\overline{M}_\lambda,\mathrm{Div}^1_{\mathcal{Y}}} \times_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{Spa}(C, C^+)$$

where the morphism $\mathrm{Spa}(C, C^+) \to \mathrm{Div}^1_{\mathcal{Y}}$ is given by $C_i^\sharp$ on the i-th component of the product. This is identified with

$$\prod_{i=1}^r X_*(\overline{M}_\lambda) \times \mathrm{Spa}(C, C^+)$$

and the weighted sum morphism $X_*(\overline{M}_\lambda)^r \to X_*(\overline{M}_\lambda)$ (weighing each term with the multiplicity of $C_i^\sharp$ as an untilt of C in the morphism $\mathrm{Spa}(C, C^+) \to \mathrm{Div}^d_{\mathcal{Y}}$) defines thus a function

$$\mathrm{Gr}_{\overline{M}_\lambda, \mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} \mathrm{Spa}(C, C^+) \longrightarrow X_*(\overline{M}_\lambda).$$

This defines the locally constant function of (VI.3.1).

For $\nu \in X_*(\overline{M}_\lambda)$ let

$$\mathrm{Gr}^\nu_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}} \subset \mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$$

be the corresponding open and closed subset obtained as the preimage.

Proposition VI.3.1. *The map*

$$\mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}} = \bigsqcup_\nu \mathrm{Gr}^\nu_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$$

is bijective on geometric points, and it is a locally closed immersion on each $\mathrm{Gr}^\nu_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$. *The union* $\bigcup_{\nu' \leq \nu} \mathrm{Gr}^{\nu'}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$ *has closed image in* $\mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$. *The action of* $\mathbb{G}_m$ *via* $L^+\lambda$ *on* $\mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$ *extends to an action of the monoid* $\mathbb{A}^1$, *and the* $\mathbb{G}_m$*-fixed points agree with* $\mathrm{Gr}_{M_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$.

Applying this proposition also in the case of the inverse $\mathbb{G}_m$-action, and pulling back to a relative Schubert variety, verifies Hypothesis IV.6.1 in this situation.

Proof. The action of $\mathbb{G}_m$ on P_λ via conjugation extends to an action of the monoid $\mathbb{A}^1$. Applying loop spaces to this observation and the observation that the map $L_{\mathrm{Div}^d_{\mathcal{Y}}} P_\lambda \to \mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$ is equivariant for the action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} \mathbb{G}_m$ on the source via conjugation and on the target via the given action (as we quotient by the right action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} P_\lambda$) gives the action of the monoid $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} \mathbb{A}^1$, and thus of $\mathbb{A}^1$ via restricting to Teichmüller elements. As everything is separated, this also shows that $\mathbb{G}_m$-fixed points necessarily lie in the image of $L_{\mathrm{Div}^d_{\mathcal{Y}}} M_\lambda$, thus the $\mathbb{G}_m$-fixed points agree with $\mathrm{Gr}_{M_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$.

Bijectivity of the map

$$\mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$$

on geometric points follows from the Iwahori decomposition. It remains to prove that the map is a locally closed immersion on each $\mathrm{Gr}^\nu_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$, and the union over $\nu' \leq \nu$ is closed. Picking a closed embedding into GL_n, this reduces to the case $G = \mathrm{GL}_n$, and by writing any standard parabolic as an intersection of maximal parabolics, we can assume that $P_\lambda \subset \mathrm{GL}_n$ is a maximal parabolic. Passing to a higher exterior power, we can even assume that $P_\lambda \subset \mathrm{GL}_n$ is the mirabolic, fixing a one-dimensional quotient of the standard representation. In that case, $\mathrm{Gr}_{\mathrm{GL}_n, \mathrm{Div}^d_{\mathcal{Y}}}$ parametrizes finite projective $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}$-modules M with an identification $M \otimes_{B^+_{\mathrm{Div}^d_{\mathcal{Y}}}} B_{\mathrm{Div}^d_{\mathcal{Y}}} \cong B^n_{\mathrm{Div}^d_{\mathcal{Y}}}$, and $\mathrm{Gr}_{P_\lambda, \mathrm{Div}^d_{\mathcal{Y}}}$

parametrizes such M for which the image $L \subset B_{\mathrm{Div}^d_{\mathcal{Y}}}$ of M in the quotient $B^n_{\mathrm{Div}^d_{\mathcal{Y}}} \to B_{\mathrm{Div}^d_{\mathcal{Y}}}$: $(x_1, \ldots, x_n) \mapsto x_n$ is a line bundle over $B^+_{\mathrm{Div}^d_{\mathcal{Y}}}$. It also suffices to prove the result after pullback along $(\mathrm{Div}^1_{\mathcal{Y}})^d \to \mathrm{Div}^d_{\mathcal{Y}}$. Now the result follows from the next lemma. □

In the following, the "relative position" of a B^+-lattice $L \subset B$ to the standard lattice $B^+ \subset B$ refers to the image under the map

$$\mathrm{Gr}_{\mathbb{G}_m, \mathrm{Div}^d_{\mathcal{Y}}} \to X_*(\mathbb{G}_m) = \mathbb{Z}$$

defined above.

Lemma VI.3.2. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$ with untilts $S^\sharp_i = \mathrm{Spa}(R^\sharp_i, R^{\sharp +}_i)$ over $\mathcal{O}_E$ for $i = 1, \ldots, n$. Let $\xi_i \in W_{\mathcal{O}_E}(R^+)$ generate the kernel of $\theta_i : W_{\mathcal{O}_E}(R^+) \to R^{\sharp +}_i$ and let $\xi = \xi_1 \cdots \xi_n$. Let B^+ be the ξ-adic completion of $W_{\mathcal{O}_E}(R^+)[\frac{1}{[\varpi]}]$ where $\varpi \in R$ is a pseudouniformizer, and let $B = B^+[\frac{1}{\xi}]$. Finally, let*

$$L \subset B$$

be a finitely generated B^+-module that is open and bounded, i.e. there is some integer N such that $\xi^N B^+ \subset L \subset \xi^{-N} B^+ \subset B$.

For any $m \in \mathbb{Z}$, let $S_m \subset S$ be the subset of those points at which the relative position of L to $B^+ \subset B$ is given by m. Then $\bigcup_{m' \geq m} S_{m'}$ is closed, and if $S_m = S$ then the B^+-module L is a line bundle.

Proof. We can assume that $L \subset B^+$ via multiplying by a power of ξ. Let $s \in S$ be any point, corresponding to a map $\mathrm{Spa}(K(s), K(s)^+) \to S_m$. Let B^+_s be the version of B^+ constructed from $(K(s), K(s)^+)$. Then B^+_s is a finite product of discrete valuation rings, and the image L_s of $L \otimes_{B^+} B^+_s$ in B^+_s is necessarily free of rank 1. Then $s \in S_m$ if and only if the length of B^+_s / L_s as B^+_s-module is given by m. Localizing on S if necessary, we can find an element $l \in L \subset B$ whose image in L_s is a generator. In a neighborhood of s, the element l generates a submodule $L' = B^+ \cdot l \subset L$ whose relative position to B^+ is bounded above by m at all points by the next lemma, and then the relative position of $L \subset B$ is also bounded above by m at all points as $L' \subset L$. This gives the desired semicontinuity of the stratification (noting that as L is open and bounded, only finitely many values of m can appear). If $S_m = S$, then the containment $L' \subset L$ has to be an equality, and hence $L = L'$ is generated by l, so L is a line bundle. □

Lemma VI.3.3. *In the situation of the previous lemma, let $f \in B^+$ be any element, and consider the map*

$$|S| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

sending any point s of S to the length of B^+_s / f as B^+_s-module. This map is semicontinuous in the sense that for any $m \geq 0$, the locus where it is $\leq m$ is open.

Proof. For any $i = 1, \ldots, n$, one can look at the closed subspace $S_i \subset S$ where the image of f in $R^\sharp_i$ vanishes. On the open complement of all S_i, the function is identically 0. By induction, we can thus pass to a closed subspace $S_i \subset S$, where we can consider the function $f_i = \frac{f}{\xi_i}$; the length function for f is then the length function for f_i plus one. This gives the result. □

Example VI.3.4. Suppose $\lambda \in X_*(T)$ is regular dominant. Then $P_\lambda = B$. We then obtain the stratification by semi-infinite orbits

$$S_\nu = \mathrm{Gr}^\nu_{B,\mathrm{Div}^d_{\mathcal{Y}}}$$

for $\nu \in X_*(T)$. One has $S_\nu \hookrightarrow \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$, a locally closed immersion, and

$$\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} = \bigcup_{\nu \in X_*(T)} \mathrm{Gr}^\nu_{B,\mathrm{Div}^d_{\mathcal{Y}}}$$

(disjoint union) at the level of points.

We can now apply Theorem IV.6.5. Here, we also use the opposite parabolic $P_\lambda^- \subset G$. If $S \to \mathrm{Div}^d_{\mathcal{Y}}$ is a small v-stack, we denote

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = \mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$$

and similarly

$$\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} S.$$

For any $A \in D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$, we call A *bounded* if it arises via pushforward from some finite union $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq \mu_\bullet} \subset \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$. We let

$$D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}} \subset D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

be the corresponding full subcategory.

Corollary VI.3.5. *Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be any small v-stack. Consider the diagram*

$$\begin{tikzcd}
& \mathrm{Gr}_{P_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\
\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} & & \mathrm{Gr}_{M_\lambda,\mathrm{Div}^d_{\mathcal{Y}}} \\
& \mathrm{Gr}_{P_\lambda^-,\mathrm{Div}^d_{\mathcal{Y}}} \arrow[ul, "q^-"] \arrow[ur, "p^-"'] &
\end{tikzcd}$$

and denote by q_S^+ etc. the base change along $S \to \mathrm{Div}^d_{\mathcal{Y}}$. Consider the full subcategory

$$D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathbb{G}_m\text{-mon},\mathrm{bd}} \subset D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

of all $A \in D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ that are bounded and $\mathbb{G}_m$-monodromic in the sense of Definition IV.6.11.

On $D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathbb{G}_m\text{-mon},\mathrm{bd}}$, the natural map

$$R(p_S^-)_* R(q_S^-)^! \to R(p_S^+)_! (q_S^+)^*$$

is an equivalence, inducing a "constant term" functor

$$\mathrm{CT}_{P_\lambda} : D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathbb{G}_m\text{-mon},\mathrm{bd}} \to D_{\text{ét}}(\mathrm{Gr}_{M_\lambda,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

This functor commutes with any base change in S and preserves the condition of being universally locally acyclic over S (which is well-defined for bounded A).

Proof. This follows from Proposition VI.3.1 and Theorem IV.6.5, Proposition IV.6.12 and Proposition IV.6.14. □

Example VI.3.6 (Follow-up to Example VI.3.4). In the context of Example VI.3.4, suppose $d = 1$. Then, $\mathrm{Gr}_{T,S/\mathrm{Div}^1_{\mathcal{Y}}} = X_*(T) \times S$, and the corresponding semi-infinite orbits are denoted by $S_\nu \subset \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$ for $\nu \in X_*(T)$. Thus,

$$\mathrm{CT}_B(A) = \bigoplus_{\nu \in X_*(T)} R(p_\nu)_!(A|_{S_\nu})$$

with $p_\nu : S_\nu \times_{\mathrm{Div}^1_{\mathcal{Y}}} S \to S \subset \mathrm{Gr}_{T,S/\mathrm{Div}^1_{\mathcal{Y}}}$ the embedding indexed by ν.

As a final topic here, let us analyze more closely the semi-infinite orbits in the special fibre, i.e. for the Witt vector affine Grassmannian $\mathrm{Gr}_G^{\mathrm{Witt}}$ (so that $(\mathrm{Gr}_G^{\mathrm{Witt}})^\diamond \cong \mathrm{Gr}_{G,\mathrm{Spd}\,\mathbb{F}_q/\mathrm{Div}^1_{\mathcal{Y}}}$), which is an increasing union of perfections of projective varieties over $\mathbb{F}_q$ by [**BS17**], cf. also [**Zhu17**]. For any $\lambda \in X_*(T)$ as above, we have the semi-infinite orbit

$$S_\lambda = LU \cdot [\lambda] \subset \mathrm{Gr}_G^{\mathrm{Witt}}.$$

Proposition VI.3.7. *For any $\mu \in X_*(T)^+$, the intersection $S_\lambda \cap \mathrm{Gr}_{G,\leq\mu}^{\mathrm{Witt}}$ is representable by an affine scheme.*

Proof. Picking a closed immersion $G \hookrightarrow \mathrm{GL}_n$, one can reduce to $G = \mathrm{GL}_n$. In that case, there is an ample line bundle $\mathcal{L}$ on $\mathrm{Gr}_G^{\mathrm{Witt}}$ constructed in [**BS17**]. We first claim that the pullback of $\mathcal{L}$ to $\mathrm{Gr}_B^{\mathrm{Witt}}$ is trivial. Indeed, recall that if $\mathrm{Spec}\, R \to \mathrm{Gr}_G^{\mathrm{Witt}}$ corresponds to a lattice $\Xi \subset W_{\mathcal{O}_E}(R)[\frac{1}{\pi}]^n$, then $\mathcal{L}$ is given by $\det(\pi^{-m} W_{\mathcal{O}_E}(R)/\Xi)$ for any large enough m, using the determinant

$$\det : \mathrm{Perf}(W_{\mathcal{O}_E}(R) \text{ on } R) \to \mathrm{Pic}(R),$$

which is multiplicative in exact triangles. On $\mathrm{Gr}_B^{\mathrm{Witt}}$, one has a universal filtration of Ξ compatible with the standard filtration on the standard lattice, which induces a similar filtration on $\Xi/\pi^m W_{\mathcal{O}_E}(R)$, where all the graded quotients are locally constant (and constant on S_λ). This means that the line bundle is naturally trivialized over each connected component S_λ of $\mathrm{Gr}_B^{\mathrm{Witt}}$.

We claim that this section over S_λ extends uniquely to a section over the closed subset $\bigcup_{\lambda' \leq \lambda} S_{\lambda'}$ that vanishes over the complement of S_λ, showing that the intersection of S_λ with each $\mathrm{Gr}_{G,\leq\mu}^{\mathrm{Witt}}$ must be affine. To see this, by the v-descent results of [**BS17**], it suffices to check that for any rank 1 valuation ring V with a map $\mathrm{Spec}\, V \to \mathrm{Gr}_G^{\mathrm{Witt}}$ whose generic point $\mathrm{Spec}\, K$ maps into S_λ, the section of $\mathcal{L}$ over $\mathrm{Spec}\, K$ extends to $\mathrm{Spec}\, V$ and is nonzero in the special fibre precisely when all of $\mathrm{Spec}\, V$ maps into S_λ. Now the filtration

$$0 = \Xi_{K,0} \subset \Xi_{K,1} \subset \ldots \subset \Xi_{K,n} = \Xi_K$$

with

$$\Xi_{K,i} = \Xi_K \cap W_{\mathcal{O}_E}(K)[\tfrac{1}{\pi}]^i \subset W_{\mathcal{O}_E}(K)[\tfrac{1}{\pi}]^i$$

has the property that

$$\Xi_{K,i}/\Xi_{K,i-1} = \pi^{\lambda_i} W_{\mathcal{O}_E}(K)$$

for the cocharacter $\lambda = (\lambda_1, \ldots, \lambda_n)$. Moreover, the filtration by the $\Xi_{K,i}$ extends integrally to the filtration

$$0 = \Xi_0 \subset \Xi_1 \subset \ldots \subset \Xi_n = \Xi$$

with

$$\Xi_i = \Xi \cap W_{\mathcal{O}_E}(V)[\tfrac{1}{\pi}]^i \subset W_{\mathcal{O}_E}(V)[\tfrac{1}{\pi}]^i,$$

which is still a filtration by finite projective $W_{\mathcal{O}_E}(V)$-modules by [**SW20**, Lemma 14.2.3]. The injection of Ξ_i/Ξ_{i-1} into $W_{\mathcal{O}_E}(V)[\frac{1}{\pi}]$ (projecting to the i-th coordinate) has image contained in

$$W_{\mathcal{O}_E}(V)[\tfrac{1}{\pi}] \cap \pi^{\lambda_i} W_{\mathcal{O}_E}(K) = \pi^{\lambda_i} W_{\mathcal{O}_E}(V),$$

so we get natural injections $\Xi_i/\Xi_{i-1} \hookrightarrow \pi^{\lambda_i} W_{\mathcal{O}_E}(V)$, that are isomorphisms after inverting π or $[a]$ for a pseudouniformizer $a \in V$. Now the relevant line bundle can be written as the tensor product of the line bundles given by the determinants of the complexes

$$\pi^{\lambda_i} W_{\mathcal{O}_E}(V)/(\Xi_i/\Xi_{i-1}) \in \mathrm{Perf}(W_{\mathcal{O}_E}(R) \text{ on } R).$$

These line bundles are indeed naturally trivial over K as the perfect complex is acyclic there. Now this complex is concentrated in degree 0, and is torsion, so admits a filtration by complexes of the form $V/a \cong [aV \hookrightarrow V]$ for pseudouniformizers $a \in V$. The associated line bundle on V is then given by the alternating tensor product $V \otimes_V (aV)^{-1} = a^{-1}V$, and the natural section by $1 \in a^{-1}V$. We see that the section is indeed integral, and that it is nonzero in the special fibre if and only if all the above complexes are acyclic, equivalently if $\Xi_i/\Xi_{i-1} \to \pi^{\lambda_i} W_{\mathcal{O}_E}(V)$ is an isomorphism. But this is precisely the condition that all of $\operatorname{Spec} V$ maps into S_λ. □

The union $\bigcup_{\lambda, \langle 2\rho, \lambda\rangle \le d} S_\lambda \subset \mathrm{Gr}_G^{\mathrm{Witt}}$ is closed, thus so is

$$\bigcup_{\lambda, \langle 2\rho, \lambda\rangle \le d} S_\lambda \cap \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}} \subset \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}}.$$

For $d = \langle 2\rho, \mu\rangle$, this is all of $\mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}}$, while for $d = -\langle 2\rho, \mu\rangle$ it contains only a point, corresponding to $[\lambda]$ for λ the antidominant representative of the Weyl orbit of μ. Also, only d of the same parity as $\langle 2\rho, \mu\rangle$ are relevant. By Proposition VI.3.7, the successive complements

$$\bigcup_{\lambda, \langle 2\rho, \lambda\rangle \le d} S_\lambda \cap \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}} \setminus \bigcup_{\lambda, \langle 2\rho, \lambda\rangle \le d-2} S_\lambda \cap \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}} = \bigsqcup_{nu, \langle 2\rho, \lambda\rangle = d} S_\lambda \cap \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}}$$

are affine. This means that at each step, the dimension can drop by at most 1. However, in $\langle 2\rho, \mu\rangle$ steps, it drops by $\langle 2\rho, \mu\rangle$. We get the following corollary on Mirković–Vilonen cycles, cf. [**MV07**, Theorem 3.2], and [**GHKR10**], [**Zhu17**, Corollary 2.8] for a different proof based on point counting, the classical Satake isomorphism, and the Kato-Lusztig formula [**Kat82**], [**Lus83**].

Corollary VI.3.8. *The scheme $S_\lambda \cap \mathrm{Gr}_{G,\le\mu}^{\mathrm{Witt}}$ is equidimensional of dimension $\langle \rho, \mu + \lambda\rangle$.*

VI.4. Equivariant sheaves

Now we go back to the setting of general reductive groups G over $\mathcal{O}_E$ (resp. over E if we work over Div_Y^d or Div_X^d). As usual, let Λ be some coefficient ring killed by some integer n prime to p. We want to study $D_{\text{ét}}(-, \Lambda)$ for the local Hecke stack

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}_{\mathcal{Y}}^d} = L^+_{\mathrm{Div}_{\mathcal{Y}}^d} G \backslash \mathrm{Gr}_{G,\mathrm{Div}_{\mathcal{Y}}^d}$$

or its versions for Div^d_Y and Div^d_X. Neither this nor its bounded versions $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet}$ (say, when G is split) is an Artin stack as $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G$ is not finite-dimensional. However, Proposition VI.2.8 shows that on the bounded version, the action factors over a finite-dimensional quotient.

First, we observe that on the level of $D_{\text{ét}}(-,\Lambda)$, one can then forget about the rest of the action.

Proposition VI.4.1. *Let H be a group small v-sheaf over a small v-sheaf S that admits a filtration $H^{\geq m}\subset H$ by closed subgroups such that, v-locally on S, for each $m\geq 1$ each quotient $H^{\geq m}/H^{\geq m+1}$ admits a further finite filtration with graded pieces given by $(\mathbb{A}^1_{S^\sharp})^\diamond$ for some untilt $S^\sharp$ of S (that may vary). Let X be some small v-sheaf over S with an action of H that factors over $H^{<m}=H/H^{\geq m}$ for some $m>0$. Then the pullback functor*

$$D_{\text{ét}}(H^{<m}\backslash X,\Lambda)\to D_{\text{ét}}(H\backslash X,\Lambda)$$

is an equivalence.

Proof. Both stacks live over the classifying stack $[H^{<m}\backslash S]$ of $H^{<m}$ over S. Applying descent along $S\to[H^{<m}\backslash S]$, one reduces to the case that $H^{<m}$ is trivial. In that case, the map $X/H\to X/H^{<m}=X$ admits a section $s:X\to X/H$, and it is enough to prove that s^* is fully faithful. Doing descent once more, it is enough to prove that for any affinoid perfectoid space S' over S over which a filtration by $\mathbb{A}^1$'s exists, pullback

$$D_{\text{ét}}(S',\Lambda)\to D_{\text{ét}}(S'\times H,\Lambda)$$

is fully faithful. Replace S by S' and let $f:H\to S$ be the projection. We need to see that for all $A\in D_{\text{ét}}(S,\Lambda)$, the map

$$A\to Rf_*f^*A$$

is an isomorphism; doing this for all S, it is enough to check it on global sections, i.e.

$$R\Gamma(S,A)\to R\Gamma(S\times H,f^*A)$$

is an isomorphism. Using Postnikov towers, we can assume that $A\in D^+_{\text{ét}}(S,\Lambda)$. We can write H as a filtered colimit of subgroups $H_j\subset H$ such that each H_j is a successive extension as before, but now the quotients are balls inside each $\mathbb{A}^1_{S^\sharp}$. In particular, each H_j is a spatial diamond, and it is enough to prove that

$$R\Gamma(S,A)\to R\Gamma(S\times H_j,f^*A)$$

is an isomorphism for all j. Now each $H_j=\varprojlim_m H_j^{<m}$ is an inverse limit of spatial diamonds, so by [**Sch17a**, Proposition 14.9] it is enough to prove that

$$R\Gamma(S,A)\to R\Gamma(S\times H_j^{<m},f_n^*A)$$

is an isomorphism for all m and j. But this follows easily from each $H_j^{<m}$ being a successive extension of balls inside $\mathbb{A}^1_{S^\sharp}$. □

Using hyperbolic localization, we can prove the following important conservativity result.

Proposition VI.4.2. *Assume that $B\subset G$ is a Borel. Let $S\to\mathrm{Div}^d_{\mathcal{Y}}$ be any small v-sheaf. Let $A\in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$ with support quasicompact over S. Assume that the hyperbolic localization $\mathrm{CT}_B(A)=0$ of the pullback of A to $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ vanishes. Then $A=0$.*

The similar assertion holds with Div^d_Y and Div^d_X in place of $\mathrm{Div}^d_{\mathcal{Y}}$.

Proof. Note that the formation of CT_B commutes with any base change in S, by Corollary VI.3.5. We can thus assume that $S = \mathrm{Spa}(C, C^+)$ is strictly local. Up to replacing d by a smaller integer, removing double points, we can assume that the map $S \to \mathrm{Div}^d_{\mathcal{Y}}$ is given by d distinct untilts $S_i^\sharp$ over $\mathcal{O}_E$, $i = 1, \ldots, d$. Let $E'|E$ be an extension splitting G, assumed unramified in our situation where we work over $\mathcal{Y}$. We can then lift all $S_i^\sharp$ to $\mathcal{O}_{E'}$, and thereby reduce to the case of split G. The corresponding geometric fibre

$$\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

has a stratification enumerated by $\mu_1, \ldots, \mu_d \in X_*(T)^+$, with strata

$$[S/(\prod_{i=1}^{d}(L^+_{\mathrm{Div}^1_{\mathcal{Y}}}G)_{\mu_i} \times_{\mathrm{Div}^1_{\mathcal{Y}}} S)].$$

If A is nonzero, we can find a maximal such stratum on which A is nonzero. Now we apply Corollary VI.3.5, see Example VI.3.6. One has an isomorphism

$$S \times X_*(T)^d \cong \mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}}.$$

Over the copy of S enumerated by the antidominant representatives of (the Weyl group orbits of) $\mu_1, \ldots, \mu_n$ the functor CT_B is the pullback of A to a section of the stratum corresponding to $\mu_1, \ldots, \mu_n \in X_*(T)^+$ (which, as we recall, correspond to a maximal stratum where A is nonzero). This shows that the restriction of A to a section over this maximal stratum is zero. This gives the desired contradiction, so $A = 0$. □

VI.5. Affine flag variety

At a few isolated spots, it will be useful to use the affine flag variety, the main point being that the Schubert varieties in the affine flag variety admit explicit resolutions of singularities, given by Demazure resolutions (also known as Bott–Samelson resolutions). It will be enough to appeal to these in the setting of a split reductive group G, with a reductive model over $\mathcal{O}_E$ and Borel $B \subset G$ defined over $\mathcal{O}_E$, for $d = 1$, and for a small v-stack $S \to \mathrm{Div}^1_{\mathcal{Y}}$ factoring over $\mathrm{Spd}\,\mathcal{O}_C$ where $C = \widehat{\overline{E}}$, so we restrict attention to this setting.

Consider the base change G_A of G to $A = W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$. We have Fontaine's map $\theta : A \to \mathcal{O}_C$, and we can define an "Iwahori" group scheme $\mathcal{I} \to G_A$, flat over A, whose points in a $\ker\theta$-torsionfree A-algebra R are given those elements $g \in G(R)$ such that $\theta(g) \in G(R \otimes_A \mathcal{O}_C)$ lies in $B(R \otimes_A \mathcal{O}_C)$. Similarly, for any parabolic $P \subset G$ containing B, we get a "parahoric" group scheme $\mathcal{P} \to G_A$, flat over A, whose points in a $\ker\theta$-torsionfree A-algebra R are those $g \in G(R)$ such that $\theta(g) \in P(R \otimes_A \mathcal{O}_C)$. In particular, this applies to the parabolics P_i corresponding to the simple reflections s_i; let $\mathcal{P}_i$ be the corresponding parahorics. Still more generally, for any affine simple reflection s_i, one can define a parabolic $\mathcal{P}_i \to G$ flat over A, and such that $\mathcal{I} \to G$ factors over $\mathcal{P}_i$. (The construction of these parahoric group schemes over A can be reduced to the case of $W_{\mathcal{O}_E}(k)[[u]]$ via a faithfully flat embedding $W_{\mathcal{O}_E}(k)[[u]] \hookrightarrow A$ along which everything arises via base change, and then one can appeal to the work of Bruhat–Tits [**BT84**, Section 3.9.4].)

Definition VI.5.1. *In the situation above, including a small v-stack S over $\operatorname{Spd}\mathcal{O}_C$, mapping to $\operatorname{Div}^1_{\mathcal{Y}}$, let*

$$\mathcal{F}\ell_{G,S} \to S$$

be the étale quotient $LG/L^+\mathcal{I}$, where $L^+\mathcal{I}(R,R^+) = \mathcal{I}(B^+_{\mathrm{dR}}(R^\sharp))$.

Note here that as S lives over $\operatorname{Spd}\mathcal{O}_C$, any $\operatorname{Spa}(R,R^+)$ over S comes with an untilt $R^\sharp$ over $\mathcal{O}_C$, in which case $B^+_{\mathrm{dR}}(R^\sharp)$ is an $A = W_{\mathcal{O}_E}(\mathcal{O}_C)$-algebra, so that $\mathcal{I}(B^+_{\mathrm{dR}}(R^\sharp))$ is well-defined.

Proposition VI.5.2. *There is a natural projection map*

$$\mathcal{F}\ell_{G,S} \to \mathrm{Gr}_{G,S/\operatorname{Div}^1_{\mathcal{Y}}}$$

that is a torsor under $(G/B)^\diamond$. In particular, it is proper, representable in spatial diamonds, and cohomologically smooth.

Proof. This follows from the identification $L^+G/L^+\mathcal{I} \cong (G/B)^\diamond$, which follows from the definition, and the similar properties of $(G/B)^\diamond \to \operatorname{Spd}\mathcal{O}_E$. □

We analyze the stratification of $\mathcal{F}\ell_{G,S}$ into $L^+\mathcal{I}$-orbits. Let $N(T) \subset T$ be the normalizer of T, and

$$\widetilde{W} = N(T)(B_{\mathrm{dR}}(C'))/T(B^+_{\mathrm{dR}}(C'))$$

be the affine Weyl group, for any complete algebraically closed field C' over $\mathcal{O}_E$ with a map $C^\flat \to C'^\flat$; this is naturally independent of the choice of C'. As $T(B_{\mathrm{dR}}(C'))/T(B^+_{\mathrm{dR}}(C')) \cong X_*(T)$, there is a short exact sequence

$$0 \to X_*(T) \to \widetilde{W} \to W \to 0,$$

where W is the usual Weyl group of G.

Proposition VI.5.3. *The decomposition of $\mathcal{F}\ell_{G,S}(C')$ into $L^+\mathcal{I}(C')$-orbits is given by*

$$\mathcal{F}\ell_{G,S}(C') = \bigsqcup_{w\in\widetilde{W}} L^+\mathcal{I}(C)\cdot w.$$

Proof. If C' lives over E, we can choose an isomorphism $B_{\mathrm{dR}}(C') \cong C'((\xi))$ and the result follows from the classical result. If C' lives over the residue field $\mathbb{F}_q$ of E, this reduces to the assertion for the Witt vector affine flag variety, for which we refer to [**Zhu17**]. □

Recall that $\widetilde{W}$ acts on $X_*(T)$. Fixing the alcove $\mathfrak{a}$ corresponding to the Iwahori group $\mathcal{I}$, one gets a set of affine simple reflections s_i as the reflections along the faces of the alcove; these generate a normal subgroup $W_{\mathrm{aff}} \subset \widetilde{W}$. Letting $\Omega \subset \widetilde{W}$ denote the stabilizer of the alcove, there is a split short exact sequence

$$1 \to W_{\mathrm{aff}} \to \widetilde{W} \to \Omega \to 1.$$

One gets the Bruhat order on $\widetilde{W}$: If $w_i = w_{i,0}\omega_i \in \widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$ for $i = 1,2$ are two elements, then $w_1 \leq w_2$ if $\omega_1 = \omega_2$ and in one (hence every) presentation of w_2 as a product of affine simple reflections, w_1 is obtained by removing some factors.

Definition VI.5.4. *For $w \in \widetilde{W}$, the affine Schubert cell is the subfunctor $\mathcal{F}\ell_{G,w,S} \subset \mathcal{F}\ell_{G,S}$ of all maps $\mathrm{Spa}(R, R^+) \to \mathcal{F}\ell_{G,S}$ that on all geometric points lie in the $L^+\mathcal{I}$-orbit of w. The affine Schubert variety is the subfunctor $\mathcal{F}\ell_{G,\leq w,S}$ of all maps $\mathrm{Spa}(R, R^+) \to \mathcal{F}\ell_{G,S}$ that on all geometric points lie in the $L^+\mathcal{I}$-orbit of w' for some $w' \leq w$.*

Theorem VI.5.5. *For each $w \in \widetilde{W}$, the subfunctor $\mathcal{F}\ell_{G,\leq w,S} \subset \mathcal{F}\ell_{G,S}$ is closed, and $\mathcal{F}\ell_{G,\leq w,S} \to \mathrm{Spd}\,\mathcal{O}_C$ is proper and representable in spatial diamonds, of finite* dim. trg. *The subfunctor $\mathcal{F}\ell_{G,w,S} \subset \mathcal{F}\ell_{G,\leq w,S}$ is open and dense.*

Proof. We will prove the theorem by constructing the Demazure resolution of $\mathcal{F}\ell_{G,\leq w,S}$. Write $w = w_0\omega \in \widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$, and fix a decomposition $w_0 = \prod_{j=1}^{l} s_{i_j}$ as a product of affine simple reflections of minimal length, so $l(w) = l(w_0) = l$. We write $\dot{w}$ for the element w with such a choice of decomposition.

For each affine simple reflection s_i, we have a corresponding parahoric group $\mathcal{P}_i \to G_A$ corresponding to the face of $\mathfrak{a}$; one has $L^+\mathcal{P}_i/L^+\mathcal{I} \cong (\mathbb{P}^1)^\diamond$.

Definition VI.5.6. *The Demazure variety corresponding to $\dot{w}$ is the étale sheaf*

$$\mathrm{Dem}_{\dot{w},S} = L^+\mathcal{P}_{s_{i_1}} \times^{L^+\mathcal{I}} L^+\mathcal{P}_{s_{i_2}} \times^{L^+\mathcal{I}} \ldots \times^{L^+\mathcal{I}} L^+\mathcal{P}_{s_{i_l}}/L^+\mathcal{I} \to S,$$

equipped with the left $L^+\mathcal{I}$-action and the $L^+\mathcal{I}$-equivariant map

$$\mathrm{Dem}_{\dot{w},S} \to \mathcal{F}\ell_{G,S}$$

given by $(p_1, \ldots, p_l) \mapsto p_1 \cdots p_l \cdot \omega$.

It is clear from the definition that $\mathrm{Dem}_{\dot{w}} \to S$ is a successive $(\mathbb{P}^1)^\diamond$-fibration over S, and in particular is a spatial diamond, proper over S of finite dim. trg. As $\mathcal{F}\ell_{G,S} \to S$ is partially proper, it follows that the image of $\mathrm{Dem}_{\dot{w},S} \to \mathcal{F}\ell_{G,S}$ is proper. Moreover, the image can be identified on geometric points, and we see that $\mathrm{Dem}_{\dot{w},S} \to \mathcal{F}\ell_{G,\leq w,S}$ is surjective, $\mathcal{F}\ell_{G,\leq w,S} \subset \mathcal{F}\ell_{G,S}$ is closed, and $\mathcal{F}\ell_{G,\leq w,S}$ is proper over S. In particular, $\mathcal{F}\ell_{G,w,S} \subset \mathcal{F}\ell_{G,\leq w,S}$ is open, as the complement is a finite union of closed subfunctors. As $\mathcal{F}\ell_{G,S} \to \mathrm{Gr}_{G,S}$ is locally a product with $(G/B)^\diamond$, it follows from [**SW20**, Theorem 19.2.4] that $\mathcal{F}\ell_{G,S} \times_{\mathrm{Gr}_{G,S}} \mathrm{Gr}_{G,\leq\mu,S}$ is a spatial diamond, and thus so is $\mathcal{F}\ell_{G,\leq w,S}$, as it is a closed subspace for μ large enough.

Also, by checking on geometric points and reducing to the classical case, the map $\mathrm{Dem}_{\dot{w},S} \to \mathcal{F}\ell_{G,\leq w,S}$ is an isomorphism over $\mathcal{F}\ell_{G,w,S}$ whose preimage is given by

$$(L^+\mathcal{P}_{s_{i_1}} \setminus L^+\mathcal{I}) \times^{L^+\mathcal{I}} (L^+\mathcal{P}_{s_{i_2}} \setminus L^+\mathcal{I}) \times^{L^+\mathcal{I}} \ldots \times^{L^+\mathcal{I}} (L^+\mathcal{P}_{s_{i_l}} \setminus L^+\mathcal{I})/L^+\mathcal{I}.$$

This implies that $\mathcal{F}\ell_{G,w,S} \subset \mathcal{F}\ell_{G,\leq w,S}$ is dense, as desired. As usual, a consequence of this discussion is that the Bruhat order is independent of the choice of $\dot{w}$. □

Using Demazure resolutions, one can prove that the standard sheaves on the affine flag variety are universally locally acyclic.

Proposition VI.5.7. *For any $w \in \widetilde{W}$, let $j_w : \mathcal{F}\ell_{G,w,S} \hookrightarrow \mathcal{F}\ell_{G,\leq w,S}$ be the open embedding. Then $j_{w!}\Lambda \in D_{\text{ét}}(\mathcal{F}\ell_{G,\leq w,S}, \Lambda)$ is universally locally acyclic over S.*

Proof. Using Proposition IV.2.11, it suffices to prove the same for $\widetilde{j}_w : \mathcal{F}\ell_{G,w,S} \hookrightarrow \mathrm{Dem}_{\dot{w},S}$ and $\widetilde{j}_{w!}\Lambda$. Then $\widetilde{j}_{w!}\Lambda$ can be resolved in terms of Λ and all $i_{\dot{w}',\dot{w},*}\Lambda$ for

$$i_{\dot{w}',\dot{w}} : \mathrm{Dem}_{\dot{w}',S} \to \mathrm{Dem}_{\dot{w},S}$$

the closed immersion from another Demazure variety, corresponding to a subword of $\dot{w}'$ of $\dot{w}$; note that combinatorially, we are dealing with the situation of a normal crossing divisor at the boundary. By cohomological smoothness of all $\mathrm{Dem}_{\dot{w}',S} \to S$ and Proposition IV.2.11, the result follows. □

VI.6. ULA sheaves

We will be interested in universally locally acyclic sheaves on the local Hecke stack.

Definition VI.6.1. *Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be any small v-stack. An object*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is universally locally acyclic over S if it is bounded, and its pullback to

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

is universally locally acyclic over S.

Let

$$D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

be the corresponding full subcategory.

This definition is a priori not symmetric in the two bundles $\mathcal{E}_1$, $\mathcal{E}_2$ parametrized by the local Hecke stack. However, we can check that it actually is.

Proposition VI.6.2. *Consider the automorphism*

$$\mathrm{sw} : \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \cong \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

switching $\mathcal{E}_1$ and $\mathcal{E}_2$. Then $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ is universally locally acyclic over S if and only if sw^*A is universally locally acyclic over S.

Proof. Fix any large enough substack $U \subset \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ quasicompact over S containing the support of A. Let $(L_{\mathrm{Div}^d_{\mathcal{Y}}}G)_U \subset L_{\mathrm{Div}^d_{\mathcal{Y}}}G$ be the preimage of U. Universal local acyclicity after pullback to $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ is equivalent to universal local acyclicity after pullback to

$$(L_{\mathrm{Div}^d_{\mathcal{Y}}}G)_U/(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)^{\geq m}$$

for any $m > 0$, by Proposition VI.1.11 and Proposition VI.1.12. We need to see that this is equivalent to universal local acyclicity after pullback to

$$(L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G)^{\geq m}\backslash(L_{\mathrm{Div}^d_{\mathcal{Y}}}G)_U$$

for any $m > 0$. For this, we note that these two pro-systems in m are pro-isomorphic. By the next lemma, the transition maps back and forth are also cohomologically smooth, which implies the desired equivalence. □

In the following lemma, we call a map f universally locally acyclic if Λ is f-universally locally acyclic.

Lemma VI.6.3. *Let*

$$X_4 \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

be surjective maps of locally spatial diamonds that are compactifiable and of locally finite dim. trg. *Assume that $f_0 \circ f_1$ and $f_1 \circ f_2$ are cohomologically smooth. Then f_0 is universally locally acyclic. If f_1 is universally locally acyclic and $f_0 \circ f_1$ is cohomologically smooth, then f_1 is cohomologically smooth. Thus, if $f_0 \circ f_1$, $f_1 \circ f_2$ and $f_2 \circ f_3$ are cohomologically smooth, then f_0 and f_1 are cohomologically smooth.*

We would expect that $f_3 : X_4 \to X_3$ should be unnecessary in order for f_0 to be cohomologically smooth.

Proof. We claim that for any map $g_0 : Y_0 \to X_0$, with pullbacks $g_i : Y_i \to X_i$ and $\widetilde{f_i} : Y_{i+1} \to Y_i$, the natural transformation

$$f_0^* R g_0^! \to R g_1^! \widetilde{f_0^*}$$

is an isomorphism. Indeed, we have natural maps

$$f_2^* f_1^* f_0^* R g_0^! \to f_2^* f_1^* R g_1^! \widetilde{f_0^*} \to f_2^* R g_2^! \widetilde{f_1^*} \widetilde{f_0^*} \to R g_3^! \widetilde{f_2^*} \widetilde{f_1^*} \widetilde{f_0^*}$$

and the composite of any two maps is an isomorphism. By the two-out-of-six-lemma, this implies that all maps are isomorphisms. By surjectivity of f_1 and f_2, this implies that $f_0^* R g_0^! \to R g_1^! \widetilde{f_0^*}$ is an isomorphism. Applying this with $Y_0 = X_1$ and to the constant sheaf Λ then shows, by the criterion of Theorem IV.2.23, that Λ is f_0-universally locally acyclic.

Now assume that f_1 is universally locally acyclic and $f_0 \circ f_1$ is cohomologically smooth, then

$$R(f_0 \circ f_1)^! \Lambda \cong f_1^* R f_0^! \Lambda \otimes_\Lambda^{\mathbb{L}} R f_1^! \Lambda$$

is invertible. This implies that both tensor factors are invertible, and in particular $R f_1^! \Lambda$ is invertible, so f_1 is cohomologically smooth. For the final statement, we now know that the hypotheses imply that f_0 and f_1 are universally locally acyclic, so the displayed equation implies that f_0 and f_1 are cohomologically smooth. □

Using the conversativity result Proposition VI.4.2, we can characterize universally locally acyclic sheaves in terms of their hyperbolic localization. Note that we can always reduce to the case of quasisplit G by étale localization on S.

Proposition VI.6.4. *Let $B \subset G$ be a Borel with torus quotient T. Let S be a small v-stack with a map $S \to \mathrm{Div}^d_{\mathcal{Y}}$, and let*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Then A is universally locally acyclic over S if and only if the hyperbolic localization

$$\mathrm{CT}_B(A) \in D_{\text{ét}}(\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

is universally locally acyclic over S. This, in turn, is equivalent to the property that

$$R\pi_{T,S,*}\mathrm{CT}_B(A) \in D_{\text{ét}}(S, \Lambda)$$

is locally constant with perfect fibres.

Here

$$\pi_{T,S} : \mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}} \to S$$

is the projection.

Proof. The forward direction follows from Corollary VI.3.5 and the ind-properness of $\pi_{T,S}$ and Corollary IV.2.12. For the converse direction, we may assume that S is strictly totally disconnected and G is split. Note that to prove universal local acyclicity of A, it is enough to prove that the map

$$p_1^* R\mathscr{H}\mathrm{om}(A, R\pi^!_{G,S}\Lambda) \otimes^{\mathbb{L}}_{\Lambda} p_2^* A \to R\mathscr{H}\mathrm{om}(p_1^* A, Rp_2^! A)$$

is an isomorphism (by Theorem IV.2.23). (Implicitly, we pass here to a bounded part of $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ and replace the quotient by $L^+_{\mathrm{Div}^d_{\mathcal{Y}}} G$ by a finite-dimensional quotient in order to be in the setting of Artin stacks.) By Proposition VI.4.2 applied to $G \times G$, it is enough to prove this after applying $\mathrm{CT}_{B^- \times B}$, where B^- is the opposite Borel. Using that hyperbolic localization commutes with exterior tensor products, and Proposition IV.6.13, this translates exactly into the similar isomorphism characterizing universal local acyclicity of $\mathrm{CT}_B(A)$. The final statement follows from Proposition IV.2.28. □

In the case of one leg, one can completely characterize universally locally acyclic sheaves.

Proposition VI.6.5. *Assume that G is split. Let $S \to \mathrm{Div}^1_{\mathcal{Y}}$ be any small v-stack. Consider*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Then A is universally locally acyclic over S if and only if for all $\mu \in X_*(T)^+$, the restriction of A to the section $[\mu] : S \to \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_{\mathcal{Y}}}$ is locally constant with perfect fibres in $D_{\text{ét}}(S, \Lambda)$.

If G is not split, a similar characterization holds, by applying the result étale locally to reduce to the case of split G. Again, there is also the obvious version for Div^1_Y and Div^1_X.

Proof. First, we prove that if all fibres are locally constant with perfect fibres, then A is universally locally acyclic. This easily reduces to the case of $j_{\mu!}\Lambda$ where

$$j_\mu : \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

is the inclusion of an open Schubert cell, and $S = \mathrm{Div}^1_{\mathcal{Y}}$. We can also argue v-locally on $\mathrm{Div}^1_{\mathcal{Y}}$ and so base change to the case $S = \mathrm{Spd}\,\mathcal{O}_C$. In that case, Proposition VI.5.2 and Proposition IV.2.13 show that it suffices to prove the similar assertion for the affine flag variety, where it follows from Proposition VI.5.7.

Now for the converse, we argue by induction on the support of A. On a maximal Schubert cell $\mathrm{Gr}_{G,S/\mathrm{Div}^1_{\mathcal{Y}},\mu}$ where A is nonzero, its restriction is universally locally acyclic, and as on the Hecke stack this stratum is the classifying space of a (pro-)cohomologically smooth group, it follows that the restriction of A along the section $[\mu] : S \to \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_{\mathcal{Y}}}$ is locally constant with perfect fibres. Replacing A by the cone of

$$j_{\mu!} A|_{\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_{\mathcal{Y}},\mu}} \to A,$$

the claim follows. □

In the following corollaries, we no longer assume that G is split.

Corollary VI.6.6. *Let $S \to \mathrm{Div}^1_{\mathcal{Y}}$ be any small v-stack. Then*

$$D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)$$

is stable under Verdier duality and $- \otimes^{\mathbb{L}}_{\Lambda} -$, $R\mathscr{H}\mathrm{om}_{\Lambda}(-,-)$ as well as $j_! j^*$, $Rj_* j^*$, $j_! Rj^!$, $Rj_* Rj^!$ where j is the locally closed immersion of a Schubert cell. Moreover, all of these operations commute with all pullbacks in S.

Proof. Stability under Verdier duality and compatibility with base change in S follow from Corollary IV.2.25. For the other assertions, one can reduce to the case that G is split by working locally on S, where it follows from the previous proposition, and juggling with the six functors, and the stalkwise characterization of the previous proposition. □

Corollary VI.6.7. *For a complete algebraically closed extension C of E with residue field k, taking $S = \mathrm{Spd}\,\mathcal{O}_C$, $S = \mathrm{Spd}\,C$ and $S = \mathrm{Spd}\,k$, the functors*

$$D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\,C/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda) \leftarrow D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\,\mathcal{O}_C/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda) \to D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)$$

are equivalences.

Proof. Use that the formation of $R\mathscr{H}\mathrm{om}$ commutes with any base change in S, and that the category of locally constant sheaves with perfect fibres on any such S is equivalent to the category of perfect Λ-modules. □

In fact, the previous results extend to the case of general d as long as $S \to \mathrm{Div}^d_{\mathcal{Y}}$ has image in the open subset $(\mathrm{Div}^d_{\mathcal{Y}})_{\neq} \subset \mathrm{Div}^d_{\mathcal{Y}}$ where all untilts are distinct. After passing to a finite étale cover of S, we can then in fact assume that S maps to $(\mathrm{Div}^1_{\mathcal{Y}})^d_{\neq}$.

Proposition VI.6.8. *Assume that G is split. Let $S \to (\mathrm{Div}^1_{\mathcal{Y}})^d_{\neq} \to \mathrm{Div}^d_{\mathcal{Y}}$ be a small v-stack. Consider*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Then A is universally locally acyclic over S if and only if for all $\mu_1, \ldots, \mu_d \in X_*(T)^+$, the restriction of A to the section $[\mu_\bullet] : S \to \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ is locally constant with perfect fibres in $D_{\text{ét}}(S, \Lambda)$.

The category

$$D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is stable under Verdier duality and $- \otimes^{\mathbb{L}}_{\Lambda} -$, $R\mathscr{H}\mathrm{om}_{\Lambda}(-,-)$ as well as $j_! j^*$, $Rj_* j^*$, $j_! Rj^!$, $Rj_* Rj^!$ where j is the locally closed immersion of a Schubert cell. Moreover, all of these operations commute with all pullbacks in S.

Proof. We have the decomposition

$$\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \cong \prod_{i=1}^{d} \mathcal{H}\mathrm{ck}_{G,S/_{\pi_i}\mathrm{Div}^1_{\mathcal{Y}}}$$

where $\pi_1, \ldots, \pi_d : S \to \mathrm{Div}^1_{\mathcal{Y}}$ are the d projections, and the product on the right is taken over S. One can then stratify according to Schubert cells parametrized by tuples $\mu_\bullet = (\mu_1, \ldots, \mu_d)$ and the above

arguments imply the result. Here, in the beginning, to see that $j_{\mu_\bullet !}\Lambda$ is universally locally acyclic, one uses that exterior tensor products preserve universal local acyclicity, see Corollary IV.2.25, to reduce to the case of one leg. $\square$

VI.7. Perverse Sheaves

For any small v-stack $S \to \mathrm{Div}^d_{\mathcal{Y}}$, we define a (relative) perverse t-structure on

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Definition/Proposition VI.7.1. *Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be a small v-stack. There is a unique t-structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ such that*

$$A \in {}^pD^{\leq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

if and only if for all geometric points $\mathrm{Spa}(C, C^+) \to S$ and open Schubert cells of

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}}},$$

parametrized by some $\mu_1, \ldots, \mu_r \in X_*(T)^+$ (where r is the number of distinct untilts at $\mathrm{Spa}(C, C^+) \to \mathrm{Div}^d_{\mathcal{Y}}$), the pullback of A to this open Schubert cell sits in cohomological degrees $\leq -\sum_{i=1}^r \langle 2\rho, \mu_i\rangle$.

Proof. We note that on any bounded closed subset of $Z \subset \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ there is a presentable stable ∞-category $\mathcal{D}_{\text{ét}}(Z \times_{\mathrm{Div}^d_{\mathcal{Y}}} S, \Lambda)$ refining the derived category, and the given class of objects is stable under all colimits and extensions (and is generated by a set of objects). Thus, the existence and uniqueness of the t-structure follow from [**Lur16**, Proposition 1.4.4.11]. Moreover, one easily checks that when enlarging Z, the inclusion functors are t-exact, so these glue to a t-structure in the direct limit. $\square$

Let

$$\mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

be the heart of the perverse t-structure. On it, pullback to the affine Grassmannian is fully faithful.

Proposition VI.7.2. *The pullback functor*

$$\mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

is fully faithful.

Moreover, if

$$A \in {}^pD^{\leq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}} \text{ and } B \in {}^pD^{\geq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}},$$

then $R\mathscr{H}\mathrm{om}_\Lambda(A, B) \in D^{\geq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$.

Proof. For the final statement, we need to see that if $C \in D^{\leq -1}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$, then there are no nonzero maps $C \to R\mathscr{H}\mathrm{om}_\Lambda(A, B)$; equivalently, there are no nonzero maps $C \otimes^{\mathbb{L}}_\Lambda A \to B$. But this follows from the simple observation that $C \otimes^{\mathbb{L}}_\Lambda A$ lies in ${}^pD^{\leq -1}$.

Now using this property of $R\mathscr{H}\mathrm{om}_\Lambda(A,B)$, descent implies that it is enough to see that if $A,B \in \mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$, then any map between their pullbacks to $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ is automatically invariant under the action of $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G$. This follows from Lemma VI.7.3 applied to a finite-dimensional approximation of

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G \to \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}\,. \qquad \square$$

We used the following lemma on actions of connected groups on étale sheaves.

Lemma VI.7.3. *Let $f : Y \to X$ be a compactifiable cohomologically smooth map of locally spatial diamonds with a section $s : X \to Y$. Assume that all geometric fibres of f are connected. Then for all $A \in D^{\geq 0}_{\text{ét}}(X,\Lambda)$, the map*

$$\mathcal{H}^0(Rf_*f^*A) \to \mathcal{H}^0(A)$$

given by evaluation at the section s is an isomorphism.

Proof. Note that

$$Rf_*f^*A \cong R\mathscr{H}\mathrm{om}(Rf_!Rf^!\Lambda, A)$$

where $Rf_!Rf^!\Lambda$ sits in cohomological degrees ≤ 0 with $\mathcal{H}^0 \cong \Lambda$. Indeed, this reduces easily to the case of discrete Λ, and then to $\Lambda = \mathbb{F}_\ell$, and can be checked on geometric stalks. But if $S = \mathrm{Spa}(C, C^+)$ and $i : \{s\} \hookrightarrow S$ is the closed point, then $i^*Rf_!Rf^!\mathbb{F}_\ell \in D(\mathbb{F}_\ell)$ with dual

$$R\,\mathrm{Hom}(i^*Rf_!Rf^!\mathbb{F}_\ell, \mathbb{F}_\ell) \cong R\Gamma(Y, f^*i_*\mathbb{F}_\ell)$$

which sits in degrees ≥ 0 and is equal to $\mathbb{F}_\ell$ in degree 0, as the geometric fibres are connected. Using the section, we get $Rf_!Rf^!\Lambda \cong \Lambda \oplus B$ for some B that sits in cohomological degrees ≤ -1, and the lemma follows. $\square$

Unfortunately, it is a priori not easy to describe the category ${}^pD^{\geq 0}$. It is however possible to describe it via hyperbolic localization. This also implies that pullbacks in S are t-exact.

Proposition VI.7.4. *For any $S' \to S \to \mathrm{Div}^d_{\mathcal{Y}}$, pullback along*

$$\mathcal{H}\mathrm{ck}_{G,S'/\mathrm{Div}^d_{\mathcal{Y}}} \to \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

is t-exact for the perverse t-structure. Moreover, if G is split, then

$$\mathrm{CT}_B : D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{\mathrm{bd}} \to D_{\text{ét}}(\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$$

satisfies the following exactness property. There is the natural locally constant map $\mathrm{Gr}_{T,\mathrm{Div}^d_{\mathcal{Y}}} \to X_*(T)$ measuring the sum of relative positions, and by pairing with 2ρ, we get a locally constant map $\deg : \mathrm{Gr}_{T,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathbb{Z}$. Then $\mathrm{CT}_B[\deg]$ is t-exact for the perverse t-structure on the source, and the standard t-structure on the right. As $\mathrm{CT}_B[\deg]$ is conservative, this implies in particular that

$$A \in {}^pD^{\leq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{\mathrm{bd}} \text{ (resp. } A \in {}^pD^{\geq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{\mathrm{bd}})$$

if and only if

$$\mathrm{CT}_B(A)[\deg] \in D^{\leq 0}_{\text{ét}}(\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda) \text{ (resp. } \mathrm{CT}_B(A)[\deg] \in D^{\geq 0}_{\text{ét}}(\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)).$$

Proof. To prove t-exactness of pullbacks, we need to see that pullback commutes with t-truncations. By descent, it is enough to check that this holds v-locally on S; this allows us to reduce to the case that G is split. It is then enough to prove t-exactness of $\mathrm{CT}_B[\deg]$, as by conservativity of $\mathrm{CT}_B[\deg]$ (see Proposition VI.4.2) this gives the characterization in terms of the t-structure in terms of $\mathrm{CT}_B[\deg]$, and the latter characterization is clearly preserved under pullback (as hyperbolic localization commutes with pullback, see Corollary VI.3.5).

We can assume that $S \to \mathrm{Div}^d_{\mathcal{Y}}$ lifts to $S \to (\mathrm{Div}^1_{\mathcal{Y}})^d$. We have a stratification of S into finitely many strata $i_a : S_a \hookrightarrow S$, obtained by pulling back the partial diagonals of $(\mathrm{Div}^1_{\mathcal{Y}})^d$. Accordingly, we get triangles expressing A as a successive extension of $i_{a!}i_a^*A$ resp. $Ri_{a*}Ri_a^!A$; if $A \in {}^pD^{\leq 0}$ (resp. $A \in {}^pD^{\geq 0}$), then also all $i_{a!}i_a^*A \in {}^pD^{\leq 0}$ (resp. $Ri_{a*}Ri_a^!A \in {}^pD^{\geq 0}$). This allows us to reduce to the cases of $i_{a!}i_a^*A$ and $Ri_{a*}Ri_a^!A$. As hyperbolic localization commutes with all functors by Proposition IV.6.12, we can then reduce to the case that $S = S_a$ for some a. Reducing d if necessary, we can then assume that S maps into the locus of distinct untilts $(\mathrm{Div}^1_{\mathcal{Y}})^d_{\neq} \subset (\mathrm{Div}^1_{\mathcal{Y}})^d$. There is then a stratification in terms of open Schubert cells

$$j_{\mu_\bullet} : \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\mu_\bullet} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

parametrized by $\mu_\bullet = (\mu_1, \ldots, \mu_d)$, $\mu_i \in X_*(T)^+$. Now $A \in {}^pD^{\leq 0}$ if and only if all

$$j_{\mu_\bullet}^* A \in D^{\leq -d_{\mu_\bullet}}$$

for $d_{\mu_\bullet} = \sum_{i=1}^d \langle 2\rho, \mu_i \rangle$, and dually $A \in {}^pD^{\geq 0}$ if and only if all

$$Rj_{\mu_\bullet}^! A \in D^{\geq -d_{\mu_\bullet}}.$$

Using excision triangles, we can then assume that

$$A = j_{\mu_\bullet !} A_{\mu_\bullet},\ A_{\mu_\bullet} \in D^{\leq -d_{\mu_\bullet}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\mu_\bullet}, \Lambda)$$

resp.

$$A = Rj_{\mu_\bullet *} A_{\mu_\bullet},\ A_{\mu_\bullet} \in D^{\geq -d_{\mu_\bullet}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\mu_\bullet}, \Lambda).$$

Moreover, filtering by cohomology sheaves, we can actually assume that $A_{\mu_\bullet}$ is concentrated in degree $-d_{\mu_\bullet}$. Recall that

$$[\mu_\bullet] : S \to \mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\mu_\bullet}$$

is a v-cover, and the automorphism group of the stratum is an inverse limit of smooth and connected groups (as follows from Proposition VI.2.4 and the Künneth formula); this implies that for complexes concentrated in one degree, pullback under $[\mu_\bullet]^*$ is fully faithful, cf. Lemma VI.7.3. We can thus assume that $A_{\mu_\bullet}$ comes via pullback from some $B \in D(S, \Lambda)$ concentrated in cohomological degree $-d_{\mu_\bullet}$. Note that at this point, the desired statement (that $\mathrm{CT}_B(A)[\deg]$ sits in the correct degrees) can be checked after pullback along $\mathrm{Spa}(C, C^+) \to S$, so we can assume $S = \mathrm{Spa}(C, C^+)$ is strictly local, and it is enough to check that $\mathrm{CT}_B(A)[\deg]$ sits in the correct degrees in the fibre over the closed point of S. This fibre in turn depends only on the restriction of A to the fibre over the closed point of S, by Proposition IV.6.12. We can thus assume that B is in fact constant. We can assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for some n prime to p, and then by dévissage that $\Lambda = \mathbb{F}_\ell$ for some $\ell \neq p$. One can then further reduce to the case $B = \mathbb{F}_\ell[d_{\mu_\bullet}]$. Also, by the Künneth formula, we can then reduce to the case $d = 1$. Thus, finally

$$A = j_{\mu !}\mathbb{F}_\ell[\langle 2\rho, \mu \rangle]$$

resp.

$$A = Rj_{\mu *}\mathbb{F}_\ell[\langle 2\rho, \mu\rangle],$$

and we want to see that $\mathrm{CT}_B(A)[\deg] \in D^{\leq 0}$ (resp. $\mathrm{CT}_B(A)[\deg] \in D^{\geq 0}$). By Proposition IV.6.13, it suffices to handle the first case. Note that A is now universally locally acyclic, and the claim can be checked in the universal case $S = \mathrm{Div}^1_{\mathcal{Y}}$. As $\mathrm{Gr}_{T,\mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^1_{\mathcal{Y}}$ is a disjoint union of $X_*(T)$ many copies of $\mathrm{Div}^1_{\mathcal{Y}}$, and the image is universally locally acyclic, thus locally constant, it is in fact enough to check the result after pullback to the special fibre $S = \mathrm{Spd}\,\mathbb{F}_q \to \mathrm{Div}^1_{\mathcal{Y}}$, where

$$\mathrm{Gr}_{G,\mathrm{Spd}\,\mathbb{F}_q/\mathrm{Div}^1_{\mathcal{Y}}} \cong (\mathrm{Gr}_G^{\mathrm{Witt}})^\diamond.$$

Using [**Sch17a**, Section 27], we can now translate all computations to the setting of schemes. Let $\lambda \in X_*(T)$ be any element, giving rise to the semi-infinite orbit $S_\lambda \subset \mathrm{Gr}_G^{\mathrm{Witt}}$, i.e.

$$S_\lambda = \mathrm{Gr}_B^{\mathrm{Witt}} \times_{\mathrm{Gr}_T^{\mathrm{Witt}}} [\lambda].$$

By Corollary VI.3.8, the dimension of $S_\lambda \cap \mathrm{Gr}_{G,\mu}^{\mathrm{Witt}}$ is bounded by $\langle \rho, \lambda + \mu\rangle$. The restriction of

$$\mathrm{CT}_B(j_{\mu!}\mathbb{F}_\ell[\langle 2\rho, \mu\rangle])$$

to $[\lambda] \in \mathrm{Gr}_T^{\mathrm{Witt}}$ is given by

$$R\Gamma_c((S_\lambda \cap \mathrm{Gr}_{G,\mu}^{\mathrm{Witt}})_{\overline{\mathbb{F}}_q}, \mathbb{F}_\ell)[\langle 2\rho, \mu\rangle]$$

and thus sits in degrees $\leq 2\langle \rho, \lambda + \mu\rangle - \langle 2\rho, \mu\rangle = \langle 2\rho, \lambda\rangle$, giving the desired bound. □

We note that if $d = 1$, G is split, and $S = \mathrm{Spd}\,k \to \mathrm{Div}^1_{\mathcal{Y}}$ for $k = \overline{\mathbb{F}}_q$, then under the full inclusion

$$\mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda) \subset D_{\text{ét}}(\mathrm{Gr}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}},$$

the identification $\mathrm{Gr}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}} \cong (\mathrm{Gr}_{G,k}^{\mathrm{Witt}})^\diamond$ and the full embedding

$$D_{\text{ét}}(\mathrm{Gr}_{G,k}^{\mathrm{Witt}}, \Lambda)^{\mathrm{bd}} \hookrightarrow D_{\text{ét}}(\mathrm{Gr}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

from [**Sch17a**, Proposition 27.2], the category

$$\mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\,k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)$$

identifies with the full subcategory

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}_{G,k}^{\mathrm{Witt}}, \Lambda) \subset D_{\text{ét}}(\mathrm{Gr}_{G,k}^{\mathrm{Witt}}, \Lambda)^{\mathrm{bd}}$$

of L^+G-equivariant perverse sheaves on $\mathrm{Gr}_{G,k}^{\mathrm{Witt}}$; this was considered by Zhu [**Zhu17**] and Yu [**Yu19**]. In particular, this discussion implies the following result that we will need later.

Proposition VI.7.5. *Assume that G is split, so that for any $\mu \in X_*(T)^+$ we have the inclusion*

$$j_\mu : \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

of the open Schubert cell, of dimension $d_\mu = \langle 2\rho, \mu\rangle$. Then

$${}^p j_{\mu!}\Lambda[d_\mu] = {}^p\mathcal{H}^0(j_{\mu!}\Lambda[d_\mu]),\ {}^p Rj_{\mu *}\Lambda[d_\mu] = {}^p\mathcal{H}^0(Rj_{\mu *}\Lambda[d_\mu])$$

are universally locally acyclic, and their image under $\mathrm{CT}_B[\deg]$ is locally finite free over Λ. Their formation commutes with any base change in Λ. The natural map

$${}^p Rj_{\mu *}\Lambda[d_\mu](d_\mu) \to \mathbb{D}({}^p j_{\mu!}\Lambda[d_\mu])$$

is an isomorphism.

Moreover, if Λ is a $\mathbb{Z}_\ell$-algebra, then there is some integer $a = a(\mu)$ (independent of Λ) such that the kernel and cokernel of the map

$$^p\mathcal{H}^0(j_{\mu!}\Lambda[d_\mu]) \to {}^p\mathcal{H}^0(Rj_{\mu*}\Lambda[d_\mu])$$

are killed by ℓ^a.

We remark that the final statement ultimately makes use of the decomposition theorem (and thus requires the degeneration to the Witt vector affine Grassmannian).

Proof. Consider $A = j_{\mu!}\Lambda[d_\mu] \in {}^pD^{\leq 0}$, which is universally locally acyclic. Then $\mathrm{CT}_B(A)[\deg]$ sits in degrees ≤ 0, and is universally locally acyclic. Moreover, its degree 0 part is locally finite free over Λ. Indeed, this can be computed in terms of the top compactly supported cohomology group of the Mirković–Vilonen cycles $S_\lambda \cap \mathrm{Gr}_{G,\mu}$, which (as for any separated variety) is finite free over Λ. As $\mathrm{CT}_B(A)[\deg]$ is t-exact, this implies that $A' = {}^p\mathcal{H}^0(A)$ has the property that $\mathrm{CT}_B(A')[\deg]$ is locally finite free over Λ. Applying Verdier duality and using Proposition IV.6.13, we see that $\mathrm{CT}_B(\mathbb{D}(A))[\deg] \cong \mathbb{D}(w_0^*\mathrm{CT}_B(A)[\deg])$ (where w_0 is the longest Weyl group element), which then sits in cohomological degrees ≥ 0, and is finite free in degree 0, with degree 0 parts also under Verdier duality. This shows that the natural map

$$^pRj_{\mu*}\Lambda[d_\mu](d_\mu) \to \mathbb{D}(^pj_{\mu!}\Lambda[d_\mu])$$

is an isomorphism. The proof also shows that the formation commutes with any base change in Λ.

For the final statement, we can first of all reduce by universal local acyclicity and Corollary VI.6.7 to the same statement on $\mathrm{Gr}_{G,k}^{\mathrm{Witt}}$. By base change, we can assume that $\Lambda = \mathbb{Z}/\ell^N\mathbb{Z}$, and we can even formally pass to the inverse limit over N, and then invert ℓ; it is thus enough to show that on the perfectly projective scheme $\mathrm{Gr}_{G,k,\leq\mu}^{\mathrm{Witt}}$, the map

$$^pj_{\mu!}\mathbb{Q}_\ell[d_\mu] \to {}^pRj_{\mu*}\mathbb{Q}_\ell[d_\mu]$$

is an isomorphism. This follows from [**Zhu17**, Lemma 2.1], cf. also [**Gai01**, Proposition 1], [**Lus83**]. Let us recall the argument. It is enough to prove injectivity, as then surjectivity follows by Poincaré duality (as the two sheaves are Verdier dual, as we have already proved), using that we are working with field coefficients now. Let $j_{\mu!*}\mathbb{Q}_\ell[d_\mu]$ be the image of the displayed map (i.e., the intersection complex of $\mathrm{Gr}_{G,k,\leq\mu}^{\mathrm{Witt}}$). It is enough to see that for $i : \mathrm{Gr}_{G,k,<\mu}^{\mathrm{Witt}} \hookrightarrow \mathrm{Gr}_{G,k,\leq\mu}^{\mathrm{Witt}}$ the complementary closed, that $i^*j_{\mu!*}\mathbb{Q}_\ell[d_\mu]$ lies in ${}^pD^{\leq -2}$. Indeed, we have a short exact sequence

$$0 \to i_*K \to {}^pj_{\mu!}\mathbb{Q}_\ell[d_\mu] \to j_{\mu!*}\mathbb{Q}_\ell[d_\mu] \to 0$$

for some perverse sheaf K on $\mathrm{Gr}_{G,k,<\mu}^{\mathrm{Witt}}$; but this gives a map $i^*j_{\mu!*}\mathbb{Q}_\ell[d_\mu] \to K[1]$, so if $i^*j_{\mu!*}\mathbb{Q}_\ell[d_\mu] \in {}^pD^{\leq -2}$, then necessarily $K = 0$.

To prove that $i^*j_{\mu!*}\mathbb{Q}_\ell[d_\mu] \in {}^pD^{\leq -2}$, it suffices to prove that all geometric fibres of $j_{\mu!*}\mathbb{Q}_\ell[d_\mu]$ are concentrated in degrees of the same parity as d_μ; indeed, any other stratum in $\mathrm{Gr}_{G,k,\leq\mu}$ has dimension of the same parity as d_μ, so the trivial bound $i^*j_{\mu!*}\mathbb{Q}_\ell[d_\mu] \in {}^pD^{\leq -1}$ gets amplified by one on each stratum for parity reasons. This parity claim about the intersection complex can be checked smooth locally. We have the smooth map $\mathcal{F}\ell_{G,k}^{\mathrm{Witt}} \to \mathrm{Gr}_{G,k}^{\mathrm{Witt}}$ from the Witt vector affine flag variety, and choosing the element w in the Iwahori-Weyl group corresponding to the generic

stratum on the preimage of the Schubert cell, we get the smooth map $\mathcal{F}\ell^{\mathrm{Witt}}_{G,k,\leq w} \to \mathrm{Gr}^{\mathrm{Witt}}_{G,k,\leq \mu}$. It is thus enough to prove the similar claim about the intersection complex of $\mathcal{F}\ell^{\mathrm{Witt}}_{G,k,\leq w}$. Choosing a reduced expression $\dot{w} = s_1 \cdots s_r \cdot \omega$ as above, we get the Demazure-Bott-Samuelson resolution

$$\pi_{\dot{w}} : \mathrm{Dem}^{\mathrm{Witt}}_{\dot{w}} \to \mathcal{F}\ell^{\mathrm{Witt}}_{G,k,\leq w}.$$

This has the property that all geometric fibres admit stratifications into affine spaces, cf. [**Zhu17**, Section 1.4.2]. In particular, all geometric fibres of $R\pi_{\dot{w}*}\mathbb{Q}_\ell$ sit only in even degrees. On the other hand, by the decomposition theorem, the intersection complex is a direct summand of $R\pi_{\dot{w}*}\mathbb{Q}_\ell[d_\mu]$, giving the claim. □

As a consequence of Proposition VI.7.4, we see, perhaps surprisingly, that containment in ${}^pD^{\geq 0}$ can be checked in geometric fibres over S. (Note however that we are using a relative perverse t-structure.) This gives a complete justification for calling it a relative perverse t-structure.

Corollary VI.7.6. *Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be any small v-stack and let*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Then $A \in {}^pD^{\geq 0}_{\text{ét}}$ if and only if this holds true after pullback to all strictly local $\mathrm{Spa}(C, C^+) \to S$. In particular,

$$A \in \mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

if and only if for all strictly local $\mathrm{Spa}(C, C^+) \to S$, the pullback of A to $\mathcal{H}\mathrm{ck}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}}}$ is perverse.

Also note that over geometric points, we are simply considering the usual perverse t-structure corresponding to the stratification in terms of open Schubert cells, and then ${}^pD^{\geq 0}$ admits its usual characterization in terms of !-restriction to the open Schubert cells.

Proof. It suffices to check after a cover, as pullback is t-exact. This allows us to reduce to the case that G is split. But then it follows from the condition in terms of the hyperbolic localization. □

VI.7.1. The Satake category. We also get the following characterization. The condition asked here is stronger than perversity.

Proposition VI.7.7. *Let S be any small v-stack over $\mathrm{Div}^d_{\mathcal{Y}}$ and assume that G is split. Then*

$$A \in D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

has the property that A is a flat perverse sheaf (in the sense that $A \otimes^{\mathbb{L}}_{\Lambda} M$ is perverse for all Λ-modules M) if and only if

$$R\pi_{T*}\mathrm{CT}_B(A)[\deg] \in D_{\text{ét}}(S, \Lambda)$$

is étale locally on S isomorphic to a finite projective Λ-module in degree 0.

Proof. The functor $R\pi_{T*}\mathrm{CT}_B(A)[\deg]$ preserves universally locally acyclic sheaves and hence takes values in sheaves that are locally constant with perfect fibres. By Proposition VI.7.4 and as any bounded part of $\mathrm{Gr}_{T,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Div}^d_{\mathcal{Y}}$ is finite over the base, the condition $A \in {}^pD^{\leq 0}$ is equivalent

to $R\pi_{T*}\mathrm{CT}_B(A)[\deg] \in D^{\leq 0}$. The flatness then ensures that this is locally isomorphic to a perfect complex of Tor-amplitude $[0,0]$, i.e. a finite projective Λ-module in degree 0. $\square$

In the following definition, $S \to \mathrm{Div}^d_{\mathcal{Y}}$ is any small v-stack, and G is general.

Definition VI.7.8. *Let*

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \subset D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

be the full subcategory of all objects that are universally locally acyclic and flat perverse.

This definition has the virtue that it is invariant under switching sw^*. Let us give some examples of objects in the Satake category, when $d = 1$. Assume for simplicity that G is split. For any $\mu \in X_*(T)^+$, we get the open Schubert cell

$$j_\mu : \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

of dimension $d_\mu = \langle 2\rho, \mu\rangle$. The following proposition gives the analogue of highest weight modules in the Satake category.

Proposition VI.7.9. *The perverse sheaves*

$${}^p j_{\mu!}\Lambda[d_\mu] = {}^p\mathcal{H}^0(j_{\mu!}\Lambda[d_\mu]),\ {}^p Rj_{\mu*}\Lambda[d_\mu] = {}^p\mathcal{H}^0(Rj_{\mu*}\Lambda[d_\mu])$$

lie in the Satake category $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)$.

Proof. This follows from Proposition VI.7.7 and Proposition VI.7.5. $\square$

Definition/Proposition VI.7.10. *The functor*

$$R\pi_{G,S*} : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to D_{\text{ét}}(S, \Lambda)$$

of pullback to $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ *and pushforward along* $\pi_{G,S} : \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \to S$ *takes values in complexes* $C \in D_{\text{ét}}(S, \Lambda)$ *such that all* $\mathcal{H}^i(C)$ *are local systems of finite projective* Λ*-modules, and each functor*

$$\mathcal{H}^i(R\pi_{G,S*}) : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to \mathrm{LocSys}(S, \Lambda).$$

is exact.

Let

$$F_{G,S} = \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(R\pi_{G,S*}) : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to \mathrm{LocSys}(S, \Lambda).$$

The functor $F_{G,S}$ *is exact, faithful, and conservative. Moreover, if* $f : A \to B$ *is a map in*

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

such that $\ker F_{G,S}(f)$ *is a direct summand of* $F_{G,S}(A)$*, then* f *admits a kernel in* $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$*; similarly for cokernels.*

The final statement in particular ensures the condition of "existence of coequalizers of $F_{G,S}$-split parallel pairs" appearing in the Barr–Beck theorem.

Remark VI.7.11. It is not clear whether there are natural isomorphisms $F_{G,S}(\mathrm{sw}^* A) \cong F_{G,S}(A)$, so this fibre functor is (at least a priori) destroying part of the symmetry. What makes this question slightly delicate is that it is asking for extra structure, and ideally one would like to produce this structure in a clean way; it is conceivable that one can reduce to geometric points and then use the affirmative answers we give later under stronger assumptions on S.

Proof. Localize on S to reduce to the case that G is split, and fix a Borel $B \subset G$ with torus T. Using the stratification of Gr_G into the strata S_ν, we get a filtration on $R\pi_{G,S*}A$ whose associated graded is given by $\bigoplus_\nu Rp_{\nu!}A|_{S_\nu}$. Restricting to connected components of Gr_G and for A in the Satake category, these are concentrated in degrees of the same parity, so the corresponding spectral sequence necessarily degenerates. Thus, most of this follows from Proposition VI.7.7 and Proposition VI.4.2. Faithfulness of $F_{G,S}$ reduces to conservativity and the Barr–Beck type assertion, so it remains to prove the Barr–Beck type assertion. For this, consider the kernel of f in the category of all perverse sheaves on $\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$. We need to see that this is still universally locally acyclic, and flat perverse. These properties can be checked after applying hyperbolic localization, shift by deg, and pushforward to S (using various t-exactness properties), where they follow from the assumption of being a direct summand. □

The Satake category also carries a Verdier duality functor. Again, it is not clear that this functor commutes naturally with sw^* (we will settle it later under stronger assumptions on S).

Proposition VI.7.12. *The image of the fully faithful functor*

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is stable under Verdier duality $\mathbb{D}_{\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}/S}$. *The induced functor*

$$\mathbb{D} : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{op}} \to \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is an equivalence, with $\mathbb{D}^2 = \mathrm{id}$. *Moreover, it makes the diagram*

$$\begin{tikzcd}
\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{op}} \arrow[r, "\mathbb{D}"] \arrow[d, "F_{G,S}"] & \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \arrow[d, "F_{G,S}"] \\
\mathrm{LocSys}(S, \Lambda)^{\mathrm{op}} \arrow[r, "V \mapsto V^*"] & \mathrm{LocSys}(S, \Lambda)
\end{tikzcd}$$

commute naturally.

Proof. The Verdier dual $\mathbb{D}(A) \in D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ can actually be defined already in $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ by using Verdier duality along bounded subsets of $\mathcal{H}\mathrm{ck}_G \to [*/L^+G]$. It follows from Verdier duality that it commutes with the passage to cohomology, i.e. the functor $F_{G,S}$, and from this one can deduce that it is flat perverse and hence lies in the Satake category. Biduality follows from Corollary IV.2.25. □

Moreover, the formation of the Satake category is compatible with constant term functors. We define a locally constant function

$$\deg_P : \mathrm{Gr}_{M,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathbb{Z}$$

as the composite of the projection to $X_*(\overline{M})$ considered before and the map $X_*(\overline{M}) \to \mathbb{Z}$ given by pairing with $2\rho_G - 2\rho_M$.

PROPOSITION VI.7.13. *Let $P \subset G$ be a parabolic with Levi M. Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be any small v-stack. Consider the diagram*

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \xleftarrow{q_S} \mathrm{Gr}_{P,S/\mathrm{Div}^d_{\mathcal{Y}}} \xrightarrow{p_S} \mathrm{Gr}_{M,S/\mathrm{Div}^d_{\mathcal{Y}}}.$$

Then the functor $Rp_{S!}q_S^*[\deg_P]$ defines a functor

$$\mathrm{CT}_{P,S}[\deg_P] : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{M,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda).$$

These functors are compatible with composition, i.e. if $P' \subset P$ is a further parabolic with image $Q \subset M$ and Levi M', then there is a natural equivalence

$$\mathrm{CT}_{P',S}[\deg_{P'}] \cong \mathrm{CT}_{Q,S}[\deg_Q] \circ \mathrm{CT}_{P,S}[\deg_P] : \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{M',S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

(and for triple compositions, the obvious diagram commutes).

PROOF. Let $\lambda : \mathbb{G}_m \to G$ be a cocharacter such that $P = P_\lambda$. This induces in particular a Levi splitting $M \hookrightarrow P$ as the centralizer of λ. We can then divide the diagram

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \xleftarrow{q_S} \mathrm{Gr}_{P,S/\mathrm{Div}^d_{\mathcal{Y}}} \xrightarrow{p_S} \mathrm{Gr}_{M,S/\mathrm{Div}^d_{\mathcal{Y}}}.$$

by $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}M$ to see that one can refine $Rp_{S!}q_S^*$ into a functor

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \to D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{M,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

via first pulling back to $L^+_{\mathrm{Div}^d_{\mathcal{Y}}}M \backslash \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$. It is clear that these functors are compatible with composition.

We want to see that the image is contained in the Satake category. First, by Proposition IV.6.14, we see that the image is universally locally acyclic. Now the claim follows from Proposition VI.7.7 and the compatibility with composition (used for the Borel $B \subset P$), after passing to an étale cover to assume that G is split. □

VI.8. Convolution

For any d and small v-stack $S \to \mathrm{Div}^d_{\mathcal{Y}}$, the category

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

is naturally a monoidal category. Indeed, with all loop groups taken over $\mathrm{Div}^d_{\mathcal{Y}}$, there is a convolution morphism

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \xleftarrow{a} L^+G\backslash LG \times^{L^+G} LG/L^+G \xrightarrow{b} L^+G\backslash LG/L^+G = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$$

where the morphism a is an L^+G-torsor, and the right morphism is ind-proper (its fibres are the fibres of $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Div}^d_{\mathcal{Y}}$). If one denotes by a_S and b_S the pullbacks along $S \to \mathrm{Div}^d_{\mathcal{Y}}$, one can then define the convolution product $\star$ on

$$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$$

via $A_1 \star A_2 = Rb_{S*}a_S^*(A_1 \boxtimes A_2)$ for $A_1, A_2 \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$. It is easy to see that this is associative by writing out the corresponding convolution diagrams with multiple factors.

In fact, modulo the problem that $[\mathrm{Div}^d_{\mathcal{Y}}/L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G] \to [\mathrm{Div}^d_{\mathcal{Y}}/L_{\mathrm{Div}^d_{\mathcal{Y}}}G]$ is not representable in locally spatial diamonds (only ind-representable), the category $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ is precisely the category of endomorphisms of $[\mathrm{Div}^d_{\mathcal{Y}}/L^+_{\mathrm{Div}^d_{\mathcal{Y}}}G]\times_{\mathrm{Div}^d_{\mathcal{Y}}} S$ in the 2-category $\mathcal{C}_T$ for $T = [\mathrm{Div}^d_{\mathcal{Y}}/L_{\mathrm{Div}^d_{\mathcal{Y}}}G]\times_{\mathrm{Div}^d_{\mathcal{Y}}} S$. This problem is corrected by passing to bounded sheaves – one can extend the formalism to the case of maps that are ind-representable in locally spatial diamonds, with closed immersions in the ind-system, using categories of bounded sheaves as morphisms.

Convolution interacts nicely with the classes of sheaves we have previously singled out. In particular, it preserves (flat) perverse sheaves; this observation goes back to Lusztig [**Lus83**].

Proposition VI.8.1. *Let $A_1, A_2 \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$.*

(i) *If A_1 and A_2 are universally locally acyclic, then $A_1 \star A_2$ is universally locally acyclic.*
(ii) *If A_1 and A_2 lie in ${}^pD^{\leq 0}$, then $A_1 \star A_2 \in {}^pD^{\leq 0}$.*
(iii) *If $A_1, A_2 \in \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$, then also $A_1 \star A_2 \in \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$.*

Proof. Part (i) follows from Proposition IV.2.11 and Proposition IV.2.26. For part (ii), we first make some reductions. Namely, the claim can be checked if $S = \mathrm{Spa}(C, C^+)$ is strictly local and G split. Moreover, by a dévissage one can assume that A_1 and A_2 are the !-extensions of the constant sheaves on open Schubert cells; in particular, these are universally locally acyclic. By the Künneth formula one can then reduce to the case $d = 1$. In that case, we can pass to the universal case $S = \mathrm{Div}^1_{\mathcal{Y}}$. Over $(\mathrm{Div}^1_{\mathcal{Y}})^2$, we can consider the moduli space $\widetilde{\mathcal{H}\mathrm{ck}}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2}$ of G-bundles $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$ over $B^+_{(\mathrm{Div}^1_{\mathcal{Y}})^2}$ together with isomorphisms between $\mathcal{E}_0$ and $\mathcal{E}_1$ over $B^+_{(\mathrm{Div}^1_{\mathcal{Y}})^2}[\frac{1}{\mathcal{I}_1}]$ and between $\mathcal{E}_1$ and $\mathcal{E}_2$ over $B^+_{(\mathrm{Div}^1_{\mathcal{Y}})^2}[\frac{1}{\mathcal{I}_2}]$, where $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_{\mathcal{Y}_S}$ are the ideal sheaves parametrizing the two Cartier divisors. Away from the diagonal, this is isomorphic to $\mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2_{\neq}/\mathrm{Div}^2_{\mathcal{Y}}}$, while over the diagonal it is isomorphic to

$$L^+_{\mathrm{Div}^1_{\mathcal{Y}}}G\backslash L_{\mathrm{Div}^1_{\mathcal{Y}}}G \times^{L^+_{\mathrm{Div}^1_{\mathcal{Y}}}G} L_{\mathrm{Div}^1_{\mathcal{Y}}}G/L^+_{\mathrm{Div}^1_{\mathcal{Y}}}G.$$

There are two natural projections

$$p_1, p_2 : \widetilde{\mathcal{H}\mathrm{ck}}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2} \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

keeping track of $\mathcal{E}_0$ and $\mathcal{E}_1$ resp. $\mathcal{E}_1$ and $\mathcal{E}_2$, and a projection

$$m : \widetilde{\mathcal{H}\mathrm{ck}}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2} \to \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2/\mathrm{Div}^2_{\mathcal{Y}}}$$

keeping track of $\mathcal{E}_0$ and $\mathcal{E}_2$. One can the form $B = Rm_*(p_1^*A_1 \otimes^{\mathbb{L}}_{\Lambda} p_2^*A_2)$. Recall that we reduced to the case that A_1 and A_2 are moreover universally locally acyclic. By Proposition IV.2.11 and Proposition IV.2.26, one sees that also

$$B \in D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2/\mathrm{Div}^2_{\mathcal{Y}}}, \Lambda).$$

Away from the diagonal, this is simply the exterior tensor product of A_1 and A_2 and in particular lies in ${}^pD^{\leq 0}$. Looking at $\mathrm{CT}_B(B)[\deg]$, we get a universally locally acyclic sheaf on (a bounded

subset of) $\mathrm{Gr}_{T,(\mathrm{Div}^1_{\mathcal{Y}})^2/\mathrm{Div}^2_{\mathcal{Y}}}$ whose restriction away from the diagonal lies in degrees ≤ 0. This implies that the whole sheaf lies in degrees ≤ 0: As any bounded subset of $\mathrm{Gr}_{T,(\mathrm{Div}^1_{\mathcal{Y}})^2/\mathrm{Div}^2_{\mathcal{Y}}}$ is finite over $(\mathrm{Div}^1_{\mathcal{Y}})^2$, it suffices to check this for the pushforward to $(\mathrm{Div}^1_{\mathcal{Y}})^2$. But this pushforward is locally constant with perfect fibres, and the complement of the diagonal is dense.

Thus, using Proposition VI.7.4, the restriction of B to the diagonal lies in ${}^pD^{\leq 0}$. But this restriction is precisely $A_1 \star A_2$, giving the desired result. Finally, part (iii) easily follows from (i), (ii), and the observation that convolution commutes with Verdier duality. □

VI.8.1. Dualizability. Next, we observe that all objects are dualizable.

Proposition VI.8.2. *All objects of the monoidal category* $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$ *are (left and right) dualizable. The right dual of* $A \in \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$ *is given by* $\mathrm{sw}^*\mathbb{D}(A)$ *where* $\mathrm{sw} : \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \cong \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ *is the switching isomorphism (induced by inversion on* $L_{\mathrm{Div}^d_{\mathcal{Y}}}G$*).*

Remark VI.8.3. In the classical setting, this is asserted without indication of proof in [**MV07**, end of Section 11].

Proof. All objects of $D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)$ are left dualizable, with right dual given by $\mathrm{sw}^*\mathbb{D}(A)$: This follows from Proposition IV.2.24 (modulo the technical nuisance that everything is only ind-representable here; everything adapts to that setting). Here, sw simply arises by swapping source and target. Also note that the condition of being universally locally acyclic is invariant under sw^* by Proposition VI.6.2, so also using Proposition VI.7.12, the functor $\mathrm{sw}^*\mathbb{D}(A)$ preserves the Satake category. □

Remark VI.8.4. Again, we stress that all results above also hold if G is reductive over E, and we replace $\mathrm{Div}^d_{\mathcal{Y}}$ with Div^d_Y or Div^d_X. Indeed, the case of Div^d_Y follows from the case of $\mathrm{Div}^d_{\mathcal{Y}}$ as it is an open subset, at least if G admits a reductive model over $\mathcal{O}_E$. In general, this happens étale locally, making it possible to reduce to this case. Then the case of Div^d_X follows from the case of Div^d_Y as any map $S \to \mathrm{Div}^d_X$ can locally be lifted to a map $S \to \mathrm{Div}^d_Y$ in such a way that the corresponding pullbacks of the local Hecke stacks are isomorphic.

VI.9. Fusion

Now let G be a reductive group over E. From now on, we fix the base field $k = \overline{\mathbb{F}}_q$ and work on the category Perf_k. For brevity, we define for any finite set I with $d = |I|$ the local Hecke stack

$$\mathcal{H}\mathrm{ck}^I_G = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_X} \times_{\mathrm{Div}^d_X} (\mathrm{Div}^1_X)^I$$

and correspondingly

$$\mathrm{Gr}^I_G = \mathrm{Gr}_{G,\mathrm{Div}^d_X} \times_{\mathrm{Div}^d_X} (\mathrm{Div}^1_X)^I.$$

Definition VI.9.1. *For any finite set I, the Satake category*

$$\mathrm{Sat}^I_G(\Lambda)$$

is the category $\mathrm{Sat}(\mathcal{H}\mathrm{ck}^I_G,\Lambda)$ *of all*

$$A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}^I_G,\Lambda)$$

that are universally locally acylic and flat perverse over $(\mathrm{Div}^1_X)^I$.

By Proposition VI.7.10, we get a (not yet monoidal) fibre functor

$$F^I : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda).$$

The target category $\mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda)$ is in fact very explicit.

Proposition VI.9.2. *The category* $\mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda)$ *is naturally equivalent to the category* $\mathrm{Rep}_{W_E^I}(\Lambda)$ *of continuous representations of* W_E^I *on finite projective* Λ*-modules.*

Proof. This is a consequence of Proposition IV.7.3. □

For any map $f : I \to J$ of finite sets, there is a natural monoidal functor $\mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^J(\Lambda)$. Indeed, there is a natural closed immersion $\mathcal{H}\mathrm{ck}_G^J \times_{(\mathrm{Div}_X^1)^J} (\mathrm{Div}_X^1)^I \hookrightarrow \mathcal{H}\mathrm{ck}_G^I$, so pull-push along

$$\mathcal{H}\mathrm{ck}_G^I \leftarrow \mathcal{H}\mathrm{ck}_G^J \times_{(\mathrm{Div}_X^1)^J} (\mathrm{Div}_X^1)^I \to \mathcal{H}\mathrm{ck}_G^J$$

defines the desired functor. It is easy to see that this functor is compatible with composition of maps of finite sets. Moreover, the functors $\mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^J(\Lambda)$ make the diagram

$$\begin{tikzcd} \mathrm{Sat}_G^I(\Lambda) \arrow[r] \arrow[d, "F^I"] & \mathrm{Sat}_G^J(\Lambda) \arrow[d, "F^J"] \\ \mathrm{Rep}_{W_E^I}(\Lambda) \arrow[r] & \mathrm{Rep}_{W_E^J}(\Lambda) \end{tikzcd}$$

commute naturally, where the lower functor is pullback under $W_E^J \to W_E^I$.

Actually, the functor $I \mapsto \mathrm{Sat}_G^I(\Lambda)$ has further functoriality, given by the fusion product. Namely, for finite sets $I_1, \ldots, I_k$ with disjoint union $I = I_1 \sqcup \ldots \sqcup I_k$, there is a natural monoidal functor

$$\mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda),$$

functorial in $I_1, \ldots, I_k$ and compatible with composition. To construct this, let

$$j : (\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k} \subset (\mathrm{Div}_X^1)^I$$

be the open subset where $x_i \neq x_{i'}$ whenever $i, i' \in I = I_1 \sqcup \ldots \sqcup I_k$ lie in different I_j's, and let

$$\mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda) \subset D_{\text{ét}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_G^I \times_{(\mathrm{Div}_X^1)^I} (\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}, \Lambda)$$

be defined similarly as $\mathrm{Sat}_G^I(\Lambda)$.

Proposition VI.9.3. *The restriction functor*

$$j^* : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda)$$

is fully faithful. Similarly,

$$j^* : \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda) \to \mathrm{LocSys}((\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}, \Lambda)$$

is fully faithful.

Proof. For the first part, it suffices to prove that for all $A \in \mathrm{Sat}_G^I(\Lambda)$, the natural map

$$A \to {}^p\mathcal{H}^0(Rj_*j^*A)$$

is an isomorphism. Let $i : Z \hookrightarrow (\mathrm{Div}_X^1)^I$ be the complementary closed. It suffices to see that $i_*i^!A \in {}^pD^{\geq 2}$. Working locally to reduce to the case G split, and applying the t-exact hyperbolic localization functor $R\pi_{T*}\mathrm{CT}_B[\deg]$, taking values in local systems of finite projective Λ-modules on $\mathrm{Sat}_G^I(\Lambda)$, this follows from the observation that $i_*i^!\Lambda \in D^{\geq 2}$, which follows from the observation that Z admits a stratification (by partial diagonals) with smooth strata of ℓ-codimension ≥ 1 inside the smooth $(\mathrm{Div}_X^1)^I$.

This final argument in fact proves directly the second part. $\square$

On the other hand, over $(\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}$, one has

$$\mathcal{H}\mathrm{ck}_G^I \times_{(\mathrm{Div}_X^1)^I} (\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k} \cong \prod_{j=1}^k \mathcal{H}\mathrm{ck}_G^{I_j} \times_{\prod_j (\mathrm{Div}_X^1)^{I_j}} (\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k},$$

so there is a natural monoidal functor

$$\mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda)$$

given by exterior product. Actually, recall that when forming symmetric monoidal tensor products, there are implicit sign rules when commuting factors. We change these here by hand. Namely, note that each

$$\mathcal{H}\mathrm{ck}_G^I = (\mathcal{H}\mathrm{ck}_G^I)^{\mathrm{even}} \sqcup (\mathcal{H}\mathrm{ck}_G^I)^{\mathrm{odd}}$$

decomposes into open and closed subsets given by the even and the odd part; the even part contains those Schubert varieties for which $d_{\mu_\bullet} = \sum_i \langle 2\rho, \mu_i \rangle$ is even, while the odd part contains those for which $d_{\mu_\bullet}$ is odd. Note that the dominance order can only nontrivially compare elements with the same parity, so these are really open and closed subsets. Also note that it follows from Proposition VI.7.4 that for sheaves concentrated on the even (resp. odd) part, the functor F^I is concentrated in even (resp. odd) degrees. Now we impose that when forming the above exterior product, we introduce a minus sign whenever we commute two sheaves concentrated on the odd parts. A different way to say it is that there is a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \arrow[r] \arrow[d, "(F^{I_1},\ldots,F^{I_k})"] & \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda) \arrow[d, "F^{I;I_1,\ldots,I_k}"] \\
\mathrm{LocSys}((\mathrm{Div}_X^1)^{I_1},\Lambda) \times \ldots \times \mathrm{LocSys}((\mathrm{Div}_X^1)^{I_k},\Lambda) \arrow[r, "\boxtimes"] & \mathrm{LocSys}((\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k},\Lambda)
\end{tikzcd}$$

functorial in $I_1, \ldots, I_k$, and under permutations of the sets $I_1, \ldots, I_k$. Indeed, note that the functors F^I invoke a shift by deg, which exactly introduces this sign rule. This in fact pins down this choice of signs by faithfulness of the functors.

Definition/Proposition VI.9.4. *The image of*

$$\mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \to \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda)$$

lands in $\mathrm{Sat}_G^I(\Lambda) \subset \mathrm{Sat}_G^{I;I_1,\ldots,I_k}(\Lambda)$*, defining the fusion product*

$$* : \mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda),$$

a functor of monoidal categories, functorial in $I_1, \ldots, I_k$. It makes the diagram

$$\begin{tikzcd} \mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \arrow[r, "*"] \arrow[d, "(F^{I_1},\ldots,F^{I_k})"] & \mathrm{Sat}_G^{I}(\Lambda) \arrow[d, "F^I"] \\ \mathrm{LocSys}((\mathrm{Div}_X^1)^{I_1}, \Lambda) \times \ldots \times \mathrm{LocSys}((\mathrm{Div}_X^1)^{I_k}, \Lambda) \arrow[r, "\boxtimes"] & \mathrm{LocSys}((\mathrm{Div}_X^1)^{I}, \Lambda) \end{tikzcd}$$

commute functorially in $I_1, \ldots, I_k$ and permutations of $I_1, \ldots, I_k$.

Proof. We can define a convolution local Hecke stack

$$\mathcal{H}\mathrm{ck}_G^{I;I_1,\ldots,I_k} \to (\mathrm{Div}_X^1)^I$$

as follows. It parametrizes G-bundles $\mathcal{E}_0, \ldots, \mathcal{E}_k$ over $B^+_{(\mathrm{Div}_X^1)^I}$ together with isomorphisms of $\mathcal{E}_{j-1}$ and $\mathcal{E}_j$ after inverting $\mathcal{I}_i$ for all $i \in I_j$, for $j = 1, \ldots, k$. Here $\mathcal{I}_i \subset \mathcal{O}_{X_S}$ is the ideal defining the i-th Cartier divisor. There are natural projections

$$p_j : \mathcal{H}\mathrm{ck}_G^{I;I_1,\ldots,I_k} \to \mathcal{H}\mathrm{ck}_G^{I_j},\ j = 1, \ldots, k$$

remembering $\mathcal{E}_{j-1}$ and $\mathcal{E}_j$, and

$$m : \mathcal{H}\mathrm{ck}_G^{I;I_1,\ldots,I_k} \to \mathcal{H}\mathrm{ck}_G^{I}$$

remembering $\mathcal{E}_0$ and $\mathcal{E}_k$. Given $A_j \in \mathrm{Sat}_G^{I_j}(\Lambda)$, one can then define

$$B = Rm_*(p_1^* A_1 \otimes_\Lambda^{\mathbb{L}} \ldots \otimes_\Lambda^{\mathbb{L}} p_k^* A_k) \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_G^I, \Lambda).$$

This is still universally locally acyclic, by Proposition IV.2.11 and Proposition IV.2.26. After pullback to $(\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}$, the map m is an isomorphism, and we simply get the exterior product of all A_i. Moreover, working locally to reduce to the case G is split, we see that $R\pi_{T*}\mathrm{CT}_B(B)[\deg]$ is a local system of finite projective Λ-modules, as it is locally constant with perfect fibres, and over the dense open subset $(\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}$, the perfect complex is a finite projective module in degree 0. This means that $B \in \mathrm{Sat}_G^I(\Lambda)$, as desired. □

In particular, for any finite set I, this structure makes $\mathrm{Sat}_G^I(\Lambda)$ into an E_∞-monoid object in monoidal categories, functorially in I, by using the composite

$$\mathrm{Sat}_G^I(\Lambda) \times \ldots \times \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I \sqcup \ldots \sqcup I}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda),$$

using the functor corresponding to the natural map $I \sqcup \ldots \sqcup I \to I$. Recall that E_∞-monoid structures on monoidal categories are the same as symmetric monoidal category structures refining the given monoidal category structure. Thus, each $\mathrm{Sat}_G^I(\Lambda)$ has become naturally a symmetric monoidal category with the fusion product, refining the monoidal convolution product; and everything is functorial in I. Moreover, by the final part of Definition/Proposition VI.9.4, the functor

$$F^I : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda) \cong \mathrm{Rep}_{W_E^I}(\Lambda)$$

is a symmetric monoidal functor, functorially in I.

A consequence of these symmetric monoidal structures are the following natural isomorphisms.

Corollary VI.9.5. *For $A \in \mathrm{Sat}_G^I(\Lambda)$, there are natural isomorphisms*

$$F^I(\mathrm{sw}^* A) \cong F^I(A),\ \mathbb{D}(\mathrm{sw}^* A) \cong \mathrm{sw}^* \mathbb{D}(A).$$

Moreover, $\mathbb{D}$ is naturally a symmetric monoidal functor, and $\mathbb{D} \circ F^I \cong (F^I)^*$ as symmetric monoidal functors.

Proof. By Proposition VI.8.2, all $A \in \mathrm{Sat}_G^I(\Lambda)$ are dualizable, with dual $\mathrm{sw}^*\mathbb{D}(A)$. In a symmetric monoidal category, this means that there are natural isomorphisms

$$\mathrm{sw}^* \mathbb{D} \mathrm{sw}^* \mathbb{D}(A) \cong A.$$

As both sw^* and $\mathbb{D}$ are self-inverse, this amounts to the commutation of $\mathbb{D}$ and sw^*.

Also, as F^I is symmetric monoidal, it follows that $F^I(\mathrm{sw}^*\mathbb{D}(A))$ and $F^I(A)$ are naturally dual. But by Proposition VI.7.12, the dual of $F^I(A)$ is also $F^I(\mathbb{D}(A))$. Replacing $\mathbb{D}(A)$ by A, we find a natural isomorphism $F^I(\mathrm{sw}^* A) \cong F^I(A)$.

Finally, it is easy to see that the whole construction of the fusion product is compatible with Verdier duality, making Verdier duality a symmetric monoidal functor, compatibly with the fibre functor. □

Moreover, the constant term functors are compatible with the fusion product. More precisely, given a parabolic $P \subset G$ with Levi M, we have the constant term functors

$$\mathrm{CT}_P^I[\deg_P] : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_M^I(\Lambda).$$

Proposition VI.9.6. *For any finite set I decomposed into finite sets $I = I_1 \sqcup \ldots \sqcup I_k$, the diagram*

$$\begin{tikzcd}
\mathrm{Sat}_G^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_G^{I_k}(\Lambda) \arrow[r] \arrow[d, "(\mathrm{CT}_P^{I_1}[\deg_P],\ldots,\mathrm{CT}_P^{I_k}[\deg_P])"] & \mathrm{Sat}_G^I(\Lambda) \arrow[d, "\mathrm{CT}_P^I[\deg_P]"] \\
\mathrm{Sat}_M^{I_1}(\Lambda) \times \ldots \times \mathrm{Sat}_M^{I_k}(\Lambda) \arrow[r] & \mathrm{Sat}_M^I(\Lambda)
\end{tikzcd}$$

commutes functorially in I and permutations of $I_1, \ldots, I_k$.

Proof. After passing to the open subset $(\mathrm{Div}_X^1)^{I;I_1,\ldots,I_k}$, this follows from the Künneth formula, so Proposition VI.9.3 gives the result. □

In particular, the functor

$$\mathrm{CT}_P^I[\deg_P] : \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_M^I(\Lambda)$$

is naturally symmetric monoidal with respect to the fusion product. Moreover, everything is compatible with composition, for another parabolic $P' \subset P$.

VI.10. Tannakian reconstruction

Our next goal is to construct a group scheme whose category of representations recovers the symmetric monoidal category $\mathrm{Sat}_G^I(\Lambda)$. More precisely, we want to use some relative Tannaka duality over $\mathrm{Rep}_{W_E^I}(\Lambda)$. To achieve this, we need the following proposition. Given any finite and Galois-stable subsets $W_i \subset X_*(T)^+$, $i \in I$, closed under the dominance order, we have a quasicompact closed substack

$$\mathcal{H}\mathrm{ck}^I_{G,(W_i)_i} \subset \mathcal{H}\mathrm{ck}^I_G$$

and we get a corresponding full subcategory

$$\mathrm{Sat}^I_{G,(W_i)_i}(\Lambda) \subset \mathrm{Sat}^I_G(\Lambda).$$

Proposition VI.10.1. *The functor*

$$F^I : \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda) \to \mathrm{Rep}_{W_E^I}(\Lambda)$$

admits a left adjoint $L^I_{(W_i)_i}$, satisfying the following properties.

(i) *There is a natural isomorphism*

$$L^I_{(W_i)_i}(V) \cong L^I_{(W_i)_i}(1) \otimes V,\ V \in \mathrm{Rep}_{W_E^I}(\Lambda),$$

where $1 \in \mathrm{Rep}_{W_E^I}(\Lambda)$ is the tensor unit, and we use that $\mathrm{Sat}^I_G(\Lambda)$ is tensored over $\mathrm{Rep}_{W_E^I}(\Lambda)$.

(ii) *There is a natural isomorphism*

$$L^I_{(W_i)_i}(1) \cong *_{i\in I} L^{\{i\}}_{W_i}(1)$$

as the fusion of $L^{\{i\}}_{W_i}(1) \in \mathrm{Sat}^{\{i\}}_{G,W_i}(\Lambda)$.

(iii) *If $I = \{i\}$ has one element and $W = W_i$, then the left adjoint is the restriction of the left adjoint to*

$$F' = \bigoplus_m \mathcal{H}^m(R\pi_{G*}) : \mathrm{Perv}(\mathcal{H}\mathrm{ck}^{\{i\}}_{G,W}, \Lambda) \to \mathrm{Mod}_{W_E}(\Lambda).$$

Proof. It is enough to find the value $L^I_{(W_i)_i}(1)$ satisfying (ii) and (iii). Indeed, then the formula in (i) defines the left adjoint in general. Assume now that (iii) holds, and let us denote $P_{W_i} = L^{\{i\}}_{W_i}(1)$. Then for part (ii) we first observe that

$$F'^I = \bigoplus_m \mathcal{H}^m(R\pi_{G*}) : \mathrm{Perv}(\mathcal{H}\mathrm{ck}^I_{G,(W_i)_i}, \Lambda) \to \mathrm{Shv}_{\text{ét}}((\mathrm{Div}^1_X)^I, \Lambda)$$

admits a left adjoint, and this left adjoint evaluated on the unit, $P_{(W_i)_i}$, is generically on $(\mathrm{Div}^1_X)^I$ given by an exterior tensor product of the corresponding left adjoints for I being a singleton. Indeed, note that there is a natural map

$$P_{(W_i)_i} \to *_{i\in I} P_{W_i}$$

adjoint to the section of

$$F'^I(*_{i\in I} P_{W_i}) \cong \boxtimes_{i\in I} F'(P_{W_i})$$

given by the exterior tensor product of the classes given by (iii). To check that this is an isomorphism generically, we can by étale descent reduce to the case that G is split. In that case, one can make

the left adjoint explicit in terms of the left adjoint to hyperbolic localization. Writing hyperbolic localization as a composite of !-pullback and $*$-pushforward, this left adjoint is then given in terms of $*$-pullback and !-pushforward, and the perverse ${}^p\mathcal{H}^0$. As generically, everything decomposes geometrically into a product, it follows from the Künneth formula that the left adjoint commutes with exterior products. But as any $B \in \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda)$ is equal to ${}^p\mathcal{H}^0(Rj_*j^*B)$ as in the discussion of the fusion product, we see that

$$\begin{aligned} F'^I(B) &\cong \mathrm{Hom}(P_{(W_i)_i}, B) \cong \mathrm{Hom}(P_{(W_i)_i}, {}^p\mathcal{H}^0(Rj_*j^*B)) \\ &\cong \mathrm{Hom}(P_{(W_i)_i}, Rj_*j^*B) \\ &\cong \mathrm{Hom}(j^*P_{(W_i)_i}, j^*B) \\ &\cong \mathrm{Hom}(j^* *_{i\in I} P_{W_i}, j^*B) \\ &\cong \mathrm{Hom}(*_{i\in I} P_{W_i}, {}^p\mathcal{H}^0(Rj_*j^*B)) \cong \mathrm{Hom}(*_{i\in I} P_{W_i}, B). \end{aligned}$$

It remains to prove part (iii). We can assume that $\Lambda = \mathbb{Z}/\ell^c\mathbb{Z}$, using base change. Note first that

$$F' = \bigoplus_m \mathcal{H}^m(R\pi_{G*}) : \mathrm{Perv}(\mathcal{H}\mathrm{ck}^{\{i\}}_{G,W}, \Lambda) \to \mathrm{Mod}_{W_E}(\Lambda)$$

admits a left adjoint L'_W, by the adjoint functor theorem. We need to see that when evaluated at the unit, $P_W := L'_W(1)$ is universally locally acyclic, and flat perverse. By the characterization of these properties, it suffices to show that $F'(P_W) \in \mathrm{Mod}_{W_E}(\Lambda)$ is a representation on a finite projective Λ-module. This does not depend on the W_E-action, so we can check these things after pullback along $\mathrm{Spd}\, C \to \mathrm{Div}^1_X$, where C is a completed algebraic closure of E. In particular, we can assume that G is split. For any open Schubert cell

$$j_\mu : \mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C,\mu} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C,W}$$

for $\mu \in W$, of dimension $d_\mu = \langle 2\rho, \mu\rangle$, we can compute

$$\mathrm{Hom}(P_W, {}^pRj_{\mu*}\Lambda[d_\mu]) = F'({}^pRj_{\mu*}\Lambda[d_\mu]).$$

By Proposition VI.7.9, this is a finite free Λ-module. Using adjunction, we thus see that

$$\mathrm{Hom}({}^pj^*_\mu P_W, \Lambda[d_\mu])$$

is a finite free Λ-module for all $\mu \in W$. Now ${}^pj^*_\mu P_W$ is concentrated on an open Schubert cell $\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C,\mu}$, which is covered by $\mathrm{Spd}\, C$, and concentrated in degree $-d_\mu$. It is thus given by the constant sheaf $M[d_\mu]$ for some Λ-module M, and we know that $\mathrm{Hom}(M, \Lambda)$ is finite free over Λ. As we reduced to $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$, this implies that M is free.

Now argue by induction on W, and take a maximal element $\mu \in W$; let $\overline{W} = W \setminus \{\mu\}$. We get an exact sequence

$$0 \to K \to {}^pj_{\mu!}j^*_\mu P_W \to P_W \to Q \to 0$$

in $\mathrm{Perv}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C}, \Lambda)$ supported on W. In fact, we necessarily have $Q = P_{\overline{W}}$ (as they represent the same functor), for which we know by induction that $F'(Q)$ is a finite free Λ-module. We claim that $K = 0$. As K lies in the kernel of ${}^pj_{\mu!}j^*_\mu P_W \to {}^pRj_{\mu*}j^*_\mu P_W$, it follows from Proposition VI.7.5 that $\ell^a K = 0$ for some a independent of Λ. Using functoriality of the construction for $\Lambda' = \mathbb{Z}/\ell^{a+c}\mathbb{Z} \to \Lambda = \mathbb{Z}/\ell^c\mathbb{Z}$ and that ${}^pj_{\mu!}j^*_\mu P_W$ lies in the Satake category (so in particular it is flat over Λ), we see the image of K' in K is equal to 0. On the other hand, as all constructions are compatible with base

change, the map $K' \to K$ had to be surjective. It follows that $K = 0$, as desired. (Alternatively, we could have reduced to $\mathbb{Z}_\ell$-coefficients, in which case ${}^p j_{\mu!} j_\mu^* P_W \to {}^p R j_{\mu*} j_\mu^* P_W$ is injective (as the kernel is both ℓ-torsion free and killed by ℓ^a), implying $K = 0$ directly.) $\square$

Now we use the following general Tannakian reconstruction result. This is essentially an axiomatization of [**MV07**, Proposition 11.1]. Recall that a symmetric monoidal category is rigid if all of its objects are dualizable.

Proposition VI.10.2. *Let $\mathcal{A}$ be a rigid symmetric monoidal category, and let $\mathcal{C}$ be a symmetric monoidal category with a tensor action of $\mathcal{A}$. Moreover, let*

$$F : \mathcal{C} \to \mathcal{A}$$

be a symmetric monoidal $\mathcal{A}$-linear conservative functor, such that $\mathcal{C}$ admits and F reflects coequalizers of F-split parallel pairs. Assume that $\mathcal{C}$ can be written as a filtered union of full subcategories $\mathcal{C}_i$, stable under coequalizers of F-split parallel pairs and the $\mathcal{A}$-action, such that $F|_{\mathcal{C}_i}$ is representable by some $X_i \in \mathcal{C}$.

Then

$$\mathcal{H} = \varinjlim_i F(X_i)^\vee \in \mathrm{Ind}(\mathcal{A})$$

admits a natural structure as a bialgebra (with commutative multiplication and associative comultiplication), and $\mathcal{C}$ is naturally equivalent to the symmetric monoidal category of representations of $\mathcal{H}$ in $\mathcal{A}$. If $\mathcal{C}$ is rigid, then $\mathcal{H}$ admits an inverse, i.e. is a Hopf algebra.

Here, the symmetric monoidal category of representations of $\mathcal{H}$ is the category of comodules over $\mathcal{H}$ as a coalgebra, endowed with the symmetric monoidal structure coming from the commutative multiplication on $\mathcal{H}$.

Proof. Consider $F_i = F|_{\mathcal{C}_i} : \mathcal{C}_i \to \mathcal{A}$. This admits the left adjoint $A \mapsto A \otimes X_i$, as

$$\mathrm{Hom}_{\mathcal{C}_i}(A \otimes X_i, Y) \cong \mathrm{Hom}_{\mathcal{C}_i}(X_i, A^\vee \otimes Y) \cong F(A^\vee \otimes Y) \cong A^\vee \otimes F(Y) \cong \mathrm{Hom}(A, F(Y)).$$

By the Barr–Beck monadicity theorem, it follows that $\mathcal{C}_i$ is equivalent to the category of modules over the monad

$$A \mapsto F(A \otimes X_i) \cong A \otimes F(X_i).$$

Note that the monad structure here is equivalently turning $F(X_i)$ into an associative algebra $A_i \in \mathcal{A}$, and its category of modules is the category of modules over A_i. Passing to duals, we note that $F(X_i)^\vee$ is a coalgebra, and its category of comodules is equivalent to the category of modules over A_i, i.e. to $\mathcal{C}_i$. Now we can take a colimit over i and see that

$$\mathcal{H} = \varinjlim_i F(X_i)^\vee$$

is naturally a coalgebra whose category of comodules in $\mathcal{A}$ is equivalent to $\mathcal{C}$. The functor is the following: Any $X \in \mathcal{C}$ defines the object $F(X) \in \mathcal{A}$ and for any i large enough so that $X \in \mathcal{C}_i$ a map $F(X) \otimes X_i \to X$ (by adjunction), thus a map

$$F(X) \otimes F(X_i) \cong F(F(X) \otimes X_i) \to F(X),$$

and hence dually we get the map

$$F(X) \to F(X) \otimes F(X_i)^\vee \to F(X) \otimes \mathcal{H}.$$

Moreover, for any i, j there is some k such that

$$\mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_k :$$

indeed, $\mathcal{C}_i$ (resp. $\mathcal{C}_j$) is generated by X_i (resp. X_j) under tensors with $\mathcal{A}$ and coequalizers of F-split parallel pairs, so $\mathcal{C}_i \otimes \mathcal{C}_j$ is generated by $X_i \otimes X_j$ under these operations. Thus, for any k such that $X_i \otimes X_j \in \mathcal{C}_k$, we actually have $\mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_k$. Let $X_k \in \mathcal{C}_k$ represent $F|_{\mathcal{C}_k}$; then we have a natural map

$$X_k \to X_i \otimes X_j.$$

Indeed, this is adjoint to a map $1 \to F(X_i \otimes X_j) = F(X_i) \otimes F(X_j)$, for which we use the tensor product of the unit maps $1 \to F(X_i)$, $1 \to F(X_j)$. This means that there is a natural map

$$\mathcal{H} \otimes \mathcal{H} = \varinjlim_{i,j} F(X_i)^\vee \otimes F(X_j)^\vee \cong \varinjlim_{i,j} F(X_i \otimes X_j)^\vee \to \varinjlim_{k} F(X_k)^\vee = \mathcal{H},$$

which turns $\mathcal{H}$ into a commutative algebra, where the unit is induced by the maps $X_i \to 1$ adjoint to $1 = F(1) \in \mathcal{A}$ (inducing maps $1 = F(1) \to F(X_i)^\vee$).

It is a matter of unraveling definitions that this makes $\mathcal{H}$ into a Hopf algebra whose symmetric monoidal category of representations in $\mathcal{A}$ is exactly $\mathcal{C}$. If $\mathcal{C}$ is rigid, one also sees that $\mathcal{A}$ admits an inverse. Indeed, one can write

$$F(X_i)^\vee \cong F(X_i^\vee) \cong \mathscr{H}\mathrm{om}(X_i, X_i^\vee) \cong \mathscr{H}\mathrm{om}(X_i \otimes X_i, 1)$$

and the switching of the two factors defines the desired involution on $\mathcal{A}$. Here $\mathscr{H}\mathrm{om} \in \mathcal{A}$ denotes the internal Hom over $\mathcal{A}$. □

We can apply Proposition VI.10.2 to $\mathrm{Sat}_G^I(\Lambda)$ to get Hopf algebras

$$\mathcal{H}_G^I(\Lambda) \in \mathrm{Ind}(\mathrm{Rep}_{W_E^I}(\Lambda)).$$

Proposition VI.10.3. *The exterior tensor product*

$$\boxtimes_{i \in I} : \prod_{i \in I} \mathrm{Sat}_G^{\{i\}}(\Lambda) \to \mathrm{Sat}_G^I(\Lambda)$$

induces an isomorphism

$$\bigotimes_{i \in I} \mathcal{H}_G^{\{i\}}(\Lambda) \cong \mathcal{H}_G^I(\Lambda).$$

Proof. This is a consequence of the construction of the Hopf algebras together with Proposition VI.10.1 (ii), noting that

$$F^I(*_{i \in I} A_i) \cong \bigotimes_{i \in I} F(A_i).$$ □

We see that all information about the categories $\mathrm{Sat}_G^I(\Lambda)$ is in the Hopf algebra

$$\mathcal{H}_G(\Lambda) = \mathcal{H}_G^{\{*\}}(\Lambda) \in \mathrm{Ind}(\mathrm{Rep}_{W_E}(\Lambda)).$$

Note also that the construction of $\mathcal{H}_G(\Lambda)$ is compatible with base change in Λ, so it is enough to consider the case $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n prime to p. In fact, note that we can formally take the inverse limit over n to define

$$\mathrm{Sat}_G(\widehat{\mathbb{Z}}^p) = \varprojlim_n \mathrm{Sat}_G(\mathbb{Z}/n\mathbb{Z})$$

with a fibre functor into

$$\mathrm{Rep}^{\mathrm{cont}}_{W_E}(\widehat{\mathbb{Z}}^p) = \varprojlim_n \mathrm{Rep}_{W_E}(\mathbb{Z}/n\mathbb{Z}),$$

the category of continuous representations of W_E on finite free $\widehat{\mathbb{Z}}^p = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$-modules, yielding a Hopf algebra

$$\mathcal{H}_G \in \mathrm{Ind}(\mathrm{Rep}^{\mathrm{cont}}_{W_E}(\widehat{\mathbb{Z}}^p)).$$

This can equivalently be thought of as an affine group scheme $\breve{G}$ over $\widehat{\mathbb{Z}}^p$, with an action of W_E that is in a suitable sense continuous.

VI.11. Identification of the dual group

Our goal is to identify $\breve{G}$ with the Langlands dual group of G. Recall that the universal Cartan of G defines a cocharacter group X_* as an étale sheaf on $\mathrm{Spec}(E)$, i.e. equivalently a finite free abelian group X_* together with an action of the absolute Galois group of E, and in particular of W_E. It comes with the W_E-stable set of coroots $\Phi^\vee \subset X_*$, and the subset of positive roots $\Phi^\vee_+$. Dually, we have the cocharacters X^* and dominant Weyl chamber $(X^*)^+ \subset X^*$, and the roots $\Phi \subset X^*$, containing the positive roots $\Phi_+ \subset \Phi$. These data give rise to a pinned Chevalley group scheme $\widehat{G}$ over $\widehat{\mathbb{Z}}^p$ (or already over $\mathbb{Z}$, but we will only consider it over $\widehat{\mathbb{Z}}^p$) corresponding to the dual root data $(X^*, \Phi, X_*, \Phi^\vee)$. Being pinned, there are distinguished torus and Borel $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, isomorphisms $X^*(\widehat{T}) \cong X_*$ under which the positive coroots $\Phi^\vee_+$ correspond to the weights of $\widehat{T}$ on $\mathrm{Lie}\,\widehat{B}/\mathrm{Lie}\,\widehat{T}$, so

$$\mathrm{Lie}\,\widehat{B}/\mathrm{Lie}\,\widehat{T} = \bigoplus_{a \in \Phi^\vee_+} \mathrm{Lie}\,\widehat{U}_a$$

for root subgroups $\widehat{U}_a \subset \widehat{B}$. Moreover, one has fixed pinnings

$$\psi_a : \mathrm{Lie}\,\widehat{U}_a \cong \widehat{\mathbb{Z}}^p$$

for all simple roots a. We want to endow $\widehat{G}$ with a W_E-action. We already have the W_E-action on $(X^*, \Phi, X_*, \Phi^\vee)$, but we need to twist the action on the pinning. More precisely, let us write the pinning instead with a Tate twist as

$$\psi_a : \mathrm{Lie}\,\widehat{U}_a \cong \widehat{\mathbb{Z}}^p(1).$$

Then W_E acts naturally on the pinning as well, and thereby induces an action of W_E on $\widehat{G}$.

We aim to prove the following theorem. Recall that we write $\breve{G}$ for the Tannaka group arising from the Satake category. Generally, we will denote by $\breve{-}$ various objects defined via the Satake category, while by $\widehat{-}$ we will denote objects formally defined as Langlands duals.

Theorem VI.11.1. *There is a canonical W_E-equivariant isomorphism $\breve{G} \cong \widehat{G}$.*

We note that the formulation of this theorem is slightly more precise than the formulation in [**MV07**], where no canonical isomorphism is given. Also, we handle the case of non-split groups. Note that in particular, $\breve{G}$ only depends on G up to inner automorphisms; this is not clear.

To prove the theorem, we can work over $\mathbb{Z}_\ell$ for some $\ell \neq p$: Indeed, the statement of the theorem is equivalent to having isomorphisms over $\mathbb{Z}/n\mathbb{Z}$ for all n prime to p (by the Tannakian perspective), so the reduction follows from the Chinese remainder theorem.

We will now first prove the theorem when the group G is split; more precisely, if we have fixed a split torus and Borel $T \subset B \subset G$ and trivializations of all simple root groups $U_a \subset B$. Afterwards, we will verify that the isomorphism does not depend on this pinning (essentially, as pinnings vary algebraically, while automorphisms of $\widehat{G}/\mathbb{Z}_\ell$ form an ℓ-adic group), and finally use Galois descent to deduce the result in general.

Note first that if $G = T$ is a torus, then $\mathrm{Gr}_{T,\mathrm{Div}^1_X} \cong X_*(T) \times \mathrm{Div}^1_X$, and it is clear that Sat_T is just the category of $X_*(T)$-graded objects in $\mathrm{Rep}^{\mathrm{cont}}_{W_E}(\mathbb{Z}_\ell)$. This implies that $\breve{T} \cong \widehat{T}$ is the dual torus with $X^*(\widehat{T}) = X_*(T)$.

We have the symmetric monoidal constant term functor

$$\mathrm{CT}_B[\deg] : \mathrm{Sat}_G \to \mathrm{Sat}_T,$$

and it commutes with the fibre functors by the identity $\bigoplus_i \mathcal{H}^i(R\pi_{G*}) \cong \mathcal{H}^0(R\pi_{T*}\mathrm{CT}_B[\deg])$. This gives rise to a W_E-equivariant map $\widehat{T} = \breve{T} \to \breve{G}$. Using the objects $A_\mu = {}^p j_{\mu!}\mathbb{Z}_\ell$, whose μ-weight space is 1-dimensional, we see that the map $\breve{T} \to \breve{G}$ must be a closed immersion.

We have the following information about the generic fibre $\breve{G}_{\mathbb{Q}_\ell}$, following [**MV07**, Section 7]. First, it follows from Proposition VI.7.5 that its category of representations $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ is given by

$$\mathrm{Sat}_G(\mathbb{Q}_\ell) \cong \bigoplus_\mu \mathrm{Rep}^{\mathrm{cont}}_{W_E}(\mathbb{Q}_\ell) \otimes A_\mu.$$

(Here

$$\mathrm{Sat}_G(\mathbb{Q}_\ell) = \mathrm{Sat}_G(\mathbb{Z}_\ell)[\tfrac{1}{\ell}],$$

where $\mathrm{Sat}_G(\mathbb{Z}_\ell) = \varprojlim_m \mathrm{Sat}_G(\mathbb{Z}/\ell^m\mathbb{Z})$.) The category of representations of $\breve{G}_{\mathbb{Q}_\ell}$ as an abstract group scheme is then given by

$$\mathrm{Sat}_G(\mathbb{Q}_\ell) \otimes_{\mathrm{Rep}^{\mathrm{cont}}_{W_E}(\mathbb{Q}_\ell)} \mathrm{Vect}(\mathbb{Q}_\ell) \cong \bigoplus_\mu \mathrm{Vect}(\mathbb{Q}_\ell) \otimes A_\mu,$$

and in particular is semisimple. As $A_\mu * A_{\mu'}$ contains $A_{\mu+\mu'}$ as a direct summand and X_*^+ is finitely generated as a monoid, we see that $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ has a finite number of tensor generators. This implies that $\breve{G}_{\mathbb{Q}_\ell}$ is of finite type by [**DM82**, Proposition 2.20]. Moreover, it is connected as $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ does not have nontrivial finite tensor subcategories (as for any A_μ with $\mu \neq 0$, the tensor category generated by A_μ contains all $A_{n\mu}$), cf. [**DM82**, Corollary 2.22]. As $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ is semisimple, we even know that $\breve{G}_{\mathbb{Q}_\ell}$ is reductive by [**DM82**, Proposition 2.23]. For any simple object A_μ, the weights of A_μ on $\widehat{T}_{\mathbb{Q}_\ell} \to \breve{G}_{\mathbb{Q}_\ell}$ are contained in the set of all $\lambda \in X_* = X^*(\widehat{T})$ such that the dominant representative of λ is bounded by μ in the dominance order, and contains μ (with weight 1). This implies that $\breve{T}_{\mathbb{Q}_\ell} \to \breve{G}_{\mathbb{Q}_\ell}$ is a maximal torus of $\breve{G}_{\mathbb{Q}_\ell}$. We can also define a subgroup $\breve{B} \subset \breve{G}$ as the stabilizer of the filtration associated to the cohomological grading of F (stabilizing the filtration $\bigoplus_{m \leq i} R^m\pi_{G*}$ on the fibre functor $F = \bigoplus_m R^m\pi_{G*}$); then $\breve{B}_{\mathbb{Q}_\ell} \subset \breve{G}_{\mathbb{Q}_\ell}$ is a Borel.

Now we analyze the case $G = \mathrm{PGL}_2$. In that case, we have the minuscule cocharacter $\mu : \mathbb{G}_m \to G$ giving rise to the minuscule Schubert cell $\mathrm{Gr}_{G,\mathrm{Div}^1_X,\mu} \cong \mathbb{P}^1_{\mathrm{Div}^1_X}$. Then

$$F(A_\mu) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1) = \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1)$$

as W_E-representation. This is a representation of $\breve{G}$, giving a natural map $\breve{G} \to \mathrm{GL}(\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1))$. We claim that this is a closed immersion, with image given by $\mathrm{SL}(\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1))$. Note that $\breve{T}$ acts on $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1)$ with weight ± 1, and in particular lands inside $\mathrm{SL}(\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1))$. As $\breve{G}_{\mathbb{Q}_\ell}$ is reductive of rank 1, it necessarily follows that

$$\breve{G}_{\mathbb{Q}_\ell} \to \mathrm{SL}(\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(-1))$$

is an isomorphism, and integrally we get a map $\breve{G} \to \mathrm{SL}(\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1))$. This gives a map $\breve{G}_{\mathbb{F}_\ell} \to \mathrm{SL}(\mathbb{F}_\ell \oplus \mathbb{F}_\ell(-1))$. Let $H \subset \mathrm{SL}(\mathbb{F}_\ell \oplus \mathbb{F}_\ell(-1))$ be the closed subgroup that is the image of $\breve{G}_{\mathbb{F}_\ell}$. Note that the irreducible representations of $\breve{G}_{\mathbb{F}_\ell}$ (as an abstract group) are in bijection with dominant cocharacters, corresponding to the simple objects $B_\mu = j_{\mu!*}\mathbb{F}_\ell$ on $\mathrm{Gr}_{G,\mathrm{Spd}\,C}$; each B_μ has a highest weight vector given by weight μ. It follows that H satisfies the hypothesis of the next lemma.

Lemma VI.11.2. *Let H be a closed subgroup of $\mathrm{SL}_2/\mathbb{F}_\ell$ containing the diagonal torus such that its set of irreducible representations injects into $\mathbb{Z}_{\geq 0}$ via consideration of highest weight vectors. Then $H = \mathrm{SL}_2$.*

Proof. Using a power of Frobenius, one can assume that H is reduced, and thus smooth. By [**DM82**, Corollary 2.22] and consideration of highest weight vectors, one also sees that H must be connected. Then H is either the torus, or a Borel, or SL_2. The first cases lead to too many irreducible representations. □

Thus, the map $\breve{G}_{\mathbb{F}_\ell} \to \mathrm{SL}(\mathbb{F}_\ell \oplus \mathbb{F}_\ell(-1))$ is surjective. Together with the isomorphism on the generic fibre, this implies formally that $\breve{G} \to \mathrm{SL}(\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1))$ is an isomorphism by the following lemma (used on the level of the corresponding Hopf algebras).

Lemma VI.11.3. *Let $f : M \to N$ be a map of flat $\mathbb{Z}_\ell$-modules such that $M/\ell \to N/\ell$ is injective and $M[\frac{1}{\ell}] \to N[\frac{1}{\ell}]$ is an isomorphism. Then f is an isomorphism.*

Proof. As M is flat, $f : M \to N$ is injective; moreover, for any $x \in N$ there is some minimal k such that $\ell^k n = f(m)$ lies in the image of M. But if $k > 0$, then m lies in the kernel of $M/\ell \to N/\ell$, a contradiction. □

The subgroup $\breve{B} \subset \breve{G}$ is then given by the Borel stabilizing the line $\mathbb{Z}_\ell \subset \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1)$. Its unipotent radical is the space of maps $\mathbb{Z}_\ell(-1) \to \mathbb{Z}_\ell$, which is canonically isomorphic to $\mathbb{Z}_\ell(1)$. This finishes the proof of the theorem for $G = \mathrm{PGL}_2$.

If now G is of rank 1, we get the map $G \to G_{\mathrm{ad}} \cong \mathrm{PGL}_2$, where the isomorphism $G_{\mathrm{ad}} \cong \mathrm{PGL}_2$ is uniquely determined by our choice of pinning. The map

$$\mathrm{Gr}_{G,\mathrm{Div}^1_X} \to \mathrm{Gr}_{G_{\mathrm{ad}},\mathrm{Div}^1_X}$$

is an isomorphism when restricted to each connected component, inducing an isomorphism

$$\mathrm{Gr}_{G,\mathrm{Div}^1_X} \cong \pi_1(G) \times_{\pi_1(G_{\mathrm{ad}})} \mathrm{Gr}_{G_{\mathrm{ad}},\mathrm{Div}^1_X}.$$

Here of course $\pi_1(G_{\mathrm{ad}}) \cong \mathbb{Z}/2\mathbb{Z}$. This implies that Sat_G can be equivalently described as the category of $A \in \mathrm{Sat}_{G_{\mathrm{ad}}}$ together with a refinement of the $\mathbb{Z}/2\mathbb{Z}$-grading to a $\pi_1(G)$-grading. This implies that $\breve{G} = \widetilde{G_{\mathrm{ad}}} \times^{\mu_2} \breve{Z}$ where $\breve{Z}$ is the split torus with character group $\pi_1(G)$. Thus, one gets an isomorphism $\breve{G} \cong \widehat{G}$ also in this case, including the isomorphism ψ_a on the root group.

Coming back to a general split group G, let a be any simple coroot. We now look at the corresponding minimal Levi subgroups $M_a \subset G$ properly containing T, with parabolic $P_a \subset B$. We have the symmetric monoidal constant term functor

$$\mathrm{CT}_{P_a}[\deg_{P_a}] : \mathrm{Sat}_G \to \mathrm{Sat}_{M_a},$$

commuting with the functors to Sat_T. This induces a map $\widetilde{M_a} \to \breve{G}$, commuting with the inclusion of $\breve{T}$ into both. In particular, passing to Lie algebras, we see that $a \in X_* = X^*(\breve{T})$ is a root of $\breve{G}$, and $a^\vee \in X^* = X_*(\breve{T})$ is a coroot of $\breve{G}$. Moreover, if $s_a \in W$ is the corresponding simple reflection for G, we also see that $s_a \in \breve{W}$, the Weyl group of the reductive group $\breve{G}_{\mathbb{Q}_\ell}$. Using this information for all a, we see that $W \subset \breve{W}$, and that under $X_* = X^*(\breve{T})$ resp. $X^* = X_*(\breve{T})$, we have

$$\Phi^\vee \subset \Phi(\breve{G}_{\mathbb{Q}_\ell}),\ \Phi \subset \Phi^\vee(\breve{G}_{\mathbb{Q}_\ell}).$$

Moreover, for any irreducible object $A_\mu \in \mathrm{Sat}_G(\mathbb{Q}_\ell)$, the weights of A_μ are contained in the convex hull of the W-orbit of μ. This implies that these inclusions must be isomorphisms — indeed, the directions of the edges emanating from μ, for μ regular, correspond to $\Phi(\breve{G}_{\mathbb{Q}_\ell})$. Together with the isomorphisms on simple root groups, we get a unique isomorphism

$$\breve{G}_{\mathbb{Q}_\ell} \cong \widehat{G}_{\mathbb{Q}_\ell}.$$

Under this isomorphism, the map $\widehat{M_a} \cong \widetilde{M_a} \to \breve{G}$ is compatible with the map $\widehat{M_a} \to \widehat{G}$ induced by Langlands duality. It follows that

$$\breve{G}(\breve{\mathbb{Z}}_\ell) \subset \breve{G}(\breve{\mathbb{Q}}_\ell) \cong \widehat{G}(\breve{\mathbb{Q}}_\ell)$$

is a subgroup containing all $\widehat{M_a}(\breve{\mathbb{Z}}_\ell)$. But these generate $\widehat{G}(\breve{\mathbb{Z}}_\ell)$, so $\widehat{G}(\breve{\mathbb{Z}}_\ell) \subset \breve{G}(\breve{\mathbb{Z}}_\ell)$. Now pick a representation $\breve{G} \to \mathrm{GL}_N$ (given by some object of Sat_G) that is a closed immersion over $\mathbb{Q}_\ell$. By the inclusion $\widehat{G}(\breve{\mathbb{Z}}_\ell) \subset \breve{G}(\breve{\mathbb{Z}}_\ell)$, we see that the map $\widehat{G}_{\mathbb{Q}_\ell} \cong \breve{G}_{\mathbb{Q}_\ell} \to \mathrm{GL}_N$ extends to a map $\widehat{G} \to \mathrm{GL}_N$. By Lemma VI.11.4, this is necessarily a closed immersion, at least if $\ell \neq 2$ or $\widehat{G}$ is simply connected. We can always reduce to the case that $\widehat{G}$ is simply connected by arguing with the adjoint group G_{ad} (whose dual group $\widehat{G_{\mathrm{ad}}}$ is simply connected) first, as in the discussion of rank-1-groups above. It then follows that $\breve{G} \to \mathrm{GL}_N$ factors over $\widehat{G} \hookrightarrow \mathrm{GL}_N$, giving a map $\breve{G} \to \widehat{G}$ that is an isomorphism in the generic fibre, and surjective in the special fibre (as any $\overline{\mathbb{F}}_\ell$-point of $\widehat{G}$ lifts to $\breve{\mathbb{Z}}_\ell$, and then to $\breve{G}(\breve{\mathbb{Z}}_\ell)$), and hence an isomorphism by Lemma VI.11.3.

Lemma VI.11.4 ([**PY06**, Corollary 5.2]). *Let H be a reductive group over $\mathbb{Z}_\ell$, H' some affine group scheme of finite type over $\mathbb{Z}_\ell$, and let $\rho : H \to H'$ be a homomorphism that is a closed immersion in the generic fibre. Assume that $\ell \neq 2$, or that no almost simple factor of the derived group of $H_{\overline{\mathbb{Q}}_\ell}$ is isomorphic to SO_{2n+1} (e.g., the derived group of H is simply connected). Then ρ is a closed immersion.*

This finishes the proof of Theorem VI.11.1 when G is split, and endowed with a splitting. Now we prove independence of the choice of splitting. For this, we note that in fact the cohomological

grading on F alone determines $\breve{T} \subset \breve{G}$ as its stabilizer, and $\breve{B} \subset \breve{G}$ as the stabilizer of the associated filtration. It remains to check that the isomorphisms

$$\psi_a : \operatorname{Lie} \widehat{U}_a \cong \mathbb{Z}_\ell(1)$$

are independent of the choices. For this, consider the flag variety $\mathcal{F}\ell$ over E, parametrizing Borels $B \subset G$. Each such Borel comes with its torus T, which is the universal Cartan and thus descends to E. Equivalently, note that tori over $\mathcal{F}\ell$ are equivalent to étale $\mathbb{Z}$-local systems, and as $\mathcal{F}\ell$ is simply connected all of them come via pullback from E; this then gives the so-called universal Cartan T over E, which is split as G is split. Let a be a simple coroot of G. At each point of $\mathcal{F}\ell$, we get the corresponding parabolic $P_a \supset B$, with Levi M_a. Let $\widetilde{\mathcal{F}\ell}_a \to \mathcal{F}\ell$ parametrize pinnings of M_a, i.e. isomorphisms of U_a with the additive group; this is a $\mathbb{G}_m$-torsor. Over $\widetilde{\mathcal{F}\ell}_a$, the universal group M_a is constant, with adjoint group $M_{a,\mathrm{ad}} \cong \mathrm{PGL}_2$. Consider

$$S = \widetilde{\mathcal{F}\ell}^{\diamond}/\varphi^{\mathbb{Z}} \to \operatorname{Spd} E/\varphi^{\mathbb{Z}} = \mathrm{Div}^1_X.$$

Applying the constant term functor for P_a over S gives a symmetric monoidal functor

$$\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1_X} \times_{\mathrm{Div}^1_X} S, \mathbb{Z}_\ell) \to \mathrm{Sat}(\mathcal{H}\mathrm{ck}_{M_a,\mathrm{Div}^1_X} \times_{\mathrm{Div}^1_X} S, \mathbb{Z}_\ell);$$

here, being symmetric monoidal is verified by repeating the construction of the fusion product after the smooth pullback $S \to \mathrm{Div}^1_X$. Both sides admit fibre functors to $\mathrm{LocSys}(S, \mathbb{Z}_\ell)$; this contains $\mathrm{LocSys}(\mathrm{Div}^1_X, \mathbb{Z}_\ell) = \mathrm{Rep}^{\mathrm{cont}}_{W_E}(\mathbb{Z}_\ell)$ fully faithfully, and we can consider the symmetric monoidal full subcategories on which the fibre functors land in this subcategory. As the constant term functor is compatible with fibre functors, it induces a symmetric monoidal functor on these full subcategories, which are then easily seen to be equivalent to Sat_G and Sat_M (reconstructing both starting from Schubert cells). This shows that the constant term functor $\mathrm{Sat}_G \to \mathrm{Sat}_{M_a}$ is naturally independent of the choice of Borel, reducing us to the rank 1 case. In the rank 1 case, we can then further reduce to PGL_2, and we have the minuscule Schubert variety, which is the flag variety $\mathcal{F}\ell \cong \mathbb{P}^1$ of $G \cong \mathrm{PGL}_2$. There are canonical isomorphisms

$$H^0(\mathcal{F}\ell) = \mathbb{Z}_\ell,\ H^2(\mathcal{F}\ell) = \mathbb{Z}_\ell(-1),$$

and $\widehat{U}_a$ is canonically isomorphic to $\mathrm{Hom}(H^2(\mathcal{F}\ell), H^0(\mathcal{F}\ell)) \cong \mathbb{Z}_\ell(1)$.

Thus, we have shown that if G is split, the isomorphism $\breve{G} \cong \widehat{G}$ is canonical. Finally, the general case follows by Galois descent from a finite Galois extension $E'|E$ splitting G.

VI.12. Cartan involution

Any Chevalley group scheme $\widehat{G}$ comes with the Cartan involution, induced by the map on root data which on X_* is given by $\mu \mapsto -w_0(\mu)$ where w_0 is the longest Weyl group element. Under the geometric Satake equivalence, this has a geometric interpretation: Namely, it essentially corresponds to the switching equivalence sw^*. Note that one can upgrade

$$\mathrm{sw}^* : \mathrm{Sat}^I_G(\Lambda) \to \mathrm{Sat}^I_G(\Lambda)$$

to a symmetric monoidal functor by writing it as the composition of Verdier duality and the duality functor $\mathrm{sw}^*\mathbb{D}$ in $\mathrm{Sat}^I_G(\Lambda)$; moreover, this symmetric monoidal functor commutes with the fibre functor F^I (as symmetric monoidal functors), cf. Corollary VI.9.5. Thus, sw^* induces an automorphism of the Tannaka group $\breve{G}$, commuting with the W_E-action.

PROPOSITION VI.12.1. *Under the isomorphism $\breve{G} \cong \widehat{G}$ with the dual group, the isomorphism* sw^* *is given by the Cartan involution, up to conjugation by $\widehat{\rho}(-1) \in \widehat{G}_{\mathrm{ad}}(\mathbb{Z}_\ell)$.*

REMARK VI.12.2. There is a different construction of the commutativity constraint on Sat_G, not employing the fusion product, that relies on the Cartan involution — this is essentially a categorical version of the classical Gelfand trick to prove commutativity of the Satake algebra. For the Satake category, this construction was first proposed by Ginzburg [**Gin90**], who however overlooked the sign $\widehat{\rho}(-1)$. Zhu's proof [**Zhu17**] of the geometric Satake equivalence for $\mathrm{Gr}_G^{\mathrm{Witt}}$ used this approach, taking careful control of the signs; these are related to the work of Lusztig–Yun [**LY13**]. We remark that Zhu gives a different construction of the commutation of sw^* with the fibre functor, using instead that the two actions (on left and right) of $H^*([*/L^+G], \mathbb{Q}_\ell)$ on $H^*(\mathcal{H}\mathrm{ck}_G, A)$ agree for $A \in \mathrm{Sat}_G(\mathbb{Q}_\ell)$.

PROOF. We note that this is really a proposition: The statement only asks about the commutation of a certain diagram, not some extra structure. For the statement, we can also forget about the W_E-action. In particular, enlarging E, we can assume that G is split. As in the proof of Theorem VI.11.1, one can reduce to the case that G is adjoint, so $\widehat{G}$ is semisimple and simply connected. We also fix a pinning of G.

Now, being pinned, G has its own Cartan involution $\theta : G \to G$, and by the functoriality of all constructions under isomorphisms, the induced automorphism of Sat_G corresponds to the Cartan involution of $\widehat{G}$. In other words, we need to see that the automorphism $\theta^*\mathrm{sw}^* : \mathrm{Sat}_G \to \mathrm{Sat}_G$ (which is symmetric monoidal, and commutes with the fibre functors) induces conjugation by $\breve{\rho}(-1)$ on $\breve{G}$.

We claim that the natural cohomological grading on the fibre functor $F : \mathrm{Sat}_G(\Lambda) \to \mathrm{Rep}_{W_E}(\Lambda)$ is compatible with sw^*. In other words, we need to see that in Corollary VI.9.5, the isomorphism $F(A) \cong F(\mathrm{sw}^*A)$ is compatible with the grading, which follows from its construction. In particular, it follows that the automorphism of $\breve{G}$ restricts to the identity on the corresponding cocharacter $2\breve{\rho} : \mathbb{G}_m \subset \breve{G}$. This implies already that it preserves $\breve{T}$ and the Borel $\breve{B}$ (as the centralizer and dynamical parabolic). Any such automorphism of $\breve{G}$ is given by conjugation by some element $s \in \breve{T}_{\mathrm{ad}} \subset \breve{G}_{\mathrm{ad}}$. We need to see that $s = \breve{\rho}(-1)$. Equivalently, the automorphism acts by negation on any simple root space $\breve{U}_a$ of $\breve{G}$.

We claim that the symmetric monoidal automorphism $\theta^*\mathrm{sw}^* : \mathrm{Sat}_G \to \mathrm{Sat}_G$ (commuting with the fibre functor) is compatible with the constant term functors CT_P, for any standard parabolic $P \supset B$, and the similar functor on Levi subgroups. Let θ' be the composition of θ with conjugation by w_0. We know, by the proof of Theorem VI.11.1, that any inner automorphism of G induces the identity on $\breve{G}$. Thus, it suffices to prove the similar claim for $\theta'^*\mathrm{sw}^* : \mathrm{Sat}_G \to \mathrm{Sat}_G$. Let $P^- \subset G$ be the opposite parabolic of P; then $\theta'(P^-) = P$, and the induced automorphism of the Levi M is given by the corresponding automorphism θ'_M defined similarly as θ'. Now Proposition IV.6.13 and the fusion definition of the symmetric monoidal structure (along with the definition of sw^* as the composite of Verdier duality and internal duality) give the claim.

These observations reduce us to the case $G = \mathrm{PGL}_2$. We note that in this case the Cartan involution is the identity, so we can ignore θ. We have the minuscule Schubert variety $i_\mu : \mathrm{Gr}_{G,\mu} = \mathbb{P}^1 \subset \mathrm{Gr}_G$ and the sheaf $A = i_{\mu*}\mathbb{Z}_\ell[1](\frac{1}{2}) \in \mathrm{Sat}_G$ (assuming without loss of generality $\sqrt{q} \in \Lambda$ to

introduce a half-Tate twist), and we know $\check{G} = \mathrm{SL}(F(A))$, where

$$F(A) = F(A)_1 \oplus F(A)_{-1} = H^0(\mathbb{P}^1)(\tfrac{1}{2}) \oplus H^2(\mathbb{P}^1)(\tfrac{1}{2}).$$

The image of A under sw^* is isomorphic to A itself; fix an isomorphism. Then on the one hand

$$F(A) \cong F(\mathrm{sw}^* A)$$

as the functor $\mathrm{sw}^* : \mathrm{Sat}_G \to \mathrm{Sat}_G$ commutes with the fibre functor F, while on the other hand

$$F(\mathrm{sw}^* A) \cong F(A)$$

as the two objects are isomorphic. We need to see that the composite isomorphism is given by the diagonal action of $(u, -u)$ for some $u \in \mathbb{Z}_\ell^\times$ (this claim is independent of the chosen isomorphism between A and $\mathrm{sw}^* A$). We already know that the isomorphism is graded, so it is given by diagonal multiplication by (u_1, u_2) for some units $u_1, u_2 \in \mathbb{Z}_\ell^\times$.

Recall that the first isomorphism is constructed as the composite of Verdier duality and internal duality in Sat_G. Now the Verdier dual of A is A itself (because of the half-Tate twist), and the Verdier duality pairing

$$F(A) \otimes F(A) \to \mathbb{Z}_\ell$$

is the tautological pairing; in particular, restricted to $F(A)_{-1} \otimes F(A)_1$ and $F(A)_1 \otimes F(A)_{-1}$ it is the same map, up to the natural commutativity constraint on $\mathbb{Z}_\ell$-modules. It follows that the internal dual $A^\vee$ of A is also isomorphic to A, and picking such an identification we need to understand the induced pairing

$$F(A) \otimes F(A) \to F(1) = \mathbb{Z}_\ell,$$

and show that when restricted to $F(A)_{-1} \otimes F(A)_1$ and $F(A)_1 \otimes F(A)_{-1}$, the two induced maps differ by a sign (up to the natural commutativity constraint); this claim is again independent of the chosen isomorphism between $A^\vee$ and A. But this is a question purely internal to the symmetric monoidal category $\mathrm{Sat}_G \cong \mathrm{Rep}(\mathrm{SL}_2)$ with its fibre functor. In there, we have the tautological representation $V = \mathbb{Z}_\ell^2 = \mathbb{Z}_\ell e_1 \oplus \mathbb{Z}_\ell e_2$, and it has the determinant pairing $V \otimes V \to \mathbb{Z}_\ell$ as SL_2-representation, realizing the internal duality. The determinant pairing is alternating, so takes opposite signs on (e_1, e_2) and (e_2, e_1), as desired. $\square$

CHAPTER VII

$D_{\blacksquare}(X)$

In order to deal with smooth representations of $G(E)$ on $\overline{\mathbb{Q}}_\ell$-vector spaces (not Banach spaces), we extend (a modified form of) the 6-functor formalism from [**Sch17a**] to the larger class of solid pro-étale sheaves. The results in this chapter were obtained in discussions with Clausen, and Mann has obtained analogues of some of these results in the case of schemes. (Strangely enough, in some respects the formalism actually works better for diamonds than for schemes.)

More precisely, we want to find a "good" category of $\mathbb{Q}_\ell$-sheaves on $[*/\underline{G(E)}]$ that corresponds to smooth representations of $G(E)$ with values in $\mathbb{Q}_\ell$-vector spaces, and extends to a category of $\mathbb{Q}_\ell$-sheaves on Bun_G with a good formalism of six operations that allows us to extend the preceding results for étale torsion coefficients. The first idea would be to take pro-systems of étale torsion sheaves as $\mathbb{Z}_\ell$-coefficients and invert formally ℓ; this formalism is easy to construct, see [**Sch17a**, Section 26]. This would give rise to continuous representations of $G(E)$ in $\mathbb{Q}_\ell$-Banach spaces, and we do not want that:

- supercuspidal representations of $G(E)$ in $\overline{\mathbb{Q}}_\ell$-vector spaces are defined over a finite degree extension of $\mathbb{Q}_\ell$, and after twist admit an invariant lattice that allows us to complete them ℓ-adically, but we do not want to make such a choice.
- we want to construct semi-simple Langlands parameters using the Bernstein center and not some ℓ-adic completion of it.
- in usual discussions of the cohomology of the Lubin–Tate tower, or more general Rapoport–Zink spaces, it *is* possible to use $\overline{\mathbb{Q}}_\ell$-coefficients while talking about usual smooth representations. We want to be able to achieve the same on the level of Bun_G.

We could take $\mathbb{Q}_\ell$-pro-étale sheaves. This would give rise to representations of $G(E)$ (seen as a condensed group) with values in condensed $\mathbb{Q}_\ell$-vector spaces. This category is too big; there is no hope to obtain a formalism of six operations in this too general context. We need to ask for some "completeness" of the sheaves, for which we take inspiration from the theory of solid abelian groups developed in [**CS**].

The idea is the following. We define a category of solid pro-étale $\mathbb{Q}_\ell$-sheaves on Bun_G with a good formalism of (a modified form of) six operations. More precisely, for any small v-stack, we define a full subcategory

$$D_{\blacksquare}(X, \mathbb{Z}_\ell) \subset D(X_v, \mathbb{Z}_\ell),$$

compatible with pullback, and equipped with a symmetric monoidal tensor product (for which pullback is symmetric monoidal). A complex is solid if and only if each cohomology sheaf is solid, and this can be checked v-locally. The subcategory $D_{\blacksquare}(X, \mathbb{Z}_\ell)$ is stable under all (derived) limits and colimits, and the inclusion into $D(X_v, \mathbb{Z}_\ell)$ admits a left adjoint. If X is a diamond, then $D_{\blacksquare}(X, \mathbb{Z}_\ell)$

is also a full subcategory of $D(X_{\text{qproét}}, \mathbb{Z}_\ell)$. If X is a spatial diamond, then on the abelian level, the category of solid $\mathbb{Z}_\ell$-sheaves is the Ind-category of the Pro-category of constructible étale sheaves killed by some power of ℓ. In this way, one can bootstrap many results from the usual étale case.

For any map $f : Y \to X$ of small v-stacks, the pullback functor f^* admits a right adjoint $Rf_* : D_\blacksquare(Y, \mathbb{Z}_\ell) \to D_\blacksquare(X, \mathbb{Z}_\ell)$ that in fact commutes with *any* base change, see Proposition VII.2.4. Similarly, the formation of $R\mathscr{H}\mathrm{om}$ commutes with any base change. Both of these operations can a priori be taken in all v-sheaves, but turn out to preserve solid sheaves. This already gives us four operations.

Unfortunately, $Rf_!$ does not have the same good properties as usual. In particular, if f is proper (and finite-dimensional), Rf_* does not in general satisfy a projection formula. As a remedy, it turns out that for *all* f, the functor f^* admits a left adjoint

$$f_\natural : D_\blacksquare(Y, \mathbb{Z}_\ell) \to D_\blacksquare(X, \mathbb{Z}_\ell),$$

given by "relative homology". This is a completely novel feature, and already for closed immersions this takes usual étale sheaves to complicated solid sheaves. Again $f_\natural$ commutes with any base change, and also satisfies the projection formula (which is just a condition here, as there is automatically a natural map).

When f is "proper and smooth", one can moreover relative $f_\natural$ ("homology") and Rf_* ("cohomology") in the expected way. One also gets a formula for the dualizing complex of f in terms of such functors. These results even extend to universally locally acyclic complexes. This solid 5-functor formalism thus has some excellent formal properties. We are somewhat confused about exactly how expressive it is, and whether it is preferable over the standard 6-functor formalism. One advantage is certainly that $f_\natural$ is more canonical, and even defined much more generally, than $Rf_!$ (whose construction for stacky maps would require the resolution of very subtle homotopy coherence issues, and also can only be defined for *certain* (finite-dimensional) maps). The main problem with the solid formalism is that a stratification of a stack does not lead to a semi-orthogonal decomposition on the level of $D_\blacksquare$.

On the other hand, for our concerns here, $D_\blacksquare(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ is much too large. On $[*/\underline{G(E)}]$ this gives rise to representations of $G(E)$ (as a condensed group) with values in solid $\mathbb{Z}_\ell$-modules. The category of discrete $\mathbb{Q}_\ell$-vector spaces injects into the category of solid $\mathbb{Z}_\ell$-modules. In fact, $\mathbb{Q}_\ell$-vector spaces are the same as Ind-finite dimensional vector spaces. Any finite dimensional $\mathbb{Q}_\ell$-vector space is complete and thus "solid". This means that if V is a $\mathbb{Q}_\ell$-vector space then it gives rise to the solid condensed sheaf $V \otimes_{\mathbb{Q}_\ell^{disc}} \mathbb{Q}_\ell$ whose value on the profinite set S is

$$\varinjlim_{\substack{W \subset V \\ \text{finite dim.}}} \mathrm{Cont}(S, W).$$

The category of smooth representations of $G(E)$ with values in discrete $\mathbb{Q}_\ell$-vector spaces injects in the category of solid $\mathbb{Q}_\ell$-pro-étale shaves on $[*/\underline{G}(E)]$. In fact, since $\ell \neq p$ and $G(E)$ is locally pro-p, representations of the condensed group $G(E)$ on the condensed $\mathbb{Q}_\ell$-vector space $V \otimes_{\mathbb{Q}_\ell^{disc}} \mathbb{Q}_\ell$ are the same as smooth representations of $G(E)$ on V.

We then cut out a subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Q}_\ell)$ of $D_\blacksquare(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ that gives back the category of smooth representations of $G(E)$ in $\mathbb{Q}_\ell$-vector spaces when we look at $[*/\underline{G(E)}]$. (Of course, we can also stick with $\mathbb{Z}_\ell$-coefficients.)

VII.1. Solid sheaves

In the following, $\widehat{\mathbb{Z}}$ always denotes the pro-étale sheaf $\underline{\widehat{\mathbb{Z}}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ where n runs over nonzero integers. We will quickly restrict attention to $\widehat{\mathbb{Z}}^p = \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z}$, allowing only n prime to p.

Let X be a spatial diamond. For any quasi-pro-étale $j : U \to X$ that can be written as a cofiltered inverse limit of qcqs étale $j_i : U_i \to X$, we let

$$j_\natural \widehat{\mathbb{Z}} = \varprojlim_i j_{i!} \widehat{\mathbb{Z}};$$

as the pro-system of the U_i is unique, this is well-defined. Note that there is a tautological section of $j_\natural \widehat{\mathbb{Z}}$ over U. Equivalently, if one writes $\widehat{\mathbb{Z}}[U]$ for the free pro-étale sheaf of $\widehat{\mathbb{Z}}$-modules generated by U (noting that j^* admits a left adjoint on pro-étale sheaves, being a slice), there is a natural map $\widehat{\mathbb{Z}}[U] \to \varprojlim_i \widehat{\mathbb{Z}}[U_i] = j_\natural \widehat{\mathbb{Z}}$.

When $X = \mathrm{Spa}(C)$, $j_\natural \widehat{\mathbb{Z}}$ is the sheaf denoted $\widehat{\mathbb{Z}}[U]^\blacksquare$ in [**CS**], and the same notation will be appropriate here in general.

Definition VII.1.1. *Let $\mathcal{F}$ be a pro-étale sheaf of $\widehat{\mathbb{Z}}$-modules on X. Then $\mathcal{F}$ is solid if for all $j : U \to X$ as above, the map*

$$\mathrm{Hom}(j_\natural \widehat{\mathbb{Z}}, \mathcal{F}) \to \mathcal{F}(U)$$

is an isomorphism.

Let us begin with the following basic example. We note $\nu : X_{\mathrm{qpro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ the projection to the étale site.

Proposition VII.1.2. *For any étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $X_{\mathrm{\acute{e}t}}$, $\nu^*\mathcal{F}$ is solid.*

Proof. This is a consequence of [**Sch17a**, Proposition 14.9]. □

The notion of solid sheaf is well-behaved:

Theorem VII.1.3. *The category of solid $\widehat{\mathbb{Z}}$-sheaves on X is an abelian subcategory of all pro-étale $\widehat{\mathbb{Z}}$-sheaves on X, stable under all limits, colimits, and extensions. It is generated by the finitely presented objects $j_\natural \widehat{\mathbb{Z}}$ for quasi-pro-étale $j : U \to X$ as above, and the inclusion admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}^\blacksquare$ that commutes with all colimits.*

Let $\mathcal{F}$ be a pro-étale $\widehat{\mathbb{Z}}$-sheaf on X. The following conditions are equivalent.

(1) The $\widehat{\mathbb{Z}}$-sheaf $\mathcal{F}$ is finitely presented in the category of all pro-étale $\widehat{\mathbb{Z}}$-sheaves, and is solid.

(2) The $\widehat{\mathbb{Z}}$-sheaf $\mathcal{F}$ is solid, and finitely presented in the category of solid $\widehat{\mathbb{Z}}$-sheaves.

(3) The $\widehat{\mathbb{Z}}$-sheaf $\mathcal{F}$ can be written as a cofiltered inverse limit of torsion constructible étale sheaves.

For any such $\mathcal{F}$, the underlying pro-étale sheaf is representable by a spatial diamond. The category of $\mathcal{F}$ satisfying (1) – (3) is stable under kernels, cokernels, and extensions, in particular an abelian category, and is equivalent to the Pro-*category of torsion constructible étale sheaves.*

Moreover, the category of all solid $\widehat{\mathbb{Z}}$-sheaves on X is equivalent to the Ind-*category of the* Pro-*category of torsion constructible étale sheaves.*

Question VII.1.4. *If $\mathcal{F}$ is a pro-étale $\widehat{\mathbb{Z}}$-sheaf on X whose underlying pro-étale sheaf is representable by a spatial diamond, or even is just qcqs, is $\mathcal{F}$ necessarily solid? If so, these conditions would also be equivalent to (1) – (3).*

Let us remark the following lemma.

Lemma VII.1.5. *Any torsion constructible étale sheaf on the spatial diamond X is represented by a spatial diamond.*

Proof. Let $\mathcal{F} \to X$ be such a sheaf. We can find a surjection of étale sheaves $\mathcal{F}' = \bigoplus_i j_{i!}\mathbb{Z}/n_i\mathbb{Z} \to \mathcal{F}$ for some quasicompact separated étale maps $j_i : U_i \to X$ and nonzero integers n_i (where the direct sum is finite). Then $\mathcal{F}'$ is quasicompact separated étale over X, and thus a spatial diamond; and the surjective map $\mathcal{F}' \to \mathcal{F}$ is also quasicompact separated étale, in particular universally open, and so also $\mathcal{F}$ is a spatial diamond. $\square$

Before starting the proof, we record a key proposition. Its proof is a rare instance that requires w-contractible objects — in most proofs, strictly totally disconnected objects suffice.

Proposition VII.1.6. *Let X be a spatial diamond and let $\mathcal{F}_i$, $i \in I$, be a cofiltered system of torsion constructible étale sheaves. Then for all $j > 0$ the higher inverse limit*

$$R^j \varprojlim_i \mathcal{F}_i = 0,$$

taken in the category of pro-étale sheaves on X, vanishes.

Proof. The pro-étale site of X has a basis given by the w-contractible $Y \to X$, that is strictly totally disconnected perfectoid spaces Y such that the closed points in $|Y|$ are a closed subset and $\pi_0 Y$ is an extremally disconnected profinite set; equivalently, any pro-étale cover $\widetilde{Y} \to Y$ splits, cf. [**BS15**, Section 2.4] for a discussion of w-contractibility. Thus, it suffices to check sections on Y. In other words, we may assume that X is w-contractible, and prove that

$$R^j \varprojlim_i \mathcal{F}_i(X) = 0$$

for all $j > 0$. As X is strictly totally disconnected, sheaves on $X_{\text{ét}}$ are equivalent to sheaves on $|X|$. Moreover, we have the closed immersion $f : \pi_0 X \to |X|$ given by the closed points, and pullback along this map induces isomorphisms $\mathcal{F}_i(X) \cong (f^*\mathcal{F}_i)(\pi_0 X)$. Let $S = \pi_0 X$ be the extremally disconnected profinite set.

Then any torsion constructible sheaf on S is locally constant with finite fibres. In particular, $f^*\mathcal{F}_i$ maps isomorphically to $\mathscr{H}\mathrm{om}(\mathscr{H}\mathrm{om}(f^*\mathcal{F}_i, \underline{S^1}), \underline{S^1})$ where $\underline{S^1} = \underline{\mathbb{R}/\mathbb{Z}}$ is the sheaf of continuous

maps to the circle. (We could for the moment also use $\mathbb{Q}/\mathbb{Z}$, but it will become critical that S^1 is compact.) It follows that

$$R^j \varprojlim_i \mathcal{F}_i(X) = \mathrm{Ext}^j_S(\varinjlim_i \mathscr{H}\mathrm{om}(f^*\mathcal{F}_i, \underline{S^1}), \underline{S^1}).$$

Thus, the result follows from the injectivity of $\underline{S^1}$ as stated in the next lemma. □

Lemma VII.1.7. *Let S be an extremally disconnected profinite set. Then the abelian sheaf $\underline{\mathbb{R}/\mathbb{Z}}$ on S is injective.*

Proof. First, we note that $\underline{\mathbb{R}/\mathbb{Z}}$ is flasque in a strong sense. Namely, if $U \subset S$ is any open subset with closure $\overline{U} \subset S$, then $\overline{U}$ is the Stone-Čech compactification of U (as $\overline{U} \subset S$ is open by one definition of extremally disconnected spaces, and then $\beta U \sqcup (S \setminus \overline{U}) \to S$ admits a splitting, which in particular gives a splitting of the surjection $\beta U \to \overline{U}$ that is the identity on U, thus implying that $\beta U \cong \overline{U}$) and hence any section of $\underline{\mathbb{R}/\mathbb{Z}}$ over U extends uniquely to $\overline{U}$ as $\mathbb{R}/\mathbb{Z}$ is compact Hausdorff. Also, all sections over $\overline{U}$ extend to S, as $\overline{U}$ is open and closed in S.

Let $\mathcal{F} \hookrightarrow \mathcal{G}$ be an injection of sheaves on S with a map $\mathcal{F} \to \underline{\mathbb{R}/\mathbb{Z}}$. Using Zorn's lemma, choose a maximal subsheaf of $\mathcal{G}$ containing $\mathcal{F}$ with an extension of the map to $\underline{\mathbb{R}/\mathbb{Z}}$. Replacing $\mathcal{F}$ by this maximal subsheaf, we can assume that $\mathcal{F}$ is maximal already. If $\mathcal{F} \to \mathcal{G}$ is not an isomorphism, then it is not an isomorphism on global sections (any local section not in the image can be extended by zero to form a global section not in the image), so we can find a map $\mathbb{Z} \to \mathcal{G}$ such that $\mathcal{F}' = \mathcal{F} \times_{\mathcal{G}} \mathbb{Z} \subset \mathbb{Z}$ is a proper subsheaf, and we can replace $\mathcal{G}$ by $\mathbb{Z}$ and assume that $\mathcal{F}$ is a proper subsheaf of $\mathbb{Z}$.

For each integer n, we can look at the open subset $j_n : U_n \subset S$ where $n \in \mathbb{Z}$ lies in $\mathcal{F}$. On this open subset, we have a map $n\mathbb{Z} \to \underline{\mathbb{R}/\mathbb{Z}}$, and by the above this extends uniquely to $\overline{j}_n : \overline{U_n} \hookrightarrow S$. The extension $\overline{j}_{n!}\mathbb{Z} \to \underline{\mathbb{R}/\mathbb{Z}}$ necessarily agrees with the restriction of the given map $\mathcal{F} \to \underline{\mathbb{R}/\mathbb{Z}}$ on the intersection $\overline{j}_{n!}\mathbb{Z} \cap \mathcal{F} \subset \overline{j}_{n!}\mathbb{Z}$, as this contains the dense subset $j_{n!}\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ is separated.

Thus, by maximality of $\mathcal{F}$, we see that necessarily all U_n are open and closed, hence so are all $V_n = U_n \setminus \bigcup_{m<n} U_m$. Thus $V_1, V_2, \ldots \subset S$ are pairwise disjoint open and closed subsets such that $\mathcal{F} = \bigoplus_n n\mathbb{Z}|_{V_n}$. But one can then extend to $\bigoplus_n \mathbb{Z}|_{V_n}$ as the continuous maps from V_n to $\mathbb{R}/\mathbb{Z}$ form a divisible group. By maximality of $\mathcal{F}$, this means that V_n is empty for all $n > 1$, and hence $\mathcal{F} = \mathbb{Z}|_{V_1}$ is a direct summand of $\mathbb{Z}$, in which case the possibility of extension is clear. □

Now we can give the proof of Theorem VII.1.3.

Proof of Theorem VII.1.3. The Pro-category of torsion constructible étale sheaves is an abelian category, and by Proposition VII.1.6 the functor to pro-étale $\widehat{\mathbb{Z}}$-sheaves is exact. It is also fully faithful: For this, it suffices to see that if $\mathcal{F}_i$, $i \in I$, is a cofiltered inverse system of torsion constructible étale sheaves and $\mathcal{G}$ is any étale sheaf, then

$$\varinjlim_i \mathrm{Hom}(\mathcal{F}_i, \mathcal{G}) \to \mathrm{Hom}(\varprojlim_i \mathcal{F}_i, \mathcal{G})$$

is an isomorphism. But the underlying pro-étale sheaf of each $\mathcal{F}_i$ is a spatial diamond over X, so by [**Sch17a**, Proposition 14.9] (applied with $j = 0$) we see that the similar result holds true when then taking homomorphisms of pro-étale sheaves (without the abelian group structure). Enforcing

compatibility with addition amounts to a similar diagram for $\varprojlim_i \mathcal{F}_i \times \varprojlim_i \mathcal{F}_i$ to which the same argument applies.

We see that the Pro-category of torsion constructible étale sheaves is a full subcategory $\mathcal{C}$ of all pro-étale $\widehat{\mathbb{Z}}$-modules on X, stable under the formation of kernels and cokernels. All of these sheaves are solid: As the condition of being solid is stable under all limits, it suffices to see that any étale sheaf is solid; this is Proposition VII.1.2.

Also, by [**Sch17a**, Lemma 11.22] and Lemma VII.1.5 all objects of $\mathcal{C}$ have as underlying pro-étale sheaf a spatial diamond. One also checks that $\mathcal{C}$ is stable under extensions, by reduction to constructible sheaves (cf. proof of Proposition VII.1.12 below).

Next, we prove that (3) implies (1), so let $\mathcal{F}$ be in $\mathcal{C}$; in particular, the underlying pro-étale sheaf is a spatial diamond. Then for any pro-étale $\widehat{\mathbb{Z}}$-module $\mathcal{G}$, one can describe $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ as the maps $\mathcal{F} \to \mathcal{G}$ of pro-étale sheaves satisfying additivity and $\widehat{\mathbb{Z}}$-linearity, i.e. certain maps $\mathcal{F} \times \mathcal{F} \to \mathcal{G}$ resp. $\mathcal{F} \times \widehat{\mathbb{Z}} \to \mathcal{G}$ agree. This description commutes with filtered colimits (as for spatial diamonds Y, the functor $\mathcal{G} \mapsto \mathcal{G}(Y)$ commutes with filtered colimits).

Now we can describe the full category of solid $\widehat{\mathbb{Z}}$-sheaves. Indeed, using that $j_\natural \widehat{\mathbb{Z}}$ is finitely presented in all pro-étale $\widehat{\mathbb{Z}}$-modules by the previous paragraph, we see from the definition that the category of solid $\widehat{\mathbb{Z}}$-sheaves is stable under all filtered colimits. In particular, we get an exact functor from the Ind-category of $\mathcal{C}$ to solid $\widehat{\mathbb{Z}}$-sheaves. This is also fully faithful, as all objects of $\mathcal{C}$ are finitely presented. Moreover, $\mathrm{Ind}(\mathcal{C})$ is an abelian category for formal reasons. We see that $\mathrm{Ind}(\mathcal{C})$ is a full subcategory of the category of pro-étale $\widehat{\mathbb{Z}}$-sheaves stable under kernels and cokernels, and all of its objects are solid. Conversely, any solid $\widehat{\mathbb{Z}}$-sheaf admits a surjection from a direct sum of objects of the form $j_\natural \widehat{\mathbb{Z}} \in \mathcal{C}$, and the kernel of any such surjection is still solid, so we may write any solid $\widehat{\mathbb{Z}}$-sheaf as the cokernel of a map in $\mathrm{Ind}(\mathcal{C})$. As $\mathrm{Ind}(\mathcal{C})$ is stable under cokernels, we see that $\mathrm{Ind}(\mathcal{C})$ is exactly the category of solid $\widehat{\mathbb{Z}}$-sheaves.

As filtered colimits of solid sheaves stay solid, it is now formal that (1) implies (2), and (2) implies (3) as $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C})$ are the finitely presented objects (as $\mathcal{C}$ is idempotent-complete). This finishes the proof of the equivalences.

The identification with $\mathrm{Ind}(\mathcal{C})$ shows that the category of solid $\widehat{\mathbb{Z}}$-sheaves is stable under kernels, cokernels, and filtered colimits. The latter two imply stability under all colimits, and stability under all limits is clear from the definition. One also easily checks stability under extensions by reduction to $\mathcal{C}$ (again, cf. proof of Proposition VII.1.12). For the existence of the left adjoint, note that it exists on the free pro-étale $\widehat{\mathbb{Z}}$-modules generated by U, with value $j_\natural \widehat{\mathbb{Z}}$, i.e.

$$\widehat{\mathbb{Z}}[U]^{\blacksquare} = j_\natural \widehat{\mathbb{Z}}.$$

As these generate all pro-étale $\widehat{\mathbb{Z}}$-modules, one finds that the left adjoint $\mathcal{F} \mapsto \mathcal{F}^{\blacksquare}$ exists in general: one can write any $\mathcal{F}$ as a colimit $\varinjlim_\alpha \widehat{\mathbb{Z}}[U_\alpha]$ and $(\varinjlim_\alpha \widehat{\mathbb{Z}}[U_\alpha])^{\blacksquare} = \varinjlim_\alpha \widehat{\mathbb{Z}}[U_\alpha]^{\blacksquare}$. $\square$

We have the following proposition on the functorial behaviour of the notion of solid $\widehat{\mathbb{Z}}$-sheaves.

Proposition VII.1.8. *Let $f : Y \to X$ be a map of spatial diamonds. Then pullbacks of solid $\widehat{\mathbb{Z}}$-sheaves are solid, and the functor f^* commutes with solidification. Moreover, if f is surjective, and $\mathcal{F}$ is a pro-étale $\widehat{\mathbb{Z}}$-sheaf on X such that $f^*\mathcal{F}$ is solid, then $\mathcal{F}$ is solid.*

Proof. Recall that f^* commutes with all limits (and of course colimits) by [**Sch17a**, Lemma 14.4]. To check that f^* commutes with solidification, it suffices to check on the pro-étale $\widehat{\mathbb{Z}}$-modules $\widehat{\mathbb{Z}}[U]$ generated by some quasi-pro-étale $j : U \to X$, and in that case the claim follows from f^* commuting with all limits,

$$f^*(\widehat{\mathbb{Z}}[U]^{\blacksquare}) = f^*(\varprojlim_i \widehat{\mathbb{Z}}[U_i]) = \varprojlim_i f^*\widehat{\mathbb{Z}}[U_i] = \varprojlim_i \widehat{\mathbb{Z}}[f^*U_i] = \widehat{\mathbb{Z}}[f^*U]^{\blacksquare}.$$

In particular, applied to solid $\widehat{\mathbb{Z}}$-sheaves on X, this implies that their pullback to Y is already solid.

Now assume that f is surjective and $\mathcal{F}$ is a pro-étale $\widehat{\mathbb{Z}}$-sheaf on X such that $f^*\mathcal{F}$ is solid. Let $j : U = \varprojlim_i U_i \to X$ be a quasi-pro-étale map. We first check that

$$\mathrm{Hom}(j_\natural\widehat{\mathbb{Z}}, \mathcal{F}) \to \mathcal{F}(U)$$

is injective. Indeed, assume that $f : j_\natural\widehat{\mathbb{Z}} \to \mathcal{F}$ lies in the kernel. Then after pullback to Y, this map vanishes since $f^*\widehat{\mathbb{Z}}[U]^{\blacksquare} = \widehat{\mathbb{Z}}[f^*U]^{\blacksquare}$. But for any quasi-pro-étale $V \to X$, the map $\mathcal{F}(V) \to (f^*\mathcal{F})(V \times_X Y)$ is injective (using [**Sch17a**, Proposition 14.7] one has $f^*\mathcal{F}(V \times_X Y) = (\lambda_X^*\mathcal{F})(V \times_X Y)$ and we conclude since $V \times_X Y \to V$ is a v-cover), so it follows that $f = 0$.

We see that an element of $\mathcal{F}(U)$ determines at most one map $j_\natural\widehat{\mathbb{Z}} \to \mathcal{F}$, and this assertion stays true after any pullback. By [**Sch17a**, Proposition 14.7], it suffices to construct the map v-locally; but it exists after pullback to $Y \to X$, thus proving existence. □

In particular, it makes sense to make the following definition.

Definition VII.1.9. *Let Y be a small v-stack and let $\mathcal{F}$ be a v-sheaf of $\widehat{\mathbb{Z}}$-modules on Y. Then $\mathcal{F}$ is solid if for all maps $f : X \to Y$ from a spatial diamond X, the pullback $f^*\mathcal{F}$ comes via pullback from a solid $\widehat{\mathbb{Z}}$-sheaf on $X_{\mathrm{qproét}}$.*

Regarding passage to the derived category, we make the following definition.

Definition VII.1.10. *Let X be a small v-stack. Let $D_{\blacksquare}(X, \widehat{\mathbb{Z}}) \subset D(X_v, \widehat{\mathbb{Z}})$ be the full subcategory of all A such that each cohomology sheaf $\mathcal{H}^i(A)$ is solid.*

As being solid is stable under kernels, cokernels, and extensions, this defines a triangulated subcategory.

If X is a diamond, one could alternatively define a full subcategory of $D(X_{\mathrm{qproét}}, \widehat{\mathbb{Z}})$ by the same condition, and pullback from the quasi-pro-étale to the v-site defines a functor. This functor is an equivalence, by repleteness (to handle Postnikov towers, cf. [**BS15**, Section 3]) and the following proposition that is an amelioration of [**Sch17a**, Proposition 14.7] for solid sheaves.

Proposition VII.1.11. *Let X be a diamond and let $\mathcal{F}$ be a sheaf of $\widehat{\mathbb{Z}}$-modules on $X_{\mathrm{qproét}}$ that is solid. Let $\lambda : X_v \to X_{\mathrm{qproét}}$ be the map of sites. Then $\mathcal{F} \to R\lambda_*\lambda^*\mathcal{F}$ is an isomorphism.*

Proof. We may assume that X is spatial (or strictly totally disconnected). Then $\mathcal{F}$ is a filtered colimit of finitely presented solid $\widehat{\mathbb{Z}}$-sheaves, and the functor $R\lambda_*$ commutes with filtered colimits in $D^{\geq 0}$. We may thus assume that $\mathcal{F}$ is finitely presented; in that case $\mathcal{F}$ is a cofiltered limit of torsion constructible étale sheaves, and λ^* commutes with all limits by [**Sch17a**, Lemma 14.4]. Thus, we can assume that $\mathcal{F}$ is an étale sheaf, where the claim is [**Sch17a**, Proposition 14.7]. □

Moreover, solid objects in the derived category satisfy a derived and internal version of Definition VII.1.1.

Proposition VII.1.12. *Let X be a spatial diamond. For all $A \in D_\blacksquare(X, \widehat{\mathbb{Z}})$, the map*

$$R\mathscr{H}\mathrm{om}(j_\natural \widehat{\mathbb{Z}}, A) \to Rj_* A|_U$$

is an isomorphism for all quasi-pro-étale $j : U \to X$.

Proof. By taking a Postnikov limit, we can assume that $A \in D^+_\blacksquare(X, \widehat{\mathbb{Z}})$, and then one reduces to the case that $A = \mathcal{F}[0]$ is concentrated in degree 0. Now by a resolution of Breen [**Bre78**, Section 3] (appropriately sheafified), there is a resolution of any $\widehat{\mathbb{Z}}$-sheaf $\mathcal{G}$ where all terms are finite direct sums of sheaves of the form $\widehat{\mathbb{Z}}[\mathcal{G}^i \times \widehat{\mathbb{Z}}^j]$. If $\mathcal{G}$ is a spatial diamond, then all $\mathcal{G}^i \times \widehat{\mathbb{Z}}^j$ are spatial diamonds, hence

$$R\mathscr{H}\mathrm{om}(\widehat{\mathbb{Z}}[\mathcal{G}^i \times \widehat{\mathbb{Z}}^j], \mathcal{F})$$

commutes with all filtered colimits. Applied to $\mathcal{G} = j_\natural \widehat{\mathbb{Z}}$, Breen's resolution then implies that $R\mathscr{H}\mathrm{om}(\mathcal{G}, -)$ commutes with all filtered colimits.

We may thus assume that $\mathcal{F}$ is finitely presented. But then Theorem VII.1.3 implies that $\mathcal{F}$ is a limit of constructible étale sheaves, so one can reduce to the case that $\mathcal{F}$ is an étale sheaf. But then Breen's resolution shows that

$$R\mathscr{H}\mathrm{om}(j_\natural \widehat{\mathbb{Z}}, \mathcal{F}) = \varinjlim_i R\mathscr{H}\mathrm{om}(j_{i!} \widehat{\mathbb{Z}}, \mathcal{F}) = \varinjlim_i Rj_{i*} \mathcal{F}|_{U_i}$$

and [**Sch17a**, Proposition 14.9] shows that this identifies with $Rj_* \mathcal{F}|_U$. □

Proposition VII.1.13. *Let X be a spatial diamond. The inclusion*

$$D_\blacksquare(X, \widehat{\mathbb{Z}}) \subset D(X_{\mathrm{qproét}}, \widehat{\mathbb{Z}})$$

admits a left adjoint

$$A \mapsto A^\blacksquare : D(X_{\mathrm{qproét}}, \widehat{\mathbb{Z}}) \to D_\blacksquare(X, \widehat{\mathbb{Z}}).$$

Moreover, $D_\blacksquare(X, \widehat{\mathbb{Z}})$ identifies with the derived category of solid $\widehat{\mathbb{Z}}$-sheaves on X, and $A \mapsto A^\blacksquare$ with the left derived functor of $\mathcal{F} \mapsto \mathcal{F}^\blacksquare$. The formation of $A \mapsto A^\blacksquare$, for $A \in D(X_{\mathrm{qproét}}, \widehat{\mathbb{Z}})$, commutes with any base change $X' \to X$ of spatial diamonds.

Proof. This follows easily from Proposition VII.1.12. □

Proposition VII.1.14. *Let X be a spatial diamond. The kernel of $A \mapsto A^\blacksquare$ is a tensor ideal. In particular, there is a unique symmetric monoidal structure $- \overset{\blacksquare}{\otimes}^{\mathbb{L}} -$ on $D_\blacksquare(X, \widehat{\mathbb{Z}})$ making $A \mapsto A^\blacksquare$ symmetric monoidal. It is the left derived functor of the induced symmetric monoidal structure on solid $\widehat{\mathbb{Z}}$-sheaves. This symmetric monoidal structure commutes with all colimits (in each variable) and any pullback.*

Proof. To check that the kernel is a tensor ideal, take any quasi-pro-étale $j : U \to X$ written as a cofiltered inverse limit of separated étale $j_i : U_i \to X$, and any further quasi-pro-étale $j' : U' \to X$. Then for any solid $A \in D_\blacksquare(X, \widehat{\mathbb{Z}})$, we know by Proposition VII.1.12 that the map

$$R\mathscr{H}\mathrm{om}(j_\natural \widehat{\mathbb{Z}}, A) \to Rj_* A|_U$$

is an isomorphism. Taking sections over $U' \to X$, this translates into

$$R\,\mathrm{Hom}(j_\natural \widehat{\mathbb{Z}} \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}[U'], A) \to R\,\mathrm{Hom}(\widehat{\mathbb{Z}}[U \times_X U'], A)$$

being an isomorphism. In other words, taking the tensor product of $\widehat{\mathbb{Z}}[U] \to j_\natural \widehat{\mathbb{Z}}$ with $\widehat{\mathbb{Z}}[U']$ still lies in the kernel, but these generate the tensor ideal generated by the kernel.

It is now formal that there is a unique symmetric monoidal structure $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} -$ on $D_\blacksquare(X, \widehat{\mathbb{Z}})$ making $A \mapsto A^\blacksquare$ symmetric monoidal (given by the solidification of the tensor product in all solid pro-étale sheaves). As solidification commutes with all colimits, so does this tensor product. On generators $j : U \to X$, $j' : U' \to X$ as above, it is given by $j_\natural \widehat{\mathbb{Z}} \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} j'_\natural \widehat{\mathbb{Z}} = (j \times_X j')_\natural \widehat{\mathbb{Z}}$, which still sits in degree 0; this implies that the functor is a left derived functor. Moreover, this description commutes with any base change. □

Moreover, the inclusion into all v-sheaves also admits a left adjoint, if X is a diamond. We will later improve on this proposition when working with $\widehat{\mathbb{Z}}^p$-coefficients.

Proposition VII.1.15. *For any diamond X, the fully faithful embedding*

$$D_\blacksquare(X, \widehat{\mathbb{Z}}) \subset D(X_v, \widehat{\mathbb{Z}})$$

admits a left adjoint $A \mapsto A^\blacksquare$. The formation of $A^\blacksquare$ commutes with quasi-pro-étale base change $X' \to X$.

Proof. Assume first that X is strictly totally disconnected. It suffices to construct the left adjoint on a set of generators, such as the pro-étale sheaves of $\widehat{\mathbb{Z}}$-modules generated by some strictly totally disconnected $Y \to X$. By [**Sch17a**, Lemma 14.5], there is a strictly totally disconnected affinoid pro-étale $j : Y' \to X$ such that $Y \to X$ factors over a map $Y \to Y'$ that is surjective and induces a bijection of connected components. Then for any $B \in D_\blacksquare(Y', \widehat{\mathbb{Z}})$, the map

$$R\Gamma(Y', B) \to R\Gamma(Y, B)$$

is an isomorphism. Indeed, by Postnikov limits this easily reduces to $B = \mathcal{F}[0]$ for a solid sheaf of $\widehat{\mathbb{Z}}$-modules, and then to a finitely presented solid sheaf, and finally to a constructible étale sheaf, for which the result is proved at the end of the proof of [**Sch17a**, Lemma 14.4]. This means that the left adjoint $A \mapsto A^\blacksquare$ when evaluated on $\widehat{\mathbb{Z}}[Y]$ exists and is given by $j_\natural \widehat{\mathbb{Z}}$.

The formation of $Y' \to X$ from $Y \to X$ commutes with any quasi-pro-étale base change of strictly totally disconnected $X' \to X$. This implies that $A \mapsto A^\blacksquare$ commutes with such base changes. By descent, this implies the existence of the left adjoint in general, and its commutation with quasi-pro-étale base change. □

As usual, we also want to have a theory with coefficients in a ring Λ. As before, we assume that Λ is constant in the sense that it comes via pullback from the point. In our case, this means that

it comes via pullback from the pro-étale site of a point, i.e. is a condensed ring [**CS**], and we need to assume that it is solid over $\widehat{\mathbb{Z}}$; in other words, we allow as coefficients any solid $\widehat{\mathbb{Z}}$-algebra Λ. Via pullback, this gives rise to a v-sheaf of $\widehat{\mathbb{Z}}$-algebras on any small v-stack X, and we can consider $D(X_v, \Lambda)$.

Example VII.1.16. We may consider $\Lambda = \overline{\mathbb{Z}}_\ell$ as the solid condensed ring $\varinjlim_{L|\mathbb{Q}_\ell \text{ finite}} \mathcal{O}_L$.

Definition VII.1.17. *Let $D_\blacksquare(X, \Lambda) \subset D(X_v, \Lambda)$ be the full subcategory of all $A \in D(X_v, \Lambda)$ such that the image of A in $D(X_v, \widehat{\mathbb{Z}})$ is solid.*

On the level of ∞-categorical enrichments, we thus see that $\mathcal{D}_\blacksquare(X, \Lambda)$ is the category of $\underline{\Lambda}$-modules in $\mathcal{D}_\blacksquare(X, \widehat{\mathbb{Z}})$. It is then formal that the inclusion $D_\blacksquare(X, \Lambda) \subset D(X_v, \Lambda)$ admits a symmetric monoidal left adjoint $A \mapsto A^\blacksquare$, compatible with forgetting the Λ-structure.

Remark VII.1.18. Let us briefly compare the present theory with the one developed in [**CS**]. Over a geometric point $X = \operatorname{Spa} C$, $D_\blacksquare(X, \widehat{\mathbb{Z}})$ is the derived category of solid $\widehat{\mathbb{Z}}$-modules in the sense of [**CS**]. For general Λ, we are now simply considering Λ-modules in $\mathcal{D}_\blacksquare(X, \widehat{\mathbb{Z}})$. This is in general different from the theory of $\Lambda_\blacksquare$-modules, which would ask for a stronger completeness notion relative to Λ. Our present theory corresponds to the analytic ring structure on Λ induced from $\widehat{\mathbb{Z}}_\blacksquare$.

One might wonder whether for any analytic ring $\mathcal{A}$ in the sense of [**CS**] one can define a category $D(X, \mathcal{A})$ of "$\mathcal{A}$-complete" pro-étale sheaves on any spatial diamond X. This does not seem to be the case; it is certainly not formal. In fact, already for $\mathcal{A} = \mathbb{Z}_\blacksquare$, problems occur and there is certainly no abelian category; it is still possible to define a nice derived category, though. For general $\mathcal{A}$, defining $D(X, \mathcal{A})$ also seems to require extra data beyond the analytic ring structure on $\mathcal{A}$.

VII.2. Four functors

Now we discuss some functors on solid sheaves. For this, *we assume from now on that we work with coefficients Λ given by a solid $\widehat{\mathbb{Z}}^p$-algebra* (so we stay away from p-adic coefficients). For any map $f : Y \to X$ of small v-stacks, we have the pullback functor $f^* : D_\blacksquare(X, \Lambda) \subset D_\blacksquare(Y, \Lambda)$. This admits a right adjoint Rf_*; in fact, one can simply import Rf_* from the full $D(Y, \Lambda)$:

Proposition VII.2.1. *Let $f : Y \to X$ be a map of small v-stacks and let $A \in D_\blacksquare(Y, \Lambda) \subset D(Y, \Lambda)$. Then $Rf_{v*}A \in D(X_v, \Lambda)$ lies in $D_\blacksquare(X, \Lambda)$. In particular, $Rf_{v*} : D(Y_v, \Lambda) \to D(X_v, \Lambda)$ restricts to a functor $Rf_* : D_\blacksquare(Y, \Lambda) \to D_\blacksquare(X, \Lambda)$ that is right adjoint to f^*.*

Proof. We can formally reduce to the case $\Lambda = \widehat{\mathbb{Z}}^p$. The formation of Rf_{v*} commutes with any pullback (as everything is a slice in the v-site), so using Proposition VII.1.8 we can assume that X is a spatial diamond. Moreover, taking a simplicial resolution of Y by disjoint unions of spatial diamonds, and using that $D_\blacksquare(X, \widehat{\mathbb{Z}}^p) \subset D(X_v, \widehat{\mathbb{Z}}^p)$ is stable under all derived limits (as it is stable under all products), we can also assume that Y is a spatial diamond.

We may assume $A \in D^+_\blacksquare(Y, \widehat{\mathbb{Z}}^p)$ by a Postnikov limit, then that $A = \mathcal{F}[0]$ is concentrated in degree 0, then that $\mathcal{F}$ is finitely presented by writing it as a filtered colimit, and finally that $\mathcal{F}$ is

a constructible étale sheaf by writing it as a cofiltered limit. Now the result follows from [**Sch17a**, Proposition 17.6]. □

PROPOSITION VII.2.2. *For any small v-stack X, the inclusion*

$$D_{\blacksquare}(X,\Lambda) \subset D(X_v,\Lambda)$$

admits a left adjoint

$$A \mapsto A^{\blacksquare} : D(X_v,\Lambda) \to D_{\blacksquare}(X,\Lambda).$$

The functor $A \mapsto A^{\blacksquare}$ commutes with any base change.

The kernel of $A \mapsto A^{\blacksquare}$ is a tensor ideal. In particular, there is a unique symmetric monoidal structure $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$ on $D_{\blacksquare}(X,\Lambda)$ making $A \mapsto A^{\blacksquare}$ a symmetric monoidal functor. The functor $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$ commutes with all colimits (in each variable) and with all pullbacks $f : Y \to X$.

We note that in the case of overlap with previous definitions of $A \mapsto A^{\blacksquare}$ and $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$, the definitions agree, by uniqueness of the previous definitions.

PROOF. Again, one can formally reduce to the case $\Lambda = \widehat{\mathbb{Z}}^p$. By descent, we can reduce to the case that X is strictly totally disconnected. (Note that $Y \mapsto \mathcal{D}_{\blacksquare}(Y,\widehat{\mathbb{Z}}^p)$ is a v-sheaf of ∞-categories — this is clear for $\mathcal{D}(Y_v,\widehat{\mathbb{Z}}^p)$, and follows for $\mathcal{D}_{\blacksquare}$ as being solid can be checked v-locally by Proposition VII.1.8.) In this case, we already know existence of the left adjoint $A \mapsto A^{\blacksquare}$ by Proposition VII.1.15.

We check that the left adjoint $A \mapsto A^{\blacksquare}$ commutes with any base change $f : Y \to X$. We already know that pullbacks of solid objects stay solid, so we have to see that if $A \in D(X_v,\widehat{\mathbb{Z}}^p)$ satisfies $A^{\blacksquare} = 0$, then also $(f_v^* A)^{\blacksquare} = 0$. But this statement is adjoint to the statement that Rf_{v*} preserves $D_{\blacksquare}$, i.e. Proposition VII.2.1.

We need to see that the class of all $A \in D(X_v,\widehat{\mathbb{Z}}^p)$ with $A^{\blacksquare} = 0$ is a $\otimes$-ideal. But we have seen that for all $f : Y \to X$, also $f_v^* A$ lies in the corresponding class for Y, and then so does $f_{v\natural} f_v^* A$ (as pullback preserves $D_{\blacksquare}$), where we write $f_{v\natural}$ for the left adjoint of f_v^* (which exists as it is a slice). But $f_{v\natural} f_v^* A = A \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}^p} f_{v\natural}\widehat{\mathbb{Z}}^p$ by the projection formula for slices, so this gives the desired claim. □

It turns out that for $\Lambda = \widehat{\mathbb{Z}}^p$, the functor $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} -$ is actually almost exact. If one would work with $\Lambda = \mathbb{F}_\ell$-coefficients, it would even be exact.

PROPOSITION VII.2.3. *Let X be a small v-stack and $A, B \in D_{\blacksquare}(X,\widehat{\mathbb{Z}}^p)$ be concentrated in degree 0. Then $A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B$ sits in cohomological degrees -1 and 0.*

If X is a spatial diamond and $\mathcal{F} = \varprojlim_i \mathcal{F}_i$ and $\mathcal{G} = \varprojlim_j \mathcal{G}_j$ are finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves written as cofiltered limits of constructible étale sheaves killed by some integer prime to p, then the natural map

$$\mathcal{F} \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} \mathcal{G} \to R\varprojlim_{i,j} \mathcal{F}_i \otimes^{\mathbb{L}} \mathcal{G}_j$$

is an isomorphism.

Proof. It suffices to prove the final assertion, as the statement on $A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B$ can be checked after pullback to spatial diamonds, and then A and B can be written as filtered colimits of finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves (and $\mathcal{F}_i \otimes^{\mathbb{L}} \mathcal{G}_j$ sits in degrees -1 and 0 as $\widehat{\mathbb{Z}}^p$ has global dimension 1). Resolving $\mathcal{F}$ and $\mathcal{G}$, we can reduce to the case $\mathcal{F} = j_\natural \widehat{\mathbb{Z}}^p$, $\mathcal{G} = j'_\natural \widehat{\mathbb{Z}}^p$. But their solid tensor product is indeed given by $(j \times_X j')_\natural \widehat{\mathbb{Z}}^p$. $\square$

At this point, we have defined $D_\blacksquare(X, \Lambda) \subset D(X_v, \Lambda)$ for any small v-stack X, and this subcategory is preserved by pullback and pushforward, and in particular this gives such functors for $D_\blacksquare(X, \Lambda)$. Moreover, $D_\blacksquare(X, \Lambda)$ has a natural symmetric monoidal structure $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$, commuting with colimits in both variables, and with pullbacks. Moreover, we have a functor

$$R\mathscr{H}\mathrm{om}_\Lambda(-,-) : D_\blacksquare(X, \Lambda)^{\mathrm{op}} \times D_\blacksquare(X, \Lambda) \to D_\blacksquare(X, \Lambda),$$

a partial right adjoint to $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$ as usual. Again, it can be obtained from the corresponding functor on $D(X_v, \Lambda)$ via restriction. In fact, for all $A \in D(X_v, \Lambda)$ and $B \in D_\blacksquare(X, \Lambda)$, one has $R\mathscr{H}\mathrm{om}_\Lambda(A, B) \in D_\blacksquare(X, \Lambda)$. This can be reduced to $\Lambda = \widehat{\mathbb{Z}}^p$ and the case $A = f_\natural \widehat{\mathbb{Z}}^p$ for some $f : Y \to X$, and then it amounts to $Rf_{v*} f_v^* B \in D_\blacksquare(X, \widehat{\mathbb{Z}}^p)$, which follows from Proposition VII.2.1.

There is the following general base change result. We stress the absence of *any* conditions.

Proposition VII.2.4. *Let*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

be a cartesian diagram of small v-stacks. For all $A \in D_\blacksquare(Y, \Lambda)$, the base change map

$$g^* R f_* A \to R f'_* g'^* A$$

in $D_\blacksquare(X', \Lambda)$ is an isomorphism.

Similarly, for any map $f : Y \to X$ of small v-stacks and all $A, B \in D_\blacksquare(X, \Lambda)$, the map

$$f^* R\mathscr{H}\mathrm{om}(A, B) \to R\mathscr{H}\mathrm{om}(f^* A, f^* B)$$

in $D_\blacksquare(Y, \Lambda)$ is an isomorphism.

Proof. The base change is a direct consequence of Proposition VII.2.1, noting that in the v-site, everything is a slice (and hence satisfies base change). The statement about $R\mathscr{H}\mathrm{om}$ follows similarly from the compatibility with the $R\mathscr{H}\mathrm{om}$ as formed on the v-site, as was noted above. $\square$

The projection formula, however, fails to hold.

Warning VII.2.5. If $f : Y \to X$ is a proper map of small v-stacks that is representable in spatial diamonds with $\dim.\operatorname{trg} f < \infty$, the map

$$A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} R f_* B \to R f_*(f^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B)$$

may fail to be an isomorphism for $A \in D_\blacksquare(X, \widehat{\mathbb{Z}}^p)$ and $B \in D_\blacksquare(Y, \widehat{\mathbb{Z}}^p)$. In fact, already if $X = \mathbb{B}_C$ is a perfectoid ball and $f = j : Y = \operatorname{Spa} C \to X$ is the inclusion of a point (which is quasi-pro-étale),

then this fails for $A = j_\natural \widehat{\mathbb{Z}}^p$ and $B = \widehat{\mathbb{Z}}^p$. In fact, the map becomes $j_\natural \widehat{\mathbb{Z}}^p \to Rj_* \widehat{\mathbb{Z}}^p$, which is far from an isomorphism: For example, on global sections the left-hand side becomes $\widehat{\mathbb{Z}}^p[-2]$, while the right-hand side becomes $\widehat{\mathbb{Z}}^p$.

There is the following result on change of algebraically closed base field, an analogue of [**Sch17a**, Theorem 19.5].

Proposition VII.2.6. *Let X be a small v-stack.*

(i) *Assume that X lives over k, where k is a discrete algebraically closed field of characteristic p, and k'/k is an extension of discrete algebraically closed base fields, $X' = X \times_k k'$. Then the pullback functor*

$$D_\blacksquare(X, \Lambda) \to D_\blacksquare(X', \Lambda)$$

is fully faithful.

(ii) *Assume that X lives over k, where k is a discrete algebraically closed field of characteristic p. Let C/k be an algebraically closed complete nonarchimedean field, and $X' = X \times_k \mathrm{Spa}(C, C^+)$ for some open and bounded valuation subring $C^+ \subset C$ containing k. Then the pullback functor*

$$D_\blacksquare(X, \Lambda) \to D_\blacksquare(X', \Lambda)$$

is fully faithful.

(iii) *Assume that X lives over $\mathrm{Spa}(C, C^+)$, where C is an algebraically closed complete nonarchimedean field with an open and bounded valuation subring $C^+ \subset C$, C'/C is an extension of algebraically closed complete nonarchimedean fields, and $C'^+ \subset C'$ an open and bounded valuation subring containing C^+, such that $\mathrm{Spa}(C', C'^+) \to \mathrm{Spa}(C, C^+)$ is surjective. Then for $X' = X \times_{\mathrm{Spa}(C,C^+)} \mathrm{Spa}(C', C'^+)$, the pullback functor*

$$D_\blacksquare(X, \Lambda) \to D_\blacksquare(X', \Lambda)$$

is fully faithful.

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. As in [**Sch17a**, Theorem 19.5], it suffices to prove (iii) and the restricted case of (ii) where C is the completed algebraic closure of $k(\!(t)\!)$ (and hence $C^+ = \mathcal{O}_C$).

Let $f : X' \to X$ be the map. We have to see that for all $A \in D_\blacksquare(X, \widehat{\mathbb{Z}}^p)$, the map

$$A \to Rf_* f^* A$$

is an equivalence. This can be checked locally in the v-topology, so we can assume that $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space. By Postnikov limits, we can also assume that $A \in D^+_\blacksquare(X, \widehat{\mathbb{Z}}^p)$, and then that A is concentrated in degree 0. In case (iii), we can now conclude by writing A as a filtered colimit of finitely presented solid $\widehat{\mathbb{Z}}^p$-modules, and these as cofiltered limits of constructible étale sheaves, noting that both operations commute with Rf_* and f^* (as f is qcqs in case (iii)), and hence reducing us to [**Sch17a**, Theorem 19.5].

It remains to handle case (ii) when C is the completed algebraic closure of $k(\!(t)\!)$. In that case X' lives over a punctured open unit disc $\mathbb{D}^*_X$ over X, and fixing a pseudouniformizer $\varpi \in R$, this can be written as the increasing union of quasicompact open subspaces $X'_n = \{|t|^n \leq |\varpi| \leq |t|^{1/n}\} \subset X'$, with maps $f_n : X'_n \to X$. It suffices to prove that for all n, the map

$$A \to Rf_{n*} f_n^* A$$

is an isomorphism. These functors commute again with filtered colimits of sheaves, and hence the previous reductions apply and reduce the assertion to the étale case, which was handled in the proof of [**Sch17a**, Theorem 19.5]. □

As an application, let us record the following versions of Proposition IV.7.1 and Corollary IV.7.2, where we fix an algebraically closed field $k|\mathbb{F}_q$ and work on Perf_k.

Corollary VII.2.7. *For any small v-stack X, the functor*

$$\psi_X^* : D_\blacksquare(X \times [*/\underline{W_E}], \Lambda) \to D_\blacksquare(X \times \mathrm{Div}^1, \Lambda)$$

is fully faithful. If the natural pullback functor

$$D_\blacksquare(X, \Lambda) \to D_\blacksquare(X \times \mathrm{Spd}\,\widehat{\overline{E}}, \Lambda)$$

is an equivalence, then ψ_X^* is also an equivalence.

Proof. By descent along $X \to X \times [*/\underline{W_E}]$ this reduces to Proposition VII.2.6. □

Corollary VII.2.8. *For any small v-stack X and finite set I, pullback along $X \times (\mathrm{Div}^1)^I \to X \times [*/\underline{W_E^I}]$ induces a fully faithful functor*

$$D_\blacksquare(X \times [*/\underline{W_E^I}], \Lambda) \to D_\blacksquare(X \times (\mathrm{Div}^1)^I, \Lambda).$$

Proof. This follows inductively from Corollary VII.2.7. □

We also need a solid analogue of Theorem IV.5.3; we only prove a restricted variant, however. As there, work over Perf_k, and let X be a spatial diamond such that $X \to *$ is proper, of finite dim. trg, and take any spatial diamond S. As before, one can introduce the doubly-indexed ind-system $\{U_{a,b}\}_{(a,b)} \subset X \times S$, well-defined up to ind-isomorphism; and then $U_a = \bigcup_{b<\infty} U_{a,b}$ and $U_b = \bigcup_{a>0} U_{a,b}$.

Definition VII.2.9. *The functors*

$$R\beta_{!+}, R\beta_{!-} : D_\blacksquare(X \times S, \Lambda) \to D_\blacksquare(S, \Lambda)$$

are defined by

$$R\beta_{!+}C := \varinjlim_a R\beta_*(j_{a!}C|_{U_a}),$$
$$R\beta_{!-}C := \varinjlim_b R\beta_*(j_{b!}C|_{U_b})$$

for $C \in D_\blacksquare(X \times S, \Lambda)$.

Here $j_{a!}$ and $j_{b!}$ denote the left adjoints to j_a^* and j_b^*. Let $\alpha : X \times S \to X$ be the projection.

Theorem VII.2.10. *Assume that $C = \alpha^* A$ for $A \in D_\blacksquare(X, \Lambda)$, and assume that either $A \in D_\blacksquare^+(X, \Lambda)$, or that $X \to *$ is cohomologically smooth. Then*

$$R\beta_{!+}C = 0 = R\beta_{!-}C.$$

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. All operations commute with any base change; we can thus assume that $S = \operatorname{Spa} K$ where K is the complete algebraic closure of $k(\!(t)\!)$. We observe that if $X \to *$ is cohomologically smooth, then $R\beta_* : D_\blacksquare(X \times S, \Lambda) \to D_\blacksquare(S, \Lambda)$ has finite cohomological dimension; this is a statement about sheaves concentrated in degree 0. Any such B can be written as the countable limit of $Rj_{a,b,*}j_{a,b}^*B$ for the open immersions $j_{a,b} : U_{a,b} \subset X \times S$; it is thus enough to show that pushforward along $U_{a,b} \to S$ has finite cohomological dimension on solid sheaves. As $U_{a,b} \to S$ is qcqs, we can reduce to finitely presented sheaves; these are cofiltered limits of constructible sheaves. For constructible sheaves, the cohomological dimension is bounded, and each cohomology group (recall that $S = \operatorname{Spa} K$ is a geometric point) is finite by [**Sch17a**, Theorem 25.1]. Thus, the cofiltered limit stays in the same range of degrees.

It follows that we can assume that $A \in D_\blacksquare^+(X, \widehat{\mathbb{Z}}^p)$. Arguing as in the proof of Theorem IV.5.3, we can then reduce to the case that $X = \operatorname{Spa}(R, R^+)$ is an affinoid perfectoid space with no nonsplit finite étale covers, and then to $X = \operatorname{Spa} K$ where K is still the completed algebraic closure of $k(\!(t)\!)$. In that case, as in the proof of Theorem IV.5.3, one can make a more precise assertion on actual annuli; this statement is compatible with passage to filtered colimits, reducing us to the case that A is a finitely presented solid sheaf. For $A \in D_\blacksquare(\operatorname{Spa} K, \widehat{\mathbb{Z}}^p)$, this means that A is a cofiltered limit of finite abelian groups killed by integers prime to p. We can also pull this cofiltered limit through, reducing us to Theorem IV.5.3. $\square$

VII.3. Relative homology

A unique feature of the formalism of solid sheaves is the existence of a general left adjoint to pullback, with excellent properties. We continue to work with coefficients in a solid $\widehat{\mathbb{Z}}^p$-algebra Λ.

Proposition VII.3.1. *Let $f : Y \to X$ be any map of small v-stacks.*

(i) *The functor $f^* : D_\blacksquare(X, \Lambda) \to D_\blacksquare(Y, \Lambda)$ admits a left adjoint*

$$f_\natural : D_\blacksquare(Y, \Lambda) \to D_\blacksquare(X, \Lambda).$$

The natural map

$$f_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda f^*B) \to f_\natural A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B$$

is an isomorphism for all $A \in D_\blacksquare(Y, \Lambda)$ and $B \in D_\blacksquare(X, \Lambda)$. Similarly, the map

$$R\mathscr{H}\mathrm{om}(f_\natural A, B) \to Rf_* R\mathscr{H}\mathrm{om}(A, f^*B)$$

is an isomorphism.

(ii) *The formation of $f_\natural$ commutes with restriction of coefficients along a map $\Lambda' \to \Lambda$.*

(iii) *For any cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of small v-stacks, the natural map

$$f'_\natural g'^* A \to g^* f_\natural A$$

is an isomorphism for all $A \in D_\blacksquare(Y, \Lambda)$.

Proof. As f is a slice in the v-site, it is tautological that $f_v^* : D(X_v, \Lambda) \to D(Y_v, \Lambda)$ admits a left adjoint $f_{v\natural} : D(Y_v, \Lambda) \to D(X_v, \Lambda)$. One can then define $f_\natural$ as the solidification of $f_{v\natural}$. By general properties of slices, the map

$$f_{v\natural}(A \otimes^{\mathbb{L}}_\Lambda f^*B) \to f_{v\natural}A \otimes^{\mathbb{L}}_\Lambda B$$

is an isomorphism. Passing to solidifications, using that that this is symmetric monoidal, then gives that

$$f_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda f^*B) \to f_\natural A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B$$

is an isomorphism. The isomorphism

$$R\mathscr{H}\mathrm{om}(f_\natural A, B) \cong Rf_* R\mathscr{H}\mathrm{om}(A, f^*B)$$

then follows by adjointness.

For part (ii), we can assume $\Lambda' = \widehat{\mathbb{Z}}^p$, and check on generators. These are given by $j_\natural \widehat{\mathbb{Z}}^p \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}^p} \Lambda$ for $j : Y' \to Y$. The claim then follows from the projection formula.

Part (iii) is obtained by passing to left adjoints in Proposition VII.2.4. □

Now if f is "proper and smooth", we want to relate the left adjoint $f_\natural$ ("homology") and the right adjoint Rf_* ("cohomology"). Thus, assume that $f : Y \to X$ is a proper map of small v-stacks that is representable in spatial diamonds with $\dim.\operatorname{trg} f < \infty$, and cohomologically smooth, i.e. ℓ-cohomologically smooth for all $\ell \neq p$ (or just all ℓ relevant for Λ). In this case, we want to express Rf_* in terms of $f_\natural$. As a first step, we show that Rf_* has bounded cohomological dimension.

Proposition VII.3.2. *Let $f : Y \to X$ be a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\operatorname{trg} f < \infty$*, and cohomologically smooth. Then $Rf_* : D_\blacksquare(Y, \Lambda) \to D_\blacksquare(X, \Lambda)$ has bounded cohomological dimension and commutes with arbitrary direct sums. If X is a spatial diamond (thus Y is) and $\mathcal{F}$ is a finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaf on Y, then $Rf_*\mathcal{F}$ is a bounded complex all of whose cohomology sheaves are finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves on X.*

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. The commutation with arbitrary direct sums follows from bounded cohomological dimension, as one can then reduce to the case of complexes concentrated in degree 0, where Rf_* commutes with all direct sums as f is qcqs. For the claim about bounded cohomological dimension, we can argue v-locally, and hence assume that X is a spatial diamond. It suffices to prove that for all solid $\widehat{\mathbb{Z}}^p$-sheaves $\mathcal{F}$ on Y, the complex $Rf_*\mathcal{F}$ is bounded; this reduces to the case of finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves as Rf_* commutes with filtered colimits of sheaves. Now if $\mathcal{F}$ is finitely presented, it is a cofiltered limit of constructible étale sheaves killed by some integer prime to p. As Rf_* commutes with this limit, it is now enough to see that Rf_* preserves constructible complexes and has bounded amplitude. But this follows from cohomological smoothness, cf. [**Sch17a**, Proposition 23.12 (ii)]. □

Next, we prove a projection formula for Rf_*.

Proposition VII.3.3. *Let $f : Y \to X$ be a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\operatorname{trg} f < \infty$*, and cohomologically smooth. Then for all $A \in D_\blacksquare(Y, \Lambda)$ and $B \in D_\blacksquare(X, \Lambda)$, the projection map*

$$Rf_*A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B \to Rf_*(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} f^*B)$$

is an isomorphism.

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. We note that Rf_* and $\overset{\blacksquare}{\otimes}^{\mathbb{L}}$ both have bounded cohomological dimension, so one easily reduces to the case that A and B are concentrated in degree 0. We can also assume that X is a spatial diamond (thus Y is, too). Then we can write A and B as filtered colimits of finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves, and reduce to the case that A and B are cofiltered limits of constructible étale sheaves killed by some integer prime to p. In that case, it follows from Proposition VII.2.3 and Proposition VII.3.2 that all operations commute with these cofiltered limits, and one reduces to the case that A and B are constructible étale sheaves killed by some integer prime to p. Now it follows from [**Sch17a**, Proposition 22.11]. □

Moreover, the functor Rf_* interacts well with $g_\natural$ for maps $g : X' \to X$.

Proposition VII.3.4. *Let*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

be a cartesian diagram of small v-stacks, where $f : Y \to X$ is proper, representable in spatial diamonds, with $\dim.\operatorname{trg} f < \infty$ *and cohomologically smooth. Then the natural transformation*

$$g_\natural Rf'_* A \to Rf_* g'_\natural A$$

is an isomorphism for all $A \in D_\blacksquare(Y', \Lambda)$.

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. By Proposition VII.3.2 both sides commute with Postnikov limits, so we can assume $A \in D^+$, and then reduce to the case that A is concentrated in degree 0. We may assume that X is a spatial diamond, and one can also reduce to the case X' is a spatial diamond, by writing A as the geometric realization of $h'_{\bullet\natural} h'^{\natural}_\bullet A$ for some simplicial hypercover $h_\bullet : X'_\bullet \to X'$ by disjoint unions of spatial diamonds, and its pullback $h'_\bullet : Y'_\bullet \to Y'$ (and using Proposition VII.3.2 to commute the geometric realization with pushforward). Under these circumstances, one can write A as a filtered colimit of finitely presented solid $\widehat{\mathbb{Z}}^p$-modules, and hence reduce to the case that A is a cofiltered limit of constructible étale sheaves killed by some integer prime to p. By Proposition VII.3.2 the complex $Rf'_* A$ is then bounded with all cohomology sheaves finitely presented solid $\widehat{\mathbb{Z}}^p$-modules. As $g_\natural$ preserves pseudocoherent objects, it follows that the map $g_\natural Rf'_* A \to Rf_* g'_\natural A$ is a map of bounded to the right complexes in $D_\blacksquare(X, \widehat{\mathbb{Z}}^p)$ all of whose cohomology sheaves are finitely presented solid $\widehat{\mathbb{Z}}^p$-modules. If the cone of this map is nonzero, then by looking at its first nonzero cohomology sheaf, we find some nonzero map to a constructible étale sheaf B on X, killed by some integer prime to p. Note that, using the usual étale $Rf^!$ functor, there is a natural adjunction

$$R\operatorname{Hom}(Rf_* g'_\natural A, B) \cong R\operatorname{Hom}(g'_\natural A, Rf^! B) :$$

Indeed, it suffices to check this when $g'_\natural A$ is replaced by a finitely presented solid $\widehat{\mathbb{Z}}^p$-module, by a Postnikov tower (and as all cohomology sheaves of $g'_\natural A$ are of this form). Writing this as a cofiltered

limit of constructible étale sheaves killed by some integer prime to p, both sides turn this cofiltered limit into a filtered colimit, so the claim reduces to the usual étale adjunction.

Applying $R\operatorname{Hom}(-,B)$ to the map $g_\natural Rf'_*A \to Rf_*g'_\natural A$ will thus produce $R\operatorname{Hom}(A,-)$ applied to the base change map

$$Rf'^! g^* B \leftarrow g'^* Rf^! B,$$

which is an isomorphism by [**Sch17a**, Proposition 23.12 (iii)]. □

Now we can describe the functor Rf_*. Indeed, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta_f"] & Y \times_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"] & X. \end{tikzcd}$$

Then, under our assumption that $f : Y \to X$ is a proper map of small v-stacks that is representable in spatial diamonds with $\dim.\operatorname{trg} f < \infty$ and cohomologically smooth, we have

$$\begin{aligned} Rf_*A &\cong Rf_*\pi_{2\natural}\Delta_{f\natural}A \\ &\cong f_\natural R\pi_{1*}\Delta_{f\natural}A \\ &\cong f_\natural R\pi_{1*}\Delta_{f\natural}\Delta_f^*\pi_1^*A \\ &\cong f_\natural R\pi_{1*}(\Delta_{f\natural}\Lambda \overset{\blacksquare\mathbb{L}}{\otimes}_\Lambda \pi_1^*A) \\ &\cong f_\natural(R\pi_{1*}\Delta_{f\natural}\Lambda \overset{\blacksquare\mathbb{L}}{\otimes}_\Lambda A). \end{aligned}$$

We combine this with the following observation.

Proposition VII.3.5. *Let $f : Y \to X$ be a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\operatorname{trg} f < \infty$ *and cohomologically smooth. Then*

$$R\pi_{1*}\Delta_{f\natural}\Lambda \in D_\blacksquare(Y,\Lambda)$$

is invertible, and its inverse is canonically isomorphic to

$$Rf^!\Lambda := \varprojlim_n Rf^!\mathbb{Z}/n\mathbb{Z} \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}^p} \Lambda.$$

Thus, there is a canonical isomorphism

$$f_\natural A \cong Rf_*(A \overset{\blacksquare\mathbb{L}}{\otimes}_\Lambda Rf^!\Lambda) : D_\blacksquare(Y,\Lambda) \to D_\blacksquare(X,\Lambda).$$

Thus, we get a somewhat unusual formula for the dualizing complex. We remark that the fibres of $R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p$ are given by the limit of $R\Gamma_c(U,\widehat{\mathbb{Z}}^p)$ over all étale neighborhoods U of the given geometric point.

Remark VII.3.6. We see here that an important instance of $Rf^!$ admits an alternative description in terms of $g_\natural$ functors. We are a bit confused about exactly how expressive the present 5-functor formalism is. So far, we were always able to translate any argument in terms of a 6-functor formalism into this 5-functor formalism, although it is often a nontrivial matter and there seems to be no completely general recipe.

Proof. We can assume $\Lambda = \widehat{\mathbb{Z}}^p$. By the isomorphism $Rf_* \cong f_\natural(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} A)$, it follows that $Rf_* : D_\blacksquare(Y, \widehat{\mathbb{Z}}^p) \to D_\blacksquare(X, \widehat{\mathbb{Z}}^p)$ admits a right adjoint, given by

$$A \mapsto R\mathscr{H}\mathrm{om}(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p, f^*A).$$

We claim that this right adjoint maps $D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$ into $D_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z})$ for any n prime to p, and thus agrees with the right adjoint $Rf^!$ in that setting. This claim can be checked v-locally, so we can assume that X is a spatial diamond. Then $R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p \in D_\blacksquare(Y, \widehat{\mathbb{Z}}^p)$ is a bounded complex all of whose cohomology sheaves are finitely presented solid, by Proposition VII.3.2 and as Δ_f is quasi-pro-étale (so $\Delta_{f\natural}\widehat{\mathbb{Z}}^p$ is finitely presented solid). This implies that $R\mathscr{H}\mathrm{om}(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p, -)$ preserves $D_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z})$.

Thus, for any $A \in D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$, there is a natural isomorphism

$$Rf^!A \cong R\mathscr{H}\mathrm{om}(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p, f^*A).$$

Applied with $A = \mathbb{Z}/n\mathbb{Z}$, this gives isomorphisms

$$Rf^!\mathbb{Z}/n\mathbb{Z} \cong R\mathscr{H}\mathrm{om}(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p, \mathbb{Z}/n\mathbb{Z}).$$

It remains to see that $R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p$ is invertible; more precisely, we already get a natural map

$$R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p \to (Rf^!\widehat{\mathbb{Z}}^p)^{-1}$$

that we want to prove is an isomorphism. This can again be checked v-locally, so we can assume that X is a spatial diamond. Then $R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p$ is a bounded complex all of whose cohomology sheaves are finitely presented solid; so as in the proof of Proposition VII.3.4, it is enough to check that one gets isomorphisms after applying $R\mathscr{H}\mathrm{om}(-, B)$ for any $B \in D_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z})$. But

$$\begin{aligned} R\mathscr{H}\mathrm{om}(R\pi_{1*}\Delta_{f\natural}\widehat{\mathbb{Z}}^p, B) &\cong R\pi_{1*}R\mathscr{H}\mathrm{om}(\Delta_{f\natural}\widehat{\mathbb{Z}}^p, R\pi_1^!B) \\ &\cong \Delta_f^* R\pi_1^! B \\ &\cong B \otimes^{\mathbb{L}}_{\mathbb{Z}/n\mathbb{Z}} Rf^!\mathbb{Z}/n\mathbb{Z}, \end{aligned}$$

giving the result.

The final statement follows formally from the identification of $R\pi_{1*}\Delta_{f\natural}\Lambda$ and the discussion leading up to the proposition. □

VII.4. Relation to $D_{\text{ét}}$

Assume now that Λ is discrete. In particular, also being a $\widehat{\mathbb{Z}}^p$-algebra, we have $n\Lambda = 0$ for some n prime to p. We wish to understand the relation between $D_{\text{ét}}(X, \Lambda)$ and $D_\blacksquare(X, \Lambda)$, and the functors defined on them.

VII.4.1. Naive embedding. For any small v-stack X, we have a fully faithful embedding

$$D_{\text{ét}}(X, \Lambda) \hookrightarrow D_\blacksquare(X, \Lambda)$$

as full subcategories of $D(X_v, \Lambda)$. As usual, the adjoint functor theorem implies that this admits a right adjoint $R_{X\text{ét}} : D_\blacksquare(X, \Lambda) \to D_{\text{ét}}(X, \Lambda)$. The full inclusion $D_{\text{ét}}(X, \Lambda) \subset D_\blacksquare(X, \Lambda)$ is symmetric

monoidal, and compatible with pullback. Moreover, by [**Sch17a**, Proposition 17.6], it also commutes with Rf_* if $f : Y \to X$ is qcqs and one restricts to D^+; or in general f is qcqs and of finite cohomological dimension. Moreover, one always has

$$R_{X\text{ét}} Rf_* \cong Rf_* R_{Y\text{ét}}.$$

Similarly, passing to right adjoints in the commutation with tensor products, we also have

$$R_{X\text{ét}} R\mathscr{H}\mathrm{om}_{D_{\text{ét}}(X,\Lambda)}(A, B) \cong R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A, R_{X\text{ét}} B)$$

if $A \in D_{\text{ét}}(X, \Lambda)$ and $B \in D_\blacksquare(X, \Lambda)$. If A is perfect-constructible, then for all $B \in D_{\text{ét}}(X, \Lambda)$, one actually has

$$R\mathscr{H}\mathrm{om}_{D_{\text{ét}}(X,\Lambda)}(A, B) \cong R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A, B) :$$

by descent, it suffices to check this when X is spatial diamond, and then one reduces to $A = j_!\Lambda$ for some quasicompact separated étale map $j : U \to X$. In that case, it follows from Rj_* commuting with the embedding $D_{\text{ét}}(X, \Lambda) \to D_\blacksquare(X, \Lambda)$, as it is qcqs and has cohomological dimension 0.

VII.4.2. Dual embedding. For a small v-stack X, let $D^\dagger_{\text{ét}}(X, \Lambda) \subset D_{\text{ét}}(X, \Lambda)$ be the full subcategory of overconvergent objects. Recall that $A \in D_{\text{ét}}(X, \Lambda)$ is overconvergent if for any strictly local $\mathrm{Spa}(C, C^+) \to X$, the map

$$R\Gamma(\mathrm{Spa}(C, C^+), A) \to R\Gamma(\mathrm{Spa}(C, \mathcal{O}_C), A)$$

is an isomorphism.

Proposition VII.4.1. *Assume that $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n prime to p. For any overconvergent $A \in D^\dagger_{\text{ét}}(X, \Lambda)$, let*

$$A^\vee = R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A, \Lambda) \in D_\blacksquare(X, \Lambda).$$

Then the functor

$$D^\dagger_{\text{ét}}(X, \Lambda)^{\mathrm{op}} \to D_\blacksquare(X, \Lambda) : A \mapsto A^\vee$$

is fully faithful, t-exact (for the standard t-structure), compatible with pullback, and the map

$$A \to R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, \Lambda)$$

is an isomorphism.

Proof. As the formation of $R\mathscr{H}\mathrm{om}$ in the solid context commutes with any base change, all assertions can be proved by v-descent, so we can assume that X is strictly totally disconnected. Then $D^\dagger_{\text{ét}}(X, \Lambda) \cong D(\pi_0 X, \Lambda)$. The heart of the standard t-structure is then an abelian category with compact projective generators $i_*\Lambda$ for open and closed subsets $i : S \subset \pi_0 X$, and the whole category is the Ind-category of the constructible complexes of Λ-modules on $\pi_0 X$ (which are locally constant with finite fibres). Passage to the naive dual is an autoequivalence on constructible complexes (as Λ is selfinjective), and thus embeds the whole Ind-category fully faithfully into the Pro-category of constructible complexes of Λ-modules on $\pi_0 X$, which sits fully faithfully inside the category of finitely presented solid sheaves on X. This already establishes that the functor is fully faithful and t-exact, and we already observed at the beginning that it commutes with any pullback.

It remains to prove that

$$A \to R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, \Lambda)$$

is an isomorphism. Again, we can assume $\Lambda = \mathbb{F}_\ell$ so that all operations are t-exact. Again, the statement is clear if A is constructible, and in general it follows from Breen's resolution that the Pro-structure on $A^\vee$ dualizes to a filtered colimit on applying $R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(-,B)$. $\square$

The functor $A \mapsto A^\vee$ is also close to being symmetric monoidal. Note that it is lax-symmetric monoidal, i.e. there is a natural functorial map

$$A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B^\vee \to (A \otimes^{\mathbb{L}}_\Lambda B)^\vee.$$

Proposition VII.4.2. *Assume that $A \in D^\dagger_{\text{ét}}(X,\Lambda)$ has finite Tor-amplitude over $\Lambda = \mathbb{Z}/n\mathbb{Z}$, i.e. for all quotients $\Lambda \to \mathbb{F}_\ell$, the complex $A \otimes^{\mathbb{L}}_\Lambda \mathbb{F}_\ell \in D^\dagger_{\text{ét}}(X,\mathbb{F}_\ell)$ is bounded. Then for all $B \in D^\dagger_{\text{ét}}(X,\Lambda)$, the maps*

$$A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B^\vee \to (A \otimes^{\mathbb{L}}_\Lambda B)^\vee \ , \ A \otimes^{\mathbb{L}}_\Lambda B \to R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, B)$$

are isomorphisms.

Proof. The second follows from the first: Using Proposition VII.4.1,

$$R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee,B) \cong R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(B^\vee,\Lambda)) \cong R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B^\vee, \Lambda),$$

which one can further rewrite to $A \otimes^{\mathbb{L}}_\Lambda B$ assuming the first isomorphism.

We can assume $\Lambda = \mathbb{F}_\ell$, and that A is concentrated in degree 0. Now as functors of B, all operations are t-exact, so we can reduce to the case that also B is concentrated in degree 0. We can assume that X is strictly totally disconnected, and then $D^\dagger_{\text{ét}}(X,\mathbb{F}_\ell) \cong D(\pi_0 X, \mathbb{F}_\ell)$. Then A and B are filtered colimits of constructible sheaves on $\pi_0 X$, and $R\mathscr{H}\mathrm{om}(-,\mathbb{F}_\ell)$ is a contravariant autoequivalence on constructible $\mathbb{F}_\ell$-sheaves on $\pi_0 X$. Then the result follows by observing that $A \mapsto A^\vee$ simply exchanges the Ind-category of constructible $\mathbb{F}_\ell$-sheaves on $\pi_0 X$ with its Pro-category. $\square$

As noted above, the functor $A \mapsto A^\vee$ is compatible with pullback. Regarding pushforward, we have the following result.

Proposition VII.4.3. *Let $f : Y \to X$ be a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\operatorname{trg} f < \infty$. *Let $A \in D^\dagger_{\text{ét}}(Y,\Lambda)$ with dual $A^\vee \in D_\blacksquare(Y,\Lambda)$. Then there is a natural isomorphism*

$$(Rf_* A)^\vee \cong f_\natural A^\vee.$$

Note that Rf_*A is again overconvergent, by proper base change.

Proof. One has

$$R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(f_\natural A^\vee, \Lambda) \cong Rf_* R\mathscr{H}\mathrm{om}_{D_\blacksquare(Y,\Lambda)}(A^\vee,\Lambda) \cong Rf_* A,$$

so by biduality one gets a natural map

$$f_\natural A^\vee \to (Rf_* A)^\vee;$$

we claim that this is an isomorphism. This can be checked v-locally on X, so we can assume that X is w-contractible. One can assume A is bounded above (i.e. $A \in D^-$) as both functors take

very coconnective objects to very connective objects; by shifting, we can assume $A \in D^{\leq 0}$. Now using a Postnikov limit and the assumption $\dim.\operatorname{trg} f < \infty$, we can also assume that $A \in D^+$, and hence reduce to A sitting in degree 0. Now we can choose a hypercover of Y by perfectoid spaces Y_i that are the canonical compactifications (relative to X) of w-contractible spaces. One can then replace Y by one of the Y_i, so assume that Y is the canonical compactification of a w-contractible space. In particular, $D^\dagger_{\text{ét}}(Y, \Lambda) \cong D(\pi_0 Y, \Lambda)$, and all operations can be computed on the level of $\pi_0 f : \pi_0 Y \to \pi_0 X$ instead. Here, the result amounts again to the duality between Ind- and Pro-objects in the category of constructible sheaves on profinite sets. $\square$

VII.5. Dualizability

It turns out that most of the results above on Poincaré duality hold verbatim if the assumption that f is cohomologically smooth is relaxed to the assumption that $\mathbb{F}_\ell$ is f-universally locally acyclic for all $\ell \neq p$. In fact, even more generally, one can obtain certain results comparing twisted forms of $f_\natural$ and Rf_* for any f-universally locally acyclic complex A.

Assume that Λ is a quotient of $\widehat{\mathbb{Z}}^p$ of the form $\varprojlim_n \mathbb{Z}/n\mathbb{Z}$ where n now runs only over some integers prime to p. If $f : X \to S$ is a compactifiable map of small v-stacks that is representable in locally spatial diamonds with locally $\dim.\operatorname{trg} f < \infty$, we define the category $D^{\mathrm{ULA}}(X/S, \Lambda)$ of f-universally locally acyclic complexes with coefficients Λ as the limit of the full subcategories

$$D^{\mathrm{ULA}}(X/S, \mathbb{Z}/n\mathbb{Z}) \subset D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$$

of f-universally locally acyclic objects in $D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$, for n running over the same set of integers prime to p. Equivalently, $D^{\mathrm{ULA}}(X/S, \Lambda)$ is the category of all $A \in D_\blacksquare(X, \Lambda)$ such that $A_n = A \otimes^{\mathbb{L}}_\Lambda \mathbb{Z}/n\mathbb{Z}$ lies in $D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$ for all such n, is f-universally locally acyclic, and A is the derived limit of the A_n.

Given such an A, in particular all A_n are overconvergent, and the functor

$$A \mapsto A^\vee = R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A, \Lambda) = R\varprojlim_n A_n^\vee \in D_\blacksquare(X, \Lambda)$$

defines another fully faithful (contravariant) embedding

$$D^{\mathrm{ULA}}(X/S, \Lambda)^{\mathrm{op}} \hookrightarrow D_\blacksquare(X, \Lambda)$$

of f-universally locally acyclic complexes into $D_\blacksquare(X, \Lambda)$. We can also precompose with Verdier duality $\mathbb{D}_{X/S}$ to obtain a covariant fully faithful embedding

$$D^{\mathrm{ULA}}(X/S, \Lambda) \hookrightarrow D_\blacksquare(X, \Lambda) : A \mapsto \mathbb{D}_{X/S}(A)^\vee.$$

Example VII.5.1. Assume that $S = \operatorname{Spa} C$ is a geometric point, and X is the analytification of an algebraic variety $X^{\mathrm{alg}}/\operatorname{Spec} C$. Then any constructible complex on X^{alg} is universally locally acyclic over S, yielding a fully faithful embedding

$$D^b_c(X^{\mathrm{alg}}, \mathbb{Z}_\ell) \hookrightarrow D^{\mathrm{ULA}}(X/S, \mathbb{Z}_\ell) \hookrightarrow D_\blacksquare(X, \mathbb{Z}_\ell),$$

embedding the usual bounded derived category of constructible $\mathbb{Z}_\ell$-sheaves on X^{alg} into $D_\blacksquare(X, \mathbb{Z}_\ell)$. The image lands in bounded complexes with finitely presented solid cohomology sheaves; in fact, in compact objects. Thus, this fully faithful embedding extends to a fully faithful embedding

$$\operatorname{Ind} D^b_c(X^{\mathrm{alg}}, \mathbb{Z}_\ell) \hookrightarrow D_\blacksquare(X, \mathbb{Z}_\ell).$$

The category on the left is the one customarily associated to X^{alg}. This functor takes the sheaf $i_*\mathbb{Z}_\ell$, for a point $i : \operatorname{Spec} C \to X^{\mathrm{alg}}$, to the solid sheaf $i_\natural \mathbb{Z}_\ell$.

In many papers in geometric Langlands and related fields, one often finds the following construction. If Y is a stack on the category of schemes over $\operatorname{Spec} C$, let

$$\mathcal{D}(Y, \mathbb{Z}_\ell) := \varprojlim_{X^{\mathrm{alg}} \to Y} \operatorname{Ind} \mathcal{D}_c^b(X^{\mathrm{alg}}, \mathbb{Z}_\ell)$$

where X^{alg} runs over schemes of finite type over $\operatorname{Spec} C$, and the transition functors are given by $Rf^!$. This, in fact, embeds naturally into $D_\blacksquare(Y^\diamondsuit, \mathbb{Z}_\ell)$ via the previous embedding, noting that it intertwines $Rf^!$ with the usual pullback f^* on solid sheaves. In fact,

$$\mathbb{D}_{X'/S}(Rf^! A)^\vee \cong (f^* \mathbb{D}_{X/S}(A))^\vee \cong f^* \mathbb{D}_{X/S}(A)^\vee$$

for a map $f : X'^{\mathrm{alg}} \to X^{\mathrm{alg}}$ of algebraic varieties over $\operatorname{Spec} C$.

Now for $A \in D^{\mathrm{ULA}}(X/S, \Lambda)$, we analyze the functor

$$f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda -) : D_\blacksquare(X, \Lambda) \to D_\blacksquare(S, \Lambda).$$

We note that from the definition one sees that this functor commutes with all colimits, the formation of this functor commutes with any base change, and it satisfies the projection formula. In fact, this functor extends the functor $Rf_!(A \otimes^{\mathbb{L}}_\Lambda -)$.

Proposition VII.5.2. *Assume that $A \in D^{\mathrm{ULA}}(X/S, \Lambda)$ has bounded Tor-amplitude. Let $\mathbb{Z}/n\mathbb{Z}$ be a discrete quotient of Λ. For $B \in D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$, there is a natural equivalence*

$$f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \cong Rf_!(A_n \otimes^{\mathbb{L}}_{\mathbb{Z}/n\mathbb{Z}} B) \in D_{\text{ét}}(S, \mathbb{Z}/n\mathbb{Z}).$$

Proof. We can assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Note that for any $C \in D_{\text{ét}}(S, \Lambda)$, one has

$$\begin{aligned}
R\operatorname{Hom}_\Lambda(f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B), C) &\cong R\operatorname{Hom}_\Lambda(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B, f^*C) \\
&\cong R\operatorname{Hom}_\Lambda(B, R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(\mathbb{D}_{X/S}(A)^\vee, f^*C)) \\
&\cong R\operatorname{Hom}_\Lambda(B, \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda f^*C) \\
&\cong R\operatorname{Hom}_\Lambda(B, R\mathscr{H}\mathrm{om}_{D_{\text{ét}}(X,\Lambda)}(A, Rf^! C)) \\
&\cong R\operatorname{Hom}_\Lambda(A \otimes^{\mathbb{L}}_\Lambda B, Rf^! C) \\
&\cong R\operatorname{Hom}_\Lambda(Rf_!(A \otimes^{\mathbb{L}}_\Lambda B), C).
\end{aligned}$$

Here, we use Proposition VII.4.1 and Proposition IV.2.19. In particular, there is a natural map

$$f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \to Rf_!(A \otimes^{\mathbb{L}}_\Lambda B).$$

We claim that this is an isomorphism. This can be checked v-locally, so we can assume that S is strictly totally disconnected. We can assume that X is a spatial diamond by localization. As the functor commutes with all colimits in B, we can also assume that $B = j_!\Lambda$ for some quasicompact separated étale $j : V \to X$. Replacing X by V, we can then even assume $B = \Lambda$.

Now $\mathbb{D}_{X/S}(A)$ lies in D^+ and then again $\mathbb{D}_{X/S}(A)^\vee$ in D^-. It follows that $\mathbb{D}_{X/S}(A)^\vee$ is a complex that is bounded above, and finitely presented solid in each degree. Thus $f_\natural \mathbb{D}_{X/S}(A)^\vee$ is

of the same form, and so is the cone Q of $f_\natural \mathbb{D}_{X/S}(A)^\vee \to Rf_! A$. If Q is nonzero, we can look at the largest i such that $\mathcal{H}^i(Q)$ is nonzero. This is finitely presented solid, so a cofiltered limit of constructible étale sheaves. But $R\operatorname{Hom}(Q, C) = 0$ for all $C \in D_{\text{ét}}(S, \Lambda)$, so it follows that indeed $Q = 0$. $\square$

If f is moreover proper, one can also prove the following version of A-twisted Poincaré duality.

Proposition VII.5.3. *Assume that $f : X \to S$ is a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\operatorname{trg} f < \infty$. *Let $A \in D^{\mathrm{ULA}}(X/S, \Lambda)$ with bounded Tor-amplitude. Then there is a natural equivalence*

$$f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda -) \cong Rf_* R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, -)$$

of functors $D_\blacksquare(X, \Lambda) \to D_\blacksquare(S, \Lambda)$.

Proof. First, we construct the natural transformation. Let $\pi_1, \pi_2 : X \times_S X \to X$ be the two projections. Giving a map

$$f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \to Rf_* R\mathscr{H}\mathrm{om}_{D_\blacksquare(X,\Lambda)}(A^\vee, B)$$

is equivalent to giving a map

$$f^* f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda A^\vee \to B.$$

But

$$\begin{aligned} f^* f_\natural(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda A^\vee &\cong \pi_{1\natural}\pi_2^*(\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda A^\vee \\ &\cong \pi_{1\natural}(\pi_2^*\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_2^* B \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_1^* A^\vee). \end{aligned}$$

Thus, it suffices to construct a functorial map

$$\pi_2^*\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_1^* A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_2^* B \to \pi_1^* B.$$

For this in turn it suffices to construct a natural map

$$\pi_2^*\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_1^* A^\vee \to \Delta_\natural \Lambda$$

where $\Delta : X \hookrightarrow X \times_S X$ is the diagonal. Here $\Delta_\natural \Lambda \cong (\Delta_* \Lambda)^\vee$ by Proposition VII.4.3 and $\pi_2^*\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \pi_1^* A^\vee \cong (\pi_2^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda \pi_1^* A)^\vee$ by Proposition VII.4.2. Thus, we have to find a map

$$\Delta_* \Lambda \to \pi_2^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda \pi_1^* A$$

or equivalently a section of $R\Delta^!(\pi_2^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda \pi_1^* A)$. But $\pi_2^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda \pi_1^* A \cong R\mathscr{H}\mathrm{om}(\pi_2^* A, R\pi_1^! A)$ as A is f-universally locally acyclic, and then

$$R\Delta^!(\pi_2^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda \pi_1^* A) \cong R\Delta^! R\mathscr{H}\mathrm{om}(\pi_2^* A, R\pi_1^! A) \cong R\mathscr{H}\mathrm{om}(A, A),$$

where we find the identity section.

To show that the map is an isomorphism, we can now localize on S, and in particular assume that S is strictly totally disconnected. By the bounded assumption on A and finite cohomological dimension of f, the functor $Rf_* R\mathscr{H}\mathrm{om}_\Lambda(A^\vee, -)$ commutes with all direct sums, and hence we can assume that B is finitely presented solid (concentrated in degree 0). Then we can write B

as a cofiltered limit of constructible étale sheaves, and the left-hand side commutes with such limits; so we can reduce to B being a constructible étale sheaf, where the result follows from Proposition VII.5.2 and Proposition VII.4.1. $\square$

From the perspective of using sheaves as kernels of induced functors, we have the following picture. We can introduce a variant of the category $\mathcal{C}_S$ introduced above. Namely, for any small v-stack S, let us consider the 2-category $\mathcal{C}_{S,\blacksquare}$ whose objects are relatively 0-truncated small v-stacks X over S, and whose categories of morphisms

$$\mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}}(X,Y) = D_\blacksquare(X \times_S Y, \Lambda)$$

are given by solid complexes. Again, to any $X \in \mathcal{C}_{S,\blacksquare}$, we can associate the triangulated category $D_\blacksquare(X,\Lambda)$ and to any $A \in D_\blacksquare(X \times_S Y, \Lambda)$ the functor

$$p_{2\natural}(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda p_1^*) : D_\blacksquare(X,\Lambda) \to D_\blacksquare(Y,\Lambda)$$

with kernel A. The composition in $\mathcal{C}_{S,\blacksquare}$ is defined by the convolution

$$D_\blacksquare(X \times_S Y, \Lambda) \times D_\blacksquare(Y \times_S Z, \Lambda) \to D_\blacksquare(X \times_S Z, \Lambda) : (A,B) \mapsto A \star B = p_{13\natural}(p_{12}^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda p_{23}^* B).$$

We wish to compare $\mathcal{C}_S$ and $\mathcal{C}_{S,\blacksquare}$. Note that the naive embedding $D_{\text{ét}}(X \times_S Y, \Lambda) \hookrightarrow D_\blacksquare(X \times_S Y, \Lambda)$ is not compatible with the convolution (as one employs $R\pi_{13!}$ while the other employs $\pi_{13\natural}$). On the other hand, we can restrict to the sub-2-category $\mathcal{C}_S^\dagger \subset \mathcal{C}_S$ whose objects are only the proper X/S representable in spatial diamonds of finite dim. trg, and with

$$\mathrm{Fun}_{\mathcal{C}_S^\dagger}(X,Y) = D^\dagger_{\text{ét},\mathrm{ftor}}(X \times_S Y, \Lambda),$$

where the subscript ftor stands for finite Tor-dimension over Λ. Then there is a fully faithful embedding $\mathcal{C}_S^\dagger \hookrightarrow \mathcal{C}_{S,\blacksquare}^{\mathrm{co}}$ where the superscript co means that we change the direction of the arrows within each $\mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}}(X,Y)$. Indeed, for any $X, Y \in \mathcal{C}_S^\dagger$, the functor $A \mapsto A^\vee$ defines a fully faithful embedding

$$\mathrm{Fun}_{\mathcal{C}_S^\dagger}(X,Y) = D^\dagger_{\text{ét},\mathrm{ftor}}(X \times_S Y, \Lambda) \hookrightarrow D_\blacksquare(X \times_S Y, \Lambda)^{\mathrm{op}} = \mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}^{\mathrm{co}}}(X,Y).$$

This is compatible with composition by Proposition VII.4.2 and Proposition VII.4.3. This discussion leads to another proof of Proposition VII.5.3:

Corollary VII.5.4. *Let $f : X \to S$ be a proper map of small v-stacks that is representable in spatial diamonds with* $\dim.\mathrm{trg}\, f < \infty$. *Let $A \in D_{\text{ét}}(X,\Lambda)$ be f-universally locally acyclic and of finite Tor-dimension over Λ. Then $A^\vee \in D_\blacksquare(X,\Lambda) = \mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}}(X,S)$ is right adjoint to*

$$\mathbb{D}_{X/S}(A)^\vee \in D_\blacksquare(X,\Lambda) = \mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}}(S,X).$$

In particular, the functor

$$f_\natural(A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda -) : D_\blacksquare(X,\Lambda) \to D_\blacksquare(S,\Lambda)$$

is right adjoint to the functor

$$\mathbb{D}_{X/S}(A)^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda f^* - : D_\blacksquare(S,\Lambda) \to D_\blacksquare(X,\Lambda),$$

so

$$f_\natural(A^\vee \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda -) \cong Rf_* R\mathscr{H}\mathrm{om}_\Lambda(\mathbb{D}_{X/S}(A)^\vee, -) : D_\blacksquare(X,\Lambda) \to D_\blacksquare(S,\Lambda).$$

Moreover, when applied to the Satake category, we get a fully faithful embedding

$$(\mathrm{Sat}_G^I)^{\mathrm{op}} \hookrightarrow D_\blacksquare(\mathcal{H}\mathrm{ck}_G^I, \widehat{\mathbb{Z}}^p) : A \mapsto A^\vee$$

compatible with the monoidal structure (and functorially in I), where the right-hand side is given by

$$\mathrm{Fun}_{\mathcal{C}_{S,\blacksquare}}([(\mathrm{Div}^1)_X^I/L^+_{(\mathrm{Div}^1_X)^I}G], [(\mathrm{Div}^1)_X^I/L^+_{(\mathrm{Div}^1_X)^I}G])$$

for $S = [(\mathrm{Div}^1_X)^I/L_{(\mathrm{Div}^1_X)^I}G]$. Precomposing with Verdier duality, we get a covariant fully faithful embedding

$$\mathrm{Sat}_G^I \hookrightarrow D_\blacksquare(\mathcal{H}\mathrm{ck}_G^I, \widehat{\mathbb{Z}}^p) : A \mapsto \mathbb{D}_{X/S}(A)^\vee.$$

By Proposition VII.5.2, when one uses objects in the Satake category as kernels to define Hecke operators, this fully faithful embedding makes it possible to extend Hecke operators from $D_{\text{ét}}$ to $D_\blacksquare$.

VII.6. Lisse sheaves

The category $D_\blacksquare(X, \Lambda)$ is huge: Already if X is a point and $\Lambda = \mathbb{F}_\ell$, it is the derived category of solid $\mathbb{F}_\ell$-vector spaces, which is much larger than the category of usual discrete $\mathbb{F}_\ell$-vector spaces. When applied to Bun_G, we would however really like to study smooth representations on discrete Λ-modules.

As coefficients, we will from now on choose a discrete $\mathbb{Z}_\ell$-algebra Λ for some $\ell \neq p$, or rather the corresponding condensed ring $\Lambda := \mathbb{Z}_\ell \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \Lambda_{\mathrm{disc}}$. (For a technical reason, we have to restrict attention to a particular prime ℓ.)

It turns out that when X is an Artin v-stack, one can define a full subcategory $D_{\mathrm{lis}}(X, \Lambda) \subset D_\blacksquare(X, \Lambda)$ that when specialized to $X = \mathrm{Bun}_G$ has the desired properties. Here the subscript "lis" is an abbreviation of "lisse" (french smooth), and is not meant to evoke lisse sheaves in the sense of locally constant sheaves, but lisse-étale sheaves in the sense of Artin stacks [**LMB00**].

Definition VII.6.1. *Let X be an Artin v-stack. The full subcategory $D_{\mathrm{lis}}(X, \Lambda) \subset D_\blacksquare(X, \Lambda)$ is the smallest triangulated subcategory stable under all direct sums that contains $f_\natural\Lambda$ for all maps $f : Y \to X$ that are separated, representable in locally spatial diamonds, and ℓ-cohomologically smooth.*

In principle, one could give this definition even when X is any small v-stack, but in that case there might be very few objects.

Proposition VII.6.2. *Let X be an Artin v-stack. The full subcategory $D_{\mathrm{lis}}(X, \Lambda) \subset D_\blacksquare(X, \Lambda)$ is stable under $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$. Moreover, if $f : Y \to X$ is a map of Artin v-stacks, then f^* maps $D_{\mathrm{lis}}(X, \Lambda) \subset D_\blacksquare(X, \Lambda)$ into $D_{\mathrm{lis}}(Y, \Lambda) \subset D_\blacksquare(Y, \Lambda)$.*

Proof. As tensor products and pullbacks commute with all direct sums, it suffices to check the claim on the generators $g_\natural\Lambda$ for maps $g : Z \to X$ that are separated, representable in locally spatial diamonds, and ℓ-cohomologically smooth. Now the result follows as pullbacks and products of such maps are of the same form. $\square$

Proposition VII.6.3. *Let X be an Artin v-stack. The inclusion $D_{\mathrm{lis}}(X,\Lambda) \subset D_{\blacksquare}(X,\Lambda)$ admits a right adjoint*

$$A \mapsto A^{\mathrm{lis}} : D_{\blacksquare}(X,\Lambda) \to D_{\mathrm{lis}}(X,\Lambda).$$

The kernel of $A \mapsto A^{\mathrm{lis}}$ is the class of all $A \in D_{\blacksquare}(X,\Lambda)$ such that $A(Y) = 0$ for all $f : Y \to X$ that are separated, representable in locally spatial diamonds, and ℓ-cohomologically smooth.

Proof. The existence of the right adjoint is formal. We note that the ∞-category $\mathcal{D}_{\blacksquare}(X,\Lambda)$ is not itself presentable, but rather is the large filtered colimit of presentable ∞-categories $\mathcal{D}_{\blacksquare}(X_\kappa,\Lambda)$ for uncountable strong limit cardinals κ (restricting the v-site to κ-small perfectoid spaces). Also note that $D_{\mathrm{lis}}(X,\Lambda)$ is contained in $D_{\blacksquare}(X_\kappa,\Lambda)$ for some κ: This can be checked when X is a spatial diamond and for the generators $f_\natural\Lambda \cong f_\natural\mathbb{Z}_\ell \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \Lambda$ when $f : Y \to X$ is in addition quasicompact, in which case $f_\natural\mathbb{Z}_\ell$ is the limit of $f_\natural\mathbb{Z}/\ell^m\mathbb{Z}$ all of which lie in $D_{\text{ét}}(X,\mathbb{Z}/\ell^m\mathbb{Z})$, so we conclude by [**Sch17a**, Remark 17.4]. It follows that the right adjoints to $D_{\mathrm{lis}}(X,\Lambda) \to D_{\blacksquare}(X_\kappa,\Lambda)$ for all large enough κ glue to the desired right adjoint.

The description of the kernel is formal. □

Using Proposition VII.6.3, we can then also define $R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(A,B) \in D_{\mathrm{lis}}(X,\Lambda)$ for $A,B \in D_{\mathrm{lis}}(X,\Lambda)$ and $Rf_{\mathrm{lis}*} : D_{\mathrm{lis}}(Y,\Lambda) \to D_{\mathrm{lis}}(X,\Lambda)$ for a map $f : Y \to X$ of Artin v-stacks, satisfying the usual adjunction to the tensor product and pullback.

The goal of passing to D_{lis} is to make sheaves "discrete" again. Recall the following result.

Proposition VII.6.4. *For any condensed ring A with underlying ring $A(*)$, the functor $M \mapsto M \otimes_{A(*)} A$ induces a fully faithful functor*

$$D(A(*)) \hookrightarrow D(A)$$

from the derived category of usual $A(*)$-modules to the derived category of condensed modules over the condensed ring A.

Proof. We need to see that for any $M,N \in D(A(*))$, the map

$$R\operatorname{Hom}_{A(*)}(M,N) \to R\operatorname{Hom}_A(M \otimes_{A(*)} A, N \otimes_{A(*)} A)$$

is an isomorphism. The class of all M for which this happens is triangulated and stable under all direct sums, so it suffices to consider $M = A(*)$. Then it amounts to

$$N(*) \to (N \otimes_{A(*)} A)(*)$$

being an isomorphism, which follows from evaluation at $*$ being symmetric monoidal. □

In particular, we have the following result for a geometric point.

Proposition VII.6.5. *Let $X = \operatorname{Spa} C$ for some complete algebraically closed nonarchimedean field C. Then $D_{\mathrm{lis}}(X,\Lambda) \cong D(\Lambda)$, the derived category of (relatively) discrete Λ-modules.*

Proof. We need to see that for all separated ℓ-cohomologically smooth maps $f : Y \to X$ of spatial diamonds, one has $f_\natural\Lambda \in D(\Lambda)$. This reduces to $\Lambda = \mathbb{Z}_\ell$. In that case, $f_\natural\mathbb{Z}_\ell = \varprojlim_m f_\natural\mathbb{Z}/\ell^m\mathbb{Z}$, where by Proposition VII.5.2 each

$$f_\natural\mathbb{Z}/\ell^m\mathbb{Z} \cong Rf_!Rf^!\mathbb{Z}/\ell^m\mathbb{Z},$$

which is a perfect complex of $\mathbb{Z}/\ell^m\mathbb{Z}$-modules, in particular discrete. Taking the limit over m, we get a perfect complex of $\mathbb{Z}_\ell$-modules, which is in particular (relatively) discrete over $\mathbb{Z}_\ell$. $\square$

When working with torsion coefficients, one recovers $D_{\text{ét}}$.

Proposition VII.6.6. *Let X be an Artin v-stack, and assume that Λ is killed by a power of ℓ. Then $D_{\text{lis}}(X,\Lambda) \subset D_\blacksquare(X,\Lambda)$ is contained in the image of the naive embedding $D_{\text{ét}}(X,\Lambda) \hookrightarrow D_\blacksquare(X,\Lambda)$. If there is a separated ℓ-cohomologically smooth surjection $U \to X$ from a locally spatial diamond U, such that $U_{\text{ét}}$ has a basis with bounded ℓ-cohomological dimension, then it induces an equivalence $D_{\text{lis}}(X,\Lambda) \cong D_{\text{ét}}(X,\Lambda)$.*

Proof. If $f : Y \to X$ is separated, representable in locally spatial diamonds, and ℓ-cohomologically smooth, then $f_\natural\Lambda = Rf_!Rf^!\Lambda$ lies in $D_{\text{ét}}(X,\Lambda)$, hence $D_{\text{lis}}(X,\Lambda) \subset D_{\text{ét}}(X,\Lambda)$. To check equality, we can work on an atlas, so by the assumption we can reduce to the case that X is a locally spatial diamond for which $X_{\text{ét}}$ has a basis with bounded ℓ-cohomological dimension. In that case $D_{\text{ét}}(X,\Lambda) \cong D(X_{\text{ét}},\Lambda)$ by [**Sch17a**, Proposition 20.17] (the proof only needs a basis with bounded cohomological dimension), which is generated by $j_!\Lambda$ for $j : U \to X$ quasicompact separated étale, which is thus also contained in $D_{\text{lis}}(X,\Lambda)$. $\square$

The most severe problem with the general formalism of solid sheaves is that stratifications of a space do not lead to corresponding decompositions of sheaves into pieces on the individual strata. This problem is somewhat salvaged by $D_{\text{lis}}(X,\Lambda)$: We expect that it holds true if X and its stratification are sufficiently nice. Here is a simple instance that will be sufficient for our purposes.

Proposition VII.6.7. *Let X be a locally spatial diamond with a closed point $x \in X$, giving a corresponding closed subdiamond $i : Z \subset X$ with complement $j : U \subset X$. Assume that $Z = \operatorname{Spa} C$ is representable, with C an algebraically closed nonarchimedean field. Moreover, assume that Z can be written as a cofiltered intersection of quasicompact open neighborhoods $V \subset X$ such that $R\Gamma(V,\mathbb{F}_\ell) \cong \mathbb{F}_\ell$.*

Then one has a semi-orthogonal decomposition of $D_{\text{lis}}(X,\Lambda)$ into $D_{\text{lis}}(U,\Lambda)$ and $D_{\text{lis}}(Z,\Lambda) \cong D(\Lambda)$.

Proof. We may assume that X is spatial. We analyze the quotient of $D_{\text{lis}}(X,\Lambda)$ by $j_!D_{\text{lis}}(U,\Lambda)$. This is equivalently the subcategory of all $A \in D_{\text{lis}}(X,\Lambda)$ with $j^*A = 0$. It is generated by the images of $f_\natural\Lambda$ for $f : Y \to X$ cohomologically smooth separated map of spatial diamonds; under the embedding of the quotient category back into $D_{\text{lis}}(X,\Lambda)$, this corresponds to the cone of $j_!j^*f_\natural\Lambda \to f_\natural\Lambda$. Let $M = i^*f_\natural\Lambda \in D_{\text{lis}}(Z,\Lambda) \cong D(\Lambda)$, which in fact is a perfect complex of Λ-modules (by the proof of Proposition VII.6.5). Then we claim that there is an isomorphism

$$\operatorname{cone}(j_!j^*f_\natural\Lambda \to f_\natural\Lambda) \cong \operatorname{cone}(j_!M \to M).$$

To see this, it suffices to prove that there is some open neighborhood V of Z such that $f_\natural\Lambda|_V \cong M$, the constant sheaf associated with M. We can reduce to $\Lambda = \mathbb{Z}_\ell$. As $f_\natural\mathbb{F}_\ell$ is constructible, we can find some such V for which $f_\natural\mathbb{F}_\ell|_V \cong M/\ell$. Picking such an isomorphism reducing to the identity at x, and choosing V with the property $R\Gamma(V,\mathbb{F}_\ell) \cong \mathbb{F}_\ell$, we see that in fact the isomorphism lifts uniquely to $\mathbb{Z}/\ell^m\mathbb{Z}$ for each m, and thus by taking the limit over m to the desired isomorphism $f_\natural\Lambda|_V \cong M$.

Thus, the quotient of $D_{\mathrm{lis}}(X,\Lambda)$ by $j_!D_{\mathrm{lis}}(U,\Lambda)$ is generated by the constant sheaf Λ. Moreover, the endomorphisms of Λ in the quotient category are given by the cone of

$$R\Gamma(X, j_!\Lambda) \to R\Gamma(X,\Lambda).$$

This is equivalently the filtered colimit of $R\Gamma(V,\Lambda)$ over all open neighborhoods V of Z; we can restrict to those for which $R\Gamma(V,\mathbb{F}_\ell) \cong \mathbb{F}_\ell$. This implies formally that $R\Gamma(V,\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ by passing to limits and then $R\Gamma(V,\Lambda) \cong \Lambda$ by passing to filtered colimits. Thus, we get the desired semi-orthogonal decomposition. □

VII.7. $D_{\mathrm{lis}}(\mathrm{Bun}_G)$

Our goal now is to extend the results of Chapter V to the case of $D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$. This will notably include the case $\Lambda = \overline{\mathbb{Q}}_\ell$.

Thus, let again be E any nonarchimedean local field with residue field $\mathbb{F}_q$ and G a reductive group over E. We work with Perf_k where $k = \overline{\mathbb{F}}_q$, and fix a complete algebraically closed nonarchimedean field C/k.

Proposition VII.7.1. *Let $b \in B(G)$. The pullback functors*

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda) \to D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda) \to D_{\mathrm{lis}}([\mathrm{Spa}\,C/\underline{G_b(E)}],\Lambda),$$
$$D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G^b \times \mathrm{Spa}\,C,\Lambda) \to D_{\mathrm{lis}}([\mathrm{Spa}\,C/\underline{G_b(E)}],\Lambda)$$

are equivalences, and all categories are naturally equivalent (as symmetric monoidal categories) to the derived category $D(G_b(E),\Lambda)$ of smooth representations of $G_b(E)$ on discrete Λ-modules.

Proof. Recall that the map $s : [*/\underline{G_b(E)}] \to \mathrm{Bun}_G^b$ is cohomologically smooth and surjective; in fact, its fibres are successive extensions of positive Banach–Colmez spaces. This implies that $s_\natural\Lambda \cong \Lambda$. This, in turn, implies by the projection formula for $s_\natural$ that $s_\natural s^*A \cong A$ for all $A \in D_\blacksquare(\mathrm{Bun}_G^b,\Lambda)$, thus giving fully faithfulness. The same applies after base change to $\mathrm{Spa}\,C$. Moreover, using pullback under the projection $\mathrm{Bun}_G^b \to [*/\underline{G_b(E)}]$, we see that s^* is also necessarily essentially surjective.

It remains to show that the pullback $D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda) \to D_{\mathrm{lis}}([\mathrm{Spa}\,C/\underline{G_b(E)}],\Lambda)$ is an equivalence, and identify this symmetric monoidal category with $D(G_b(E),\Lambda)$. By Proposition VII.2.6, the functor $D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda) \to D_{\mathrm{lis}}([\mathrm{Spa}\,C/\underline{G_b(E)}],\Lambda)$ is fully faithful. One can easily build a functor $D(G_b(E),\Lambda) \to D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda)$, and it is enough to see that the composite functor

$$D(G_b(E),\Lambda) \to D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda) \to D_{\mathrm{lis}}([\mathrm{Spa}\,C/\underline{G_b(E)}],\Lambda)$$

is an equivalence. Using that $D(G_b(E),\Lambda)$ is generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ for $K \subset G_b(E)$ open pro-p, one easily sees that the functor is fully faithful, so it remains to prove essential surjectivity. Using descent along $\mathrm{Spa}\,C \to [\mathrm{Spa}\,C/\underline{G_b(E)}]$ and the equivalence $D_{\mathrm{lis}}(\mathrm{Spa}\,C,\Lambda) \cong D(\Lambda)$, the target maps fully faithfully into the derived category of representations of the condensed group $G_b(E)$ on the full subcategory of condensed Λ-modules of the form $M \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$ for Λ-modules M. As $G_b(E)$ is locally pro-p, any such action in fact comes from a smooth action on M: For $K \subset G_b(E)$ pro-p, the K-orbit of any $m \in M$ lies in some compact submodule, thus in $M \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$ for some finitely generated $\mathbb{Z}_\ell$-submodule $M' \subset M$. The action of K on m then gives a continuous map

$K \to \mathrm{GL}(M')$. As the target is locally pro-ℓ, this map has finite image, so that the action of K on m is locally constant. $\square$

Recall that for any $b \in B(G)$, we have the cohomologically smooth chart $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ near Bun_G^b. This comes with a projection $q_b : \mathcal{M}_b \to [*/\underline{G_b(E)}]$ which has a natural section, given by the preimage of $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ in $\mathcal{M}_b$. Over $\mathcal{M}_b$, we have the $\underline{G_b(E)}$-torsor $\widetilde{\mathcal{M}}_b \to \mathcal{M}_b$, and for any complete algebraically closed field C over $k = \overline{\mathbb{F}}_q$, the base change

$$\widetilde{\mathcal{M}}_{b,C} = \widetilde{\mathcal{M}}_b \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C$$

is representable by a locally spatial diamond, endowed with a distinguished point $i : \mathrm{Spa}\, C \hookrightarrow \widetilde{\mathcal{M}}_{b,C}$. Recall that $\widetilde{\mathcal{M}}_{b,C}$ is a successive extension of negative Banach–Colmez spaces. Iteratively restricting to small quasicompact balls inside these negative Banach–Colmez spaces, we see that the closed subset $i : \mathrm{Spa}\, C \hookrightarrow \widetilde{\mathcal{M}}_{b,C}$ can be written as cofiltered intersection of quasicompact open subsets V for which $R\Gamma(V, \mathbb{F}_\ell) \cong \mathbb{F}_\ell$.

Proposition VII.7.2. *For any $b \in B(G)$ with locally closed immersion $i^b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$, the functor*

$$i^{b*} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \cong D_{\mathrm{lis}}([*/\underline{G_b(E)}], \Lambda)$$

admits a left adjoint, given by

$$\pi_{b\natural} q_b^* : D_{\mathrm{lis}}([*/\underline{G_b(E)}], \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

The unit of the adjunction is given by the equivalence $\mathrm{id} \cong i^{b*}\pi_{b\natural} q_b^*$ arising from base change, and the identification of the pullback of i^b along π_b with $[*/\underline{G_b(E)}] \subset \mathcal{M}_b$.

Proof. As $D([*/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$ is generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ for open pro-p subgroups $K \subset G_b(E)$, and as we already determined the unit of the adjunction, it suffices to verify the adjunction on these objects. Let $\mathcal{M}_{b,K} = \widetilde{\mathcal{M}}_b/\underline{K} \to \mathcal{M}_b$. This comes with a closed immersion $i_K : [*/\underline{K}] \to \mathcal{M}_{b,K}$. It suffices to see that for all $A \in D_{\mathrm{lis}}(\mathcal{M}_{b,K}, \Lambda)$, the map

$$R\Gamma(\mathcal{M}_{b,K}, A) \to R\Gamma([*/\underline{K}], A)$$

is an isomorphism, where we continue to denote by A any of its pullbacks. Assume first that $A = j_! A_0$ for some $A_0 \in D_{\mathrm{lis}}(\mathcal{M}_{b,K}^\circ, \Lambda)$. Then the result follows from Theorem VII.2.10. In general we can then replace A by the cone of $j_! A \to A$ in the displayed formula. For this statement, we can even base change to $\mathrm{Spa}\, C$ for some complete algebraically closed nonarchimedean field $C|k$, and allow more generally any $A \in D_{\mathrm{lis}}(\mathcal{M}_{b,K,C}, \Lambda)$. We can then assume that $A = f_\natural \mathbb{Z}_\ell$ for some ℓ-cohomologically smooth separated qcqs map $f : Y \to \mathcal{M}_{b,K,C}$. Then as in the proof of Proposition VII.6.7, A is constant in a neighborhood of $[\mathrm{Spa}\, C/\underline{K}]$, which implies the result (as $\mathrm{Spa}\, C \subset \widetilde{\mathcal{M}}_{b,C}$ is a cofiltered intersection of quasicompact open V's with trivial cohomology). $\square$

Proposition VII.7.3. *For any quasicompact open substack $U \subset \mathrm{Bun}_G$, the category $D_{\mathrm{lis}}(U, \Lambda)$ admits a semi-orthogonal decomposition into the categories $D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ for $b \in |U| \subset B(G)$. Moreover, for any not necessarily quasicompact U, the functor $D_{\mathrm{lis}}(U, \Lambda) \to D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ is an equivalence.*

Proof. We argue by induction on $|U|$, so take some closed element $b \in |U| \subset B(G)$ and let $i : \mathrm{Bun}_G^b \to U$ and $j : V \to U$ be the closed and complementary open substacks. We know that $D_{\mathrm{lis}}(U, \Lambda) \to D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ is fully faithful by Proposition VII.2.6, and by induction $D_{\mathrm{lis}}(V, \Lambda) \to D_{\mathrm{lis}}(V \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ is an equivalence.

Now by the previous proposition, i^{b*} admits the left adjoint

$$\pi_{b\natural} q_b^* : D_{\mathrm{lis}}([*/\underline{G_b(E)}], \Lambda) \to D_{\mathrm{lis}}(U, \Lambda),$$

and in fact the proof of that proposition shows (using our standing induction assumption) that, composed with the embedding into $D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$, it continues to be a left adjoint to $i^{b*} : D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda) \to D([\mathrm{Spa}\, C/\underline{G_b(E)}], \Lambda) \cong D([*/\underline{G_b(E)}], \Lambda)$.

The unit $\mathrm{id} \to i^{b*}\pi_{b\natural} q_b^*$ of the adjunction is an equivalence. We see that $D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ has full subcategories given by $j_! D_{\mathrm{lis}}(V, \Lambda)$ and the essential image of $\pi_{b\natural} q_b^*$ (both of which lie in $D_{\mathrm{lis}}(U, \Lambda)$). To see that one has a semi-orthogonal decomposition, it suffices to see that if $A \in D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ with $i^* A = j^* A = 0$, then $A = 0$. This can be checked after pullback to $\widetilde{\mathcal{M}}_{b,C}$, where it follows from Proposition VII.6.7. This also shows that $D_{\mathrm{lis}}(U, \Lambda) \to D_{\mathrm{lis}}(U \times_{\mathrm{Spd}\, k} \mathrm{Spa}\, C, \Lambda)$ is an equivalence. □

Now we also want to analyze the compact objects as well as the universally locally acyclic objects, and various dualities. We start with the compact objects.

Proposition VII.7.4. *The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is compactly generated. An object $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is compact if and only if it has finite support and $i^{b*} A \in D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ is compact for all $b \in B(G)$, i.e. lies in the thick triangulated subcategory generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ for open pro-p subgroups $K \subset G_b(E)$.*

Moreover, for each b and $K \subset G_b(E)$ pro-p, letting

$$f_K : \mathcal{M}_{b,K} \to \mathrm{Bun}_G$$

be the natural map, the object $A_K^b = f_{K\natural}\Lambda \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is compact, and these generate $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Proof. By Proposition VII.7.3, the left adjoints $\pi_{b\natural} q_b^*$ to i^{b*} generate $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$; as i^{b*} commutes with colimits, these left adjoints also preserve compact objects. As each $D(G_b(E), \Lambda)$ is compactly generated, it follows that $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is compactly generated, with compact generators A_K^b.

To see that the given property characterizes compact objects, we argue by induction over quasicompact open substacks $U \subset \mathrm{Bun}_G$. Pick any closed $b \in |U| \subset B(G)$, and assume the result for the complementary open $j : V \subset U$. We first show that all of the given compact generators (coming from $b' \in |U| \subset B(G)$) have the property that all of their stalks are compact. This is clear by induction if $b' \in |V|$, so we can assume $b' = b$. Then we need to see that $j^* \pi_{b\natural} q_b^*$ preserves compact objects. But this follows from Lemma VII.7.5 below. Using the semi-orthogonal decomposition structure, it now follows that conversely, all A with compact stalks are compact. □

Lemma VII.7.5. *For $K \subset G_b(E)$ an open pro-p subgroup, the functor*

$$R\Gamma(\mathcal{M}_{b,K}^\circ, -) : D_\blacksquare(\mathcal{M}_{b,K}^\circ, \Lambda) \to D(\Lambda)$$

has finite cohomological dimension and commutes with all direct sums.

Proof. As $\mathcal{M}_{b,K}^\circ = \widetilde{\mathcal{M}}_b^\circ/\underline{K}$ where $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond, it suffices to prove that the functor has finite cohomological dimension. It suffices to prove this for $\widetilde{\mathcal{M}}_b^\circ$ (as taking K-invariants is exact). One can formally reduce to $\Lambda = \mathbb{Z}_\ell$ and then to finitely presented solid $\mathbb{Z}_\ell$-sheaves $\mathcal{F}$ on $\widetilde{\mathcal{M}}_b^\circ$. Now these can be written as cofiltered inverse limits of constructible $\mathcal{F}_i$. The $R\Gamma(\widetilde{\mathcal{M}}_b^\circ, \mathcal{F}_i)$ are uniformly bounded; to see that their derived limit is also bounded, it is then sufficient to see that each $H^j(\widetilde{\mathcal{M}}_b^\circ, \mathcal{F}_i)$ is finite. By Theorem IV.5.3, this is isomorphic to $H_c^{j+1}(\widetilde{\mathcal{M}}_b^\circ, \mathcal{F}_i)$. But $R\Gamma_c(\widetilde{\mathcal{M}}_b^\circ, -)$ preserves compact objects as its right adjoint commutes with all colimits (as $\widetilde{\mathcal{M}}_b^\circ$ is cohomologically smooth over $\operatorname{Spd} k$, being open in a successive extension of negative Banach–Colmez spaces). $\square$

Next, we study Bernstein–Zelevinsky duality. Denoting $\pi : \mathrm{Bun}_G \to *$ the projection, the pullback π^* has a left adjoint

$$\pi_\natural : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(*, \Lambda) \cong D(\Lambda).$$

This induces a pairing

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \times D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda) : (A, B) \mapsto \pi_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B).$$

Proposition VII.7.6. *For any compact object $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, there is a unique compact object $\mathbb{D}_{\mathrm{BZ}}(A) \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ with a functorial identification*

$$R\operatorname{Hom}(\mathbb{D}_{\mathrm{BZ}}(A), B) \cong \pi_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B)$$

for $B \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Moreover, the functor $\mathbb{D}_{\mathrm{BZ}}$ is a contravariant autoequivalence of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$, and $\mathbb{D}_{\mathrm{BZ}}^2$ is naturally isomorphic to the identity.

If $U \subset \mathrm{Bun}_G$ is an open substack and A is concentrated on U, then so is $\mathbb{D}_{\mathrm{BZ}}(A)$. In particular, $\mathbb{D}_{\mathrm{BZ}}$ restricts to an autoequivalence of the compact objects in $D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ for $b \in B(G)$ basic, and in that setting it is the usual Bernstein–Zelevinsky involution.

Proof. The existence of $\mathbb{D}_{\mathrm{BZ}}$ follows as in Theorem V.5.1, using the left adjoint given by Proposition VII.7.2; this construction also shows that $\mathbb{D}_{\mathrm{BZ}}$ preserves $D_{\mathrm{lis}}(U, \Lambda)$, and for basic b it recovers the usual Bernstein–Zelevinsky involution by the same argument as in Theorem V.5.1.

We also formally get a morphism $\mathbb{D}_{\mathrm{BZ}}^2(A) \to A$ by adjunctions. We need to see that this is an isomorphism. It suffices to check on generators. The Bernstein–Zelevinsky dual of $i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ is $A_K^b = f_{K\natural}\Lambda$, for $f_K : \mathcal{M}_{b,K} \to \mathrm{Bun}_G$ the natural map. Its restriction to Bun_G^b is again $i_!^b c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$, so one easily checks that the map $\mathbb{D}_{\mathrm{BZ}}^2(A) \to A$ is an isomorphism over Bun_G^b. To see that it is an isomorphism everywhere, one needs to see that if $B = Rj_*B'$, $B' \in D_{\mathrm{lis}}(U, \Lambda)$ for some open substack $j : U \subset \mathrm{Bun}_G$ not containing Bun_G^b, then

$$\pi_\natural(A_K^b \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) = 0.$$

Twisting a few things away and using the definition of $A_K^b = f_{K\natural}\Lambda$, this follows from the assertion that for all $A' \in D_{\mathrm{lis}}(\mathcal{M}_{b,K}^\circ, \Lambda)$, with $j_K : \mathcal{M}_{b,K}^\circ \hookrightarrow \mathcal{M}_{b,K}$ the open immersion, one has

$$R\Gamma_c(\mathcal{M}_{b,K}, Rj_{K*}A') = 0.$$

Using the trace map for $\widetilde{\mathcal{M}}_b \to \mathcal{M}_{b,K}$, this follows from Theorem VII.2.10. □

As in Theorem V.6.1, this has the following consequence for Verdier duality.

PROPOSITION VII.7.7. *Let $j : V \hookrightarrow U$ be an open immersion of open substacks of Bun_G. For any $A \in D_{\mathrm{lis}}(V, \Lambda)$, the natural map*

$$j_! R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(A, \Lambda) \to R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(Rj_{\mathrm{lis}*}A, \Lambda)$$

is an isomorphism in $D_{\mathrm{lis}}(U, \Lambda)$.

PROOF. The proof is identical to the proof of Theorem V.6.1. □

Using this, one can characterize the reflexive objects as in Theorem V.6.2; we omit it here.

Finally, one can also characterize the universally locally acyclic $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Note that we have not defined a notion of universal local acyclicity for lisse sheaves, but in our present situation we can simply import the characterization from Proposition IV.2.32 and make the following definition.

DEFINITION VII.7.8. *A complex $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is universally locally acyclic (with respect to $\mathrm{Bun}_G \to *$) if the natural map*

$$p_1^* R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(A, \Lambda) \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} p_2^* A \to R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(p_1^* A, p_2^* A)$$

is an isomorphism, where $p_1, p_2 : \mathrm{Bun}_G \times \mathrm{Bun}_G \to \mathrm{Bun}_G$ are the two projections.

We get the following version of Theorem V.7.1.

PROPOSITION VII.7.9. *Let $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Then A is universally locally acyclic if and only if for all $b \in B(G)$, the pullback $i^{b*}A$ to $i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ corresponds under $D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ to a complex M_b of smooth $G_b(E)$-representations for which M^K is a perfect complex of Λ-modules for all open pro-p subgroups $K \subset G_b(E)$.*

The proof is identical to the proof of Theorem V.7.1, and proceeds by proving first the following proposition.

PROPOSITION VII.7.10. *Let G_1 and G_2 be two reductive groups over E, and let $G = G_1 \times G_2$. Consider the exterior tensor product*

$$- \boxtimes - : D_{\mathrm{lis}}(\mathrm{Bun}_{G_1}, \Lambda) \times D_{\mathrm{lis}}(\mathrm{Bun}_{G_2}, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

For all compact objects $A_i \in D_{\mathrm{lis}}(\mathrm{Bun}_{G_i}, \Lambda)$, $i = 1, 2$, the exterior tensor product $A_1 \boxtimes A_2 \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is compact, these objects form a class of compact generators, and for all further objects $B_i \in D_{\mathrm{lis}}(\mathrm{Bun}_{G_i}, \Lambda)$, $i = 1, 2$, the natural map

$$R\,\mathrm{Hom}(A_1, B_1) \otimes^{\mathbb{L}}_{\Lambda} R\,\mathrm{Hom}(A_2, B_2) \to R\,\mathrm{Hom}(A_1 \boxtimes A_2, B_1 \boxtimes B_2)$$

is an isomorphism.

PROOF. The proof is identical to the proof of Proposition V.7.2. □

CHAPTER VIII

L-parameter

It is time to understand the other side of the correspondence: In this chapter, we define, and study basic properties of, the stack of L-parameters. These results have recently been obtained by Dat–Helm–Kurinczuk–Moss [**DHKM20**], and also Zhu [**Zhu20**]; previous work in a related direction includes [**Hel16**], [**HH20**], [**BG19**], [**BP19**], [**LTX$^+$19**, Appendix E].

In this chapter, we fix again a nonarchimedean local field E with residue field $\mathbb{F}_q$ of characteristic p, and a reductive group G over E, as well as a prime $\ell \neq p$. We get the dual group $\widehat{G}/\mathbb{Z}_\ell$, which we endow with its usual "algebraic" action by W_E; the action thus factors over a finite quotient Q of W_E, and we fix such a quotient Q of W_E. (The difference to the cyclotomically twisted W_E-action disappears after base change to $\mathbb{Z}_\ell[\sqrt{q}]$, and we could thus obtain analogues of all results below for this other action by a simple descent along $\mathbb{Z}_\ell[\sqrt{q}]/\mathbb{Z}_\ell$.) We define a scheme whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the condensed 1-cocycles

$$\varphi : W_E \to \widehat{G}(\Lambda),$$

where $\Lambda = \Lambda_{\mathrm{disc}} \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$ is regarded as a relatively discrete condensed $\mathbb{Z}_\ell$-module.

Theorem VIII.0.1 (Theorem VIII.1.3). *There is a scheme $Z^1(W_E, \widehat{G})$ over $\mathbb{Z}_\ell$ whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the condensed 1-cocycles*

$$\varphi : W_E \to \widehat{G}(\Lambda).$$

The scheme $Z^1(W_E, \widehat{G})$ is a union of open and closed affine subschemes $Z^1(W_E/P, \widehat{G})$ as P runs through open subgroups of the wild inertia subgroup of W_E, and each $Z^1(W_E/P, \widehat{G})$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $\dim G$.

To prove the theorem, following [**DHKM20**] and [**Zhu20**] we define discrete dense subgroups $W \subset W_E/P$ by discretizing the tame inertia, and the restriction $Z^1(W_E/P, \widehat{G}) \to Z^1(W, \widehat{G})$ is an isomorphism, where the latter is clearly an affine scheme.

We can also prove further results about the $\widehat{G}$-action on $Z^1(W_E, \widehat{G})$, or more precisely each $Z^1(W_E/P, \widehat{G})$. For this result, we need to exclude some small primes, but if $G = \mathrm{GL}_n$, all primes ℓ are allowed; for classical groups, all $\ell \neq 2$ are allowed. More precisely, we say that ℓ is very good for $\widehat{G}$ if the following conditions are satisfied.

(i) The (algebraic) action of W_E on $\widehat{G}$ factors over a finite quotient Q of order prime to ℓ.

(ii) The order of the fundamental group of the derived group of $\widehat{G}$ is prime to ℓ (equivalently, $\pi_0 Z(G)$ is of order prime to ℓ).

(iii) If G has factors of type B, C, or D, then $\ell \neq 2$; if it has factors of type E, F, or G, then $\ell \neq 2, 3$; and if it has factors of type E_8, then $\ell \neq 2, 3, 5$.

THEOREM VIII.0.2 (Theorem VIII.5.1). *Assume that ℓ is a very good prime for $\widehat{G}$. Then $H^i(\widehat{G}, \mathcal{O}(Z^1(W_E/P, \widehat{G}))) = 0$ for $i > 0$ and the formation of the invariants $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ commutes with any base change. The algebra $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ admits an explicit presentation in terms of excursion operators,*

$$\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}} = \operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}}$$

where the colimit runs over all maps from a free group F_n to $W \subset W_E/P$, and $Z^1(F_n, \widehat{G}) \cong \widehat{G}^n$ with the simultaneous twisted $\widehat{G}$-conjugation.

Moreover, the ∞-category $\operatorname{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$ is generated under cones and retracts by the image of $\operatorname{Rep}(\widehat{G}) \to \operatorname{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$, and $\operatorname{Ind}\operatorname{Perf}(Z^1(W_E/P, \widehat{G}))$ is equivalent to the ∞-category of modules over $\mathcal{O}(Z^1(W_E/P, \widehat{G}))$ in $\operatorname{Ind}\operatorname{Perf}(B\widehat{G})$.

All of these results also hold with $\mathbb{Q}_\ell$-coefficients, without any assumption on ℓ.

With $\mathbb{Q}_\ell$-coefficients, these results are simple, as the representation theory of $\widehat{G}$ is semisimple. However, with $\mathbb{Z}_\ell$-coefficients, these results are quite subtle, and we need to dive somewhat into modular representation of reductive groups. Very roughly, the proof of the theorem proceeds by analyzing the closed $\widehat{G}$-orbits in the stack of L-parameters first, and then use a deformation to the normal cone to understand the behaviour near any $\widehat{G}$-orbit. We make critical use of some results of Touzé–van der Kallen [**TvdK10**]. Let us make some further remarks about the closed $\widehat{G}$-orbits.

First, the closed $\widehat{G}$-orbits in $Z^1(W_E/P, \widehat{G})_L$, for any algebraically closed field L over $\mathbb{Z}_\ell$, correspond to the *semisimple* L-parameter $\varphi : W_E \to \widehat{G}(L)$, and the geometric points of $\operatorname{Spec} \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$. Here, an L-parameter is semisimple if, whenever it factors over a parabolic $\widehat{P} \subset \widehat{G}$, it also factors over a Levi $\widehat{M} \subset \widehat{P}$. Any semisimple parameter is in fact continuous for the discrete topology on L, i.e. trivial on an open subgroup of I_E. If $L = \overline{\mathbb{F}}_\ell$ and Q is of order prime to ℓ, then φ is semisimple if and only if it factors over a finite quotient of order prime to ℓ. Its orbit in $Z^1(W_E/P, \widehat{G})_L$ is then given by $\widehat{G}/S_\varphi$ where $S_\varphi \subset \widehat{G}$ is the centralizer of φ, which is in that case the fixed points under the action of a finite solvable group F of automorphisms of $\widehat{G}$, where the order of F is prime to ℓ. We need the following result.

THEOREM VIII.0.3 (Theorem VIII.5.14). *Let H be a reductive group over an algebraically closed field L of characteristic ℓ. Let F be a finite group of order prime to ℓ acting on H. Then H^F is a smooth linear algebraic group whose connected component is reductive, and with $\pi_0 H^F$ of order prime to ℓ. If F is solvable, the image of $\operatorname{Perf}(BH) \to \operatorname{Perf}(BH^F)$ generates the whole category under cones and retracts.*

The last part of this theorem is proved by a very explicit (and exhausting) analysis of all possible cases. It would be desirable to have a more enlightening argument, possibly also removing the assumption that F is solvable. In fact, we would expect similar results to hold true in the case where W is replaced by the fundamental group of a Riemann surface. Our arguments are not general enough to handle that case.

Remark VIII.0.4. While the hypotheses imposed on ℓ are surely not optimal, we are quite sure that *some* hypothesis on ℓ is required in Theorem I.8.2. For example, if $\widehat{G} = \mathrm{PGL}_2$ and $\ell = 2$, we expect problems to arise. For example, one can show that for $X = \widehat{G}$ with the conjugation action by $\widehat{G}$, the ∞-category $\mathrm{Perf}(X/\widehat{G})$ is not generated under cones and retracts by the image of $\mathrm{Rep}(\widehat{G})$. Our guess would be that the condition that ℓ does not divide the order of $\pi_1(\widehat{G}_{\mathrm{der}})$ is essential.

On the other hand, we expect that, for example by the use of z-embeddings [**Kal14**, Section 5], one can reduce all relevant questions (like the general construction of maps on Bernstein centers discussed below, or the construction of the spectral action) to the case where ℓ does not divide the order of $\pi_1(\widehat{G}_{\mathrm{der}})$. We are not taking this up here.

VIII.1. The stack of L-parameters

VIII.1.1. Definition and representability. Recall that for a reductive group G over a nonarchimedean local field E, we have the dual group $\widehat{G}$ over $\mathbb{Z}_\ell$, equipped with an action of the Weil group W_E. In this chapter, we use the standard action.

Now let Λ be any $\mathbb{Z}_\ell$-algebra. As in the last chapter, we regard it as a condensed $\mathbb{Z}_\ell$-algebra, as $\Lambda_{\mathrm{disc}} \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$. Its value on a profinite set S is the ring of maps $S \to \Lambda$ that take values in a sub-$\mathbb{Z}_\ell$-module of finite type and are continuous. For example, if $\Lambda = \overline{\mathbb{Q}}_\ell$ then $\Lambda(S) = \varinjlim_{L \subset \overline{\mathbb{Q}}_\ell} \mathrm{Cont}(S, L)$ with $L|\mathbb{Q}_\ell$ finite.

Definition VIII.1.1. *An L-parameter for G, with coefficients in Λ, is a section*

$$\varphi : W_E \to \widehat{G}(\Lambda) \rtimes W_E$$

of the natural map of condensed groups

$$\widehat{G}(\Lambda) \rtimes W_E \to W_E.$$

Equivalently, an L-parameter for G with coefficients in Λ is a (condensed) 1-cocycle

$$\varphi : W_E \to \widehat{G}(\Lambda)$$

for the given W_E-action on $\widehat{G}$.

More concretely, an L-parameter with values in Λ is a 1-cocycle $\varphi : W_E \to \widehat{G}(\Lambda)$ such that if $\widehat{G} \hookrightarrow \mathrm{GL}_N$, the associated map $W_E \to \mathrm{GL}_N(\Lambda)$ is continuous. The preceding means the matrix coefficients of its restriction to I_E are maps $I_E \to \Lambda$ that take values in finite type $\mathbb{Z}_\ell$-modules and are continuous.

Remark VIII.1.2. The standard action of W_E factors over a finite quotient Q. This means that L-parameters are also equivalent to maps $W_E \to \widehat{G}(\Lambda) \rtimes Q$ lifting $W_E \to Q$.

The first main result is the following.

Theorem VIII.1.3. *There is a scheme $Z^1(W_E, \widehat{G})$ over $\mathbb{Z}_\ell$ parametrizing L-parameters for G, which is a disjoint union of affine schemes of finite type over $\mathbb{Z}_\ell$. It is flat and a relative complete intersection of dimension* $\dim G = \dim \widehat{G}$.

Proof. Any condensed 1-cocycle $\varphi : W_E \to \widehat{G}(\Lambda)$ is trivial on an open subgroup of the wild inertia subgroup P_E; note also that P_E acts on $\widehat{G}$ through a finite quotient. Moreover, for any $\gamma \in P_E$ acting trivially on $\widehat{G}$, the locus where $\varphi(\gamma) = 1$ is open and closed: Taking a closed embedding $\widehat{G} \hookrightarrow \mathrm{GL}_N$, this follows from $A = 1$ being a connected component of the locus of all $A \in \mathrm{GL}_N$ such that $A^{p^r} = 1$, as can be checked by observing that the tangent space at $A = 1$ is trivial. It follows that the moduli space of L-parameters decomposes as a disjoint union of open and closed subspaces according to the kernel of φ on P_E.

Thus, fix now some quotient $W_E \to W'_E$ by an open subgroup of P_E such that the action of W_E on $\widehat{G}$ factors over W'_E. We are interested in the moduli space of condensed 1-cocycles $W'_E \to \widehat{G}(\Lambda)$. Inside W'_E, we look at the discrete dense subgroup $W \subset W'_E$ generated by the image of P_E, a choice of generator of the tame inertia τ, and a choice of Frobenius σ. Thus, W sits in an exact sequence

$$0 \to I \to W \to \sigma^{\mathbb{Z}} \to 0$$

where I in turn sits in an exact sequence

$$0 \to P \to I \to \tau^{\mathbb{Z}[\frac{1}{p}]} \to 0$$

where P is a finite p-group. Moreover, in W/P, the elements τ and σ satisfy the commutation $\sigma^{-1}\tau\sigma = \tau^q$.

Now observe that any condensed 1-cocycle $W'_E \to \widehat{G}(\Lambda)$ is already determined by its restriction to the discrete group W, as $\widehat{G}(\Lambda)$ is quasiseparated and $W \subset W'_E$ is dense. Conversely, we claim that any 1-cocycle $W \to \widehat{G}(\Lambda)$ extends uniquely to a condensed 1-cocycle $W'_E \to \widehat{G}(\Lambda)$. To check this, we may replace E by a finite extension; we can thus pass to a setting where the action of W'_E on $\widehat{G}$ is trivial, and where $P = 1$. Taking a closed immersion $\widehat{G} \hookrightarrow \mathrm{GL}_N$, it then suffices to see that any representation of $\tau^{\mathbb{Z}[\frac{1}{p}]} \rtimes \sigma^{\mathbb{Z}}$ on a finite free Λ-module extends uniquely to a representation of the condensed group $\widehat{\mathbb{Z}}^p \rtimes \sigma^{\mathbb{Z}}$. For this, in turn, it suffices to see that for any $A \in \mathrm{GL}_N(\Lambda)$ such that A is conjugate to A^q, the map

$$\mathbb{Z} \to \mathrm{GL}_N(\Lambda) : n \mapsto A^n$$

extends uniquely to $\widehat{\mathbb{Z}}^p$. The assumption on A implies that all eigenvalues of A at all geometric points of $\operatorname{Spec}\Lambda$ are roots of unity of order prime to p; replacing A by a prime-to-p-power (as we may) we can thus reduce to the case that A is unipotent, i.e. $A - 1$ is nilpotent. But then $n \mapsto X^n$ extends to a continuous map

$$n \mapsto X^n = (1 + (X-1))^n = \sum_{i \geq 0} \binom{n}{i} (X-1)^i,$$

defining a map $\mathbb{Z}_\ell \to \mathrm{GL}_N(\Lambda)$ (and hence $\widehat{\mathbb{Z}}^p \to \mathbb{Z}_\ell \to \mathrm{GL}_N(\Lambda)$).

Thus, we need to see that the space $X = Z^1(W, \widehat{G})$ of all 1-cocycles $\varphi : W \to \widehat{G}(\Lambda)$ is an affine scheme of finite type over $\mathbb{Z}_\ell$ that is flat and a relative complete intersection of dimension $\dim \widehat{G}$. It is clear that it is an affine scheme of finite type over $\mathbb{Z}_\ell$ as W is discrete and finitely generated.

To prove the geometric properties, we find it slightly more convenient to argue with the Artin stack $[X/\widehat{G}]$, which we aim to prove is flat and a relative complete intersection of dimension 0 over $\mathbb{Z}_\ell$.

We can understand the deformation theory of $[X/\widehat{G}]$: If Λ is a field, then the obstruction group is $H^2(W, \widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda)$ (where $\widehat{\mathfrak{g}}$ is the Lie algebra of $\widehat{G}$), the tangent space is $H^1(W, \widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda)$, and the infinitesimal automorphisms are $H^0(W, \widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda)$, where in all cases the action of W is twisted by the local 1-cocycle φ. Now note that by direct computation the prime-to-p cohomological dimension of W is 2, and the Euler characteristic of any representation is equal to 0. Thus, this analysis shows that we only have to prove that all fibres of $[X/\widehat{G}] \to \operatorname{Spec} \mathbb{Z}_\ell$ are of dimension at most 0.

Note that X is actually naturally defined over $\mathbb{Z}[\frac{1}{p}]$ (as $\widehat{G}$ is, and the discretization W of W'_E is independent of ℓ). It follows that it suffices to bound the dimension of the fibre over $\mathbb{F}_\ell$ (as if we can do this for all closed points of $\operatorname{Spec} \mathbb{Z}[\frac{1}{p}]$, it follows over the generic fibre by constructibility of the dimension of fibers). To do this, we switch back to the picture of condensed 1-cocycles on W_E. From now on, we work over $\overline{\mathbb{F}}_\ell$.

The stack $[Z^1(W_E, \widehat{G})_{\overline{\mathbb{F}}_\ell}/\widehat{G}]$ maps to the similar stack parametrizing 1-cocycles $\varphi^{I^\ell} : I^\ell \to \widehat{G}(\Lambda)$ of the prime-to-ℓ inertia subgroup I^ℓ, up to conjugation. By deformation theory, that stack is smooth and each connected component is a quotient of $\operatorname{Spec} \overline{\mathbb{F}}_\ell$ by the centralizer group $C_{\varphi^{I^\ell}} \subset \widehat{G}_{\overline{\mathbb{F}}_\ell}$, which is a smooth group, whose identity component is reductive by [**PY02**, Theorem 2.1]. We may thus fix $\varphi^{I^\ell} : I^\ell \to \widehat{G}(\overline{\mathbb{F}}_\ell)$ and consider the open and closed subscheme $X_{\varphi^{I^\ell}} \subset Z^1(W_E, \widehat{G})_{\overline{\mathbb{F}}_\ell}$ of all 1-cocycles $\varphi : W_E \to \widehat{G}(\Lambda)$ whose restriction to I^ℓ is equal to φ^{I^ℓ}. Our goal is to show that $X_{\varphi^{I^\ell}}$ is of dimension at most $\dim C_{\varphi^{I^\ell}}$.

Consider the centralizer $\widetilde{C}$ of φ^{I^ℓ} inside $\widehat{G}_{\overline{\mathbb{F}}_\ell} \rtimes Q$. Then $X_{\varphi^{I^\ell}}$ maps with finite fibres to the space of maps

$$f : W_E/I^\ell \cong \mathbb{Z}_\ell \rtimes \sigma^{\mathbb{Z}} \longrightarrow \widetilde{C}/\varphi^{I^\ell}(I^\ell).$$

Note that, by representability of $X_{\varphi^{I^\ell}}$, the universal map f factors over a quotient of the form $\mathbb{Z}/\ell^m\mathbb{Z} \rtimes \sigma^{\mathbb{Z}}$. Finally, we have reduced to Lemma VIII.1.4 below. □

Lemma VIII.1.4. *Let H be a smooth group scheme over $\overline{\mathbb{F}}_\ell$ whose identity component is reductive. Then the affine scheme parametrizing maps of groups*

$$\mathbb{Z}/\ell^m\mathbb{Z} \rtimes \sigma^{\mathbb{Z}} \to H,$$

where σ acts on $\mathbb{Z}/\ell^m\mathbb{Z}$ via multiplication by q, is of dimension at most $\dim H$.

Proof. The image of the generator of $\mathbb{Z}/\ell^m\mathbb{Z}$ is a unipotent element of H. By finiteness of the number of unipotent conjugacy classes, cf. [**Lus76**], [**FG12**, Corollary 2.6], we can stratify the scheme according to the conjugacy class of the image of τ. But for each fixed conjugacy class, one has to choose the image of σ so as to conjugate τ into τ^q: This bounds the dimension of each stratum by the dimension of the conjugacy class of τ (giving the choices for τ) plus the codimension of the conjugacy class of τ (giving the choices for σ, for any given τ), which is the dimension of H. □

We observe the following corollary of the proof.

Proposition VIII.1.5. *Assume that W_E acts on $\widehat{G}$ via a finite quotient Q of order prime to ℓ. Let $L = \overline{\mathbb{F}}_\ell$. Let $\varphi : W_E \to \widehat{G}(L)$ be any L-parameter. Then the $\widehat{G}$-orbit of φ in $Z^1(W_E, \widehat{G})_L$ is closed if and only if the map φ factors over a finite quotient of W_E of order prime to ℓ.*

Proof. In the notation of the proof, the group $C_{\varphi^{I^\ell}}$ has connected components of order prime to ℓ, see Proposition VIII.5.11 below, so one will apply Lemma VIII.1.4 to a group H with $\pi_0 H$ of order prime to ℓ. This implies that the unipotent element is actually an element of H°, and its orbit is closed only if it is trivial. Moreover, the image of the Frobenius σ defines a closed orbit if and only if it is semisimple, i.e. of order prime to ℓ, giving the claim. □

VIII.2. The singularities of the moduli space

The following proposition was already implicitly noted in the proof of Theorem VIII.1.3.

Proposition VIII.2.1. *For any parameter $\varphi : W_E \to \widehat{G}(\Lambda) \rtimes Q$ corresponding to $x : \mathrm{Spec}(\Lambda) \to \mathrm{LocSys}_{\widehat{G}}$,*

$$x^* \mathbb{L}^\vee_{Z^1(W_E, \widehat{G})/\widehat{G}} = R\Gamma(W_E, (\widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi)[1]$$

where $(\widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi$ is $\widehat{\mathfrak{g}} \otimes_{\mathbb{Z}_\ell} \Lambda$ equipped with the twisted action of W_E deduced from φ.

Proof. This would be clear if we defined the moduli problem on all animated $\mathbb{Z}_\ell$-algebras, by deformation theory. Then the cohomological dimension of W_E would imply that this moduli problem is a derived local complete intersection, of expected dimension 0. However, we proved that $Z^1(W_E, \widehat{G})/\widehat{G}$ is a local complete intersection Artin stack of dimension 0, hence it represents the correct moduli problem even on all animated $\mathbb{Z}_\ell$-algebras, thus giving the result. □

Proposition VIII.2.2. *Let M be a free Λ-module of finite rank equipped with a condensed action of W_E. Then $R\Gamma(W_E, M)$ is a perfect complex of Λ-modules and there is a canonical isomorphism*

$$R\Gamma(W_E, M)^* \cong R\Gamma(W_E, M^*(1))[2].$$

Proof. This follows from Poincaré duality applied to $\mathrm{Div}^1 \to *$, using Proposition VII.3.5 and the discussion before. It can also be proved by hand, by comparing the W_E-cohomology with the W-cohomology, for a discretization W of W_E/P as before. □

Corollary VIII.2.3. *For any parameter $\varphi : W_E \to \widehat{G}(\Lambda) \rtimes W_E$ corresponding to $x : \mathrm{Spec}(\Lambda) \to \mathrm{LocSys}_{\widehat{G}}$,*

$$x^* \mathbb{L}_{\mathrm{LocSys}_{\widehat{G}/\mathbb{Z}_\ell}} = R\Gamma(W_E, (\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi(1))[1]$$

where $(\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi$ is $\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda$ equipped with the twisted action of W_E deduced from φ.

VIII.2.1. The characteristic zero case. Fix an isomorphism $I_E/P_E \cong \widehat{\mathbb{Z}}^p$. There is a $\widehat{G}_{\mathbb{Q}_\ell}$-equivariant "unipotent monodromy" morphism

$$\mathcal{M} : Z^1(W_E, \widehat{G})_{\mathbb{Q}_\ell} \longrightarrow \mathcal{N}_{\widehat{G}_{\mathbb{Q}_\ell}}$$

where $\mathcal{N}_{\widehat{G}_{\mathbb{Q}_\ell}}$ is the nilpotent cone inside $\widehat{\mathfrak{g}} \otimes \mathbb{Q}_\ell$.

In fact, one can lift the inclusion $\mathbb{Z}_\ell \hookrightarrow \widehat{\mathbb{Z}}^p \cong I_E/P_E$ to a morphism $\mathbb{Z}_\ell \to I_E$. Now, if $\varphi : W_E \to \widehat{G}(\Lambda)$, with Λ a $\mathbb{Q}_\ell$-algebra, is a parameter, then $\varphi|_{\mathbb{Z}_\ell} : \mathbb{Z}_\ell \to \widehat{G}(\Lambda)$ is such that for $n \gg 0$, $\varphi|_{\ell^n \mathbb{Z}_\ell}$ is a morphism of condensed groups satisfying

$$\varphi(\sigma^m)\varphi|_{\ell^n \mathbb{Z}_\ell}\varphi(\sigma^m)^{-1} = \varphi^{q^m}_{|\ell^n \mathbb{Z}_\ell}$$

for $m \gg 0$. One deduces, using an embedding of $\widehat{G}$ in GL_N, that there is a unique $N \in \mathcal{N}_{\widehat{G}}(\Lambda)$ such that for $n \gg 0$ and $x \in \ell^n \mathbb{Z}_\ell$,

$$\varphi(x) = \exp(xN).$$

Using these observations, we get a comparison to Weil–Deligne L-parameters.

Definition VIII.2.4. *For Λ a $\mathbb{Q}_\ell$-algebra one defines $\mathrm{Par}^{\mathrm{WD}}_{\widehat{G}}(\Lambda)$ to be the set of pairs (φ_0, N) where*

(i) $\varphi_0 : W_E \to \widehat{G}(\Lambda^{disc})$ *is a condensed* 1*-cocycle,*
(ii) $N \in \mathcal{N}_{\widehat{G}} \otimes \mathbb{Q}_\ell$ *satisfies* $\mathrm{Ad}(\varphi_0(\sigma)).\sigma N = q^{|\sigma|} N$ *for all* $\sigma \in W_E$.

Then we have the following result, which is essentially Grothendieck's quasi-unipotence theorem.

Proposition VIII.2.5 ([**Zhu20**, Lemma 3.1.8]). *There is a $\widehat{G}$-equivariant isomorphism*

$$Z^1(W_E, \widehat{G}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{Par}^{\mathrm{WD}}_{\widehat{G}}.$$

However, we warn the reader that this isomorphism depends on some auxiliary choices, such as that of a Frobenius element.

VIII.2.2. The singular support.

VIII.2.2.1. *General construction.* Recall the following construction, see for example [**AG15**]. Let $A \to B$ be a flat map of commutative rings. One has the Hochschild cohomology

$$HH^\bullet(B/A) = \mathrm{Ext}^\bullet_{B \otimes_A B}(B, B).$$

Note that any $M \in D(B \otimes_A B)$ induces a functor $D(B) \to D(B)$, via $N \mapsto M \otimes^{\mathbb{L}}_B N$ (with the "left" B-module structure). Here, $M = B \in D(B \otimes_A B)$, via the multiplication $B \otimes_A B \to B$, induces the identity functor. It follows that there is a natural map

$$HH^i(B/A) = \mathrm{Ext}^i_{B \otimes_A B}(B, B) \to \mathrm{Ext}^i_B(N, N)$$

for any $N \in D(B)$. Moreover, Hochschild cohomology is naturally a graded algebra, and this map is a map of algebras

$$HH^\bullet(B/A) \to \mathrm{Ext}^\bullet_B(N, N).$$

There is an identification ([**ML95**, Theorem X.3.1])

$$HH^2(B/A) = \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, B)$$

which itself is nothing else than $\mathcal{E}xalcom_A(B, B)$ ([**Gro64**, Chap.0, Sec. 18.4]). We thus have an identification

$$HH^2(B/A) = H^1(\mathbb{L}^\vee_{B/A}).$$

Suppose now that $A \to B$ is syntomic, i.e. flat and a local complete intersection. Let $X = \mathrm{Spec}\, B \to S = \mathrm{Spec}\, A$ be the associated map of affine schemes.

Definition VIII.2.6. *The scheme*

$$\mathrm{Sing}_{X/S} \longrightarrow X$$

represents the functor $T/X \mapsto H^{-1}(\mathbb{L}_{X/S} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_T)$.

In fact, locally on X, $\mathbb{L}_{X/S}$ is isomorphic to a complex of vector bundles $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ and then $\mathrm{Sing}_{X/S}$ is the kernel of $\mathbb{V}(\mathcal{E}^{-1}) \to \mathbb{V}(\mathcal{E}^0)$. Explicitly, $\mathrm{Sing}_{X/S}$ is the affine scheme with

$$\mathcal{O}(\mathrm{Sing}_{X/S}) = \mathrm{Sym}_B^\bullet H^1(\mathbb{L}_{B/A}^\vee).$$

This is an X-group scheme equipped with an action of $\mathbb{G}_m$. *The image of* $\mathrm{Sing}_{X/S} \setminus \{0\} \to X$ *is the closed subset complementary of the smooth locus of* $X \to S$.

Consider now any

$$N \in D^b_{coh}(X),$$

and the graded B-algebra $\mathrm{Ext}_B^\bullet(N, N)$. Using the map

$$H^1(\mathbb{L}_{B/A}^\vee) = HH^2(B/A) \to \mathrm{Ext}_B^2(N, N),$$

this is in fact naturally a (graded) $\mathcal{O}(\mathrm{Sing}_{X/S})$-algebra. This defines a $\mathbb{G}_m$-equivariant quasi-coherent sheaf

$$\mu\,\mathrm{End}(N)$$

on $\mathrm{Sing}_{X/S}$.

Suppose now moreover that S is regular.

THEOREM VIII.2.7 ([**Gul74**, Theorem 3.1],[**AG15**, Appendix D]). *For $N \in D^b_{coh}(X)$, the quasi-coherent sheaf $\mu\,\mathrm{End}(N)$ on $\mathrm{Sing}_{X/S}$ is coherent.*

DEFINITION VIII.2.8. *The singular support of N, $\mathrm{SingSupp}(N)$, is the support of $\mu\,\mathrm{End}(\mathcal{F})$ as a closed conical subset of $\mathrm{Sing}_{X/S}$.*

Of course, the image of $\mathrm{SingSupp}(N) \to X$ is contained in $\mathrm{Supp}(N)$.

THEOREM VIII.2.9 ([**AG15**, Theorem 4.2.6]). *The following are equivalent:*

(i) *N is a perfect complex,*
(ii) $\mathrm{SingSupp}(N)$ *is contained in the zero section of* $\mathrm{Sing}_{X/S}$.

PROOF. We have to prove that if $\mathrm{Ext}_B^i(N, N) = 0$ for $i \gg 0$ then N is a perfect complex. This is for example a consequence of [**Jor08**]. Since S is regular X is Gorenstein. According to [**Jor08**], if N is a B-module of finite type that satisfies $\mathrm{Ext}_B^i(N, N) = 0$ for $i > n$, then $\mathrm{pd}_B N \leq n$. In general, up to taking a shift of N, we can find a map $N \to N'$, where N' is a finitely generated B-module concentrated in degree 0, such that the cone C of $N \to N'$ is perfect. Suppose that $\mathrm{Ext}_B^i(N, N) = 0$ for $i \gg 0$. In the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_B^i(C, N) \longrightarrow \mathrm{Ext}_B^i(N', N) \longrightarrow \mathrm{Ext}_B^i(N, N) \longrightarrow \cdots$$

one has $\mathrm{Ext}_B^i(C, N) = 0$ for $i \gg 0$ since C is perfect and $\mathrm{Ext}_B^i(N, N) = 0$ for $i \gg 0$ by hypothesis. We deduce that $\mathrm{Ext}_B^i(N', N) = 0$ for $i \gg 0$. In the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_B^i(N', N) \longrightarrow \mathrm{Ext}_B^i(N', N') \longrightarrow \mathrm{Ext}_B^i(N', C) \longrightarrow \cdots$$

we have $\mathrm{Ext}_B^i(N', C) = 0$ for $i \gg 0$ since C is perfect and A has finite injective dimension over itself since it is Gorenstein. Thus, for $i \gg 0$, $\mathrm{Ext}_B^i(N', N') \xrightarrow{\sim} \mathrm{Ext}_B^i(N', N)$ and this vanishes. We can thus apply Jorgensen's theorem to N' to conclude that N', and hence N, is perfect. $\square$

Let us note the following corollary.

COROLLARY VIII.2.10. *The image of* $\mathrm{SingSupp}(N) \setminus \{0\} \to X$ *is the complementary of the biggest open subset of* X *on which* N *is a perfect complex.*

VIII.2.2.2. *The case of* $Z^1(W_E, \widehat{G})$. Now we apply the preceding theory in the case $A = \mathbb{Z}_\ell$ and $X = Z^1(W_E, \widehat{G})$ (which is only a union of affine schemes, but this is not a problem). We can also pass to the quotient stack $Z^1(W_E, \widehat{G})/\widehat{G}$. According to Corollary VIII.2.3, there is an embedding

$$\begin{tikzcd} \mathrm{Sing}_{[Z^1(W_E,\widehat{G})/\widehat{G}]/\mathbb{Z}_\ell} \arrow[r, hook] \arrow[d] & {[\widehat{\mathfrak{g}}^*/\widehat{G}] \times_{B\widehat{G}} [Z^1(W_E,\widehat{G})/\widehat{G}]} \arrow[dl] \\ {[Z^1(W_E,\widehat{G})/\widehat{G}]} & \end{tikzcd}$$

where $\widehat{\mathfrak{g}} = \mathrm{Lie}\,\widehat{G}$ and $[\widehat{\mathfrak{g}}^*/\widehat{G}]$ is seen here as a vector bundle on $B\widehat{G} = [\mathrm{Spec}\,\mathbb{Z}_\ell/\widehat{G}]$. Let $\mathcal{N}^*_{\widehat{G}} \subset \widehat{\mathfrak{g}}^*$ be the nilpotent cone; by this we mean the closed subset of all $\widehat{G}$-orbits whose closure contains the origin. (If there is a $\widehat{G}$-equivariant isomorphism between $\widehat{\mathfrak{g}}^*$ and $\widehat{\mathfrak{g}}$, this would identify with the usual nilpotent cone.) Since this is stable under the adjoint action this defines a Zariski closed substack

$$\begin{tikzcd} {[\mathcal{N}^*_{\widehat{G}}/\widehat{G}] \times_{B\widehat{G}} [Z^1(W_E,\widehat{G})/\widehat{G}]} \arrow[r, hook] \arrow[d] & {[\widehat{\mathfrak{g}}^*/\widehat{G}] \times_{B\widehat{G}} [Z^1(W_E,\widehat{G})/\widehat{G}]} \arrow[dl] \\ {[Z^1(W_E,\widehat{G})/\widehat{G}].} & \end{tikzcd}$$

PROPOSITION VIII.2.11. *For a* $\mathbb{Z}_\ell$*-field* L *and a point* $x : \mathrm{Spec}(L) \to \mathrm{LocSys}_{\widehat{G}}$ *we have*

$$x^*\mathrm{Sing}_{[Z^1(W_E,\widehat{G})/\widehat{G}]/\mathbb{Z}_\ell} \subset \mathcal{N}^*_{\widehat{G}} \otimes_{\mathbb{Z}_\ell} L$$

in the following two cases:

(i) $L|\mathbb{Q}_\ell$,

(ii) ℓ *does not divide the order of the fundamental group of the adjoint form of* G*, and if* $n = f_{E'/E}$ *with* $W_{E'} = \ker(W_E \to \mathrm{Out}(\widehat{G}))$, $\ell \not| q^{en} - 1$ *for any exponent* e *of* $\widehat{G}$.

PROOF. If x corresponds to the parameter φ then $x^*\mathrm{Sing}_{[Z^1(W_E,\widehat{G})/\widehat{G}]/\mathbb{Z}_\ell} = H^0(W_E, \widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} L(1))$ where the W_E action on $\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} L(1)$ is twisted by φ. For an element $v \in \widehat{\mathfrak{g}} \cong \widehat{\mathfrak{g}}^*$ in this subspace we thus have that $\sigma.v$ and qv are in the same orbits under the adjoint action of $\widehat{G}(L)$ (here $\sigma.v$ is given by the action of W_E on $\widehat{\mathfrak{g}}^*$ defining the L-group). We thus obtain that v is conjugated under the adjoint action to $q^n v$. There is a morphism

$$\widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{g}} /\!\!/ \widehat{G} = \widehat{\mathfrak{t}} /\!\!/ W \cong \mathbb{A}^m_{\mathbb{Z}_\ell}$$

given by m homogeneous polynomials of degrees the exponents of the root system. This implies that the image of v in $\mathbb{A}^m(L)$ is zero and thus v lies in the nilpotent cone. $\square$

The supremum of the exponents of $\widehat{G}$ is the Coxeter number h of G. The preceding condition is satisfied if for example $\ell > q^{hn} - 1$. We refer to [**DHKM20**, Section 5.3] for finer definitions and results about $\widehat{G}$-banal primes; we have not tried to optimize the condition above, and it is likely that with their results one can obtain a much better condition on ℓ.

Remark VIII.2.12. In the non-banal case things become more complicated and the Arinkin–Gaitsgory condition of nilpotent singular support becomes important. This is also the case when interesting congruences modulo ℓ between smooth irreducible representations of $G(E)$ occur, cf. [**DHKM20**, Section 1.5].

VIII.3. The coarse moduli space

Let us now describe the corresponding coarse moduli space, i.e. we consider the quotient

$$Z^1(W_E, \widehat{G}) \mathbin{/\!\!/} \widehat{G}$$

taken in the category of schemes. Concretely, for every connected component $\operatorname{Spec} A \subset Z^1(W_E, \widehat{G})$, we get a corresponding connected component $\operatorname{Spec} A^{\widehat{G}} \subset Z^1(W_E, \widehat{G}) \mathbin{/\!\!/} \widehat{G}$.

VIII.3.1. Geometric points. For any algebraically closed field L over $\mathbb{Z}_\ell$, the L-valued points of $Z^1(W_E, \widehat{G})_L \mathbin{/\!\!/} \widehat{G}$ are in bijection with the closed $\widehat{G}$-orbits in $Z^1(W_E, \widehat{G})_L$.

We want to describe L-valued points with closed $\widehat{G}$-orbit as the "semisimple" parameters. For this, recall (cf. [**Bor79**]) that parabolic subgroups of $\widehat{G}_L \rtimes W_E$ surjecting onto W_E are up to $\widehat{G}(L)$-conjugation given by $\widehat{P}_L \rtimes W_E$ for a standard parabolic $P \subset G^*$ of the quasisplit inner form G^* of G. A Levi subgroup is given by $\widehat{M_L} \rtimes W_E$ where $M \subset P$ is the standard Levi. *We now call them the parabolic subgroups of $\widehat{G} \rtimes W_E$ i.e. we always suppose they surject to W_E.* If $\widehat{\Delta}$ are the simple roots of $\widehat{G}$ then the standard parabolic subgroups are in bijection with the finite W_E-stable subsets of $\widehat{\Delta}$.

Definition VIII.3.1. *Let L be an algebraically closed field over $\mathbb{Z}_\ell$. An L-parameter $\varphi : W_E \to \widehat{G}(L) \rtimes W_E$ is semisimple if whenever the image of φ is contained in a parabolic subgroup of $\widehat{G} \rtimes W_E$ then it is contained in a Levi subgroup of this parabolic subgroup.*

In terms of the standard parabolic subgroups this means that if some $\widehat{G}(L)$-conjugate φ' of φ factorizes through $\widehat{P}(L) \rtimes W_E$, then there exists $g \in \widehat{P}(L)$ such that $g\varphi' g^{-1} = \mathrm{pr}_{{}^L M} \circ \varphi'$, where $\mathrm{pr}_{{}^L M} : \widehat{P}(L) \rtimes W_E \to \widehat{M}(L) \rtimes W_E$ is the projection onto the standard Levi subgroup.

Proposition VIII.3.2 ([**DHKM20**, Proposition 4.13]). *Let L be an algebraically closed field over $\mathbb{Z}_\ell$ and $\varphi : W_E \to \widehat{G}(L) \rtimes W_E$ a parameter. The following are equivalent:*

(i) *The $\widehat{G}$-orbit of φ in $Z^1(W_E, \widehat{G})_L$ is closed.*

(ii) *For any conjugate φ' of φ such that $\varphi' : W_E \to \widehat{P}(L) \rtimes W_E$ factors over a standard parabolic subgroup, φ is $\widehat{G}(L)$-conjugate to $\mathrm{pr}_{{}^L M} \circ \varphi'$.*

(iii) *φ is semi-simple.*

Proof. We use the Hilbert–Mumford–Kempf theorem, cf. [**Kem78**, Corollary 3.5]. Let $\lambda : \mathbb{G}_m \to \widehat{G}_L$. Up to conjugation we can assume $\lambda \in X_*(\widehat{T})_+$. For each $\tau \in W_E$ there is a morphism $\mathrm{ev}_\tau : Z^1(W_E, \widehat{G})_L \to \widehat{G}$ given by evaluating a parameter on τ. Thus, if $\lim_{t\to 0} \lambda(t) \cdot \varphi$ exists, i.e. the associated morphism $\mathbb{G}_{m,L} \to Z^1(W_E, \widehat{G})_L$ extends to $\mathbb{A}^1_L$, for each $\tau \in W_E$ one has $\lambda^\tau = \lambda$ and $\varphi(\tau) \in Q_\lambda(L) \rtimes \tau$, cf. Lemma VIII.3.3. One thus has $Q_\lambda = \widehat{P}$ for P a standard parabolic subgroup of G^*, and $\varphi : W_E \to \widehat{P} \rtimes W_E$.

For $g \in \widehat{P}$, $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ is the projection onto the standard Levi subgroup $\widehat{M}$. Thus, using the evaluation morphism ev_τ for each τ we deduce that $\lim_{t\to 0} \lambda(t) \cdot \varphi$, if it exists, is given by the composite $W_E \xrightarrow{\varphi} \widehat{P}(L) \rtimes W_E \xrightarrow{\mathrm{proj}} \widehat{M}(L) \rtimes W_E$. Reciprocally, since the morphism $\mathbb{G}_m \times \widehat{P} \to \widehat{P}$, given by $(t, g) \mapsto \lambda(t) g \lambda(t)^{-1}$ extends to $\mathbb{A}^1 \times \widehat{P}$ with fiber over $0 \in \mathbb{A}^1$ given by the projection to $\widehat{M}$, for any $\varphi : W_E \to \widehat{P} \rtimes W_E$, $\lim_{t\to 0} \lambda(t) \cdot \varphi$ exists.

From this analysis we deduce the equivalence between (1) and (2). It is clear that (3) implies (2). For the proof of (2) implies (3) we use the results of [**BMR05**] and [**Ric88**]. For this we see parameters as morphisms $W \to \widehat{G}(\Lambda) \rtimes Q$ where W is discrete finitely generated as in the proof of Theorem VIII.1.3, and Q is a finite quotient of W. Let $\varphi : W \to \widehat{G}(L) \rtimes Q$ satisfying (2). Let $H \subset \widehat{G}_L \rtimes Q$ be the Zariski closure of the image of φ. Then if $(x_1, \ldots, x_n) \in (\widehat{G}(L) \times Q)^n$ are the images of a set of generators of W, applying the Hilbert–Mumford–Kempf criterion we see that the $\widehat{G}_L$-orbit of $(x_1, \ldots, x_n)$ via the diagonal action is closed, cf. the proof of [**BMR05**, Lemma 2.17]. We can then apply [**Ric88**], cf. [**BMR05**, Proposition 2.16], to deduce that H is strongly reductive in $\widehat{G}_L \rtimes Q$ and thus $\widehat{G}_L$-completely reducible. Strictly speaking, since we are working in a non-connected situation, we use in fact [**BMR05**, Section 6]. □

Lemma VIII.3.3. *For $\lambda : \mathbb{G}_m \to \widehat{G}_L$ and $g \rtimes \tau \in \widehat{G}(L) \rtimes W_E$, the limit $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-\tau}$ exists if and only if $g \in Q_\lambda(L)$, the parabolic subgroup attached to λ, and $\lambda^\tau = \lambda$.*

Proof. Up to conjugation we can suppose $\lambda \in X_*(T)^+$. Then, Q_λ and Q_{λ^τ} are standard parabolic subgroups of $\widehat{G}$. Let us write $g = g' \dot{w} g''$ with $g' \in Q_\lambda(L)$, $g'' \in Q_{\lambda^\tau}(L)$ and $w \in \widehat{W}$. Then, writing

$$\lambda(t) g \lambda(t)^{-\tau} = (\lambda(t) g' \lambda(t)^{-1})(\lambda(t) \dot{w} \lambda(t)^{-\tau})(\lambda(t)^\tau g'' \lambda(t)^{-\tau}),$$

one deduces that $\lim_{t\to 0} \lambda(t) \dot{w} \lambda(t)^{-\tau}$ exists. Thus, $\lim_{t\to 0} (\lambda(\lambda^{-\tau})^w)(t)$ exists and thus $\lambda = (\lambda^\tau)^w$. Since $\lambda^\tau \in X_*(\widehat{T})^+$ we deduce $\lambda = \lambda^\tau$ and $\lambda^w = \lambda$. □

The proof shows that up to replacing $\widehat{G}(L) \rtimes W_E$ by $\widehat{G}(L) \rtimes Q$ for some finite quotient Q of W_E (as we can), semisimplicity of φ is equivalent to the Zariski closure of the image of φ being completely reducible in the terminology of [**BMR05**, Section 6].

VIII.3.2. A presentation of $\mathcal{O}(Z^1(W_E, \widehat{G}))$. It will be useful to have a presentation of the algebra $\mathcal{O}(Z^1(W_E, \widehat{G}))$, or rather of the finite type $\mathbb{Z}_\ell$-algebras $\mathcal{O}(Z^1(W_E/P, \widehat{G}))$ for open subgroups P of the wild inertia (with the property that the action of W_E on $\widehat{G}$ factors over W_E/P). Pick a discrete dense subgroup $W \subset W_E/P$ as above, so that $Z^1(W_E/P, \widehat{G}) = Z^1(W, \widehat{G})$. For any $n \geq 0$ with a map $F_n \to W$, we get a $\widehat{G}$-equivariant map

$$\mathcal{O}(Z^1(F_n, \widehat{G})) \to \mathcal{O}(Z^1(W, \widehat{G})),$$

where the source is isomorphic to $\mathcal{O}(\widehat{G}^n)$ with appropriately twisted diagonal $\widehat{G}$-conjugation. Consider the category $\{(n, F_n \to W)\}$ consisting of maps from finite free groups to W, with maps given by commutative diagrams $F_n \to F_m \to W$; this is a sifted index category (as it admits coproducts). The map

$$\operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G})) \to \mathcal{O}(Z^1(W, \widehat{G}))$$

is an isomorphism of algebras with $\widehat{G}$-action (as 1-cocycles from W to $\widehat{G}$ are uniquely specified by compatible collections of 1-cocycles $F_n \to \widehat{G}$ for all $F_n \to W$). By Haboush's theorem on geometric reductivity [**Hab75**] it follows that the map

$$\operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \to \mathcal{O}(Z^1(W, \widehat{G}))^{\widehat{G}}$$

on $\widehat{G}$-invariants is a universal homeomorphism of finite type $\mathbb{Z}_\ell$-algebras, and an isomorphism after inverting ℓ.

Definition VIII.3.4. *The algebra of excursion operators (for $Z^1(W, \widehat{G})$) is*

$$\operatorname{Exc}(W, \widehat{G}) = \operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}}.$$

We see in particular that the geometric points of $\operatorname{Exc}(W, \widehat{G})$ and $Z^1(W, \widehat{G})$ agree.

Actually, the following higher-categorical variant is true.

Proposition VIII.3.5. *Working in the derived ∞-category $\mathcal{D}(\mathbb{Z}_\ell)$, the map*

$$\operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G})) \to \mathcal{O}(Z^1(W, \widehat{G}))$$

is an isomorphism.

Proof. The left-hand side defines an animated $\mathbb{Z}_\ell$-algebra, in fact the universal animated $\mathbb{Z}_\ell$-algebra A with a condensed 1-cocycle $W \to \widehat{G}(A)$, and the right-hand side is given by $\pi_0 A$. Now the deformation-theoretic arguments from the proof of Theorem VIII.1.3 show that A is a derived complete intersection, but as $\pi_0 A$ has the correct dimension, we get $A = \pi_0 A$. □

We will later prove an even finer version, incorporating the $\widehat{G}$-action; we defer the proof to Section VIII.5.

Theorem VIII.3.6. *Assume that ℓ is a very good prime for $\widehat{G}$. Then the map*

$$\operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G})) \to \mathcal{O}(Z^1(W, \widehat{G}))$$

is an isomorphism in the presentable stable ∞-category $\operatorname{Ind}\operatorname{Perf}(B\widehat{G})$.

In particular, the map

$$\operatorname{Exc}(W, \widehat{G}) = \operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \to \mathcal{O}(Z^1(W, \widehat{G}))^{\widehat{G}}$$

is an isomorphism.

In particular, we see that the algebra of excursion operators plays a central role. In the next subsection, we analyze it more explicitly.

VIII.3.3. The algebra of excursion operators. Fix a finite quotient Q of W_E over which the W_E-action on $\widehat{G}$ factors. Let $(\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}$ be the quotient of $(\widehat{G} \rtimes Q)^n$ under simultaneous conjugation by $\widehat{G}$.

PROPOSITION VIII.3.7. *The algebra of excursion operators* $\mathrm{Exc}(W, \widehat{G})$ *is the universal* $\mathbb{Z}_\ell$*-algebra* A *equipped with maps*

$$\Theta_n : \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) \to \mathrm{Map}(W^n, A)$$

for $n \geq 1$*, linear over* $\mathcal{O}(Q^n) \to \mathrm{Map}(W^n, A)$*, subject to the following relations. If* $g : \{1, \ldots, m\} \to \{1, \ldots, n\}$ *is any map, the induced diagram*

$$\begin{array}{ccc} \mathcal{O}((\widehat{G} \rtimes Q)^m /\!\!/ \widehat{G}) & \longrightarrow & \mathrm{Map}(W^m, A) \\ \downarrow & & \downarrow \\ \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) & \longrightarrow & \mathrm{Map}(W^n, A) \end{array}$$

commutes, where both vertical maps are the natural pullback maps. On the other hand, g *also induces a map* $(\widehat{G} \rtimes Q)^m \to (\widehat{G} \rtimes Q)^n$*, multiplying in every fibre over* $i = 1, \ldots, n$ *the terms in* $g^{-1}(i)$ *(ordered by virtue of their ordering as a subset of* $\{1, \ldots, m\}$*). This map is equivariant under diagonal* $\widehat{G}$*-conjugation, and hence descends to the quotient. Similarly,* g *induces a map* $W^m \to W^n$*. Then also the induced diagram*

$$\begin{array}{ccc} \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) & \longrightarrow & \mathrm{Map}(W^n, A) \\ \downarrow & & \downarrow \\ \mathcal{O}((\widehat{G} \rtimes Q)^m /\!\!/ \widehat{G}) & \longrightarrow & \mathrm{Map}(W^m, A) \end{array}$$

commutes.

The ℓ*-torsion free quotient of* $\mathrm{Exc}(W, \widehat{G})$ *is also the universal flat* $\mathbb{Z}_\ell$*-algebra* A' *equipped with maps*

$$\Theta'_n : \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) \to \mathrm{Map}((W_E/P)^n, A')$$

for $n \geq 1$*, linear over* $\mathcal{O}(Q^n) \to \mathrm{Map}((W_E/P)^n, A')$*, satisfying the same relations as in Proposition VIII.3.7, where the right-hand side* $\mathrm{Map}((W_E/P)^n, A')$ *denotes the maps of condensed sets (where as usual* A' *is considered as relatively discrete over* $\mathbb{Z}_\ell$*). In particular, the* ℓ*-torsion free quotient of* $\mathrm{Exc}(W, \widehat{G})$ *is independent of the discretization* W *of* W_E/P*.*

We do not know whether it is necessary to pass to the ℓ-torsion free quotient for the final assertion. Note that if ℓ is very good for $\widehat{G}$, then $\mathrm{Exc}(W, \widehat{G}) \cong \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ is flat over $\mathbb{Z}_\ell$. Moreover note that the ℓ-torsion in $\mathrm{Exc}(W, \widehat{G})$ is always nilpotent, so passing to this quotient is a universal homeomorphism.

PROOF. The datum of the Θ_n is equivalent the datum of a map of algebras

$$\mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \to A$$

for each map $F_n \to W$. The relations encode the relations arising in the diagram category $(n, F_n \to W)$ corresponding to maps $F_m \to F_n$ (over W) either sending generators to generators, or

multiplying subsets of generators. If one would also allow the inversion of elements, then this would generate all required relations. We leave it as an exercise to see that this relation, corresponding to $F_n \to F_n$ which is the identity on the first $n-1$ generators and inverts the n-th generator, is in fact enforced by the others. (Hint: Look at the part of Θ_{n+1} corresponding to $(\gamma_1, \ldots, \gamma_n, \gamma_n^{-1})$ and use that under multiplication of the last two variables, this maps to $(\gamma_1, \ldots, \gamma_{n-1}, 1)$, which arises from $(\gamma_1, \ldots, \gamma_{n-1})$.)

The second description is a priori stronger as $\mathrm{Map}((W_E/P)^n, A')$ injects into $\mathrm{Map}(W^n, A')$ as $W \subset W_E/P$ is dense. The ℓ-torsion free quotient of $\mathrm{Exc}(W, \widehat{G})$ injects into $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ (as we have an isomorphism after inverting ℓ), and it is clear that the elements of $\mathrm{Map}(W^n, \mathrm{Exc}(W, \widehat{G}))$ map to elements of $\mathrm{Map}((W_E/P)^n, \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}})$. Thus, this already happens on the ℓ-torsion free quotient of $\mathrm{Exc}(W, \widehat{G})$, which thus has the desired universal property. $\square$

Regarding the passage to W_E in place of W_E/P, where there is no natural (finite type) algebra anymore, we still have the following result.

Proposition VIII.3.8. *Let L be an algebraically closed field over $\mathbb{Z}_\ell$. Then the following are in canonical bijection.*

(i) *Semisimple L-parameters $\varphi : W_E \to \widehat{G}(L) \rtimes W_E$, up to $\widehat{G}(L)$-conjugation.*

(ii) *L-valued points of $Z^1(W_E, \widehat{G}) /\!\!/ \widehat{G}$.*

(iii)) *Collections of maps of $\mathbb{Z}_\ell$-algebras*

$$\Theta_n : \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) \to \mathrm{Map}(W_E^n, L)$$

for $n \geq 1$, linear over $\mathcal{O}(Q^n) \to \mathrm{Map}(W_E^n, L)$, such that for any map $g : \{1, \ldots, m\} \to \{1, \ldots, n\}$, the diagrams

$$\begin{array}{ccc} \mathcal{O}((\widehat{G} \rtimes Q)^m /\!\!/ \widehat{G}) & \xrightarrow{\Theta_m} & \mathrm{Map}(W_E^m, L) \\ \downarrow & & \downarrow \\ \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) & \xrightarrow{\Theta_n} & \mathrm{Map}(W_E^n, L) \end{array}$$

induced by pullback, and

$$\begin{array}{ccc} \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) & \xrightarrow{\Theta_n} & \mathrm{Map}(W_E^n, L) \\ \downarrow & & \downarrow \\ \mathcal{O}((\widehat{G} \rtimes Q)^m /\!\!/ \widehat{G}) & \xrightarrow{\Theta_m} & \mathrm{Map}(W_E^m, L) \end{array}$$

induced by multiplication, commute.

Proof. We already know that (i) and (ii) are in natural bijection. The recipe above gives a canonical map from (ii) to (iii). Now take data as in (iii). Forgetting the continuity of all maps, we see that data as in (iii) gives rise to a semisimple 1-cocycles $\varphi : W_E \to \widehat{G}(L)$ (of discrete groups), up to conjugation. We need to see that if the data in (iii) are maps of condensed sets, then φ is also a map of condensed sets (this condition does not depend on the representative of its conjugacy class). Consider the corresponding map $\varphi' : W_E \to \widehat{G}(L) \rtimes Q$ and let $Z \subset \widehat{G}_L \rtimes Q$ be

the Zariski closure of its image; by semisimplicity, Z is completely reductive. We can find finitely many elements $\gamma_1, \ldots, \gamma_n \in W_E$ such that Z is also the normalizer of the $\varphi'(\gamma_i)$. By a theorem of Richardson [**Ric88**], see [**BMR05**, Corollary 3.7] in the positive characteristic case, it follows that the orbit of $(\varphi'(\gamma_1), \ldots, \varphi'(\gamma_n))$ under simultaneous $\widehat{G}$-conjugation is closed. Moreover, for any further element γ, the $\widehat{G}$-orbit of $(\varphi'(\gamma_1), \ldots, \varphi'(\gamma_n), \varphi'(\gamma))$ maps isomorphically to the $\widehat{G}$-orbit of $(\varphi'(\gamma_1), \ldots, \varphi'(\gamma_n))$ as both are given by $\widehat{G}/Z$. It follows that $\varphi'(\gamma) \in \widehat{G}(L) \rtimes Q$ is determined uniquely by the condition that $(\varphi'(\gamma_1), \ldots, \varphi'(\gamma_n), \varphi'(\gamma))$ lies in the correct $\widehat{G}$-orbit (which is closed). This is determined by the map

$$\mathcal{O}((\widehat{G} \rtimes Q)^{n+1} /\!\!/ \widehat{G}) \to \mathrm{Map}(W_E^{n+1}, L)$$

when evaluated at $(\gamma_1, \ldots, \gamma_n, \gamma)$. As we required this to be a map of condensed sets, the result follows. □

VIII.4. Excursion operators

One can use Proposition VIII.3.8 to construct L-parameters in the following general categorical situation. In order to avoid topological problems, we work in the setting of the discrete subgroup $W \subset W_E/P$; in fact, we can take here any discrete group W. Let Λ be a discrete $\mathbb{Z}_\ell$-algebra and let $\mathcal{C}$ be a $\mathbb{Z}_\ell$-linear category. Assume that functorially in finite sets I, we are given a monoidal $\mathrm{Rep}_{\mathbb{Z}_\ell}(Q^I)$-linear functor

$$\mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G} \rtimes Q)^I \to \mathrm{End}(\mathcal{C})^{BW^I} : V \mapsto T_V$$

where $\mathrm{End}(\mathcal{C})$ is the category of endomorphisms of $\mathcal{C}$, and $\mathrm{End}(\mathcal{C})^{BW_E^I}$ is the category of $F \in \mathrm{End}(\mathcal{C})$ equipped with a map of groups $W^I \to \mathrm{Aut}(F)$.

The goal of this section is to prove the following theorem; this is essentially due to V. Lafforgue [**Laf18**].

Theorem VIII.4.1. *Given the above categorical data, there is a natural map of algebras*

$$\mathrm{Exc}(W, \widehat{G}) = \mathrm{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \to \mathrm{End}(\mathrm{id}_{\mathcal{C}})$$

to the Bernstein center of $\mathcal{C}$ (i.e., the algebra of endomorphisms of the identity of $\mathcal{C}$).

To prove Theorem VIII.4.1, we construct explicit "excursion operators". These are associated to the following data.

Definition VIII.4.2. *An excursion datum is a tuple $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$ consisting of a finite set I, an object $V \in \mathrm{Rep}_{\mathbb{Z}_\ell}((\widehat{G} \rtimes Q)^I)$ with maps $\alpha : 1 \to V|_{\mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G})}$, $\beta : V|_{\mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G})} \to 1$ and elements $\gamma_i \in W$, $i \in I$.*

Here, the restriction $\mathrm{Rep}_{\mathbb{Z}_\ell}((\widehat{G} \rtimes Q)^I) \to \mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G})$ is the restriction to the diagonal copy of $\widehat{G} \subset \widehat{G}^I \subset (\widehat{G} \rtimes Q)^I$.

Now consider excursion data $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$. These give rise to an endomorphism of the identity functor of $\mathcal{C}$, as follows.

$$S_{\mathcal{D}} : \mathrm{id} = T_1 \xrightarrow{T_\alpha} T_V \xrightarrow{(\gamma_i)_{i \in I}} T_V \xrightarrow{T_\beta} T_1 = \mathrm{id}.$$

Varying the γ_i, this gives a map of groups

$$W^I \to \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$$

to the automorphisms of the identity functor on $\mathcal{C}$.

We note that if we have two excursion data $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ and $\mathcal{D}' = (I, V', \alpha', \beta', (\gamma_i)_{i\in I})$ with same finite set I and elements $\gamma_i \in W$, and a map $g : V \to V'$ taking α to α' and β' to β (by post- and pre-composition), then $S_{\mathcal{D}} = S_{\mathcal{D}'}$. Indeed, the diagram

$$\begin{tikzcd}
T_1 \arrow[r, "T_\alpha"] \arrow[d, equal] & T_V \arrow[r, "(\gamma_i)_{i\in I}"] \arrow[d, "T_g"] & T_V \arrow[r, "T_\beta"] \arrow[d, "T_g"] & T_1 \arrow[d, equal] \\
T_1 \arrow[r, "T_{\alpha'}"] & T_{V'} \arrow[r, "(\gamma_i)_{i\in I}"] & T_{V'} \arrow[r, "T_{\beta'}"] & T_1
\end{tikzcd}$$

commutes. Now note that (V, α, β) give rise to an element

$$f = f(V, \alpha, \beta) \in \mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^I/\widehat{G}),$$

the quotient under diagonal left and right multiplication. Indeed, given any $g_i \in \widehat{G} \rtimes Q$, $i \in I$, one can form the composite

$$1 \xrightarrow{\alpha} V \xrightarrow{(g_i)_{i\in I}} V \xrightarrow{\beta} 1,$$

giving an element of the base ring; as α and β are equivariant for the diagonal $\widehat{G}$-action, this indeed gives an element

$$f = f(V, \alpha, \beta) \in \mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^I/\widehat{G}).$$

Conversely, given f we can look at the $(\widehat{G} \rtimes Q)^I$-representation $V = V_f \subset \mathcal{O}((\widehat{G} \rtimes Q)^I/\widehat{G})$ generated by f. This comes with a map $\alpha_f : 1 \to V_f|_{\mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G})}$ induced by the element f, and a map $\beta_f : V_f|_{\mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G})} \to 1$ given by evaluation at $1 \in (\widehat{G} \rtimes Q)^I$. If we replace V by the subrepresentation generated by α, then there is a natural map $V \to V_f$ taking α to α_f and β_f to β. The above commutative diagrams then imply that $S_{\mathcal{D}}$ depends on (V, α, β) only through f, and we get a map (a priori, of $\mathbb{Z}_\ell$-modules)

$$\Theta^I : \mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^I/\widehat{G}) \to \mathrm{Map}(W^I, \mathrm{End}(\mathrm{id}_{\mathcal{C}})).$$

Restricted to $\mathcal{O}(Q^I)$, this is given by the natural map $\mathcal{O}(Q^I) \to \mathrm{Map}(W^I, \Lambda)$ (and $\Lambda \to \mathrm{End}(\mathrm{id}_{\mathcal{C}})$). Also, it follows from the definitions that for any map $g : I \to J$, the diagram

$$\begin{tikzcd}
\mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^I/\widehat{G}) \arrow[r, "\Theta^I"] \arrow[d] & \mathrm{Map}(W^I, \mathrm{End}(\mathrm{id}_{\mathcal{C}})) \arrow[d] \\
\mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^J/\widehat{G}) \arrow[r, "\Theta^J"] & \mathrm{Map}(W^J, \mathrm{End}(\mathrm{id}_{\mathcal{C}}))
\end{tikzcd},$$

induced by pullback along g, is cartesian.

We want to check that Θ^I is a map of algebras. For this, we use a version of "convolution product = fusion product" in this situation. Namely, given $f_1, f_2 \in \mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^I/\widehat{G})$, we can build their exterior product $f_1 \boxtimes f_2 \in \mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^{I\sqcup I}/\widehat{G})$. Then one easily checks

$$\Theta^{I\sqcup I}(f_1 \boxtimes f_2)((\gamma_i, \gamma_i')_{i\in I}) = \Theta^I(f_1)((\gamma_i)_{i\in I})\Theta^I(f_2)((\gamma_i')_{i\in I}).$$

Applying now functoriality for pullback under $I \sqcup I \to I$, it follows that indeed $\Theta^I(f_1 f_2) = \Theta^I(f_1)\Theta^I(f_2)$.

For any $n \geq 0$, we can identify

$$\mathcal{O}(\widehat{G}\backslash(\widehat{G} \rtimes Q)^{\{0,\dots,n\}}/\widehat{G}) \otimes_{\mathcal{O}(Q^{\{0,\dots,n\}})} \mathcal{O}(Q^{\{1,\dots,n\}}) \cong \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G})$$

via pullback under $(g_1, \dots, g_n) \mapsto (1, g_1, \dots, g_n)$. This translates $\Theta^{\{0,\dots,n\}}$ into maps of $\mathbb{Z}_\ell$-algebras

$$\Theta_n : \mathcal{O}((\widehat{G} \rtimes Q)^n /\!\!/ \widehat{G}) \to \mathrm{Map}(W^n, \mathrm{End}(\mathrm{id}_{\mathcal{C}}))$$

over $\mathcal{O}(Q^n) \to \mathrm{Map}(W^n, \Lambda)$, still satisfying compatibility with pullback under maps $g : \{1, \dots, m\} \to \{1, \dots, n\}$.

Arguing also as in [**Laf18**, Lemma 10.1, equation (10.5)] and the resulting [**Laf18**, Proposition 10.8 (iii), Definition-Proposition 11.3 (d)], one sees that the maps Θ_n are also compatible with the multiplication maps induced by such maps g, thus finishing the proof of Theorem VIII.4.1.

In particular, using the description of geometric points, Theorem VIII.4.1 implies the following proposition.

Corollary VIII.4.3. *Assume that $\Lambda = L$ is an algebraically closed field and $X \in \mathcal{C}$ is an object with $\mathrm{End}(X) = L$. Then there is, up to $\widehat{G}(L)$-conjugation, a unique semisimple L-parameter*

$$\varphi_X : W \to \widehat{G}(L) \rtimes W$$

such that for all excursion data $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$, the endomorphism $S_{\mathcal{D}}(X) \in \mathrm{End}(X) = L$,

$$X = T_1(X) \xrightarrow{\alpha} T_V(X) \xrightarrow{(\gamma_i)_{i \in I}} T_V(X) \xrightarrow{\beta} T_1(X) = X,$$

is given by the composite

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_X(\gamma_i))_{i \in I}} V \xrightarrow{\beta} L.$$

VIII.5. Modular representation theory

The goal of this section is to give a proof of Theorem VIII.3.6. In fact, we prove a slight refinement of it, concerning perfect complexes, that will be useful in the construction of the spectral action.

Theorem VIII.5.1. *Assume that ℓ is a very good prime for $\widehat{G}$. Then the map*

$$\mathrm{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G})) \to \mathcal{O}(Z^1(W, \widehat{G}))$$

is an isomorphism in the presentable stable ∞-category $\mathrm{Ind}\,\mathrm{Perf}(B\widehat{G})$. Moreover, the ∞-category $\mathrm{Perf}(Z^1(W, \widehat{G})/\widehat{G})$ is generated under cones and retracts by $\mathrm{Perf}(B\widehat{G})$, and $\mathrm{Ind}\,\mathrm{Perf}(Z^1(W, \widehat{G})/\widehat{G})$ identifies with the ∞-category of modules over $\mathcal{O}(Z^1(W, \widehat{G}))$ in $\mathrm{Ind}\,\mathrm{Perf}(B\widehat{G})$.

The difficulties in this theorem all arise on the special fibre. Indeed, we will show below that we can reduce to the following version in characteristic ℓ.

THEOREM VIII.5.2. *Assume that ℓ is a very good prime for $\widehat{G}$, and let $L = \overline{\mathbb{F}}_\ell$. Then the map*

$$\operatorname{colim}_{(n,F_n\to W)} \mathcal{O}(Z^1(F_n,\widehat{G})_L) \to \mathcal{O}(Z^1(W,\widehat{G})_L)$$

is an isomorphism in the presentable stable ∞-category $\operatorname{Ind}\operatorname{Perf}(B\widehat{G})$. *Moreover, the ∞-category* $\operatorname{Perf}(Z^1(W,\widehat{G})_L/\widehat{G})$ *is generated under cones and retracts by* $\operatorname{Perf}(B\widehat{G})$.

Then we have the following reduction:

THEOREM VIII.5.2 IMPLIES THEOREM VIII.5.1. For the colimit claim, we need to see that for all representations V of $\widehat{G}$, the map

$$\operatorname{colim}_{(n,F_n\to W)} R\Gamma(\widehat{G},\mathcal{O}(Z^1(F_n,\widehat{G}))\otimes V) \to R\Gamma(\widehat{G},\mathcal{O}(Z^1(W,\widehat{G}))\otimes V)$$

is an isomorphism. It is an isomorphism after inverting ℓ, as then the representation theory of $\widehat{G}$ is semisimple, and it is true on underlying complexes by Proposition VIII.3.5. Thus, it suffices to show that it is an isomorphism after inverting ℓ, or even after base change to L, which follows from Theorem VIII.5.2.

For the other half, note first that if $\operatorname{Perf}(B\widehat{G})$ generates $\operatorname{Perf}(Z^1(W,\widehat{G})/\widehat{G})$, then it follows by Barr–Beck–Lurie [**Lur16**, Theorem 4.7.4.5] that $\operatorname{Ind}\operatorname{Perf}(Z^1(W,\widehat{G})/\widehat{G})$ is the ∞-category of modules over $\mathcal{O}(Z^1(W,\widehat{G}))$ in $\operatorname{Ind}\operatorname{Perf}(B\widehat{G})$. Now take any $V \in \operatorname{Perf}(Z^1(W,\widehat{G})/\widehat{G})$. As its lowest cohomology group is finitely generated, we can find some surjection from an induced vector bundle onto it, and by passing to cones reduce the perfect amplitude until V is a $\widehat{G}$-equivariant vector bundle on $Z^1(W,\widehat{G})$. We may then again find a representation W of $\widehat{G}$ and a surjection $W \otimes \mathcal{O}(Z^1(W,\widehat{G})) \to V$. This map splits after inverting ℓ, showing that V is a retract of an induced vector bundle up to a power of ℓ. Thus, it suffices to show that V/ℓ lies in this subcategory, and this follows from Theorem VIII.5.2. □

Thus, we concentrate now on Theorem VIII.5.2, which takes place over an algebraically closed base field L of characteristic ℓ. For the proof, we need many preparations on the modular representation theory of reductive groups, for $\widehat{G}$ and many of its subgroups. As everything here happens on the dual side but we do not want to clutter notation, we will change notation, only for this section, and write G for reductive groups over L.

VIII.5.1. Good filtrations. First, we need to recall the notion of good filtrations. Let G be a reductive group over L; as disconnected groups will appear frequently below, we stress that reductive groups are always connected for us, so in particular G is assumed to be connected here. Let $T \subset B \subset G$ be a torus and Borel for G. For any dominant cocharacter λ of T, we have the induced representation

$$\nabla_\lambda = H^0(G/B,\mathcal{O}(\lambda)).$$

A representation V of G has a good filtration if it admits an exhaustive filtration

$$0 = V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V$$

such that each V_i/V_{i-1} is isomorphic to a direct sum of ∇_λ's. If one picks a total ordering $0 = \lambda_0, \lambda_1, \ldots$ of the dominant cocharacters, compatible with their dominance order, one can choose

$V_i \subset V$ to be the maximal subrepresentation admitting only weights λ_j with $j \leq i$. In that case, V_i/V_{i-1} is generated by its weight space W_i of weight λ_i, and by adjunction there is a map

$$V_i/V_{i-1} \to W_i \otimes \nabla_{\lambda_i};$$

then V admits a good filtration if and only if all of these maps are isomorphisms. For this, it is in fact enough that $V_i \to W_i \otimes \nabla_{\lambda_i}$ is surjective: The kernel is necessarily given by V_{i-1}, as it has only smaller weights.

A key result is that if V and W admit good G-filtrations, then so does $V \otimes W$; this is a theorem of Donkin [**Don85**] and Mathieu [**Mat90**] in general. Moreover, if V admits a good filtration, then $H^i(G, V) = 0$ for $i > 0$: This clearly reduces to the case of $V = \nabla_\lambda$, in which case it follows from Kempf's vanishing theorem [**Kem76**]. These results imply the following standard characterization of modules admitting a good filtration.

Proposition VIII.5.3 ([**Don81**]). *A G-representation V admits a good filtration if and only if for all λ, one has*

$$H^i(G, V \otimes \nabla_\lambda) = 0$$

for $i > 0$.

Using this, one can define a well-behaved notion of a "good filtration dimension" of V, referring to the minimal i such that $H^j(G, V \otimes \nabla_\lambda) = 0$ for all λ and $j > i$. Equivalently, there is a resolution of length i by representations with a good filtration. We will use the following deep result of Touzé–van der Kallen [**TvdK10**, Corollary 1.5].

Theorem VIII.5.4. *Let A be a finitely generated L-algebra with an action of G. Let M be a G-equivariant finitely generated A-module. Then $H^i(G, M)$ is a finitely generated A^G-module for all $i \geq 0$. Moreover, if A has a good filtration, then M has finite good filtration dimension.*

Another key result we need is the following.

Theorem VIII.5.5 ([**Kop84**], [**Don88**]). *The $G \times G$-representation $\mathcal{O}(G)$ (via left and right multiplication) admits a good filtration.*

In particular, we have the following corollary. For any $n \geq 0$, let F_n be the free group on n letters.

Corollary VIII.5.6. *For any map $F_n \to \mathrm{Aut}(G)$, the G-representation $\mathcal{O}(Z^1(F_n, G))$ admits a good filtration.*

Proof. Note that $Z^1(F_n, G) = G^n$, where the G-action is that of simultaenous twisted conjugation (by the n given automorphisms of G). But $\mathcal{O}(G^n)$ admits a good filtration as representation of G^{2n}, and restricting to $G \subset G^{2n}$ it remains good by stability under tensor products (and as the induced representations of G^{2n} are tensor products of induced representations of each factor). $\square$

Another property of relevance to us is the following.

Theorem VIII.5.7 ([**AJ84**, 4.4]). *Assume that ℓ is very good for G. Then all symmetric powers $\mathrm{Sym}^n \mathfrak{g}^*$ have a good filtration.*

VIII.5.2. Reductions. To prove Theorem VIII.5.2, we will use the following criterion.

THEOREM VIII.5.8. *Let X be an affine scheme of finite type over L equipped with an action of a reductive group G. Assume the following conditions.*

(i) *The scheme X is a local complete intersection.*

(ii) *For any closed G-orbit $Z \cong G/H \subset X$, the algebra $\mathcal{O}(G/H)$ has a good G-filtration, and the image of* $\mathrm{Perf}(BG) \to \mathrm{Perf}(BH)$ *generates under cones and retracts.*

(iii) *With Z as in (ii), the G-equivariant $\mathbb{L}_{Z/X}[-1] \in \mathcal{D}^{[-1,0]}(\mathcal{O}(Z))$ has the property that all $\mathrm{Sym}^n_{\mathcal{O}(Z)}(\mathbb{L}_{Z/X}[-1]) \in \mathcal{D}^{[-n,0]}(\mathcal{O}(Z))$ have good filtration dimension 0; i.e., for all induced representations V, one has*

$$H^i(G, \mathrm{Sym}^n_{\mathcal{O}(Z)}(\mathbb{L}_{Z/X}[-1]) \otimes V) = 0$$

for $i \geq 1$.

Then $\mathcal{O}(X)$ has a good G-filtration, and $\mathrm{Perf}(X/G)$ *is generated (under cones and retracts) by* $\mathrm{Perf}(BG)$.

PROOF. To see that $A = \mathcal{O}(X)$ has a good filtration, we need to show that for every G-representation V with a good filtration, $H^i(G, V \otimes \mathcal{O}(X)) = 0$ for $i > 0$. Note that this cohomology group is a finitely generated A^G-module by Theorem VIII.5.4. If it is not zero, pick some closed point of $\mathrm{Spec}\, A^G$ in its support, and let $Z \subset X$ be the corresponding closed G-orbit. There is a (derived) filtration $I^\bullet \subset A$ such that $I^0 = A$, I^1 is the ideal of Z, and in general $I^n/^{\mathbb{L}} I^{n+1} \cong \mathrm{Sym}^n_{\mathcal{O}(Z)}(\mathbb{L}_{Z/X}[-1])$. Indeed, if X was smooth, we could simply take the filtration by powers of the ideal I. In general, we may pick a G-equivariant simplicial resolution of A by smooth algebras, and totalize the corresponding filtrations. Consider the G-equivariant animated A-algebra

$$\widetilde{A} = \bigoplus_{n \in \mathbb{Z}} I^n$$

where by definition $I^n = A$ for $n \leq 0$. (Again, this can be defined by passage to G-equivariant smooth resolutions.) Let $t \in \widetilde{A}$ be the generator in degree -1. Then

$$\widetilde{A}/^{\mathbb{L}} t = \bigoplus_{n \geq 0} I^n/^{\mathbb{L}} I^{n+1} = \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O}(Z)}(\mathbb{L}_{Z/X}[-1])$$

has good filtration dimension ≤ 0. We also note that it is concentrated in a bounded range of degrees, bounded by the number of generators for the ideal defining the (local) complete intersection A. This implies that for any V with a good G-filtration, $H^i(G, V \otimes \widetilde{A})$ are coherent $H^0(G, \widetilde{A})$-modules whose reduction modulo t vanishes for $i > 0$. Moreover, after inverting t, one has $\widetilde{A}[t^{-1}] = A[t^{\pm 1}]$. Thus, the support of $H^i(G, V \otimes \widetilde{A})$ in $\mathrm{Spec}\, H^0(G, \widetilde{A})$ contains $\mathrm{Spec}\, \mathcal{O}(Z)[t^{\pm 1}]$. As it is closed, it contains the spectrum of $\bigoplus_{n \leq 0} (A/I)^G = \mathcal{O}(Z)[t]$, and thus meets the locus $t = 0$, in contradiction to its vanishing modulo t. We note that it follows now that also $\widetilde{A}$ has good filtration dimension ≤ 0, as all $H^i(G, V \otimes \widetilde{A})$ vanish after inverting t and after reduction modulo t.

Now we prove that $\mathrm{Perf}(X/G)$ is generated by $\mathrm{Perf}(BG)$. Given $N \in \mathrm{Perf}(\mathrm{Spec}\, A/G)$, let $i \in \mathbb{Z}$ be maximal such that $\mathcal{H}^i(N) \neq 0$. We may pick a G-representation V with a map $V[-i] \to N|_{\mathrm{Perf}(BG)}$ so that $A \otimes_{\mathbb{Z}_\ell} V[-i] \to N$ has the property that the induced map on $\mathcal{H}^i$ is surjective.

Passing to the fibre, we may then induct on the perfect amplitude of N to reduce to the case that $N = M[0]$ is a vector bundle.

Now we use the theorem of Touzé–van der Kallen, Theorem VIII.5.4, which ensures that M has finite good filtration dimension, i.e. there is some integer i such that for all λ and all $j > i$ one has $H^j(G, M \otimes \nabla_\lambda) = 0$ for $j > i$. By Proposition VIII.5.9, when we choose V above, we can assume that V^* admits a good filtration. Choosing such a V for $N = M^*[0]$, we can thus find a strict injection $M \to A \otimes V^*$ of vector bundles such that V^* admits a good filtration. Passing to the cokernel then decrases the good filtration dimension of M, and we can reduce to the case that M has a good filtration.

We have a G-equivariant resolution

$$\ldots \to A \otimes A \otimes M \to A \otimes M \to M$$

of G-equivariant A-modules, which is canonically split in $\operatorname{Ind}\operatorname{Perf}(BG)$. The class of all M for which this is a resolution in $\operatorname{Ind}\operatorname{Perf}(X/G)$ is exactly the class of all M generated under cones and retracts by $\operatorname{Perf}(BG)$ – in one direction, it is a resolution in $\operatorname{Ind}\operatorname{Perf}(X/G)$ when M arises via pullback from BG (as it can then be split), and thus for all M generated by $\operatorname{Perf}(BG)$. In the other direction, if it is a resolution in $\operatorname{Ind}\operatorname{Perf}(X/G)$, then it follows that M is generated by $\operatorname{Perf}(BG)$ as all other terms arise via pullback from BG. Now note that this is a resolution in $\operatorname{Ind}\operatorname{Perf}(X/G)$ if and only if for all $N \in \operatorname{Perf}(X/G)$, which we may again assume to be a G-equivariant finite projective A-module with a good G-filtration, the complex

$$\ldots \to (N \otimes A \otimes M)^G \to (N \otimes M)^G \to (N \otimes_A M)^G \to 0$$

is exact, and $H^i(G, N \otimes_A M) = 0$ for $i > 0$. (Note that all terms $N \otimes M$, $N \otimes A \otimes M$ etc. have a good G-filtration, hence there is no higher G-cohomology.) In particular, these observations show that if M has a good filtration and lies in the full subcategory of $\operatorname{Perf}(X/G)$ generated by $\operatorname{Perf}(BG)$, then for all G-equivariant finite projective A-modules N with a good filtration, also $M \otimes_A N$ has a good G-filtration.

We note that the previous arguments also apply in case $A = \mathcal{O}(Z)$ for an orbit Z as in (ii), in which case the assumption of (ii) implies that for two G-equivariant $\mathcal{O}(Z)$-perfect complexes $M_1, M_2 \in \operatorname{Perf}(\operatorname{Spec}\mathcal{O}(Z)/G)$ with G-good filtration dimension ≤ 0, also their tensor product is of G-good filtration dimension ≤ 0.

Coming back to a general G-equivariant finite projective A-module M, we can now find some large enough $V \subset M$ with a good filtration inducing a surjection $A \otimes V \to M$. Moreover, as $(M \otimes L[G/U])^G$ is finitely generated over $(A \otimes L[G/U])^G$ (for example, by Theorem VIII.5.4), we can even ensure that the map stays surjective after tensoring with $L[G/U] = \bigoplus_\lambda \nabla_\lambda$ and taking G-invariants. This implies that the kernel M' of $A \otimes V \to M$ still has a good filtration.

In other words, we can find an infinite resolution

$$\ldots \to A \otimes V_1 \to A \otimes V_0 \to M \to 0$$

such that all V_i are G-representations with a good filtration, and the kernel M_i of $A \otimes V_i \to A \otimes V_{i-1}$ is always a G-equivariant finite projective A-module that admits a good filtration. In particular, it is a resolution $\operatorname{Ind}\operatorname{Perf}(BG)$. It suffices to see that this is in fact a resolution in $\operatorname{Ind}\operatorname{Perf}(X/G)$. To see this, we need to see that for any $N \in \operatorname{Perf}(X/G)$, which we may assume to be a G-equivariant

vector bundle admitting a good G-filtration, also the complex

$$\ldots \to N \otimes V_1 \to N \otimes V_0 \to N \otimes_A M \to 0$$

is a resolution in $\operatorname{Ind}\operatorname{Perf}(BG)$. We note that this is automatic if $N \otimes_A M_i$ has a good G-filtration for all i.

Now assume that for any closed point x of $\operatorname{Spec} A^G$, with localization $(A^G)_x$ at x, that $M \otimes_{A^G} A^G_x \in \operatorname{Perf}(\operatorname{Spec}(A \otimes_{A^G} A^G_x)/G)$ is generated by $\operatorname{Perf}(BG)$. Then in particular, also $M_i \otimes_{A^G} A^G_x$ is generated by $\operatorname{Perf}(BG)$, and hence $N \otimes_A M_i \otimes_{A^G} A^G_x$ has a good G-filtration. This then implies that the A^G-module $H^{>0}(G, N \otimes_A M_i \otimes V)$ vanishes locally on $\operatorname{Spec} A^G$ for any representation V with good G-filtration, and hence vanishes, thus showing that $N \otimes_A M_i$ has a good G-filtration. By the above, this then implies that M is also globally generated by $\operatorname{Perf}(BG)$.

We can thus work locally on $\operatorname{Spec} A^G$, and we fix a closed point, corresponding to a closed G-orbit $Z \subset X$. With

$$\widetilde{A} = \bigoplus_{n \in \mathbb{Z}} I^n$$

as above, which we endow with an action of $\widetilde{G} = G \times \mathbb{G}_m$ (where $\mathbb{G}_m$ acts via the grading), we consider $\widetilde{G}$-equivariant finite projective $\widetilde{A}$-module $\widetilde{M} = M \otimes_A \widetilde{A}$. By Theorem VIII.5.4, the $\mathcal{O}(Z)$-module $M \otimes_A A/I$ has finite good filtration dimension, and as all $I^n/^{\mathbb{L}} I^{n+1}$ have good filtration dimension ≤ 0, it follows that $\widetilde{M}/t = \bigoplus_{n \geq 0} (M \otimes_A A/I) \otimes_{A/I} I^n/^{\mathbb{L}} I^{n+1}$ has finite good filtration dimension. By finite generation arguments on cohomology groups, this then also implies that $\widetilde{M}$ has finite good filtration dimension. Arguing as for the A-module M, we can then first arrange that $\widetilde{M}$ has good filtration dimension ≤ 0, and then find $V_0, V_1, \ldots \in \operatorname{Perf}^{[-1,0]}(BG)$ with good filtration dimension ≤ 0 such that

$$\ldots \to \widetilde{A} \otimes V_1 \to \widetilde{A} \otimes V_0 \to \widetilde{M} \to 0$$

is a resolution in $\operatorname{Ind}\operatorname{Perf}(B\widetilde{G})$, with the limit $\widetilde{M}_i$ of $[\widetilde{A} \otimes V_i \to \widetilde{A} \otimes V_{i-1} \to \ldots \to \widetilde{M}]$ being a $\widetilde{G}$-equivariant finite projective $\widetilde{A}$-module with good filtration dimension ≤ 0. Reducing modulo t, the sequence

$$\ldots \to \widetilde{A}/t \otimes V_1 \to \widetilde{A}/t \otimes V_0 \to \widetilde{M}/t \to 0$$

is a resolution in $\operatorname{Ind}\operatorname{Perf}(\operatorname{Spec}(\widetilde{A}/t)/G)$, as in fact $\widetilde{M}/t = (M \otimes_A A/I) \otimes_{A/I} \widetilde{A}/t$ lies in the subcategory generated by $\operatorname{Perf}(BG)$, as this is true for $M \otimes_A A/I \in \operatorname{Perf}(Z/G) = \operatorname{Perf}(BH)$ by assumption (ii). Indeed, reversing some arguments above, this implies that for all G-equivariant finite projective $\widetilde{A}/t$-modules N with good filtration dimension ≤ 0, also $N \otimes_{\widetilde{A}/t} \widetilde{M}_i/t$ has good filtration dimension ≤ 0, which gives that the displayed resolution is indeed a resolution in $\operatorname{Ind}\operatorname{Perf}(\operatorname{Spec}(\widetilde{A}/t)/G)$.

Now consider any $\widetilde{N} \in \operatorname{Perf}(\operatorname{Spec} \widetilde{A}/\widetilde{G})$. We are interested to what extent

$$\ldots \to (\widetilde{N} \otimes V_1)^G \to (\widetilde{N} \otimes V_0)^G \to (\widetilde{N} \otimes_{\widetilde{A}} \widetilde{M})^G \to 0$$

is a resolution. All the cohomology groups are finitely generated $H^0(G, \widetilde{A})$-modules, and their reduction modulo t vanishes. This implies, as above, that their support is disjoint from the $\widetilde{G}$-orbit $\operatorname{Spec} \mathcal{O}(Z)[t^{\pm 1}] \to \operatorname{Spec} \widetilde{A}$ (as the closure of this orbit meets the fibre $t = 0$).

We can apply this in particular to $\widetilde{N} = N \otimes_A \widetilde{A}$ for some G-equivariant finite projective A-module with good filtration. Inverting t in the above sequence, we get the complex

$$\ldots \to (N \otimes V_1)^G[t^{\pm 1}] \to (N \otimes V_0)^G[t^{\pm 1}] \to (N \otimes_A M)^G[t^{\pm 1}] \to 0$$

such that the support of its cohomology groups is disjoint from $\operatorname{Spec} \mathcal{O}(Z)[t^{\pm 1}]$. In other words, we get that

$$\ldots \to (N \otimes V_1)^G \otimes_{A^G} A_x^G \to (N \otimes V_0)^G \otimes_{A^G} A_x^G \to (N \otimes_A M)^G \otimes_{A^G} A_x^G \to 0$$

is exact, which implies that

$$\ldots \to A \otimes_{A^G} A_x^G \otimes V_1 \to A \otimes_{A^G} A_x^G \otimes V_0 \to M \otimes_{A^G} A_x^G \to 0$$

is a resolution in $\operatorname{Ind}\operatorname{Perf}(\operatorname{Spec}(A \otimes_{A^G} A_x^G)/G)$. (Note that we can repeat the arguments after replacing A^G by some localization of it, and N there.) In particular, $M \otimes_{A^G} A_x^G \in \operatorname{Perf}(\operatorname{Spec}(A \otimes_{A^G} A_x^G)/G)$ is generated by $\operatorname{Perf}(BG)$. As we discussed above, this result for all closed points $x \in \operatorname{Spec} A^G$ gives the claim. □

Proposition VIII.5.9. *Let V be a representation of G. Then there is a surjection $W \to V$ such that W^* admits a good filtration.*

Proof. We can look at the subcategory of $\operatorname{Rep}(G)$ where all weights are within some finite set (closed under the dominance order). It is well-known that this category admits a projective generator P, and the projective generator has the property that its dual P^* has a good filtration; this follows for example from $H^1(G, P^* \otimes_{\mathbb{Z}_\ell} \nabla_\lambda) = \operatorname{Ext}^1(P, \nabla_\lambda) = 0$ for all relevant ∇, and the characterization of Proposition VIII.5.3. This implies the proposition. □

To produce examples of algebras satisfying Theorem VIII.5.8, we will use the following proposition.

Proposition VIII.5.10. *Let $A_\bullet \to A$ be a G-equivariant simplicial resolution of algebras of finite type over L equipped with an action of G, such that A and all $A_\bullet$ are local complete intersections. Assume that A_i has a good G-filtration for all i, and that for all closed orbits $Z \cong G/H \subset \operatorname{Spec} A$, $\mathcal{O}(Z)$ admits a good G-filtration and the image of $\operatorname{Perf}(BG) \to \operatorname{Perf}(BH)$ generates under cones and retracts. Moreover, assume that for all $i \geq 0$ and $n \geq 0$,*

$$\operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A_i}[-1]) \in \operatorname{Perf}(\mathcal{O}(Z)/G)$$

has good filtration dimension ≤ 0, and that for varying i, $\mathbb{L}_{Z/\operatorname{Spec} A_i} \in \operatorname{Perf}(\mathcal{O}(Z)/G) = \operatorname{Perf}(BH)$ involves only finitely many irreducible representations of H. Then the map

$$\operatorname{colim} A_\bullet \to A$$

is an isomorphism in $\operatorname{Ind}\operatorname{Perf}(BG)$, and for all $n \geq 0$ the map

$$\operatorname{colim} \operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A_\bullet}[-1]) \to \operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A}[-1]) \in \operatorname{Perf}(\mathcal{O}(Z)/G)$$

is an isomorphism, so the right-hand side has good filtration dimension ≤ 0.

Proof. First, note that the map

$$\operatorname{colim} \mathbb{L}_{Z/\operatorname{Spec} A_\bullet} \to \mathbb{L}_{Z/\operatorname{Spec} A},$$

which is necessarily an isomorphism in $\mathcal{D}(L)$, is also an isomorphism in $\operatorname{Ind}\operatorname{Perf}(\mathcal{O}(Z)/G) = \operatorname{Ind}\operatorname{Perf}(BH)$ as it involves only finitely many irreducible representations of H. Similarly,

$$\operatorname{colim}\operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A_\bullet}[-1]) \to \operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A}[-1])$$

is an isomorphism in $\operatorname{Ind}\operatorname{Perf}(\mathcal{O}(Z)/G)$, and in particular in $\operatorname{Ind}\operatorname{Perf}(BG)$. As all terms on the left have good filtration dimension ≤ 0, this implies that also $\operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A}[-1])$ has good filtration dimension ≤ 0.

In particular, Theorem VIII.5.8 implies that A has a good G-filtration. It remains to see that

$$\operatorname{colim} A_\bullet \to A$$

is an isomorphism in $\operatorname{Ind}\operatorname{Perf}(BG)$. Equivalently, for any representation V of G with a good filtration, the map

$$\operatorname{colim}(A_\bullet \otimes V)^G \to (A \otimes V)^G$$

is an isomorphism.

Assume first that $V = L$ is trivial. Then $A_\bullet^G$ is a simplicial finite type L-algebra, and thus its colimit B is an animated L-algebra that is almost of finite type; in other words, $\pi_0 B$ is a finite type L-algebra, and $\pi_i B$ is a finitely generated $\pi_0 B$-module for $i > 0$. Moreover, it follows from Haboush's theorem that the map $\pi_0 B \to A^G$ is a universal homeomorphism. Thus, if the map $B \to A^G$ is not an isomorphism, the homotopy groups of its cone are finitely generated $\pi_0 B$-algebras, and we can take some closed point of $\operatorname{Spec} \pi_0 B \cong \operatorname{Spec} A^G$ in its support, corresponding to some closed G-orbit $Z \cong G/H \subset \operatorname{Spec} A$. We can then form $\widetilde{A}$ and $\widetilde{A}_\bullet$ as in the proof of Theorem VIII.5.8, and look at the map

$$\operatorname{colim} \widetilde{A}_\bullet^G \to \widetilde{A}^G.$$

Again, the homotopy groups of the cone are finitely generated, and so as in the proof of Theorem VIII.5.8 it suffices to show that it is an isomorphism after reduction modulo t. But modulo t, we get $\widetilde{A}_i/t = \bigoplus_{n\geq 0} \operatorname{Sym}^n(\mathbb{L}_{Z/\operatorname{Spec} A_i}[-1])$ and similarly for $\widetilde{A}/t$, and the claim follows from the first paragraph of the proof.

To see that for all V with a good filtration, the map

$$\operatorname{colim}(A_\bullet \otimes V)^G \to (A \otimes V)^G$$

is an isomorphism, we note that now both sides are in $\mathcal{D}^{\leq 0}(A^G)$ with finitely generated cohomology groups. We can then pick Z as before, and reduce to showing that

$$\operatorname{colim}(\widetilde{A}_\bullet \otimes V)^G \to (\widetilde{A} \otimes V)^G$$

is an isomorphism after reduction modulo t, where it follows again from the direct sum decomposition of $\widetilde{A}_\bullet/t$ and $\widetilde{A}/t$. □

VIII.5.3. Fixed point groups. We will need to know some properties of the fixed points $H = G^F$ of reductive groups G under a (finite) group F of automorphisms of G. These will in general not stay connected, so to facilitate inductive arguments, we will also not assume that G is connected here. We will, however, usually make the assumption that $\pi_0 G$ is of order prime to the characteristic ℓ of the base field, and we will also assume that F is of order prime to ℓ.

First, we have the following structural result.

PROPOSITION VIII.5.11. *Let L be an algebraically closed field of characteristic $\ell > 0$, and let G be a smooth linear algebraic group over L such that G° is reductive and $\pi_0 G$ is of order prime to ℓ. Assume that F is a finite group of order prime to ℓ acting on G and let $H = G^F$ be the fixed points. Then H is a smooth linear algebraic group, H° is reductive, and $\pi_0 H$ is of order prime to ℓ.*

We note that our proof of the last part of this proposition probably uses unnecessarily heavy machinery. Under the assumption that F is solvable, we will later give a completely explicit proof.

PROOF. It is a standard fact that the fixed points of a smooth affine scheme under a finite group of order prime to the characteristic is still affine and smooth. Moreover, by [**PY02**, Theorem 2.1], H° is reductive.

For the final statement, we can first assume that G is connected (as $\pi_0 G$ is of order prime to ℓ), and thus reductive. We consider the action of G on $\prod_F G$, where it acts on the factor enumerated by $\Theta \in F$ through Θ-twisted conjugation. Note that $\mathcal{O}(\prod_F G)$ has a good G-filtration. By Theorem VIII.5.4, for any G-equivariant finitely generated $\mathcal{O}(\prod_F G)$-module M, the good filtration dimension of M is finite, and in particular $H^i(G, M) = 0$ for all large enough i.

Now G/H is a closed orbit of G acting on $\prod_F G$ (the orbit of the identity element). Moreover, if $\pi_0 H$ has an element of order ℓ, then we get a subgroup $H' \subset H$ with $\pi_0 H' \cong \mathbb{Z}/\ell\mathbb{Z}$. In that case,

$$R\Gamma(G, \mathcal{O}(G/H')) \cong R\Gamma(H', L) \cong R\Gamma(\mathbb{Z}/\ell\mathbb{Z}, L)$$

has cohomology in all positive degrees, while $\mathcal{O}(G/H')$ correponds to a G-equivariant coherent sheaf on $\prod_F G$ (equipped with its Θ-conjugation), so this contradicts the previous paragraph. $\square$

We will also need the following result.

PROPOSITION VIII.5.12. *In the situation of Proposition VIII.5.11, assume that the simply connected cover G' of G°_{der} has the property that all $\mathrm{Sym}^n \mathfrak{g}'^*$ have a good G'-filtration. Then $\mathcal{O}(G/H)$ has a good G-filtration.*

We recall that the assumption on symmetric powers of $\mathfrak{g}'^*$ is satisfied for all ℓ in type A, for all $\ell \neq 2$ in classical types, for all $\ell \neq 2, 3$ in exceptional types except for E_8, where one needs $\ell \neq 2, 3, 5$.

REMARK VIII.5.13. The result reduces easily to the case that G is connected, semisimple and simply connected. In that case, the result is related, but apparently somewhat weaker, than the assertion that $H^\circ \subset G$ is a Donkin subgroup, meaning that for any representation V of G with a good G-filtration, its restriction to H° has a good H°-filtration; here we only prove that $H^i(H^\circ, V) = 0$ for $i > 0$. This stronger statement that $H^\circ \subset G$ is a Donkin subgroup is actually known in most cases, at least when F is solvable so that one can reduce to automorphisms of prime order. In particular, involutions are handled completely in [**Bru98**, Section 2], [**vdK01**], and the (few) remaining cases are mostly done in [**HM13**, Theorem 4.3.3]. In particular, assuming F solvable, one can show this way that the conclusion of the proposition is true for all ℓ in classical types or G_2. The missing cases are the centralizers of semisimple elements of order 3 in F_4, E_6 and E_7, and of order 3 and 5 in E_8. These cases are not handled for small ℓ in [**HM13**, Theorem 4.3.3].

Proof. We need to see that for all G-representations V with a good filtration, one has $H^i(G, \mathcal{O}(G/H) \otimes V) = 0$ for $i > 0$. This translates into $H^i(H, V|_H) = 0$ for $i > 0$. The latter can be checked after replacing H by H°_{der} (as $\pi_0 H$ is of order prime to ℓ), or even a finite cover of it. From these remarks, one sees that one can assume that G is connected, semisimple and simply connected.

As in the proof of Proposition VIII.5.11, we can embed G/H into $\prod_F G$. Moreover, F acts in the following way on $\prod_F G$: It takes a tuple $(g_f)_{f \in F}$ to $(g_{ff'} f(g_{f'})^{-1})_f$. This action commutes with the simultaneous twisted G-conjugation. Moreover, the fixed points of F are exactly the 1-cocycles $Z^1(F, G) \subset \prod_F G$, with their G-conjugation. The map $G/H \hookrightarrow Z^1(F, G)$ is the inclusion of a connected component. Note that

$$\mathbb{L}_{\prod_F G}|_{Z^1(F,G)} \to \mathbb{L}_{Z^1(F,G)}$$

is a map of G-equivariant vector bundles on $Z^1(F, G)$, and it moreover carries a commuting F-action, where the target is the F-orbits of the source. As F is of order prime to ℓ, it follows that the map splits. In particular, restricting to G/H, we see that

$$\mathbb{L}_{\prod_F G}|_{G/H} \to \mathbb{L}_{G/H}$$

splits G-equivariantly. Using the transitivity triangle, it follows that if $I \subset \mathcal{O}(\prod_F G)$ is the ideal sheaf of G/H, then I/I^2 is a G-equivariant direct summand of $\mathbb{L}_{\prod_F G}|_{G/H}$. Note that G-equivariant vector bundles on G/H are equivalent to representations of H, and $\mathbb{L}_{\prod_F G}|_{G/H}$ corresponds to the representation $\bigoplus_F \mathfrak{g}^*|_H$. Taking symmetric powers, it follows that for all n, $I^n/^{\mathbb{L}} I^{n+1}$ is a G-equivariant direct summand of the G-equivariant vector bundle on G/H corresponding to a direct sum of copies of $\mathrm{Sym}^n g^*|_H$.

Now let $A = \mathcal{O}(\prod_F G)$ with its G-action and F-action and G- and F-stable ideal $I \subset A$. We consider the complex

$$\ldots \to A^{\otimes F^2} \to A^{\otimes F} \to A \to \mathcal{O}(Z^1(F, G)) \to 0$$

computing the F-orbits on A, in the sense of animated L-algebras. This still computes $\mathcal{O}(Z^1(F, G))$, as F is of order prime to ℓ and A is smooth. Let $A_\bullet = A^{\otimes F^\bullet}$ be the corresponding simplicial L-algebra, with ideal $I_\bullet \subset A_\bullet$ of G/H. Let

$$\widetilde{A}_\bullet = \bigoplus_{n \in \mathbb{Z}} \widetilde{I}^n_\bullet,$$

where $\widetilde{I}^n_\bullet = A_\bullet$ for $n \leq 0$. This defines a simplicial $L[t]$-algebra, where t is the natural degenerator in degree -1. Then $\widetilde{A}_\bullet[t^{-1}] = A_\bullet[t^{\pm 1}]$ and $\widetilde{A}_\bullet/t$ is the symmetric algebra on $\widetilde{I}_\bullet/\widetilde{I}^2_\bullet$ over $\mathcal{O}(G/H)$. In particular, this is a resolution of $\mathcal{O}(G/H)$, and in fact $\widetilde{A}_\bullet$ is a resolution of $\mathcal{O}(G/H)[t]$.

Note that the good filtration dimension of each $\widetilde{A}_n$ is bounded by the good filtration dimension of $\mathcal{O}(G/H)$ plus 1. To see this, note that $\widetilde{A}_n[t^{-1}] = A_n[t^{\pm 1}]$ has a good filtration, so it is enough to show that the good filtration dimension of $\widetilde{A}_n/t$ is bounded by the good filtration dimension of $\mathcal{O}(G/H)$. This is because all symmetric powers of $\widetilde{I}_n/\widetilde{I}^2_n$ are direct summands of direct sums of copies of symmetric powers of $\mathfrak{g}^*$ restricted to H. We checked this for $n = 0$, but for $n > 0$ one simply gets more copies of $\mathfrak{g}^*$ (or applies the reasoning with F replaced by F^{n+1}).

We claim that for all G-representations V, the map

$$\operatorname{colim} R\Gamma(G, \widetilde{A}_\bullet \otimes V) \to R\Gamma(G, \mathcal{O}(G/H)[t] \otimes V)$$

is an isomorphism. Consider the simplicial L-algebra $\widetilde{A}_\bullet^G$, and its colimit B, an animated L-algebra (with $\pi_0 B$ finitely generated, and all $\pi_i B$ finitely generated $\pi_0 B$-modules). Then for all i, $H^i(G, \widetilde{A}_\bullet \otimes V)$ defines a degree-wise finitely generated $\widetilde{A}_\bullet^G$-module (by Theorem VIII.5.4), and hence its colimit defines an animated B-module whose π_j is finitely generated over $\pi_0 B$, for all j. By the good filtration bounds, we also know that these cohomology groups vanish for i sufficiently large. This implies that each cohomology group of the left-hand side is finitely generated over $\pi_0 B$. Moreover, by Haboush's theorem, the map $\pi_0 B \to \mathcal{O}(G/H)[t]$ is a universal homeomorphism, and hence also the right-hand side is finitely generated over $\pi_0 B$. In particular, the cohomology groups of the cone have support which is a closed G-invariant subset of $\operatorname{Spec} \pi_0 B \cong \operatorname{Spec} \mathcal{O}(G/H)[t]$. All in all, it follows that it suffices to prove that the map is an isomorphism modulo t. But modulo t, we consider

$$\operatorname{colim} R\Gamma(G, \widetilde{A}_\bullet/t \otimes V) \to R\Gamma(G, \mathcal{O}(G/H) \otimes V)$$

and all $\widetilde{A}_\bullet/t$ are given by the symmetric algebras on $\widetilde{I}_\bullet/\widetilde{I}_\bullet^2$ over $\mathcal{O}(G/H)$, and hence everything decomposes into a direct sum over n of the degree n parts. In degree 0, it is the constant simplicial object given by $\mathcal{O}(G/H)$. In degree 1, the corresponding simplicial G-equivariant $\mathcal{O}(G/H)$-module

$$\widetilde{I}_\bullet/\widetilde{I}_\bullet^2$$

is contractible, as it comes from some computation of F-homology, and F is of order prime to ℓ. Passing to symmetric powers, the simplicial objects stay contractible, and stay so after taking G-cohomology.

Inverting t, we find that

$$\operatorname{colim} R\Gamma(G, A_\bullet \otimes V)[t^{\pm 1}] \to R\Gamma(G, \mathcal{O}(G/H) \otimes V)[t^{\pm 1}]$$

is an isomorphism; we can then omit the variable t. Applying this to V with a good filtration, we note that all $A_\bullet$ have a good G-filtration, hence so does $A_\bullet \otimes V$, and the left-hand side sits in cohomological degrees ≤ 0. Thus, the same is true for the right-hand side, showing that $\mathcal{O}(G/H)$ has a good G-filtration. □

We will need to know that in the situation of Proposition VIII.5.11, the image of $\operatorname{Perf}(BG) \to \operatorname{Perf}(BH)$ generates (under cones and retracts), at least if F is solvable. It is very likely that this assumption is superfluous.

Theorem VIII.5.14. *Let L be an algebraically closed field of characteristic $\ell > 0$, and let G be a smooth linear algebraic group over L such that G° is reductive, and $\pi_0 G$ is of order prime to ℓ. Assume that F is a finite group of order prime to ℓ acting on G and let $H = G^F$ be the fixed points. Then H is a smooth linear algebraic group such that H° is reductive, $\pi_0 H$ is of order prime to ℓ, and the image of* $\operatorname{Perf}(BG) \to \operatorname{Perf}(BH)$ *generates under cones and retracts.*

Remark VIII.5.15. The following example shows that the hypothesis that F is of order prime to ℓ is important, and cannot even be weakened to for example "quasi-semisimple" automorphisms (preserving a Borel and a torus); also, the example shows that the precise form of the center is critical in the theorem. If $G = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ with the automorphism switching the two factors, then $H = \mathrm{PGL}_2 \times (\mu_2 \times \mu_2)/\mu_2$. If we had $\ell = 2$ – this is excluded by the hypotheses – then

one can show that for all objects $A \in \mathrm{Perf}(BH)$ in the image of $\mathrm{Perf}(BG)$, the summand A_1 of A with nontrivial central character has the property that the (homotopy) invariants of the $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{PGL}_2$-action on A_1 are a perfect complex. This implies that the nontrivial character of H is not generated by $\mathrm{Perf}(BG)$ under cones and retracts. On the other hand, the conclusion of the theorem is (trivially) true for the simply connected and adjoint form of G, as then the inclusion $H \subset G$ admits a retraction.

Before starting the proof of Theorem VIII.5.14, let us recall the following lemma.

Lemma VIII.5.16. *Let $\mathcal{C}$ be an idempotent-complete stable ∞-category, and let $\mathcal{D} \subset \mathcal{C}$ be a full idempotent-complete stable ∞-subcategory, with idempotent-completed Verdier quotient $\mathcal{E}$. Let $S \subset \mathcal{C}$ be a collection of objects of $\mathcal{C}$. Then S generates $\mathcal{C}$ under cones, shifts, and retracts if and only if the image of S in $\mathcal{E}$ generates $\mathcal{E}$ under cones, shifts, and retracts, and all of objects of $\mathcal{D}$ are in the full ∞-subcategory of $\mathcal{C}$ generated by S under cones, shifts, and retracts.*

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full idempotent-complete stable ∞-category generated by S; this is, equivalently, the full ∞-subcategory generated by S under cones, shifts, and retracts. We want to see that $\mathcal{C}' = \mathcal{C}$. Passing to Ind-categories, we have a Verdier quotient sequence

$$\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{E})$$

and the quotient $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{E})$ has a right adjoint $R : \mathrm{Ind}(\mathcal{E}) \to \mathrm{Ind}(\mathcal{C})$; see, for example [**NS18**, Theorem I.3.3, Proposition I.3.5]. In particular, for any $X \in \mathrm{Ind}(\mathcal{C})$, with image $\overline{X} \in \mathrm{Ind}(\mathcal{E})$, the cone of $X \to R(\overline{X})$ lies in $\mathrm{Ind}(\mathcal{D}) \subset \mathrm{Ind}(\mathcal{C})$.

To see that $\mathcal{C}' \to \mathcal{C}$ is essentially surjective, we need to see that for all $X \in \mathrm{Ind}(\mathcal{C})$ with $\mathrm{Hom}(X', X) = 0$ for all $X' \in \mathcal{C}'$, one has $X = 0$. Now by assumption all objects of $\mathcal{D}$ lie in $\mathcal{C}'$, so in particular $\mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathcal{D}$. As $\mathrm{Hom}(Y, R(\overline{X})) = \mathrm{Hom}(\overline{Y}, \overline{X}) = 0$ for all such Y (as $\overline{Y} = 0 \in \mathcal{E}$), we see that $\mathrm{Hom}(Y, \mathrm{cone}(X \to R(\overline{X}))) = 0$ for all $Y \in \mathcal{D}$. As the cone lies in $\mathrm{Ind}(\mathcal{D})$, this implies that the cone is equal to 0, so $X = R(\overline{X})$. But then $\mathrm{Hom}(X', R(\overline{X})) = \mathrm{Hom}(\overline{X'}, \overline{X}) = 0$ for all $X' \in S$, which implies that $\overline{X} = 0$ by the assumption that S generates $\mathcal{E}$ under cones, shifts, and retracts. $\square$

An application of this is the following result.

Lemma VIII.5.17. *Let G be a connected reductive group and $H \subset G$ be a connected reductive subgroup of the same rank. Then the image of* $\mathrm{Perf}(BG) \to \mathrm{Perf}(BH)$ *generates under cones and retracts.*

Proof. Let $T \subset H$ be a maximal torus. Then it is also a maximal torus of G. For any subset $W \subset X^*(T)$ stable under the Weyl group of G and the dominance order, we have the subcategory $\mathrm{Perf}_W(BG)$ consisting of those representations whose weights lie in T, and similarly the subcategory $\mathrm{Perf}_W(BH)$; obviously, restriction gives a functor $\mathrm{Perf}_W(BG) \to \mathrm{Perf}_W(BH)$. We claim by induction on W that the image generates under cones and retracts. We may always write $W = W' \cup W''$ where W'' is a single Weyl group orbit, and W' is still stable under the dominance order. Then $\mathrm{Perf}_W(BG)/\mathrm{Perf}_{W'}(BG)$ is isomorphic to $\mathrm{Perf}(L)$, and is generated by the highest weight representation V of weight given by W''. Note that that all the weight spaces of V with weights in W'' have multiplicity exactly 1. The quotient $\mathrm{Perf}_W(BH)/\mathrm{Perf}_{W'}(BH)$ is isomorphic

to a product of copies of $\mathrm{Perf}(L)$, enumerated by all the orbits of W'' under the Weyl group W_H of H. The induced map on quotients $\mathrm{Perf}(L) \to \prod_{W''/W_H} \mathrm{Perf}(L)$ is the diagonal embedding, by the multiplicity 1 observation. Thus, its image generates under retracts, and we conclude by induction. $\square$

A variation of Lemma VIII.5.16 is the following.

Lemma VIII.5.18. *Let G be a linear algebraic group over L and let $G' \subset G$ be a normal subgroup, with quotient $\overline{G} = G/G'$, and assume that the map $L \to \mathcal{O}(\overline{G})$ splits as $\overline{G}$-representations. Let $\mathcal{C} \subset \mathrm{Perf}(BG)$ be an idempotent-complete stable ∞-subcategory such that for all $X, Y \in \mathcal{C}$ also $X \otimes Y \in \mathcal{C}$. Assume that $\mathcal{C}$ contains $\mathrm{Perf}(B\overline{G})$ and the image of $\mathcal{C}$ in $\mathrm{Perf}(BG')$ generates under cones and retracts. Then $\mathcal{C} = \mathrm{Perf}(BG)$.*

Proof. The functor $\mathrm{Ind}\,\mathrm{Perf}(BG) \to \mathrm{Ind}\,\mathrm{Perf}(BG')$ admits a right adjoint $R : \mathrm{Ind}\,\mathrm{Perf}(BG') \to \mathrm{Ind}\,\mathrm{Perf}(BG)$, given by pushforward along $f : BG' \to BG$. Note that these functors preserve the coconnective part of these ∞-categories, and $\mathrm{Ind}\,\mathrm{Perf}^{\geq 0}(BG) \cong \mathcal{D}^{\geq 0}(BG)$. In particular, this induces a monad on $\mathrm{Ind}\,\mathrm{Perf}(BG)$, given by Rf_*f^*. By the projection formula, this monad is given by $V \mapsto V \otimes \mathcal{O}(\overline{G})$. By the assumption that $L \to \mathcal{O}(\overline{G})$ splits as $\overline{G}$-representations, it follows that for all $V \in \mathrm{Ind}\,\mathrm{Perf}(BG)$, V is a direct summand of $V \otimes \mathcal{O}(\overline{G})$.

Now take any $V \in \mathrm{Ind}\,\mathrm{Perf}(BG)$; we want to see that it lies in $\mathrm{Ind}(\mathcal{C})$. Let $V' \in \mathrm{Ind}\,\mathrm{Perf}(BG')$ be the image of V. The functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}\,\mathrm{Perf}(BG')$ is surjective, so we can write V' as the image of some $X \in \mathrm{Ind}(\mathcal{C}) \subset \mathrm{Ind}\,\mathrm{Perf}(BG)$. Then V is a direct summand of $R(V') = X \otimes \mathcal{O}(\overline{G})$. As $\mathcal{O}(\overline{G}) \in \mathrm{Ind}\,\mathrm{Perf}(B\overline{G})$ lies in $\mathrm{Ind}(\mathcal{C})$ by assumption, and tensor products preserve $\mathrm{Ind}(\mathcal{C})$, we get the result. $\square$

Proof of Theorem VIII.5.14. By induction, we can assume that F is cyclic of prime order $p \neq \ell$. The following argument actually proves simultaneously that $\pi_0 H$ is of order prime to ℓ, and H° is reductive, but this is already part of Proposition VIII.5.11.

For the statement that $\mathrm{Perf}(BG) \to \mathrm{Perf}(BH)$ generates, we first reduce to the case that G is connected and semisimple. Indeed, we have an exact sequence

$$1 \to G^\circ_{\mathrm{der}} \to G \to D \to 1$$

where D° is a torus and $\pi_0 D$ is of order prime to ℓ; in particular D has semisimple representation theory and $L \to \mathcal{O}(D)$ splits as D-representations. Intersecting with H, we get a similar exact sequence, and then Lemma VIII.5.18 reduces us to the case of G°_{der}.

Now we claim that we can reduce to the case that Θ permutes the almost simple factors of G transitively. Let $G_1 \to G$ be the simply connected cover, to which Θ lifts, and let $H_1 = G_1^\Theta$ be the corresponding fixed points.

Lemma VIII.5.19. *Let G be a connected semisimple group, and let $\Theta : G \to G$ be an automorphism of order $p \neq \ell$; it also induces an automorphism $\Theta : G_1 \to G_1$ of the simply connected cover. Let $H = G^\Theta \subset G$ be the fixed points, and similarly $H_1 = G_1^\Theta$. Then H_1 is connected, and $\pi_0 H$ is an elementary abelian p-group which injects naturally into the coinvariants $\ker(Z(G_1) \to Z(G))_\Theta$.*

Proof. For the claim that H_1 is connected, see [**Ste68**, Theorem 8.1]. We define a map $\psi : H \to \ker(Z(G_1) \to Z(G))_\Theta$. Given $h \in H$, lift to $\widetilde{h} \in G_1$. Then $\widetilde{h}\Theta(\widetilde{h})^{-1}$ must lie in the kernel

of $G_1 \to G$, i.e. in the kernel of $Z(G_1) \to Z(G)$. Taking a different choice of $\widetilde{h}$ multiplies $\widetilde{h}\Theta(\widetilde{h})^{-1}$ by $z\Theta(z)^{-1}$, so the image in the coinvariants is well-defined. Moreover, it is clear that the image is 0 when $h \in H$ lifts to H_1. Conversely, if the image in the coinvariants is 0, then we can find a choice of $\widetilde{h}$ so that $\widetilde{h} = \Theta(\widetilde{h})$, i.e. we can lift h to $\widetilde{h} \in H_1$. This shows that $\pi_0 H = H/\mathrm{im}(H_1 \to H)$ injects into $\ker(Z(G_1) \to Z(G))_\Theta$. It is also easy to check that it is a group homomorphism, so $\pi_0 H$ is an abelian group.

Moreover, all elements of $\pi_0 H$ are of order p. Indeed, consider $\widetilde{h}\Theta(\widetilde{h})^{-1}$ above. Its image in the coinvariants agrees with $\Theta^i(\widetilde{h})\Theta^{i+1}(\widetilde{h})^{-1}$ for all $i = 0, \dots, p-1$, so its p-th power is $\widetilde{h}\Theta^p(\widetilde{h})^{-1} = 1$, as Θ is of order p. $\square$

Now assume that the theorem holds when Θ permutes the almost simple factors of G transitively. This implies that it holds for all simply connected groups: Indeed, we can decompose these into factors. Now for general G, we induct on the order of $\ker(Z(G_1) \to Z(G))$. Assume that there is some normal Θ-stable connected semisimple subgroup $G' \subset G$ such that $\overline{G} = G/G'$ has almost simple factors permuted transitively by Θ, and $\ker(Z(\overline{G}_1) \to Z(\overline{G}))_\Theta \neq 0$. In that case, let $\overline{G}' \to \overline{G}$ be the cover with $\ker(Z(\overline{G}') \to Z(\overline{G})) = \ker(Z(\overline{G}_1) \to Z(\overline{G}))_\Theta$, and let $G' \to G$ be its pullback. Then Θ also acts on G', and the action on $W = \ker(Z(G') \to Z(G))$ is trivial. In particular, if $H' = (G')^\Theta$, then H' contains $W = \ker(Z(G') \to Z(G))$. By induction, we can assume that $\mathrm{Perf}(BG') \to \mathrm{Perf}(BH')$ generates under cones and retracts. Both categories decompose into direct sums over the characters of W, so it follows that also $\mathrm{Perf}(BG) \to \mathrm{Perf}(B(H'/W))$ generates under cones and retracts. Now $H'/W \subset H$ is a union of connected components, so by Lemma VIII.5.18, it suffices to shows that $\mathcal{O}(H/H')$ is generated by $\mathrm{Perf}(BG)$. But as $G' = G \times_{\overline{G}} \overline{G}'$, we also similarly have $H' = H \times_{\overline{H}} \overline{H}'$, and hence $H/H' = \overline{H}/\overline{H}'$, and hence this part follows from $\mathcal{O}(\overline{H}/\overline{H}')$ being generated by $\mathrm{Perf}(B\overline{G})$.

Thus, we can assume that for all such quotients $\overline{G}$, one has $\ker(Z(\overline{G}_1) \to Z(\overline{G}))_\Theta = 0$. This actually implies that $\ker(Z(\overline{G}_1) \to Z(\overline{G}))$ is of order prime to p. If this happens for all such quotients $\overline{G}$, then in fact $\ker(Z(G_1) \to Z(G))$ is of order prime to p. By Lemma VIII.5.19, it is then the case that H is connected. If $\ker(Z(G_1) \to Z(G))_\Theta \neq 0$, then we can pass to the corresponding cover $G' \to G$ and the previous arguments reduce us to G'. Thus, we can further assume that $\ker(Z(G_1) \to Z(G))_\Theta = 0$. In that case also the Θ-invariants are trivial (as the group is prime to p), and the map $H_1 \to H$ is an isomorphism. Moreover, we can find a maximal quotient $G \to G'$ such that $\ker(Z(G_1) \to Z(G'))$ is prime to p and $\ker(Z(G_1) \to Z(G'))_\Theta = 0$, and this decomposes into factors. **This finishes the reduction to the case when Θ permutes the almost simple factors of G transitively.**

Assume that there is more than one simple factor. Then Θ permutes them cyclically, with Θ^p acting as the identity, so $G_1 = \prod_{\mathbb{Z}/p\mathbb{Z}} H_1$. There is nothing to do in the simply connected types E_8, F_4 and G_2. **Consider next the types B, C, D and E_7 where the fundamental group is a 2-group. Assume first that also $p > 2$.** Then as in the previous paragraph, we can assume that $G_1 \to G$ is the maximal Θ-stable quotient such that $H_1 = H = G^\Theta$; then $Z(H) \to Z(G)$ is an isomorphism. Let V be any representation of H. Then V splits off $V \otimes V^* \otimes V$, and inductively it splits off $V^{\otimes m} \otimes V^{*\otimes(m-1)}$ for any $m \geq 1$. In type B, C, D_n with n even and E_7, take $n = p$: Then $V^{\otimes(n-1)}$ is a representation of the adjoint form, hence generated by $\mathrm{Perf}(BG)$, and $V^{\otimes p}$ lifts to a representation of G, whence lies in the subcategory of $\mathrm{Perf}(BH)$ generated by $\mathrm{Perf}(BG)$, showing

that also V lies in there. In type D_n with n odd, the same argument works if $p \equiv 1 \mod 4$; if $p \equiv 3 \mod 4$, take $m = p+1$: Then $V^{\otimes m}$ is a representation of the adjoint group, and $V^{*\otimes(m-1)}$ extends to G. **Now consider the case $p = 2$ in type B, C, D and E_7, so necessarily $\ell \neq 2$.** In type B, C and E_7, one has $Z(G_1) = (\mathbb{Z}/2\mathbb{Z})^2$, and besides the simply connected and adjoint form where the claim is clear, there is only one Θ-stable subgroup. In that case $G = (H_1 \times H_1)/\mu_2$, with μ_2 diagonally embedded, and the fixed points are given by $H = H_1/\mu_2 \times (\mu_2 \times \mu_2)/\mu_2$. By Lemma VIII.5.18, it suffices to generate the nontrivial character of $(\mu_2 \times \mu_2)/\mu_2$. In type B, the exterior tensor product of two copies of the spinor representation defines a representation W of G of dimension a power of 2 whose restriction to H has nontrivial central character, and admits a character as a summand, thus generating the nontrivial character of $\mu_2 \times \mu_2/\mu_2$. In type C, one can argue similarly with two copies of the i-th exterior power of the standard representation of Sp_{2n}, with $0 < i < 2n$ odd such that $\binom{2n}{i}$ is not divisible by ℓ – for example, take for i the exact power of ℓ dividing n. In type E_7, there is an embedding into Sp_{56}, reducing to type C. In type D, we make a case distinction according to the size of $\pi_1(G)$, whose size is a power of 2 between 1 and 16, where the extreme cases are clear. If it is of size 2, then $G = (\mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n})/\mu_2$ where the μ_2 is diagonally embedded, and in each copy is the kernel of $\mathrm{Spin}_{2n} \to \mathrm{SO}_{2n}$. In that case $H = \mathrm{SO}_{2n} \times (\mu_2 \times \mu_2)/\mu_2$, which can be handled similarly to type B. If $\pi_1(G)$ is of size 8, then $G = (\mathrm{SO}_{2n} \times \mathrm{SO}_{2n})/\mu_2$ and $H = \mathrm{PSO}_{2n} \times (\mu_2 \times \mu_2)/\mu_2$. This case can be handled like the case of type C. There remains the case that $\pi_1(G)$ is of size 4, where either $G = \mathrm{SO}_2 n \times \mathrm{SO}_{2n}$ where the claim is clear, or else $G \cong (\mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n})/Z$ where Z is the center of Spin_{2n}, embedded either diagonally, or twisted via the outer automorphism of Spin_{2n} in one copy. In this case we also need to distinguish between even and odd n. Assume first that n is odd and Z is embedded diagonally. Then $H = \mathrm{PSO}_{2n} \times (\mu_2 \times \mu_2)/\mu_2$. This case can be reduced to the case that $\pi_1(G)$ is of order 2 by passing to a cover. If n is odd and Z is embedded twisted via the outer automorphism, then $H \cong (\mathrm{SO}_{2n} \times \mu_4)/\mu_2$, so $H^\circ \cong \mathrm{SO}_{2n}$ and $\pi_0 H \cong \mathbb{Z}/2\mathbb{Z}$. The set of connected components actually maps isomorphically to the components in the case where $\pi_1(G)$ is of order 8, so $\mathcal{O}(\pi_0 H)$ is generated by $\mathrm{Perf}(BG)$ via reduction to that case. Moreover, taking an exterior tensor product of two copies of a spinor representation gives a representation of G whose restriction to $H^\circ = \mathrm{SO}_{2n}$ generates the part with nontrivial central character. Now assume that n is even, and Z is embedded diagonally. Then $H = \mathrm{PSO}_{2n} \times (Z \times Z)/Z$, and again one can generate the characters of $(Z \times Z)/Z$ via restrictions of exterior tensor products of spinor representations. Finally, if n is even and Z is embedded twisted via the outer automorphism, then $H = \mathrm{SO}_{2n}$. Taking an exterior tensor product of the two distinct spinor representations defines a representation of G whose restriction to H generates the part of $\mathrm{Perf}(BH)$ with nontrivial central character. **This finishes types B, C, D and E_7. Consider next type E_6, where the fundamental group is $\mathbb{Z}/3\mathbb{Z}$. Assume first $p \neq 3$.** As before, one can reduce to the case $G_1 \to G$ is the maximal Θ-stable quotient such that $H_1 = H = G^\Theta$ is the simply connected form of E_6. In that case, we use again that any V can be split off from $V^{\otimes m} \otimes V^{*\otimes(m-1)}$. If $p \equiv 1 \mod 3$, we can take $m = p$, otherwise $m = p+1$: In the first case $V^{\otimes m}$ extends to G and $V^{*\otimes(m-1)}$ is a representation of the adjoint group, in the second case it is the other way around. **Now consider $p = 3$, so necessarily $\ell \neq 3$.** Then $G_1 = E_6^3$ and $Z(G_1) = (\mathbb{Z}/3\mathbb{Z})^3$. The order of $\pi_1(G)$ is then a power of 3 between 1 and 27, with the extreme cases being trivial, and in both other cases there is exactly one possibility. For $\pi_1(G)$ of order 3, we get $H = E_{6,\mathrm{ad}} \times Z$ where $Z \cong \mu_3$ is a central subgroup, given as the kernel of the sum map $\mu_3^3 \to \mu_3$ quotient by the diagonal copy of μ_3. Consider the 27-dimensional representation V of E_6. Then $W = V \otimes V^* \otimes 1$ defines a representation of G on which Z acts via a nontrivial character, and

the restriction of W to H splits off this character (as the dimension of W is a power of 3, hence invertible modulo ℓ). If $\pi_1(G)$ is of order 9, then $H = E_{6,\mathrm{ad}} \times Z'$, where $Z' \cong \mu_3$ is the quotient of μ_3^3 by the kernel of the sum map. In this case, $V \otimes V \otimes V$ defines a representation of G on which Z' acts via a nontrivial character, and the same argument applies.

It remains to handle type A. Thus, $G = (\prod_{\mathbb{Z}/p\mathbb{Z}} \mathrm{SL}_n)/Z$. Let n' be the maximal divisor of n that is prime to p. We can decompose $Z = Z_p \times Z^p$ where Z_p is of order a power of p, and Z^p is prime to p. We can assume that $Z^p \subset \mu_{n'}^p$ is the set of all elements with product 1 (which is the maximal Θ-invariant complement to the diagonal copy of $\mu_{n'}$). On the other hand, letting p^a be the exact power of p dividing n, Z_p is a subgroup of $\mu_{p^a}^p \cong (\mathbb{Z}/p^a\mathbb{Z})^p$ that is stable under the the Θ-action. One can then show that $\pi_0 H \cong H^1(\Theta, Z_p)$. We will now inductively reduce to the case that Z_p is trivial, by replacing Z_p by a (certain) subgroup $Z_p' \subset Z_p$ of index p and arguing that the result for $G' = (\prod_{\mathbb{Z}/p\mathbb{Z}} \mathrm{SL}_n)/Z_p' Z^p$ implies the result for G. First note that $Z_p'/Z_p \subset G'$ is necessarily contained in $H' = (G')^\Theta$, so if $\mathrm{Perf}(BG') \to \mathrm{Perf}(BH')$ generates under cones and retracts, the same is true for $\mathrm{Perf}(BG) \to \mathrm{Perf}(B(\mathrm{im}(H' \to H)))$. By Lemma VIII.5.18, it remains to generate $\mathcal{O}(H/H')$. This is given by $H^1(\Theta, Z_p)/H^1(\Theta, Z_p')$, which injects into $H^1(\Theta, Z_p/Z_p') \cong \mathbb{Z}/p\mathbb{Z}$. Moreover, if a' is chosen maximal so that $Z_p \subset (p^{a'}\mathbb{Z}/p^a\mathbb{Z})^p$, we can assume that Z_p' contains $(p^{a'+1}\mathbb{Z}/p^a\mathbb{Z})^p \cap Z_p$. With this choice, we can assume that $(p^{a'+1}\mathbb{Z}/p^a\mathbb{Z})^p \subset Z_p' \subset Z_p \subset (p^{a'}\mathbb{Z}/p^a\mathbb{Z})^p$, which then determines Z_p' uniquely (given Z_p), as $\mathbb{F}_p^p$ has a unique Θ-stable filtration. Now choose i, $0 < i < n$, with i not divisible by p, such that $\binom{n}{i}$ is not divisible by ℓ – for example, take for i the exact power of ℓ dividing n, which works as long as n is not itself a power of ℓ, but then $Z_p = 0$ anyway. Then consider the $n'p^{a-a'-1}$-th tensor power V of the i-th exterior tensor power of the standard representation; this V is of dimension prime to ℓ, and a representation of $\mathrm{SL}_n/\mu_{n'p^{a-a'-1}}$ whose central character on $\mu_{p^{a-a'}}/\mu_{p^{a-a'-1}}$ is nontrivial. Taking tensor products of V, one can then define a representation of G whose central character factors over a nontrivial character of Z_p/Z_p', from which one gets the desired result. Thus, in type A, we can assume that $Z_p = 0$. Then $G = (\prod_{\mathbb{Z}/p\mathbb{Z}} \mathrm{SL}_n)/Z^p$ and $H = \mathrm{SL}_n$. We show that the standard representation V of H is a retract of a representation of G; this implies the same for its exterior powers, and these generated $\mathrm{Perf}(BH)$. We can split off V from any $V^{\otimes m} \otimes V^{*\otimes(m-1)}$. Choose a positive integer q such that $pq \equiv 1 \mod n'$ and let $m = pq$. Then $V^{\otimes m} = (V^{\otimes q})^{\otimes p}$ extends to a representation of G, while $V^{*\otimes(m-1)}$ is a representation of PGL_n, which also extends to G. **This finishes the reduction to the case that G is almost simple.**

Thus, from now on assume that G is almost simple. Assume first that Θ is an inner automorphism. If G is simply connected, then $H \subset G$ is the centralizer of a semisimple element, and thus connected and of the same rank as G, so the claim follows from Lemma VIII.5.17. Moreover, for inner automorphisms Lemma VIII.5.19 shows that the connected components of H always inject into the ones for the adjoint form, and moreover the center of G is always contained in H. **So for inner automorphisms, it remains to handle the case of adjoint G.** Moreover, by Lemma VIII.5.18, it suffices to show that $\mathcal{O}(\pi_0 H)$ is in the subcategory of $\mathrm{Perf}(BH)$ generated under cones and retracts by the image of $\mathrm{Perf}(BG) \to \mathrm{Perf}(BH)$. As a general observation, we note that the adjoint representation $\mathfrak{g}$ is a representation of G, and it restricts to a representation of $H \times \mathbb{Z}/p\mathbb{Z}$. We may decompose according to the (semisimple) action of $\Theta \in \mathbb{Z}/p\mathbb{Z}$, and in particular

the fixed points $\mathfrak{h} = \mathfrak{g}^\Theta$ are a retract of $\mathfrak{g}|_H$. This implies that also all $\bigwedge^i \mathfrak{h} \in \mathrm{Perf}(BH)$ are retracts of $\bigwedge^i \mathfrak{g}|_H$.

Now we go through the list of types. Note that by Lemma VIII.5.19, $\pi_0 H$ is an elementary abelian p-group injecting into $Z(G_{\mathrm{sc}})$. In particular, for $p > 3$, only $G = \mathrm{PGL}_{pn}$ is relevant. Moreover, the only relevant H is given by $\mathrm{GL}_n^p / \mathbb{G}_m \rtimes \mathbb{Z}/p\mathbb{Z}$, which occurs for the centralizer of the semisimple element that is the image of the diagonal element of GL_{pn} with n occurences of each p-th root of unity. Changing the center, we can also look at restriction from GL_{pn} to $\mathrm{GL}_n^p \rtimes \mathbb{Z}/p\mathbb{Z}$. For any i, consider the representation $\bigwedge^i(\mathrm{std})$ of GL_{pn}. Its restriction to GL_n^p decomposes as a direct sum (as the standard representation decomposes, and a wedge power of a direct sum decomposes into various pieces), in particular a canonical direct summand $\bigoplus_{j=1}^p \bigwedge^i(\mathrm{std}_j)$ where std_j is the standard representation of the j-th factor of GL_n^p. In particular, as a representation of H, the restriction of $\bigwedge^i(\mathrm{std})$ contains $\bigoplus_{j=1}^p \bigwedge^i(\mathrm{std}_j)$, which is naturally a representation of H. If we take $i = n$, then each $\bigwedge^i(\mathrm{std}_j)$ is just the determinant of the standard representation. Thus, we get a representation W of $\mathbb{G}_m^p \rtimes \mathbb{Z}/p\mathbb{Z}$. This group has semisimple representation theory, and $W \otimes W^*$ admits $\mathcal{O}(\mathbb{Z}/p\mathbb{Z})$ as a direct summand, so the claim follows. This handles the case of type A, in fact for all $p \geq 2$.

Let us next do the remaining case for $p = 3$, which is E_6. In that case, H° is of the form SL_3^3/μ_3^2 (for some embedding of μ_3^2 into the center μ_3^3), and $\pi_0 H = \mathbb{Z}/3\mathbb{Z}$ permutes the three factors. In particular, $\mathfrak{h}|_{H^\circ}$ decomposes into the direct sum of the adjoint representations of the three factors (each of which is of dimension 8), permuted cyclically by $\pi_0 H$. Taking $\bigwedge^8 \mathfrak{h}$, it contains a direct summand that restricted to H° is the direct sum of $\bigwedge^8 \mathfrak{sl}_3 \cong L$ for each factor, permuted cyclically by $\pi_0 H$; in other words, the regular representation of $\pi_0 H$.

The remaining types are now B, C, D and E_7, all for $p = 2$ (in particular, $\ell \neq 2$). In type B, $G = \mathrm{SO}_{2n+1}$, and we are looking at the centralizer of a diagonal element with entries 1 and -1. This is of the form $\mathrm{S}(\mathrm{O}_m \times \mathrm{O}_{2n+1-m})$ for some $m = 1, \ldots, n$. This maps naturally to O_m, identifying connected components. The natural representation decomposes into the sum of the natural representations of O_m and O_{2n+1-m}. As before, it follows that all exterior powers of the natural representation of O_m lie in the relevant subcategory of $\mathrm{Perf}(BH)$, but its determinant defines a representation of O_m that restricts to the nontrivial character of $\pi_0 H$.

In type C, $G = \mathrm{PSp}_{2n}$, and we are looking at the centralizer of an element of GSp_{2n} that has diagonal entries 1 and -1. There are two cases where $\pi_0 H$ is nontrivial. On the one hand, if $n = 2m$, one can get $H^\circ = \mathrm{Sp}_{2m}^2/\mu_2$ with $\pi_0 H = \mathbb{Z}/2\mathbb{Z}$; this case can be handled like the case of E_6, taking $\bigwedge^d \mathfrak{h}$ for d the dimension of Sp_{2m}. In the other case, we get $H = \mathrm{GL}_n/\mu_2 \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action is given by $A \mapsto {}^tA^{-1}$. In that case, we again enlarge the center as in type A, so consider $\widetilde{H} = (\mathrm{GL}_n \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z} \subset \widetilde{G} = \mathrm{GSp}_{2n}$, where $(A, \lambda) \in \mathrm{GL}_n \times \mathbb{G}_m$ embeds GSp_{2n} via the block diagonal matrix with entries A and $\lambda {}^tA^{-1}$, and $\mathbb{Z}/2\mathbb{Z}$ maps to the block matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Restricting the natural representation of GSp_{2n} to $\widetilde{H}$, its restriction to $\widetilde{H}^\circ$ is a sum of two n-dimensional representations, permuted by $\pi_0 \widetilde{H}$. Passing to the n-th exterior power, we get as a direct summand a 2-dimensional representation W of the torus quotient $\mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ of $\widetilde{H}$, and $W \otimes W^*$ contains $\mathcal{O}(\mathbb{Z}/2\mathbb{Z})$ as a direct summand.

In type D, $G = \mathrm{PSO}_{2n}$, and we are looking at the centralizer of a diagonal element of $\widetilde{G} = \mathrm{GSO}_{2n}$ with entries ± 1. In the following, we describe the preimage of $H \subset G$ in $\widetilde{G}$. Unless the number of 1's and -1's agrees, this is of the form $\mathrm{GSO}_{2n} \cap (\mathrm{GL}_m \times \mathrm{GL}_{2n-m})$ for some even m, $0 < m < n$. In that case, H has two connected components, and these two connected components are still seen when taking the centralizer $\mathrm{S}(\mathrm{O}_m \times \mathrm{O}_{2n-m})$ in SO_{2n}, so we may replace the adjoint group by SO_{2n} in this example. Then the same argument as for type B works. There remains the case that the number of 1's and -1's agree, in which case there are two conjugacy classes. In one case, one gets $\widetilde{H} = \mathrm{GSO}_{2n} \cap (\mathrm{GL}_n \times \mathrm{GL}_n \rtimes \mathbb{Z}/2\mathbb{Z})$. If n is odd, then this intersection agrees with $\mathrm{GSO}_{2n} \cap (\mathrm{GL}_n \times \mathrm{GL}_n)$, and the previous argument applies. If n is even, then $\pi_0 \widetilde{H}$ agrees with $Z(G_{\mathrm{sc}}) = (\mathbb{Z}/2\mathbb{Z})^2$. Restricting $\bigwedge^n$ of the standard representation of GSO_{2n} to $\widetilde{H}$ admits a 2-dimensional summand W which on $\widetilde{H}^\circ$ factors over the torus quotient $\mathbb{G}_m$. Let us identify this W: We can also first restrict from GL_{2n} to $\mathrm{GL}_n^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, in which case we get a representation of $\mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ that on $\mathbb{G}_m^2$ is the direct sum of the two characters given by the two projections. In particular, $W \otimes W^*$ already admits $\mathcal{O}(\mathbb{Z}/2\mathbb{Z})$ as a direct summand for $\mathrm{GL}_n^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, and hence continues to do so after restriction to $\widetilde{H}$. We may then apply Lemma VIII.5.18 to reduce to the intersection of GSO_{2n} with GL_n^2, where we can as before reduce to the cover SO_{2n} of $G = \mathrm{PSO}_{2n}$, and the argument of type B applies. Finally, there is an inner involution for which $\widetilde{H}^\circ = \mathrm{GL}_n \times \mathbb{G}_m$ with $\pi_0 \widetilde{H} = \mathbb{Z}/2\mathbb{Z}$. This can be handled like the similar case in type C.

For inner automorphisms, it remains to handle the case of E_7. In that case, there is an embedding $E_7 \to \widetilde{G} = \mathrm{Sp}_{56}$ mapping the center to the center. In particular, any semisimple element of $E_{7,\mathrm{ad}}$ maps to a semisimple element of PSp_{56}, and we get an inclusion $H \subset \widetilde{H}$ of centralizers. Lemma VIII.5.19 implies that $\pi_0 H \to \pi_0 \widetilde{H}$ is injective, so this case reduces to type C.

Now we consider the case Θ is an outer automorphism. Let us first treat the case of triality, so $p = 3$ (and $\ell \neq 3$). Then G is of type D_4 and either simply connected or adjoint. By Lemma VIII.5.19, the fixed points H are always connected. Moreover, H is unchanged by taking G adjoint, so we can assume G adjoint. There are two conjugacy of order 3 outer automorphisms of Spin_8 (see e.g. [**Tit59**]): The diagram automorphism with fixed points G_2, and an automorphism whose fixed points agree with the fixed points of the order 3 semisimple element of G_2 with centralizer $\mathrm{PGL}_3 \subset G_2$. The second case thus follows from the first. For the first, note first that any highest weight for G_2 lifts to a highest weight for Spin_8, so it follows easily from highest weight theory that $\mathrm{Perf}(B\mathrm{Spin}_8) \to \mathrm{Perf}(BG_2)$ generates under cones and retracts. Now for the adjoint form, there is the problem that there is a fundamental weight for G_2 that does not lift to a highest weight for PSpin_8; rather, if would lift to either of the three fundamental weights for Spin_8 giving the two spinor representations V_1, V_2 and the standard representation V_3 (which are permuted under triality). But then $V = V_1 \otimes V_2 \otimes V_3$ has trivial central character, and upon restriction to G_2 all three of these are isomorphic and selfdual. Thus $V_1|_{G_2}$ splits off $V|_{G_2}$ (as in general W splits off $W \otimes W^* \otimes W$), giving the desired result.

It remains to handle the case of outer involutions, so $p = 2$ (and $\ell \neq 2$). In continuing the tradition of handling exceptional cases first for outer involutions, consider E_6. By Lemma VIII.5.19, the fixed points are necessarily connected, and moreover are unchanged when going to the adjoint form of E_6. We can thus assume that the group is adjoint. There are two conjugacy classes of outer automorphisms of E_6: The diagram automorphism, with fixed points

F_4, and one automorphism with fixed points PSp_8. Consider first the diagram automorphism. If G is simply connected, any highest weight of F_4 or PSp_8 lifts to a highest weight for E_6, and the claim is easy. Otherwise, if G is adjoint, let W be an irreducible representation of the simply connected form of E_6 with nontrivial central character. Then $W|_{F_4}$ is selfdual, so $W|_{F_4}$ splits off $(W \otimes W \otimes W)|_{F_4}$, and $W \otimes W \otimes W$ is always a representation of the adjoint form G.

In type A, if the simply connected cover is SL_n with n odd, there is just one conjugacy class of outer involutions, with fixed points SO_n (independently of the center of G). The worst case is thus $G = \mathrm{PGL}_n$. But for any irreducible representation V of SL_n, the restriction of V to SO_n is selfdual, and hence $V|_{\mathrm{SO}_n}$ splits off $V^{\otimes n}|_{\mathrm{SO}_n}$ (noting that n is odd), where $V^{\otimes n}$ is a representation of the adjoint form G. This reduces us back to the simply connected case, where the claim follows easily from highest weights. If the simply connected cover is SL_{2n}, there are two conjugacy classes of outer involutions, with fixed points Sp_{2n} resp. SO_{2n} (for the simply connected form). Arguing as for SL_n with n odd, we can assume that the fundamental group of G is a 2-group (but the 2-part of $\pi_1(G)$ changes the structure of H), and so $G = \mathrm{SL}_{2n}/\mu_{2^a}$ for some $a \leq 0$ such that 2^a divides $2N$. In the simply connected form, $a = 0$, both cases follow from consideration of highest weights. In general, we reduce to the simply connected case by induction on a. Unless a is the exact power of 2 dividing $2n$ (and assuming $a > 0$), one has $H = H^\circ \times \mu_{2^{a+1}}/\mu_{2^a}$ where H° is PSp_{2n} resp. PSO_{2n}. Moreover, $H' \to H$ factors over a μ_2-cover $H' \to H^\circ$, so by Lemma VIII.5.18, it suffices to generate the nontrivial character of $\mu_{2^{a+1}}/\mu_{2^a}$. For this, take V the i-th exterior power of the standard representation where i is the exact power of ℓ dividing n, so that V is of dimension prime to ℓ. Consider $V^{\otimes 2^a}$. This is a representation of G such that the central character of $\mu_{2^{a+1}}/\mu_{2^a} \subset G$ is nontrivial. The restriction of V to H will then split off the nontrivial character of $\mu_{2^{a+1}}/\mu_{2^a}$, noting that the restriction of $V^{\otimes 2^a}$ to PSp_{2n} resp. PSO_{2n} splits off the trivial representation (as $a > 0$ and V becomes selfdual). It remains to consider the case that a is the exact power of 2 dividing $2n$. In the symplectic case, H is connected, and the reduction to H' works immediately. In the orthogonal case, H is given by PO_{2n}, and we need to construct the nontrivial character of $\pi_0 H$. Here, we observe as above that the subcategory of $\mathrm{Perf}(BH)$ generated by $\mathrm{Perf}(BG)$ under cones and retracts contains all wedge powers of $\mathfrak{h} = \mathfrak{so}_{2n}$. Taking the determinant of $\mathfrak{h}$, we observe that it gives the nontrivial character of $\pi_0 H$, giving the claim.

In type D, we consider first type D_n with n odd. In the simply connected form $H \subset \mathrm{Spin}_{2n}$, the centralizer is the diagonal μ_2-cover $H \to \mathrm{SO}_m \times \mathrm{SO}_{2n-m}$ for some odd m. This case is again handled by highest weights. For the intermediate form $G = \mathrm{SO}_{2n}$, the centralizer is $\mathrm{S}(\mathrm{O}_m \times \mathrm{O}_{2n-m}) = \mathrm{SO}_m \times \mathrm{SO}_{2n-m} \times \mu_4/\mu_2$. In that case, let V be a spinor representation of Spin_{2n}, and consider $V^{\otimes 2}$. Then V becomes selfdual upon restriction to $\mathrm{Spin}_m \times \mathrm{Spin}_{2n-m}$ (as there is only one spinor representation in type B), and hence $V^{\otimes 2}$ splits off the trivial representation as a representation of $\mathrm{SO}_m \times \mathrm{SO}_{2n-m}$. On the other hand, the central character is the nontrivial character of μ_4/μ_2, so $V^{\otimes 2}$ restricted to H splits off the nontrivial character of μ_4/μ_2 and hence Lemma VIII.5.18 handles this case again. Finally, if $G = \mathrm{PSO}_{2n}$ is adjoint, then the centralizer is $\mathrm{SO}_m \times \mathrm{SO}_{2n-m}$ and connected as long as $m \neq n$, when the reduction to the previous case is immediate. If on the other hand $m = n$, then the centralizer H has $H^\circ = \mathrm{SO}_n \times \mathrm{SO}_n$ and $\pi_0 H = \mathbb{Z}/2\mathbb{Z}$, switching the two factors. Here we use again that the adjoint representation $\mathfrak{h}$ lies in the subcategory of $\mathrm{Perf}(BH)$ generated by $\mathrm{Perf}(BH)$ under cones and retracts. This is given by $\mathfrak{so}_n \times \mathfrak{so}_n$, where $\pi_0 H$ switches the two factors. Passing to the d-th exterior power, where d is the dimension of $\mathfrak{so}_n$, we get a two-dimensional summand given by the sum of two copies of the determinants of $\mathfrak{so}_n$, with

$\pi_0 H$ acting by switching. This generates the regular representation of $\pi_0 H$, as desired. Finally, consider the case of D_n with n even. In that case, the discussion works in exactly the same way (except that μ_4 is replaced by $\mu_2 \times \mu_2$, which however causes no difference), except that as m is still odd the case $m = n$ cannot occur, so the most difficult case does not arise. $\square$

VIII.5.4. End of proof. Finally, we can prove Theorem VIII.5.2. We switch back to the notation employed there.

Note that W can be written (inside animated groups) as a geometric realization of a simplicial group which is finite free in each degree. In particular, the sifted colimit

$$\operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G})_L)$$

can be understood to be a geometric realization of a simplicial diagram.

We want to use Proposition VIII.5.10 to show that

$$\operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G})_L) \to \mathcal{O}(Z^1(W, \widehat{G})_L)$$

is an isomorphism in $\operatorname{Ind}\operatorname{Perf}(B\widehat{G})$, and that $Z^1(W, \widehat{G})_L$ satisfies the hypotheses of Theorem VIII.5.8. To do so, it suffices to see that for all closed $\widehat{G}$-orbits $Z \cong \widehat{G}/H \subset Z^1(W, \widehat{G})_L$, the $\widehat{G}$-representation $\mathcal{O}(Z)$ has a good filtration, the image of $\operatorname{Perf}(B\widehat{G}) \to \operatorname{Perf}(BH)$ generates under cones and retracts, and for any closed immersion

$$Z \cong Z^1(W, \widehat{G}) \subset Z^1(F_n, \widehat{G}) \cong \widehat{G}^n,$$

the conormal sheaf I/I^2 defines a representation of H that is a direct summand of a direct sum of copies of $\widehat{\mathfrak{g}}^*|_H$.

Note that by Proposition VIII.1.5, any such closed orbit Z has the property that $\varphi : W_E \to \widehat{G}(L)$ factors over a finite quotient F of order prime to ℓ. Then $H = \operatorname{Cent}_{\widehat{G}}(\varphi) \subset \widehat{G}$ is the centralizer of a group of automorphisms of $\widehat{G}$ of order prime to ℓ. By Proposition VIII.5.12, $\mathcal{O}(\widehat{G}/H)$ has a good $\widehat{G}$-filtration, and by Theorem VIII.5.14, the map $\operatorname{Perf}(B\widehat{G}) \to \operatorname{Perf}(BH)$ generates under cones and retracts. Finally, the claim about the conormal sheaf for the closed immersion

$$Z \subset Z^1(F_n, \widehat{G}) \cong \widehat{G}^n$$

can be reduced to the case that $F_n \to W_E$ has the property that the images of the generators give a set of representatives for the quotient $W_E \to F$ in W_E, where it follows from the argument in the proof of Proposition VIII.5.12.

CHAPTER IX

The Hecke action

The time has come to put everything together. As before, let E be any nonarchimedean local field with residue field $\mathbb{F}_q$ of residue characteristic p, and let G be a reductive group over E. For any $\mathbb{Z}_\ell$-algebra Λ, we have defined $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, we have the geometric Satake equivalence relating $\widehat{G}$ to perverse sheaves on the Hecke stack, and we have studied the stack of L-parameters.

Our first task is to use the geometric Satake equivalence to define the Hecke operators on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. As in the last chapter, we work over a $\mathbb{Z}_\ell[\sqrt{q}]$-algebra Λ in order to trivialize the cyclotomic twist in the geometric Satake equivalence; let Q be a finite quotient of W_E over which the action on $\widehat{G}$ factors. If Λ is killed by a power of ℓ, then we can define Hecke operators in the following standard way. For any finite set I and $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$, we get a perverse sheaf $\mathcal{S}_V$ on $\mathcal{H}\mathrm{ck}_G^I$, which we can pull back to the global Hecke stack Hck_G^I; we denote its pullback still by $\mathcal{S}_V$. Using the correspondence

$$\begin{tikzcd} & \mathrm{Hck}_G^I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

we get the Hecke operator

$$T_V : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \to D_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda) : A \mapsto Rp_{2*}(p_1^* A \otimes_\Lambda^{\mathbb{L}} \mathcal{S}_V).$$

By Corollary IV.7.2, the target has $D_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda)$ as a full subcategory, and we will see below that T_V will factor over this subcategory. Working ∞-categorically in order to have descent, and using a little bit of condensed formalism in order to deal with W_E^I not being discrete, we can in fact rewrite

$$\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda) \cong \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$$

as the W_E^I-equivariant objects of the condensed ∞-category $\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$; we will discuss the condensed structure below.

The following theorem summarizes the properties of the Hecke operators. In particular, it asserts that these functors are defined even when Λ is not torsion.

Theorem IX.0.1 (Theorem IX.2.2; Corollary IX.2.4, Proposition IX.5.1). *For any $\mathbb{Z}_\ell[\sqrt{q}]$-algebra Λ, any finite set I, and any $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$, there is a natural Hecke operator*

$$T_V : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

(i) *Forgetting the W_E^I-action, i.e. as an endofunctor of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$, the functor T_V is functorial in $V \in \mathrm{Rep}_\Lambda \widehat{G}^I$. Moreover, T_V commutes with all limits and colimits, and preserves compact objects and universally locally acyclic objects. Letting $\mathrm{sw}^* : \mathrm{Rep}_\Lambda \widehat{G}^I \to \mathrm{Rep}_\Lambda \widehat{G}^I$ be the involution of Proposition VI.12.1, there are natural isomorphisms*

$$\mathbb{D}_{\mathrm{BZ}}(T_V(A)) \cong T_{\mathrm{sw}^* V^\vee}(\mathbb{D}_{\mathrm{BZ}}(A)) \ , \ R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(T_V(A),\Lambda) \cong T_{\mathrm{sw}^* V^\vee} R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(A,\Lambda).$$

(ii) *As a functor of V, it induces an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega)^{BW_E^I}.$$

Moreover, for any compact object $X \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega$, there is some open subgroup P of the wild inertia subgroup of W_E such that for all I and V, the P^I-action on $T_V(X)$ is trivial. In particular, one can write $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega$ as an increasing union of full stable ∞-subcategories $\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega$ such that the Hecke action defines functors

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega)^{B(W_E/P)^I}.$$

(iii) *Varying also I, the functors of (ii) are functorial in I.*

In particular, the categories $D^P_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)^\omega$ fit the bill of the discussion of Section VIII.4, so Theorem VIII.4.1 gives a construction of excursion operators. To state the outcome, we make the following definitions as in the introduction.

Definition IX.0.2.

(i) *The Bernstein center of $G(E)$ is*

$$\mathcal{Z}(G(E),\Lambda) = \pi_0\mathrm{End}(\mathrm{id}_{\mathcal{D}(G(E),\Lambda)}) = \varprojlim_{K \subset G(E)} \mathcal{Z}(\Lambda[K\backslash G(E)/K])$$

where K runs over open pro-p subgroups of $G(E)$, and $\Lambda[K\backslash G(E)/K] = \mathrm{End}_{G(E)}(c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda)$ is the Hecke algebra of level K.

(ii) *The geometric Bernstein center of G is*

$$\mathcal{Z}^{\mathrm{geom}}(G,\Lambda) = \pi_0\mathrm{End}(\mathrm{id}_{\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}).$$

Inside $\mathcal{Z}^{\mathrm{geom}}(G,\Lambda)$, we let $\mathcal{Z}^{\mathrm{geom}}_{\mathrm{Hecke}}(G,\Lambda)$ be the subring of all endomorphisms $f : \mathrm{id} \to \mathrm{id}$ commuting with Hecke operators, in the sense that for all $V \in \mathrm{Rep}(\widehat{G}^I)$ and $A \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$, one has $T_V(f(A)) = f(T_V(A)) \in \mathrm{End}(T_V(A))$.

(iii) *The spectral Bernstein center of G is*

$$\mathcal{Z}^{\mathrm{spec}}(G,\Lambda) = \mathcal{O}(Z^1(W_E,\widehat{G})_\Lambda)^{\widehat{G}},$$

the ring of global functions on $Z^1(W_E,\widehat{G})_\Lambda /\!\!/ \widehat{G}$.

The inclusion $\mathcal{D}(G(E),\Lambda) \hookrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ induces a map of algebra $\mathcal{Z}^{\mathrm{geom}}(G,\Lambda) \to \mathcal{Z}(G(E),\Lambda)$.

Corollary IX.0.3. *Assume that ℓ is invertible in Λ, or ℓ is a very good prime for $\widehat{G}$. Then there is a canonical map*

$$\mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \to \mathcal{Z}^{\mathrm{geom}}_{\mathrm{Hecke}}(G,\Lambda) \subset \mathcal{Z}^{\mathrm{geom}}(G,\Lambda),$$

and in particular a map

$$\Psi_G : \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda).$$

In general, there is such a map, up to replacing $Z^1(W_E, \widehat{G})_\Lambda /\!\!/ \widehat{G}$ *by a universally homeomorphic scheme.*

In particular, if $\Lambda = L$ is an algebraically closed field over $\mathbb{Z}_\ell[\sqrt{q}]$, we get the following construction of L-parameters.

Definition IX.0.4. *Let* L *be an algebraically closed field over* $\mathbb{Z}_\ell[\sqrt{q}]$*, and let* $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ *be a Schur-irreducible object, i.e.* $\mathrm{End}(A) = L$ *as condensed algebras. Then there is a unique semisimple* L*-parameter*

$$\varphi_A : W_E \to \widehat{G}(L) \rtimes Q$$

such that for all excursion data $(I, V, \alpha, \beta, (\gamma_i)_{i \in I})$ *consisting of a finite set* I*,* $V \in \mathrm{Rep}((\widehat{G} \rtimes Q)^I)$*,* $\alpha : 1 \to V|_{\widehat{G}}$*,* $\beta : V|_{\widehat{G}} \to 1$ *and* $\gamma_i \in W_E$ *for* $i \in I$*, the endomorphism*

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i \in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

is given by the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i))_{i \in I}} V \xrightarrow{\beta} L.$$

We can apply this in particular in the case of irreducible smooth representations π of $G(E)$. Concerning the L-parameters we construct, we can prove the following basic results. (In fact, we prove slightly finer results on the level of Bernstein centers.)

Theorem IX.0.5 (Sections IX.6, IX.7).

(i) *If* $G = T$ *is a torus, then* $\pi \mapsto \varphi_\pi$ *is the usual Langlands correspondence.*

(ii) *The correspondence* $\pi \mapsto \varphi_\pi$ *is compatible with twisting.*

(iii) *The correspondence* $\pi \mapsto \varphi_\pi$ *is compatible with central characters.*

(iv) *The correspondence* $\pi \mapsto \varphi_\pi$ *is compatible with passage to congradients.*

(v) *If* $G' \to G$ *is a map of reductive groups inducing an isomorphism of adjoint groups,* π *is an irreducible smooth representation of* $G(E)$ *and* π' *is an irreducible constitutent of* $\pi|_{G'(E)}$*, then* $\varphi_{\pi'}$ *is the image of* φ_π *under the induced map* $\widehat{G} \to \widehat{G'}$*.*

(vi) *If* $G = G_1 \times G_2$ *is a product of two groups and* π *is an irreducible smooth representation of* $G(E)$*, then* $\pi = \pi_1 \boxtimes \pi_2$ *for irreducible smooth representations* π_i *of* $G_i(E)$*, and* $\varphi_\pi = \varphi_{\pi_1} \times \varphi_{\pi_2}$ *under* $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$*.*

(vii) *If* $G = \mathrm{Res}_{E'|E} G'$ *is the Weil restriction of scalars of a reductive group* G' *over some finite separable extension* $E'|E$*, so that* $G(E) = G'(E')$*, then* L*-parameters for* $G|E$ *agree with* L*-parameters for* $G'|E'$*.*

(viii) *The correspondence* $\pi \mapsto \varphi_\pi$ *is compatible with parabolic induction.*

(ix) *For* $G = \mathrm{GL}_n$ *and supercuspidal* π*, the correspondence* $\pi \mapsto \varphi_\pi$ *agrees with the usual local Langlands correspondence* [**LRS93**], [**HT01**], [**Hen00**].

IX.1. Condensed ∞-categories

In order to meaningfully talk about W_E^I-equivariant objects in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, we need to give $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ the structure of a condensed ∞-category. This is in fact easy to do: We can associate to any extremally disconnected profinite set S the ∞-category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G \times S, \Lambda)$. This is a full condensed ∞-subcategory of the condensed ∞-category $\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$, taking any profinite S to $\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G \times S, \Lambda)$. The latter defines a hypersheaf in S, by v-hyperdescent of $\mathcal{D}_{\blacksquare}(X, \Lambda)$ (as follows from the case of $\mathcal{D}(X_v, \Lambda)$). With this definition, it becomes a direct consequence of descent that

$$\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda) \cong \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)^{BW_E^I},$$

where the latter is the evaluation of the condensed ∞-category $\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$ on the condensed anima BW_E^I. More concretely, this is the ∞-category of objects $A \in \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$ together with a map of condensed animated groups $W_E^I \to \mathrm{Aut}(A)$. We see in particular that to define $\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$, we do not need to know the full structure as a condensed ∞-category. Rather, we only need the structure as an ∞-category enriched in condensed anima. This structure on $\mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)$ induces a similar structure on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

For the discussion of Hecke operators, we observe in particular the following result, that follows directly from the discussion above.

Proposition IX.1.1. *Pullback under* $\mathrm{Bun}_G \times (\mathrm{Div}^1)^I \to \mathrm{Bun}_G \times [*/\underline{W_E^I}]$ *induces a fully faithful functor*

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I} \hookrightarrow \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G, \Lambda)^{BW_E^I} \cong \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda) \hookrightarrow \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda).$$

The essential image consists of all objects $A \in \mathcal{D}_{\blacksquare}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda)$ *whose pullback to* Bun_G *lies in* $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. $\square$

In fact, this structure of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ as an ∞-category enriched in condensed anima, in fact condensed animated Λ-modules, can be obtained in the following way from its structure as a Λ-linear stable ∞-category.

Proposition IX.1.2. *For* $A \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ *and* $B \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$*, the condensed animated* Λ*-module* $\mathrm{Hom}_{\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)}(A, B)$ *is relatively discrete over* $\mathbb{Z}_\ell$*.*

In other words, the condensed structure on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ can also be defined as the relatively discrete condensed structure when restricted to compact objects, and in general induced from this. In particular, when restricting attention to the compact objects $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$, it is simply the relatively discrete condensed structure.

Proof. Take some $b \in B(G)$ and $K \subset G_b(E)$ an open pro-p-subgroup, and let $f_K : \widetilde{\mathcal{M}}^b/\underline{K} \to \mathrm{Bun}_G$ be the local chart. We can assume $A = f_{K\natural}\mathbb{Z}_\ell$, as these form a family of generators. By adjunction, it is enough to show that for any $B' \in \mathcal{D}_{\mathrm{lis}}(\widetilde{\mathcal{M}}^b/\underline{K}, \Lambda)$, the global sections $R\Gamma(\widetilde{\mathcal{M}}^b/\underline{K}, B')$ have the relatively discrete condensed $\mathbb{Z}_\ell$-module structure. We claim that the restriction map $R\Gamma(\widetilde{\mathcal{M}}^b/\underline{K}, B'') \to R\Gamma([*/\underline{K}], B')$ is an isomorphism, where $[*/\underline{K}] \subset \widetilde{\mathcal{M}}^b/\underline{K}$ is the base point. Without the condensed structure, this was proved in the proof of Proposition VII.7.2, but actually the proof applies with condensed structure (as Theorem VII.2.10 remembers the condensed

structure). But $R\Gamma([*/\underline{K}], B')$ is a direct summand of the stalk of B' at $*$, which has the relatively discrete condensed $\mathbb{Z}_\ell$-module structure (as this is true for all objects of $\mathcal{D}_{\mathrm{lis}}(*, \Lambda)$). $\square$

IX.2. Hecke operators

The geometric Satake equivalence gives exact $\mathrm{Rep}_{\mathbb{Z}_\ell[\sqrt{q}]}(Q^I)$-linear monoidal functors

$$\mathrm{Rep}_{\mathbb{Z}_\ell[\sqrt{q}]}(\widehat{G} \rtimes Q)^I \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_G^I, \mathbb{Z}_\ell[\sqrt{q}]) : V \mapsto \mathcal{S}_V.$$

Moreover, this association is functorial in I. We can compose with the functor $A \mapsto \mathbb{D}(A)^\vee$ (where the Verdier duality is relative to the projection $\mathcal{H}\mathrm{ck}_G^I \to [(\mathrm{Div}^1)^I/L^+G]$) to get exact $\mathrm{Rep}_{\mathbb{Z}_\ell[\sqrt{q}]}(Q^I)$-linear monoidal functors

$$\mathrm{Rep}_{\mathbb{Z}_\ell[\sqrt{q}]}(\widehat{G} \rtimes Q)^I \to \mathcal{D}_\blacksquare(\mathcal{H}\mathrm{ck}_G^I, \mathbb{Z}_\ell[\sqrt{q}]),$$

functorially in I. Here, the functor $A \mapsto \mathbb{D}(A)^\vee$ is monoidal with respect to the usual convolution on perverse sheaves, and the convolution of Section VII.5 on the right. We note that as the convolution on $\mathcal{D}_\blacksquare$ makes use only of pullback, tensor product, and $\pi_\natural$-functors, all of which are defined naturally on ∞-categories, this monoidal structure is actually a monoidal structure on the ∞-category $\mathcal{D}_\blacksquare(\mathcal{H}\mathrm{ck}_G^I, \mathbb{Z}_\ell[\sqrt{q}])$. (We would have to work harder to obtain this structure when employing lower-!-functors, as we have not defined them in a sufficiently structured way.) Also, the functor from $\mathrm{Rep}_{\mathbb{Z}_\ell[\sqrt{q}]}(\widehat{G} \rtimes Q)^I$ is monoidal in this setting, as on perverse sheaves there are no higher coherences to take care of.

This extends by linearity uniquely to an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathcal{D}_\blacksquare(\mathcal{H}\mathrm{ck}_G^I, \Lambda) : V \mapsto \mathcal{S}'_V;$$

here, we implicitly use highest weight theory to show

$$\mathrm{Perf}(B(\widehat{G} \rtimes Q)^I_{\mathbb{Z}_\ell[\sqrt{q}]}) \otimes_{\mathrm{Perf}(BQ^I_{\mathbb{Z}_\ell[\sqrt{q}]})} \mathrm{Perf}(BQ^I_\Lambda) \cong \mathrm{Perf}(B(\widehat{G} \rtimes Q)^I_\Lambda),$$

and that the free stable ∞-category with an exact functor from $\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$ is $\mathrm{Perf}(B(\widehat{G} \rtimes Q)^I_\Lambda)$.

Pulling back to the global Hecke stack, we get exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functors

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathcal{D}_\blacksquare(\mathrm{Hck}_G^I, \Lambda).$$

On the other hand, there is a natural exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$$\mathcal{D}_\blacksquare(\mathrm{Hck}_G^I, \Lambda) \to \mathrm{End}_{\mathcal{D}_\blacksquare((\mathrm{Div}^1)^I, \Lambda)}(\mathcal{D}_\blacksquare(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)),$$

where the right-hand side denotes the $\mathcal{D}_\blacksquare((\mathrm{Div}^1)^I, \Lambda)$-linear endofunctors. In particular, any $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$ gives rise to a functor

$$T_V : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)$$

via

$$T_V(A) = p_{2\natural}(p_1^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \mathcal{S}'_V)$$

where we consider the usual diagram

$$\begin{tikzcd} & \mathrm{Hck}_G^I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I. \end{tikzcd}$$

Note that we have thus essentially used the translation of Proposition VII.5.2 to extend the Hecke operators from the case of torsion rings Λ to all Λ.

We note that if we pull back to the diagonal geometric point $\mathrm{Spd}\, C \to (\mathrm{Div}^1)^I$, where $C = \widehat{\overline{E}}$, then this functor depends only on the composite

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathcal{D}_\blacksquare(\mathrm{Hck}_G^I, \Lambda) \to \mathcal{D}_\blacksquare(\mathrm{Hck}_G^I \times_{(\mathrm{Div}^1)^I} \mathrm{Spd}\, C, \Lambda),$$

and this composite factors naturally over $\mathrm{Rep}_\Lambda(\widehat{G}^I)$.

Proposition IX.2.1. *For any $V \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, the functor*

$$T_V : \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda) \to \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda)$$

restricts to a functor

$$T_V : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

Proof. By highest weight theory, one can reduce to the case that V is an exterior tensor product of representations of $\widehat{G}$, and then by using that $V \mapsto T_V$ is monoidal, we can reduce to the tensor factors, which reduces us to the case $I = \{*\}$. Consider the Hecke diagram

$$\mathrm{Bun}_{G,C} \xleftarrow{h_1} \mathrm{Hck}_{G,C} \xrightarrow{h_2} \mathrm{Bun}_{G,C}$$

where $\mathrm{Hck}_{G,C}$ parametrizes over $S \in \mathrm{Perf}_C$ pairs of G-torsors $\mathcal{E}_1, \mathcal{E}_2$ on X_S together with an isomorphism over $X_S \backslash S^\sharp$ meromorphic along $S^\sharp$. It suffices to see that for all $B \in D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C/\mathrm{Div}^1_X}, \mathbb{Z}_\ell)$, the object

$$h_{2\natural}(h_1^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} q^* B^\vee) \in D_{\mathrm{lis}}(\mathrm{Bun}_{G,C}, \Lambda).$$

Now the category of such B is generated (under colimits) by the objects $Rf_{\dot{w}*}\mathbb{Z}_\ell$ for

$$f_{\dot{w}} : L^+\mathcal{I} \backslash \mathrm{Dem}_{\dot{w}} \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\, C/\mathrm{Div}^1_X}$$

a Demazure resolution (modulo action of Iwahori) of some Schubert variety in the affine flag variety. Using Proposition VII.4.3, it thus suffices to see that for the corresponding push-pull correspondence on $\mathrm{Bun}_{G,C}$ with kernel given by the Demazure resolution, one has preservation of $D_{\mathrm{lis}}(\mathrm{Bun}_{G,C}, \Lambda)$. But this is a proper and cohomologically smooth correspondence. □

Theorem IX.2.2. *For any $V \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, the action of T_V on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ preserves all limits and colimits, and the full subcategories of compact objects, and of universally locally acyclic objects. Moreover, for the automorphism sw^* of $\mathrm{Rep}_\Lambda(\widehat{G}^I)$ given by Proposition VI.12.1, there are natural isomorphisms*

$$\mathbb{D}_{\mathrm{BZ}}(T_V(A)) \cong T_{\mathrm{sw}^* V^\vee}(\mathbb{D}_{\mathrm{BZ}}(A)) \ , \ R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(T_V(A), \Lambda) \cong T_{\mathrm{sw}^* V^\vee} R\mathscr{H}\mathrm{om}_{\mathrm{lis}}(A, \Lambda).$$

Proof. The functor $V \mapsto T_V$ is monoidal. As V is dualizable in the Satake category, with dual $V^\vee$, it follows that T_V has a left and a right adjoint, given by $T_{V^\vee}$, and hence it follows formally that it preserves all limits and colimits, and compact objects. Now recall that $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is universally locally acyclic if and only if for all compact $B \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, the object $R\mathrm{Hom}_\Lambda(B, A) \in D(\Lambda)$ is perfect, by Proposition VII.7.9. Thus, the preservation of universally locally acyclic objects follows by adjointness from the preservation of compact objects.

For the duality statements, we note that, for $\pi : \mathrm{Bun}_G \to *$ the projection, there are natural isomorphisms

$$\pi_\natural(T_V(A) \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda B) \cong \pi_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda T_{\mathrm{sw}^* V}(B)),$$

as follows from the definition of the Hecke operator, and Proposition VI.12.1: Both sides identify with the homology of $\mathrm{Hck}^I_G \times_{(\mathrm{Div}^1)^I} \mathrm{Spd}\, C$ with coefficients in $h_1^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda h_2^* B \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \mathcal{S}'_V$. The displayed equation implies the statement for Bernstein–Zelevinsky duals by also using that $T_{\mathrm{sw}^* V^\vee}$ is right adjoint to $T_{\mathrm{sw}^* V}$, and the statement for naive duals by using that $T_{\mathrm{sw}^* V^\vee}$ is left adjoint to $T_{\mathrm{sw}^* V}$. $\square$

Composing Hecke operators, we get the following corollary.

Corollary IX.2.3. *For any $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$, the functor*

$$T_V : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times (\mathrm{Div}^1_X)^I, \Lambda)$$

takes image in the full subcategory $\mathcal{D}_\blacksquare(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda)$; moreover, all objects in the image have the property that their pullback to $\mathcal{D}_\blacksquare(\mathrm{Bun}_G, \Lambda)$ lies in $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, so by Proposition IX.1.1 the functor T_V induces a functor

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

Proof. We only need to see that the image lands in $\mathcal{D}_\blacksquare(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda)$; the rest follows from Proposition IX.2.1. One can reduce to the case that V is an exterior tensor product of $|I|$ representations $V_i \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)$ — one can always find a, possibly infinite, resolution by such exterior tensor products that involves only finitely many weights of $\widehat{G}^I$, and thus induces a resolution in $\mathcal{D}_\blacksquare(\mathcal{H}\mathrm{ck}^I_G, \Lambda)$ — and thus reduce to $I = \{*\}$. By Corollary VII.2.7, it suffices to see that the pullback to $\mathcal{D}_\blacksquare(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda)$ lies in $\mathcal{D}_\blacksquare(\mathrm{Bun}_G, \Lambda)$. But by Proposition IX.2.1, we know that it lies in $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda)$, and $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda)$ is an equivalence by Proposition VII.7.3. $\square$

Finally, we get the following Hecke action.

Corollary IX.2.4. *Endowing the stable $\mathbb{Z}_\ell$-linear ∞-category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ with the relatively discrete condensed structure, the Hecke action defines exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functors*

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega)^{BW_E^I},$$

functorially in I.

IX.3. Cohomology of local Shimura varieties

Theorem IX.2.2 encodes strong finiteness properties for the cohomology of local Shimura varieties, giving unconditional proofs, and refinements, of the results of [**RV14**, Section 6]. For this, we first specialize to $E = \mathbb{Q}_p$ as this is the standard setting of local Shimura varieties. Consider any local Shimura datum, consisting of a reductive group G over $\mathbb{Q}_p$, a conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ with field of definition $E|\mathbb{Q}_p$ and some element $b \in B(G,\mu) \subset B(G)$. (Beware that we are making a small sin here in changing the meaning of the letter E, using it now in its usual meaning as a reflex field.) In [**SW20**, Lecture 24], we construct a tower of partially proper smooth rigid-analytic spaces

$$(\mathcal{M}_{(G,b,\mu),K})_{K \subset G(\mathbb{Q}_p)}$$

over $\breve{E}$, equipped with a Weil descent datum. Each object in the tower carries an action of $G_b(\mathbb{Q}_p)$, and the tower carries an action of $G(\mathbb{Q}_p)$. Following Huber [**Hub98**], one defines

$$R\Gamma_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell) = \varinjlim_U R\Gamma_c(U, \mathbb{Z}_\ell)$$

where $U \subset \mathcal{M}_{(G,b,\mu),K,C}$ runs through quasicompact open subsets, and one defines $R\Gamma_c(U, \mathbb{Z}_\ell) = \varprojlim_m R\Gamma_c(U, \mathbb{Z}/\ell^m\mathbb{Z})$. This carries an action of $G_b(\mathbb{Q}_p)$ as well as an action of the Weil group W_E.

Theorem IX.3.1. *The complex $R\Gamma_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell)$ is naturally a complex of smooth $G_b(\mathbb{Q}_p)$-representations, and, if K is pro-p, a compact object in $D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell)$. Moreover, the action of W_E is continuous.*

In particular, each $H^i_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell)$ is a finitely generated smooth $G_b(\mathbb{Q}_p)$-representation. By descent, this is true even for all K (not necessarily pro-p).

Proof. Let $f_K : \mathcal{M}_{(G,b,\mu),K,C} \to \operatorname{Spa} C$ be the projection. Up to shift, we can replace $\mathbb{Z}_\ell$ by the dualizing complex $Rf_K^! \mathbb{Z}_\ell$. Now by Proposition VII.5.2, one has

$$Rf_{K!} Rf_K^! \mathbb{Z}_\ell|_U \cong f_{K\natural} \mathbb{Z}_\ell|_U$$

for any quasicompact $U \subset \mathcal{M}_{(G,b,\mu),K,C}$. As the left-hand side is perfect, it is given by its limit over reductions modulo ℓ^m. We see that $H^i_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell)$ can be identified with $H^i(f_{K\natural}\mathbb{Z}_\ell)$ up to shift.

Now μ gives rise to a Hecke operator $T_\mu = T_{V_\mu}$ where V_μ is the highest weight representation of weight μ. It corresponds to the Hecke correspondence on $\mathrm{Bun}_{G,C}$ parametrizing modifications of type μ; this Hecke correspondence is proper and smooth over both factors. We apply T_μ to the compact object

$$A = j_! c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)} \mathbb{Z}_\ell \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$$

where $j : \mathrm{Bun}_G^1 \cong [*/\underline{G(\mathbb{Q}_p)}] \hookrightarrow \mathrm{Bun}_G$ is the open immersion. By Theorem IX.2.2, also $T_\mu(A)$ is compact. By Proposition VII.7.4, it follows that also $i^{b*}T_\mu(A) \in D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \mathbb{Z}_\ell) \cong D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell)$ is compact. But this is, up to shift again, precisely $f_{K\natural}\mathbb{Z}_\ell$, by the identification of $\mathcal{M}_{(G,b,\mu),K,C}$ with the space of modifications of G-torsors of type μ from the G-bundle $\mathcal{E}_b$ to the G-bundle $\mathcal{E}_1$, up to the action of K (cf. [**SW20**, Lecture 23, 24]).

Descending to E, note that T_μ can be defined with values in $D_\blacksquare(\mathrm{Bun}_G \times \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}, \mathbb{Z}_\ell)$, and takes values in those sheaves whose pullback to $\mathrm{Bun}_{G,C}$ lies in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$. Thus W_E, as a condensed group, acts on $i^{b*}T_\mu(A) \in D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \mathbb{Z}_\ell) \cong D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell)$ considered as representations on condensed $\mathbb{Z}_\ell$-modules. In classical language, this means that the action is continuous. $\square$

In particular, for each admissible representation ρ of $G_b(\mathbb{Q}_p)$ on a $\mathbb{Z}_\ell$-algebra Λ, the complex

$$R\,\mathrm{Hom}_{G_b(\mathbb{Q}_p)}(R\Gamma_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell), \rho)$$

is a perfect complex of Λ-modules. Passing to the colimit over K, one obtains at least on each cohomology group an admissible $G(\mathbb{Q}_p)$-representation. In fact, as T_μ is left adjoint to $T_{\mu^\vee}$, we see that this is (up to shift) given by

$$i^{1*}T_{\mu^\vee}(Ri^b_*[\rho]) \in D_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda) \cong D(G(\mathbb{Q}_p), \Lambda).$$

Here $i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ is the inclusion, and $[\rho] \in D(G_b(\mathbb{Q}_p), \Lambda) \cong D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)$ can be a complex of smooth $G_b(\mathbb{Q}_p)$-representations. This shows in particular that there is in fact a natural complex of admissible $G(\mathbb{Q}_p)$-representations underlying

$$R\,\mathrm{Hom}_{G_b(\mathbb{Q}_p)}(R\Gamma_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell), \rho).$$

Assuming again that ρ is admissible, one can pull through Verdier duality,

$$\begin{aligned} i^{1*}T_{\mu^\vee}(Ri^b_*[\rho]) &\cong i^{1*}T_{\mu^\vee}(Ri^b_*\mathbb{D}([\rho^\vee])) \\ &\cong i^{1*}T_{\mu^\vee}(\mathbb{D}(i^b_![\rho^\vee])) \\ &\cong i^{1*}\mathbb{D}(T_{\mathrm{sw}^*\mu}(i^b_![\rho^\vee])) \\ &\cong \mathbb{D}(i^{1*}T_{\mathrm{sw}^*\mu}(i^b_![\rho^\vee])). \end{aligned}$$

As $T_{\mathrm{sw}^*\mu}$ also preserves compact objects, it follows that [**RV14**, Remark 6.2 (iii)] has a positive answer: If $\Lambda = \overline{\mathbb{Q}}_\ell$ and ρ has finite length, then also each cohomology group of

$$R\,\mathrm{Hom}_{G_b(\mathbb{Q}_p)}(R\Gamma_c(\mathcal{M}_{(G,b,\mu),K,C}, \mathbb{Z}_\ell), \rho)$$

has finite length. Indeed, with $\overline{\mathbb{Q}}_\ell$-coefficients, the category of smooth representations has finite global dimension, and hence being compact is equivalent to each cohomology group being finitely generated. Compact objects are preserved under the Hecke operators, and so we see that each cohomology group is finitely generated. Being also admissible, it is then of finite length by Howe's theorem [**Ren10**, VI.6.3].

The same arguments apply to prove Corollary I.7.3. Let us recall the setup. We start with a general E now. As in [**SW20**, Lecture XXIII], for any collection $\{\mu_i\}_i$ of conjugacy classes of cocharacters with fields of definition E_i/E and $b \in B(G)$, there is a tower of moduli spaces of local shtukas

$$f_K : (\mathrm{Sht}_{(G,b,\mu_\bullet),K})_{K \subset G(E)} \to \prod_{i \in I} \mathrm{Spd}\, \breve{E}_i$$

as K ranges over compact open subgroups of $G(E)$, equipped with compatible étale period maps

$$\pi_K : \mathrm{Sht}_{(G,b,\mu_\bullet),K} \to \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i,\leq\mu_\bullet}.$$

Here, $\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i} \to \prod_{i\in I}\mathrm{Spd}\,\breve{E}$ is a certain twisted form of the convolution affine Grassmannian, cf. [**SW20**, Section 23.5]. Let W be the exterior tensor product $\boxtimes_{i\in I}V_{\mu_i}$ of highest weight representations, and $\mathcal{S}_W$ the corresponding sheaf on $\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i}$. More precisely, away from Frobenius-twisted partial diagonals, $\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i}$ is isomorphic to the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i}$, and we have defined $\mathcal{S}_W$ on this locus. One can uniquely extend over these Frobenius-twisted partial diagonals to universally locally acyclic, necessarily perverse, sheaves, as in the discussion of the fusion product. We continue to write $\mathcal{S}_W$ for its pullback to $\mathrm{Sht}_{(G,b,\mu_\bullet),K}$. Let $\mathcal{S}'_W = \mathbb{D}(\mathcal{S}_W)^\vee$ be the corresponding solid sheaf. By Proposition VII.5.2, with torsion coefficients $f_{K\natural}\mathcal{S}'_W$ agrees with $Rf_{K!}\mathcal{S}_W$, but $f_{K\natural}\mathcal{S}'_W$ is well-defined in general.

Proposition IX.3.2. *The sheaf*

$$f_{K\natural}\mathcal{S}'_W \in \mathcal{D}_\blacksquare([*/\underline{G_b(E)}] \times \prod_{i\in I}\mathrm{Spd}\,\breve{E}_i, \Lambda)$$

is equipped with partial Frobenii, thus descends to an object of

$$\mathcal{D}_\blacksquare([*/\underline{G_b(E)}] \times \prod_{i\in I}\mathrm{Spd}\,\breve{E}_i/\varphi_i^{\mathbb{Z}}, \Lambda).$$

This object lives in the full ∞-subcategory

$$\mathcal{D}(G_b(E),\Lambda)^{B\prod_{i\in I}W_{E_i}} \subset \mathcal{D}_\blacksquare([*/\underline{G_b(E))}] \times \prod_{i\in I}\mathrm{Spd}\,\breve{E}_i/\varphi_i^{\mathbb{Z}}, \Lambda),$$

and its restriction to $\mathcal{D}(G_b(E),\Lambda)$ is compact. In particular, for any admissible representation ρ of $G_b(E)$, the object

$$R\,\mathrm{Hom}_{G_b(E)}(f_{K\natural}\mathcal{S}'_W, \rho) \in \mathcal{D}(\Lambda)^{B\prod_{i\in I}W_{E_i}}$$

is a representation of $\prod_{i\in I}W_{E_i}$ on a perfect complex of Λ-modules. Taking the colimit over K, this gives rise to a complex of admissible $G(E)$-representations

$$\varinjlim_K R\,\mathrm{Hom}_{G_b(E)}(f_{K\natural}\mathcal{S}'_W, \rho)$$

equipped with a $\prod_{i\in I}W_{E_i}$-action.

If ρ is compact, then so is

$$\varinjlim_K R\,\mathrm{Hom}_{G_b(E)}(f_{K\natural}\mathcal{S}'_W, \rho)$$

as a complex of $G(E)$-representations.

Proof. The key observation is that $f_{K\natural}\mathcal{S}'_W$ can be identified with $T_W(j_!{[c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda]})|_{\mathrm{Bun}_G^b}$. A priori, for the latter, we have to look at the moduli space $\mathcal{M}$ of modifications of type bounded by $\mu_\bullet$ from $\mathcal{E}_b$ to the trivial vector bundle, up to the action of K, and take the homology of $\mathcal{M}$ with coefficients in $\mathcal{S}'_W$; more precisely, the relative homology of $\mathcal{M} \to \prod_{i\in I}\mathrm{Spd}\,\breve{E}_i/\varphi_i^{\mathbb{Z}}$. After pull back to $\prod_{i\in I}\mathrm{Spd}\,\breve{E}_i$, there is a natural map from $\mathcal{M}$ to $\mathrm{Sht}_{(G,b,\mu_\bullet,K)}$ that is an isomorphism away from Frobenius-twisted partial diagonals. Indeed, $\mathrm{Sht}_{(G,b,\mu_\bullet,K)}$ parametrizes G-torsors over Y_S together with an isomorphism with their Frobenius pullback away from the given points, together with a level-K-trivialization of the G-bundle near $\{\pi = 0\}$. This induces two vector bundles on X_S, given by the bundles near $\{\pi = 0\}$ and near $\{[\varpi] = 0\}$, and these are identified away from the images of

the punctures in X_S. As long as their images in X_S are disjoint, one can reverse this procedure. Now the fusion compatibility of $\mathcal{S}_W$ (and thus $\mathcal{S}'_W$) implies the desired result.

In particular, this shows that $f_{K\natural}\mathcal{S}'_W$ admits natural partial Frobenius operators. The rest of the proof is now as before. □

IX.4. L-parameter

We can now define L-parameter.

Definition/Proposition IX.4.1. *Let L be an algebraically closed field over $\mathbb{Z}_\ell[\sqrt{q}]$, and let $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ be a Schur-irreducible object, i.e. $\mathrm{End}(A) = L$ as condensed algebras. Then there is a unique semisimple L-parameter*

$$\varphi_A : W_E \to \widehat{G}(L) \rtimes Q$$

such that for all excursion data $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ consisting of a finite set I, $V \in \mathrm{Rep}((\widehat{G} \rtimes Q)^I)$, $\alpha : 1 \to V|_{\widehat{G}}$, $\beta : V|_{\widehat{G}} \to 1$ and $\gamma_i \in W_E$ for $i \in I$, the endomorphism

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

is given by the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i))_{i\in I}} V \xrightarrow{\beta} L.$$

Proof. By the arguments of Section VIII.4, we can build excursion data as required for Proposition VIII.3.8. □

IX.5. The Bernstein center

As before, there is the problem that the stack $Z^1(W_E, \widehat{G})$ of L-parameters is not quasicompact, but an infinite disjoint union. We can now actually decompose $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ into a direct product according to the connected components of $Z^1(W_E, \widehat{G})$. We start with the following observation.

Proposition IX.5.1. *Let $A \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ be any compact object. Then there is an open subgroup $P \subset W_E$ of the wild inertia subgroup such that for all finite sets I and all $V \in \mathrm{Rep}((\widehat{G} \rtimes Q)^I)$, the object*

$$T_V(A) \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$$

lies in the full ∞-subcategory

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{B(W_E/P)^I} \subset \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

Proof. First, note that indeed the functor

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{B(W_E/P)^I} \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

is fully faithful; this follows from fully faithfulness of the pullback functor

$$f^* : \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times [*/\underline{(W_E/P)^I}], \Lambda) \to \mathcal{D}_\blacksquare(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda),$$

which in turn follows from $f_\natural\Lambda \cong \Lambda$ (and the projection formula for $f_\natural$), which can be deduced via base change from the case of $[*/\underline{W_E^I}] \to [*/\underline{(W_E/P)^I}]$, or after pullback to a v-cover $\mathrm{Spa}\, C \to$

$[*/\underline{(W_E/P)^I}]$, for $[\mathrm{Spa}\, C/\underline{P^I}] \to \mathrm{Spa}\, C$, where it amounts to the vanishing of the Λ-homology of P^I.

Now note that if P^I acts trivially on $T_V(A)$ and on $T_W(A)$ for two $V, W \in \mathrm{Rep}_{\mathbb{Z}_\ell}((\widehat{G} \rtimes Q)^I)$, then it also acts trivially on $T_{V \otimes W}(A) = T_V(T_W(A)) = T_W(T_V(A))$: Indeed, the $W_E^{I \sqcup I}$-action on $T_V(T_W(A)) \cong T_{V \boxtimes W}(A) \cong T_W(T_V(A))$ is trivial on $P^{I \sqcup \emptyset}$ and $P^{\emptyset \sqcup I}$, thus on $P^{I \sqcup I}$, and hence the diagonal W_E^I-action is trivial on P^I. Using reductions to exterior tensor products, we can also reduce to $I = \{*\}$. Then if $V \in \mathrm{Rep}_{\mathbb{Z}_\ell}(\widehat{G} \rtimes Q)$ is a $\otimes$-generator, it follows that it suffices that P acts trivially on $T_V(A)$. But

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_E} = \bigcup_P \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{B(W_E/P)}$$

as for any relatively discrete condensed animated $\mathbb{Z}_\ell$-module M with a map $W_E \to \mathrm{Aut}(M)$, the map factors over W_E/P for some P. Indeed, one can reduce to M concentrated in degree 0, and then to M finitely generated, so that $\mathrm{Aut}(M)$ is profinite, and locally pro-ℓ. $\square$

Fix some open subgroup P of the wild inertia subgroup of W_E, and let

$$\mathcal{D}_{\mathrm{lis}}^P(\mathrm{Bun}_G, \Lambda)^\omega \subset \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$$

be the full ∞-subcategory of all A such that P^I acts trivially on $T_V(A)$ for all $V \in \mathrm{Rep}((\widehat{G} \rtimes Q)^I)$. Pick $W \subset W_E/P$ a discrete dense subgroup, by discretizing the tame inertia, as before. Then Theorem VIII.4.1 gives a canonical map of algebras

$$\mathrm{Exc}(W, \widehat{G}) \to \mathcal{Z}(\mathcal{D}_{\mathrm{lis}}^P(\mathrm{Bun}_G, \Lambda)^\omega) = \pi_0 \mathrm{End}(\mathrm{id}_{\mathcal{D}_{\mathrm{lis}}^P(\mathrm{Bun}_G, \Lambda)^\omega}).$$

As $\mathrm{Exc}(W, \widehat{G}) \otimes \Lambda \to \mathcal{O}(Z^1(W_E/P, \widehat{G})_\Lambda)^{\widehat{G}}$ is a universal homeomorphism, there are in particular idempotents corresponding to the connected components of $Z^1(W_E/P, \widehat{G})_\Lambda$. Their action on $\mathcal{D}_{\mathrm{lis}}^P(\mathrm{Bun}_G, \Lambda)^\omega$ then induces a direct sum decomposition

$$\mathcal{D}_{\mathrm{lis}}^P(\mathrm{Bun}_G, \Lambda)^\omega = \bigoplus_{c \in \pi_0 Z^1(W_E/P, \widehat{G})_\Lambda} \mathcal{D}_{\mathrm{lis}}^c(\mathrm{Bun}_G, \Lambda)^\omega.$$

Taking now a union over all P, we get a direct sum decomposition

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega = \bigoplus_{c \in \pi_0 Z^1(W_E, \widehat{G})_\Lambda} \mathcal{D}_{\mathrm{lis}}^c(\mathrm{Bun}_G, \Lambda)^\omega.$$

On the level of Ind-categories, this gives a direct product

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) = \prod_{c \in \pi_0 Z^1(W_E, \widehat{G})_\Lambda} \mathcal{D}_{\mathrm{lis}}^c(\mathrm{Bun}_G, \Lambda).$$

Note in particular that any Schur-irreducible object $A \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ necessarily lies in one of these factors, given by some connected component c of $Z^1(W_E, \widehat{G})_\Lambda$; and then the L-parameter φ_A of A necessarily lies in this connected component.

Using excursion operators, we get the following result on the "Bernstein center". Note that for $G = \mathrm{GL}_n$, all ℓ are allowed, so this is a generalization of results of Helm–Moss, [**HM18**], noting that by the fully faithful functor $\mathcal{D}(G(E), \Lambda) \hookrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, there is a map of algebras

$$\mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda)$$

to the usual Bernstein center of smooth $G(E)$-representations on Λ-modules.

THEOREM IX.5.2. *Assume that ℓ is invertible in Λ, or that ℓ is a very good prime for $\widehat{G}$. Then there is a natural map*

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}^{\mathrm{geom}}(G, \Lambda)$$

compatible with the above decomposition into connected components. Moreover, for all finite sets I, all $V \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, and all $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, the diagram

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \arrow[r] \arrow[dr] & \mathrm{End}(A) \arrow[d] \\ & \mathrm{End}(T_V(A)) \end{tikzcd}$$

commutes, so the map factors over $\mathcal{Z}^{\mathrm{geom}}_{\mathrm{Hecke}}(G, \Lambda) \subset \mathcal{Z}^{\mathrm{geom}}(G, \Lambda)$.

Of course, as noted above, in general we still have maps

$$\mathrm{Exc}(W, \widehat{G}) \to \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega)$$

and $\mathrm{Exc}(W, \widehat{G}) \to \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ is a universal homeomorphism. The commutation with the Hecke action is true here as well.

PROOF. This follows from the decomposition into connected components, the map $\mathrm{Exc}(W, \widehat{G})_\Lambda \to \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega)$ above, and Theorem VIII.3.6. The statement about commutation with Hecke operators follows from the construction of excursion operators and the commutation of Hecke operators. □

Before going on, we make the following observation regarding duality. The Bernstein–Zelevinsky duality functor $\mathbb{D}_{\mathrm{BZ}}$ on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ induces an involution D^{geom} of $\mathcal{Z}^{\mathrm{geom}}(G, \Lambda)$. On the other hand, on $Z^1(W_E, \widehat{G})$, the Cartan involution of $\widehat{G}$ induces an involution; after passing to the quotient by the conjugation action of $\widehat{G}$, we can also forget about the inner automorphism appearing in Proposition VI.12.1. Let D^{spec} be the induced involution of $\mathcal{Z}^{\mathrm{spec}}(G, \Lambda)$.

PROPOSITION IX.5.3. *Assume that ℓ is very good for $\widehat{G}$, or ℓ is invertible in Λ. Then the diagram*

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \arrow[r] \arrow[d, "D^{\mathrm{spec}}"] & \mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \arrow[d, "D^{\mathrm{geom}}"] \\ \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \arrow[r] & \mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \end{tikzcd}$$

commutes.

The formation of L-parameters for irreducible smooth representations of $G(E)$ is compatible with passage to Bernstein–Zelevinsky duals, and to smooth duals.

Proof. The commutation follows easily from the construction of excursion operators and Proposition VI.12.1. For the final part, it now follows that the formation of L-parameters is compatible with passage to Bernstein–Zelevinsky duals. For supercuspidal representations, this agrees with the smooth dual. In general, the claim for smooth duals follows from the compatibility with parabolic induction proved below. $\square$

IX.6. Properties of the correspondence

As usual, we fix an open subgroup P of the wild inertia of W_E (mapping trivially to Q), and a discretization $W \subset W_E/P$.

IX.6.1. Isogenies.

Theorem IX.6.1. *Let $G' \to G$ be a map of reductive groups inducing an isomorphism of adjoint groups, inducing a dual map $\widehat{G} \to \widehat{G'}$, and $\pi : \mathrm{Bun}_{G'} \to \mathrm{Bun}_G$. Then for any $A \in D^P_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ the diagram*

$$\begin{tikzcd} \mathrm{Exc}(W, \widehat{G'})_\Lambda \arrow[r] \arrow[d] & \mathrm{End}(\pi^* A) \\ \mathrm{Exc}(W, \widehat{G})_\Lambda \arrow[r] & \mathrm{End}(A) \arrow[u] \end{tikzcd}$$

commutes. In particular, if L is an algebraically closed field, A is Schur-irreducible and A' is a Schur-irreducible constituent of $\pi^* A$, then $\varphi_{A'}$ is the composite of φ_A with $\widehat{G} \to \widehat{G'}$. Moreover, if either ℓ is invertible in Λ or ℓ is very good for $\widehat{G}$ and $\widehat{G'}$, the diagram

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(\widehat{G'}, \Lambda) \arrow[r] \arrow[d] & \mathrm{End}(\pi^* A) \\ \mathcal{Z}^{\mathrm{spec}}(\widehat{G}, \Lambda) \arrow[r] & \mathrm{End}(A) \arrow[u] \end{tikzcd}$$

commutes.

Proof. Consider any excursion operator for G', given by some finite set I, a representation $V' \in \mathrm{Rep}_\Lambda((\widehat{G'} \rtimes Q)^I)$, maps $\alpha : 1 \to V'|_{\widehat{G'}}$, $\beta : V'|_{\widehat{G'}} \to 1$ and elements $\gamma_i \in \Gamma$ as usual. Consider the diagram

$$\begin{tikzcd} \mathrm{Bun}_{G'} \arrow[d, "\pi"] & \mathrm{Hck}^I_{G'} \arrow[l, "h'_1"'] \arrow[r, "h'_2"] \arrow[d, "\pi_H"] & \mathrm{Bun}_{G'} \times (\mathrm{Div}^1)^I \arrow[d, "\pi"] \\ \mathrm{Bun}_G & \mathrm{Hck}^I_G \arrow[l, "h_1"'] \arrow[r, "h_2"] & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I. \end{tikzcd}$$

Then

$$T_{V'}(\pi^* A) = h'_{2\natural}(h_1'^* \pi^* A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_\Lambda \mathcal{S}'_{V'}).$$

We are interested in computing an endomorphism of π^*A; in particular, it is enough to compute $\pi_\natural T_{V'}(\pi^*A)$. But

$$\begin{aligned}\pi_\natural T_{V'}(\pi^*A) &= \pi_\natural h'_{2\natural}(h_1'^*\pi^*A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}\mathcal{S}'_{V'})\\ &\cong h_{2\natural}\pi_{\mathcal{H}\natural}(\pi_H^*h_1^*A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}\mathcal{S}_{V'})\\ &\cong h_{2\natural}(h_1^*A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}\pi_{H\natural}\mathcal{S}'_{V'})\\ &\cong h_{2\natural}(h_1^*A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}\mathcal{S}'_{V}) = T_V(A).\end{aligned}$$

This identification is functorial in V' and I, and is over $\mathrm{Bun}_G \times (\mathrm{Div}^1)^I$, hence implies the desired equality of excursion operators. Here, to identify $\pi_{H\natural}\mathcal{S}_{V'}$, we use that the diagram

$$\begin{tikzcd}\mathrm{Hck}^I_{G'} \arrow[r] \arrow[d, "\pi_H"] & \mathcal{H}\mathrm{ck}^I_{G'} \arrow[d]\\ \mathrm{Hck}^I_G \arrow[r] & \mathcal{H}\mathrm{ck}^I_G\end{tikzcd}$$

is cartesian, and the compatibility of the geometric Satake equivalence with the map $G \to G'$ inducing isomorphisms of adjoint groups, as in the proof of Theorem VI.11.1. □

IX.6.2. Products.

Proposition IX.6.2. *If $G = G_1 \times G_2$ is a product of two groups, then the diagram*

$$\begin{tikzcd}\mathrm{Exc}(W, \widehat{G_1})_\Lambda \otimes_\Lambda \mathrm{Exc}(W, \widehat{G_2})_\Lambda \arrow[r] \arrow[d, "\cong"] & \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_{G_1}, \Lambda)) \otimes_\Lambda \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_{G_2}, \Lambda)) \arrow[d]\\ \mathrm{Exc}(W, \widehat{G}) \arrow[r] & \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda))\end{tikzcd}$$

commutes.

In particular, if $\Lambda = L$ is an algebraically closed field and $A_1, A_2 \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ are Schur-irreducible, and A is a Schur-irreducible constituent of $A_1 \boxtimes A_2$, then

$$\varphi_A = (\varphi_{A_1}, \varphi_{A_2}) : W_E \to \widehat{G}(L) \cong \widehat{G_1}(L) \times \widehat{G_2}(L).$$

If ℓ is invertible in Λ or is very good for $\widehat{G_1}$ and $\widehat{G_2}$, then the diagram

$$\begin{tikzcd}\mathcal{Z}^{\mathrm{spec}}(G_1, \Lambda) \otimes_\Lambda \mathcal{Z}^{\mathrm{spec}}(G_2, \Lambda) \arrow[r] \arrow[d, "\cong"] & \mathcal{Z}^{\mathrm{geom}}(G_1, \Lambda) \otimes_\Lambda \mathcal{Z}^{\mathrm{geom}}(G_1, \Lambda) \arrow[d]\\ \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \arrow[r] & \mathcal{Z}^{\mathrm{geom}}(G, \Lambda)\end{tikzcd}$$

commutes.

Proof. The statement can be checked using excursion operators, and the proof is a straightforward diagram chase, noting that everything decomposes into products. □

IX.6.3. Weil restriction.

PROPOSITION IX.6.3. *If $G = \mathrm{Res}_{E'|E} G'$ is a Weil restriction of scalars of some reductive group G' over some finite separable extension E' of E. Choose P to be an open subgroup of the wild inertia of $W_{E'} \subset W_E$, and let $W' \subset W_{E'}/P$ be the preimage of $W \subset W_E/P$. Then there are canonical identifications $\mathrm{Bun}_{G'} \cong \mathrm{Bun}_G$, $Z^1(W_E, \widehat{G}) \cong Z^1(W_{E'}, \widehat{G'})$ and $\mathrm{Exc}(W, \widehat{G}) \cong \mathrm{Exc}(W', \widehat{G'})$, and the diagram*

$$\begin{tikzcd} \mathrm{Exc}(W', \widehat{G'}) \arrow[r] \arrow[d, "\cong"] & \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_{G'}, \Lambda)) \arrow[d, "\cong"] \\ \mathrm{Exc}(W, \widehat{G}) \arrow[r] & \mathcal{Z}(\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)) \end{tikzcd}$$

commutes. In particular, L-parameters are compatible with Weil restriction. If ℓ is invertible in Λ or is very good for $\widehat{G}$ and $\widehat{G'}$, then the diagram

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(G', \Lambda) \arrow[r] \arrow[d, "\cong"] & \mathcal{Z}^{\mathrm{geom}}(G', \Lambda) \arrow[d, "\cong"] \\ \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \arrow[r] & \mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \end{tikzcd}$$

commutes.

PROOF. The most nontrivial of these identifications is the identification

$$\mathrm{Exc}(W, \widehat{G}) \cong \mathrm{Exc}(W', \widehat{G'}).$$

One way to understand this is to use the presentation

$$\mathrm{Exc}(W, \widehat{G}) = \mathrm{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}}$$

(and the similar presentation for $\mathrm{Exc}(W', \widehat{G'})$) and the natural isomorphism $Z^1(F_n, \widehat{G}) /\!\!/ \widehat{G} \cong Z^1(F_n \times_W W', \widehat{G'}) /\!\!/ \widehat{G'}$ of affine schemes (and then passing to global sections), noting that $F_n \times_W W' \subset F_n$ is a subgroup of finite index, and thus itself a finitely generated free group. This shows in fact that restricting to those maps $F_n \to W$ factoring over W' produces the same colimit, and so

$$\begin{aligned} \mathrm{Exc}(W, \widehat{G}) &= \mathrm{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \\ &\xleftarrow{\sim} \mathrm{colim}_{(n, F_n \to W')} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}} \\ &\cong \mathrm{colim}_{(n, F_n \to W')} \mathcal{O}(Z^1(F_n, \widehat{G'}))^{\widehat{G'}} \\ &= \mathrm{Exc}(W', \widehat{G'}). \end{aligned}$$

Now consider an excursion operator for G', including a representation V' of $(\widehat{G'} \rtimes W_{E'})^I$. Note that $\widehat{G} \rtimes W_E$ contains $\widehat{G} \rtimes W_{E'}$ as a subgroup, and this admits a surjection onto $\widehat{G'} \rtimes W_{E'}$ (noting that $\widehat{G} = \prod_{E' \hookrightarrow \overline{E}} \widehat{G'}$, where we picked out an embedding $E' \hookrightarrow \overline{E}$ and hence a projection $\widehat{G} \to \widehat{G'}$ when we regarded $W_{E'} \subset W_E$ as a subgroup). In this way, one can inflate V' to a representation of

$(\widehat{G} \rtimes W_{E'})^I$ and then induce to $(\widehat{G} \rtimes W_E)^I$ to get a representation V of $(\widehat{G} \rtimes W_E)^I$. Geometrically, this procedure amounts to the commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_{G'} \arrow[d, "\cong"] & \mathrm{Hck}^I_{G'} \arrow[l, "h'_1"'] \arrow[r, "h'_2"] \arrow[d, "\psi"] & \mathrm{Bun}_{G'} \times (\mathrm{Div}'^1)^I \arrow[d] \\ \mathrm{Bun}_G & \mathrm{Hck}^I_G \arrow[l, "h_1"'] \arrow[r, "h_2"] & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

and taking ψ_* on sheaves. More precisely, we note that

$$\mathrm{Hck}^I_{G'} \to \mathrm{Hck}^I_G \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}'^1)^I$$

is a closed immersion (and arises via pullback from a similar closed immersion of local Hecke stacks). Now the claim follows from a diagram chase. □

IX.6.4. Tori. If $G = T$ is a torus, then

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda) \cong \prod_{b \in B(T) = \pi_1(T)_\Gamma} \mathcal{D}(T(E), \Lambda)$$

and in particular

$$\mathcal{Z}^{\mathrm{geom}}(T, \Lambda) = \prod_{b \in B(T)} \mathcal{Z}(T(E), \Lambda)$$

where $\mathcal{Z}(T(E), \Lambda)$ is the Bernstein center of $T(E)$; explicitly, this is

$$\mathcal{Z}(T(E), \Lambda) = \varprojlim_{K \subset T(E)} \Lambda[T(E)/K]$$

where K runs over open subgroups of $T(E)$.

For tori, the representation theory of $\widehat{T}$ is semisimple even integrally, so we get a map

$$\mathcal{Z}^{\mathrm{spec}}(T, \Lambda) \to \mathcal{Z}^{\mathrm{geom}}(T, \Lambda).$$

Proposition IX.6.4. *There is a natural isomorphism*

$$\mathcal{Z}^{\mathrm{spec}}(T, \Lambda) \cong \varprojlim_{K \subset T(E)} \Lambda[T(E)/K].$$

Proof. One can resolve T by products of induced tori and then reduce to the case that T is induced, and then by Weil restrictions of scalars to $T = \mathbb{G}_m$. In that case $Z^1(W_E, \mathbb{G}_m) = \mathrm{Hom}(E^*, \mathbb{G}_m)$ by local class field theory, giving the result. □

Proposition IX.6.5. *Under the above identifications*

$$\mathcal{Z}^{\mathrm{spec}}(T, \Lambda) = \varprojlim_{K \subset T(E)} \Lambda[T(E)/K]$$

and

$$\mathcal{Z}^{\mathrm{geom}}(T, \Lambda) = \prod_{b \in B(T)} \varprojlim_{K \subset T(E)} \Lambda[T(E)/K],$$

the map

$$\mathcal{Z}^{\mathrm{spec}}(T, \Lambda) \to \mathcal{Z}^{\mathrm{geom}}(T, \Lambda)$$

is the diagonal embedding.

Proof. We may resolve T by induced tori and use Theorem IX.6.1, Proposition IX.6.2 and Proposition IX.6.3 to reduce to the case of $T = \mathbb{G}_m$. It is enough to compute the excursion operators corresponding to $I = \{1, 2\}$, $V = \mathrm{std} \boxtimes \mathrm{std}^\vee$ and the tautological maps $\alpha : 1 \to \mathrm{std} \otimes \mathrm{std}^\vee$ and $\beta : \mathrm{std} \otimes \mathrm{std}^\vee \to 1$. It is then an easy consequence of Section II.2.1. $\square$

Proposition IX.6.5 in particular shows that the L-parameters we construct for tori are the usual L-parameters, and together with Theorem IX.6.1 and Proposition IX.6.2 implies that L-parameters are compatible with central characters and twisting, by applying Theorem IX.6.1 to the maps $Z \times G \to G$ and $G \to G \times D$ where $Z \subset G$ is the center and $G \to D$ is the quotient by the derived group.

IX.7. Applications to representations of $G(E)$

Finally, we apply the preceding results to representations of $G(E)$. To simplify the statements, we assume from now on that either ℓ is invertible in Λ, or that ℓ is very good for $\widehat{G}$; in general, similar statements hold true up to replacing $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ by the algebra $\mathrm{Exc}(W, \widehat{G})$ as above.

We get the following map to the Bernstein center.

Definition IX.7.1. *The map*

$$\Psi_G : \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda)$$

is the composite

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}^{\mathrm{geom}}(G, \Lambda) \to \mathcal{Z}(G(E), \Lambda)$$

induced by the fully faithful functor

$$j_! : \mathcal{D}(G(E), \Lambda) \cong \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

More generally, for any $b \in B(G)$, we can define a map

$$\Psi_G^b : \mathcal{Z}^{\mathrm{spec}}(G, \Lambda) \to \mathcal{Z}(G_b(E), \Lambda)$$

to the Bernstein center for $G_b(E)$ by using the fully faithful embedding

$$\mathcal{D}(G_b(E), \Lambda) \cong \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

determined for example by the left adjoint to i^{b*}, where $i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ is the locally closed embedding (see Proposition VII.7.2). (Recall that in the D_{lis}-setting, we do not have a general $i_!^b$-functor, although it can be defined in the present situation. All these maps will induce the same map to the Bernstein center.)

IX.7.1. Compatibility with G_b. One can describe the maps Ψ_G^b for $b \neq 1$ in terms of the maps Ψ_{G_b}. Note that $\widehat{G_b}$ is naturally a Levi subgroup of $\widehat{G}$, as $G_{b,\breve{E}} \subset G_{\breve{E}}$ is the centralizer of the slope morphism $\nu_b : \mathbb{D} \to G_{\breve{E}}$. This extends naturally to a morphism of L-groups

$$\widehat{G_b} \rtimes Q \to \widehat{G} \rtimes Q$$

where as usual Q is a finite quotient of W_E over which the action on $\widehat{G}$ factors. However, from geometric Satake we rather get the natural inclusion

$$\breve{\widehat{G_b}} \rtimes W_E \to \breve{\widehat{G}} \rtimes W_E$$

where the W_E-actions include the cyclotomic twist. The latter induces a map

$$Z^1(W_E, \widehat{G_b}) \to Z^1(W_E, \widehat{G})$$

that in terms of the usual W_E-action is given by sending a 1-cocycle $\varphi : W_E \to \widehat{G_b}(A)$ to the 1-cocycle

$$W_E \to \widehat{G}(A) : w \mapsto (2\rho_{\widehat{G}} - 2\rho_{\widehat{G_b}})(\sqrt{q})^{|w|}\varphi(w)$$

where $|\cdot| : W_E \to W_E/I_E \cong \mathbb{Z}$ is normalized as usual by sending a geometric Frobenius to 1.

THEOREM IX.7.2. *For all G and $b \in B(G)$, the diagram*

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \arrow[r, "\Psi_G^b"] \arrow[d] & \mathcal{Z}(\mathcal{D}(G_b(E),\Lambda)) \\ \mathcal{Z}^{\mathrm{spec}}(G_b,\Lambda) \arrow[ur, "\Psi_{G_b}"'] & \end{tikzcd}$$

commutes.

PROOF. We note that to prove the theorem, we can assume that Λ is killed by power of ℓ (if ℓ is not very good for $\widehat{G}$, replacing the left-hand side with an algebra of excursion operators), as the result for $\Lambda = \mathbb{Z}_\ell[\sqrt{q}]$ implies it in general, and the right-hand side

$$\mathcal{Z}(\mathcal{D}(G_b(E),\Lambda)) = \varprojlim_{K \subset G_b(E)} \mathcal{Z}(\Lambda[G_b(E) /\!\!/ K])$$

is ℓ-adically separated in that case. This means we can avoid the subtleties of $\mathcal{D}_{\mathrm{lis}}$ in place of $\mathcal{D}_{\text{ét}}$.

If b is basic, the theorem follows from the identification $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_b}$ of Corollary III.4.3, which is equivariant for the Hecke action.

In general, we first reduce to the case that G is quasisplit. Take a z-embedding $G \hookrightarrow G'$ as in [**Kal18**, Section 5], with quotient a torus D, so that the center $Z(G')$ is connected. Then $\mathrm{Bun}_G = \mathrm{Bun}_{G'} \times_{\mathrm{Bun}_D} \{*\}$ and the map $B(G) \to B(G')$ is injective. To see the latter, by the description of the stacks, it suffices to see that for all $b \in B(G)$ with image $b' \in B(G')$, the map $G'_{b'}(E) \to D(E)$ is surjective. But for any $b \in B(G)$, the map $G_b \to G'_{b'}$ is a z-embedding with quotient D, and $Z'(E) \to D(E)$ is surjective by [**Kal18**, Fact 5.5], where also $Z' \subset G'_{b'}$, so in particular $G'_{b'}(E) \to D(E)$ is surjective. An element of

$$\mathcal{Z}(\mathcal{D}(G_b(E),\Lambda)) = \varprojlim_{K \subset G_b(E)} \mathcal{Z}(\Lambda[G_b(E) /\!\!/ K])$$

of the Bernstein center of $G_b(E)$ is determined by its action on $\pi'|_{G_b(E)}$ for representations π' of $G'_{b'}(E)$. By Theorem IX.6.1, we can thus reduce to G' in place of G, i.e. that the center of G is connected. When $Z(G)$ is connected, there is some basic $b_0 \in B(G)$ such that G_{b_0} is quasisplit. Using the Hecke-equivariant isomorphism $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_{b_0}}$ we can thus assume that G is quasisplit.

Now if G is quasisplit, fix a Borel $B \subset G$. Any $b \in B(G)$ then admits a reduction to a canonical parabolic $P = P_b \subset G$ containing B. Pick a cocharacter $\mu : \mathbb{G}_m \to G$ with dynamical parabolic P. For any $N \geq 0$, let $b_N = b\mu(\pi^N)$: This is a sequence of elements of $B(G)$ associated to the same parabolic P but increasingly instable. Moreover, $G_b = G_{b_N}$. We note that the diagram

$$\begin{array}{ccc} \mathcal{Z}^{\mathrm{spec}}(G,\Lambda) & \xrightarrow{\Psi_G^b} & \mathcal{Z}(G_b(E),\Lambda) \\ \downarrow = & & \downarrow = \\ \mathcal{Z}^{\mathrm{spec}}(G,\Lambda) & \xrightarrow{\Psi_G^{b_N}} & \mathcal{Z}(G_{b_N}(E),\Lambda) \end{array}$$

commutes. For this, take any representation σ of $G_b(E)$ and consider the sheaf $A_N \in \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ concentrated on $\mathrm{Bun}_G^{b_N}$, corresponding to the representation σ. Let $V \in \mathrm{Rep}\,\widehat{G}$ be the highest weight representation with weight μ^N. We claim that $T_V(A_N)|_{\mathrm{Bun}_G^b}$ is given by the representation σ. As Hecke operators commute with excursion operators, this implies the desired result. To compute $T_V(A_N)|_{\mathrm{Bun}_G^b}$, we have to analyze the moduli space of modifications of $\mathcal{E}_b$ of type bounded by μ^N that are isomorphic to $\mathcal{E}_{b_N}$. There is in fact precisely one such modification, given by pushout of the standard modification of line bundles from $\mathcal{O}$ to $\mathcal{O}(1)$ via $\mu^N : \mathbb{G}_m \to G$; its type is exactly μ^N. This gives the claim.

Now to prove the theorem, we have to prove the commutativity of the diagram for any excursion operator, given by excursion data $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$. For any such excursion data, we can pick N large enough so that any modification of $\mathcal{E}_{b_N}$ to itself, of type bounded by V, is automatically compatible with the Harder–Narasimhan reduction to P. In that case, for σ and A_N as above, to analyze the excursion operators

$$A_N = T_1(A_N) \xrightarrow{\alpha} T_V(A_N) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A_N) \xrightarrow{\beta} T_1(A_N) = A_N,$$

we have to analyze the moduli space of modifications of $\mathcal{E}_{b_N}$, at I varying points, of type bounded by V, and that are isomorphic to $\mathcal{E}_{b_N}$. By assumption on N, this is the same as the moduli space of such modifications as P-bundles. This maps to the similar moduli space parametrizing modifications as M-bundles, where M is the Levi of P (and $G_b = M_{b_M}$ for a basic $b_M \in B(M)$). We want to compute $T_V(A_N)$. Note that A_N comes from $A'_N \in D_{\text{ét}}(\mathrm{Bun}_P, \Lambda)$ as it is concentrated on $\mathrm{Bun}_G^{b_N} \cong \mathrm{Bun}_P^{b_N} \subset \mathrm{Bun}_P$, by the Harder–Narasimhan reduction.

Consider the diagram

$$\begin{tikzcd}
\mathrm{Bun}_M \arrow[d, "\psi"] & \mathrm{Hck}^I_{M,P} \arrow[l, "h''_1"'] \arrow[r, "h''_2"] \arrow[d, "\psi_H"] & \mathrm{Bun}_P \times (\mathrm{Div}^1)^I \arrow[d, "="] \\
\mathrm{Bun}_P \arrow[d, "\pi"] & \mathrm{Hck}^I_P \arrow[l, "h'_1"'] \arrow[r, "h'_2"] \arrow[d, "\pi_H"] & \mathrm{Bun}_P \times (\mathrm{Div}^1)^I \arrow[d, "\pi"] \\
\mathrm{Bun}_M & \mathrm{Hck}^I_M \arrow[l, "h_1"'] \arrow[r, "h_2"] & \mathrm{Bun}_M \times (\mathrm{Div}^1)^I
\end{tikzcd}$$

where $\mathrm{Hck}^I_{M,P}$ is defined as the fibre product $\mathrm{Hck}^I_P \times_{\mathrm{Bun}_P} \mathrm{Bun}_M$, and thus parametrizes modifications from an M-bundle to a P-bundle.

We need to compute $T_V(A_N)|_{\mathrm{Bun}_G^{b_N}}$, which by the above argument that any modification of $\mathcal{E}_{b_N}$ to itself of type bounded by V is a modification as P-bundles, can be computed in terms of the middle diagram, as

$$Rh'_{2!}(h'^*_1 A'_N \otimes^{\mathbb{L}}_\Lambda \mathcal{S}_V)|_{\mathrm{Bun}_P^{b_N}}$$

where $\mathcal{S}_V \in \mathcal{H}\mathrm{ck}^I_G$ is the perverse sheaf determined by V under the geometric Satake equivalence, and we continue to denote by $\mathcal{S}_V$ any of its pullbacks.

There is some $B_N \in D_{\text{ét}}(\mathrm{Bun}_M, \Lambda)$ such that $A'_N = R\psi_! B_N$. In fact, one can take $B_N = R\pi_! A'_N$, noting that on the support of A'_N, the map $\pi : \mathrm{Bun}_P \to \mathrm{Bun}_M$ is (cohomologically) smooth, so $R\pi_!$ is defined on A'_N (although π is a stacky map). (Indeed, everything is concentrated on one stratum, and the relevant categories are all equivalent to $D(G_b(E), \Lambda)$.) Moreover, to compute the restriction to $\mathrm{Bun}_P^{b_N}$ it is enough to do the computation after applying $R\pi_!$. We compute:

$$\begin{aligned}
R\pi_! Rh'_{2!}(h'^*_1 A'_N \otimes^{\mathbb{L}}_\Lambda \mathcal{S}_V) &= R\pi_! Rh'_{2!} R\psi_{\mathcal{H}!}(h''^*_1 B_N \otimes^{\mathbb{L}}_\Lambda \mathcal{S}_V) \\
&= Rh_{2!}(h^*_1 B_N \otimes^{\mathbb{L}}_\Lambda Rg_! \mathcal{S}_V)
\end{aligned}$$

where $g : \mathrm{Hck}^I_{M,P} \to \mathrm{Hck}^I_M$ is the projection. But this is the pullback of the map $L^+M \backslash \mathrm{Gr}^I_P \to L^+M \backslash \mathrm{Gr}^I_M = \mathcal{H}\mathrm{ck}^I_M$ under $\mathrm{Hck}^I_M \to \mathcal{H}\mathrm{ck}^I_M$. This means that $Rg_! \mathcal{S}_V$ arises via pullback from $\mathrm{CT}_P(\mathcal{S}_V) \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}^I_M, \Lambda)$. Up to the shift $[\deg_P]$, this agrees with $\mathcal{S}_V|_{(\widehat{M} \rtimes Q)^I}$, where the restriction involves a cyclotomic twist, as above. (It is the canonical restriction along $\widetilde{M} \to \widetilde{G}$ for the canonical W_E-actions arising geometrically.) Now the excursion operators, which involve maps from and to the sheaf corresponding to $V = 1$, require only the connected component where $\deg_P = 0$, so we can ignore the shift.

With these translations, we see that the excursion operators on $\mathrm{Bun}_G^{b_N}$ and on $\mathrm{Bun}_M^{b_N}$ agree, giving the desired result. □

IX.7.2. Parabolic induction. A corollary of this result is compatibility with parabolic induction.

Corollary IX.7.3. *Let G be a reductive group with a parabolic $P \subset G$ and Levi $P \to M$. Then for all representations σ of $M(E)$ with (unnormalized) parabolic induction $\mathrm{Ind}_{P(E)}^{G(E)}\sigma$, the diagram*

$$\begin{tikzcd} \mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \arrow[r] \arrow[d] & \mathrm{End}(\mathrm{Ind}_{P(E)}^{G(E)}\sigma) \\ \mathcal{Z}^{\mathrm{spec}}(M,\Lambda) \arrow[r] & \mathrm{End}(\sigma) \arrow[u] \end{tikzcd}$$

commutes. In particular, the formation of L-parameters is compatible with parabolic induction: If $\Lambda = L$ is an algebraically closed field, σ is irreducible and $\widetilde{\sigma}$ is an irreducible subquotient of $\mathrm{Ind}_{P(E)}^{G(E)}\sigma$, then $\varphi_{\widetilde{\sigma}}$ is conjugate to the composite

$$W_E \xrightarrow{\varphi_\sigma} \widehat{M}(L) \rtimes W_E \to \widehat{G}(L) \rtimes W_E$$

where the map $\widehat{M} \rtimes W_E \to \widehat{G} \rtimes W_E$ is defined as above, involving the cyclotomic twist.

Proof. It suffices to prove the result for $\sigma = c\text{-}\mathrm{Ind}_K^{M(E)}\Lambda$ for $K \subset M(E)$ an open pro-p-subgroup, and then one can assume $\Lambda = \mathbb{Z}_\ell[\sqrt{q}]$, where one can further by ℓ-adic separatedness reduce to torsion coefficients.

Let $\mu : \mathbb{G}_m \to G$ be a cocharacter with dynamical parabolic P and let $b = \mu(\pi) \in B(G)$. Then $G_b = M$, and we can build a sheaf $A \in \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ concentrated on Bun_G^b, given by the representation σ. Let $V \in \mathrm{Rep}(\widehat{G})$ be a highest weight representation of weight μ. Then $T_V(A)|_{\mathrm{Bun}_G^1}$ is given by $\mathrm{Ind}_{P(E)}^{G(E)}\sigma$: To see this, we have to understand the moduli space of modification of the trivial G-torsor of type bounded by μ that are isomorphic to $\mathcal{E}_b$. This is in fact given by $\underline{G(E)/P(E)}$, the $G(E)$-orbit of the pushout of the modification from $\mathcal{O}$ to $\mathcal{O}(1)$ via μ. All of these modifications are of type exactly μ. This easily gives the claim on $T_V(A)|_{\mathrm{Bun}_G^1}$. Now as Hecke operators commute with excursion operators, the excursion operators on $\mathrm{Ind}_{P(E)}^{G(E)}\sigma$ agree with those on A, and these are determined by Theorem IX.7.2, giving the result. □

IX.7.3. The case $G = \mathrm{GL}_n$. For the group $G = \mathrm{GL}_n$, we can identify the L-parameters with the usual L-parameters of [**LRS93**], [**HT01**], [**Hen00**]. This is the only place of this paper where we rely on previous work on the local Langlands correspondence, or (implicitly) rely on global arguments. More precisely, we use the identification of the cohomology of the Lubin–Tate and Drinfeld tower, see [**Boy99**], [**Har97**], [**HT01**], [**Hau05**], [**Dat07**]. In the proof, we use the translation between Hecke operators and local Shimura varieties as Section IX.3, together with the description of these as the Lubin–Tate tower and Drinfeld tower in special cases, see [**SW13**].

Theorem IX.7.4. *Let π be any irreducible smooth representation of $\mathrm{GL}_n(E)$ over some algebraically closed field L over $\mathbb{Q}_\ell(\sqrt{q})$. Then the L-parameter φ_π agrees with the usual (semisimplified) L-parameter.*

Proof. By Corollary IX.7.3, we can assume that π is supercuspidal. We only need to evaluate the excursion operators for the excursion data given by $I = \{1, 2\}$, the representation $V = \mathrm{std} \boxtimes \mathrm{std}^\vee$

of $\widehat{\mathrm{GL}_n}^2$, and the tautological maps $\alpha : 1 \to \mathrm{std} \otimes \mathrm{std}^\vee$ and $\beta : \mathrm{std} \otimes \mathrm{std}^\vee \to 1$, as these excursion operators determine the trace of the representation (and thus the semisimplified representation).

First, we analyze these excursion operators on the sheaf B which is the sheaf on $\mathrm{Bun}^b_{\mathrm{GL}_n}$ for b corresponding to the bundle $\mathcal{O}(-\frac{1}{n})$, given by the representation $\sigma = \mathrm{JL}(\pi)$ of $D^\times$; here D is the division algebra of invariant $\frac{1}{n}$. The Hecke operator T_V is the composite of two operators. The first Hecke operator, corresponding to std, takes minuscule modifications $\mathcal{O}(-\frac{1}{n}) \subset \mathcal{E}$ with cokernel a skyscraper sheaf of rank 1. Such an $\mathcal{E}$ is necessarily isomorphic to $\mathcal{O}^n$, and the Hecke operator will then produce the σ-isotypic part of the cohomology of the Lubin–Tate tower, which is $\pi \otimes \rho_\pi$, where ρ_π is the irreducible n-dimensional W_E-representation associated to π by the local Langlands correspondence. (Note that the shift $[n-1]$, as well as the cyclotomic twist $(\frac{n-1}{2})$ that usually appears, is hidden inside the normalization of the perverse sheaf corresponding to the standard representation.) Now the second Hecke operator, when restricted to $\mathrm{Bun}^b_{\mathrm{GL}_n}$, produces the π-isotypic component of the cohomology of the Drinfeld tower, which is $\sigma \otimes \rho_\pi^*$. In total, we see that $T_V(B)|_{\mathrm{Bun}^b_{\mathrm{GL}_n}}$ is given by $\pi \otimes \rho_\pi \otimes \rho_\pi^*$ as representation of $D^\times \times W_E \times W_E$. By irreducibility of ρ_π, the W_E-equivariant map $\sigma \to \sigma \otimes \rho_\pi \otimes \rho_\pi^*$ induced by α (and the similar backwards induced by β) must agree up to scalar with the obvious map. The scalar of the total composite can be identified by taking both elements of W_E to be equal to 1. This shows that B has the correct L-parameter. Now use that the sheaf corresponding to π appears as a summand of $T_{\mathrm{std}}(B)$ (after forgetting the W_E-action) to conclude the same for π. $\square$

In particular, it follows that the map

$$\mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \to \mathcal{Z}(\mathrm{GL}_n(E), \overline{\mathbb{Q}}_\ell)$$

to the Bernstein center agrees with the usual map. We recall that we have already proved that this map refines to a map

$$\mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \Lambda) \to \mathcal{Z}(\mathrm{GL}_n(E), \Lambda)$$

to the integral Bernstein center (where $\Lambda = \mathbb{Z}_\ell[\sqrt{q}]$), recovering a result of Helm–Moss [**HM18**].

CHAPTER X

The spectral action

As a final topic, we construct the spectral action. We will first construct it with characteristic 0 coefficients, and then explain refinements with integral coefficients.

Let Λ be the ring of integers in a finite extension of $\mathbb{Q}_\ell(\sqrt{q})$. We have the stable ∞-category $\mathcal{C} = \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ of compact objects, which is linear over Λ, and functorially in the finite set I an exact monoidal functor $\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{C})^{BW_E^I}$ that is linear over $\mathrm{Rep}_\Lambda(Q^I)$. A first version of the following theorem is due to Nadler–Yun [**NY19**] in the context of Betti geometric Langlands, and a more general version appeared in the work of Gaitsgory–Kazhdan–Rozenblyum–Varshavsky [**GKRV19**]. Both references, however, effectively assume that G is split, work only with characteristic 0 coefficients, and work with a discrete group in place of W_E. At least the extension to $\mathbb{Z}_\ell$-coefficients is a nontrivial matter.

Note that $Z^1(W_E, \widehat{G})$ is not quasicompact, as it has infinitely many connected components; it can be written as the increasing union of open and closed quasicompact subschemes $Z^1(W_E/P, \widehat{G})$. We say that an action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})$ on a stable ∞-category $\mathcal{C}$ is compactly supported if for all $X \in \mathcal{C}$ the functor $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G}) \to \mathcal{C}$ (induced by acting on X) factors over some $\mathrm{Perf}(Z^1(W_E/P, \widehat{G})/\widehat{G})$.

The goal of this chapter is to prove the following theorem. Recall that for $G = \mathrm{GL}_n$, all ℓ are very good, and for classical groups, all $\ell \neq 2$ are very good.

THEOREM X.0.1. *Assume that ℓ is a very good prime for $\widehat{G}$. Let $\mathcal{C}$ be a small idempotent-complete Λ-linear stable ∞-category. Then giving, functorially in the finite set I, an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{C})^{BW_E^I}$$

is equivalent to giving a compactly supported Λ-linear action of

$$\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}).$$

Here, given a Λ-linear action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$, one can produce such an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{C})^{BW_E^I}$$

functorially in I by composing the exact $\mathrm{Rep}_\Lambda(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E^I}$$

with the action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})$.

The same result holds true if Λ is a field over $\mathbb{Q}_\ell(\sqrt{q})$, for any prime ℓ.

Here, the exact $\mathrm{Rep}_\Lambda(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E^I}$$

is induced by tensor products and the exact $\mathrm{Rep}_\Lambda(Q)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q) \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E}$$

corresponding to the universal $\widehat{G} \rtimes Q$-torsor, with the universal W_E-equivariance as parametrized by $Z^1(W_E, \widehat{G})/\widehat{G}$.

Before starting the proof, we note that the proof of Proposition IX.5.1 shows that we may replace W_E by W_E/P in the statement of Theorem X.0.1. Choosing moreover a discretization $W \subset W_E/P$, we reduce to the following variant.

THEOREM X.0.2. *Assume that ℓ is a very good prime for $\widehat{G}$. Let $\mathcal{C}$ be a small idempotent-complete Λ-linear stable ∞-category. Then giving, functorially in the finite set I, an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda(\mathcal{C})^{BW^I}$$

is equivalent to giving a Λ-linear action of

$$\mathrm{Perf}(Z^1(W, \widehat{G})_\Lambda/\widehat{G}),$$

with the same compatibility as above. The same result holds true if Λ is a field over $\mathbb{Q}_\ell$, for any prime ℓ.

X.1. Rational coefficients

With rational coefficients, we can prove a much more general result, following [**GKRV19**]. Consider a reductive group H over a field L of characteristic 0 (like $\widehat{G}$ over $\overline{\mathbb{Q}}_\ell$) with an action of a finite group Q. Let S be any anima over BQ (like BW, where $W \subset W_E/P$ is a discretization of W_E/P for an open subgroup of the wild inertia, as usual). We can then consider the (derived) stack $\mathrm{Map}_{BQ}(S, B(H \rtimes Q))$ over L, whose values in an animated L-algebra A are the maps of anima $S \to B(H \rtimes Q)(A)$ over BQ. This recovers the stack $[Z^1(W, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}]$ in the above example, using Proposition VIII.3.5.

In general, $\mathrm{Map}_{BQ}(S, B(H \rtimes Q))$ is the fpqc quotient of an affine derived scheme by a power of H. Indeed, pick a surjection $S' \to S \times_{BQ} *$ from a set S'. Then $\mathrm{Map}_{BQ}(S, B(H \rtimes Q))$ maps to $BH^{S'}$; we claim that the fibre is an affine derived scheme, i.e. representable by an animated L-algebra. For this, note that

$$\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) \to \mathrm{Map}(S \times_{BQ} *, BH)$$

is relatively representable, as it is given by the Q-fixed points. To show that the right-hand side is relatively representable over $BH^{S'}$, we can replace $S \times_{BQ} *$ by a connected anima T, and S' by a point. Then $\mathrm{Map}(T, BH) \times_{BH} *$ parametrizes pointed maps $T \to BH$, which are equivalent to maps of $\mathbb{E}_1$-groups $\Omega(T) \to H$. Writing $\Omega(T)$ as a sifted colimit of finite free groups F_n, one reduces to representability of maps of groups $F_n \to H$, which is representable by H^n.

THEOREM X.1.1. *Let $\mathcal{C}$ be an idempotent-complete small stable L-linear ∞-category. Giving, functorially in finite sets I, an exact $\mathrm{Rep}_L(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathrm{End}_L(\mathcal{C})^{S^I}$$

is equivalent to giving an L-linear action of $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$ on $\mathcal{C}$. Here, given such an action of $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$, one gets exact $\mathrm{Rep}_L(Q^I)$-linear monoidal functors

$$\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathrm{End}_L(\mathcal{C})^{S^I}$$

by precomposing the exact monoidal functor $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q))) \to \mathrm{End}_L(\mathcal{C})$ with the natural exact $\mathrm{Rep}_L(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))^{S^I}$$

given by I-fold tensor product of the exact $\mathrm{Rep}_L(Q)$-linear symmetric monoidal functor

$$\mathrm{Rep}_L(H \rtimes Q) \to \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))^S$$

assigning to each $s \in S$ pullback along evaluation at s, $\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) \to B(H \rtimes Q)$.

PROOF. Note first that, for any L-linear idempotent-complete small stable ∞-category $\mathcal{C}$, giving an exact L-linear functor $\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathcal{C}$ is equivalent to giving an exact L-linear functor of stable ∞-categories $\mathrm{Perf}((H \rtimes Q)^I) \to \mathcal{C}$, as the ∞-category of perfect complexes is freely generated by the exact category of representations. Indeed, such functors extend to the ∞-category obtained by inverting quasi-isomorphisms in $\mathrm{Ch}^b(\mathrm{Rep}_L(H \rtimes Q)^I)$, and this is $\mathrm{Perf}(B(H \rtimes Q)^I)$.

For any S, we have the anima $F_1(S)$ of L-linear actions of $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$ on $\mathcal{C}$, and the anima $F_2(S)$ of functorial exact monoidal functors

$$\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathrm{End}_L(\mathcal{C})^{S^I}$$

linear over $\mathrm{Rep}_L(Q^I)$, and a natural map $F_1(S) \to F_2(S)$ functorial in S (where both F_1 and F_2 are contravariant functors of S). Both functors take sifted colimits in S to limits. This is clear for F_2 (as $S \mapsto S^I$ commutes with sifted colimits). For F_1, it is enough to see that taking S to $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$ commutes with sifted colimits (taken in idempotent-complete stable ∞-categories), which is Lemma X.1.2 below.

Therefore it suffices to handle the case that S is a finite set, for which the map $S \to BQ$ can be factored over $*$. Then $\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) \cong BH^S$. Similarly, exact monoidal functors

$$\mathrm{Rep}_L((H \rtimes Q)^I) \to \mathrm{End}_L(\mathcal{C})^{S^I}$$

linear over $\mathrm{Rep}_L(Q^I)$ are equivalent to exact monoidal functors

$$\mathrm{Rep}_L(H^I) \to \mathrm{End}_L(\mathcal{C})^{S^I}$$

linear over L. Here, we use $\mathrm{Perf}(B(H \rtimes Q)^I) \otimes_{\mathrm{Perf}(BQ^I)} \mathrm{Perf}(L) \cong \mathrm{Perf}(BH^I)$, which follows easily from highest weight theory.

The latter data is equivalent to maps

$$\mathrm{Hom}(I, S) = S^I \to \mathrm{Fun}^{\mathrm{mon}}_{\mathrm{ex},L}(\mathrm{Rep}_L(H^I), \mathrm{End}_L(\mathcal{C}))$$

functorially in I, where $\mathrm{Fun}^{\mathrm{mon}}_{\mathrm{ex},L}$ denotes the exact L-linear monoidal functors. Both sides here are functors in I, and on the left-hand side we have a representable functor. By the Yoneda lemma, it follows that this data is equivalent to L-linear exact monoidal functors

$$\mathrm{Rep}_L(H^S) \to \mathrm{End}_L(\mathcal{C}).$$

Such actions extend uniquely to $\mathrm{Perf}(BH^S)$, giving the desired result. □

Lemma X.1.2. *The functor taking an anima S over BQ to $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$, regarded as an idempotent-complete stable ∞-category, commutes with sifted colimits. More precisely, as a functor into L-linear symmetric monoidal idempotent-complete stable ∞-categories, it commutes with all colimits.*

Proof. We first check that it commutes with filtered colimits. For this, let S_i, $i \in I$, be a filtered diagram of anima over BQ, and choose compatible surjections $S_i' \to S_i \times_{BQ} *$ from sets S_i'. Let $S = \mathrm{colim}_i S_i$ and $S' = \mathrm{colim}_i S_i'$, which is a set surjecting onto $S \times_{BQ} *$. Letting $G_i = H^{S_i'}$ and $G = H^{S'}$, we get presentations $\mathrm{Map}_{BQ}(S_i, B(H \rtimes Q)) = X_i/G_i$ as quotients of affine derived L-schemes X_i by the pro-reductive group G_i, and similarly $\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) = X/G$, with $X = \varprojlim_i X_i$ and $G = \varprojlim_i G_i$. We claim that in this generality

$$\varinjlim_i \mathrm{Perf}(X_i/G_i) \to \mathrm{Perf}(X/G)$$

is an isomorphism of idempotent-complete stable ∞-categories.

Assume first that all $X = \mathrm{Spec}\, L$ are a point. Then note that $\mathrm{Perf}(BG)$ is generated by $\mathrm{Rep}(G)$, which is easily seen to be the filtered colimit $\varinjlim_i \mathrm{Rep}(G_i)$, and (by writing it as limit of reductive groups) is seen to be semisimple. The claim is easily checked in this case.

In general, $\mathrm{Perf}(X/G)$ is generated by $\mathrm{Rep}(G)$ as an idempotent complete stable ∞-category. Indeed, given any perfect complex $A \in \mathrm{Perf}(X/G)$, we can look at the largest n for which the cohomology sheaf $H^n(A)$ is nonzero; after shift, $n = 0$. Pick $V \in \mathrm{Rep}(G)$ with a map $V \to H^0(A)$ such that $V \otimes_L \mathcal{O}_{X/G} \to H^0(A)$ is surjective. By semisimplicity of $\mathrm{Rep}(G)$, we can lift $V \to H^0(A)$ to $V \to A$, and then pass to the cone of $V \otimes_L \mathcal{O}_{X/G} \to A$ to reduce the projective amplitude until A is a vector bundle. In that case the homotopy fibre B of $V \otimes_L \mathcal{O}_{X/G} \to A$ is again a vector bundle, and the map $V \otimes_L \mathcal{O}_{X/G} \to A$ splits, as the obstruction is $H^1(X/G, A^\vee \otimes_{\mathcal{O}_{X/G}} B)$, which vanishes by semisimplicity of $\mathrm{Rep}(G)$.

This already proves essential surjectivity. For fully faithfulness, it suffices by passage to internal Hom's to show that for all $A_{i_0} \in \mathrm{Perf}(X_{i_0}/G_{i_0})$ (for some chosen i_0) with pullbacks $A_i \in \mathrm{Perf}(X_i/G_i)$ for $i \to i_0$ and $A \in \mathrm{Perf}(X/G)$, the map $\varinjlim_i R\Gamma(X_i/G_i, A_i) \to R\Gamma(X/G, A)$ is an isomorphism. By semisimplicity of $\mathrm{Rep}(G_i)$ and $\mathrm{Rep}(G)$, it suffices to see that $\varinjlim_i R\Gamma(X_i, A_i) \to R\Gamma(X, A)$ is an isomorphism, which is clear by affineness.

This handles the case of filtered colimits. For the more precise claim, it is also easy to see that it commutes with disjoint unions. It is now enough to handle pushouts, so consider a diagram $S_1 \leftarrow S_0 \to S_2$ of anima over BQ, with pushout S. We can assume that the maps $S_0 \to S_1$ and $S_0 \to S_2$ are surjective, as otherwise we can use compatibility with disjoint unions (replacing S_2 by the disjoint union of the image of S_0 and its complement). Then choose a surjection $S' \to S_0 \times_{BQ} *$,

which induces similar surjections in the other cases. Thus, we get affine derived L-schemes $X_1 \to X_0 \leftarrow X_2$ with actions by $G = H^{S'}$, and $X = X_1 \times_{X_0} X_2$, and we want to see that the functor

$$\mathrm{Perf}(X_1/G) \otimes_{\mathrm{Perf}(X_0/G)} \mathrm{Perf}(X_2/G) \to \mathrm{Perf}(X_1 \times_X X_2/G)$$

is an equivalence. On the level of Ind-categories, $\mathrm{Ind}\,\mathrm{Perf}(X_i/G)$ is the ∞-category of $\mathcal{O}(X_i)$-modules in $\mathrm{Ind}\,\mathrm{Perf}(BG)$: This is a consequence of Barr–Beck and the fact observed above that $\mathrm{Perf}(BG)$ generates $\mathrm{Perf}(X_i/G)$, so that the forgetful functor $\mathrm{Ind}\,\mathrm{Perf}(X_i/G) \to \mathrm{Ind}\,\mathrm{Perf}(BG)$ is conservative. It follows that the tensor product is the ∞-category of $\mathcal{O}(X_1) \otimes_{\mathcal{O}(X_0)} \mathcal{O}(X_2)$-modules in $\mathrm{Ind}\,\mathrm{Perf}(BG)$, the tensor product taken in the symmetric monoidal stable ∞-category $\mathrm{Ind}\,\mathrm{Perf}(BG)$. The map $\mathcal{O}(X_1) \otimes_{\mathcal{O}(X_0)} \mathcal{O}(X_2) \to \mathcal{O}(X)$ is an isomorphism in $\mathrm{Ind}\,\mathrm{Perf}(BG)$: This can be checked after the forgetful functor $\mathrm{Ind}\,\mathrm{Perf}(BG) \to \mathcal{D}(L)$ as it is conservative (using that G is pro-reductive, hence $\mathrm{Rep}(G)$ is semisimple), and then it amounts to $X = X_1 \times_{X_0} X_2$. $\square$

In particular, we get the following corollary.

Corollary X.1.3. *Let L be a field over $\mathbb{Q}_\ell(\sqrt{q})$. There is a natural compactly supported L-linear action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_L/\widehat{G})$ on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)^\omega$, uniquely characterized by the requirement that by restricting along the $\mathrm{Rep}_L(Q^I)$-linear maps*

$$\mathrm{Rep}_L((\widehat{G} \rtimes Q)^I) \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_L/\widehat{G})^{BW_E^I}$$

it induces the Hecke action, which gives functorially in the finite set I exact $\mathrm{Rep}_L(Q^I)$-linear functors

$$\mathrm{Rep}_L((\widehat{G} \rtimes Q)^I) \to \mathrm{End}_L(\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)^\omega)^{BW_E^I}.$$

Proof. We can reduce to the subcategories $\mathcal{D}^P_{\mathrm{lis}}(\mathrm{Bun}_G, L)^\omega \subset \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ for open subgroups P of the wild inertia of W_E, acting trivially on $\widehat{G}$. Then we can replace W_E by W_E/P throughout. In that case, restricting the given Hecke action to $W \subset W_E/P$, Theorem X.1.1 gives an action of $\mathrm{Perf}(Z^1(W, \widehat{G})_L/\widehat{G})$, and $Z^1(W, \widehat{G}) = Z^1(W_E/P, \widehat{G})$, so we get the desired action of $\mathrm{Perf}(Z^1(W_E/P, \widehat{G})_L/\widehat{G})$. $\square$

With this action, we can formulate the main conjecture, "the categorical form of the geometric Langlands conjecture on the Fargues–Fontaine curve". Recall that for a quasisplit reductive group G over E, Whittaker data consist of a choice of a Borel $B \subset G$ with unipotent radical $U \subset B$, together with a generic character $\psi : U(E) \to \overline{\mathbb{Q}}_\ell^\times$. As usual, we also fix $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$.

Conjecture X.1.4. *There is an equivalence of $\overline{\mathbb{Q}}_\ell$-linear small stable ∞-categories*

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)^\omega \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})$$

linear over $\mathrm{Perf}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})$, under which the structure sheaf on the right corresponds to the Whittaker sheaf $\mathcal{W}_\psi$, which is the sheaf concentrated on $\mathrm{Bun}^1_G \subset \mathrm{Bun}_G$ corresponding to the representation $c\text{-}\mathrm{Ind}_{U(E)}^{G(E)}\psi$ of $G(E)$.

To be precise, $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})$ refers here to the ∞-category of those bounded complexes with coherent cohomology that also have quasicompact support, i.e. only live on finitely many connected components, and the final compatibility statement is really on the level of Ind-objects

(or after restriction to connected components). The linearity condition over $\mathrm{Perf}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})$ means that this equivalence is compatible with the spectral action.

Remark X.1.5. There is an orthogonal decomposition

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^\omega = \bigoplus_{\alpha\in\pi_1(G)_\Gamma} \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^{c_1=\alpha},\overline{\mathbb{Q}}_\ell)^\omega$$

given by the connected components of Bun_G. There is a morphism $Z(\widehat{G})^\Gamma \to \mathrm{Id}_{Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}}$, as $Z(\widehat{G})^\Gamma \subset \widehat{G}$ acts trivially on $Z^1(W_E,\widehat{G})$. There is an associated "eigenspace" decomposition

$$\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}) = \bigoplus_{\chi\in X^*(Z(\widehat{G})^\Gamma)} \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})_\chi.$$

Compatibility with the spectral action implies that via the identification $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$ those two decompositions should match.

Another way to phrase the preceding conjecture is to say that, noting $*$ the spectral action, the "non-abelian Fourier transform"

$$\begin{aligned}\mathrm{Perf}^{\mathrm{qc}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}) &\longrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^\omega\\ M &\longmapsto M * \mathcal{W}_\psi\end{aligned}$$

is fully faithful and extends to an equivalence of $\overline{\mathbb{Q}}_\ell$-linear small stable ∞-categories

$$\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}) \xrightarrow{\sim} \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)^\omega.$$

Example X.1.6. Fully faithfulness in the categorical conjecture, applied to the structure sheaf, implies that

$$\mathcal{Z}^{\mathrm{spec}}(G,\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{End}(c\text{-}\mathrm{Ind}_{U(E)}^{G(E)}\psi).$$

Example X.1.7 (Kernel of functoriality). Conjecture X.1.4 implies the existence of a kernel of functoriality for the local Langlands correspondence in the following way. Let

$$f : {}^LH \to {}^LG$$

be an L-morphism between the L-groups of two quasi-split reductive groups H and G over E. This defines a morphism of stacks

$$Z^1(W_E,\widehat{H})_{\overline{\mathbb{Q}}_\ell}/\widehat{H} \longrightarrow Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G},$$

and pushforward along this map induces a functor

$$\mathrm{Ind}\,\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{H})_{\overline{\mathbb{Q}}_\ell}/\widehat{H}) \to \mathrm{Ind}\,\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}).$$

(There may be slightly different ways of handling the singularities here. One way to argue is to observe that pushforward is naturally a functor

$$\mathcal{D}^{\geq 0}_{\mathrm{qcoh}}(Z^1(W_E,\widehat{H})_{\overline{\mathbb{Q}}_\ell}/\widehat{H}) \to \mathcal{D}^{\geq 0}_{\mathrm{qcoh}}(Z^1(W_E,\widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}),$$

and $\mathcal{D}^{\geq 0}_{\mathrm{qcoh}} = \mathrm{Ind}\mathcal{D}^{b,\mathrm{qc},\geq 0}_{\mathrm{coh}}$, and then extend by shifts.) The categorical equivalence then leads to a canonical functor

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_H,\overline{\mathbb{Q}}_\ell) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell).$$

By the self-duality of $\mathcal{D}_{\mathrm{lis}}$ coming from Bernstein–Zelevinsky duality, and Proposition VII.7.10, any such functor is given by a kernel

$$A_f \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_H \times \mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell).$$

One could, in fact, identify the image of A_f under the categorical equivalence for $H \times G$; up to minor twists, it should be given by the structure sheaf of the graph of $Z^1(W_E, \widehat{H})_{\overline{\mathbb{Q}}_\ell}/\widehat{H} \longrightarrow Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}$. It would be very interesting if some examples of such kernels A_f can be constructed explicitly.

Since $D(H(E), \overline{\mathbb{Q}}_\ell)$, resp. $D(G(E), \overline{\mathbb{Q}}_\ell)$, are direct factors of $D_{\mathrm{lis}}(\mathrm{Bun}_H, \overline{\mathbb{Q}}_\ell)$, resp. $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, this should give rise to the "classical" Langlands functoriality $D(H(E), \overline{\mathbb{Q}}_\ell) \to D(G(E), \overline{\mathbb{Q}}_\ell)$.

Remark X.1.8. A kernel of functoriality should only exist in the quasi-split case. In fact, as is well-known, the local Jacquet–Langlands correspondence can not be functorial since any discrete series representation of $\mathrm{GL}_n(E)$ is defined over its field of moduli but this is not the case for smooth irreducible representations of $D^\times$ if D is a division algebra over E.

Let us now explain how *Fargues's original conjecture* fits into this context. Let $\varphi : W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be a Langlands parameter. Consider the map $i : \mathrm{Spec}\, \overline{\mathbb{Q}}_\ell \to Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}$ corresponding to φ, and let

$$\mathcal{E}_\varphi = i_* \overline{\mathbb{Q}}_\ell \in \mathcal{D}_{\mathrm{qcoh}}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}) = \mathrm{Ind}\,\mathrm{Perf}^{\mathrm{qc}}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}).$$

Factoring the map i via $[\mathrm{Spec}\, \overline{\mathbb{Q}}_\ell/S_\varphi]$, one actually sees that $\mathcal{E}_\varphi$ carries naturally an action of S_φ. Moreover, if one acts via tensoring with a representation V of $\widehat{G} \rtimes Q$, then by the projection formula the sheaf $\mathcal{E}_\varphi$ gets taken to itself, tensored with the W_E-representation $V \circ \varphi$. Using the spectral action, we find an S_φ-equivariant "automorphic complex"

$$\mathrm{Aut}_\varphi = \mathcal{E}_\varphi * \mathcal{W}_\psi \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell).$$

It already follows that $\mathrm{Aut}_\varphi \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ *is* a Hecke eigensheaf, with eigenvalue φ, so the spectral action produces Hecke eigensheaves. Except, it is not clear whether $\mathrm{Aut}_\varphi \neq 0$. Under the fully faithfulness part of the categorical conjecture, one sees that it must be nonzero, and moreover have some of the properties stated in [**Far16**], in particular regarding the relation to L-packets. The particular case of elliptic parameters is further spelled out in the next section.

X.2. Elliptic parameters

Let us make explicit what the spectral action, and Conjecture X.1.4, entails in the case of elliptic parameters. As coefficients, we take $L = \overline{\mathbb{Q}}_\ell$ for simplicity.

Definition X.2.1. *An L-parameter $\varphi : W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is elliptic if it is semisimple and the centralizer $S_\varphi \subset \widehat{G}_{\overline{\mathbb{Q}}_\ell}$ has the property that $S_\varphi/Z(\widehat{G})^{\Gamma}_{\overline{\mathbb{Q}}_\ell}$ is finite.*

By deformation theory, it follows that the unramified twists of φ define a connected component

$$C_\varphi \hookrightarrow [Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}].$$

Thus, the spectral action (in fact, the excursion operators are enough for this, see the discussion around Theorem IX.5.2) implies that there is a corresponding direct summand

$$\mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \subset \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega.$$

For any Schur-irreducible $A \in \mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, the excursion operators act via scalars on A, as determined by an unramified twist of φ. In particular, they act in this way on $i^{b*}A$ for any $b \in B(G)$. By compatibility with parabolic induction, it follows that for any $A \in \mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, the restriction $i^{b*}A$ is equal to 0 if b is not basic (if it was not zero, one could find an irreducible subquotient to which this argument applies). Thus,

$$\mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathcal{D}^{C_\varphi}(G_b(E), \overline{\mathbb{Q}}_\ell)^\omega.$$

Moreover, all $A \in \mathcal{D}^{C_\varphi}(G_b(E), \overline{\mathbb{Q}}_\ell)^\omega$ must lie in only supercuspidal components of the Bernstein center, again by compatibility with parabolic induction. If $Z(\widehat{G})^\Gamma$ is finite (equivalently, if the connected split center of G is trivial), then $C_\varphi = [*/S_\varphi]$ is a point and it follows that all A are finite direct sums of shifts supercuspidal representations of $G_b(E)$, and so

$$\mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \bigoplus_{\pi} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi,$$

where π runs over supercuspidal $\overline{\mathbb{Q}}_\ell$-representations of $G_b(E)$ with L-parameter $\varphi_\pi = \varphi$.

In general, acting on $\mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, we have the direct summand

$$\mathrm{Perf}(C_\varphi)$$

of

$$\mathrm{Perf}([Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}]).$$

If $Z(\widehat{G})^\Gamma$ is finite, one has $C_\varphi = [*/S_\varphi]$, and hence we get an action of $\mathrm{Rep}(S_\varphi)$ on $\mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$. In general, one can get a similar picture by fixing central characters; let us for simplicity only spell out the case when $Z(\widehat{G})^\Gamma$ is finite, i.e. the connected split center of G is trivial.

If π_b is a supercuspidal representation of some $G_b(E)$ with $\varphi_{\pi_b} = \varphi$, and $W \in \mathrm{Rep}(S_\varphi)$ then acting via W on π_b we get some object

$$\mathrm{Act}_W(\pi_b) \in \bigoplus_{b' \in B(G)_{\mathrm{basic}}} \bigoplus_{\pi} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi.$$

Assume that $W|_{Z(\widehat{G})^\Gamma}$ is isotypic, given by some character $\chi : Z(\widehat{G})^\Gamma \to \overline{\mathbb{Q}}_\ell^\times$. As $Z(\widehat{G})^\Gamma$ is the diagonalizable group with characters $\pi_1(G)_\Gamma$, it follows that we get an element $b_\chi \in \pi_1(G)_\Gamma = B(G)_{\mathrm{basic}}$. Then $\mathrm{Act}_W(\pi_b)$ is concentrated on $b' = b + b_\chi$, and so

$$\mathrm{Act}_W(\pi_b) \cong \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$$

for a certain multiplicity space $V_{\pi_{b'}} \in \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)$, where $\pi_{b'}$ runs over supercuspidal representations of $G_{b'}(E)$, $b' = b + b_\chi$, with L-parameter $\varphi_{\pi_{b'}} = \varphi$.

The conjectural description of L-packets [**Kal14**] then suggests the following conjecture, which is (up to the added t-exactness) the specialization of Conjecture X.1.4 to the case of elliptic L-parameters.

Conjecture X.2.2. *Assume that G is quasisplit, with a fixed Whittaker datum, and that the connected split center of G is trivial. Then there is a unique generic supercuspidal representation π of $G(E)$ with L-parameter $\varphi_\pi = \varphi$, and the functor*

$$\mathrm{Perf}([\mathrm{Spec}\,\overline{\mathbb{Q}}_\ell/S_\varphi]) \to \mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega : W \mapsto \mathrm{Act}_W(\pi)$$

is an equivalence. In particular, the set of irreducible supercuspidal representations of some $G_b(E)$ with L-parameter φ is in bijection with the set of irreducible representations of S_φ.

Moreover, the equivalence is t-exact for the standard t-structures on source and target.

Thus, the conjecture gives an explicit parametrization of L-packets.

Let us explain what the compatibility of the spectral action with Hecke operators entails in this case. Given $V \in \mathrm{Rep}(\widehat{G} \rtimes Q)$, the restriction of V to S_φ admits a commuting W_E-action given by φ. This defines a functor

$$\mathrm{Rep}(\widehat{G} \rtimes Q) \to \mathrm{Rep}(S_\varphi)^{BW_E}.$$

Now the diagram of monoidal functors

$$\begin{tikzcd} \mathrm{Rep}(\widehat{G} \rtimes Q) \arrow[r] \arrow[d] & \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell))^{BW_E} \\ \mathrm{Rep}(S_\varphi)^{BW_E} \arrow[ur] & \end{tikzcd}$$

commutes; this follows from the compatibility of the spectral action with the Hecke action.

Concretely, given π as above and $V \in \mathrm{Rep}(\widehat{G} \rtimes Q)$, decompose the image of V in $\mathrm{Rep}(S_\varphi)^{BW_E}$ as a direct sum $\bigoplus_{i \in I} W_i \otimes \sigma_i$ where $W_i \in \mathrm{Rep}(S_\varphi)$ is irreducible and σ_i is some continuous representation of W_E on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-space. Then

$$T_V(\pi) \cong \bigoplus_{i \in I} \mathrm{Act}_{W_i}(\pi) \otimes \sigma_i.$$

Recall that $T_V(\pi)$ can be calculated concretely through the cohomology of local Shimura varieties, or in general moduli spaces of local shtukas. Noting that the functor Act_{W_i} is realizing a form of the Jacquet–Langlands correspondence relating different inner forms, the formula above is essentially the conjecture of Kottwitz [**RV14**, Conjecture 7.3]. In fact, assuming Conjecture X.2.2, it is an easy exercise to deduce [**RV14**, Conjecture 7.3], assuming that the parametrization of the Conjecture X.2.2 agrees with the parametrization implicit in [**RV14**, Conjecture 7.3].

X.3. Integral coefficients

We want to construct the spectral action with integral coefficients. Unfortunately, the naive analogue of Theorem X.1.1 is not true, the problem being that the analogue of Lemma X.1.2 fails. However, the rest of the argument still works, and gives the following result.

Consider a split reductive group H over a discrete valuation ring R with an action of a finite group Q. Let S be any anima over BQ. As before, we can define a derived stack $\mathrm{Map}_{BQ}(S, B(H \rtimes Q))$ over R, whose values in an animated R-algebra A are the maps of anima $S \to B(H \rtimes Q)(A)$ over BQ. In general, the functor $S \mapsto \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$ does not commute with sifted colimits in S.

However, we can consider the best approximation to it that does commute with sifted colimits. Note that the ∞-category of anima over BQ is the animation of the category of sets equipped with a Q-torsor; it is freely generated under sifted colimits by the category of finite sets equipped with a Q-torsor. Thus, the sifted-colimit approximation to $S \mapsto \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$ is the animation of its restriction to finite sets with Q-torsors; we denote it by

$$S \mapsto \mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))),$$

with the idea in mind that it is like the ∞-category of perfect complexes on some (nonexistent) derived stack $\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))$, gotten as a (co-)sifted limit approximation to $\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))$. The symbol Σ here is in reference to the notation used in [**Lur09**, Section 5.5.8] in relation to sifted colimits. Thus $\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$ is an R-linear idempotent-complete small stable ∞-category, mapping to $\mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$.

Proposition X.3.1. *Let $\mathcal{C}$ be an R-linear idempotent-complete small stable ∞-category. Giving, functorially in finite sets I, an exact $\mathrm{Rep}_R(Q^I)$-linear monoidal functor*

$$\mathrm{Rep}_R((H \rtimes Q)^I) \to \mathrm{End}_R(\mathcal{C})^{S^I}$$

is equivalent to giving an R-linear action of $\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$ on $\mathcal{C}$. Here, given such an action of $\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$, one gets exact $\mathrm{Rep}_R(Q^I)$-linear monoidal functors

$$\mathrm{Rep}_R((H \rtimes Q)^I) \to \mathrm{End}_R(\mathcal{C})^{S^I}$$

by composing the exact monoidal functor $\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))) \to \mathrm{End}_R(\mathcal{C})$ with the natural exact $\mathrm{Rep}_R(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_R((H \rtimes Q)^I) \to \mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))^{S^I}$$

given by I-fold tensor product of the exact $\mathrm{Rep}_R(Q)$-linear symmetric monoidal functor

$$\mathrm{Rep}_R(H \rtimes Q) \to \mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))^{S}$$

assigning to each $s \in S$ pullback along evaluation at s, $\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) \to B(H \rtimes Q)$; more precisely, it is defined in this way if S is a finite set, and in general by animation.

Proof. This follows from the proof of Theorem X.1.1. □

To make use of Proposition X.3.1, we need to find sufficiently many situations in which the functor

$$\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))) \to \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$$

is an equivalence, and specifically we need to prove this for $\mathrm{Map}_{BQ}(BW, B(\widehat{G} \rtimes Q)) = Z^1(W, \widehat{G})/\widehat{G}$.

First, we have the following result.

PROPOSITION X.3.2. *The functor* $S \mapsto \mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$ *from anima over* BQ *to symmetric monoidal idempotent-complete stable* R*-linear* ∞*-categories commutes with all colimits.*

PROOF. As the functor commutes with sifted colimits by definition, it suffices to show that when restricted to finite sets S equipped with Q-torsors, it commutes with disjoint unions. But for such S, the map $S \to BQ$ can be factored over a point, and then $\mathrm{Map}_{BQ}(S, B(H \rtimes Q)) = BH^S$. Thus, one has to see that for two finite sets S_1, S_2, the functor

$$\mathrm{Perf}(BH^{S_1}) \otimes_{\mathrm{Perf}(R)} \mathrm{Perf}(BH^{S_2}) \to \mathrm{Perf}(BH^{S_1 \sqcup S_2})$$

is an equivalence. But this follows easily from highest weight theory, which for any split reductive group H filters $\mathrm{Perf}(BH)$ in terms of copies of $\mathrm{Perf}(R)$ enumerated by highest weights. □

PROPOSITION X.3.3. *Assume that* $S = BF_n$ *is the classifying space of a free group. Then the functor*

$$\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q))) \to \mathrm{Perf}(\mathrm{Map}_{BQ}(S, B(H \rtimes Q)))$$

is fully faithful, and the essential image is the idempotent-complete stable ∞*-subcategory generated by the image of* $\mathrm{Rep}_R(H)$*.*

PROOF. Represent $BF_n \to BQ$ by a map $F_n \to Q$, and let $\sigma_1, \ldots, \sigma_n \in Q$ be the images of the generators. Then $\mathrm{Map}_{BQ}(S, B(H \rtimes Q))$ can be identified with $[H^n/H]$, where H acts on H^n via the $(\sigma_1, \ldots, \sigma_n)$-twisted diagonal conjugation action. We claim that

$$\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$$

is the ∞-category of compact objects in the ∞-category of modules over $\mathcal{O}(H^n)$ in $\mathrm{Ind}\,\mathrm{Perf}(BH)$; in fact, this is equivalent to the claim, as by Barr–Beck this gives a description of the full ∞-subcategory of $\mathrm{Perf}([H^n/H])$ generated by $\mathrm{Perf}(BH)$.

As $\mathcal{O}(H^n) = \mathcal{O}(H) \otimes \ldots \otimes \mathcal{O}(H)$ in $\mathrm{Ind}\,\mathrm{Perf}(BH)$, one reduces to the case $n = 1$. In that case $S = BF_1$ is a circle, which we can present as a pushout of $* \sqcup * \rightrightarrows *$. Thus, we have to compute

$$\mathrm{Perf}(BH) \otimes_{\mathrm{Perf}(BH^2)} \mathrm{Perf}(BH)$$

where the two implicit maps $H \to H^2$ are given by the diagonal and the σ-twisted diagonal, respectively. As the pullback functors $\mathrm{Perf}(BH^2) \to \mathrm{Perf}(BH)$ generate the image, we can write $\mathrm{Perf}(BH) = \mathrm{Perf}(H/H^2)$ as the compact objects in the ∞-category of $\mathcal{O}(H)$-modules in $\mathrm{Ind}\,\mathrm{Perf}(BH^2)$. Similarly, the expected answer $\mathrm{Perf}([H/H]) = \mathrm{Perf}([H \times H/H^2])$ is given by the compact objects in the ∞-category of modules over $\mathcal{O}(H \times H) = \mathcal{O}(H) \otimes \mathcal{O}(H)$ in $\mathrm{Ind}\,\mathrm{Perf}(BH^2)$, thus implying the result. □

PROPOSITION X.3.4. *Let* $S = B\Gamma$*, where* Γ *is any discrete group, and lift the map* $S \to BQ$ *to a map* $\Gamma \to Q$*. Note that one can write*

$$B\Gamma = \mathrm{colim}_{(n, F_n \to \Gamma)} BF_n$$

as a sifted colimit (in anima). Then $\mathrm{Perf}(\mathrm{Map}^{\Sigma}_{BQ}(S, B(H \rtimes Q)))$ *is the* ∞*-category of compact objects in the* ∞*-category of modules over*

$$\mathrm{colim}_{(n, F_n \to \Gamma)} \mathcal{O}(H^n)$$

in $\mathrm{Ind}\,\mathrm{Perf}(BH)$*, where* $\mathcal{O}(H^n)$ *is equipped with the twisted (via the map* $F_n \to \Gamma \to Q$*) diagonal conjugation of* H*.*

Proof. As $\mathbb{E}_1$-groups in anima are equivalent to animated groups, with compact projective generators the free groups F_n, it follows that $B\Gamma$ is the sifted colimit $\operatorname{colim}_{(n,F_n\to\Gamma)} BF_n$. Now the result follows from the previous proposition (and its proof), together with the commutation with sifted colimits. □

Combining this with Theorem VIII.5.1, we have finished the proof of Theorem X.0.2. In particular, this gives the spectral action on $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G)$.

Let us end by stating again the main conjecture with integral coefficients. Here, we again presume that the spectral action is defined for all ℓ.

Conjecture X.3.5. *Assume that G is quasisplit and choose Whittaker data consisting of a Borel $B \subset G$ and generic character $\psi : U(E) \to \mathcal{O}_L^\times$ of the unipotent radical $U \subset B$, where $L/\mathbb{Q}_\ell$ is some algebraic extension; also fix $\sqrt{q} \in \mathcal{O}_L$. Then there is an equivalence*

$$\mathcal{D}(\mathrm{Bun}_G, \mathcal{O}_L)^\omega \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G})$$

of stable ∞-categories equipped with actions of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G})$. Under this correspondence, the structure sheaf of $Z^1(W_E, \widehat{G})_{\mathcal{O}_L}/\widehat{G}$ maps to the Whittaker sheaf, which is the sheaf concentrated on Bun_G^1 corresponding to the Whittaker representations $c\text{-}\mathrm{Ind}_{U(E)}^{G(E)}\psi$.

The notion of nilpotent singular support here is defined via the results of Section VIII.2.2.

Bibliography

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615.

[AB10] Dmitry Arinkin and Roman Bezrukavnikov, *Perverse coherent sheaves*, Mosc. Math. J. **10** (2010), no. 1, 3–29, 271. MR 2668828

[AG15] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. (N.S.) **21** (2015), no. 1, 1–199.

[AJ84] H. H. Andersen and J. C. Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann. **269** (1984), no. 4, 487–525.

[Ans18] J. Anschütz, *Extending torsors on the punctured* $\mathrm{Spec}(A_{\mathrm{inf}})$, arXiv:1804.06356, 2018.

[Ans19] ———, *Reductive group schemes over the Fargues-Fontaine curve*, Math. Ann. **374** (2019), no. 3-4, 1219–1260.

[BFH$^+$17] C. Birkbeck, T. Feng, D. Hansen, S. Hong, Q. Li, A. Wang, and L. Ye, *Extensions of vector bundles on the Fargues-Fontaine curve*, arXiv:1705.00710, to appear in J. Inst. Math. Jussieu, 2017.

[BG19] R. Bellovin and T. Gee, *G-valued local deformation rings and global lifts*, Algebra Number Theory **13** (2019), no. 2, 333–378.

[BKV15] R. Bezrukavnikov, D. Kazhdan, and Y. Varshavsky, *A categorical approach to the stable center conjecture*, Astérisque (2015), no. 369, 27–97.

[BM18] B. Bhatt and A. Mathew, *The arc-topology*, arXiv:1807.04725, 2018.

[BMR05] M. Bate, B. Martin, and G. Röhrle, *A geometric approach to complete reducibility*, Invent. Math. **161** (2005), no. 1, 177–218.

[BMS18] B. Bhatt, M. Morrow, and P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397.

[Bor67] K. Borsuk, *Theory of retracts*, Monografie Matematyczne, Tom 44, Państwowe Wydawnictwo Naukowe, Warsaw, 1967.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[Bor98] M. Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50.

[Boy99] P. Boyer, *Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale*, Invent. Math. **138** (1999), no. 3, 573–629.

[Boy09] ———, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. **177** (2009), no. 2, 239–280.

[BP19] J. Booher and S. Patrikis, *G-valued Galois deformation rings when $\ell \neq p$*, Math. Res. Lett. **26** (2019), no. 4, 973–990.

[Bra03] T. Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), no. 3, 209–216.

[Bre78] L. Breen, *Extensions du groupe additif*, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 39–125.

[Bru98] J. Brundan, *Dense orbits and double cosets*, Algebraic groups and their representations (Cambridge, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 517, Kluwer Acad. Publ., Dordrecht, 1998, pp. 259–274.

[BS15] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque (2015), no. 369, 99–201.

[BS17] ———, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.

[BS19] ———, *Prisms and prismatic cohomology*, arXiv:1905.08229, 2019.

[BT84] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376.

[Col02] P. Colmez, *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu **1** (2002), no. 3, 331–439.

[CS] D. Clausen and P. Scholze, *Condensed Mathematics*, in preparation, see `www.math.uni-bonn.de/people/scholze/Condensed.pdf`, `www.math.uni-bonn.de/people/scholze/Analytic.pdf` for an account.

[CS17] A. Caraiani and P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766.

[CS19a] ———, *On the generic part of the cohomology of non-compact unitary Shimura varieties*, arXiv:1909.01898, 2019.

[ČS19b] K. Česnavičius and P. Scholze, *Purity for flat cohomology*, arXiv:1912.10932, 2019.

[Dat07] J.-F. Dat, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. **169** (2007), no. 1, 75–152.

[DHKM20] J.-F. Dat, D. Helm, R. Kurinczuk, and G. Moss, *Moduli of Langlands Parameters*, arXiv:2009.06708, 2020.

[dJvdP96] J. de Jong and M. van der Put, *Étale cohomology of rigid analytic spaces*, Doc. Math. **1** (1996), No. 01, 1–56 (electronic).

[DM82] P. Deligne and J. S. Milne, *Tannakian categories*, Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982), 1982.

[Dol80] A. Dold, *Lectures on algebraic topology*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 200, Springer-Verlag, Berlin-New York, 1980.

[Don81] Stephen Donkin, *A filtration for rational modules*, Math. Z. **177** (1981), no. 1, 1–8.

[Don85] ———, *Rational representations of algebraic groups*, Lecture Notes in Mathematics, vol. 1140, Springer-Verlag, Berlin, 1985, Tensor products and filtration.

[Don88] ———, *Skew modules for reductive groups*, J. Algebra **113** (1988), no. 2, 465–479.

[Dri80] V. G. Drinfeld, *Langlands' conjecture for* GL(2) *over functional fields*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 565–574.

[DS95] V. G. Drinfeld and C. Simpson, *B-structures on G-bundles and local triviality*, Math. Res. Lett. **2** (1995), no. 6, 823–829.

[Far04] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, no. 291, Astérisque, 2004, pp. 1–199.

[Far08] ———, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Progr. Math., vol. 262, Birkhäuser, Basel, 2008, pp. 1–325.

[Far10] ———, *Talk at the conference in honor of Jean-Marc Fontaine*, `https://webusers.imj-prg.fr/~laurent.fargues/ExposeConfFontaine.pdf`, 2010.

[Far16] ———, *Geometrization of the local Langlands correspondence: an overview*, arXiv:1602.00999, 2016.

[Far18a] ———, *From local class field to the curve and vice versa*, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 181–198.

[Far18b] ———, *G-torseurs en théorie de Hodge p-adique*, to appear, Comp. Math., 2018.

[FF18] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque (2018), no. 406, xiii+382, With a preface by Pierre Colmez.

[FG12] J. Fulman and R. Guralnick, *Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements*, Trans. Amer. Math. Soc. **364** (2012), no. 6, 3023–3070.

[Gai01] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), no. 2, 253–280.

[Gai07] ———, *On de Jong's conjecture*, Israel J. Math. **157** (2007), 155–191.

[Gai16] ———, *From geometric to function-theoretic Langlands (or how to invent shtukas)*, arXiv:1606.09608, 2016.

[GHKR10] U. Görtz, T. J. Haines, R. E. Kottwitz, and D. C. Reuman, *Affine Deligne-Lusztig varieties in affine flag varieties*, Compos. Math. **146** (2010), no. 5, 1339–1382.

[Gin90] V. A. Ginzburg, *Sheaves on a loop group, and Langlands duality*, Funktsional. Anal. i Prilozhen. **24** (1990), no. 4, 76–77.

[GKRV19] D. Gaitsgory, D. Kazhdan, N. Rozenblyum, and Y. Varshavsky, *A toy model for the Drinfeld-Lafforgue shtuka construction*, arXiv:1908.05420, 2019.

[GL17] A. Genestier and V. Lafforgue, *Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale*, arXiv:1709.00978, 2017.

[GR03] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Mathematics, vol. 1800, Springer-Verlag, Berlin, 2003.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167.

[Gro64] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 20, 259.

[Gul74] T. H. Gulliksen, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. **34** (1974), 167–183.

[Hab75] W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. (2) **102** (1975), no. 1, 67–83.

[Hai14] T. J. Haines, *The stable Bernstein center and test functions for Shimura varieties*, Automorphic forms and Galois representations. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 415, Cambridge Univ. Press, Cambridge, 2014, pp. 118–186.

[Han17] D. Hansen, *Degenerating vector bundles in p-adic Hodge theory*, http://www.davidrenshawhansen.com/degen.pdf, to appear in J. Inst. Math. Jussieu, 2017.

[Har97] M. Harris, *Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol's program*, Invent. Math. **129** (1997), no. 1, 75–119.

[Hau05] T. Hausberger, *Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 4, 1285–1371.

[HC66] Harish-Chandra, *Discrete series for semisimple Lie groups. II. Explicit determination of the characters*, Acta Math. **116** (1966), 1–111.

[Hel16] D. Helm, *The Bernstein center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$-modules*, Forum Math. Sigma **4** (2016), Paper No. e11, 98.

[Hel20] E. Hellmann, *On the derived category of the iwahori-hecke algebra*, arXiv:2006.03013, 2020.

[Hen00] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455.

[HH20] U. Hartl and E. Hellmann, *The universal family of semistable p-adic Galois representations*, Algebra Number Theory **14** (2020), no. 5, 1055–1121.

[HK20] D. Hansen and K. S. Kedlaya, *Sheafiness criteria for Huber rings*, https://kskedlaya.org/papers/criteria.pdf, 2020.

[HM13] C. Hague and G. McNinch, *Some good-filtration subgroups of simple algebraic groups*, J. Pure Appl. Algebra **217** (2013), no. 12, 2400–2413.

[HM18] D. Helm and G. Moss, *Converse theorems and the local Langlands correspondence in families*, Invent. Math. **214** (2018), no. 2, 999–1022.

[HP04] U. Hartl and R. Pink, *Vector bundles with a Frobenius structure on the punctured unit disc*, Compos. Math. **140** (2004), no. 3, 689–716.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Hub96] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hub98] ———, *A comparison theorem for ℓ-adic cohomology*, Compositio Math. **112** (1998), no. 2, 217–235.

[Ill06] L. Illusie, *Vanishing cycles over general bases, after P. Deligne, O. Gabber, G. Laumon and F. Orgogozo*, `https://www.imo.universite-paris-saclay.fr/~illusie/vanishing1b.pdf`, 2006.

[Iva20] A. B. Ivanov, *On ind-representability of loop Deligne–Lusztig sheaves*, arXiv:2003.04399, 2020.

[Jor08] D. A. Jorgensen, *Finite projective dimension and the vanishing of* $\mathrm{Ext}_R(M, M)$, Comm. Algebra **36** (2008), no. 12, 4461–4471.

[Kal14] T. Kaletha, *Supercuspidal L-packets via isocrystals*, Amer. J. Math. **136** (2014), no. 1, 203–239.

[Kal18] ———, *Rigid inner forms vs isocrystals*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 1, 61–101.

[Kat82] S. Kato, *Spherical functions and a q-analogue of Kostant's weight multiplicity formula*, Invent. Math. **66** (1982), no. 3, 461–468.

[Ked04] K. S. Kedlaya, *A p-adic local monodromy theorem*, Ann. of Math. (2) **160** (2004), no. 1, 93–184.

[Ked16] ———, *Noetherian properties of Fargues-Fontaine curves*, Int. Math. Res. Not. IMRN (2016), no. 8, 2544–2567.

[Kem76] G. R. Kempf, *Linear systems on homogeneous spaces*, Ann. of Math. (2) **103** (1976), no. 3, 557–591.

[Kem78] ———, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316.

[KL15] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: foundations*, Astérisque (2015), no. 371, 239pp.

[KL16] ———, *Relative p-adic Hodge theory, II: Imperfect period rings*, arXiv:1602.06899, 2016.

[Kop84] M. Koppinen, *Good bimodule filtrations for coordinate rings*, J. London Math. Soc. (2) **30** (1984), no. 2, 244–250.

[Kot85] R. E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220.

[Kot97] ———, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339.

[KW17] T. Kaletha and J. Weinstein, *On the Kottwitz conjecture for local Shimura varieties*, arXiv:1709.06651, 2017.

[Lab99] J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque (1999), no. 257, vi+161pp, Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[Laf18] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891.

[Lan89] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., vol. 31, Amer. Math. Soc., Providence, RI, 1989, pp. 101–170.

[Lau04] E. Lau, *On generalized $\mathcal{D}$-shtukas*, Thesis, 2004.

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

[LRS93] G. Laumon, M. Rapoport, and U. Stuhler, *$\mathscr{D}$-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), no. 2, 217–338.

[LTX+19] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, *On the Beilinson-Bloch-Kato conjecture for Rankin-Selberg motives*, arXiv:1912.11942, 2019.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur16] ———, *Higher algebra*, `http://www.math.harvard.edu/~lurie/papers/HA.pdf`, 2016.

[Lus76] G. Lusztig, *On the finiteness of the number of unipotent classes*, Invent. Math. **34** (1976), no. 3, 201–213.

[Lus83] ———, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229.

[LY13] G. Lusztig and Z. Yun, *A $(-q)$-analogue of weight multiplicities*, J. Ramanujan Math. Soc. **28A** (2013), 311–340.

[LZ19] Q. Lu and W. Zheng, *Duality and nearby cycles over general bases*, Duke Math. J. **168** (2019), no. 16, 3135–3213.

[LZ20] ———, *Categorical traces and a relative Lefschetz-Verdier formula*, arXiv:2005.08522, 2020.

[Mat90] O. Mathieu, *Filtrations of G-modules*, Ann. Sci. École Norm. Sup. (4) **23** (1990), no. 4, 625–644.

[ML95] S. Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition.

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[Nee96] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[NS18] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409.

[NY19] D. Nadler and Z. Yun, *Spectral action in Betti geometric Langlands*, Israel J. Math. **232** (2019), no. 1, 299–349.

[PY02] G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. **147** (2002), no. 3, 545–560.

[PY06] ———, *On quasi-reductive group schemes*, J. Algebraic Geom. **15** (2006), no. 3, 507–549, With an appendix by Brian Conrad.

[Ren10] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés, vol. 17, Société Mathématique de France, Paris, 2010.

[Ric88] R. W. Richardson, *Conjugacy classes of n-tuples in Lie algebras and algebraic groups*, Duke Math. J. **57** (1988), no. 1, 1–35.

[Ric19] T. Richarz, *Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles*, J. Algebraic Geom. **28** (2019), no. 2, 251–289.

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181.

[RV14] M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[Sch11] P. Scholze, *Perfectoid Spaces and the Weight-Monodromy Conjecture*, 2011, Cours d'Arithmétique et de Géométrie Algébrique, `http://www.ihes.fr/~abbes/CAGA/scholze.html`.

[Sch12] ———, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

[Sch13] ———, *The local Langlands correspondence for* GL_n *over p-adic fields*, Invent. Math. **192** (2013), no. 3, 663–715.

[Sch14] ———, *The local Langlands correspondence and equivariant sheaves on projective space*, 2014, `https://www.msri.org/workshops/731/schedules/17615`.

[Sch15] ———, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066.

[Sch17a] ———, *Étale cohomology of diamonds*, `www.math.uni-bonn.de/people/scholze/EtCohDiamonds.pdf`, 2017.

[Sch17b] ———, *On the local Langlands conjectures for reductive groups*, 2017, Hadamard lectures at the IHÉS.

[Ser94] J.-P. Serre, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994.

[SR72] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin-New York, 1972.

[SS13] P. Scholze and S. W. Shin, *On the cohomology of compact unitary group Shimura varieties at ramified split places*, J. Amer. Math. Soc. **26** (2013), no. 1, 261–294.

[Sta] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`.

[Ste68] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

[SW13] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics **1** (2013), no. 2, 145–237.

[SW20] ———, *Berkeley lectures on p-adic geometry*, vol. 207, Princeton, NJ: Princeton University Press, 2020.

[Tem21] M. Temkin, *Topological transcendence degree*, J. Algebra **568** (2021), 35–60.

[Tit59] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent*, Inst. Hautes Études Sci. Publ. Math. (1959), no. 2, 13–60.

[TvdK10] A. Touzé and W. van der Kallen, *Bifunctor cohomology and cohomological finite generation for reductive groups*, Duke Math. J. **151** (2010), no. 2, 251–278.

[vdK01] W. van der Kallen, *Steinberg modules and Donkin pairs*, Transform. Groups **6** (2001), no. 1, 87–98.

[Vie21a] E. Viehmann, *On newton strata in the B^+_{dR}-grassmannian*, `https://arxiv.org/abs/2101.07510`, 2021.

[Vie21b] ———, *On the Newton strata in the B^+_{dr}-Grassmannian*, arXiv:2101.07510, 2021.

[Vog93] D. Vogan, Jr., *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.

[Wei17] J. Weinstein, $\mathrm{G}al(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *as a geometric fundamental group*, Int. Math. Res. Not. IMRN (2017), no. 10, 2964–2997.

[Xue20] C. Xue, *Smoothness of cohomology sheaves of stacks of shtukas*, arXiv:2012.12833, 2020.

[XZ17] L. Xiao and X. Zhu, *Cycles on Shimura varieties via geometric Satake*, arXiv:1707.05700, 2017.

[Yu19] J. Yu, *The integral geometric Satake equivalence in mixed characteristic*, arXiv:1903.11132, 2019.

[Zhu17] X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), no. 2, 403–492.

[Zhu20] ———, *Coherent sheaves on the stack of Langlands parameters*, arXiv:2008.02998, 2020.

[Zie15] P. Ziegler, *Graded and filtered fiber functors on Tannakian categories*, J. Inst. Math. Jussieu **14** (2015), no. 1, 87–130.