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Constructive exact controls for some semilinear PDEs

ARNAUD MÜNCH

Let Ω be a bounded domain of \mathbb{R}^d , $d \in \{2, 3\}$ with $C^{1,1}$ boundary and $\omega \subset\subset \Omega$ be a non empty open set. Let $T > 0$ and denote $Q_T := \Omega \times (0, T)$, $q_T := \omega \times (0, T)$ and $\Sigma_T := \partial\Omega \times (0, T)$. We consider the semilinear wave equation

$$(1) \quad \begin{cases} \square y + g(y) = f1_\omega, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1), & \text{in } \Omega, \end{cases}$$

where $(u_0, u_1) \in \mathbf{V} := H_0^1(\Omega) \times L^2(\Omega)$ is the initial state of y , $f \in L^2(q_T)$ is a control function and $\square y := \partial_{tt}y - \Delta y$. $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 such that $|g(r)| \leq C(1 + |r|) \ln(2 + |r|)$ for every $r \in \mathbb{R}$ and some $C > 0$. (1) has a unique global weak solution in $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ (see [2]). The exact controllability for (1) in time T is formulated as follows: for any $(u_0, u_1), (z_0, z_1) \in \mathbf{V}$, find a control function $f \in L^2(q_T)$ such that the weak solution of (1) satisfies $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$. This problem has been solved by Fu, Yong and Zhang:

Theorem 1. [Fu, Yong, Zhang, 2007] For any $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$, let $\Gamma_0 = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\}$ and, for any $\epsilon > 0$, let $\mathcal{O}_\epsilon(\Gamma_0) = \{y \in \mathbb{R}^d \mid |y - x| \leq \epsilon \text{ for } x \in \Gamma_0\}$. Assume

$$(\mathbf{H}_0) \quad T > 2 \max_{x \in \bar{\Omega}} |x - x_0| \text{ and } \omega = \mathcal{O}_\epsilon(\Gamma_0) \cap \Omega \text{ for some } \epsilon > 0,$$

$$(\mathbf{H}_1) \quad \limsup_{|r| \rightarrow \infty} \frac{|g(r)|}{|r| \ln^{1/2}|r|} = 0,$$

then (1) is exactly controllable in time T .

Γ_0 is the star-shaped part of the whole boundary of Ω introduced in [11]. Theorem 1 extends to the multi-dimensional case the result of [15] devoted to the one dimensional case. The proof given in [5] is based on a fixed point argument introduced in [15]: it is shown that the operator $K : L^\infty(0, T; L^d(\Omega)) \rightarrow L^\infty(0, T; L^d(\Omega))$ where $y_\xi := K(\xi)$ is a controlled solution through the control function f_ξ (of minimal $L^2(q_T)$ -norm) of the linear boundary value problem

$$\begin{cases} \square y_\xi + y_\xi \widehat{g}(\xi) = -g(0) + f_\xi 1_\omega, & \text{in } Q_T, \\ y_\xi = 0, & \text{on } \Sigma_T, \\ (y_\xi(\cdot, 0), \partial_t y_\xi(\cdot, 0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \quad \widehat{g}(r) := \begin{cases} \frac{g(r) - g(0)}{r} & r \neq 0, \\ g'(0) & r = 0 \end{cases},$$

satisfying $(y_\xi(\cdot, T), y_{\xi,t}(\cdot, T)) = (z_0, z_1)$ has a fixed point. The existence of a fixed point for the compact operator K is obtained by using the Leray-Schauder's degree theorem: it is shown under the growth assumption (\mathbf{H}_1) that there exists a constant $M = M(\|u_0, u_1\|_{\mathbf{V}}, \|z_0, z_1\|_{\mathbf{V}})$ such that K maps the ball $B_{L^\infty(0, T; L^d(\Omega))}(0, M)$ into itself.

Our goal is to construct an explicit sequence $(f_k)_{k \in \mathbb{N}}$ that converges strongly to an exact control for (1). The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see [3]). However, as far as we know, few are concerned with the approximation of exact controls for

nonlinear partial differential equations, and the construction of convergent control approximations for nonlinear equations remains a challenge.

A first idea that comes to mind is to consider the Picard iterations $(y_k)_{k \in \mathbb{N}}$ associated with the operator K defined by $y_{k+1} = K(y_k)$, $k \geq 0$ initialized with any element $y_0 \in L^\infty(0, T; L^d(\Omega))$. Such a strategy usually fails since the operator K is in general not contracting, even if g is globally Lipschitz.

Given any initial data $(u_0, u_1) \in \mathbf{V}$, we design an algorithm providing a sequence $(y_k, f_k)_{k \in \mathbb{N}}$ converging to a controlled pair for (1), under assumptions on g that are slightly stronger than (\mathbf{H}_1) . Moreover, after a finite number of iterations, the convergence is super-linear. This is done by introducing a least-squares functional measuring how much a pair $(y, f) \in \mathcal{A}$ is close to a controlled solution for (1) and then by determining a particular minimizing sequence enjoying the announced property. We define the Hilbert space \mathcal{H}

$$\mathcal{H} = \left\{ (y, f) \in L^2(Q_T) \times L^2(Q_T) \mid \square y \in L^2(Q_T), (y(\cdot, 0), \partial_t y(\cdot, 0)) \in \mathbf{V}, y = 0 \text{ on } \Sigma_T \right\}.$$

Then, for any $(u_0, u_1), (z_0, z_1) \in \mathbf{V}$, we define the subspaces of \mathcal{H}

$$\mathcal{A} = \left\{ (y, f) \in \mathcal{H} \mid (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), (y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1) \right\},$$

$$\mathcal{A}_0 = \left\{ (y, f) \in \mathcal{H} \mid (y(\cdot, 0), \partial_t y(\cdot, 0)) = (0, 0), (y(\cdot, T), \partial_t y(\cdot, T)) = (0, 0) \right\}$$

and consider the following non convex extremal problem

$$(2) \quad \inf_{(y, f) \in \mathcal{A}} E(y, f), \quad E(y, f) := \frac{1}{2} \|\square y + g(y) - f \mathbf{1}_\omega\|_2^2$$

observing that any zero $(y, f) \in \mathcal{A}$ of E is a solution of the controllability problem. Our main result is

Theorem 2. [Lemoine, Münch, 2021] Assume for some $s \in (0, 1]$

$$(\overline{\mathbf{H}}_s) \quad [g']_s := \sup_{\substack{a, b \in \mathbb{R} \\ a \neq b}} \frac{|g'(a) - g'(b)|}{|a - b|^s} < +\infty,$$

$$(\mathbf{H}_2) \quad \text{There exists } \alpha \geq 0 \text{ and } \beta \in [0, \sqrt{\frac{s}{2C(2s+1)}}) \text{ such that } |g'(r)| \leq \alpha + \beta \ln^{1/2}(1 + |r|) \text{ for every } r \text{ in } \mathbb{R}.$$

Then, for any $(y_0, f_0) \in \mathcal{A}$, the sequence $(y_k, f_k)_{k \in \mathbb{N}}$ defined by

$$(3) \quad \begin{cases} (y_0, f_0) \in \mathcal{A}, \\ (y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k (Y_k^1, F_k^1), \quad k \in \mathbb{N}, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, 1]} E((y_k, f_k) - \lambda (Y_k^1, F_k^1)), \end{cases}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$(4) \quad \begin{cases} \square Y_k^1 + g'(y_k) Y_k^1 = F_k^1 \mathbf{1}_\omega + \square y_k + g(y_k) - f_k \mathbf{1}_\omega, & \text{in } Q_T, \\ Y_k^1 = 0, & \text{on } \Sigma_T, \\ (Y_k^1(\cdot, 0), \partial_t Y_k^1(\cdot, 0)) = (0, 0), & \text{in } \Omega \end{cases}$$

strongly converges to a pair $(\bar{y}, \bar{f}) \in \mathcal{A}$ satisfying (1). Moreover, the convergence is at least linear and is at least of order $1 + s$ after a finite number of iterations.

As far as we know, the method described here is the first one providing an explicit, algorithmic construction of exact controls for semilinear wave equations with non Lipschitz nonlinearity and defined over multi-dimensional bounded domains. It extends the one-dimensional study addressed in [14]. The parabolic case can be addressed as well: for semilinear heat equation, we mention [6] for $d \in \{2, 3\}$ with Lipschitz nonlinearity and [10] for $d = 1$ and non Lipschitz nonlinearity. These works devoted to controllability problems takes their roots in [9, 7] concerned with the direct problem for the Navier-Stokes equation: they refine the analysis performed in [8, 12] inspired from the seminal contribution [1].

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