Existence in critical spaces for the magnetohydrodynamical system in 3D bounded Lipschitz domains

Sylvie Monniaux

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Abstract

Existence of mild solutions for the 3D MHD system in bounded Lipschitz domains is established in critical spaces with the *absolute* boundary conditions.

1 Introduction

The magnetohydrodynamical system in a domain $\Omega \subset \mathbb{R}^3$ on a time interval $(0,T)$ $(0 < T \leq \infty)$ as considered in [17] (with all constants equal to 1) reads

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla \pi + (u\cdot\nabla)u &=& (\operatorname{curl} b)\times b & \text{in} \quad (0,T)\times\Omega \\ \partial_t b - \Delta b &=& \operatorname{curl}(u\times b) & \text{in} \quad (0,T)\times\Omega \\ \operatorname{div} u &=& 0 & \text{in} \quad (0,T)\times\Omega \\ \operatorname{div} b &=& 0 & \text{in} \quad (0,T)\times\Omega \end{array}\right. \tag{MHD}$$

where $u : (0,T)\times\Omega \to \mathbb{R}^3$ denotes the *velocity* of the (incompressible homogeneous) fluid, the *magnetic field* (in the absence of magnetic monopole) is denoted by $b : (0,T)\times\Omega \to \mathbb{R}^3$ and $\pi : (0,T)\times\Omega \to \mathbb{R}^3$ is the pressure of the fluid. The first equation of (MHD) corresponds to *Navier-Stokes* equations subject to the *Laplace force* $(\operatorname{curl} b)\times b$ applied by the magnetic field b. Actually, the divergence-free condition on the magnetic field b comes from the fact that b is in the range of the curl operator. The second equation of (MHD) describes the evolution of the magnetic field following the so-called *induction* equation.

This system (MHD) (with $T = \infty$ and $\Omega = \mathbb{R}^3$) is invariant under the scaling $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$, $b_\lambda(t,x) = \lambda b(\lambda^2 t, \lambda x)$ and $\pi_\lambda(t,x) = \lambda^2\pi(\lambda^2 t, \lambda x)$, $\lambda > 0$. This suggests that a critical space for (u,b) is $\mathscr{C}([0,\infty);L^3(\mathbb{R}^3)^3)\times\mathscr{C}([0,\infty);L^3(\mathbb{R}^3)^3)$.

The purpose of this paper is to prove existence of solutions of this system in this critical space in a bounded Lipschitz domain under the so-called *absolute* boundary conditions, denoted by (BC1) below. This is investigated in Theorem 3.3, Theorem 3.4 in Section 3. The methods used here come from the theory developed in [9] for the absolute boundary conditions.

In Section 2 are collected results on potential operators (similar to the famous Bogovskĭi operator),the Stokes operators with Dirichlet boundary conditions and Hodge boundary conditions, as well as properties of the Hodge Laplacian in bounded Lipschitz domains. Section 3 is devoted to the existence of mild solutions of the system (MHD) under absolute boundary conditions on a bounded Lipschitz domain in critical spaces.

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†partially supported by the ANR project INFAMIE, ANR-15-CE40-0011

2 Tools

In this section are recalled some results proved in [9] which will be useful in the following. See also [11] and [12].

Notation 2.1. For an (unbounded) operator A on a Banach space X, we denote by $\mathsf{D}(A)$ its domain, $\mathsf{R}(A)$ its range and $\mathsf{N}(A)$ its null space.

2.1 Differential forms, Potential operators

We consider the *exterior derivative* $d := \nabla\wedge = \sum_{j=1}^n \partial_j e_j\wedge$ and the *interior derivative* (or co-derivative) $\delta := -\nabla\lrcorner = -\sum_{j=1}^n \partial_j e_j\lrcorner$ acting on *differential forms* on a domain $\Omega \subset \mathbb{R}^n$, i.e. acting on functions from Ω to the exterior algebra $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$ of $\mathbb{R}^n$.

We denote by $\{e_S\,;\, S \subset \{1,\dots,n\}\}$ the basis for Λ. The space of ℓ-vectors Λ^ℓ is the span of $\{e_S\,;\, |S| = \ell\}$, where

$$e_S = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_\ell} \quad \text{for} \quad S = \{e_{j_1}, \dots, e_{j_\ell}\} \quad \text{with} \;\; j_1 < j_2 < \cdots < j_\ell.$$

Remark that Λ^0, the space of complex scalars, is the span of $e_\emptyset$ ($\emptyset$ being the empty set). We set $\Lambda^\ell = \{0\}$ if $\ell < 0$ or $\ell > n$.

On the exterior algebra Λ, the basic operations are

(i) the exterior product $\wedge : \Lambda^k \times \Lambda^\ell \to \Lambda^{k+\ell}$,

(ii) the interior product $\lrcorner : \Lambda^k \times \Lambda^\ell \to \Lambda^{\ell-k}$,

(iii) the Hodge star operator $\star : \Lambda^\ell \to \Lambda^{n-\ell}$,

(iv) the inner product $\langle\cdot,\cdot\rangle : \Lambda^\ell \times \Lambda^\ell \to \mathbb{R}$.

If $a \in \Lambda^1$, $u \in \Lambda^\ell$ and $v \in \Lambda^{\ell+1}$, then

$$\langle a \wedge u, v\rangle = \langle u, a \lrcorner v\rangle.$$

For more details, we refer to, e.g., [3, Section 2] and [6, Section 2], noting that both these papers contain some historical background (and being careful that δ has the opposite sign in [3]). In particular, we note the relation between d and δ via the Hodge star operator:

$$\star\delta u = (-1)^\ell d(\star\, u) \quad \text{and} \quad \star\, du = (-1)^{\ell-1}\delta(\star\, u) \quad \text{for an } \ell\text{-form } u. \tag{2.1}$$

In dimension $n = 3$, this gives (see [6, §2]) for a vector $a \in \mathbb{R}^3$ identified with a 1-form

- u scalar, interpreted as 0-form: $a \wedge u = ua$, $a \lrcorner u = 0$;
- u scalar, interpreted as 3-form: $a \wedge u = 0$, $a \lrcorner u = ua$;
- u vector, interpreted as 1-form: $a \wedge u = a \times u$, $a \lrcorner u = a \cdot u$;
- u vector, interpreted as 2-form: $a \wedge u = a \cdot u$, $a \lrcorner u = -a \times u$.

The domains of the differential operators d and δ, denoted by $\mathsf{D}(d)$ and $\mathsf{D}(\delta)$ are defined by

$$\mathsf{D}(d) := \big\{u \in L^2(\Omega,\Lambda); du \in L^2(\Omega,\Lambda)\big\} \quad \text{and} \quad \mathsf{D}(\delta) := \big\{u \in L^2(\Omega,\Lambda); \delta u \in L^2(\Omega,\Lambda)\big\}.$$

Similarly, the L^p versions of these domains read

$$\mathsf{D}^p(d) := \big\{u \in L^p(\Omega,\Lambda); du \in L^p(\Omega,\Lambda)\big\} \;\text{ and }\; \mathsf{D}^p(\delta) := \big\{u \in L^p(\Omega,\Lambda); \delta u \in L^p(\Omega,\Lambda)\big\}.$$

The differential operators d and δ satisfiy $d^2 = d \circ d = 0$ and $\delta^2 = \delta \circ \delta = 0$. We will also consider the adjoints of d and δ in the sense of maximal adjoint operators in a Hilbert space: $\underline{\delta} := d^*$ and

$\underline{d} := \delta^*$. They are defined as the closures in $L^2(\Omega,\Lambda)$ of the closable operators $\big(d^*, \mathscr{C}_c^\infty(\Omega,\Lambda)\big)$ and $\big(\delta^*, \mathscr{C}_c^\infty(\Omega,\Lambda)\big)$.

The following proposition has been proved in [9, Proposition 4.1] in a slightly more general framework (see also [10, Theorem 1.5] and [6, Theorem 1.1, Theorem 4.6 and Remark 4.12]).

Proposition 2.2. *Suppose Ω is a bounded Lipschitz domain. Then the potential operators R_Ω, S_Ω and K_Ω defined above satisfy for all $p \in (1,\infty)$, with the convention $p^S = \frac{np}{n-p}$ if $p < n$, $p^S = +\infty$ if $p > n$ and $p^S \in [n,+\infty)$ if $p = n$,*

$$
\begin{aligned}
&R_\Omega : L^p(\Omega,\Lambda) \to L^{p^S}(\Omega,\Lambda) \cap \mathsf{D}^p(d), \quad S_\Omega : L^p(\Omega,\Lambda) \to L^{p^S}(\Omega,\Lambda) \cap \mathsf{D}^p(d^*),\\
&K_\Omega : L^p(\Omega,\Lambda) \to L^\infty(\Omega,\Lambda) \cap \mathsf{D}^p(d), \quad K_\Omega^* : L^p(\Omega,\Lambda) \to L^\infty(\Omega,\Lambda) \cap \mathsf{D}^p(d^*),\\
&K_\Omega, K_\Omega^* \text{ are compact operators in } L^p(\Omega,\Lambda),\\
&dR_\Omega + R_\Omega d = \mathrm{I} - K_\Omega, \qquad d^*S_\Omega + S_\Omega d^* = \mathrm{I} - K_\Omega^*,\\
&dK_\Omega = 0, \quad d^*K_\Omega^* = 0 \quad \text{and} \quad K_\Omega = 0 \text{ on } \mathsf{R}^p(d), \quad K_\Omega^* = 0 \text{ on } \mathsf{R}^p(d^*),\\
&dR_\Omega u = u \text{ if } u \in \mathsf{R}^p(d), \quad d^*S_\Omega u = u \text{ if } u \in \mathsf{R}^p(d^*).
\end{aligned}
$$

As direct consequence we obtain that dR_Ω and d^*S_Ω are projections from $L^p(\Omega,\Lambda)$ onto the ranges of d and d^*, $\mathsf{R}^p(d)$ and $\mathsf{R}^p(d^*)$, for all $p \in (1,\infty)$.

2.2 Hodge-Laplacian and Hodge-Stokes operators in Lipschitz domains

Definition 2.3. The *Hodge-Dirac operator* on Ω with *tangential boundary conditions* is

$$D_\parallel := d + d^*.$$

Note that $-\Delta_\parallel := D_\parallel^2 = dd^* + d^*d$ is the *Hodge-Laplacian* with *absolute* (generalised Neumann) boundary conditions.

For a scalar function $u : \Omega \to \Lambda^0$ we have that $-\Delta_\parallel u = d^*du = -\Delta_N u$, where Δ_N is the Neumann Laplacian.

Following [2, Section 4], we have that the operator $D_\parallel$ is a closed densely defined operator in $L^2(\Omega,\Lambda)$, and that

$$
\begin{aligned}
L^2(\Omega,\Lambda) =& \mathsf{R}(d) \overset{\perp}{\oplus} \mathsf{R}(d^*) \overset{\perp}{\oplus} \mathsf{N}(D_\parallel) && (H_2)\\
=& \mathsf{R}(d) \overset{\perp}{\oplus} \mathsf{N}(d^*) && (2.2)\\
=& \mathsf{N}(d) \overset{\perp}{\oplus} \mathsf{R}(d^*) && (2.3)
\end{aligned}
$$

where $\mathsf{N}(D_\parallel) = \mathsf{N}(d) \cap \mathsf{N}(d^*) = \mathsf{N}\big(\Delta_\parallel\big)$ is finite dimensional. The orthogonal projection from $L^2(\Omega,\Lambda)$ onto $\mathsf{N}(d^*)$ (see (2.2)), restricted to 1-forms, is the well-known Helmholtz (or Leray) projection denoted by $\mathbb{P}$. Restricted to 2-forms, the orthogonal projection from $L^2(\Omega,\Lambda)$ onto $\mathsf{R}(d)$ will be denoted in the sequel by $\mathbb{Q}$.

The p version of the previous Hodge decompositions can be found in [9, Theorem 4.3]: there exist Hodge exponents p_H, $p^H = p'_H$ with $1 \le p_H < 2 < p^H \le \infty$ such that

$$
\begin{aligned}
L^p(\Omega,\Lambda) =& \mathsf{R}^p(d) \oplus \mathsf{R}^p(d^*) \oplus \mathsf{N}(D_\parallel) && (H_p)\\
=& \mathsf{R}^p(d) \oplus \mathsf{N}^p(d^*) && (2.4)\\
=& \mathsf{N}^p(d) \oplus \mathsf{R}^p(d^*) && (2.5)
\end{aligned}
$$

for all $p \in (p_H, p^H)$ and the projections $\mathbb{P} : L^p(\Omega, \Lambda^1) \to \mathsf{N}^p(d^*)_{|\Lambda^1}$ and $\mathbb{Q} : L^p(\Omega, \Lambda^2) \to \mathsf{R}^p(d)_{|\Lambda^2}$ extend accordingly.

Remark 2.4. If the domain is smooth or have a Lipschitz boundary, we have the following estimates on the Hodge exponents p_H and p^H.

1. If $\Omega \subset \mathbb{R}^n$ is smooth, then $p_H = 1$ and $p^H = \infty$ (see [16, Theorems 2.4.2 and 2.4.14].
2. In the case of a bounded Lipschitz domain, $p_H < \frac{2n}{n+1}$ and consequently $p^H > \frac{2n}{n-1}$, which gives in dimension $n = 3$: $p_H < \frac{3}{2}$ and $p^H > 3$ (see [9, §7]).

Remark 2.5. Proposition 2.2 and the projections $\mathbb{P}$ and $\mathbb{Q}$ yield

$$\begin{aligned} \mathbb{P}(R_\Omega du + K_\Omega u) &= u \quad \text{for } u \in \mathsf{N}^p(d^*)_{|\Lambda^1}, \\ \mathbb{Q}(S_\Omega d^* b + K_\Omega^* b) &= b \quad \text{for } b \in \mathsf{R}^p(d)_{|\Lambda^2} \end{aligned}$$

for all $p_H < p < p^H$. The second equation comes from the fact that $\mathsf{R}^p(d) \subset \mathsf{N}^p(d)$, using (2.5).

The following results can be found partly in [11, Theorem 7.3] (sectoriality) and in [9, §8] (improvement of the interval of p for the Hodge-Stokes operator and bounded holomorphic functional calculus):

Theorem 2.6. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Define $-\Delta_\parallel = D_\parallel^2$ in $L^2(\Omega, \Lambda)$. If $p_H < p < p^H$, then the Hodge-Laplacian with absolute boundary conditions $-\Delta_\parallel$ is sectorial of angle 0 in $L^p(\Omega, \Lambda)$ and for all $\mu \in (0, \frac{\pi}{2})$, $-\Delta_\parallel$ admits a bounded $S^\circ_{\mu+}$ holomorphic functional calculus in $L^p(\Omega, \Lambda)$.*

Define the Hodge-Stokes operator by $S_\parallel := D_\parallel^2 = d^*d$ in $\mathsf{N}^2(d^*)$, restricted to 1-forms. If $\max\{1, \frac{np_H}{n+p_H}\} < p < p^H$, then $S_\parallel$ is sectorial of angle 0 in $\mathsf{N}^p(d^*)_{|\Lambda^1}$ and for all $\mu \in (0, \frac{\pi}{2})$, $S_\parallel$ admits a bounded $S^\circ_{\mu+}$ holomorphic functional calculus in $\mathsf{N}^p(d^*)_{|\Lambda^1}$. In particular, the semigroup $(e^{-tS_\parallel})_{t\geq 0}$ is bounded on $\mathsf{N}^p(d^*)_{|\Lambda^1}$ with norm denoted by $K_{p,S}$.

Define the Hodge-Maxwell operator $M_\parallel := dd^*$ in $\mathsf{N}(d)$, restricted to 2-forms. If $\max\{1, \frac{np_H}{n+p_H}\} < p < p^H$, then $M_\parallel$ is sectorial of angle 0 in $\mathsf{N}^p(d)_{|\Lambda^2}$ and for all $\mu \in (0, \frac{\pi}{2})$, $M_\parallel$ admits a bounded $S^\circ_{\mu+}$ holomorphic functional calculus in $\mathsf{N}^p(d)_{|\Lambda^2}$. In particular, the semigroup $(e^{-tM_\parallel})_{t\geq 0}$ is bounded on $\mathsf{R}^p(d)_{|\Lambda^1}$ with norm denoted by $K_{p,M}$.

Using the results stated in Remark 2.5, one can prove $L^p - L^q$ bounds for the operator $S_\parallel$ (resp. $M_\parallel$) (see [12, Theorems 3.1 and 4.1] for the dimension 3 and [8, Theorem 1.1] for the Riesz transform like estimates (2.7) and (2.9)).

Theorem 2.7. *Let $p \in \big(\max\{1, \frac{np_H}{n+p_H}\}, p^H\big)$ and $q \in [p, p^H)$ such that $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{q}$ for some $\alpha \in [0, 1]$. Then the semigroup $(e^{-tS_\parallel})_{t\geq 0}$ in $\mathsf{N}^p(d^*)_{|\Lambda^1}$ satisfies the estimates*

$$c^S_{p,q} := \sup_{t\geq 0} \big\| t^{\frac{\alpha}{2}} e^{-tS_\parallel} \big\|_{\mathsf{N}^p(d^*)_{|\Lambda^1} \to L^q} + \sup_{t\geq 0} \big\| t^{\frac{1+\alpha}{2}} d e^{-tS_\parallel} \big\|_{\mathsf{N}^p(d^*)_{|\Lambda^1} \to L^q} < \infty \tag{2.6}$$

and

$$\gamma^S_{p,q} := \| S_\parallel^{-\frac{\alpha}{2}} \|_{\mathsf{N}^p(d^*)_{|\Lambda^1} \to L^q} < \infty. \tag{2.7}$$

The semigroup $(e^{-tM_\parallel})_{t\geq 0}$ in $\mathsf{R}^p(d)_{|\Lambda^2}$ satisfies the estimate

$$c^M_{p,q} := \sup_{t\geq 0} \big\| t^{\frac{\alpha}{2}} e^{-tM_\parallel} \big\|_{\mathsf{R}^p(d)_{|\Lambda^2} \to L^q} + \sup_{t\geq 0} \big\| t^{\frac{1+\alpha}{2}} d^* e^{-tM_\parallel} \big\|_{\mathsf{R}^p(d)_{|\Lambda^2} \to L^q} < \infty \tag{2.8}$$

and

$$\gamma^M_{p,q} := \| M_\parallel^{-\frac{\alpha}{2}} \|_{\mathsf{R}^p(d)_{|\Lambda^2} \to L^q} < \infty. \tag{2.9}$$

3 Existence in the case of absolute boundary conditions

Thanks to the formula

$$(u\cdot\nabla)u=\tfrac{1}{2}\nabla|u|^2+u\times(\operatorname{curl}u)$$

for a sufficiently smooth vector field u, the system (MHD) can be reformulated as follows:

$$\left\{\begin{array}{rcll}\partial_t u-\Delta u+\nabla\pi_1-u\times(\operatorname{curl}u) &=& (\operatorname{curl}b)\times b & \text{in}\quad (0,T)\times\Omega\\ \partial_t b-\Delta b &=& \operatorname{curl}(u\times b) & \text{in}\quad (0,T)\times\Omega\\ \operatorname{div}u &=& 0 & \text{in}\quad (0,T)\times\Omega\\ \operatorname{div}b &=& 0 & \text{in}\quad (0,T)\times\Omega\end{array}\right. \tag{3.1}$$

where the pressure π has been replaced by the so-called *dynamical pressure* $\pi_1=\pi+\frac{1}{2}|u|^2$. This formulation can be translated in the language of differential forms: π_1 is a scalar function, interpreted as 0-form, u is a vector field interpreted as 1-form and b is a vector field interpreted as 2-form. Following Section 2 one can rewrite (3.1) in terms of differential forms:

$$\left\{\begin{array}{rcll}\partial_t u+S_\parallel u+d\pi_1+u\lrcorner\, du &=& -d^*b\lrcorner\, b & \text{in}\quad (0,T)\times\Omega\\ \partial_t b+M_\parallel b &=& -d(u\lrcorner\, b) & \text{in}\quad (0,T)\times\Omega\\ u(t,\cdot) &\in& \mathsf{N}(d^*)_{|_{\Lambda^1}} & \text{for all}\quad t\in(0,T)\\ b(t,\cdot) &\in& \mathsf{R}(d)_{|_{\Lambda^2}} & \text{for all}\quad t\in(0,T).\end{array}\right. \tag{MHD1}$$

The terms in the first equation are all 1-forms, in the second equation the terms are all 2-forms. The *absolute* boundary conditions associated with the previous system (MHD1) are defined by the term d^* in $-\Delta_\parallel=(dd^*+d^*d)$:

$$\left\{\begin{array}{rcll}\nu\cdot u=\nu\lrcorner\, u &=& 0 & \text{on}\quad (0,T)\times\partial\Omega\\ -\nu\times\operatorname{curl}u=\nu\lrcorner\, du &=& 0 & \text{on}\quad (0,T)\times\partial\Omega\\ -\nu\times b=\nu\lrcorner\, b &=& 0 & \text{on}\quad (0,T)\times\partial\Omega\\ \nu\operatorname{div}b=\nu\lrcorner\, db &=& 0 & \text{on}\quad (0,T)\times\partial\Omega.\end{array}\right.\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{absolute b.c. for the 1-form } u\\ \left.\begin{array}{l}\\ \\ \end{array}\right\}\text{absolute b.c. for the 2-form } b\end{array} \tag{BC1}$$

This formulation can be used, for instance, to study the magnetohydrodynamical system in dimensions greater than or equal to 2 with the same theoretical tools. Let us point out that these boundary conditions are different to those usually investigated in magnetohydrodynamical problems, starting with the paper [17]; see also [1]. The boundary conditions (BC1) in the case of Navier-Stokes equations (*i.e.* for $b=0$) have been studied in [12]; see also [14] and [15].

Remark 3.1. The last condition in (BC1) is void since $b\in\mathsf{R}(d)_{|_{\Lambda^2}}$: $db=0$ in all Ω.

Definition 3.2. Let $\Omega\subseteq\mathbb{R}^3$. A mild solution of the system (MHD1) with absolute boundary conditions (BC1) and initial conditions $u_0\in\mathsf{N}(d^*)_{|_{\Lambda^1}}$ and $b_0\in\mathsf{R}(d)_{|_{\Lambda^2}}$ is a pair (u,b) of vector fields satisfying

$$u(t)=e^{-tS_\parallel}u_0+\int_0^t e^{-(t-s)S_\parallel}\mathbb{P}\big(-u(s)\lrcorner\, du(s)\big)\,\mathrm{d}s+\int_0^t e^{-(t-s)S_\parallel}\mathbb{P}\big(-d^*b(s)\lrcorner\, b(s)\big)\,\mathrm{d}s, \tag{3.2}$$

$$b(t)=e^{-tM_\parallel}b_0+\int_0^t e^{-(t-s)M_\parallel}\Big(-d\big(u(s)\lrcorner\, b(s)\big)\Big)\,\mathrm{d}s. \tag{3.3}$$

From now on, we assume the following technical (Leibniz rule-like) property on the domain $\Omega\subset\mathbb{R}^3$: for all $q\in[3,p^H)$, there exists a constant $C_q>0$ such that

$$\|d(\omega_1\lrcorner\,\omega_2)\|_{\frac{q}{2}}\leq C_q\big(\|D_\parallel\omega_1\|_q\|\omega_2\|_q+\|\omega_1\|_q\|D_\parallel\omega_2\|_q\big) \tag{3.4}$$

for all $\omega_1\in\mathsf{D}^p(D_\parallel)\cap L^q(\Omega,\Lambda^1)$ and all $\omega_2\in\mathsf{D}^p(D_\parallel)\cap L^q(\Omega,\Lambda^2)$. This is the case if the domain Ω is smooth.

The following theorem is about the global existence of mild solutions with small initial data.

Theorem 3.3 (Global existence)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain or $\Omega = \mathbb{R}^3$. Then there exists $\varepsilon > 0$ such that for all $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$ with $\|u_0\|_3 + \|b_0\|_3 \leq \varepsilon$, the system* (MHD1) *with the boundary conditions* (BC1) *and $T = \infty$ admits a mild solution $u, b \in \mathscr{C}([0,\infty); L^3(\Omega)^3)$.*

The next result states local existence of mild solutions with no restriction on the size of the initial data.

Theorem 3.4 (Local existence)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain or $\Omega = \mathbb{R}^3$. Then for all $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$ there exists $T > 0$ such that the system* (MHD1) *with the boundary conditions* (BC1) *admits a mild solution $u, b \in \mathscr{C}([0,T); L^3(\Omega)^3)$.*

The methods to prove these two theorems are classical based on a fixed point theorem, already used for the Navier-Stokes equations in the paper by Fujita and Kato [7] (see also [13]) and in [5] (see also [4]) for the Boussinesq system. Most of the tools used here appeared in the paper [12]; see also [9].

Let $q \in \big(3, \min\{p^H, 6\}\big)$ and $\alpha \in (0,1)$ such that $\frac{1}{q} = \frac{1}{3} - \frac{\alpha}{3}$. For $0 < T \leq \infty$, we define the following spaces

$$\begin{aligned}\mathscr{U}_T :=& \big\{ u \in \mathscr{C}((0,T); \mathsf{N}^q(d^*)_{|_{\Lambda^1}}); du \in \mathscr{C}((0,T); L^q(\Omega, \Lambda^2)) : \\ & \sup_{0<t<T} \big(t^{\frac{\alpha}{2}} \|u(t)\|_q + t^{\frac{1+\alpha}{2}} \|du(t)\|_q \big) < \infty \big\}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathscr{B}_T :=& \big\{ b \in \mathscr{C}((0,T); \mathsf{R}^q(d)_{|_{\Lambda^2}}); d^*b \in \mathscr{C}((0,T); L^q(\Omega, \Lambda^1)) : \\ & \sup_{0<t<T} \big(t^{\frac{\alpha}{2}} \|b(t)\|_q + t^{\frac{1+\alpha}{2}} \|d^*b(t)\|_q \big) < \infty \big\},\end{aligned} \tag{3.6}$$

endowed with the norms

$$\|u\|_{\mathscr{U}_T} := \sup_{0<t<T} \big(t^{\frac{\alpha}{2}} \|u(t)\|_q + t^{\frac{1+\alpha}{2}} \|du(t)\|_q \big) \tag{3.7}$$

and

$$\|b\|_{\mathscr{B}_T} := \sup_{0<t<T} \big(t^{\frac{\alpha}{2}} \|b(t)\|_q + t^{\frac{1+\alpha}{2}} \|d^*b(t)\|_q \big). \tag{3.8}$$

Lemma 3.5. *For $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$, we have*

1. *$a_1 : t \mapsto e^{-tS_\parallel} u_0 \in \mathscr{U}_T$,*
2. *$a_2 : t \mapsto e^{-tM_\parallel} b_0 \in \mathscr{B}_T$,*

for all $T > 0$ Moreover, for all $\varepsilon > 0$, there exists $T > 0$ such that

$$\|a_1\|_{\mathscr{U}_T} + \|a_2\|_{\mathscr{B}_T} \leq \varepsilon. \tag{3.9}$$

Proof. By Theorem 2.7, the following bound holds for all $T > 0$:

$$\|a_1\|_{\mathscr{U}_T} + \|a_2\|_{\mathscr{B}_T} \leq c^S_{3,q} \|u_0\|_3 + c^M_{3,q} \|b_0\|_3. \tag{3.10}$$

Therefore, if $\|u_0\|_3$ and $\|b_0\|_3$ are small enough, (3.9) holds for every $T > 0$.

For any u_0 and b_0 (not necessarily small in the L^3 norm), for $\epsilon > 0$, let $u_{0,\epsilon} \in \mathsf{N}^q(d^*)_{|_{\Lambda^1}}$ and $b_{0,\epsilon} \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$ such that

$$\|u_0 - u_{0,\epsilon}\|_3 + \|b_0 - b_{0,\epsilon}\|_3 \leq \epsilon.$$

We denote by $a_{1,\epsilon}$ and $a_{2,\epsilon}$ the quantities $a_{1,\epsilon}(t) = e^{-tS_\parallel}u_{0,\epsilon}$ and $a_{2,\epsilon}(t) = e^{-tM_\parallel}b_{0,\epsilon}$. By (3.10), there holds

$$\|a_1 - a_{1,\epsilon}\|_{\mathscr{U}_T} + \|a_2 - a_{2,\epsilon}\|_{\mathscr{B}_T} \leq \epsilon\,(c^S_{3,q} + c^M_{3,q}). \tag{3.11}$$

Applying (2.6) with $p = q$ and $\alpha = 0$, we obtain

$$\|a_{1,\epsilon}\|_{\mathscr{U}_T} \leq c^s_{q,q} T^{\frac{\alpha}{2}} \|u_{0,\epsilon}\|_q.$$

The same reasoning applying (2.8) with $p = q$ and $\alpha = 0$ yields

$$\|a_{2,\epsilon}\|_{\mathscr{B}_T} \leq c^M_{q,q} T^{\frac{\alpha}{2}} \|b_{0,\epsilon}\|_q.$$

Now choosing $\epsilon > 0$ small enough and $T > 0$ small enough, we find that (3.9) holds. □

Next, we define the operators

$$B_1(u,v)(t) = \int_0^t e^{-(t-s)S_\parallel}\mathbb{P}\big(-u(s)\lrcorner\, dv(s)\big)\,\mathrm{d}s, \quad t \in [0,T),\ u,v \in \mathscr{U}_T, \tag{3.12}$$

$$B_2(b,b')(t) = \int_0^t e^{-(t-s)S_\parallel}\mathbb{P}\big(-d^*b(s)\lrcorner\, b'(s)\big)\,\mathrm{d}s, \quad t \in [0,T),\ b,b' \in \mathscr{B}_T, \tag{3.13}$$

$$B_3(u,b)(t) = \int_0^t e^{-(t-s)M_\parallel}\Big(-d\big(u(s)\lrcorner\, b(s)\big)\Big)\,\mathrm{d}s, \quad t \in [0,T),\ u \in \mathscr{U}_T,\ b \in \mathscr{B}_T. \tag{3.14}$$

The next lemma gives a precise statement about the boundedness of the bilinear operators B_1, B_2 and B_3.

Lemma 3.6. *The bilinear operators B_1, B_2 and B_3 are bounded in the following spaces:*

1. $B_1 : \mathscr{U}_T \times \mathscr{U}_T \to \mathscr{U}_T$,
2. $B_2 : \mathscr{B}_T \times \mathscr{B}_T \to \mathscr{U}_T$,
3. $B_3 : \mathscr{U}_T \times \mathscr{B}_T \to \mathscr{B}_T$

with norms independent from $T > 0$.

Proof. 1. For $u, v \in \mathscr{U}_T$, by definition of $\mathscr{U}_T$ we have that $s \mapsto s^{\frac{1}{2}+\alpha}u(s)\lrcorner\, dv(s) \in \mathscr{C}_b((0,T); L^{\frac{q}{2}}(\Omega,\Lambda^1)$ with norm less than or equal to $\|u\|_{\mathscr{U}_T}\|v\|_{\mathscr{U}_T}$. Since $\frac{3}{2} < \frac{q}{2} < 3$, $\mathbb{P}$ is bounded from $L^{\frac{q}{2}}(\Omega,\Lambda^1)$ to $\mathsf{N}^{\frac{q}{2}}(d^*)_{|_{\Lambda^1}}$ Moreover, $e^{-(t-s)S_\parallel}$ maps $\mathsf{N}^{\frac{q}{2}}(d^*)_{|_{\Lambda^1}}$ to $\mathsf{N}^{q}(d^*)_{|_{\Lambda^1}}$ with norm $c^S_{\frac{q}{2},q}(t-s)^{-\frac{1-\alpha}{2}}$ thanks to (2.6) with $p = \frac{q}{2}$. Therefore, we have for all $t \in (0,T)$

$$\|B_1(u,v)(t)\|_q \lesssim \Big(\int_0^t s^{-\frac{1}{2}-\alpha}(t-s)^{-\frac{1-\alpha}{2}}\,\mathrm{d}s\Big)\|u\|_{\mathscr{U}_T}\|v\|_{\mathscr{U}_T} \lesssim t^{-\frac{\alpha}{2}}\|u\|_{\mathscr{U}_T}\|v\|_{\mathscr{U}_T}.$$

This gives the first estimate for $B_1(u,v) \in \mathscr{U}_T$. For the second estimate, we note that $de^{-(t-s)S_\parallel}$ maps $\mathsf{N}^{\frac{q}{2}}(d^*)_{|_{\Lambda^1}}$ to $L^q(\Omega,\Lambda^2)$ with norm $c^S_{\frac{q}{2},q}(t-s)^{-1+\frac{\alpha}{2}}$ thanks to (2.6) with $p = \frac{q}{2}$. Therefore, we have for all $t \in (0,T)$

$$\|dB_1(u,v)(t)\|_q \lesssim \Big(\int_0^t s^{-\frac{1}{2}-\alpha}(t-s)^{-1+\frac{\alpha}{2}}\,\mathrm{d}s\Big)\|u\|_{\mathscr{U}_T}\|v\|_{\mathscr{U}_T} \lesssim t^{-\frac{1+\alpha}{2}}\|u\|_{\mathscr{U}_T}\|v\|_{\mathscr{U}_T},$$

which gives the second estimate for $B_1(u,v) \in \mathscr{U}_T$.

2. The proof that for $b, b' \in \mathscr{B}_T$, $B_2(b,b') \in \mathscr{U}_T$ with norm independent from $T > 0$ follows the lines of the previous point. We omit the details here.

3. Thanks to the property (3.4), the proof that for $u \in \mathscr{U}_T$ and $b \in \mathscr{B}_T$, $B_3(u,b) \in \mathscr{B}_T$ with norm independent from $T > 0$ can be copied from the proof of point 1, using the fact that $d^* e^{-(t-s)M_\parallel}$ maps $\mathsf{R}^{\frac{q}{2}}(d)_{|_{\Lambda^2}}$ to $L^q(\Omega,\Lambda^1)$ with norm $c^M_{\frac{q}{2},q}(t-s)^{-1+\frac{\alpha}{2}}$ thanks to (2.8) with $p = \frac{q}{2}$.

This proves Lemma 3.6. □

Lemma 3.7. *Let $T > 0$. Assume that $(u,b) \in \mathscr{U}_T \times \mathscr{B}_T$ is a mild solution of* (MHD1) *with absolute boundary conditions* (BC1) *with initial conditions $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$. Then $u \in \mathscr{C}_b([0,T);\mathsf{N}^3(d^*)_{|_{\Lambda^1}})$ and $b \in \mathscr{C}_b([0,T);\mathsf{R}^3(d)_{|_{\Lambda^2}})$.*

Proof. To prove this lemma, first observe that if $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$, then for all $T > 0$, $t \mapsto e^{-tS_\parallel}u_0 \in \mathscr{C}_b([0,T);\mathsf{N}^3(d^*)_{|_{\Lambda^1}})$ and $t \mapsto e^{-tM_\parallel}b_0 \in \mathscr{C}_b([0,T);\mathsf{R}^3(d)_{|_{\Lambda^2}})$. It remains to show that if $u \in \mathscr{U}_T$ and $b \in \mathscr{B}_T$, then $B_1(u,u) \in \mathscr{C}_b([0,T);\mathsf{N}^3(d^*)_{|_{\Lambda^1}})$, $B_2(b,b) \in \mathscr{C}_b([0,T);\mathsf{N}^3(d^*)_{|_{\Lambda^1}})$ and $B_3(u,b) \in \mathscr{C}_b([0,T);\mathsf{R}^3(d)_{|_{\Lambda^2}})$. The continuity is straightforward. To prove boundedness, it suffices to reproduce the proof of the previous lemma (recall that $\alpha = 1 - \frac{3}{q}$) to obtain

$$\|B_1(u,u)(t)\|_3 \lesssim \Big(\int_0^t s^{-\frac{1}{2}-\alpha}(t-s)^{-\frac{1}{2}+\alpha}\,\mathrm{d}s\Big)\|u\|^2_{\mathscr{U}_T} \lesssim \|u\|^2_{\mathscr{U}_T},$$

using the fact that $e^{-(t-s)S_\parallel}$ maps $\mathsf{N}^{\frac{q}{2}}(d^*)_{|_{\Lambda^1}}$ to $L^3(\Omega,\Lambda^1)$ with norm controlled by $c^S_{\frac{q}{2},3}(t-s)^{-\frac{3}{q}+\frac{1}{2}}$ thanks to (2.6). The terms B_2 and B_3 can be treated similarly. □

Proof of Theorems 3.3 and 3.4. The system

$$u = a_1 + B_1(u,u) + B_2(b,b) \quad \text{and} \quad b = a_2 + B_3(u,b), \quad (u,b) \in \mathscr{U}_T \tag{3.15}$$

can be reformulated as

$$\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{B}(\boldsymbol{u},\boldsymbol{u}) \tag{3.16}$$

where $\boldsymbol{u} = (u,b) \in \mathscr{U}_T \times \mathscr{B}_T$, $\boldsymbol{a} = (a_1,a_2)$ and $\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}) = (B_1(u,v) + B_2(b,b'), B_3(u,b'))$ if $\boldsymbol{u} = (u,b)$ and $\boldsymbol{v} = (v,b')$. On $\mathscr{U}_T \times \mathscr{B}_T$ we choose the norm $\|(u,b)\|_{\mathscr{U}_T\times\mathscr{B}_T} := \|u\|_{\mathscr{U}_T} + \|b\|_{\mathscr{B}_T}$. One can easily check, using Lemma 3.6, that

$$\|\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v})\|_{\mathscr{U}_T\times\mathscr{B}_T} \le C\|\boldsymbol{u}\|_{\mathscr{U}_T\times\mathscr{B}_T}\|\boldsymbol{v}\|_{\mathscr{U}_T\times\mathscr{B}_T}$$

where C is a constant independent from $T > 0$. We can then apply Picard's fixed point theorem to prove that for $u_0 \in \mathsf{N}^3(d^*)_{|_{\Lambda^1}}$ and $b_0 \in \mathsf{R}^3(d)_{|_{\Lambda^2}}$, with $T \le \infty$ such that (3.9) holds for $\varepsilon = \frac{1}{4C}$, the system (3.16) admits a unique solution $\boldsymbol{u} = (u,b) \in \mathscr{U}_T \times \mathscr{B}_T$. By Lemma 3.7, this provides a mild solution $(u,b) \in \mathscr{C}_b([0,T);\mathsf{N}^3(d^*)_{|_{\Lambda^1}}) \times \mathscr{C}_b([0,T);\mathsf{R}^3(d)_{|_{\Lambda^2}})$ of (MHD1) with boundary conditions (BC1). □

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