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# Constrained state estimation for nonlinear systems: a redesign approach based on convexity

Daniele Astolfi, Pauline Bernard, Romain Postoyan and Lorenzo Marconi

**Abstract**—Given a plant whose trajectories of interest remain in a known compact set and an associated observer, we propose a general framework to modify this observer so that its state remains in a given convex set for all times, without altering the observer guaranteed performances in terms of convergence and robustness to external disturbances. The methodology can be applied to any time-varying continuous-time, discrete-time and hybrid system/observer, for which a quadratic Lyapunov function is used for the analysis. The proposed approach is relevant, for instance, to remove the peaking phenomenon, to attenuate the effect of impulsive outliers in the measurement, to avoid aberrant estimates during transients, or to guarantee a given range for variables in embedded systems.

**Index Terms**—Nonlinear observers, Lyapunov methods, hybrid dynamical systems, discrete-time, nonlinear systems.

## I. INTRODUCTION

### A. Motivation

Numerous observer design methods are available in the literature for continuous-time systems, see, e.g., [1]–[14], discrete-time systems, see, e.g. [1], [15]–[21], and hybrid systems [22], see, e.g. [23]–[36], to mention a few. In general, the robust convergence of the state estimate to the plant state is ensured, under suitable conditions. Now, in many applications, the state of the plant is evolving in a given compact set  $\mathcal{X}$  describing the set of interest of the variables. However, unless the observation problem is formulated as a constrained optimization problem from the start [37], the estimated state generated by the observer could in principle leave the set  $\mathcal{X}$  for some time, because of transient dynamics or perturbations. These excursions outside  $\mathcal{X}$  may be significant in terms of amplitude and time, and may lead to the next issues.

- (Accuracy) When the estimate is outside  $\mathcal{X}$ , the information provided by the observer is likely to be inaccurate and so potentially unexploitable by the user. A typical example is the peaking phenomenon occurring in high-gain observers [7].

- (Aberrant estimates) The state estimate may violate the physics of the plant. To give an example, when the concentrations of Lithium-ion in the electrodes of an electrochemical battery are estimated, e.g. [38], it may occur that the estimated concentrations generated by the observer are negative for some time, which is physically impossible. This is the case in general for positive systems (see, e.g., [39]).
- (Implementation issues) When implementing an observer on an embedded systems, we often need to specify the range of the estimated states, and this range may be limited.
- (Destabilization) It may lead to instability in output feedback designs, see e.g. [7].
- (Existence of observer dynamics) The observer dynamics may not be well defined outside the set  $\mathcal{X}$ . This issue is typical when the observer is not implemented in the same coordinates of the plant dynamics, see, e.g., [40]–[43].

A simple idea to address some of these issues is to saturate or to project into  $\mathcal{X}$  the state estimate generated by the observer, while preserving its inner dynamics, as done in the output feedback stabilization context [7], [40] or in the context of discrete-time Kalman filtering [44]. This solution, however, does not address some of the previous problems (implementation issues and existence of observer dynamics) as the inner observer state can still venture outside  $\mathcal{X}$ . A natural way to overcome these issues is to constrain the observer state in a given set  $\hat{\mathcal{X}} \supset \mathcal{X}$ . This has nevertheless to be done with care to preserve the observer convergence and performance. In control problems, combining convergence with invariance constraints is typically handled through control barrier functions [45], leading to the so-called *safety control*. However, those methods do not transpose easily to estimation field because the state, and thus the value of the Lyapunov function, are unknown. Instead, several projection-based approaches have been developed for specific classes of systems and observer designs. In the context of continuous-time systems, projection-based solutions have been proposed for high-gain observers in [40], for Lur'e type systems in [46], and within the framework of *projected dynamical systems* in [47] for exponentially convergent observers with linear correction terms. Projection methods are also very well-known in the context of constrained Kalman filters (KF) for discrete-time systems with the so-called *equality constrained KF* (ecKF), which was proved to provide a smaller state error covariance than the unconstrained algorithm. An alternative method named *system projection* approach (spKF) is to implement the unconstrained KF on the projected linear system through a model reduction as proposed in [48]. More generally, nonlinear constraints have also been

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considered as extra measurements [49], but mainly treated via linearization [37], [44], [49], or by taking the first two orders of the Taylor development around the estimate leading to quadratic equality constraints in [50]. A surveys of all those methods (including inequality constraints) is available in [51]. More recently, the case of time-varying linear equality constraints was also considered in [52] for a continuous-time Kalman filter. Those techniques are however not easy to extend to other classes of observers or of constraints (nonlinear, inequality) and can be hard to implement on-line in presence of limited computational resources.

### B. Contributions

To the best of these authors' knowledge, there does not exist a generic solution addressing the problem of constraining the observer dynamics in a given compact set for continuous-time (CT in short), discrete-time (DT) or hybrid systems. All the designs proposed in literature are developed ad-hoc for specific classes of observers. In this context, the objective of this paper is to present a general methodology, that addresses the problem of constraining the state of a given observer into a prescribed compact set  $\hat{\mathcal{X}}$ , strictly containing  $\mathcal{X}$ , for all times, and preserving, at the same time, the guaranteed convergence or performance of the observer inside  $\hat{\mathcal{X}}$ . To characterize the performances of the observer, we assume the knowledge of a (possibly time-varying) quadratic Lyapunov function for the estimation error, whose evolution along the solutions to the estimation error system is bounded by a given supply rate. Hence, the performances are quantified by the supply rate associated to the considered Lyapunov function. Our goal is to modify the observer such that those bounds are not changed – we can then say that the guaranteed performances are preserved – while ensuring that the internal state of the observer remains in the given compact set  $\hat{\mathcal{X}}$ . For instance, if the derivative of the Lyapunov function along trajectories is bounded by a negative function, such negativity is preserved.

The main ideas of this paper stem from [41], [42] where a gradient redesign approach, based on a convexity assumption, is used for continuous-time high-gain observers implemented in coordinates which are different with respect to those of the plant's dynamics. Following the same spirit, this approach was then adapted to the discrete-time case in our preliminary work [53] by adding fictitious jumps in-between each (sampling) discrete times. The methodology in [53] has been then used in [43] for a specific problem of coordinate-implementation issues in continuous-time observer design.

In this paper, we generalize those constructions by presenting a unifying framework covering a wider class of observer design techniques. To this end, two different tools are developed.

- **Flow redesign.** The idea is to modify the observer flow map by adding an extra term that makes the chosen set  $\hat{\mathcal{X}} \supset \mathcal{X}$  invariant while preserving the given bounds on the time-derivative of the initial Lyapunov function along the system and the modified observer solutions.
- **Jump redesign.** We compose the observer jump map with a map that does not increase the Lyapunov function at jumps and whose output is in  $\hat{\mathcal{X}} \supset \mathcal{X}$ .

With the proposed tools, we are therefore able to address the desired constrained problem in the following cases.

- CT plant with CT observer and flow redesign, resulting in a CT redesigned observer.
- CT plant with CT/hybrid observer and jump redesign, resulting in a hybrid redesigned observer.
- DT plant with DT observer and jump redesign.
- Hybrid plant with hybrid observer and flow and/or jump redesign.

With respect to [41]–[43], we cover a larger class of CT nonlinear observers (e.g., [4]–[6]) including design with time-varying gains [1]–[3]. With respect to [53], we allow the Lyapunov functions to be time-dependent, thus enlarging the classes of discrete-time observers to which the proposed techniques can be applied, such as Kalman-like observers [1]. Moreover, a jump redesign based on a single step is proposed and the robustness of the initial multi-step strategy from [53] is shown based on a set-valued regularization. Finally, the new case of hybrid systems is addressed.

The rest of the paper is organized as follows. We formalize the problem by stating the standing assumptions in Section II. The flow redesign tool is developed in Section III for continuous-time systems with continuous-time observers. Then, the jump redesign tool is presented in Section IV for discrete-time systems with discrete-time observers. Finally, both tools are combined in Section V to address hybrid observers. In Section VI, we discuss the choice of some parameters characterizing the proposed redesign framework. Simulation results are presented in Section VII and conclusions are drawn in Section VIII.

*Notation.*  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$  and  $\mathbb{N}$  denote the set of real numbers, non-negative real numbers and positive integer numbers, respectively. We define  $\mathbb{R}_{\geq 0}^n := \mathbb{R}_{\geq 0} \times \cdots \times \mathbb{R}_{\geq 0}$ . We denote with  $|\cdot|$  the standard Euclidean norm. We say that a function  $f : X \rightarrow \mathbb{R}$  is bounded if  $\sup_{x \in X} |f(x)| < \infty$ . Given to sets  $S_1, S_2 \subseteq \mathbb{R}^n$ , we define the distance from two sets as  $d(S_1, S_2) := \inf\{|x - y| : x \in S_1, y \in S_2\}$ ,  $S_1 \subseteq S_2$  denotes an inclusion and  $S_1 \subset S_2$  denotes a strict inclusion such that  $d(S_1, \mathbb{R}^n \setminus S_2) > 0$ . For a set-valued map  $V : \mathbb{R}^p \rightrightarrows \mathbb{R}$  and a scalar  $c$ ,  $V(x) \leq c$  means that  $v \leq c$  for all  $v \in V(x)$ . For a set-valued map  $\mathcal{F} : \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ ,  $V : \mathbb{R}^p \rightarrow \mathbb{R}$ , and  $x \in \mathbb{R}^q$ ,  $L_{\mathcal{F}}V(x) = \sup\{L_vV(x) : v \in \mathcal{F}(x)\}$ , where  $L_vV(x) := \frac{\partial V}{\partial x}(x)v$  is the Lie derivative of  $V$  along the vector  $v$  at the point  $x$ . We refer to the definition of outer semi-continuity given in [22, Definition 5.9]. We refer to standard notations of class- $\mathcal{K}$  and class- $\mathcal{KL}$  functions, see, i.e. [22, Chapter 3].

## II. STANDING ASSUMPTIONS AND MAIN IDEA

We consider a plant  $\mathcal{P}$  modelled as a finite-dimensional time-varying dynamical system with state  $x \in \mathbb{R}^{n_x}$ , that is either continuous-time, discrete-time or hybrid, with measured outputs  $y \in \mathbb{R}^{n_y}$  and known inputs  $u \in \mathbb{R}^{n_u}$ , and is affected by unknown perturbations  $d \in \mathbb{R}^{n_d}$ , with  $n_x, n_u, n_y, n_d \in \mathbb{N}$ , as formalized in the next sections.

**Standing Assumption 1 (SA1)** *There exist subsets  $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$ ,  $\mathcal{U} \subset \mathbb{R}^{n_u}$ ,  $\mathcal{D} \subset \mathbb{R}^{n_d}$ , and a compact subset  $\mathcal{X} \subset \mathbb{R}^{n_x}$ ,*

such that any trajectory of  $\mathcal{P}$  initialized in  $\mathcal{X}_0$ , with input in  $\mathcal{U}$  and perturbation in  $\mathcal{D}$ , remains in  $\mathcal{X}$  for all forward times.

No restriction is made on the “size” of the sets in SA1, these can therefore be arbitrarily large. Actually, only the knowledge of  $\mathcal{X}$  is necessary for design. In other words, we assume the plant trajectories to be estimated evolve in a known compact set  $\mathcal{X}$ , which is not required to be invariant for the dynamics. Although this assumption could seem like a limitation of this framework, boundedness of solutions is commonly verified in practical applications, where the state variables satisfy known physical bounds. Finally, note that such a boundedness property does not necessarily mean the plant model exhibits stability properties.

We then assume that we know an observer  $\mathcal{O}$  for plant  $\mathcal{P}$  with some guaranteed properties, as formalized in the next sections, whose state estimate is denoted  $\hat{x} \in \mathbb{R}^{n_x}$ . At this stage, any existing observer can be considered; detailed examples are discussed in the sequel. While the state  $x$  of plant  $\mathcal{P}$  is known to lie in  $\mathcal{X}$ , according to SA1, it may occur that the state  $\hat{x}$  of observer  $\mathcal{O}$  leaves  $\mathcal{X}$  for some time, as explained in the introduction. Therefore, given a prescribed compact set  $\hat{\mathcal{X}} \subset \mathbb{R}^n$ , with  $\hat{\mathcal{X}} \supset \mathcal{X}$ , the goal of the paper is to redesign observer  $\mathcal{O}$  so that the following holds.

- 1) The *modified* observer, denoted as  $\mathcal{O}_m$ , exhibits the same guaranteed performances as the original observer.
- 2) The state of the redesigned observer  $\mathcal{O}_m$  remains in  $\hat{\mathcal{X}}$  for all forward times when initialized in  $\mathcal{X}$ .

As it is often the case in designs with constraints (see, e.g., [40], [42], [47], [54]), the solutions presented in the sequel rely on a key convexity assumption, as formalized next.

**Standing Assumption 2 (SA2)** *The convex hull of  $\mathcal{X}$  is strictly included in  $\hat{\mathcal{X}}$ , i.e., there exists a compact convex set  $\mathcal{C} \in \mathbb{R}^{n_x}$  satisfying  $\mathcal{X} \subseteq \mathcal{C} \subset \hat{\mathcal{X}}$ .*

This assumption says that it will not be possible with our approach to keep the observer state in any non-convex set  $\hat{\mathcal{X}}$ , but only in a compact convex set containing  $\mathcal{X}$ . For instance, our approach cannot be used to keep the observer state away from singularities, or close to a non-convex manifold. Although this assumption may seem restrictive, it actually enables to keep the estimate within polytopes and hyperspheres modelling known physical bounds on the plant state, which is of significant practical interest. A more detailed discussion about the choice of  $\hat{\mathcal{X}}$  is provided in Section VI, after presenting the main results of this work.

SA2 implies the following convexity result (see an illustration in Figure 1), which plays an essential role in the sequel.

**Lemma 1** *Suppose SA2 holds. Then, there exist  $\varrho \in \mathbb{R}_{>0}$ , an integer  $n_c \in \mathbb{N}$ , and a  $C^1$  function  $c : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}^{n_c}$  such that, by defining*

$$\mathcal{C}_0 := \{x \in \mathbb{R}^{n_x} : c(x) = 0\}, \quad \mathcal{C}_\varrho := \{x \in \mathbb{R}^{n_x} : |c(x)| \leq \varrho\}, \quad (1)$$

*the following holds:*

- a)  $\mathcal{X} \subseteq \mathcal{C}_0 \subset \mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ .

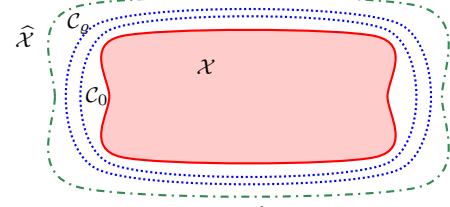


Fig. 1. Illustration of sets  $\mathcal{X}$  (in red),  $\hat{\mathcal{X}}$  (in dash-dotted green) and  $\mathcal{C}_0, \mathcal{C}_\varrho$  (in dotted blue) satisfying item a) of Lemma 1 and thus SA2.

- b) *For any  $i = 1, \dots, n_c$ , the  $i$ -th component  $c_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$  of  $c$  is convex.*

**Proof.** According to [55, (6)], any compact convex set  $\mathcal{C}$  can be outer-approximated as closely as desired by a polytope. Therefore, by continuity, and since  $\mathcal{C}$  is strictly included in  $\hat{\mathcal{X}}$ , there exist  $a_i \in \mathbb{R}^{n_x}$  and  $b_i \in \mathbb{R}$  for  $i = 1, \dots, n_c$ , with  $n_c$  large enough, such that  $\mathcal{X} \subseteq \bigcap_{i=1}^{n_c} \{x \in \mathbb{R}^{n_x} : a_i^\top x \leq b_i\} \subset \hat{\mathcal{X}}$ . Lemma 7 in Section VI shows how to build a map  $c$  verifying items a) and b). ■

A key feature of the map  $c$  is that it vanishes on  $\mathcal{X}$  where the plant trajectories evolve, since  $\mathcal{X} \subseteq \mathcal{C}_0$ . Following [41], the main idea underlying the redesign constructions proposed in this work, is to associate to any trajectory of the plant  $\mathcal{P}$  an extra “dummy” output  $y_2 = c(x)$  which is constantly zero, since  $x \in \mathcal{X}$ , and to observe that the corresponding estimated version  $\hat{y}_2 = c(\hat{x})$  is not zero whenever  $\hat{x}$  goes outside  $\mathcal{X}$ . Therefore, we can modify the observer  $\mathcal{O}$  by adding a second output injection term, driven by the measured difference  $\hat{y}_2 - y_2$ , to correct the trajectories of the observer  $\mathcal{O}$  outside the set  $\mathcal{C}_0$ . This will be done by exploiting the next convexity property of  $c$ .

**Lemma 2** *Consider a  $C^1$  function  $c : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}^{n_c}$  satisfying item b) of Lemma 1. Then, for all  $(x, \hat{x}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$  such that  $c(x) = 0$ , we have*

$$-c(\hat{x})^\top \frac{dc}{dx}(\hat{x})(\hat{x} - x) \leq -c(\hat{x})^\top c(\hat{x}) \leq 0. \quad (2)$$

**Proof.** The convexity of  $c_i$ , for  $i = 1, \dots, n_c$ , implies that  $c_i(x) \geq c_i(\hat{x}) + \frac{dc_i}{dx}(\hat{x})(x - \hat{x})$  for all  $x, \hat{x} \in \mathbb{R}^{n_x}$ , and therefore that  $-\frac{dc_i}{dx}(\hat{x})(\hat{x} - x) \leq -c_i(\hat{x})$  if  $c(x) = 0$ . Inequality (2) then follows directly by decomposing the scalar product and using the positivity of  $c_i$ . ■

As a consequence, in the rest of the paper, we suppose the elements of Lemma 1 are known and available for the design. Insights about the way  $\hat{\mathcal{X}}$ , the function  $c$  and  $\varrho$  can be chosen/built are given in Section VI.

### III. THE CONTINUOUS-TIME CASE

#### A. Problem Statement

We assume in this section that the plant  $\mathcal{P}$  is described as

$$\mathcal{P} : \quad \dot{x} = f(t, x, u, d), \quad y = h(t, x, u, d), \quad (3)$$

with state  $x \in \mathbb{R}^{n_x}$ , known input  $u \in \mathbb{R}^{n_u}$ , output  $y \in \mathbb{R}^{n_y}$ , and where  $d \in \mathbb{R}^{n_d}$  is an unknown perturbation acting on

the dynamics and/or the measurement. We consider then a continuous-time observer of the form

$$\mathcal{O} : \quad \dot{\hat{x}} = F(t, \hat{x}, u, y), \quad (4)$$

with state  $\hat{x} \in \mathbb{R}^{n_x}$ . System (3) and observer (4) satisfy some regularity properties as stated in the next assumption.

**Assumption 1** *The following is satisfied:*

- The function  $h$  in (3) is bounded in its first argument and the set  $\mathcal{Y} := h(\mathbb{R}_{\geq 0}, \mathcal{X}, \mathcal{U}, \mathcal{D})$  is compact.
- The function  $F$  is bounded on  $\mathbb{R}_{\geq 0} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{Y}$ .
- The solutions of (4) exist forward in time for any initial condition  $\hat{x}(0) \in \mathcal{X}$ , any input  $u$  in  $\mathcal{U}$  and any output  $y$  of system (4), i.e. no finite-escape time phenomenon is possible.

Furthermore, we assume that the observer (4) has been designed so that the following assumption is verified.

**Assumption 2** *There exist a (possibly time varying) known matrix  $P(t) = P(t)^\top > 0$ , differentiable in  $t$ , real numbers  $\bar{\lambda} > \underline{\lambda} > 0$ , and a function  $\beta_c : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \rightarrow \mathbb{R}$ , such that, by defining*

$$V(t, x, \hat{x}) := (\hat{x} - x)^\top P(t)(\hat{x} - x), \quad (5)$$

*the following is satisfied*

$$\underline{\lambda}|x - \hat{x}|^2 \leq V(t, x, \hat{x}) \leq \bar{\lambda}|x - \hat{x}|^2 \quad (6)$$

$$L_{\mathcal{F}}V(t, x, \hat{x}, u, d) \leq \beta_c(t, x, \hat{x}, u, d) \quad (7)$$

*for all  $(t, x, \hat{x}, u, d) \in \mathbb{R}_{\geq 0} \times \mathcal{X} \times \mathbb{R}^{n_x} \times \mathcal{U} \times \mathcal{D}$ , with*

$$\mathcal{F}(t, x, \hat{x}, u, d) := (1, f(t, x, u, d), F(t, \hat{x}, u, h(t, x, u, d))). \quad (8)$$

Assumption 2 characterizes the performances of observer (4) in terms of the function  $V$  and the supply rate  $\beta_c$ . For instance, for an exponentially stable observer which is input-to-state stable (ISS) with respect to the perturbation, the function  $\beta_c$  in (7) is typically of the form

$$\beta_c(t, x, \hat{x}, u, d) = -\rho_c V(t, x, \hat{x}) + \alpha_c(|d|), \quad (9)$$

with  $\rho_c > 0$  and  $\alpha_c \in \mathcal{K}$ . In this case,  $V$  is the Lyapunov function associated to the estimation error  $x - \hat{x}$  and  $\rho_c$  is the convergence rate. Note that, when the matrix  $P$  is constant,  $\bar{\lambda}$  and  $\underline{\lambda}$  in (6) correspond to the maximum and minimum eigenvalues of  $P$ , respectively.

Assumption 2 covers a large class of observers for the design of  $\mathcal{O}$  in (4). For instance, the following observers satisfy the conditions of Assumption 2 with  $P$  constant:

- linear Luenberger observers [4], when (3) is linear;
- high-gain nonlinear observers [7], [8] for systems in triangular forms with Lipschitz nonlinearities;
- observer designs based on linear parameter-varying-like techniques or the circle criterion, e.g., [5], [6].

Furthermore, Kalman filters [1] and Kalman-like observers [2], [3] used for state-affine normal forms, also satisfy Assumption 2 with a time-varying  $P$ . In these cases,  $P$  is solution to a differential Riccati or Lyapunov equation that is independent from the estimate  $\hat{x}$  and uniformly upper and

lower bounded for all positive times. These extra dynamics are implemented online and  $P$  is considered as an external time-varying input for observer (4). See also [14, Section VI] for further examples.

Note also that in the context of nonlinear observers, one may need to change coordinates before designing an observer based on a quadratic Lyapunov function (nonlinear Luenberger design [9], high-gain design [56]). In that case, the methods developed in this paper should be applied in those new coordinates, as done for instance in [42], [43].

The objective of this section is to modify observer (4) in order to solve the next problem.

**Problem 1** *Design a map  $F_m : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y}$  such that the following holds.*

- 1) *For any initial condition  $(x(0), \hat{x}(0)) \in \mathcal{X}_0 \times \mathcal{X}$ , any input in  $\mathcal{U}$  and perturbation in  $\mathcal{D}$ , the corresponding trajectory of the modified observer*

$$\mathcal{O}_m : \quad \dot{\hat{x}} = F_m(t, \hat{x}, u, y) \quad (10)$$

*remains in  $\hat{\mathcal{X}}$  for all  $t \geq 0$ .*

- 2) *The Lyapunov function (5) satisfies*

$$L_{\mathcal{F}_m}V(t, x, \hat{x}, u, d) \leq \beta_c(t, x, \hat{x}, u, d) \quad (11)$$

*for any  $(t, x, \hat{x}, u, d) \in \mathbb{R}_{\geq 0} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}$ , with  $\beta_c$  coming from Assumption 2 and*

$$\mathcal{F}_m(t, x, \hat{x}, u, d) := (1, f(t, x, u, d), F_m(t, \hat{x}, u, h(t, x, u, d))) \quad (12)$$

Hence, by solving Problem 1, the state estimate generated by observer  $\mathcal{O}_m$  remains in set  $\hat{\mathcal{X}}$  and property (7) is preserved for  $\mathcal{O}_m$  as desired.

## B. Observer Redesign

We modify the dynamics of observer (4) as

$$F_m(t, \hat{x}, u, y) := F(t, \hat{x}, u, y) + M(t, \hat{x}) \quad (13)$$

where function  $M$  is defined by

$$M(t, \hat{x}) := -\gamma_c P(t)^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x}) \quad (14)$$

with  $c$  defined in Lemma 1,  $P$  given by Assumption 2, and  $\gamma_c \in \mathbb{R}_{>0}$  a parameter to be chosen large enough. Note that the knowledge of the supply rate  $\beta_c$  is not needed for the design of (13). In order to state the main result of this section, we define the set  $\partial\mathcal{C}_\varrho$  and the positive real number  $\delta_{\min}$  as

$$\partial\mathcal{C}_\varrho := \{x \in \mathbb{R}^{n_x} : |c(x)| = \varrho\}, \quad (15)$$

$$\delta_{\min} := \inf_{x \in \partial\mathcal{C}_\varrho} \left| \frac{dc}{dx}(x)^\top c(x) \right|, \quad (16)$$

with  $c, \varrho$  given in Lemma 1. We have the following result.

**Theorem 1** *Suppose Assumptions 1 and 2 hold and that  $\delta_{\min}$  in (16) is strictly positive. Then, there exists  $\gamma_c^* \geq 0$  such that, for any  $\gamma_c > \gamma_c^*$ , Problem 1 is solved with  $F_m$  defined in (13), (14).*

**Proof.** First, we define  $F_{\max}, \gamma_{c,\min}$  as

$$F_{\max} := \sup_{(t,\hat{x},u,y) \in \mathbb{R}_{\geq 0} \times \partial \mathcal{C}_\varrho \times \mathcal{U} \times \mathcal{Y}} |F(t, \hat{x}, u, y)|, \quad (17)$$

$$\gamma_c^* := \frac{F_{\max} \bar{\lambda}}{\delta_{\min}}, \quad (18)$$

with  $\mathcal{Y}$  defined in Assumption 1, and  $\bar{\lambda}$  given by Assumption 2. In light of the properties of  $F$  in Assumption 1, and the fact that  $\partial \mathcal{C}_\varrho \subset \mathcal{X}$  in view of item (a) of Lemma 1, the real numbers  $F_{\max}, \gamma_c^*$  are finite. Now consider the function  $W(\hat{x}) := \frac{1}{2} c(\hat{x})^\top c(\hat{x})$  for any  $\hat{x} \in \mathbb{R}^{n_x}$ , and pick  $\hat{x} \in \partial \mathcal{C}_\varrho$ ,  $u$  in  $\mathcal{U}$  and  $x$  in  $\mathcal{X}$ . By using (16), (17), (18), we compute

$$\begin{aligned} & \langle \nabla W(\hat{x}), F_m(t, \hat{x}, u, y) \rangle \\ &= c(\hat{x})^\top \frac{dc}{dx}(\hat{x}) \left[ F(t, \hat{x}, u, y) - \gamma_c P(t)^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x}) \right] \\ &\leq \left| \frac{dc}{dx}(\hat{x})^\top c(\hat{x}) \right| \left( F_{\max} - \frac{\gamma_c}{\bar{\lambda}} \delta_{\min} \right) < 0 \end{aligned}$$

in which we used the fact that, in light of Assumption 2,  $P(t)$  satisfies  $\frac{1}{\bar{\lambda}} I \leq P(t)^{-1}$  for all  $t \geq 0$ , and  $\delta_{\min} > 0$ . This shows that the derivative of  $W$  along solutions to  $\mathcal{O}_m$  is strictly negative on  $\partial \mathcal{C}_\varrho$ , which implies that  $\mathcal{C}_\varrho$  is forward invariant. A fortiori, any trajectory initialized in  $\mathcal{C}_\varrho$  remains in  $\hat{\mathcal{X}}$  as  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$  according to Lemma 1 and therefore item 1) of Problem 1 is proved since  $\mathcal{X} \subset \mathcal{C}_\varrho \subset \hat{\mathcal{X}}$  in view of item (a) of Lemma 1.

By using the definition of  $\mathcal{F}_m$  and  $M$  in (12) and (14), and the definition of  $\beta_c$  in Assumption 2, we obtain for all  $(t, x, \hat{x}, u, d) \in \mathbb{R} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}$

$$\begin{aligned} L_{\mathcal{F}_m} V(t, x, \hat{x}, u, d) &= L_{\mathcal{F}} V(t, x, \hat{x}, u, d) + 2(\hat{x} - x)^\top P(t) M(t, \hat{x}) \\ &\leq \beta_c(t, x, \hat{x}, u, d) - 2\gamma_c (\hat{x} - x)^\top \frac{dc}{dx}(\hat{x})^\top c(\hat{x}). \end{aligned}$$

By recalling that  $c(x) = 0$  for any  $x \in \mathcal{X}$  by Lemma 1, and since  $\gamma_c > 0$ , we obtain (11) from Lemma 2. ■

The extra condition  $\delta_{\min} > 0$  in Theorem 1 is verified for the functions  $c$  proposed in Section VI (see (62) and (63) of Lemma 7 and Lemma 8). On the other hand, the role of  $\gamma_c$  in (14) is to correct the vector field  $F$  when the latter becomes too large and makes the estimate leave the set  $\mathcal{X}$ . To do that, we have chosen a constant gain  $\gamma_c$  whose magnitude is large enough with respect to the maximal norm of  $F$  on the boundary of  $\mathcal{C}_\varrho$ . A bound on  $\gamma_c$  is given in the proof of Theorem 1, see (18). This bound may be subject to some conservatism and in practice  $\gamma_c$  can be adapted online to compensate for the “current” value of  $F$ , by taking for instance

$$\gamma_c(t, \hat{x}, u, y) := \frac{2\bar{\lambda} |F(t, \hat{x}, u, y)|}{\max \left\{ \left| \frac{dc}{dx}(\hat{x})^\top c(\hat{x}) \right|, \delta_{\min} \right\}}$$

for any  $(t, \hat{x}, u, y) \in \mathbb{R} \times \mathbb{R}^{n_x} \times \mathcal{U} \times \mathcal{Y}$ , in view of the proof of Theorem 1, see [42].

**Remark 1** The correction term  $M$  defined in (14) may also be interpreted as an element of the normal cone to the level set of  $c$  in the metric defined by  $P(t)$ , thus suggesting a link with the projected dynamics used in [47] for constant metric  $P$ .

- 1) Consider a plant  $\mathcal{P}$  in (3) such that SA1 holds for some compact set  $\mathcal{X}$ .
- 2) Consider an observer  $\mathcal{O}$  in (4) such that Assumption 2 is satisfied for some known  $t \mapsto P(t)$ .
- 3) Choose a map  $c$  such that  $\mathcal{X} \subseteq \mathcal{C}_0$  along the lines of Lemma 1.
- 4) Design the function  $M$  as in (14) and select  $\gamma_c > 0$  sufficiently large.
- 5) Add the term  $M$  to the right-hand side of (4) as in (13).
- 6) Run the new observer (10) with  $F_m$  defined as in (13), and initial conditions  $\hat{x}(0) \in \mathcal{X}$ .
- 7) Problem 1 is solved with  $\hat{\mathcal{X}} \supset \mathcal{C}_\varrho$  for some  $\varrho > 0$ .

TABLE I  
REDESIGN PROCEDURE FOR CONTINUOUS-TIME OBSERVERS

To conclude, the CT redesign summarizes in the steps given in Table I, for which we only need the knowledge of the function  $c$  (which may be a degree of freedom for design, as shown later in Section VI), and the (possibly time-varying) matrix  $P$  given by Assumption 2.

#### IV. THE DISCRETE-TIME CASE

##### A. Problem Statement and Sufficient Conditions

In this section, plant  $\mathcal{P}$  is modelled as a discrete-time system of the form

$$\mathcal{P} : \begin{cases} x(k+1) &= g(k, x(k), u(k), d(k)), \\ y(k) &= h(k, x(k), u(k), d(k)), \end{cases} \quad k \in \mathbb{N}, \quad (19)$$

with state  $x \in \mathbb{R}^{n_x}$ , known input  $u \in \mathbb{R}^{n_u}$ , output  $y \in \mathbb{R}^{n_y}$ , and where  $d \in \mathbb{R}^{n_d}$  is an unknown perturbation acting on the dynamics and/or the measurement. For the sake of convenience, we will denote with  $x_k := x(k)$  the state  $x$  at discrete-time  $k$ . The same notation will be adopted for all variables and inputs. The plant (19) is therefore compactly written as

$$\mathcal{P} : \begin{cases} x_{k+1} &= g(k, x_k, u_k, d_k), \\ y_k &= h(k, x_k, u_k, d_k), \end{cases} \quad (20)$$

and the observer is given by

$$\mathcal{O} : \hat{x}_{k+1} = G(k, \hat{x}_k, u_k, y_k) \quad (21)$$

with  $G : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathcal{U} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ . Similarly to the continuous-time case in Section III, the performances of observer (21) are characterized by the next Lyapunov-like conditions.

**Assumption 3** There exist a (possibly time-varying) known matrix  $P_k = P_k^\top > 0$  for all  $k \in \mathbb{N}$ , real numbers  $\bar{\lambda} > \underline{\lambda} > 0$  and a continuous function  $\beta_d : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \rightarrow \mathbb{R}_{\geq 0}$ , such that, by defining

$$V_k(x, \hat{x}) := (\hat{x} - x)^\top P_k (\hat{x} - x), \quad (22)$$

the following inequalities hold for all  $(k, x, \hat{x}, u, d) \in \mathbb{N} \times \mathcal{X} \times \mathbb{R}^{n_x} \times \mathcal{U} \times \mathcal{D}$

$$\underline{\lambda} |\hat{x} - x|^2 \leq V_k(x, \hat{x}) \leq \bar{\lambda} |\hat{x} - x|^2, \quad (23)$$

$$V_{k+1}(x_{k+1}, \hat{x}_{k+1}) \leq \beta_d(k, x, \hat{x}, u, d) \quad (24)$$

with the compact notation  $x_{k+1} := g(k, x, u, d)$ ,  $\hat{x}_{k+1} := G(k, \hat{x}, u, h(k, x, u, d))$ .

Like Assumption 2, Assumption 3 characterizes the behavior of the estimation error  $x - \hat{x}$  and similar considerations on the ISS properties can be derived, see, e.g., [53, Corollary 1]. Moreover, the function  $\beta_d$  and the parameters  $\underline{\lambda}, \bar{\lambda}$  are likewise used only for analysis purposes and not for design. In the case of a constant matrix  $P$ ,  $\bar{\lambda}$  and  $\underline{\lambda}$  in (23) correspond to the maximum and minimum eigenvalue of  $P$ , respectively.

Assumption 3 is verified for the observers proposed in [4], [15]–[19], where  $P$  is constant; and Kalman-like observers, see [1], [20], where  $P$  is a time-varying solution to a Riccati or Lyapunov difference equation that is independent from the estimate  $\hat{x}$  and uniformly upper and lower bounded in time  $k$ . In this case the matrix  $P$  is computed online.

Similarly to the continuous-time case, we aim at solving the next problem.

**Problem 2** Design a (possibly set-valued) map  $G_m : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \rightrightarrows \mathbb{R}^{n_x}$  such that the following holds.

- 1) For any initial condition  $(x_0, \hat{x}_0) \in \mathcal{X}_0 \times \mathcal{X}$ , input in  $\mathcal{U}$  and perturbation in  $\mathcal{D}$ , the corresponding trajectory  $\hat{x}_k$  of the modified observer

$$\mathcal{O}_m : \quad \hat{x}_{k+1} \in G_m(k, \hat{x}_k, u_k, y_k) \quad (25)$$

remains inside  $\hat{\mathcal{X}}$  for all  $k \in \mathbb{N}$ .

- 2) The Lyapunov function (22) satisfies

$$V_{k+1}(x_{k+1}, \hat{x}_{k+1}) \leq \beta_d(k, x, \hat{x}, u, d) \quad (26)$$

for any  $(k, x, \hat{x}, u, d) \in \mathbb{N} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}$ ,  $x_{k+1} = g(k, x, u, d)$ , and  $\hat{x}_{k+1} \in G_m(k, \hat{x}, u, h(k, x, u, d))$ .

Differently from the continuous-time case in Section III, we do not modify the dynamics of (21) by adding a term to the map  $G$ . Instead, we build  $G_m$  by composition of the map  $G$  given in (21) with a (possibly multi-valued) time-varying map  $\psi_{k+1} : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_x}$ , namely

$$G_m(k, \cdot) := \psi_{k+1} \circ G(k, \cdot). \quad (27)$$

**Remark 2** Since the mapping  $\psi_{k+1}$  is applied after the mapping  $G(k, \cdot)$ , in the following we will assume that  $\psi_{k+1}$  may depend on  $P_{k+1}$ . This assumption is consistent with the cases in which  $P$  is obtained as solution of a Riccati difference equation for instance. It is also the reason why we index  $\psi_{k+1}$  (and all the variables involved in its definition) with  $k+1$  instead of  $k$ .

The following lemma gives a sufficient condition on the maps  $\psi_{k+1}$ , which allows solving Problem 2.

**Lemma 3** Suppose Assumption 3 hold and that for all  $k \in \mathbb{N}$ , there exist a set-valued map  $\psi_{k+1} : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_x}$  and a set  $\mathcal{S}_{k+1} \subset \mathbb{R}^{n_x}$  such that

$$V_{k+1}(x, \psi_{k+1}(\hat{x})) \leq V_{k+1}(x, \hat{x}) \quad \forall (x, \hat{x}) \in \mathcal{X} \times \mathcal{S}_{k+1} \quad (28a)$$

$$\psi_{k+1}(\mathcal{S}_{k+1}) \subseteq \hat{\mathcal{X}} \quad (28b)$$

and for any  $(k, x, \hat{x}, u, d) \in \mathbb{N} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}$ ,

$$G(k, \hat{x}, u, h(k, x, u, d)) \in \mathcal{S}_{k+1}. \quad (28c)$$

Then, Problem 2 is solved with  $G_m$  defined in (27).

**Proof.** From (28b), (28c), and the definition of  $G_m$  in (27), we derive that  $G_m(k, \hat{x}, u, h(k, x, u, d)) \subseteq \hat{\mathcal{X}}$  for any  $(k, x, \hat{x}, u, d) \in \mathbb{N} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}$ . With this, item 1) of Problem 2 is satisfied. To derive the item 2) of Problem 2, we invoke (24), (28a) and the definition of  $G_m$  in (27). ■

Set  $\mathcal{S}_{k+1}$  contains the estimate  $\hat{x}$  after applying the nominal observer (21) at iteration  $k$ . Therefore, it is sufficient to satisfy the redesign constraints (28a)–(28b) for  $\hat{x} \in \mathcal{S}_{k+1}$ .

The goal is now to find maps  $\psi_{k+1}$  that verify condition (28), namely the fact that applying  $\psi_{k+1}$  to  $G(k, \cdot)$  does not make the Lyapunov function  $V_{k+1}$  increase and that its image is contained in  $\hat{\mathcal{X}}$ . The first idea we investigate is to project  $G$  in  $\hat{\mathcal{X}}$ , as presented in Section IV-B.

#### B. About the Existence of a Single-Step Solution

The next lemma shows that the function  $\psi_{k+1}$  in Lemma 3 can be defined as a projection according to the norm induced by  $P_{k+1}$ .

**Lemma 4** For any  $k$  in  $\mathbb{N}$ , the map  $\psi_{k+1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by

$$\psi_{k+1}(\hat{x}) := \operatorname{argmin}_{x \in \mathcal{C}_0} (\hat{x} - x)^\top P_{k+1}(\hat{x} - x) \quad (29)$$

with  $\mathcal{C}_0$  given in Lemma 1, is well-defined, single-valued and verifies condition (28) with  $\mathcal{S}_{k+1} = \mathbb{R}^{n_x}$ .

**Proof.** Take  $k \in \mathbb{N}$  and define the function  $\bar{c}_{k+1} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}^{n_c}$  as  $z \mapsto \bar{c}_{k+1}(z) := c \left( P_{k+1}^{-\frac{1}{2}} z \right)$ , with  $c$  given in Lemma 1, and let  $\bar{\mathcal{C}}_{0,k+1} := \{z \in \mathbb{R}^n : \bar{c}_{k+1}(z) = 0\}$ . By linearity of the function  $z \mapsto P_{k+1}^{-\frac{1}{2}} z$ , and by convexity of  $c_i$  in Lemma 1, each  $i$ -th component  $\bar{c}_{i,k+1}$  of  $\bar{c}_{k+1}$  is convex and  $C^1$ . Therefore,  $\bar{\mathcal{C}}_{0,k+1}$  is convex and closed, so that the projection map  $\bar{\psi}_{k+1}$  defined on  $\mathbb{R}^n$  by

$$\hat{z} \mapsto \bar{\psi}_{k+1}(\hat{z}) := \operatorname{argmin}_{z \in \bar{\mathcal{C}}_{0,k+1}} (\hat{z} - z)^\top (\hat{z} - z)$$

takes finite values, is single-valued and verifies

$$(\bar{\psi}_{k+1}(\hat{z}) - z)^\top (\bar{\psi}_{k+1}(\hat{z}) - z) \leq (\hat{z} - z)^\top (\hat{z} - z)$$

for all  $z \in \bar{\mathcal{C}}_{0,k+1}$ . Since  $\psi_{k+1}(\hat{x}) = P_{k+1}^{-\frac{1}{2}} \bar{\psi}_{k+1}(P_{k+1}^{\frac{1}{2}} \hat{x})$  for all  $\hat{x}$  in  $\mathbb{R}^n$  and  $\mathcal{X} \subseteq \mathcal{C}_0$ , (28a) follows with  $\mathcal{S}_{k+1} = \mathbb{R}^n$ . Equation (28b) also holds since  $\mathcal{C}_0 \subset \hat{\mathcal{X}}$ . Finally, (28c) holds because  $\mathcal{S}_{k+1} = \mathbb{R}^n$ . ■

With Lemmas 1, and 3, Lemma 4 implies that Problem 2 can be solved by modifying the observer along (27) with  $\psi_{k+1}$  defined in (29). In other words, the estimate is brought back into  $\mathcal{C}_0$  at each iteration through a projection of the output of the observer jump map  $G$  according to the norm determined by  $P_{k+1}$ , thus leading to the procedure highlighted in Table II. This projection requires to solve a quadratic minimization problem with convex constraints at each step, which we now briefly discuss.

- 1) Consider a plant  $\mathcal{P}$  in (3) such that SA1 holds for some compact set  $\mathcal{X}$ .
- 2) Consider an observer  $\mathcal{O}$  in (21) such that Assumption 3 is satisfied for some known  $k \mapsto P_k$ .
- 3) Choose a map  $c$  such that  $\mathcal{X} \subseteq \mathcal{C}_0$  along the lines of Lemma 1.
- 4) Run the new observer (25) with  $G_m$  defined as in (27) and  $\psi_{k+1}$  the projection map defined in (29), and with an initial condition  $\hat{x}(0) \in \mathcal{X}$ .
- 5) Problem 2 is solved with  $\hat{\mathcal{X}} \supset \mathcal{C}_0$ .

TABLE II  
SINGLE-STEP REDESIGN PROCEDURE FOR DISCRETE-TIME OBSERVERS

*Equality Constraints:* If the constraint map  $c$  is linear, namely  $c(x) = Ax - b$  with  $A \in \mathbb{R}^{n_c \times n_x}$  full-rank,  $\psi_{k+1}$  is explicitly defined for all  $k \in \mathbb{N}$  on  $\mathbb{R}^{n_x}$  by

$$\psi_{k+1}(\hat{x}) = \hat{x} - P_{k+1}^{-1} A^\top (AP_{k+1}^{-1} A^\top)^{-1} (A\hat{x} - b)$$

which is an oblique projection on the constraints subspace  $\{x \in \mathbb{R}^{n_x} : Ax = b\}$  as proposed in [44] and also used in the case of time-varying linear constraints in [52]. The expression of  $\psi_{k+1}$  is obtained by solving (29) using Lagrangian multipliers, which in general gives

$$-2(\hat{x} - x)^\top P_{k+1} + \Lambda \frac{dc}{dx}(x) = 0, \quad c(x) = 0,$$

with  $\Lambda \in \mathbb{R}^{1 \times n_c}$ . Unfortunately, those cannot be solved explicitly unless  $c$  is linear. Sometimes, it can be solved numerically for instance through an iterative root search algorithm, as proposed in [50].

*Inequality Constraints:* In the case where the constraints can be written as convex inequalities, namely  $c(x) = 0 \Leftrightarrow \zeta_i(x) \leq 0$  for  $i = 1 \dots n_c$  with some convex maps  $\zeta_i : \mathbb{R}^n \rightarrow \mathbb{R}$  as a polytope (62) or an ellipse (63), then solving (29) is equivalent to solve the Karush-Kuhn-Tucker (KKT) conditions, namely finding scalars  $\lambda_1, \dots, \lambda_{n_c}$  and  $x_p \in \mathcal{C}_0$  such that

- 1)  $-2(\hat{x} - x_p)^\top P_{k+1} + \sum_{i=1}^{n_c} \lambda_i \frac{d\zeta_i}{dx}(x_p) = 0,$
- 2)  $\lambda_i \geq 0$  for all  $i = 1 \dots n_c,$
- 3)  $\lambda_1 \zeta_1(x_p) = \dots = \lambda_{n_c} \zeta_{n_c}(x_p) = 0.$

Various iterative algorithms such as interior-point methods, sub-gradient methods or semi-definite programming, are available to solve this convex optimization problem. In the particular case where  $\mathcal{C}_0$  is a polytope (62) with  $\zeta_i(x) = a_i^\top x - b_i$ , the first condition can be solved explicitly in  $x_p$  as a function of  $\hat{x}$  and of the adjoint variables  $\lambda_i$ . Injecting its expression in the second and third conditions leads to a polyhedral partition of the state space, with a particular feasible value of  $(\lambda_1, \dots, \lambda_{n_c})$  in each of the regions. It follows that  $\psi_{k+1}$  is in that case a piecewise-linear map of the form

$$\psi_{k+1}(\hat{x}) = \hat{x} - \frac{1}{2} P_{k+1}^{-1} \sum_{i=1}^{n_c} \lambda_i(\hat{x}) a_i$$

where  $\lambda_i(\hat{x})$  depends on the zone of the partition where  $\hat{x}$  is. It is interesting to note that the partition depends on the constraints  $a_i$  and on  $P_{k+1}$  only. In particular, if  $P$  is constant, the partition can be determined offline beforehand, so that the computational burden at each iteration reduces to determining

the region in which  $\hat{x}$  lies. More details about the way this can be done efficiently can be found in [57], [58].

*Towards a Multi-Step Algorithm:* According to the previous analysis, we conclude that, unless  $c$  is linear, an explicit solution to the minimization problem (29), namely a close form of  $\psi_{k+1}$ , is not available and algorithms have to be used to evaluate it. Therefore in practice,  $\psi_{k+1}$  is computed as

$$\psi_{k+1} = \psi_{k+1,\theta} \circ \dots \circ \psi_{k+1,1}, \quad (30)$$

for some  $\theta > 1$ , where each composition represents a numerical step. This implicitly requires that between any “discrete” times, we are allowed to carry out an algorithm involving several computations. As explained in [53], in the context of sampled-data systems, this means that the digital controller is fast enough with respect to the sampling frequency. Although such algorithms have been extensively studied and optimized, it may not be possible to go through all the computations in between successive sampling times when the computational power is limited or when the sampling period is small. In this case, if the algorithm is stopped prematurely, condition (28a) is not guaranteed to hold and the observer performance (26) may be destroyed. In the following section, we overcome these potential issues by proposing a map  $\psi_{k+1}$  defined as (30) with  $\theta > 1$ , but where the decrease of  $V_{k+1}$  is guaranteed after each composition. This extends the idea first proposed in [53] and constitutes the main result of Section IV.

### C. Multi-Step Solution

We introduce the map  $M_{k+1} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$  defined by<sup>1</sup>

$$M_{k+1}(\hat{x}) := -\gamma_d P_{k+1}^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x}), \quad (31)$$

with  $P_k$  coming from Assumption 3,  $c$  given by Lemma 1 and  $\gamma_d$  a strictly positive scalar to be chosen *small* enough, this time, contrary to Theorem 1 in Section III. We have the following result.

**Lemma 5** *Suppose Assumption 3 holds. For any compact set  $\hat{\mathcal{X}}_{\max} \subset \mathbb{R}^{n_x}$ , there exists  $\gamma_d^* \in \mathbb{R}_{>0} \cup \{+\infty\}$  such that, for any  $0 < \gamma_d < \gamma_d^*$ , the functions  $V_k$  and  $M_k$  defined in (22) and (31) satisfy*

$$V_k(x, \hat{x} + M_k(\hat{x})) \leq V_k(x, \hat{x}) \quad \forall (x, \hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}_{\max} \quad (32a)$$

$$V_k(x, \hat{x} + M_k(\hat{x})) \leq V_k(x, \hat{x}) - \gamma_d \varrho^2 \quad \forall (x, \hat{x}) \in \mathcal{X} \times (\hat{\mathcal{X}}_{\max} \setminus \text{int}(\mathcal{C}_\varrho)) \quad (32b)$$

for all  $k \in \mathbb{N}$ , and with  $\varrho$  given by Lemma 1.

**Proof.** Given the compact set  $\hat{\mathcal{X}}_{\max}$ , define

$$\delta_{\max} := \max_{x \in \hat{\mathcal{X}}_{\max}} \left| \frac{dc}{dx}(x) \right|, \quad (33)$$

$$\gamma_d^* := \begin{cases} \frac{\lambda}{\delta_{\max}^2} & , \text{ if } \delta_{\max} \neq 0 \\ +\infty & , \text{ otherwise} \end{cases} \quad (34)$$

<sup>1</sup>Observe that  $M_k(x)$  corresponds to  $M(t, x)$  defined in (14) with  $P_k$  instead of  $P(t)$ .



with  $c$  given by Lemma 1 and  $\underline{\lambda}$  given by Assumption 3. Note that  $\delta_{\max} < \infty$  because  $c$  is  $C^1$  and  $\hat{\mathcal{X}}_{\max}$  is compact. Take  $x$  in  $\mathcal{X}$  and  $\hat{x}$  in  $\hat{\mathcal{X}}_{\max}$ . In light of (31),

$$\begin{aligned} V_k(x, \hat{x} + M_k(\hat{x})) &= V_k(x, \hat{x}) + 2M_k(\hat{x})^\top P_k(\hat{x} - x) + M_k(\hat{x})^\top P_k M_k(\hat{x}) \\ &\leq V_k(x, \hat{x}) - 2\gamma_d c(\hat{x})^\top \frac{dc}{dx}(\hat{x})(\hat{x} - x) \\ &\quad + \gamma_d^2 c(\hat{x})^\top \frac{dc}{dx}(\hat{x}) P_k^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x}). \end{aligned}$$

Since  $c(x) = 0$  for all  $x \in \mathcal{X}$  by Lemma 1, using (2) from Lemma 2, (31), and (33), we obtain

$$\begin{aligned} V_k(x, \hat{x} + M_k(\hat{x})) &\leq V_k(x, \hat{x}) - \gamma_d c(\hat{x})^\top (2I - \gamma_d \frac{dc}{dx}(\hat{x}) P_k^{-1} \frac{dc}{dx}(\hat{x})^\top) c(\hat{x}) \\ &\leq V_k(x, \hat{x}) - \gamma_d |c(\hat{x})|^2 (2 - \gamma_d \underline{\lambda}^{-1} \delta_{\max}^2) \\ &\leq V_k(x, \hat{x}) - \gamma_d |c(\hat{x})|^2 \leq V_k(x, \hat{x}) \end{aligned}$$

in which we used  $|P^{-1}| \leq \underline{\lambda}^{-1}$  and the fact that  $\gamma_d \underline{\lambda}^{-1} \delta_{\max}^2 < 1$  for any  $\gamma_d < \gamma_d^*$ . Hence (32a) holds. Moreover, when  $\hat{x} \in \mathbb{R}^{n_x} \setminus \text{int}(\mathcal{C}_\varrho)$ , we have  $|c(\hat{x})| \geq \varrho$  and hence (32b) holds. ■

According to Lemma 5, the interest of the map  $M_{k+1}$  in (31) is that, if  $\hat{\mathcal{X}}_{\max}$  denotes a compact set where the estimate  $\hat{x}$  remains, modifying  $\hat{x}$  as  $\hat{x} + M_{k+1}(\hat{x})$  makes  $V_{k+1}(x, \hat{x})$  decrease no matter where the plant state  $x$  actually is in  $\mathcal{X}$ , provided that  $\gamma_d$  is sufficiently small (see (32a)). Furthermore, this decrease is strict when  $\hat{x}$  is outside  $\text{int}(\mathcal{C}_\varrho)$ , with  $\mathcal{C}_\varrho$  defined in (1), see (32b). Since  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ , our strategy is therefore to repetitively use  $M_{k+1}$  to bring the estimate back into  $\hat{\mathcal{X}}$  at each iteration  $k$ . The redesign thus consists in correcting the state estimate  $G(k, \hat{x}_k, u_k, y_k)$  with the map  $M_{k+1}$  as long as the estimate is outside  $\mathcal{C}_\varrho$ , namely the function  $\psi_{k+1,i}$  in (30) are selected as

$$\psi_{k+1,i}(\hat{x}) = \begin{cases} \hat{x} + M_{k+1}(\hat{x}) & \text{if } \hat{x} \in \mathbb{R}^{n_x} \setminus \mathcal{C}_\varrho \\ \hat{x} & \text{otherwise} \end{cases} \quad (35)$$

for all  $i = 1 \dots \theta$  and all  $k \in \mathbb{N}$ . As it can be noted by the expression of  $M_{k+1}$ , the correction (35) uses the gradient of the convex map  $c$ , namely  $-dc/dx$ , to bring  $\hat{x}$  back to  $\hat{\mathcal{X}}$  along level sets of  $V_{k+1}$ . The recursive algorithm (30) stops when we cross  $\mathcal{C}_\varrho$ . This strategy is depicted in Figure 2. Note that this could not be achieved in one iteration because  $\gamma_d$  needs to be sufficiently small to ensure that  $V_{k+1}$  decreases, as shown in Lemma 5. This justifies the  $\theta$  steps.

The definition of  $\psi_{k+1,i}$  in (35) is not outer-semicontinuous on the boundary of  $\mathcal{C}_\varrho$ . Although it makes sense to implement (35) in practice, namely to stop the computations as soon as the estimate reaches  $\mathcal{C}_\varrho$ , we consider here a more general algorithm where the choice is left on the boundary of  $\mathcal{C}_\varrho$  to either stop the computations or to apply  $M_{k+1}$  again. This will enable us to endow our results with some nominal robustness with respect to numerical errors. The counterpart is that the map  $\psi_{k+1}$  is then set-valued. A way to write this more general algorithm is to use a local state  $q \in \{0, 1\}$  and define on

$\mathbb{R}^{n_x} \times \{0, 1\}$  the set valued map

$$\begin{aligned} \tilde{\psi}_{k+1,i}(\hat{x}, q) &= \begin{cases} \begin{pmatrix} \hat{x} + M_{k+1}(\hat{x}) \\ 0 \end{pmatrix} & \text{if } (\hat{x}, q) \in (\mathbb{R}^{n_x} \setminus \mathcal{C}_\varrho) \times \{0\} \\ \begin{pmatrix} \hat{x} + M_{k+1}(\hat{x}) \\ 0 \end{pmatrix} \cup \begin{pmatrix} \hat{x} \\ 1 \end{pmatrix} & \text{if } (\hat{x}, q) \in \partial \mathcal{C}_\varrho \times \{0\} \\ \begin{pmatrix} \hat{x} \\ 1 \end{pmatrix} & \text{if } (\hat{x}, q) \in \text{int}(\mathcal{C}_\varrho) \times \{0\} \\ \begin{pmatrix} \hat{x} \\ 1 \end{pmatrix} & \text{if } (\hat{x}, q) \in \mathcal{C}_\varrho \times \{1\} \end{cases} \quad (36) \end{aligned}$$

where  $M_{k+1}$  is defined in (31), and  $\partial \mathcal{C}_\varrho$  in (15). By initializing  $q$  to 0 and iteratively applying  $\tilde{\psi}_{k+1,i}$ , we see that as long as the  $\hat{x}$  is outside  $\mathcal{C}_\varrho$ , it is corrected using the map  $M_{k+1}$  and  $q$  is kept equal to 0. If  $\hat{x}$  reaches the boundary of  $\mathcal{C}_\varrho$ , there is a choice between correcting it with  $M_{k+1}$  or keeping it the same. In that latter case or if the estimate reaches the interior of  $\mathcal{C}_\varrho$ ,  $q$  is switched to 1 and  $\hat{x}$  is kept constant until the end of the  $\theta$  jumps. In other words,  $\tilde{\psi}_{k+1,i}$  becomes the identity and the latest estimate is kept. The following result shows that by applying this algorithm the component  $\hat{x}$  ends up in  $\hat{\mathcal{X}}$  while making  $V_{k+1}$  decrease. To this end, we define, for  $k$  in  $\mathbb{N}$ , the maps  $\tilde{\psi}_{k+1}$  as

$$\tilde{\psi}_{k+1} = \tilde{\psi}_{k+1,\theta} \circ \dots \circ \tilde{\psi}_{k+1,1}, \quad (37)$$

with  $\tilde{\psi}_{k+1,i}$  defined in (36), and, for  $k$  in  $\mathbb{N}$ , the map  $\psi_{k+1} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$  as

$$\psi_{k+1}(\hat{x}) = \text{Proj}_{\hat{x}} \tilde{\psi}_{k+1}(\hat{x}, 0) \quad (38)$$

with  $\tilde{\psi}_{k+1}$  defined in (37). The map  $\text{Proj}_{\hat{x}}$  so defined is simply the selection of the  $n_x$  first components of  $\tilde{\psi}_{k+1}(\hat{x}, 0)$  of dimension  $n_x + 1$ , namely the removal of the inner state  $q$ .

Finally, in order to state the main result of this section, let us define the real number  $v_{\max}$  as

$$v_{\max} := \sup_{(k, x, \hat{x}, u, d) \in \mathbb{N} \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}} \beta_d(k, x, \hat{x}, u, d), \quad (39)$$

with  $\beta_d$  given by Assumption 3, and according to its definition, for any  $k \in \mathbb{N}$ , let us define the sets  $\hat{\mathcal{X}}_{k+1, \max}$

$$\hat{\mathcal{X}}_{k+1, \max} := \{\hat{x} \in \mathbb{R}^{n_x} : \exists x \in \mathcal{X}, V_{k+1}(x, \hat{x}) \leq v_{\max}\}. \quad (40)$$

**Theorem 2** Suppose Assumption 3 holds and that  $v_{\max}$  defined in (39) is finite. Then, there exists  $\gamma_d^* \in \mathbb{R}_{>0} \cup \{+\infty\}$  such that, for any  $\gamma_d$  satisfying  $0 < \gamma_d < \gamma_d^*$ , there exists a nonnegative scalar  $\theta$  such that for all  $k \in \mathbb{N}$ , the map  $\psi_{k+1}$  defined in (38) is outer semi-continuous and verifies conditions (28) of Lemma 3 with  $\mathcal{S}_{k+1} = \hat{\mathcal{X}}_{k+1, \max}$  defined in (40), namely Problem 2 is solved.

**Proof.** The proof is postponed to Appendix A. ■

In order to use  $\psi_{k+1}$  defined in (38), we need to be able to make (at most)  $\theta + 1$  recursive computations at each discrete step  $k$ . Note, however, that in this case, the Lyapunov function decreases even if we perform less than  $\theta + 1$  steps. Therefore, in case of limited computational power, it is still interesting

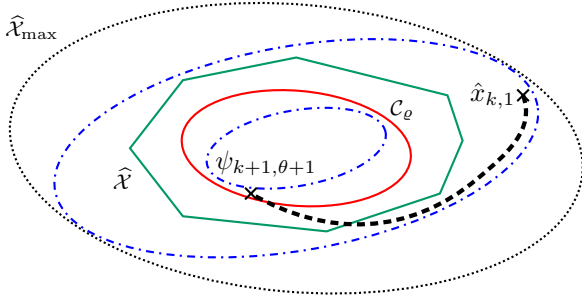


Fig. 2. Refining strategy (30) with (35), starting from  $\hat{x}_{k,1} = G(k, \hat{x}_k, u_k, y_k)$ . Dash-dotted blue ellipses: Lyapunov level sets of  $V_{k+1}$ . Red ellipsoid:  $C_0$ . Green polygon:  $\hat{\mathcal{X}}$ . Dotted black ellipse:  $\hat{\mathcal{X}}_{\max}$ , a compact set where the solution is proved to remain. Dashed black line: path of  $\hat{x}_{k,i+1} = \psi_{k,i}(\hat{x}_{k,i})$ ,  $i = 1, \dots, \theta$ .

to do as many  $\theta_m \in \{0, \dots, \theta\}$  steps as possible, as long as the estimate is outside of  $C_0$ : the Lyapunov function will decrease by  $\theta_m \gamma_d \varrho^2$  which will make the algorithm converge faster and make the estimate stay closer to  $\hat{\mathcal{X}}$ . This is a clear advantage compared to the projection-based algorithm presented in Section IV-B.

As detailed in the proof of Theorem 2, the number  $\theta$  defined in (70) gives the largest number of refining steps needed to bring  $\hat{x}$  in  $C_0$  and increases with the size of the sets  $\hat{\mathcal{X}}_{k+1, \max}$ . In practice, this number may be significantly smaller, as the analysis is made on the conservative assumption that at each step, the decrease rate of  $V_{k+1}$  is  $\gamma_d \varrho^2$ . Note that an estimate of  $\gamma_d^*$  is given in (33). According to Lemma 5, the true decrease is  $\gamma_d |c(\hat{x})|^2$ , which is larger than  $\gamma_d \varrho^2$  when  $\hat{x}$  is far from  $C_0$ , due to the convexity assumption in Lemma 1. In fact, the definition of  $\tilde{\psi}_{k+1,i}$  in (36) provides, from a computational point of view, a fast way to interrupt the algorithm once  $\hat{x}$  is in  $C_0$ .

An important observation is that although  $\hat{x}_{k+1}$  is in  $\hat{\mathcal{X}}$ , the intermediary estimates  $\hat{x}_{k,i}$ , obtained while implementing (37), can leave this set. However, according to the proof of Theorem 2, they remain in a compact set  $\hat{\mathcal{X}}_{\max}$  properly defined. Therefore,  $\hat{\mathcal{X}}_{\max}$  gives the magnitude of the numbers to be computed throughout the algorithm. This information can be useful to optimize the variables encoding and increase numerical precision in case of limited memory, in embedded systems for instance. Actually, this set could be made more precise by considering the evolution of  $V_k$  given by (24) and the maximal initial error  $e_0 = \hat{x}_0 - x_0$ .

To conclude, this multi-step DT redesign may be summed up as in Table III. Note, that, as in the CT-case, only the knowledge of  $P_k$  and  $c$  are required.

**Remark 3** We might wonder whether  $\psi_{k+1}$  defined in Theorem 2 is an iterative way of approaching  $\psi_{k+1}$  as defined in Lemma 4 when the number of iterations, namely  $\theta$ , goes to infinity. It turns out on examples that it is not. Actually, by compactness of  $C_0$  and convexity of  $c$ , it is possible to show by contradiction that,  $\psi_{k+1}(\hat{x})$  defined in (29) is not the only solution to (28a) on  $\mathbb{R}^n$ . More precisely, given  $\hat{x}$  in  $\mathbb{R}^n$ , there exists a neighborhood of  $\psi_{k+1}(\hat{x})$  such that (28a) still holds.

- 1) Consider a plant  $\mathcal{P}$  in (3) such that SA1 holds for some compact set  $\mathcal{X}$ .
- 2) Consider an observer  $\mathcal{O}$  in (21) such that Assumption 3 is satisfied for some known  $k \mapsto P_k$ .
- 3) Choose a map  $c$  such that  $\mathcal{X} \subseteq C_0$  along the lines of Lemma 1.
- 4) Design the function  $M_{k+1}$  as in (31), and select  $\gamma_d > 0$  small enough.
- 5) Design the function  $\psi_{k+1}$  as in (30) with  $\psi_{k+1,i}$  defined in (35); alternatively, if outer semi-continuity is sought, select  $\psi_{k+1}$  as in (38), with  $\tilde{\psi}_{k+1}$  defined in (37) and  $\tilde{\psi}_{k+1,i}$  in (36).
- 6) Select  $\theta$  large enough and run the new observer (25) with  $G_m$  defined as in (27) and  $\psi_{k+1}$  as in the previous step, and with an initial condition  $\hat{x}(0) \in \mathcal{X}$ .
- 7) Problem 2 is solved with  $\hat{\mathcal{X}} \supset C_0$ .

TABLE III

MULTI-STEP REDESIGN PROCEDURE FOR DISCRETE-TIME OBSERVERS

**Remark 4** The robustness of the modified observer with respect to the perturbation  $d$  is characterized by  $\beta_d$  in (26), which is the same as for the initial observer. In fact, the added computational steps due to  $\psi_{k+1}$  are not affected by perturbations, they are carried out independently from the plant and the observer: their only role is to bring the estimate back into the viable set  $\hat{\mathcal{X}}$ . Therefore, the only robustness we could be concerned about for conditions (28), is with respect to numerical errors in the recursive computation of  $\psi_{k+1}$ . In light of the outer semi-continuity of  $\tilde{\psi}_{k+1,i}$ , the maps  $\psi_{k+1}$ ,  $\psi_{k+1}$  (and thus  $F_m$ ) are outer semi-continuous and locally bounded, which ensure sequential compactness of solutions to (25), and thus robustness of the result of Theorem 2 with respect to numerical errors, see [22, Assumption 6.5].

## V. THE HYBRID CASE

### A. Problem Statement

In this section, we exploit the results of the previous two sections to address the scenario where plant  $\mathcal{P}$  is given by the hybrid model (see [22], [59])

$$\mathcal{P} : \begin{cases} \dot{\tau} \in f_{\tau}(\tau) \\ \dot{x} \in f(\tau, x, u, d) \\ \tau^+ \in g_{\tau}(\tau) \\ x^+ \in g(\tau, x, u, d) \\ y = h(\tau, x, u, d), \end{cases} \quad \begin{matrix} (\tau, x, u, d) \in C \\ (\tau, x, u, d) \in D \end{matrix} \quad (41)$$

where  $C$  and  $D$  are the flow and the jump sets respectively,  $(f_{\tau}, f)$  and  $(g_{\tau}, g)$  are the flow and jump maps respectively,  $x$  and  $y$  are respectively the state and the output of the plant. Variable  $\tau \in \mathbb{R}^{n_{\tau}}$  is another state variable<sup>2</sup>, which is supposed to be available for the observer design, and which can be a timer keeping track of the time and/or jumps to model the time-dependency of  $\mathcal{F}$  and  $\mathcal{G}$ . For instance,  $\tau$ ,  $n_{\tau} = 1$ , could model the time  $t$  with  $f_{\tau} \equiv 1$ , and  $g_{\tau} = \text{Id}$ , or a counter of jumps with  $f_{\tau} \equiv 0$  and  $g_{\tau}(\tau) = \tau + 1$ , or the time elapsed since the previous jump with  $f_{\tau} \equiv 1$  and  $g_{\tau} \equiv 0$ . Due to the presence of  $\tau$ , we need to reformulate SA1 as follows.

<sup>2</sup>Its dynamics  $f_{\tau}$  and  $g_{\tau}$  could also depend on  $(x, u, d)$  as long as  $\tau$  is available to the observer.

**Standing Assumption 3 (SA1')** *There exist subsets  $\mathcal{T}_0 \subset \mathbb{R}^{n_\tau}$ ,  $\mathcal{T} \subset \mathbb{R}^{n_\tau}$ ,  $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$ ,  $\mathcal{U} \subset \mathbb{R}^{n_u}$ ,  $\mathcal{D} \subset \mathbb{R}^{n_d}$ , and a compact subset  $\mathcal{X} \subset \mathbb{R}^{n_x}$ , such that any maximal solution ([59]) of (41) initialized in  $\mathcal{T}_0 \times \mathcal{X}_0$ , with input in  $\mathcal{U}$  and perturbation in  $\mathcal{D}$ , remains in  $\mathcal{T} \times \mathcal{X}$  at all forward times.*

Since we are interested only in solutions satisfying SA1', in the following we suppose that

$$C \subseteq \mathcal{T} \times \mathcal{X} \times \mathcal{U} \times \mathcal{D}, \quad D \subseteq \mathcal{T} \times \mathcal{X} \times \mathcal{U} \times \mathcal{D}. \quad (42)$$

Furthermore, to simplify the following statements, we denote  $\mathcal{T}_c$  and  $\mathcal{T}_d$  the projections of  $C$  and  $D$  on the first  $n_\tau$  components, namely sets verifying  $\mathcal{T}_c \cup \mathcal{T}_d \subseteq \mathcal{T}$  and

$$(\tau, x, u, d) \in C \implies \tau \in \mathcal{T}_c \quad (43a)$$

$$(\tau, x, u, d) \in D \implies \tau \in \mathcal{T}_d. \quad (43b)$$

We assume now that we know an observer whose jumps are synchronized with those of the plant, i.e., such that the complete system can be written as

$$\left. \begin{aligned} \dot{\tau} &\in f_\tau(\tau) \\ \dot{x} &\in f(\tau, x, u, d) \\ \dot{\hat{x}} &\in F(\tau, \hat{x}, u, h(\tau, x, u, d)) \\ \tau^+ &\in g_\tau(\tau) \\ x^+ &\in g(\tau, x, u, d) \\ \hat{x}^+ &\in G(\tau, \hat{x}, u, h(\tau, x, u, d)) \end{aligned} \right\} \begin{aligned} &(\tau, x, u, d) \in C, \\ &\hat{x} \in \mathbb{R}^{n_x}, \\ &(\tau, x, u, d) \in D, \\ &\hat{x} \in \mathbb{R}^{n_x}, \end{aligned} \quad (44)$$

with flow map  $\mathcal{F} := (f_\tau, f, F)$  and jump map  $\mathcal{G} := (g_\tau, g, G)$ . It is important to notice that the domain of solutions to (44) are determined by  $(\tau, x, u, d)$  only: for any solution  $(\tau, x, \hat{x})$  of (44),  $(\tau, x)$  is solution to (41) so that  $\hat{x}$  inherits the domain of  $(\tau, x)$ , at least as long as it is defined. This implicitly assumes that the jump times of (41) are known and can be used in the observer. This context covers a lot of designs, as detailed below. As before, we assume that the observer has been designed based on a quadratic Lyapunov function and we state the following assumption.

**Assumption 4** *There exist a known differentiable  $P(\tau) = P(\tau)^\top \in \mathbb{R}^{n_x \times n_x} > 0$  for  $\tau \in \mathbb{R}^{n_\tau}$ , positive real numbers  $\bar{\lambda} > \underline{\lambda} > 0$ , functions  $\beta_c : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \rightarrow \mathbb{R}$   $\beta_d : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \rightarrow \mathbb{R}_{\geq 0}$ , such that the map defined on  $\mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$  by*

$$V(\tau, x, \hat{x}) := (\hat{x} - x)^\top P(\tau)(\hat{x} - x), \quad (45)$$

*verifies for all  $(\tau, x, \hat{x}) \in \mathcal{T} \times \mathcal{X} \times \mathbb{R}^{n_x}$ ,*

$$\underline{\lambda}|\hat{x} - x|^2 \leq V(\tau, x, \hat{x}) \leq \bar{\lambda}|\hat{x} - x|^2, \quad (46)$$

*and for all  $\hat{x} \in \mathbb{R}^{n_x}$ ,*

$$L_{\mathcal{F}}V(\tau, x, \hat{x}, u, d) \leq \beta_c(\tau, x, \hat{x}, u, d) \quad \forall (\tau, x, u, d) \in C, \quad (47a)$$

$$V(\mathcal{G}(\tau, x, \hat{x}, u, d)) \leq \beta_d(\tau, x, \hat{x}, u, d) \quad \forall (\tau, x, u, d) \in D. \quad (47b)$$

The results of this section cover the following cases.

- A continuous-time plant with a sampled-data hybrid observer with  $g = \text{Id}$ , and  $\tau$  a timer determining the sampling times: see, among others, [24] for periodic sampling with  $P$

constant, [25] for aperiodic sampling with  $P$  time-varying, but also [26]–[28], [60] and references therein.

- A hybrid plant with known jump times and a hybrid synchronized observer as in the context of switched systems with common  $P$  [29] or switched  $P$  [30], [31] and  $\tau$  denoting the switching signal, impulsive systems [32], or general linear hybrid systems [33] among others.
- A continuous-time plant with a continuous-time observer as in Section III, with  $\mathcal{G} \equiv \emptyset$ ,  $D = \emptyset$ ,  $\tau = t \in \mathbb{R}$ ,  $\mathcal{F}$  defined in (8), and  $C = \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$ . In this context, we aim at redesigning the continuous-time observer by using the technique in Section III, thus remaining in the context of continuous-time observers, or by using the technique in Section IV, thus obtaining a hybrid observer. An example of the two different constructions is given in Section VII-A.
- A continuous-time plant with a hybrid observer, see for instance [34]–[36].
- A discrete-time plant with a discrete-time observer as in Section IV with  $\mathcal{F} \equiv \emptyset$ ,  $C = \emptyset$ ,  $\tau = k \in \mathbb{N}$ ,  $\mathcal{G} = (\tau + 1, f, F)$  with  $f$  and  $F$  defined in (20)–(21), and  $D = \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$ .

## B. Nominal Redesign

Using the tools previously developed for flow/jump redesign in Sections III and IV, we want to modify  $F$  and  $G$  so that the component  $\hat{x}$  of the modified system

$$\left. \begin{aligned} \dot{\tau} &\in f_\tau(\tau) \\ \dot{x} &\in f(\tau, x, u, d) \\ \dot{\hat{x}} &\in F_m(\tau, \hat{x}, u, h(\tau, x, u, d)) \\ \tau^+ &\in g_\tau(\tau) \\ x^+ &\in g(\tau, x, u, d) \\ \hat{x}^+ &\in G_m(\tau, \hat{x}, u, h(\tau, x, u, d)) \end{aligned} \right\} \begin{aligned} &(\tau, x, u, d) \in C, \\ &\hat{x} \in \mathbb{R}^{n_x}, \\ &(\tau, x, u, d) \in D, \\ &\hat{x} \in \mathbb{R}^{n_x}, \end{aligned} \quad (48)$$

with flow map  $\mathcal{F}_m = (f_\tau, f, F_m)$  and jump map  $\mathcal{G}_m = (g_\tau, g, G_m)$ , remains in  $\hat{\mathcal{X}}$  and the performances stated in Assumption 4 are preserved when  $\hat{x} \in \hat{\mathcal{X}}$ . An additional constraint we must add is that the maximal solutions of the modified observer (48) should be defined at least as long as the corresponding maximal solutions of the initial observer (44). Otherwise, a trivial solution would be to take  $F_m = F$ ,  $G_m = G$  and restrict the flow/jump maps to  $\hat{x} \in \hat{\mathcal{X}}$ , but the solutions could then die prematurely when  $\hat{x}$  points out  $\hat{\mathcal{X}}$ , thus altering the observer performances. Inspired by the previous sections, for  $(\tau, x, \hat{x}, u, y) \in \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y}$ , we define

$$F_m(\tau, \hat{x}, u, y) := F(\tau, \hat{x}, u, y) + M_c(\tau, \hat{x}) \quad (49a)$$

$$G_m(\tau, \hat{x}, u, y) := \psi(g_\tau(\tau), G(\tau, \hat{x}, u, y)) \quad (49b)$$

with  $M_c \equiv M$  defined in (14) and  $\psi$  a map to be defined. In fact, this map  $\psi(g_\tau(\tau), \cdot)$  is going to be the equivalent of  $\psi_{k+1}$  used in Section IV, which is coherent with the fact that for discrete systems,  $\tau$  stands for  $k$  and  $g_\tau(\tau)$  for  $k + 1$ .

Consider

$$F_{\max} := \sup_{(\tau, \hat{x}, u, x, d) \in \mathcal{T}_c \times \partial \mathcal{C}_\varrho \times \mathcal{U} \times \mathcal{X} \times \mathcal{D}} |F(\tau, \hat{x}, u, h(t, x, u, d))|, \quad (50)$$

with  $\mathcal{C}_\varrho$  defined in (15). Then, we have the following result.

**Lemma 6** Suppose that: Assumption 4 holds;  $\delta_{\min}$  defined in (16) is strictly positive;  $F_{\max}$  defined in (50) is finite. Fix  $\gamma_c > \gamma_c^* > 0$  with  $\gamma_c^*$  defined in (18) with  $F_{\max}$  given in (50) and  $\bar{\lambda}$  in Assumption 4. Assume there exist maps  $\psi : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_x}$  and  $\mathcal{S} : \mathbb{R}^{n_\tau} \rightrightarrows \mathbb{R}^{n_x}$  such that for all  $\tau \in \mathcal{T}_d$ ,

$$V(\tau^+, x, \psi(\tau^+, \hat{x})) \leq V(\tau^+, x, \hat{x}) \quad \forall(\tau^+, x, \hat{x}) \in g_\tau(\tau) \times \mathcal{X} \times \mathcal{S}(\tau) \quad (51a)$$

$$\psi(g_\tau(\tau), \mathcal{S}(\tau)) \subseteq \mathcal{C}_\varrho \quad (51b)$$

and moreover, for any  $(\tau, x, u, d) \in D$  and any  $\hat{x} \in \hat{\mathcal{X}}$ , we have

$$G(\tau, \hat{x}, u, h(\tau, x, u, d)) \in \mathcal{S}(\tau). \quad (51c)$$

Then, the functions  $V$ ,  $\mathcal{F}_m$ ,  $\mathcal{G}_m$ , defined by (45), (49), satisfy

$$L_{\mathcal{F}_m} V(\tau, x, \hat{x}, u, d) \leq \beta_c(\tau, x, \hat{x}, u, d) \quad \forall \hat{x} \in \hat{\mathcal{X}}, \forall(\tau, x, u, d) \in C, \quad (52a)$$

$$V(\mathcal{G}_m(\tau, x, \hat{x}, u, d)) \leq \beta_d(\tau, x, \hat{x}, u, d) \quad \forall \hat{x} \in \hat{\mathcal{X}}, \forall(\tau, x, u, d) \in D. \quad (52b)$$

In addition, any maximal solution  $(\tau, x, \hat{x})$  to (44) initialized in  $\mathcal{T}_0 \times \mathcal{X}_0 \times \mathcal{X}$  with input in  $\mathcal{U}$  and perturbation in  $\mathcal{D}$ , is such that  $\hat{x}$  remains in  $\hat{\mathcal{X}}$  at all times and  $(\tau, x)$  is a maximal solution to (41).

**Proof.** Equation (52a) follows in the same way as in the proof of Theorem 1 using (42). Equation (52b) follows directly from (49b), (51a) and (51c). Besides, combining (49b), (51a) and (51b), we get

$$G_m(\tau, \hat{x}, u, h(\tau, x, u, d)) \subseteq \mathcal{C}_\varrho \quad \forall(\tau, x, u, d) \in D, \hat{x} \in \hat{\mathcal{X}}. \quad (53)$$

Now take a solution  $(\tau, x, \hat{x})$  to (44) initialized in  $\mathcal{T}_0 \times \mathcal{X}_0 \times \mathcal{X}$ . Because of the flow/jump sets are the same as the initial plant,  $(\tau, x)$  is solution to (41) and from SA1',  $(\tau, x)$  is in  $\mathcal{T} \times \mathcal{X}$ . Since  $\mathcal{X}_0 \subset \mathcal{X} \subset \mathcal{C}_\varrho$ ,  $\hat{x}(0, 0) \in \mathcal{C}_\varrho$ . As long as (44) flows for  $t \in [0, t_1]$ ,  $\hat{x}(t, 0)$  remains in  $\mathcal{C}_\varrho$  by Theorem 1. Then, at the first jump,  $\hat{x}(t_1, 0) \in \mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ , and by definition of solutions  $(\tau, x, u, d)(t_1, 0) \in D$ , so that, from (53),  $\hat{x}(t_1, 1) \in \mathcal{C}_\varrho$ . Repeating the same reasoning from that initial condition, we deduce that  $\hat{x}$  remains in  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ . Because the flow/jump conditions of (48) are independent from  $\hat{x}$ , any maximal solution  $(\tau, x, \hat{x})$  of (44) is such that  $(\tau, x)$  is maximal for (41) unless  $\hat{x}$  explodes in finite time during flow. This latter case is impossible since  $\hat{\mathcal{X}}$  is compact. ■

Lemma 6 says that, if the map  $\psi$  used in  $G_m$  in (49b) verifies conditions (51), the Lyapunov properties of the observer are preserved and the estimate  $\hat{x}$  remains in  $\hat{\mathcal{X}}$ . More importantly, the estimate is defined as long as the underlying plant solution is, which was not necessarily the case of the initial observer (44). Therefore, the modification does not alter the properties of the solutions and keeps  $\hat{x}$  in  $\hat{\mathcal{X}}$ .

**Remark 5** Conditions (51) are trivially satisfied with  $\psi(\tau, \cdot) = \text{Id}$  if  $G(\tau, \hat{x}, u, h(\tau, x, u, d)) \in \mathcal{C}_\varrho$  for all  $(\tau, x, u, d) \in D$  and  $\hat{x} \in \hat{\mathcal{X}}$ . In that case,  $G_m = G$ , namely only the flow map is modified. Otherwise, it is necessary to

modify the jump map in order to guarantee solutions are in  $\hat{\mathcal{X}}$  after the jumps.

Similarly to Lemma 4, the map  $\psi : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x}$  defined by

$$\psi(\tau, \hat{x}) := \underset{x \in \mathcal{C}_\varrho}{\operatorname{argmin}} (\hat{x} - x)^\top P(\tau)(\hat{x} - x) \quad (54)$$

satisfies conditions (51) with  $\mathcal{S}(\tau) = \mathbb{R}^{n_x}$  for all  $\tau \in \mathbb{R}^{n_\tau}$ . However, as mentioned in Section IV-B, this map may be difficult to implement. Alternatively, we can follow the design of Section IV-C. For that, we introduce the map  $M_d : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$  defined by

$$M_d(\tau, \hat{x}) := -\gamma_d P(\tau)^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x}), \quad (55)$$

with  $\gamma_d$  a strictly positive scalar, and  $P(\tau)$  coming from Assumption 4. As in Theorem 2, if

$$v_{d,\max} := \max_{(\tau, x, \hat{x}, u, d) \in \mathcal{T}_d \times \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{U} \times \mathcal{D}} \beta_d(\tau, x, \hat{x}, u, d) \quad (56)$$

is finite and by choosing  $\gamma_d$  sufficiently small, we can bring  $\hat{x}^+ = G(\tau, \hat{x}, u, y)$  back into  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$  after each jump, without making  $V$  increase, by repetitively using  $M_d(\tau^+, \cdot)$  a finite number of times. In order to avoid heavy notations, we do not explicitly rewrite the map  $\psi$  gendered by this method which is a direct transposition of Section IV-C. Instead, we notice that, in the hybrid context, such a composition of functions can be more easily modeled by a succession of jumps that can directly be encoded in the hybrid system. This leads to the hybrid redesign presented in the following section.

### C. Hybrid Redesign

Inspired by (36), we add a toggle state  $q$  to the observer state  $\hat{x}$ , and define the map  $\tilde{\psi} : \mathbb{R}^{n_\tau} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x} \times \{0, 1\}$  as

$$\tilde{\psi}(\tau, \hat{x}) = \begin{cases} \begin{pmatrix} \hat{x} + M_d(\tau, \hat{x}) \\ 0 \end{pmatrix} & \text{if } \hat{x} \in (\mathbb{R}^{n_x} \setminus \mathcal{C}_\varrho) \\ \begin{pmatrix} \hat{x} + M_d(\tau, x) \\ 0 \end{pmatrix} \cup \begin{pmatrix} \hat{x} \\ 1 \end{pmatrix} & \text{if } \hat{x} \in \partial \mathcal{C}_\varrho \\ \begin{pmatrix} \hat{x} \\ 1 \end{pmatrix} & \text{if } \hat{x} \in \text{int}(\mathcal{C}_\varrho) \end{cases} \quad (57)$$

we consider the new observer dynamics

$$\begin{aligned} (\dot{\tau}, \dot{x}, \dot{\hat{x}}, \dot{q})^\top &\in \tilde{\mathcal{F}}(\tau, x, \hat{x}, u, d), & (\tau, x, \hat{x}, q, u, d) &\in \tilde{C}, \\ (\tau^+, x^+, \hat{x}^+, q^+)^\top &\in \tilde{\mathcal{G}}(\tau, x, \hat{x}, u, d), & (\tau, x, \hat{x}, q, u, d) &\in \tilde{D}, \end{aligned} \quad (58)$$

with  $\tilde{\mathcal{F}} = (f_\tau, f, F, 0)$ ,

$$\tilde{C} := \{(\tau, x, \hat{x}, q, u, d) : (\tau, x, u, d) \in C, \hat{x} \in \hat{\mathcal{X}}, q = 1\},$$

$\tilde{D}$  is the domain of definition of the jump map defined by

$$\tilde{\mathcal{G}} = \begin{cases} \tilde{\mathcal{G}}_{11} & \text{if } q = 1, (\tau, x, u, d) \in D, \hat{x} \in \text{int}(\hat{\mathcal{X}}) \\ \tilde{\mathcal{G}}_{11} \cup \tilde{\mathcal{G}}_{10} & \text{if } q = 1, (\tau, x, u, d) \in D, \hat{x} \in \partial \hat{\mathcal{X}} \\ \tilde{\mathcal{G}}_{10} & \text{if } q = 1, (\tau, x, u, d) \in D, \hat{x} \in \mathbb{R}^{n_x} \setminus \hat{\mathcal{X}} \\ & \text{or } q = 1, (\tau, x, u, d) \in \text{cl}(C) \setminus D, \\ & \hat{x} \in \text{cl}(\mathbb{R}^{n_x} \setminus \hat{\mathcal{X}}) \\ \tilde{\mathcal{G}}_0 & \text{if } q = 0, (\tau, x, u, d) \in \text{cl}(C) \cup D, \\ & \hat{x} \in \mathbb{R}^{n_x} \end{cases} \quad (59)$$

with

$$\tilde{\mathcal{G}}_{11}(\tau, x, \hat{x}, u, d) := (g_\tau(\tau), g(\tau, x, u, d), \\ G(\tau, \hat{x}, u, h(\tau, x, u, d)), 1) \quad (60a)$$

$$\tilde{\mathcal{G}}_{10}(\tau, x, \hat{x}, u, d) := (\tau, x, \hat{x}, 0) \quad (60b)$$

$$\tilde{\mathcal{G}}_0(\tau, x, \hat{x}, u, d) := (\tau, x, \tilde{\psi}(\tau, \hat{x})) \quad (60c)$$

Still assuming that  $v_{d,\max}$  defined in (56) is finite, we consider the compact sets

$$\hat{\mathcal{X}}_{d,\max} := \{\hat{x} \in \mathbb{R}^{n_x} : \exists \tau \in \mathcal{T}_d, \\ \exists x \in \mathcal{X}, V(\tau, x, \hat{x}) \leq v_{d,\max}\}. \quad (61)$$

**Theorem 3** *Select  $\gamma_d > 0$  satisfying  $\gamma_d < \gamma_d^*$  defined in (33)-(34) with  $\hat{\mathcal{X}}_{\max} := \hat{\mathcal{X}} \cup \hat{\mathcal{X}}_{d,\max}$ . Then, any solution  $\phi = (\tau, x, \hat{x}, q)$  of (58) initialized in  $\mathcal{T}_0 \times \mathcal{X}_0 \times \hat{\mathcal{X}} \times \{1\}$ , verifies  $\text{dom } \phi = \mathcal{D}_1 \cup \mathcal{D}_0$ , with  $\mathcal{D}_0$  and  $\mathcal{D}_1$  such that  $q(t, j) = 0$  for all  $(t, j) \in \mathcal{D}_0$  and  $q(t, j) = 1$  for all  $(t, j) \in \mathcal{D}_1$ , and the following.*

- The projection of  $\mathcal{D}_0$  on  $t$  is of null Lebesgue measure.
- The hybrid arc  $s = (\tau, x, \hat{x}, u, d)|_{\mathcal{D}_1}$  verifies  $\hat{x}(t, j) \in \hat{\mathcal{X}}$  for all  $(t, j) \in \text{dom } s$ , and, for all  $(t, j) \in \text{dom } s$ ,

$$\frac{d}{dt} V((\tau, x, \hat{x})(t, j)) \leq \beta_c((\tau, x, \hat{x}, u, d)(t, j))$$

and for all  $(t, j) \in \text{dom } s$  such that  $(t, j + 1) \in \text{dom } s$ ,

$$V((\tau, x, \hat{x})(t, j + 1)) \leq \beta_d((\tau, x, \hat{x}, u, d)(t, j)).$$

- Removing the trivial jumps from  $(\tau, x)$  gives a maximal solution of the plant (41).

The first item says that the part  $\mathcal{D}_0$  of the time domain where  $q = 0$  consists only of jumps (no flow), and the second item, that along the solutions restricted to the rest of the time domain  $\mathcal{D}_1$  where  $q = 1$ ,  $\hat{x} \in \hat{\mathcal{X}}$  and the Lyapunov properties are preserved. In other words, we are only adding jumps, “correction phases” where  $q = 0$ , designed to bring  $\hat{x}$  back into  $\hat{\mathcal{X}}$  whenever it attempts to leave  $\hat{\mathcal{X}}$ . Besides, the third item says that modulo the added jumps in  $\mathcal{D}_0$ , we indeed browse the entire domain of the underlying plant solution, i.e. our modification does not make solutions stop prematurely. The following proof further explains the behavior of the solutions.

**Proof.** As long as  $\hat{x} \in \text{int}(\hat{\mathcal{X}})$ ,  $q$  remains equal to 1, and  $(\tau, x, \hat{x})$  evolves according to the initial observer dynamics (44) with  $\tilde{\mathcal{F}} = (\mathcal{F}, 0)$  and  $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{11} = (\mathcal{G}, 1)$ . If  $\hat{x}$  reaches the boundary of  $\hat{\mathcal{X}}$  after flow or jump, it can either carry on evolving according to (44) if possible, or use  $\tilde{\mathcal{G}}_{10}$  to reset  $q$  to 0; if  $\hat{x}$  exits  $\hat{\mathcal{X}}$  after jump,  $q$  is necessarily reset to 0 through  $\tilde{\mathcal{G}}_{10}$ . In both cases, either  $\hat{x}$  comes from flow and  $\hat{x} \in \partial\hat{\mathcal{X}} \subseteq \hat{\mathcal{X}}$ , or  $\hat{x}$  is an output of  $G$  and  $\hat{x} \in \hat{\mathcal{X}}_{d,\max}$  by definition of  $\hat{\mathcal{X}}_{d,\max}$ . Therefore, at this point,  $\hat{x} \in \hat{\mathcal{X}}_{\max}$  and  $q = 0$ , so that no flow is possible, and the solution necessarily jumps via  $\tilde{\mathcal{G}}_0$ . Since  $\partial\hat{\mathcal{X}} \cap \mathcal{C}_\varrho = \emptyset$  according to Lemma 1,  $\hat{x} \in \mathbb{R}^{n_x} \setminus \mathcal{C}_\varrho$  and  $\hat{x}$  is reset to  $\hat{x} + M_d(\tau, \hat{x})$  while  $(\tau, x)$  remains unchanged and  $q$  is kept to 0. According to Lemma 5,  $V(\tau, x, \cdot)$  decreases through the jump, so that  $\hat{x}$  is still in  $\hat{\mathcal{X}}_{\max}$ . Along the same lines as the proof of Theorem 2, we can show that the solution

jumps a finite number of times via  $\tilde{\mathcal{G}}_0$ , until  $\hat{x} \in \mathcal{C}_\varrho$  and  $q$  is switched back to 1. The states  $(\tau, x)$  have not changed throughout this “correction phase” and since  $\mathcal{C}_\varrho \subset \text{int}(\hat{\mathcal{X}})$ , the solution restarts evolving according to the initial observer dynamics (44). Therefore, the domain of any trajectory of (58) initialized in  $\mathcal{T}_0 \times \mathcal{X}_0 \times \hat{\mathcal{X}} \times \{1\}$  is composed of two parts:

- a part  $\mathcal{D}_1$  where the system evolves according to the original observer dynamics (44) and where  $\hat{x} \in \hat{\mathcal{X}}$ .
- “correction phases”  $\mathcal{D}_0$  made of sequences of a finite number of successive jumps occurring at times where  $\hat{x}$  has left (or is about to leave)  $\hat{\mathcal{X}}$ . During this phase,  $(\tau, x)$  is kept unchanged and the final value of  $\hat{x}$  is back in  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ .

Therefore,  $\hat{x}$  is bounded and because  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ , the “correction phases” are always separated by flow or jump of the component  $(\tau, x)$ . This means that, after removing the trivial jumps added during the “correction phases”, we recover a maximal solution  $(\tau, x)$  of the plant (41). As for the Lyapunov performances, (47) is obviously preserved during the phases where the solution evolves according to (44), and  $V(\tau, x, \cdot)$  is non increasing throughout the “correction phase”. Therefore, the Lyapunov properties (47) are preserved along the solutions once restricted to  $\mathcal{D}_1$ . ■

Note that the same conclusions hold by taking  $\tilde{\mathcal{F}} = (f_\tau, f, F_m, 0)$  with  $F_m$  defined in (49a) since it does not change (47a). The only difference is that  $F_m$  naturally forces  $\hat{x}$  to remain in  $\mathcal{C}_\varrho$  during flow, so that fewer hybrid “correction phases” are necessary.

**Remark 6** *In the case where  $P$  is generated dynamically by Riccati/Lyapunov dynamics, the trivial jumps of  $\tau$  added in (58) during the “correction phases” where  $q = 0$  are simply traduced by trivial jumps in the dynamics of  $P$ . In other words,  $P$  is kept constant throughout the correction process.*

## VI. CHOICE OF DESIGN PARAMETERS

### A. Choice of $\hat{\mathcal{X}}$

Throughout this work,  $\hat{\mathcal{X}}$  is the set in which we want/need to constrain the observer solutions. However, in practice, we have some freedom to select this set. Often, we only want the state  $\hat{x}$  of the observer to remain as close as possible to the set  $\mathcal{X}$  where the plant state  $x$  to be estimated evolves in view of SA1. Then, the design of  $\hat{\mathcal{X}}$  and  $\mathcal{C}$  satisfying SA2 consists in choosing the convex map  $c$  of Lemma 5 such that  $\mathcal{C}_0$  is as close to  $\mathcal{X}$  as possible, and then select  $\varrho > 0$  and  $\hat{\mathcal{X}} := \mathcal{C}_{\varrho'}$  for some  $0 < \varrho < \varrho'$ .

Sets  $\mathcal{C}_0$ ,  $\mathcal{C}_\varrho$  and  $\hat{\mathcal{X}}$  can thus be arbitrarily close by choosing  $\varrho < \varrho'$  arbitrarily small. Nevertheless, a very small  $\varrho$  may induce a very large gain  $\gamma_c$  in the CT correction term (14) due to  $\delta_{\min}$  defined in (16) being very small, or a very large number of jumps  $\theta$  in the DT multi-step redesign in order to bring  $c$  under the threshold  $\varrho$ , see (70). Therefore, a compromise must be found between constraints and numerical feasibility.

It is important to mention that although item a) of Lemma 1 imposes to take a strict inclusion  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ , the observer solutions are actually shown to remain in  $\mathcal{C}_\varrho$  in the CT redesign

of Theorem 1, the multi-step DT redesign of Theorem 2 and the nominal hybrid redesign of Lemma 6. As for the single-step DT redesign of Lemma 4, solutions actually remain in  $\mathcal{C}_0$ . In fact, only in the hybrid redesign of Theorem 3 is the strict inclusion  $\mathcal{C}_\varrho \subset \hat{\mathcal{X}}$  exploited to properly define the jump logic and guarantee non-Zeno solutions.

Finally, note that, in the spirit of keeping the observer trajectories as close as possible to  $\mathcal{X}$ , the redesign can also be done without any explicit choice of  $\hat{\mathcal{X}}$  by:

- choosing  $\gamma_c$  as large as numerically acceptable in the CT redesign (Table I);
- choosing the desired threshold  $\varrho$  under which  $c$  must be brought at each iteration in the multi-step DT redesign (Table III);
- using the single-step DT redesign (Table II) thus projecting directly into  $\mathcal{C}_0$ .

### B. Construction of the map $c$

We next show how to build the map  $c$  in the common practical cases where the plant solutions are known to remain in a polytope or an ellipsoid. Those typically arise when the states are known to remain within physical bounds.

**Lemma 7** Suppose there exists vectors  $a_i \in \mathbb{R}^{n_x}$  and scalars  $b_i$ ,  $i = 1 \dots n_c$ , satisfying

$$\mathcal{X} \subseteq \bigcap_{i=1}^{n_c} \{x \in \mathbb{R}^{n_x} : a_i^\top x \leq b_i\} \subset \hat{\mathcal{X}}.$$

Then the function  $c : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}^{n_c}$  defined as

$$c(x) := (c_1(x), \dots, c_i(x), \dots, c_{n_c}(x)), \quad (62)$$

$$c_i(x) := \max \{a_i^\top x - b_i, 0\}^2$$

is  $C^1$  and satisfies items a)-b) of Lemma 1 for some  $\varrho > 0$ .

**Proof.** Consider  $\mathcal{B} := \bigcap_{i=1}^{n_c} \{x \in \mathbb{R}^{n_x} : a_i^\top x \leq b_i\}$ , and  $\bar{\mathcal{B}} := \bigcap_{i=1}^{n_c} \{x \in \mathbb{R}^{n_x} : a_i^\top x \leq \bar{b}_i\}$  satisfying  $\mathcal{X} \subset \mathcal{B} \subset \bar{\mathcal{B}} \subset \hat{\mathcal{X}}$ , for  $\bar{b}_i > b_i$  for  $i = 1, \dots, n_c$ . Pick any point  $x^\circ \in \mathcal{X} \subset \mathcal{B}$ . By construction, it satisfies  $a_i^\top x^\circ \leq b_i$  for all  $i = 1, \dots, n_c$ , so that  $c_i(x^\circ) = 0$  for all  $i = 1, \dots, n_c$ , and therefore  $c(x^\circ) = 0$ . As this holds for any point in  $\mathcal{X}$ , this shows that  $\mathcal{X} \subset \mathcal{C}_0$ , with  $\mathcal{C}_0$  defined in (1). Then, select  $\varrho := \min_i \{\bar{b}_i - b_i\}^2$ . Pick any point  $x \in \mathcal{C}_\varrho$  with  $\mathcal{C}_\varrho$  defined in (1). Hence, we have that  $|c_i(x)| \leq \varrho$  for any  $i = 1, \dots, n_c$ . It follows that  $|a_i^\top x - b_i| \leq \sqrt{\varrho} \leq \bar{b}_i - b_i$  which implies  $a_i^\top x \leq \bar{b}_i$ , for all  $i = 1, \dots, n_c$ , and thus  $x \in \bar{\mathcal{B}}$ . Since by construction  $\bar{\mathcal{B}} \subset \hat{\mathcal{X}}$ , this shows that any point in  $\mathcal{C}_\varrho$  is also in  $\hat{\mathcal{X}}$  and prove item a). Finally, in order to show item b), it is enough to observe that the functions  $c_i$  are quadratic outside their zeroing set, and therefore convex. This concludes the proof. ■

Other definitions can be considered by exploiting the shape of  $\mathcal{X}$ . For instance, when  $\mathcal{X}$  is an ellipsoid, a scalar function  $c$  can be constructed as follows.

**Lemma 8** Suppose there exists a positive definite matrix  $Q$  of dimension  $n_x \times n_x$ , a scalar  $r > 0$  and a vector  $x_0 \in \mathbb{R}^{n_x}$

satisfying  $\mathcal{X} \subseteq \{x \in \mathbb{R}^{n_x} : (x - x_0)^\top Q(x - x_0) \leq r\} \subset \hat{\mathcal{X}}$ . Then, the function  $c : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ ,  $n_c = 1$ , defined as

$$c(x) := \max \left\{ \frac{(x - x_0)^\top Q(x - x_0)}{r} - 1, 0 \right\}^2. \quad (63)$$

is  $C^1$  and verifies the items a)-b) of Lemma 1 for some  $\varrho > 0$ .

**Proof.** Similar to Lemma 7. ■

## VII. EXAMPLES

### A. Flow and Jump Redesign for Kalman-like Observers

The goal of this section is to illustrate and compare the two redesign techniques (flow or jump redesign) for a continuous-time observer. Inspired by the example in [13], we consider the case in which system (3) is given by

$$\dot{x} = A(u)x + \varphi(x), \quad y = Cx \quad (64)$$

where  $x \in \mathbb{R}^3$  is the state,  $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ ,  $A(\cdot)$  and  $\varphi(\cdot)$  are defined as

$$A(u) = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(x) = \begin{pmatrix} x_1 - \frac{1}{3}x_1^3 \\ \epsilon(3\sin(x_1) - x_2 - \eta) \\ x_3 - \frac{1}{5}x_3^5 + \frac{x_1x_2}{1+x_2^2} \end{pmatrix},$$

with  $\epsilon, \eta > 0$  some known parameters, and  $u$  an external known input. In this example we consider the case in which there are no disturbances, i.e. we consider  $d = 0$  in (3). By using a quadratic-like Lyapunov function, it can be shown that SA1 is satisfied for some  $\mathcal{X}_0, \mathcal{X}, \mathcal{U}$  and  $\mathcal{D} = \emptyset$ . See for instance Figure 3, in which we selected  $\epsilon = 0.8$ ,  $\eta = 5$ ,  $x(0) = (2, -2, -2)$ , and  $u = \sin(t + \frac{\pi}{2})$ . From simulations, we take the following numerical values for SA1:  $\mathcal{X} = \{x \in \mathbb{R}^3 : |x_i| \leq L_{\min}, i = 1, 2, 3\}$  and  $L_{\min} = 8$  for an appropriate  $\mathcal{X}_0 \subset \mathcal{X}$ . The set  $\hat{\mathcal{X}}$  in SA2 is selected then as  $\hat{\mathcal{X}} = \{x \in \mathbb{R}^3 : |x_i| \leq L_{\max}, i = 1, 2, 3\}$  with  $L_{\max} = 11$ . Finally, the function  $c$  of Lemma 1 is chosen as in (62) with  $n_c = 6$ , and  $a_1 = (1, 0, 0)$ ,  $a_2 = (0, 1, 0)$ ,  $a_3 = (0, 0, 1)$ ,  $a_4 = -a_1$ ,  $a_5 = -a_2$ ,  $a_6 = -a_3$ ,  $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = L_{\min}$ , and we choose  $\varrho = 1$ .

Following the high-gain extended-Kalman filter design in [3], we design the observer

$$\dot{\hat{x}} = A(u)\hat{x} + \varphi_s(\hat{x}) + \ell D_\ell S(t)C^\top(y - C\hat{x}) \quad (65)$$

where  $D_\ell = \text{diag}(1, \ell, \ell^2)$  with  $\ell \geq 1$  the high-gain parameter,  $\varphi_s$  a function to be properly chosen, and  $S(t)$  is computed as solution to the following differential Riccati equation

$$\dot{S} = \ell(SA(u)^\top + A(u)S + Q - 2SC^\top CS + aS), \quad (66)$$

where  $Q$  and  $S(0)$  are positive definite symmetric matrices. We have then the following result.

**Proposition 1** Let  $\varphi_s$  be any  $C^1$  function satisfying  $|\varphi(x) - \varphi_s(\hat{x})| \leq \delta|x - \hat{x}|$  for any  $(x, \hat{x}) \in \mathcal{X} \times \mathbb{R}^{n_x}$ , for some  $\delta > 0$ . Then, system (64) and observer (65), (66) verify Assumptions 1, 2 with  $P(t) = D_\ell^{-1}S(t)^{-1}D_\ell^{-1}$ ,  $\beta_c = -(a\ell - \delta)V$ , and some  $\bar{\lambda} > \underline{\lambda} > 0$ .

**Proof.** The proof follows the computations given in [3] and it is omitted for space reasons. ■

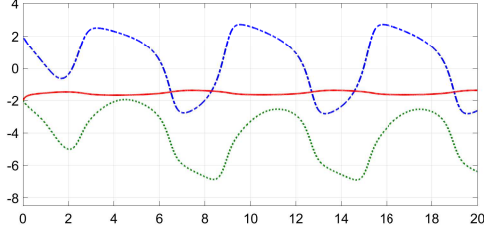
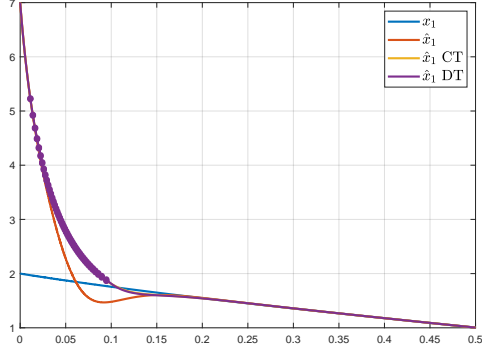
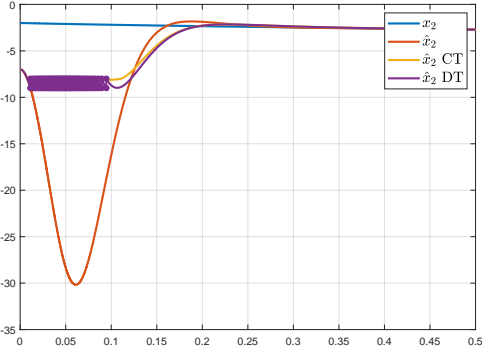


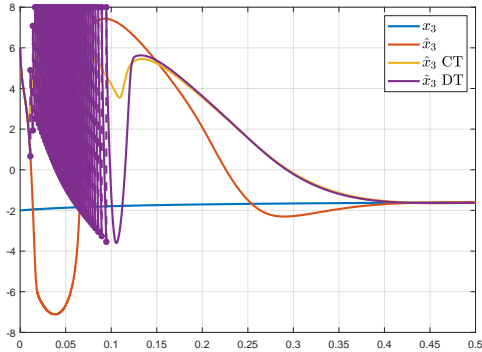
Fig. 3. Evolution of the solution to system (64) for  $t \in [0, 20]$  with initial conditions  $x_0 = (2, -2, -2)$ .



(a)  $\hat{x}_1$



(b)  $\hat{x}_2$



(c)  $\hat{x}_3$

Fig. 4. Solutions of the plant (64) (blue), the nominal observer (65), (66) (red), the modified observer with flow redesign (yellow), the modified observer with jump redesign (purple), with  $\hat{x}_0 = (7, -7, 6)$ ,  $\ell = 20$ ,  $a = 1$ .

In Figure 4, we compare the different observation strategies. With the nominal observer (65), (66), depicted in red, the peaking phenomenon [7] causes the observer state to go far away from  $\mathcal{X}$  during the transient. A first solution is to apply the flow redesign tool presented in Section III (see Table I), which enables to constrain the observer state into the set  $\mathcal{C}_\ell$  and cut away the peaking. More precisely, we add to the observer dynamics (65) the correction term  $M(t, \hat{x})$  defined in (14) with  $\gamma_c = \ell$  and  $P(t) = D_\ell^{-1} S(t)^{-1} D_\ell^{-1}$  (trajectories in yellow in Figure 4). Another solution is to apply the hybrid redesign of Section V-C (with  $D = \emptyset$ ). In other words, whenever  $\hat{x}$  attempts to leave  $\hat{\mathcal{X}}$ , jumps are triggered to bring it back into  $\hat{\mathcal{X}}$ . This can be done by successive jumps using  $M_d$  as in Section V-C, or directly by projecting  $\hat{x}$  into  $\mathcal{C}_0$  along (54). Indeed,  $\mathcal{C}_0$  being a polytope, this optimization problem reduces here to a QP with linear inequality constraints which can be solved on Matlab via the function `quadprog`. The result of simulations of this latter method appears in purple in Figure 4. Note that we have been able to use the optimization approach thanks to the polytopic form of  $\hat{\mathcal{X}}$ , but in the case of a hyper-sphere, we should probably have followed the technique proposed in Theorem 2. This is not shown for space reasons, an illustration of Theorem 2 can indeed be found in [53].

Simulations confirmed the validity of the proposed approaches (flow-redesign and hybrid redesign), and suggest to actually combine both: the flow redesign keeps  $\hat{x}$  as much as possible in  $\hat{\mathcal{X}}$ , but if for some reason ( $\gamma_c$  too small or perturbations),  $\hat{x}$  still attempts to leave  $\hat{\mathcal{X}}$ , then the jumps act as a back-up, all this without altering the Lyapunov performances. It would be necessary to investigate in details the computational load of each method to determine which is more efficient: flow or hybrid redesign? With quadratic projections or successive jumps?

### B. Hybrid Redesign for the Bouncing Ball

Consider a bouncing ball with gravity coefficient  $g > 0$  and restitution coefficient  $0 < \lambda < 1$ , modelled as a hybrid system with state  $x = (x_1, x_2)$  representing the height and velocity of the ball, flow/jump maps

$$f(x) = A_c x - \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad g(x) = A_d x$$

with  $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $A_d = \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix}$  and flow/jump sets

$$C = \mathbb{R}_{\geq 0} \times \mathbb{R}, \quad D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}.$$

We assume that the height of the ball  $h(x) = x_1 = 0$  is measured at the jumps only, i.e., the impacts of the ball can be detected by some sensor. No measurement is available during flows. In [33], an observer is designed for this system by adding a linear correction term in the jump dynamics, with an appropriately chosen gain, and by synchronizing the observer with the plant. This leads to the dynamics

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) \\ \hat{x}^+ &= g(\hat{x}) - L(\hat{x}_1 - x_1) \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} x \in C, \quad \hat{x} \in \mathbb{R}^{n_x},$$

$$\begin{aligned} x^+ &= g(x) \\ \hat{x}^+ &= g(\hat{x}) - L(\hat{x}_1 - x_1) \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} x \in D, \quad \hat{x} \in \mathbb{R}^{n_x},$$



The gain  $L$  must then be chosen depending on the available knowledge about the time elapsed between successive jumps. In this particular example, since  $\lambda < 1$ , we know that any maximal solution is Zeno, namely exists on a finite time horizon  $[0, T)$  with an infinite number of jumps, i.e., the time between two successive jumps  $t_{j+1} - t_j$  tends to 0 when  $j$  tends to  $+\infty$ . Let us fix a bounded set  $\mathcal{X}_0$  of initial conditions of interest, and let  $\tau_M \geq 0$  so that  $t_{j+1} - t_j \in [0, \tau_M]$  for all  $j \in \mathbb{N}$  along any solution initialized in  $\mathcal{X}_0$ . According to [33], exponential convergence is ensured by choosing  $L$  so that there exist  $a_c \in \mathbb{R}$  and  $a_d < 0$ , and a positive definite symmetric matrix  $P \in \mathbb{R}^{n \times n}$  such that

$$\begin{aligned} A_c^\top P + P A_c &\leq a_c P \\ (A_d - L_d H_d)^\top P (A_d - L_d H_d) &\leq e^{a_d} P \\ a_c \tau_M + a_d &< 0 \end{aligned}$$

Indeed, then the Lyapunov function  $V(x, \hat{x}) = (\hat{x} - x)^\top P (\hat{x} - x)$  decreases more at jumps than it increases during flow, so that  $V$  decreases overall. Following [33, Example 4.2], we choose  $L_d = (-1, -0.1487)$ . The corresponding observer trajectory is plotted in Figure 5(a)-5(b) (red), and the decrease of the Lyapunov function illustrated in Figure 5(c) (blue). Convergence is indeed achieved but we observe that the height estimate becomes strongly negative during the transient, which is not physically possible. In the following, we thus redesign the observer in order to ensure the estimate remains as close as possible to the set  $\mathcal{X} = [0, 10] \times [-10, 10]$ .

The set  $\mathcal{X}$  being a polytope, the function  $c$  of Lemma 1 is chosen as in (62) with  $n_c = 4$ , and  $a_1 = (1, 0)$ ,  $a_2 = (0, 1)$ ,  $a_3 = -a_1$ ,  $a_4 = -a_2$ ,  $b_1 = b_2 = b_4 = 10$ ,  $b_3 = 0$ , and  $\varrho = 1$ . Then,  $\mathcal{C}_0 = \mathcal{X}$ .

Following the hybrid redesign strategy (48)-(49), we obtain

$$\left. \begin{aligned} \dot{x} &= f(x) \\ \dot{\hat{x}} &= f(\hat{x}) + M(\hat{x}) \end{aligned} \right\} x \in C, \hat{x} \in \mathbb{R}^{n_x},$$

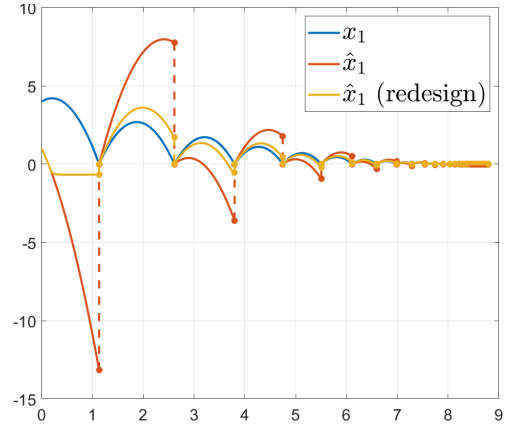
$$\left. \begin{aligned} x^+ &= g(x) \\ \hat{x}^+ &= \psi(g(\hat{x}) - L(\hat{x}_1 - y)) \end{aligned} \right\} x \in D, \hat{x} \in \mathbb{R}^{n_x},$$

(68)

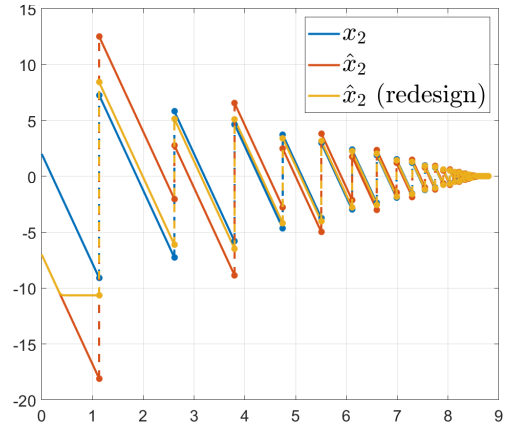
where  $\psi : \mathbb{R}^2 \rightarrow \mathcal{X}$  is the quadratic projection defined by

$$\psi(\hat{x}) = \operatorname{argmin}_{x \in \mathcal{X}} (\hat{x} - x)^\top P (\hat{x} - x)$$

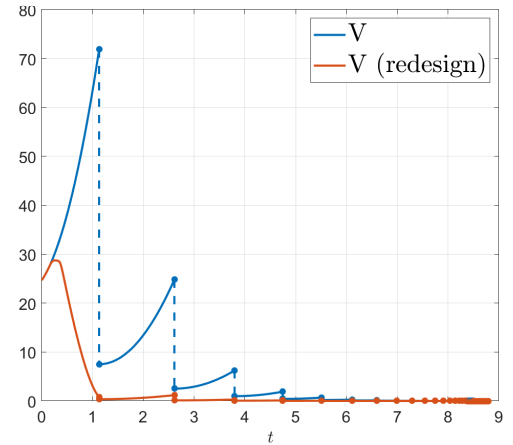
implemented with `quadprog` and  $P = \begin{pmatrix} 0.2885 & -0.0012 \\ -0.0012 & 0.2733 \end{pmatrix}$  and  $M$  is defined on  $\mathbb{R}^2$  by  $M(\hat{x}) = -\gamma_c P^{-1} \frac{dc}{dx}(\hat{x})^\top c(\hat{x})$  with  $\gamma_c = 10$ . Insights about the selection of  $P$  are given in [33] and more extensively in [61]. Here, we do not specify a particular choice of  $\hat{\mathcal{X}}$  because our goal is to keep  $\hat{x}$  close to  $\mathcal{X}$  and this is achieved by taking  $\gamma_c$  sufficiently large as explained in Section VI. The result of the simulation is provided on Figures 5(a)-5(b) (yellow). We can see that the observer actively uses the information that  $x \in \mathcal{X}$  during the first interval of flow to prevent  $\hat{x}$  from leaving. This practically acts like an extra measurement and enables the Lyapunov function to decrease during the transient (unlike in the original design) as illustrated on Figure 5(c) (red).



(a)  $x_1$ : plant ;  $\hat{x}_1$ : nominal observer ;  $\hat{x}_1$  (redesign): redesigned observer



(b)  $x_2$ : plant ;  $\hat{x}_2$ : nominal observer ;  $\hat{x}_2$  (redesign): redesigned observer



(c)  $V(x, \hat{x})$  with nominal observer (blue) and modified observer (red)

Fig. 5. Solutions of the nominal observer (67), and redesigned observer (68) with  $x_0 = (4, 2)$   $\hat{x}_0 = (1, -7)$ .



### VIII. CONCLUSION

For a given dynamical system with the state evolving in a known compact set, we supposed to know an observer with performances characterized by a quadratic Lyapunov function of the error, and we addressed the problem of constraining the observer state in a desired convex set, while preserving those performances. By exploiting the knowledge of a convex function which is zero on this desired convex set, we proposed a methodology which consists in redesigning the observer's dynamics outside such a set by means of a gradient approach, and therefore avoiding the use of on-line optimization techniques. In this work we proposed a unifying point of view covering (possibly nonlinear, time-varying) continuous time, discrete time and hybrid systems that allows to use these techniques for a large number of different observers. For this, we showed how to redesign a given observer, under the same convexity assumption, in a continuous-fashion way (flow-redesign) or discrete one (jump-redesign).

The proposed methodology may find a large number of applications such as removing peaking phenomenon, avoiding numerical issues [43], [53], improving the performances in presence of large measurement noise or output feedback stabilization [41], [42]. As shown in [41]–[43], it can be also used when the coordinates of the observer and the plant's dynamics are not the same, and such change of coordinates is not globally defined. As a future work, it would be useful to quantify and compare the computational loads of the various proposed techniques.

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### APPENDIX

#### A. Proof of Theorem 2

In light of (24), the number  $v_{\max}$  defined in (39) is the largest possible value of  $V_{k+1}$  after a jump with  $G(k, \cdot)$  from  $x_k \in \mathcal{X}$ ,  $\hat{x}_k \in \hat{\mathcal{X}}$ ,  $u_k \in \mathcal{U}$ ,  $d_k \in \mathcal{D}$  and for any  $k \in \mathbb{N}$ . Now, in light of the definition of  $V_k$  in (22), the sets  $\hat{\mathcal{X}}_{k+1, \max}$  defined in (40) are compact ellipsoids, and, by using (23) and the fact that  $v_{\max}$  is bounded, we conclude that  $\hat{\mathcal{X}}_{k+1, \max}$  are uniformly bounded. Therefore, there exists a compact set  $\hat{\mathcal{X}}_{\max}$  such that  $\bigcup_{k \in \mathbb{N}} \hat{\mathcal{X}}_{k+1, \max} \subseteq \hat{\mathcal{X}}_{\max}$ . With this  $\hat{\mathcal{X}}_{\max}$  so defined, let  $\delta_{\max}, \gamma_d^*$  be defined according to the proof of Lemma 5 as in (33) and (34). Finally, let  $v_{\min} > 0$  be defined as<sup>3</sup>

$$v_{\min} := \max \left\{ v \in \mathbb{R}_{\geq 0} : \forall x \in \mathcal{X}, \forall k \in \mathbb{N} \right. \\ \left. \left\{ \hat{x} \in \mathbb{R}^{n_x} : V_{k+1}(x, \hat{x}) \leq v \right\} \subseteq \mathcal{C}_\varrho \right\}.$$

with  $\mathcal{C}_\varrho$  defined in Lemma 1. In other words,  $v_{\min}$  is the largest positive number such that the following property holds for all  $k \in \mathbb{N}$

$$x \in \mathcal{X} \text{ and } V_{k+1}(x, \hat{x}) \leq v_{\min} \implies \hat{x} \in \mathcal{C}_\varrho \subset \hat{\mathcal{X}}. \quad (69)$$

<sup>3</sup> $v_{\min}$  is finite and positive thanks to (23) and  $\mathcal{X}$  compact.

Pick any strictly positive  $\gamma_d$  satisfying  $\gamma_d < \gamma_d^*$ , and with  $v_{\max}, v_{\min}$  defined above, let  $\theta \in \mathbb{N}$  be defined as

$$\theta := \text{ceil} \left( \frac{\max\{v_{\max} - v_{\min}, 0\}}{\gamma_d \varrho^2} \right), \quad (70)$$

with  $\varrho$  given by Lemma 1. With all the parameters so defined, the rest of the proof consists in showing that, by applying  $M_{k+1}$  at most  $\theta \in \mathbb{N}$  times, the value of  $V_{k+1}$  is brought below the threshold  $v_{\min}$ . This, in turn, implies that, in light of the previous definitions, the value  $G(k, \hat{x}_k, u_k, y_k) \in \hat{\mathcal{X}}_{k+1, \max}$  is brought back into  $\mathcal{C}_\varrho$  and therefore in  $\hat{\mathcal{X}} \supset \mathcal{C}_\varrho$  in view of item a) of Lemma 1.

To this end, let  $(k, \hat{x}) \in \mathbb{N} \times \hat{\mathcal{X}}_{k+1, \max}$ , and define  $\hat{x}_{k,1} = \hat{x}$  and  $q_{k,1} = 0$ . By definition of  $\hat{\mathcal{X}}_{k+1, \max}$  in (40), there exists  $x_k \in \mathcal{X}$  such that  $V_{k+1}(x_k, \hat{x}_{k,1}) \leq v_{\max}$ . First, let us show that for any sequence of points  $(\hat{x}_{k,i}, q_{k,i})_{i=1, \dots, \theta+1}$  such that

$$(\hat{x}_{k,i+1}, q_{k,i+1}) \in \tilde{\psi}_{k+1,i}(\hat{x}_{k,i}, q_{k,i}) \quad \forall i = 1 \dots \theta \quad (71)$$

we have  $x_{k,i+1} \in \hat{\mathcal{X}}_{k+1, \max}$  and

$$V_{k+1}(\hat{x}_{k,i+1}, x) \leq V_{k+1}(\hat{x}_{k,i}, x) \quad \forall x \in \mathcal{X}, \quad (72)$$

for all  $i = 1 \dots \theta$ . From (36), we have

$$\begin{pmatrix} \hat{x}_{k,2} \\ q_{k,2} \end{pmatrix} \in \begin{pmatrix} \hat{x}_{k,1} + M_{k+1}(\hat{x}_{k,1}) \\ 1 \end{pmatrix} \cup \begin{pmatrix} \hat{x}_{k,1} \\ 0 \end{pmatrix},$$

with the second option being only possible if  $\hat{x}_{k,1} \in \mathcal{C}_\varrho$ . By construction,  $\hat{\mathcal{X}}_{k+1, \max} \subseteq \hat{\mathcal{X}}_{\max}$ , so  $\hat{x}_{k,1} \in \hat{\mathcal{X}}_{\max}$ , and according to (32a) in Lemma 5, (72) holds in both cases for  $i = 1$ . Besides,  $V_{k+1}(x_k, \hat{x}_{k,1}) \leq v_{\max}$  so that  $V_{k+1}(x_k, \hat{x}_{k,2}) \leq v_{\max}$  and  $\hat{x}_{k,2} \in \hat{\mathcal{X}}_{k+1, \max}$ . Iterating and using the same arguments recursively, we obtain that  $\hat{x}_{k,i} \in \hat{\mathcal{X}}_{k+1, \max}$  for all  $i = 1, \dots, \theta + 1$  and that for all  $x \in \mathcal{X}$ ,  $V_{k+1}(x, \hat{x}_{k,j+1}) \leq V_{k+1}(x, \hat{x}_{k,j})$ ,  $j = 1, \dots, \theta$ , which gives  $V_{k+1}(x, \hat{x}_{k,\theta+1}) \leq V_{k+1}(x, \hat{x})$ . Observing that for any  $z \in \psi_{k+1}(\hat{x})$ , there exists a sequence  $(\hat{x}_{k,i}, q_{k,i})_{i=1, \dots, \theta+1}$  verifying (71) with  $q_{k,1} = 0$  and  $z = \hat{x}_{k,\theta+1}$ , we deduce that (28a) holds with  $\mathcal{S}_{k+1} = \hat{\mathcal{X}}_{k+1, \max}$ .

In order to prove (28b), recall that at the first step,  $q_{k,1}$  is initialized at 0, and after that, we compute  $(\hat{x}_{k,i+1}, q_{k,i+1})$  according to (36). From this definition, we can see that either  $q_{k,i} = 0$  for all  $i = 1 \dots \theta + 1$ , or there exists  $1 \leq \theta_k \leq \theta$  such that  $q_{k,i} = 0$  for all  $i = 1 \dots \theta_k$ , and  $q_{k,i} = 1$  for all  $i = \theta_k + 1 \dots \theta + 1$ . In the latter case, still from the definition of  $\tilde{\psi}_{k+1}$ , it means that  $\hat{x}_{k,\theta+1} = \hat{x}_{k,\theta_k} \in \mathcal{C}_\varrho$ , and by recalling item a) in Lemma 1, we obtain  $\hat{x}_{k,\theta+1} \in \hat{\mathcal{X}}$ . Now suppose that such  $\theta_k$  does not exist, namely,  $q_{k,i} = 0$  for all  $i = 1 \dots \theta + 1$ . From (36), this means that for all  $i = 1, \dots, \theta$ ,  $\hat{x}_{k,i} \in \mathbb{R}^{n_x} \setminus \text{int}(\mathcal{C}_\varrho)$ , and  $\hat{x}_{k,i+1} = \hat{x}_{k,i} + M_{k+1}(\hat{x}_{k,i})$ . As a consequence, we can use inequality (32b) for each  $i = 1, \dots, \theta$ , obtaining  $V_{k+1}(x_k, \hat{x}_{k,\theta+1}) \leq v_{\max} - \theta \gamma_d \varrho^2 \leq v_{\min}$ , by the definition of  $\theta$  in (31). Since  $x_k \in \mathcal{X}$ , using (69) and item a) of Lemma 1, gives  $\hat{x}_{k,\theta+1} \in \mathcal{C}_\varrho \subset \hat{\mathcal{X}}$ . This implies that  $\tilde{\psi}_{k+1}(\hat{\mathcal{X}}_{k+1, \max} \times \{0\}) \subseteq \hat{\mathcal{X}} \times \{0, 1\}$ , and therefore,  $\psi_{k+1}(\hat{\mathcal{X}}_{k+1, \max}) \subseteq \hat{\mathcal{X}}$  which is (28b). As for (28c), it holds by construction and Assumption 3.

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