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Theoretical study of the emergence of periodic solutions for the inhibitory NNLIF neuron model with synaptic delay.

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Abstract

Among other models aimed at understanding self-sustained oscillations in neural networks, the NNLIF model with synaptic delay was developed almost twenty years ago to model fast global oscillations in networks of weakly firing inhibitory neurons. Periodic solutions were numerically observed in this model, but despite its intensive study by researchers in PDEs and probability, there is up to now no analytical result on this topic. In this article, we propose to approximate formally this solution by a Gaussian wave whose periodic movement is described by an associate delay differential equation (DDE). We prove the existence of a periodic solution for this DDE and we give a rigorous asymptotic result on the solution when the connectivity parameter b goes to $-\infty$. Lastly, we provide heuristic and numerical evidence of the validity of our approximation.

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1 Introduction

Self-sustained oscillations in neural networks are key processes in the brain and several studies proved their ubiquity ([2, 37, 20, 6] among other reviews). These spontaneous (not elicited by external inputs), stable and periodic collective behaviours play a pivotal role in vital processes like respiratory rhythmogenesis [1]. In many cases, the activity arises from intrinsically oscillating neurons, but spontaneous periodic activity can occur in networks where individual noisy excitable neurons fire sporadically. This kind of collective behaviour is difficult to grasp without a self-contained mathematical model and many PDE models were studied numerically or analytically: the time-elapsed

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model [32, 33, 30], the kinetic FitzHugh-Nagumo model [29], conductance models [34, 35], . . .

One of the most striking cases of spontaneous periodic activity is fast global oscillations (gamma frequency range, see [7]) in networks of slowly firing inhibitory neurons. A milestone was reached on this topic when Brunel and Hakim proposed in [5] an approach to simulate those fast global oscillations: the Nonlinear Noisy Leaky Integrate & Fire (NNLIF[1]) neuron model.

They start from the classical Lapicque Integrate & Fire model [27]:

$$C_m \frac{dV}{dt} = -g_L(V - V_L) + I(t), \tag{1.1}$$

where C_m is the capacitance of the membrane, g_L the leak conductance and $V_L \simeq -70mV$ the leak potential. The synaptic current is a stochastic process of the form

$$I(t) = J_E \sum_{i=1}^{C_E} \sum_{j \in \mathbb{N}} \delta(t - t^i_{E_j}) - J_I \sum_{i=1}^{C_I} \sum_{j \in \mathbb{N}} \delta(t - t^i_{I_j}), \tag{1.2}$$

where δ is the Dirac measure centred at 0, $J_E > 0$ and $J_I > 0$ the strengths of excitatory and inhibitory synapses, $C_E \in \mathbb{N}$ and $C_I \in \mathbb{N}$ the numbers of excitatory and inhibitory pre-synaptic neurons and $t^i_{E_j}$, $t^i_{I_j}$ the random times of the j^{th} discharge from the i^{th} pre-synaptic excitatory or inhibitory neuron.

When a neuron reaches the discharge potential $V_F \simeq -50mV$, it emits an action potential and is reset to $V_R \simeq -60mV < V_F$. This model being hard to study in this form, many authors assume that discharges follow a Poisson law and do a diffusive approximation for a large number of neurons [5, 4]. Denoting $b = C_E J_E - C_I J_I$ and ${\sigma_C}^2 = (C_E {J_E}^2 + C_I {J_I}^2)$ and rescaling in order to have $C_m = g_L = 1$, it yields the stochastic differential equation

$$dV = (-V + V_L + b\nu)dt + \sigma_C dB_t, \qquad V < V_F, \tag{1.3}$$

where B_t is a standard Brownian motion, with the jump process and discharge intensity

$$\limsup_{t \to t_0^-} V(t) = V_F \implies \liminf_{t \to t_0^+} V(t) = V_R \qquad \text{and} \qquad \nu = \nu_{ext} + N(t-d).$$

The quantity $N(t)$ is the flux of neurons crossing the firing potential V_F at time t. The parameter $d \geq 0$ is the synaptic delay: the mean time it takes for a spike to go from one neuron to another in the network.

The so-called NNLIF model is associated to the probability density of (1.3) and it writes:

$$\frac{\partial p}{\partial t}(v,t) + \frac{\partial}{\partial v}[(-v + bN(t-d))p(v,t)] - a\frac{\partial^2 p}{\partial v^2}(v,t) = N(t)\delta(v - V_R), \qquad v \leq V_F, \tag{1.4}$$

with firing rate

$$N(t) = -a\frac{\partial p}{\partial v}(V_F, t) \geq 0, \tag{1.5}$$

and initial and boundary conditions

$$p(v,0) = p^0(v) \geq 0, \quad \int_{-\infty}^{V_F} p^0(v)dv = 1 \quad \text{and} \quad p(V_F,t) = p(-\infty,t) = 0. \tag{1.6}$$

[1] In some articles, the first N stands for Network.

The function $p(\cdot, t)$ represents the probability density of the electric potential of a randomly chosen neuron at time t. The parameter $a = \frac{\sigma_C^2}{2} > 0$ is the diffusion coefficient and b is the *connectivity parameter*. If b is positive, the neural network is said to be average-excitatory; if b is negative the network is said to be average-inhibitory.

Both problems (1.3) and (1.4) were intensively studied from a mathematical perspective. It was proved in [13] for $d = 0$ and in [10] for $d > 0$ that if the initial density p^0 is continuous on $(-\infty, V_F]$, C^1 on $(-\infty, V_R) \cup (V_R, V_F]$ and satisfies some decay and compatibility conditions, then there exists a unique strong solution $p(v, t)$ which is continuous on $(-\infty, V_F] \times [0, T^*)$ and $C^{2,1}$ on $((-\infty, V_R) \cup (V_R, V_F]) \times [0, T^*)$, where T^* is the maximal time of existence and satisfies

$$T^* = \sup\{\, t \in (0, +\infty] \mid N(t) < +\infty\}.$$

It was proved that when $b \leqslant 0$ [13] or when $d > 0$ [10] the solution is global-in-time: $T^* = +\infty$. However, it was proved in [8] that if $d = 0$ and $b > 0$, there exists initial conditions such that $T^* < +\infty$ and $\limsup_{t\to T^*} N(t) = +\infty$. If $d = 0$ and $b > 0$ is large enough, then all solutions blow-up in finite time [36]. The article [8] also characterised the stationary states of (1.4) and proved *via* an entropy method that all solutions converge towards a unique stationary state in the linear case $b = 0$. This convergence result was extended to small enough values of $|b|$ in [14] ($d = 0$) and [10] ($d > 0$).

Similar results were obtained for the stochastic counterpart (1.3): a first study of local-in-time solutions and a global solvability result when b is small enough were provided in [17]. The article [18] extended the notion of solutions, allowing continuation after a blow-up event. The article [28] studied further the link between strong solutions of (1.4) and (1.3) in the linear case. A lot of authors focused on the stochastic version of the model in the context of mathematical finance (see *e.g.* [24, 23, 31]) and some studies were devoted to a modified version of the model with a random discharge mechanism instead of the firing potential V_F [15, 16].

The emergence of periodic solutions in NNLIF-type models is a crucial question and, although it was numerically investigated there are up to our knowledge very few theoretical results on this topic. Indeed, these complex dynamics can solely occur when the strength $|b|$ of the nonlinearity is sufficient. Such a strong nonlinearity is hard to tackle mathematically. Within the scope of this article, we propose to bring new theoretical insights in the case of very inhibitory networks, *i.e.* $b \ll 0$. Before getting into the details of our methods and results, let us sum up what is already known on these periodic solutions:

- in the classical NNLIF system without delay ($d = 0$), periodic solutions have never been observed. Adding a refractory period [9] or coupling excitatory-inhibitory systems [11, 12] does not suffice to make them appear;
- in the high connectivity regime ($b > 0$ large), it is proved in [10, Th. 5.4] that there is no periodic solution neither with delay ($d > 0$) nor without delay ($d = 0$);
- in the excitatory case $b > 0$, periodic solutions were not observed when there is a delay $d > 0$, but if there is a delay and a refractory period they appear [12, 26]. They were also observed when a random discharge mechanism (without delay) is taken into account [9, 15, 16]. Periodic solutions for the random discharge model (without delay) were recently constructed analytically in [16];
- in the inhibitory case $b < 0$ with delay $d > 0$, periodic solutions are observed as soon as $|b|, d$ are large enough (regardless of the presence of a refractory period), [5, 12, 26].

In this article, we put ourselves in the context of a very inhibitory network: $b \ll 0$ with a positive delay $d > 0$. Our strategy is to write the solution as a sum of a periodic wave $\varphi(v,t)$ plus a remainder term $R(v,t)$:

$$p(v,t) = \varphi(v,t) + R(v,t), \qquad \varphi(v,t) = \phi\big(v - c(t)\big) \tag{1.7}$$

where the remainder term $R(v,t)$ is expected to vanish in some sense when $b \to -\infty$. This problem is strongly coupled since the periodic movement $c(t)$ depends upon the firing rate

$$N(t) = -a\frac{\partial p}{\partial v}(V_F,t) = -a\frac{\partial \varphi}{\partial v}(V_F,t) - a\frac{\partial R}{\partial v}(V_F,t).$$

In order to make the problem tractable, we do a second approximation, which is more involved: we assume that in the limit $b \to -\infty$ we have $bN(t) \simeq -ba\partial_v\varphi(V_F,t)$. It means we have a new firing rate that depends only upon the periodic wave $\varphi(v,t) = \phi(v - c(t))$. As we will see in more details in Proposition 2.1, the new firing rate $N(t) \simeq \mathcal{N}(c(t))$ then depends only on $c(t)$ and the function c satisfies the autonomous Delay Differential Equation (DDE):

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \tag{1.8}$$

where the new firing rate is expressed as

$$\mathcal{N}(c) = \frac{1}{\sqrt{2\pi a}}\big(V_F - c\big)e^{-\frac{(V_F-c)^2}{2a}}. \tag{1.9}$$

The properties of DDEs have been intensively studied and there are several monographs reviewing results about this topic, for example [22] which is about functional differential equations in general and [38] which focuses more on the applications of DDEs in the Life Sciences. Numerous analytical and numerical examples in the literature show that large enough delays naturally induce oscillations in DDEs modelling biological phenomena.

Implementing our strategy for the study of the PDE (1.4) requires to answer two separate questions.

- The first one, which is the hardest and which we leave open, is to justify rigorously these approximations in the limit $b \to -\infty$. Concerning the first approximation (1.7), we provide a partial answer to this question by proving that if we assume $N(t) \to 0$, then the remainder R goes to 0 in some sense. Our second approximation (1.8) is much more involved: since $bN(t) = -ba\partial_v\varphi(V_F,t) - ba\partial_v R(V_F,t)$, proving our method to be valid requires to prove that both R and $b\partial_v R(V_F,t)$ converge to 0 when $b \to -\infty$ (*i.e.* $R(v,t) = o(1)$ and $\partial_v R(V_F,t) = o(\frac{1}{b})$). This difficult question is beyond the scope of our paper.
- The second one is to study theoretically the DDE (1.8). On the one hand, we prove that there exists $(b^*,d^*) \in \mathbb{R}_-^* \times \mathbb{R}_+^*$ such that if $b < b^*, d > d^*$ there exists a periodic solution $c(t)$ (see Theorem 3.1). On the other hand, we construct in appropriate rescaled variables an explicit asymptotic profile $P(t)$ towards which the solutions converge when $b \to -\infty$ (see Theorem 4.1) This profile is periodic and gives us a precise idea of the behaviour of the periodic solutions: they are projected very fast away from V_F (at a distance $\mathcal{O}((-b)^{C_M})$, for some $C_M \in (0,1)$), then they come back at an exponential speed over a period of time in $\mathcal{O}(\log(-b))$, and so on back and forth.

In order to prove that there are periodic solutions to equation (1.8), we follow the method proposed in [21], which consists in applying Browder's fixed-point theorem on an *ad hoc* functional. Note that, as Brunel and Hakim guessed in their original article [5], the apparition of fast global oscillations in an inhibitory network is likely to be a

Hopf bifurcation. However, [5] suggests a bifurcation for some parameter value b and our result indicates a bifurcation in d when b is negative and large enough. Note that the periodic solutions of the random discharge NNLIF model constructed in [16] appear through a Hopf bifurcation in $b > 0$ (and not in d, which does not appear in their version of the model).

To carry out our study of the asymptotic profile P when the connectivity b goes to $-\infty$, we make a handy rescaling and then decompose the evolution of the solutions in different phases (growth phase, decay phase, etc.). Then, we use b-dependent estimates to prove convergence towards an explicit profile. Our numerical simulations indicate that this asymptotic profile is a good approximation.

This article is organised as follows. In Section 2 we derive (heuristically) the DDE (1.8) and we justify its relevance in a heuristic way. In Section 3, we use the method of [21] to prove the existence of periodic solutions for equation (1.8). In Section 4, we prove a result on the asymptotic profile of solutions of (1.8) when the parameter b goes to $-\infty$. Last, in Section 5 we propose a numerical study of the PDE, the DDE and the links between them.

2 An associate delay differential equation

2.1 Formal derivation of the wave-type solution

Note first that we can rewrite (1.4) on the whole real line as follows:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial v}\Big[(-v + bN(t-d))p\Big] - a\frac{\partial^2 p}{\partial v^2} = \delta_{V_R} N(t) - \delta_{V_F} N(t), \qquad v \in \mathbb{R},\ t > 0,$$

$$p(-\infty, t) = p(+\infty, t) = 0, \qquad N(t) = -a\frac{\partial p}{\partial v}(V_F, t), \qquad t > 0,$$

$$p(v, 0) = p^0(v) \geq 0, \quad v \in \mathbb{R}, \quad \int_{-\infty}^{+\infty} p^0(v)dv = 1.$$

If $p^0 \equiv 0$ on $[V_F, +\infty)$ then for all $t > 0$, $p(\cdot, t) \equiv 0$ on $[V_F, +\infty)$.

As Lemma 2.2 below and numerical simulations in the literature indicate (see *e.g.* [5] and Section 5 of this article), when $b \ll 0$ the firing rate N tends to be low. As a consequence, the term $(\delta_{V_R} - \delta_{V_F})N(t)$ is of lesser importance in the equation. Hence, we are looking for a solution composed of a periodic wave of unit mass $\varphi(v, t) = \phi\big(v - c(t)\big)$ defined on $\mathbb{R}$ plus a corrective term $R(v, t)$ needed to account for the boundary and reset conditions:

$$p(v, t) = \phi\big(v - c(t)\big) + R(v, t), \tag{2.1}$$

and it is sound to look for the wave-type periodic solution $\varphi : (v, t) \mapsto \phi\big(v - c(t)\big)$ as a solution of equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \dfrac{\partial}{\partial v}\Big[(-v + bN(t-d))\varphi\Big] - a\dfrac{\partial^2 \varphi}{\partial v^2} = 0, & v \in \mathbb{R}, t > 0, \\ \varphi(-\infty, t) = \varphi(+\infty, t) = 0, & \int_{-\infty}^{+\infty} \varphi(v, t)dv = 1. \end{cases} \tag{2.2}$$

Therefore, the remainder term $R : (v, t) \mapsto R(v, t)$ must be a solution of

$$\begin{cases} \dfrac{\partial R}{\partial t} + \dfrac{\partial}{\partial v}\Big[(-v + bN(t-d))R\Big] - a\dfrac{\partial^2 R}{\partial v^2} = \delta_{V_R} N(t) - \delta_{V_F} N(t), & v \in \mathbb{R},\ t > 0, \\ R(-\infty, t) = R(+\infty, t) = 0, \quad R(v, 0) = p^0(v) - \varphi(v, 0), & v \in \mathbb{R},\ t > 0. \end{cases} \tag{2.3}$$

Unfortunately, systems (2.2) and (2.3) are strongly coupled through the firing rate associated to $p(v, t)$:

$$N(t) = -a\partial_v \varphi(V_F, t) - a\partial_v R(V_F, t).$$

We have the following result about solutions of (2.2):

Proposition 2.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$\phi(v) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{v^2}{2a}}.$$

Let c be a solution of

$$c'(t) + c(t) = bN(t-d). \tag{2.4}$$

Then the function φ defined by $\varphi(v,t) = \phi(v - c(t))$ is a solution of (2.2).

Proof. Replacing $\varphi(v,t) = \phi(v-c(t))$ in (2.2), we get for all $v \in \mathbb{R}$,

$$-\phi(v-c(t)) - \big(v + c'(t) - bN(t-d)\big)\phi'(v-c(t)) - a\phi''(v-c(t)) = 0.$$

By change of variable $v \mapsto v + c(t)$, it yields

$$-\phi(v) - \big(v + c'(t) + c(t) - bN(t-d)\big)\phi'(v) - a\phi''(v) = 0.$$

Assume that the function c satisfies $c'(t)+c(t) = bN(t-d)$. Then, $-\phi(v)-v\phi'(v)-a\phi''(v) = 0$, that is to say $-(v\phi(v))' = a\phi''(v)$. We integrate and apply boundary conditions:

$$\phi'(v) = -\frac{v}{a}\phi(v).$$

The positive solutions of this equation are of the form $\phi(v) = C_g e^{-\frac{v^2}{2a}}$, where $C_g > 0$ is any given positive constant. Since

$$\int_{-\infty}^{+\infty} C_g e^{-\frac{v^2}{2a}} dv = \int_{-\infty}^{+\infty} \varphi(v,t) dv = 1,$$

we have $C_g = (2\pi a)^{-\frac{1}{2}}$. □

In order to make the problem autonomous and thus theoretically tractable, we make a last and more involved assumption: that $b\partial_v R(V_F, t) \simeq 0$ in an appropriate sense when $b \to -\infty$. Hence, we replace $N(t)$ by the simpler firing rate

$$\mathcal{N}(c(t)) = -a\frac{\partial \varphi}{\partial v}(V_F, t) \tag{2.5}$$

in equation (2.4) and we obtain the autonomous DDE (1.8)–(1.9).

2.2 Partial results on the remainder term R

We will give here some partial results on the remainder term $R(v,t)$ in the first approximation (2.1). Note first that as the following lemma indicates, the stationary firing rate vanishes when b tends to $-\infty$.

Lemma 2.2. *For all $b \leqslant 0$, denote N_∞^b the firing rate of the unique stationary state of (1.4) with connectivity parameter b. Then, the function $b \mapsto N_\infty^b$ is increasing on $(-\infty, 0]$ and*

$$\lim_{b \to -\infty} N_\infty^b = 0.$$

Proof. We know from [8] that when $b \leqslant 0$ there exists a unique stationary state of (1.4) and that it satisfies

$$\frac{1}{N_\infty^b} = \int_0^{+\infty} \frac{e^{-\frac{s^2}{2}}}{s} e^{-\frac{bN_\infty^b}{\sqrt{a}}s}\left(e^{s\frac{V_F}{\sqrt{a}}} - e^{s\frac{V_R}{\sqrt{a}}}\right) ds.$$

Let us define two functions:

$$I:(b,N)\mapsto \int_0^{+\infty} \frac{e^{-\frac{s^2}{2}}}{s} e^{-\frac{bN}{\sqrt{a}}s}\left(e^{s\frac{V_F}{\sqrt{a}}}-e^{s\frac{V_R}{\sqrt{a}}}\right)ds \qquad \text{and} \qquad J:(b,N)\mapsto I(b,N)-\frac{1}{N}.$$

For all $b<0$, equation $J(b,N)=0$ has a unique solution [8]. The function J is smooth on $(-\infty,0)\times(0,+\infty)$ and for all $(b,N)\in(-\infty,0)\times(0,+\infty)$,

$$\frac{\partial J}{\partial N}(b,N)=-\frac{b}{\sqrt{a}}I(b,N)+\frac{1}{N^2}>0 \qquad \text{and} \qquad \frac{\partial J}{\partial b}(b,N)=-\frac{N}{\sqrt{a}}I(b,N)<0.$$

Hence, by the implicit functions theorem applied to equation $J(b,N)=0$, the function $b\mapsto N_\infty^b$ has the following derivative:

$$b\mapsto -\frac{\dfrac{\partial J}{\partial b}(b,N)}{\dfrac{\partial J}{\partial N}(b,N)}>0.$$

The function $b\mapsto N_\infty^b$ is thus increasing on $(-\infty,0)$. Then, note that for all $N>0$, the function $b\mapsto I(b,N)$ is non-increasing and

$$\lim_{b\to-\infty} I_N(b)=+\infty.$$

Function $b\mapsto N_\infty^b$ being continuous, bounded by 0 and increasing, it has a limit $N_\infty^*\geq 0$ when b goes to $-\infty$. If we assume $N_\infty^*>0$, then

$$0=\lim_{b\to-\infty} J(b,N_\infty^*)=\lim_{b\to-\infty} I(b,N_\infty^*)-\frac{1}{N_\infty^*}=+\infty,$$

and we reached a contradiction. Thus, we must have $\lim_{b\to-\infty} N_\infty^b=0$. □

We now make the following guess:

Conjecture 2.3. *Let (p^0,N^0) be an initial condition. For all $\varepsilon>0$, there exists some time $T>0$ and some connectivity parameter $b<0$ such that any solution (p,N) of (1.4) with parameter b satisfies*

$$\forall t>T, \quad \int_0^t e^{s-t}N(s)\mathrm{d}s<\varepsilon.$$

With the previous lemma, this conjecture means that as b goes to $-\infty$, N converges towards 0 in some sense. If we assume that this conjecture holds, then the remainder term R is asymptotically small when b goes to $-\infty$. More precisely, this conjecture implies:

Proposition 2.4. *Assume that Conjecture 2.3 holds. Then, for all $\varepsilon>0$, there exists $b<0$ and $T>0$ such that for all $t>T$,*

$$\int_0^t e^{s-t}\|R(s)\|_{L^2}^2\mathrm{d}s<\varepsilon.$$

Proof. First, since the mass of p is conserved, we have

$$\int_{-\infty}^{+\infty} R(v,t)\mathrm{d}v=\int_{-\infty}^{+\infty} p(v,t)\mathrm{d}v-\int_{-\infty}^{+\infty}\varphi(v,t)\mathrm{d}v=1-1=0.$$

We know that R is solution of

$$\partial_t R+\partial_v((-v+bN(t-d))R)-a\partial_{vv}R=\delta_{v=V_R}N(t)-\delta_{v=V_F}N(t). \tag{2.6}$$

Define
$$U(v,t)=\int_{-\infty}^{v} R(w,t)\mathrm{d}w.$$
We have $U(\pm\infty,t)=0$ and the equation satisfied by U is given by
$$\partial_t U+(-v+bN(t-d))\partial_v U-a\partial_{vv}U=\mathbb{1}_{(V_R,V_F)}N(t).$$
Multiplying the above equation by U and integrating, we find that
$$\frac{d}{dt}\|U\|_{L^2}^2\leq -\|U\|_{L^2}^2-2a\|R\|_{L^2}^2+2N(t)\,\|U\|_{L^1(V_R,V_F)}\,.$$
We have
$$\begin{aligned}\|U\|_{L^1(V_R,V_F)}=\int_{V_R}^{V_F}\left|\int_{-\infty}^{v}R(w,t)dw\right|\mathrm{d}v\leqslant\int_{V_R}^{V_F}\int_{-\infty}^{v}|R(w,t)|\,\mathrm{d}w\mathrm{d}v\\ \leqslant\int_{V_R}^{V_F}\int_{-\infty}^{+\infty}\big(p(w,t)+\varphi(w,t)\big)\mathrm{d}w\mathrm{d}v\leqslant 2(V_F-V_R).\end{aligned}\tag{2.7}$$
This implies that
$$\frac{d}{dt}\|U\|_{L^2}^2\leq -\|U\|_{L^2}^2-2a\|R\|_{L^2}^2+4(V_F-V_R)N(t).$$
Hence, we obtain that
$$2ae^{-t}\int_0^t e^s\|R(s)\|_{L^2}^2\mathrm{d}s\leq\|U(0)\|_{L^2}^2e^{-t}+4(V_F-V_R)e^{-t}\int_0^t e^sN(s)\mathrm{d}s-\|U(t)\|_{L^2}^2.$$
Applying Conjecture 2.3, we choose T and $|b|$ large enough in order to have
$$\forall t>T\quad\|U(0)\|_{L^2}^2e^{-t}\leqslant a\varepsilon\qquad\text{and}\qquad 4(V_F-V_R)e^{-t}\int_0^t e^sN(s)\mathrm{d}s\leqslant a\varepsilon,$$
and thus,
$$\int_0^t e^{s-t}\|R(s)\|_{L^2}^2\mathrm{d}s<\varepsilon.$$
□

2.3 Evolution of the mean and variance of the PDE solution

We can derive evolution equations for the moments of the solution $p(v,t)$ of (1.4). Indeed, for every test function $\phi\in C^2((-\infty,V_F])$, assuming enough decay of $p(v,t)$ at $-\infty$, we have
$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F}p(v,t)\phi(v)\mathrm{d}v=\int_{-\infty}^{V_F}\Big[\big(-v+bN(t-d)\big)\phi'(v)+a\phi''(v)\Big]p(v,t)\mathrm{d}v\\ +N(t)\Big(\phi(V_R)\;-\;\phi(V_F)\Big).\end{aligned}\tag{2.8}$$
Hence, if we denote
$$m_1(t)=\int_{-\infty}^{V_F}vp(v,t)\mathrm{d}v,\qquad\text{and}\qquad m_2(t)=\int_{-\infty}^{V_F}v^2p(v,t)\mathrm{d}v\tag{2.9}$$
the first (mean) and second moment of the solution respectively, we have the following equations:
$$m_1'(t)+m_1(t)=bN(t-d)+(V_R-V_F)N(t)\tag{2.10}$$

and

$$m_2'(t) + 2m_2(t) = 2a + 2bN(t-d)m_1(t) + N(t)(V_R - V_F)(V_R + V_F). \tag{2.11}$$

Then, the variance $\mathrm{Var}(t) = m_2(t) - m_1(t)^2$ of $p(v,t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(t) + 2\,\mathrm{Var}(t) = 2a + (V_F - V_R)\big(2m_1(t) - V_R - V_F\big)N(t). \tag{2.12}$$

If we assume Conjecture 2.3 again, then we can prove that when $b \to -\infty$, the first moment $m_1(t)$ converges locally uniformly towards the solution $c(t)$ of the ODE (2.4).

Proposition 2.5. *Assume Conjecture 2.3. Let $p(v,t)$ be a classical fast-decaying solution of (1.4). Let $m_1(t)$ and $c(t)$ be the solutions of (2.10) and (2.4) respectively. Then, for all $\varepsilon > 0$, there exist $b < 0$ and $T > 0$ such that*

$$\forall t > T, \quad |c(t) - m_1(t)| < \varepsilon.$$

Proof. Denote $w_1(t) = c(t) - m_1(t)$. Then w_1 satisfies the equation

$$w_1'(t) + w_1(t) = (V_F - V_R)N(t).$$

Hence, we have

$$w_1(t) = w_1(0)e^{-t} + (V_F - V_R)\int_0^t e^{s-t}N(s)\mathrm{d}s,$$

and we get the result by applying Conjecture 2.3. □

Remark 2.6. Unfortunately, assuming Conjecture 2.3 is not enough to prove straightforwardly that $|\,\mathrm{Var}(t) - a| \to 0$, because we need to control the nonlinear term $(V_F - V_R)\big(2m_1(t) - V_R - V_F\big)N(t)$ uniformly in b. It should be possible with some work to control the product $m_1(t)N(t)$ with Conjecture 2.3, because on the one hand when $m_1(t)$ is large the solution is far from V_F and $N(t)$ is exponentially small, and on the other hand when $N(t)$ is large it means that the solution is concentrated near V_F and $m_1(t)$ will be small, or at least bounded.

3 Periodic solutions of the delay differential equation

Here we prove the existence of a periodic solution of

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \tag{3.1}$$

where the function $\mathcal{N}(c)$ is given by

$$\mathcal{N}(c) = (V_F - c)e^{-\frac{(V_F - c)^2}{2a}},$$

and $b < 0, d \geq 0, V_F \geq 0, a > 0$ are parameters. We remove in this section the constant $C_g = (2\pi a)^{-\frac{1}{2}}$ without loss of generality since the rescaling $\bar{b} = b(2\pi a)^{-\frac{1}{2}}$ allows us to come back to the general case.

Theorem 3.1. *Assume ${V_F}^2 \geq a$, then there exist $b_* < 0$ and $d_* > 0$ such that (3.1) has a non-constant periodic solution $c(t)$ for any $b < b_*$ and $d > d_*$. Moreover, given $b < 0$ there is a negative constant c_0 such that for all $d > d^*$ and for all $t > 0$, $c_0 \leq c(t) < 0$.*

We build upon the method of Hadeler and Tomiuk for a similar DDE [21], although their results are not applicable for our case directly. The main idea is to find the periodic solution as a *non-ejective fixed-point* of a functional $\mathcal{F}$ on an appropriate function space and to prove that uninteresting fixed-point (the stationary states) are *ejective*. More precisely, we recall the following definition and result.

Definition 3.2 (Ejective fixed-point, [3])**.** *Let E a Banach space and D a closed subset of E. Let $\mathcal{F} : D \to D$ a continuous map. A fixed-point $\bar{x}$ of $\mathcal{F}$ is said to be* ejective *if there exists a neighbourhood $U \subset D$ of $\bar{x}$ such that*

$$\forall x \in U \setminus \{\bar{x}\},\ \exists n \in \mathbb{N},\ \mathcal{F}^n(x) \notin U.$$

Theorem 3.3 (Browder's fixed-point theorem, [3])**.** *Let D be a closed, bounded and convex subset of an infinite dimensional Banach space and let $\mathcal{F} : D \to D$ a continuous and compact map. Then $\mathcal{F}$ admits at least one fixed-point which is not* ejective*.*

Let us describe the strategy of the proof of Theorem 3.1. We go through three different forms of the problem: original equation (3.1), equivalent equation (3.4) and the slightly modified equation (3.11).

- First we study the stationary solutions c_* of (3.1).
- We change variables and recast (3.1) into (3.4). Then we study the corresponding characteristic equation:

$$\lambda + d = bd\mathcal{N}'(c_*)e^{-\lambda}.$$

  We find that $b\mathcal{N}'(c_*) < -1$ is a necessary condition to have *unstable eigenvalues*.
- We define a functional $\mathcal{F}$ whose solutions are either stationary or periodic solutions of problem (3.11) which is very close to our renormalised problem (3.4).
- We prove that stationary solutions of (3.11) are *ejective*; hence, applying Browder's fixed-point theorem, $\mathcal{F}$ possess another fixed-point which happens to be a periodic solution of (3.11). Last, we prove that this solution yields a non-trivial periodic solution of (3.4) and equivalently of (3.1).

3.1 Stationary problem

Let us first state the existence of a stationary solution. In the following we consider two cases

$$V_F > 0 \quad (\text{Case } 1) \qquad \text{and} \qquad V_F = 0,\ b < -1 \quad (\text{Case } 2).$$

Since (3.1) describes the solution from the PDE system (1.4) on $(-\infty, V_F]$, the solution of (3.1) must be contained in $(-\infty, V_F)$. Thus, we are only looking for solutions $c(t) < V_F$.

Theorem 3.4. *There is a unique stationary solution $c_* \in (-\infty, \min(0, V_F))$ of (3.1) in Cases 1 and 2. On the other hand, there is no stationary solution of (3.1) in $(-\infty, 0)$ in the case of $V_F = 0$ and $-1 \leqslant b < 0$.*

Proof. Define $F(c, b) = c - b\mathcal{N}(c)$, which is a smooth function of (c, b) in $c < V_F$ and $b < 0$. We have $\lim_{c\to-\infty} F(c, b) = -\infty$. Moreover, $F(0, b) = -b\mathcal{N}(0) > 0$ in Case 1 while $F(0, b) = 0$ and $(\partial F/\partial c)(0, b) = 1 + b < 0$ in Case 2. Hence there exists a zero of $F(c, b)$ – a stationary solution – denoted by $c = c_* < 0$. On the other hand, let us consider the case $V_F = 0$ and $-1 \leq b \leq 0$ and assume that there is a stationary solution $c < 0$ of (3.1). Then we have a contradiction because

$$1 = -b\exp\left(-\frac{c^2}{2a}\right) < -b \leq 1.$$

Next, uniqueness of stationary solutions c^* of (3.1) follows from the following computation:

$$\frac{\partial F}{\partial c}(c_*, b) = 1 - b\mathcal{N}'(c_*) = \frac{aV_F - c_*(V_F - c_*)^2}{a(V_F - c_*)} > 0, \tag{3.2}$$

where we calculate $\mathcal{N}'(c_*)$ as

$$\mathcal{N}'(c_*) = \frac{(V_F - c_*)^2 - a}{a}\exp\left(-\frac{(V_F - c_*)^2}{2a}\right) = \frac{c_*}{b}\left(\frac{V_F - c_*}{a} - \frac{1}{V_F - c_*}\right). \tag{3.3}$$

$\square$

According to the implicit function theorem and due to (3.2), $b \mapsto c_*(b)$ is a smooth function. Note that $c_*(b)$ can be defined for $b < 0$ in Case 1, and $b < -1$ in Case 2, respectively. Next, we study some properties of $c_*(b)$.

Lemma 3.5. *The function $b \mapsto c_*(b)$ satisfies $c_*'(b) > 0$, $\lim_{b\to-\infty} c_*(b) = -\infty$ and $\lim_{b\to-\infty} c_*(b)/b = 0$. Moreover, there exists $b_* < 0$ such that $b\mathcal{N}'(c_*(b)) < -1$ for any $b < b_*$.*

Proof. In the proof we only consider Case 1 because we can prove the statement for Case 2 by the same argument. It is obvious that $\lim_{b\to 0} c_*(b) = 0$. Then, assume that there is a subsequence $(b_i)_{i\in\mathbb{N}}$ and $c_\infty \in (-\infty, 0)$ such that $b_i \to -\infty$ and $c_*(b_i) \to c_\infty$ as $i \to \infty$. We have $\mathcal{N}(c_\infty) = 0$, and then $c_\infty = V_F > 0$, which is a contradiction. Hence, $\lim_{b\to-\infty} c_*(b) = -\infty$. This fact and (3.3) imply $c_*(b)/b = \mathcal{N}(c_*(b)) \to 0$ and $b\mathcal{N}'(c_*(b)) \to -\infty$ as $b \to -\infty$.

Let F be the same function as in the previous proof. Since $(\partial F/\partial b)(c, b) = -\mathcal{N}(c) < 0$ in $c < V_F$ and $b < 0$, we differentiate both sides of $F(c_*(b), b) = 0$ with respect to b and obtain

$$c_*'(b) = -\frac{\dfrac{\partial F}{\partial b}(c_*(b), b)}{\dfrac{\partial F}{\partial c}(c_*(b), b)} > 0$$

because of (3.2). □

We now set

$$c(t) = c_* + x\left(\frac{t}{d}\right)$$

and change $\frac{t}{d}$ into t. It yields the following DDE for x:

$$x'(t) + dx(t) = bd(\mathcal{N}(c_* + x(t-1)) - \mathcal{N}(c_*)). \tag{3.4}$$

3.2 Study of the characteristic equation

The linearised problem of (3.4) around $x = 0$ has the characteristic equation

$$\lambda + d = bd\mathcal{N}'(c_*)e^{-\lambda}, \qquad \lambda \in \mathbb{C}. \tag{3.5}$$

Equations like (3.5) are classical in the theory of delay differential equations and were extensively studied; see for example Appendix of [22] or Section 4.2 of [38]. By setting $z = \lambda + d$ we can recast (3.5) into the form

$$ze^z = bdN'(c_*)e^d. \tag{3.6}$$

The solutions of Equation (3.6) can be written as the branches $W_k(bdN'(c_*)e^d)$, k integer, of the Lambert function. So we know that there exists a countable set of solutions to Equation (3.5) and by Lemma 4.2 of [38], there are either 0 or finitely many unstable eigenvalues.

We denote "Rez" and "Imz" the real and imaginary parts of a complex number z. The solution λ of (3.5) is called an *unstable* eigenvalue if Re$\lambda > 0$. If λ is an eigenvalue, then so is $\overline{\lambda}$, where $\overline{\lambda}$ is the complex conjugate of λ. Hence we can assume Im$\lambda \geq 0$ without loss of generality.

Note first that by (3.2), $1 - b\mathcal{N}'(c_*) > 0$ and thus there are no real non-negative eigenvalues. Then if we denote $\lambda = \mu + i\gamma$, and if we assume that $\mu, \gamma > 0$, then

$$\mu + d = bd\mathcal{N}'(c_*)e^{-\mu}\cos\gamma, \quad \gamma = -bd\mathcal{N}'(c_*)e^{-\mu}\sin\gamma. \tag{3.7}$$

Hence,

$$(bd\mathcal{N}'(c_*)e^{-\mu})^2 = (\mu + d)^2 + \gamma^2 \geq d^2.$$

Since $\mu > 0$, $(b\mathcal{N}'(c_*))^2 \geq e^{2\mu} \geq 1$. From (3.2), we have $b\mathcal{N}'(c_*) \leqslant -1$. This also implies that if $\gamma\ [2\pi]$ is the remainder of the division of γ by 2π, then $\gamma\ [2\pi] \in (\pi/2, \pi)$ because both right-hand sides in (3.7) must be positive.

Putting $\mu = 0$ in the first equation of (3.7), we have

$$1 = b\mathcal{N}'(c_*) \cos\gamma. \tag{3.8}$$

Equation (3.8) has denumerably infinite many solutions γ because of $b\mathcal{N}'(c_*) < -1$. We denote the minimal positive solution of (3.8) by γ_1, where γ_1 can be estimated as $\pi/2 < \gamma_1 < \pi$. Moreover we readily see that $\gamma_k = \gamma_1 + 2\pi(k-1)$ $(k = 2, 3, \ldots)$ are also solutions of (3.8). Then we define d_k by

$$d_k \equiv -\frac{\gamma_k}{b\mathcal{N}'(c_*) \sin\gamma_k}. \tag{3.9}$$

With the previous discussion and some additional arguments from the proof of Lemma 3 in [21], it is possible to prove the following result.

Lemma 3.6. *Suppose that $b\mathcal{N}'(c_*) < -1$. Let $k \geq 1$ be an integer. If $d_k < d < d_{k+1}$, then there are exactly $2k$ unstable eigenvalues λ_i and $\overline{\lambda_i}$ $(i = 1, \ldots, k)$ of (3.5). On the other hand, if $d = d_k$, (3.5) has exactly $2k-2$ unstable eigenvalues λ_i and $\overline{\lambda_i}$ $(i = 1, \ldots, k-1)$ and eigenvalues λ_k and $\overline{\lambda_k}$ with Re$\lambda_k = 0$. Moreover, λ_i satisfies $\pi/2 <$ Im$\lambda_i - 2\pi(i-1) < \pi$.*

Hence, we have the following corollary.

Corollary 3.7. *Assume $b\mathcal{N}'(c_*) < -1$, and let $d^* = d_1$, where d_1 is defined in (3.9). Then, (3.5) has an unstable eigenvalue λ with $\pi/2 < \mathrm{Im}\lambda < \pi$ if and only if $d^* < d$.*

3.3 Properties of the non-linear term

Here we summarise the properties of the nonlinear term $f_0(x) \equiv -b(\mathcal{N}(c_*+x) - \mathcal{N}(c_*))$, which appears in the right-hand side of (3.4). It is easy to see that the following lemma holds true so that we omit the details of the proof.

Lemma 3.8. *Suppose that $b\mathcal{N}'(c_*) \leq -1$. Then the smooth function $f_0(x)$ satisfies the following properties;*

(i) $f_0(0) = 0$.

(ii) $\lim_{x\to-\infty} f_0(x) = b\mathcal{N}(c_*) = c_* < 0$.

(iii) *$f_0'(x) > 0$ in $x \leq 0$ while there are exactly two zeros of $f_0'(x) = 0$ in $x > 0$, denoted by $x_1 < x_2$. In particular, $f_0'(0) = -b\mathcal{N}'(c_*) \geq 1$.*

(iv) *$f_0(x)$ is a Lipschitz function on $\mathbb{R}$, that is, there is $L_0 > 0$ such that $|f_0(x) - f_0(y)| \leq L_0|x - y|$ for any $x, y \in \mathbb{R}$.*

We can give x_1 explicitly such as $x_1 = V_F - c_* - \sqrt{a}$ by a direct calculation and see $x_1 > 0$ because of (3.3) and the assumption. Lemma 3.8 implies that the function $f_0(x)$ is monotonically increasing and bounded in $(-\infty, x_1]$. Thus we define a function $f(x)$ by

$$f(x) = \begin{cases} f_0(x), & x \leq x_1, \\ f_0(x_1), & x > x_1. \end{cases}$$

Under the assumption $b\mathcal{N}'(c_*) \leq -1$, $f \in C^1(\mathbb{R})$ satisfies the following properties;

(f1) $f(x)x > 0$ in $x \neq 0$.

(f2) $f(x) \geq c_*$ in $x \in \mathbb{R}$.

(f3) $f'(0) = -b\mathcal{N}'(c_*) \geq 1$.

(f4) There is $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ in any $x, y \in \mathbb{R}$.

From the condition (f3), it is easy to show that there exist $\alpha > 0$ and $\delta > 0$ such that

$$\forall x \in (-\delta, \delta), \qquad |f(x)| \geq \alpha|x|, \tag{3.10}$$

where δ is supposed to be less than $-c_*$ without loss of generality.

3.4 The Browder fixed-point method

We replace f_0 in the right-hand side of (3.4) into f and consider

$$x'(t) + dx(t) = -df(x(t-1)). \tag{3.11}$$

From the conditions (f1)–(f4), we can prove the existence of a periodic solution of (3.11) if d is sufficiently large. Then we will prove that this yields a periodic solution of (3.4).

Let $C[-1, 0]$ denote the space of all continuous functions on $[-1, 0]$ equipped with the usual sup norm $\|\cdot\|$. We define K by the set of all functions $\varphi \in C[-1, 0]$ such that $\varphi(-1) = 0$, and $e^{dt}\varphi(t)$ is non-decreasing on $-1 \leq t \leq 0$. Moreover, we set $K_0 = K \setminus \{0\}$ and $B_M = \{\varphi \in K \mid \|\varphi\| \leq M\}$ for a constant $M > 0$. We denote by $x(t; \varphi)$ the solution of (3.11) with an initial condition $\varphi \in C[-1, 0]$.

Lemma 3.9. *Let $\varphi \in C[-1, 0]$. Then there exists a unique global-in-time solution $x(t; \varphi)$ of (3.11). The solution $x(t; \varphi)$ is Lipschitz continuous with respect to $\varphi \in C[-1, 0]$, that is, for any $T > 0$, there exists $L_T > 0$ depending only on T such that*

$$\sup_{t \in [0,T]} |x(t; \varphi_1) - x(t; \varphi_2)| \leq L_T \|\varphi_1 - \varphi_2\|. \tag{3.12}$$

Proof. We rewrite (3.11) into an integral form

$$x(t; \varphi) = e^{-d(t-t_0)} x(t_0; \varphi) - d \int_{t_0}^{t} e^{-d(t-s)} f(x(s-1; \varphi)) \mathrm{d}s \tag{3.13}$$

for any $0 \leq t_0 \leq t$. Let $t_0 = 0$ and $t \in [0, 1]$. We have

$$\begin{aligned} |x(t; \varphi_1) - x(t; \varphi_2)| &\leq |\varphi_1(0) - \varphi_2(0)| + d \int_0^t e^{-d(t-s)} |f(\varphi_1(s-1)) - f(\varphi_2(s-1))| \mathrm{d}s \\ &\leq (1 + L)\|\varphi_1 - \varphi_2\|, \end{aligned}$$

which implies that if $T \in [0, 1]$, (3.12) holds true.

By an induction argument, we complete the proof of the lemma. □

Lemma 3.9 implies that for $T > 0$, the solution $x(t; \varphi)$ satisfies

$$|x(t; \varphi)| \leq L_T \|\varphi\| \tag{3.14}$$

on $t \in [0, T]$, which can be shown directly from (3.12) because $x(t; 0) \equiv 0$.

Next we prove that $x(t; \varphi)$ has a zero at some t.

Lemma 3.10. *Assume $(1 + \alpha)/\alpha \leq e^d$ and let $M > \delta$, where α and δ are defined in (3.10). Let $\varphi \in K_0$ with $\|\varphi\| \leq M$. Then there exists a zero of $x(t; \varphi)$, denoted by $t = z_1$, such that $z_1 > 0$ and $x'(z_1; \varphi) < 0$. Moreover, $e^{dt}x(t; \varphi)$ is decreasing on $[z_1, z_1 + 1]$.*

Proof. Denote $x(t) = x(t; \varphi)$ for simplicity. Since $\varphi \in K_0$, we see $\varphi(t) \geq 0$ on $-1 \leq t \leq 0$ and $x(0) = \varphi(0) > 0$. Let $z_1 = \inf\{t \geq 0 \mid x(t) \leq 0\}$. If $z_1 \leq 1$ and $x'(z_1) = 0$, it follows from (3.11) that $f(\varphi(z_1 - 1)) = 0$ so that $\varphi(t) = 0$ in $-1 \leq t \leq z_1 - 1$. Then we substitute

$t_0 = 0$ and $t = z_1$ into (3.13) and then have $x(0) = 0$, which is a contradiction. Hence we obtain $x'(z_1) < 0$ if $z_1 \le 1$.

Next we assume $z_1 > 1$. Then $x(t)$ decreases monotonically because $x'(t) = -dx(t) - df(x(t-1)) \le 0$ on $t \in [0, z_1]$, from which we have $x(t) \le x(0) \le M$ on $t \in [0, z_1]$. Define $t_1 = \inf\{t \ge 1 \mid x(t) \le \delta\}$. We prove that t_1 is finite. If $x(1) > \delta$, it follows from (f1) that

$$x'(t) = -dx(t) - df(x(t-1)) \le -\delta d$$

on $t \in [1, t_1]$. Integrating the both sides above on $[1, t_1]$, we obtain $t_1 \le 1 + M/(\delta d)$. Then we assume that $z_1 > t_1 + 1$. Setting $t_0 = t_1 + 1$ and $t \in [t_1 + 1, t_1 + 2]$ in (3.13), we have

$$\begin{aligned} x(t) &= e^{-d(t-(t_1+1))}x(t_1+1) - \int_{t_1+1}^{t} e^{-d(t-s)} df(x(s-1))\mathrm{d}s \\ &\le ((1+\alpha)e^{-d(t-(t_1+1))} - \alpha)x(t_1+1). \end{aligned}$$

The assumption implies $x(t_1 + 2) \le 0$, from which we have

$$z_1 \le t_1 + 2 \le 3 + \frac{M}{\delta d}. \tag{3.15}$$

In addition, we see that $x'(z_1) = -df(x(z_1 - 1)) < 0$.

Finally we see that $(e^{dt}x(t))' = -e^{dt}df(x(t-1)) \le 0$ on $t \in [z_1, z_1+1]$, which completes the proof of the lemma. □

For $\varphi \in K_0$, there is another zero of $x(t; \varphi)$.

Lemma 3.11. *Assume the same conditions as in Lemma 3.10. Then there exists a zero of $x(t; \varphi)$, denoted by $t = z_2$, such that $z_2 > z_1 + 1$ and $x'(z_2; \varphi) > 0$. Moreover, $e^{dt}x(t; \varphi)$ is increasing and $0 \le x(t; \varphi) \le -c_*$ on $t \in [z_2, z_2 + 1]$.*

Proof. Denote $x(t) = x(t; \varphi)$ for simplicity. Define $z_2 = \inf\{t > z_1 \mid x(t) > 0\}$. Substituting $t_0 = z_1$ in (3.13), we see $x(t) < 0$ in $t \in (z_1, z_1 + 1]$ and

$$x(z_1+1) = -\int_{z_1}^{z_1+1} e^{-d(z_1+1-s)} df(x(s-1))\mathrm{d}s \ge -LM. \tag{3.16}$$

Hence we have $z_1 + 1 < z_2$. Then $x(t)$ increases monotonically on $t \in [z_1 + 1, z_2]$ because

$$x'(t) = -dx(t) - df(x(t-1)) > 0.$$

Define $t_2 = \inf\{t \ge z_1 + 1 \mid x(t) > -\delta\}$. In the case of $x(z_1 + 1) \ge -\delta$, we set $t_2 = z_1 + 1$. We assume $t_2 > z_1 + 1$ and show that t_2 is finite. It follows from (f1) that

$$x'(t) = -dx(t) - df(x(t-1)) \ge \delta d$$

on $z_1 + 1 \le t \le t_2$. Integrating the inequality above over $(z_1 + 1, t_2)$ and owing to (3.15), we have

$$t_2 \le z_1 + 1 + \frac{LM}{\delta d} \le 4 + \frac{M}{\delta d}(1 + L),$$

which implies that t_2 is finite.

Suppose that $z_2 > t_2 + 1$. Setting $t_0 = t_2 + 1$ and $t \in [t_2 + 1, t_2 + 2]$ in (3.13), we see that

$$\begin{aligned} x(t) &= e^{-d(t-(t_2+1))}x(t_2+1) - \int_{t_2+1}^{t} e^{-d(t-s)} df(x(s-1))\mathrm{d}s \\ &\ge ((1+\alpha)e^{-d(t-(t_2+1))} - \alpha)x(t_2+1). \end{aligned}$$

Therefore it follows from the assumption that $x(t_2+2)$ is nonnegative, which implies $z_2 \le t_2 + 2$.

Finally we see that $(e^{dt}x(t))' = -e^{dt}df(x(t-1)) \ge 0$ from (3.11) on $z_2 \le t \le z_2+1$. By setting $t_0 = z_2$ and $t \in [z_2, z_2+1]$, it follows from (f2) that

$$0 \le x(t) = -\int_{z_2}^{t} e^{-d(t-s)} df(x(s-1))\mathrm{d}s \le -c_*,$$

which completes the proof of the lemma. □

We may emphasise the φ-dependency of z_2 and denote $z_2 = z_2(\varphi)$. Fix $M > 0$ arbitrarily. Lemmas 3.9–3.11 imply that $z_2(\varphi)$ is well-defined, continuous and uniformly bounded in $B_M \cap K_0$. In particular, we readily see that $z_2(\varphi) > z_1 + 1$ and

$$z_2(\varphi) \le t_2 + 2 \le 6 + \frac{M}{\delta d}(1+L). \tag{3.17}$$

We define the functional $\mathcal{F} : K \to C[-1,0]$ by

$$[\mathcal{F}\varphi](t) = x(z_2(\varphi)+1+t;\varphi), \qquad t \in [-1,0]$$

for $\varphi \not\equiv 0$ and $[\mathcal{F}\varphi](t) = 0$ for $\varphi \equiv 0$.

Then $\mathcal{F}$ satisfies the following lemma.

Lemma 3.12. *Assume the same conditions as in Lemma 3.10. Then the following two conditions hold true;*

(i) $\mathcal{F}(B_{-c_*}) \subset B_{-c_*}$

(ii) $\mathcal{F}$ *is continuous and compact.*

Proof. The condition (i) can be verified by Lemma 3.11. The continuity of $\mathcal{F}$ can be proved by Lemma 3.9. Set $M = -c_*$ and let $(\varphi_n)_{n\in\mathbb{N}} \in {B_M}^{\mathbb{N}}$. We can assume that $\varphi_n \not\equiv 0$ for all n without loss of generality. Denote $x_n(t) \equiv x(z_2(\varphi_n)+1+t;\varphi_n)$. By the assumption (f4), for all $n \in \mathbb{N}$, for all $t, t_1, t_2 \in [-1,0]$ with $t_2 \leqslant t_1$, we have $0 \leqslant x_n(t) \leqslant M$ and

$$|x_n(t_1) - x_n(t_2)| \le \int_{t_2}^{t_1} |x_n'(s)|\mathrm{d}s \leqslant dM(1+L)(t_1-t_2).$$

Since $\{x_n(t)\}$ is uniformly bounded and uniformly equicontinuous, by the Arzelà-Ascoli Theorem, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges in $C[-1,0]$, which implies that $\mathcal{F}$ is compact. □

According to the Schauder's fixed point theorem [19], Lemma 3.12 implies the existence of a fixed point φ_* of $\mathcal{F}$. However, since $\mathcal{F}(0) = 0$, φ_* may be identically equal to 0. Actually, we can exclude this possibility by the Browder's fixed point theorem (Theorem 1 in [3]). We first prove the next lemma.

Lemma 3.13. *Assume $b\mathcal{N}'(c_*) < -1$ and $d_1 < d$, where d_1 is defined in (3.9). Then there exists $A > 0$ such that for all $\varphi \in K_0$, for all zero z of $x(t;\varphi)$, $\sup_{t\ge z} |x(t;\varphi)| \ge A$.*

Proof. From Corollary 3.7, the characteristic equation (3.5) has an eigenvalue $\lambda = \mu + i\gamma$ with $\mu > 0$ and $\pi/2 < \gamma < \pi$. Let ε be a positive constant satisfying

$$\varepsilon < \frac{\mu}{d} e^{-d} \frac{1}{2} \cos\frac{\gamma}{2}. \tag{3.18}$$

Define $h(x) = f'(0)x - f(x)$. Since $f \in C^1(\mathbb{R})$ and $f(0) = 0$, there exists $A > 0$ such that for all $x \in [-A, A]$,

$$|h(x)| \le \int_0^{|x|} |f'(0) - f'(y)|\mathrm{d}y \leqslant \varepsilon|x|. \tag{3.19}$$

Assume that there are $\varphi \in K_0$ and a zero z of $x(t) \equiv x(t;\varphi)$ such that $\kappa \equiv \sup_{t\geq z} |x(t;\varphi)| < A$. Define $y(t) = e^{dt}x(t)$. From Lemmas 3.10 and 3.11, there is an extremum $m \in (z, z+1)$ such that $|x(m)| \geq \kappa/2$. Moreover, we assume without loss of generality that $x'(z) > 0$ and $y'(t) > 0$ in $(z, z+1)$, because the case $x'(z) < 0$ and $y'(t) < 0$ in $(z, z+1)$ can be treated by the same method.

From (3.11), we have

$$x'(t) = -dx(t) - df'(0)x(t-1) + dh(x(t-1)).$$

Set $T = z + 1$. We multiply both sides by $e^{-\lambda t}$, integrate over $(T, +\infty)$, and change variables in the delayed terms; it yields

$$\begin{aligned}\int_T^{+\infty} x'(t)e^{-\lambda t}\mathrm{d}t = -d\int_T^{+\infty} x(t)e^{-\lambda t}\mathrm{d}t - df'(0)e^{-\lambda}\int_{T-1}^{+\infty} x(t)e^{-\lambda t}\mathrm{d}t \\ + d\int_T^{+\infty} h(x(t-1))e^{-\lambda t}\mathrm{d}t.\end{aligned}$$

Using that $f'(0) = -b\mathcal{N}'(c_*)$ and that λ satisfies (3.5), we get

$$\int_T^{+\infty} x'(t)e^{-\lambda t}\mathrm{d}t = d\int_{T-1}^{T} x(t)e^{-\lambda t}\mathrm{d}t + \lambda\int_{T-1}^{+\infty} x(t)e^{-\lambda t}\mathrm{d}t + d\int_T^{+\infty} h(x(t-1))e^{-\lambda t}\mathrm{d}t.$$

Integrating by parts, we obtain

$$\int_T^{+\infty} x'(t)e^{-\lambda t}\mathrm{d}t = -x(T)e^{-\lambda T} + \lambda\int_T^{+\infty} x(t)e^{-\lambda t}\mathrm{d}t.$$

Then it follows that

$$-x(T)e^{-\lambda T} = (d+\lambda)\int_{T-1}^{T} x(t)e^{-\lambda t}\mathrm{d}t + d\int_T^{+\infty} h(x(t-1))e^{-\lambda t}\mathrm{d}t.$$

Multiplying $e^{\lambda(T-1/2)}$ to both sides, we have

$$-\int_{T-1}^{T} y'(t)e^{-dt-\lambda(t-T+1/2)}\mathrm{d}t = d\int_T^{+\infty} h(x(t-1))e^{-\lambda(t-T+1/2)}\mathrm{d}t. \tag{3.20}$$

We know that $y'(t) > 0$ in $(T-1, T)$ and $y(T-1) = e^{d(T-1)}x(T-1) = e^{dz}x(z) = 0$ and since $\gamma \in (\frac{\pi}{2}, \pi)$, the function $t \mapsto e^{-dt-\mu(t-T+1/2)}\cos\left(\gamma\left(t - T + \frac{1}{2}\right)\right)$ is decreasing on $(T-1, T)$. Hence,

$$\begin{aligned}\left|\int_{T-1}^{T} y'(t)e^{-dt-\lambda(t-T+1/2)}\mathrm{d}t\right| &\geq \int_{T-1}^{T} y'(t)e^{-dt-\mu(t-T+1/2)}\cos\left(\gamma\left(t-T+\frac{1}{2}\right)\right)\mathrm{d}t \\ &\geq \big(y(T) - y(T-1)\big)e^{-dT-\frac{\mu}{2}}\cos\frac{\gamma}{2} \\ &= x(T)e^{-\frac{\mu}{2}}\cos\frac{\gamma}{2}.\end{aligned} \tag{3.21}$$

On the other hand, we upper-bound the right-hand side of (3.20) as

$$\left|\int_T^{\infty} h(x(t-1))e^{-\lambda(t-T+\frac{1}{2})}\mathrm{d}t\right| \leq \varepsilon\kappa\frac{1}{\mu}e^{-\frac{\mu}{2}}. \tag{3.22}$$

Moreover, we set $t = T$ and $t_0 = m$ in (3.13) and then have

$$x(T) = e^{-d(T-m)}x(m) - \int_m^T e^{-d(t-s)}df(x(s-1);\varphi)\mathrm{d}s \geq e^{-d}\frac{\kappa}{2}.$$

From this inequality and (3.20)–(3.22), we see

$$e^{-\frac{\mu}{2}} \cos\frac{\gamma}{2} e^{-d}\frac{\kappa}{2} \leq d\varepsilon\kappa\frac{1}{\mu}e^{-\frac{\mu}{2}},$$

which contradicts (3.18). Hence we complete the proof of Lemma 3.13. □

From Lemma 3.13, there are infinitely many zeroes and extrema of $x(t;\varphi)$ for $\varphi \in K_0$, denoted by z_n and m_n respectively, such that $m_n \in (z_n, z_n + 1)$ and $|x(m_n)| \geq \frac{A}{2}$. This result yields the following lemma.

Lemma 3.14. *Assume $b\mathcal{N}'(c_*) < -1$ and $d_1 < d$, where d_1 is defined in (3.9). Let A be the positive constant given in Lemma 3.13. For all $\varphi \in K_0$, there are an even integer n and a positive constant $A_1 > 0$ such that $x(m_n;\varphi) \geq A_1$.*

Proof. Denote $x(t) = x(t;\varphi)$. Set $A_1 = \min\{A/(2L), A/2\}$ and suppose that $x(m_{2n}) < A_1$ for all $n \geq 1$. By putting $t_0 = z_{2n+1}$ and $t = m_{2n+1}$ in (3.13), it follows from (f4) that

$$x(m_{2n+1}) = -\int_{z_{2n+1}}^{m_{2n+1}} e^{-d(m_{2n+1}-s)} df(x(s-1))\mathrm{d}s > -\frac{A}{2},$$

which is in contrast to Lemma 3.13. □

Proof of Theorem 3.1. From Lemma 3.14, we see that for any $\varphi \in K_0$, there is a sufficiently large integer n such that $\|F^{(n)}(\varphi)\| \geq A_1$, which implies that $\varphi \equiv 0$ is an *ejective point* in the sense of Definition 3.2. Therefore, by Theorem 3.3 there exists a nonzero fixed point $\varphi_* \not\equiv 0$ of $\mathcal{F}$. From the assumption $V_F^2 \geq a$, we see that $x_1 \geq -c_*$ and then $c(t) \equiv c_* + x(\frac{t}{d};\varphi_*)$ is a periodic solution of (3.1).

In order to complete the proof of Theorem 3.1, we estimate $x(t;\varphi_*)$. From the proof of Lemma 3.10, $x(t;\varphi_*) \leq -c_*$ on $t \in [0, z_1]$. By the same argument as in the proof of Lemma 3.11, $x(t;\varphi_*)$ attains a local minimum at $t = m \in (z_1, z_1 + 1)$. Setting $t_0 = z_1$ and $t = m$ in (3.13), we have

$$x(m;\varphi_*) = -\int_{z_1}^{m} e^{-d(m-s)} df(x(s-1;\varphi))\mathrm{d}s \geq Lc_*.$$

Then it holds that $(L+1)c_* \leq c(t) \leq 0$, which also implies that $c(t)$ must be negative because $b\mathcal{N}(c(t-d))$ is negative. Since L and c_* are independent of d, t, we have proved Theorem 3.1. □

4 Asymptotic description of the periodic solution

We now prove an asymptotic result on the shape of solutions of equation

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \qquad \mathcal{N}(c) = (V_F - c)\exp\left(-\frac{(V_F - c)^2}{2a}\right). \tag{4.1}$$

We are going to describe in appropriate rescaled variables a periodic asymptotic profile $P(t)$ which is the limit of some solutions of (4.1) when b goes to $-\infty$. As a consequence, this profile is also (up to phase shift) the limit of the periodic solutions of (4.1) and hence an asymptotic description of how they behave.

In order to do so, we assume for the sake of clarity that $V_F = 0$, $d = 1$ and we make the change of variables

$$\beta = \frac{1}{\log(-b)}, \qquad u(s) = \beta\log\left(-\frac{1}{\sqrt{a}}c\left(\frac{s}{\beta}\right)\right). \tag{4.2}$$

This rescaling is meaningful only when $b < -1$, which reminds us of Case 2 ($V_F = 0$, $b < -1$) in the previous section. We come to the following equivalent system:

$$u'_\beta(t) + 1 = \exp\left(\frac{1}{\beta}(1 - u_\beta(t)) + f\big(u_\beta(t-\beta)\big)\right), \quad 1 \geq u_\beta([-\beta, 0]) > 0 \tag{4.3}$$

with

$$f(x) = \frac{1}{\beta}x - \frac{1}{2}e^{\frac{2}{\beta}x}. \tag{4.4}$$

Equation (4.3) has a unique constant positive stationary state given by

$$\overline{u_\beta} = \frac{\beta}{2}\log\left(\frac{2}{\beta}\right), \tag{4.5}$$

and we readily observe that

$$\lim_{\beta \to 0} \overline{u_\beta} = 0.$$

Note that when b goes to $-\infty$, β tends to 0. If $b \ll 0$, then $0 < \beta \ll 1$.

Theorem 4.1. *Assume that for all $\beta > 0$, $u_\beta([-\beta, 0]) \equiv 1$ and let $T > 2$. Then, there exists a constant $C_M \in (0, 1]$ such that*

$$\lim_{\beta \to 0} u_\beta = P, \qquad \text{in } L^1(0, T),$$

where P has the following form:

- $P(t) = 1 - t$ *on* $(0, 1)$;
- $P(t) = C_M + 1 - t$ *on* $(1, 1 + C_M)$;
- P *is* C_M *periodic on* $(1, T)$.

We summarise our approach in Figure 1.

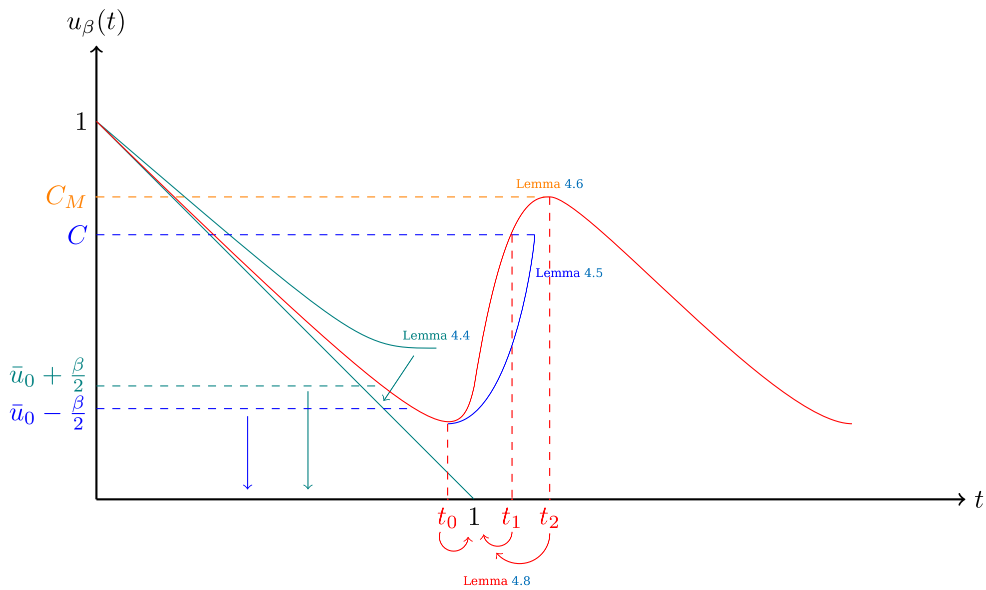

Figure 1: Strategy of the proof : Lemma 4.4 is used to prove convergence to a straight line, Lemma 4.5 gives rapid growth to a $\beta-$independent constant C, Lemma 4.8 gives $t_0, t_1, t_2 \to 1$; Lemma 4.6 helps proving periodicity of P.

Before proving this result, let us make some comments on its implications.

- First, we make the assumption $u_\beta([-\beta,0]) \equiv 1$ for the sake of clarity. A key mechanism being the unstable nature of the stationary state u_β, we expect that it is enough to take an initial condition which is not uniformly u_β to get the same convergence result after a transitory period. Our numerical simulations confirm that the hypothesis is purely technical.
- The form of our rescaling and Theorem 4.1 indicates that the period T_b of the numerically observed periodic solutions c of (1.8) evolves in $\mathcal{O}\big(\log(-b)\big)$ when $b \to -\infty$.
- More precisely, we have in the original variables

$$c(t) \simeq -\sqrt{a}\exp\left(\log(-b)P\left(\frac{t}{\log(-b)}\right)\right).$$

  In decay phases, this approximation means that c behaves like

$$c(t) \simeq -\sqrt{a}e^{C_M \log(-b)-t} = S_b e^{-t},$$

  which constitutes an exponential decay over a spatial area of length $S_b = \sqrt{a}(-b)^{C_M}$ during a time $T_b = C_M \log(-b)$. Remember that $C_M \in (0,1]$.
- The approximation of $c(t)$ by the discontinuous profile hides a very fast growth phase which makes the exponential decay appear slow in comparison. Lemma 4.5 gives an idea of this growth phase in rescaled variables.

We are now going to prove the convergence of u_β towards this periodic asymptotic profile P when β goes to 0. To do so, we rely on the instability of the stationary state $\overline{u_\beta}$. Our strategy is to study independently different phases of the solution:

1. a decay phase with asymptotic slope -1 (Lemma 4.4);
2. a rapid growth phase up to some uniform constant C (Lemma 4.5);
3. a maximal value which presents some handy stability properties from which we will derive the periodicity of the final profile (Lemma 4.6, Lemma 4.8).

4.1 Convergence of u_β to a profile P when $\beta \to 0$.

Let us first observe that the function f defined in (4.4) is decreasing on $[0,+\infty)$ with $f(0) = -\frac{1}{2}$ and $f(+\infty) = -\infty$. Indeed, we have

$$f'(x) = \frac{1}{\beta}(1 - e^{\frac{2}{\beta}x}) < 0 \quad \text{on} \quad (0,+\infty).$$

Here, we ensure that a limit profile does exist, without describing it yet.

Lemma 4.2. *Assume* $\liminf_{\beta\to 0} u_\beta(0) > 0$. *There exists* $\beta_0 > 0$ *such that for all* $\beta \in (0,\beta_0)$, *the solution of* (4.3) *satisfies : for all* $t > 0$,

$$0 < u_\beta(t) < 1$$

and for all $T > 0$,

$$\int_0^T |u_\beta'(t)|\mathrm{d}t \le 2T + 1.$$

Moreover, there exist a sequence β_n *which converges to 0 and a function* P *such that*

- $0 \le P \le 1$
- $P \in BV$, *with BV the set of functions with bounded variations*

- $\lim_{n\to+\infty} u_{\beta_n} = P$ *in* $L^1(0,T)$.

Proof. Let us first prove that $u_\beta > 0$ for all $t > 0$. Assume that there exists a first time t_0 such that $u(t_0) = 0$; we have for all $t < t_0$, $u(t) > 0$. By continuity of the derivative, $u'_\beta(t_0) \leq 0$.

At this step, there are two possibilities :

- either $u_\beta(t_0-\beta) > \beta$, and, since for all $t \geq 0$, $u'_\beta(t) > -1$, we have $u_\beta(t_0) > 0$, which is incompatible with $u(t_0) = 0$;
- or $u_\beta(t_0-\beta) \leq \beta$, and, since f is decreasing, we obtain that
$$u'_\beta(t_0) \geq -1 + e^{\frac{1}{\beta}+f(\beta)},$$
but $f(\beta) = 1 - \frac{1}{2}e^2$ and so, for β small enough,
$$u'_\beta(t_0) \geq e^{\frac{1}{2\beta}}$$
which is in contradiction with $u'_\beta(t_0) \leq 0$.

Hence, for all $t > 0$, we have $u_\beta(t) > 0$.

Let us now prove that for all $t > 0$, $u_\beta(t) < 1$. If $u_\beta(0) = 1$, then since $f < 0$, $u'_\beta(0) = -1 + e^{f\left(u_\beta(-\beta)\right)} < 0$. Hence in both the cases $u_\beta(0) = 1$ and $0 < u_\beta(0) < 1$ there exists $\varepsilon > 0$ such that for all $t \in (0,\varepsilon]$, $u_\beta(t) < 1$. Let $t_1 = \inf\{t > 0 |\ u_\beta(t_1) = 1\}$. Assume $t_1 < +\infty$, then $u'_\beta(t_1) < 0$ and there exists a small neighbourhood $(t_1-\varepsilon_1, t_1)$ such that for all $t \in (t_1-\varepsilon_1, t_1), u_\beta(t) > 1$, which constitutes a contradiction.

To prove the uniform estimate on the derivative of u_β, we first integrate Equation (4.3) between 0 and T, and obtain
$$u_\beta(T) - u_\beta(0) = -T + \int_0^T e^{\frac{1}{\beta}(1-u_\beta(t))+f(u_\beta(t-\beta))}\mathrm{d}t.$$
Hence,
$$\int_0^T e^{\frac{1}{\beta}(1-u_\beta(t))+f(u_\beta(t-\beta))}\mathrm{d}t \leq T+1$$
and so
$$\int_0^T |u'_\beta(t)|\mathrm{d}t \leq 2T+1.$$
These β-uniform bounds on the solution and its derivative imply the existence, up to extraction of a subsequence, of an L^1 limit P. □

Remark 4.3. Note that by compactness, there exists a subsequence $(\beta_n)_{n\in\mathbb{N}}$ converging to 0 and a value $u_0 \in (0,1]$ such that $\lim_{\beta_n\to 0} u_{\beta_n}(0) = u_0$. In Theorem 4.1, we prescribe $u_0 = 1$ in order to have a clear expression of the profile; another value $u_0 \in (0,1)$ would give a similar but shifted profile.

4.2 Description of the qualitative properties of u_β

Let us come back to a more precise description of the dynamics related to the delay equation satisfied by u_β.

When the solution starts at the value 1, it first decays down to a local minimum which is smaller than the stationary state. The following Lemma describes the asymptotic pace of such decay with respect to β.

Lemma 4.4. *Let I be an interval of $\mathbb{R}_+$. Assume that for all $t \in I$,*

$$u_\beta(t-\beta) \geq \overline{u_\beta} + \frac{\beta}{2}.$$

Then, for all $t \in I$, the following estimate holds

$$-1 \leq u_\beta'(t) \leq -1 + e^{\frac{1}{\beta}(1-e)+1}. \tag{4.6}$$

Proof. Let us first remark that, since we always have $u' \geq -1$, this implies that

$$u_\beta(t) \geq u_\beta(t-\beta) - \beta.$$

Hence,

$$u_\beta'(t) \leq -1 + e^{\frac{1}{\beta}(1-u_\beta(t-\beta))+1+f(u_\beta(t-\beta))}.$$

Now, if we consider the function g given by

$$g(x) = \frac{1}{\beta}(1-x) + 1 + f(x) = \frac{1}{\beta} + 1 - \frac{1}{2}e^{\frac{2}{\beta}x},$$

then $g'(x) < 0$ and thus g is decreasing. We deduce that while $u(t-\beta) \geq \overline{u_\beta} + \frac{\beta}{2}$, we have

$$u_\beta'(t) \leq -1 + e^{g(\overline{u_\beta}+\frac{\beta}{2})}.$$

But,

$$g(\overline{u_\beta} + \frac{\beta}{2}) = \frac{1}{\beta}(1-e) + 1,$$

which proves Lemma 4.4. □

Then, the solution grows rapidly. The following Lemma provides a quantitative grasp on the phenomenon.

Lemma 4.5. *Let $\varepsilon > 0$. Let I be an interval of $\mathbb{R}_+$. Assume that for all $t \in I$,*

$$u_\beta(t-\beta) \leq \overline{u_\beta} - \frac{\beta}{8} \text{ and } u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon.$$

Then, for all $t \in I$, the following estimate holds

$$u_\beta'(t) \geq -1 + e^{-\frac{1}{8}} \left(\frac{2}{\beta}\right)^{\frac{1}{2}} e^{\frac{\varepsilon}{\beta}}.$$

Proof. Since $u_\beta(t-\beta) \leq \overline{u_\beta} - \frac{\beta}{8}$ and f is decreasing, we have

$$u_\beta'(t) \geq -1 + e^{\frac{1}{\beta}(1-u_\beta(t))+f(\overline{u_\beta}-\frac{\beta}{8})}.$$

However,

$$f\left(\overline{u_\beta} - \frac{\beta}{8}\right) = \frac{1}{2}\log(\frac{2}{\beta}) - \frac{1}{8} - \frac{1}{\beta}e^{-\frac{1}{4}}.$$

Hence, while $u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon$, we have

$$u_\beta'(t) \geq -1 + e^{-\frac{1}{8}} \left(\frac{2}{\beta}\right)^{\frac{1}{2}} e^{\frac{\varepsilon}{\beta}},$$

which proves Lemma 4.5. □

Now, we are going to prove a stability estimate on Equation (4.3). As the solution goes down below the fading u_β, we want to show that small initial conditions lead to similar outcomes. The goal is to prove later that the rapid growth phase yields a stable asymptotic maximum value. More precisely, the following result holds.

Lemma 4.6. *Let u_1, u_2 be two solutions of Equation (4.3) and let t be a time such that there exist two positive constants C and α_1 independent of β and a value $\beta_0 > 0$ small enough such that for all $\beta \in (0, \beta_0)$,*

$$|u_1(s) - u_2(s)| \le Ce^{-\frac{\alpha_1}{\beta}}, \quad \forall s \in (t-\beta, t), \tag{4.7}$$

Assume that there exists $k \in \mathbb{N}$ such that for all $s \in (t-\beta, t+k\beta)$,

$$|u_1(s)| \le \overline{u_\beta} + C\beta \quad \text{or} \quad |u_2(s)| \le \overline{u_\beta} + C\beta. \tag{4.8}$$

Then, there exists $\alpha_2 > 0$ independent of β such that for all $\beta \in (0, \beta_0)$, for all $s \in (t, t+(k+1)\beta)$, the following estimate holds

$$|u_1(s) - u_2(s)| \le e^{-\frac{\alpha_2}{\beta}}. \tag{4.9}$$

Proof. Without loss of generality, with assumption (4.8), we can assume that for all $s \in (t-\beta, t+k\beta)$,

$$|u_1(s)| \le \overline{u_\beta} + C\beta.$$

We have for all $s \ge t$

$$(u_1 - u_2)'(s) = e^{\frac{1}{\beta}(1-u_1(s))} e^{f(u_1(s-\beta))} - e^{\frac{1}{\beta}(1-u_2(s))} e^{f(u_2(s-\beta))}$$

and so

$$(u_1 - u_2)'(s) = (u_1'(t)+1)(1 - e^{(f(u_2)-f(u_1))(s-\beta)}) + (u_2'(t)+1)(e^{\frac{1}{\beta}(u_2-u_1)(s)} - 1).$$

Multiplying the above equation by $u_1 - u_2$, we obtain that

$$\begin{aligned}\frac{1}{2}\left((u_1-u_2)^2\right)'(s) = (u_1'(s)+1)&(1 - e^{(f(u_2)-f(u_1))(s-\beta)})(u_1-u_2)(s) \\ &+ (u_2'(s)+1)(e^{\frac{1}{\beta}(u_2-u_1)(s)} - 1)(u_1-u_2)(s).\end{aligned}$$

As

$$(u_2'(s)+1)(e^{\frac{1}{\beta}(u_2-u_1)(s)} - 1)(u_1-u_2)(s) \le 0,$$

we deduce that

$$\left((u_1-u_2)^2\right)'(s) \le (u_1'(s)+1)\left(\left|1 - e^{f\left(u_2(s-\beta)\right) - f\left(u_1(s-\beta)\right)}\right|^2 + (u_1-u_2)^2\right).$$

But, with Lemma 4.2, we know that there exists a constant $C_1 > 0$ such that for all $s \in (t, t+(k+1)\beta)$,

$$e^{\int_t^s 1 + u_1'(w)dw} \le C_1.$$

By Gronwall inequality, we then obtain that for all $s \in (t, t+(k+1)\beta)$, the following estimate holds

$$(u_1-u_2)^2(s) \le C_1\left((u_1-u_2)^2(t) + \|1 - e^{f(u_2)-f(u_1)}\|^2_{L^\infty(t-\beta, s-\beta)}\right)$$

and so there exists a constant $C_2 > 0$ such that for all $s \in (t, t+(k+1)\beta)$,

$$(u_1-u_2)^2(s) \le C_2\left(e^{-2\frac{\alpha_1}{\beta}} + \|1 - e^{f(u_2)-f(u_1)}\|^2_{L^\infty(t-\beta, s-\beta)}\right). \tag{4.10}$$

Let us deal with the term

$$\|1-e^{f(u_1)-f(u_2)}\|_{L^\infty(t-\beta,s-\beta)}.$$

As with assumption (4.8),

$$|u_1(s)| \le \overline{u_\beta} + C\beta,$$

we obtain that for all $s \in (t-\beta, t)$,

$$|f(u_1)-f(u_2)|(s) \le \frac{C}{\beta} e^{-\frac{\alpha_1}{\beta}} + \frac{1}{2} e^{\frac{2}{\beta}(\overline{u_\beta}+C\beta)}(1-e^{\frac{2}{\beta}(u_2-u_1)}).$$

But

$$e^{\frac{2}{\beta}(\overline{u_\beta}+C\beta)} \le e^{2C}\frac{2}{\beta}.$$

Hence, for all $s \in (t-\beta, t)$, there exist constants $C_3, C_4 > 0$ such that

$$\begin{array}{rcl} \|1-e^{f(u_1)-f(u_2)}\|_{L^\infty(t-\beta,s-\beta)} & \le & C_3\|f(u_1)-f(u_2)\|_{L^\infty(t-\beta,s-\beta)} \\ & \le & C_4 \frac{1}{\beta^2} e^{-\frac{\alpha_1}{\beta}}. \end{array}$$

For β small enough we have

$$e^{-2\frac{\alpha_1}{\beta}} = e^{-\frac{\alpha_1}{\beta}} e^{-\frac{\alpha_1}{\beta}} \leqslant \frac{1}{\beta} e^{-\frac{\alpha_1}{\beta}}.$$

Hence, coming back to estimate (4.10), we obtain that for all $s \in (t, t+\beta)$, there exists a constant C_5 such that

$$|u_1-u_2|(s) \le C_5 \frac{1}{\beta^2} e^{-\frac{\alpha_1}{\beta}}.$$

By induction, we then obtain that there exists a constant C_6 such that for all $s \in (t-\beta, t+k\beta)$,

$$|f(u_1)-f(u_2)|(s) \le C_6 \left(\frac{1}{\beta}\right)^{2k+2} e^{-\frac{\alpha_1}{\beta}}$$

and so there exists a constant $C_7 > 0$ such that for all $s \in (t-\beta, t+(k+1)\beta)$,

$$|u_1-u_2|(s) \le C_7 \left(\frac{1}{\beta}\right)^{2k+2} e^{-\frac{\alpha_1}{\beta}}.$$

The constant C_7 being independent of β, we can choose β_0 so to have, for all $\beta \in (0, \beta_0)$,

$$C_7 \left(\frac{1}{\beta}\right)^{2k+2} \le 1,$$

which concludes the proof of Lemma 4.6. □

4.3 Description of the profile P.

The following theorem holds.

Theorem 4.7. *Assume that for all $\beta > 0$, $u_\beta([-\beta, 0]) \equiv 1$ and let $T > 2$. Then, there exists a constant $C_M > 0$ such that the profile P has the following form*

$$P(t) = 1 - t \text{ on } (0,1)$$

$$P(t) = C_M + 1 - t \text{ on } (1, 1+C_M)$$

and P is C_M periodic on $(1, T)$.

Proof. Step 1: proof of the shape of the profile on (0,1). We are going to prove that u'_β converges uniformly to -1 on the interval $[0, u_\beta(0) - \varepsilon]$, for all $\varepsilon > 0$.

First, note that $\overline{u_\beta}$ tends to 0 when β goes to 0. Let $\varepsilon > 0$. Since $u'_\beta \geq -1$, for all $t \in [0, u_\beta(0) - \varepsilon]$,

$$u_\beta(t) = u_\beta(0) + \int_0^t u'_\beta(s)\mathrm{d}s \geq u_\beta(0) - t \geq \varepsilon.$$

Hence we can choose β small enough in order to have for all $t \in [-\beta, u_\beta(0) - \varepsilon]$,

$$u_\beta(t) \geq \overline{u_\beta} + \frac{\beta}{2}.$$

Then, using Lemma 4.4, we deduce that for β small enough, for all $t \in [0, u_\beta(0) - \varepsilon]$

$$-1 \leq u'_\beta(t) \leq -1 + e^{\frac{1}{\beta}(1-e)+1},$$

which proves the result.

Step 2: Description of the discontinuity of P at $t = 1$ via u_β. To understand what happens at the point 1 in the limit $\beta \to 0$, let us describe more precisely some qualitative properties of the function u_β:

Lemma 4.8. *Assume that for all $\beta > 0$, $u_\beta([-\beta, 0]) \equiv 1$. The following properties hold.*

- *Let $t_0(\beta)$ be the first time such that $u'_\beta(t_0(\beta)) = 0$. Then, $u_\beta(t_0(\beta)) \leq \overline{u_\beta} - \frac{\beta}{4}$ is a local minimum. Moreover, we have*

$$1 - \overline{u_\beta} - 2\beta \leq t_0(\beta) \leq 1 - \overline{u_\beta} + 2\beta.$$

- *There exists a constant $C > 0$ independent of β and a minimal time $t_1(\beta)$ with $t_0(\beta) + \beta > t_1(\beta) > t_0(\beta)$ such that*

$$u_\beta(t_1(\beta)) \geq C.$$

- *There exists $t_2(\beta) \in [t_1, t_1 + \beta)$ such that*

$$C_1(\beta) := u_\beta(t_2(\beta)) = \sup_{t \in (t_0, t_1 + \beta)} u_\beta(t). \tag{4.11}$$

Moreover u'_β converges uniformly to -1 on $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon]$, for all $\varepsilon > 0$.

Proof of Lemma 4.8. Let us prove the first property of Lemma 4.8. First, remark that with estimate (4.6), if we consider the first time t such that $u(t - \beta) = \overline{u_\beta} + \frac{\beta}{2}$, then

$$u'_\beta(s) \leq -1 + e^{\frac{1}{\beta}(1-e)+1}$$

for all $s \in (0, t]$. Therefore, if β is small enough u'_β is close enough to -1 on $[0, t)$ and we have $u_\beta(t) \leq \overline{u_\beta} - \frac{\beta}{4}$. Hence, at the first time $t_0(\beta) > t$ such that $u'_\beta(t_0) = 0$, we have $u_\beta(t_0) < \overline{u_\beta} - \frac{\beta}{4}$. Let us now prove that $t_0(\beta)$ is a local minimum. To do so, we differentiate Equation (4.3) and we find that

$$u''_\beta(t_0) = u'_\beta(t_0 - \beta) f'(u(t_0 - \beta)) e^{\frac{1}{\beta}(1 - u_\beta(t)) + f(u_\beta(t-\beta))}.$$

As $f' < 0$ and $u'_\beta(t_0 - \beta) < 0$, we deduce that $u''_\beta(t_0) > 0$ and hence $t_0(\beta)$ is a local minimum. The proof of the fact that

$$1 - \overline{u_\beta} - 2\beta \leq t_0(\beta) \leq 1 - \overline{u_\beta} + 2\beta$$

is a direct consequence of Lemmas 4.4 and 4.5.

Let us now prove the second property of Lemma 4.8. We remark that as $u_\beta(t_0) < \overline{u_\beta} - \frac{\beta}{4}$, and $u_\beta' \geq -1$, then, for all $t \in (t_0 - \frac{\beta}{8}, t_0)$, we have $u_\beta(t) \leq \overline{u_\beta} - \frac{\beta}{8}$. this implies that on $(t_0 + \frac{7\beta}{8}, t_0 + \beta)$, we can apply Lemma 4.5. Let $\varepsilon > 0$ be small enough. While $u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon$, we have

$$u_\beta'(t) \geq -1 + Ce^{\frac{\varepsilon}{\beta}}.$$

Moreover, integrating the above estimate on an interval of size β, we conclude that there exists $t_1 \in (t_0, t_0 + \beta)$ such that $u_\beta(t) \geq 1 - e^{-\frac{1}{4}} - \varepsilon$.

To prove the third property of Lemma 4.8, we observe that as $u_\beta' \geq -1$, on $(t_1, t_1 + \beta)$,

$$u_\beta(t) \geq 1 - e^{-\frac{1}{4}} - \beta - \varepsilon.$$

Hence, using estimate (4.6), we deduce that while $u(t - \beta) \geq \overline{u_\beta} + \frac{\beta}{2}$, u_β is strictly decreasing. We deduce that there exists a time $t_2(\beta) \in [t_1, t_1 + \beta)$ such that

$$C_1 := u_\beta(t_2(\beta)) = \sup_{t \in (t_0, t_1 + \beta)} u_\beta(t)$$

with u_β satisfying estimate (4.6) on $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon(\beta)]$ with $\varepsilon(\beta) \to 0$ when β goes to 0. We then deduce that u_β' converges uniformly to -1 on $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon]$, for all $\varepsilon > 0$ which ends the proof of Lemma 4.8. $\square$

Step 3: Proof of the C_M periodicity of P. With Lemma 4.8, we know that P is a piecewise linear function with slope -1 between the jumps, that this function decreases until reaching the value 0 and that it has a jump upon reaching the value 0. We now have to prove that this jump is always the same at each step.

To do this, we first observe that, on the one hand, up to a subsequence, $C_1(\beta)$ defined in (4.11) converges to a value $C_M \in (0, 1]$. To prove that C_M is exactly the jump of P at the value 1, we observe that, since $u_\beta' \geq -1$, for all $t \in (t_2(\beta), t_1(\beta) + \beta)$,

$$C_1(\beta) - \beta \leq u_\beta(t) \leq C_1(\beta).$$

On the other hand, combining Lemmas 4.8 and 4.4, we know that there exists a first time $t \in [1, 2]$ such that $u_\beta(t) = \overline{u_\beta} + \frac{\beta}{2}$ and for all $s \in (t, t + \beta)$

$$u_\beta(s) = \overline{u_\beta} + \frac{\beta}{2} + t - s + \mathcal{O}(e^{(1-e)\beta^{-1}}).$$

Hence, to prove that P is periodic, we have to prove that by taking an initial data such that for $s \in (-\beta, 0)$,

$$u_1(s) = \overline{u_\beta} + \frac{\beta}{2} + s$$

and another initial data such that for $s \in (-\beta, 0)$,

$$u_2(s) = \overline{u_\beta} + \frac{\beta}{2} + s + \mathcal{O}(e^{(1-e)\beta^{-1}}),$$

then the maximal values $C_1^1(\beta)$ and $C_1^2(\beta)$ defined in (4.11), associated to u_1 and u_2 given by the third step of Lemma 4.8 are such that

$$\lim_{\beta \to 0} C_1^1(\beta) = \lim_{\beta \to 0} C_1^2(\beta).$$

To prove this, we first use Lemma 4.8 which implies that there exist $t_2^1(\beta)$ and $t_2^2(\beta)$ such that

$$u_1(t_2^1(\beta)) = C_1^1(\beta) \quad \text{and} \quad u_2(t_2^2(\beta)) = C_1^2(\beta).$$

Moreover, we know that there exists a constant C independent of β such that

$$|t_2^1(\beta)| + |t_2^2(\beta)| \le C\beta.$$

We can assume without loss of generality that $t_2^2(\beta) \ge t_2^1(\beta)$. By the contraposition of Lemma 4.4, we know that

$$u_2(t_2^2 - \beta) \le \overline{u_\beta} + \frac{\beta}{2}.$$

Hence, again, because $u_2' \ge -1$, this implies that for all $s \in (t_2^2 - 2\beta, t_2^2 - \beta)$,

$$u_2(s) \le \overline{u_\beta} + \frac{3}{2}\beta.$$

Hence, we are in the setting of Lemma 4.6 as soon as $t \le t_2^2 - \beta$. Moreover, using

$$|t_2^1(\beta)| + |t_2^2(\beta)| \le C\beta,$$

we obtain that there exists a constant $\alpha > 0$ such that for all $s \in (-\beta, t_2^2)$

$$|u_1(s) - u_2(s)| \le e^{-\alpha\beta^{-1}},$$

which implies that

$$\lim_{\beta\to 0} C_1^1(\beta) = \lim_{\beta\to 0} C_1^2(\beta).$$

This ends the proof of Theorem 4.7. □

Proof of Theorem 4.1. This result is a direct consequence of Theorem 4.2 and Lemma 4.7. □

5 Numerical experiments

The following numerical experiments were computed using Python. The full code is available on GitHub on the following link: https://github.com/pierreabelroux/Ikeda_Roux_Salort_Smets_2022. The figures of this article[2] can be reproduced by running the figure functions at the end of the code.

5.1 Periodic solutions of the PDE

The PDE (1.4) is challenging from a numerical point of view because of the non-local nonlinearity. Efficient numerical methods were designed specifically for this equation [8, 12, 26]. Here we use the structure preserving semi-implicit scheme from [26] with an improvement from [25] which consists in treating implicitly the firing and resetting mechanism.

More precisely, the scheme is based upon the Scharfetter-Gummel reformulation

$$\frac{\partial p}{\partial t} - a\frac{\partial}{\partial v}\left(M(v,t)\frac{\partial}{\partial v}\left(\frac{p(v,t)}{M(v,t)}\right)\right) = 0, \tag{5.1}$$

with

$$M(v,t) = e^{-\frac{(v-bN(t-d))^2}{2a}}.$$

[2] Apart from Figure 1 which is a mouse-made Microsoft Paint drawing.

Given a spatial grid $v_0 = V_{min}, v_1, \dots, v_n = V_F$, Equation (5.1) is approximated by a centre difference scheme with fluxes $F_{i+\frac{1}{2}}$ and the numerical terms $M_{i+1/2}$ between grid points v_i and v_{i+1} are computed as harmonic means of the approximations M_i and M_{i+1} of $M(v,t)$ at $v = v_i$ and $v = v_{i+1}$:

$$M_{i+\frac{1}{2}} = \left(\frac{1}{2}\left(\frac{1}{M_i} + \frac{1}{M_{i+1}}\right)\right)^{-1} = \frac{2M_i M_{i+1}}{M_i + M_{i+1}}.$$

The semi-implicit scheme from time t^j to time t^{j+1} can be written in matrix form

$$p^{j+1} = \left(I + \frac{a\Delta t}{\Delta v^2}\mathcal{M}\right)^{-1} p^j,$$

where I is the identity matrix, $\Delta t = t^{j+1} - t^j$, $\Delta v = v_{i+1} - v_i$, p_i^j is the numerical approximation of $p(v_i, t^j)$, $p^j = (p_i^j)_{0 \leqslant i \leqslant n}$ and $\mathcal{M}$ is a matrix which depends upon the M_i and the $M_{i+\frac{1}{2}}$. The firing and resetting mechanism is treated implicitly by adding appropriate coefficients in $\mathcal{M}$: the implicit flux $N(t)$ is approximated with a first order finite difference:

$$N(t^{j+1}) \simeq -a\frac{p_n^{j+1} - p_{n-1}^{j+1}}{\Delta v} = \frac{a}{\Delta v} p_{n-1}^{j+1},$$

since the boundary condition in V_F implies $p_n^{j+1} = 0$. We take constant time and space steps Δt and Δv.

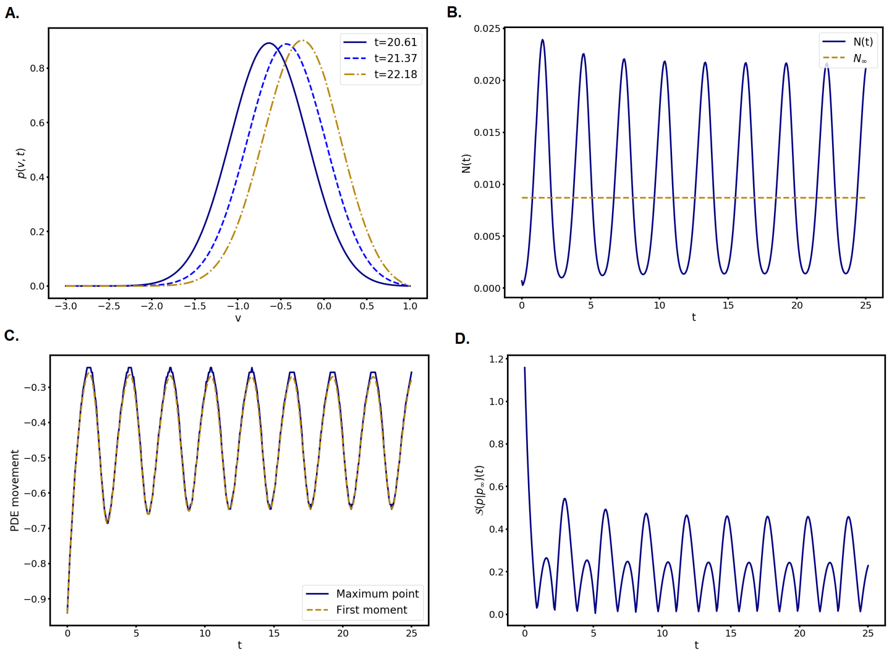

Figure 2: Simulation of the PDE (1.4) in Setup 5.1 with $b = -45$; (A.) solution at three different times; (B.) evolution in time of the firing rate $N(t)$; (C.) evolution in time of the first moment (average) and the maximum point of the solution; (D.) evolution in time of the logarithmic relative entropy $\mathcal{S}(p|p_\infty)(t)$.

This numerical scheme has many advantages: it is positivity preserving, mass conservative and when $b = 0$ it is proved to preserve the decay of the quadratic relative entropy

$$\mathcal{S}(p|p_\infty)(t) = \int_{-\infty}^{V_F} p_\infty(v) G\left(\frac{p(v,t)}{p_\infty(v)}\right) \mathrm{d}v, \tag{5.2}$$

with $G(x) = (x-1)^2$. The scheme is called semi-implicit because when $d = 0$ the matrix $\mathcal{M}$ depends on $N(t^j) \simeq \frac{a}{\Delta v} p^j_{n-1}$, but when $d > 0$ the scheme can be considered fully implicit.

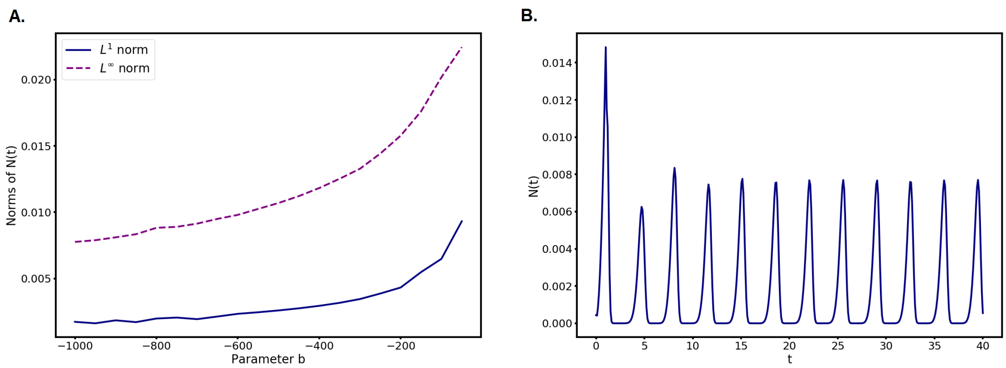

Figure 3: Simulation of the PDE (1.4) in Setup 5.1 for different values of b; (A.) L^1 and L^∞ norms of the firing rate $N(t)$; (B.) evolution in time of $N(t)$ for $b = -1000$.

In these numerical experiments, we will focus on two different settings. The first one satisfies the hypotheses of Theorem 3.1:

Setup 5.1. *We take $V_F = 1$, $V_R = 0$, $a = 0.2$, $d = 1$.*

The second setting satisfies the hypotheses of Theorem 4.1:

Setup 5.2. *We take $V_F = 0$, $V_R = -2$, $a = 0.2$, $d = 1$.*

Whenever it is possible, the colours of the curves are related to the setup we investigate.

In Figure 2 we simulate the PDE (1.4) in Setup 5.1 with $b = -45$. The solution converges rapidly towards a periodic solution. Figure 2A shows the shape of the solution at different times during the period. Figure 2B shows the evolution in time of the firing rate $N(t)$ compared to the stationary firing rate N_∞. Figure 2C shows two indicators of how the solutions moves in time:

- the first moment, which represents the centre of the solution when it is close to a Gaussian function:
$$m_1(t) = \int_{-\infty}^{V_F} v p(v,t) \mathrm{d}v;$$
- the maximum point of the solution:
$$\operatorname{argmax} \{\, p(v,t) \mid v \in (-\infty, V_F] \,\}.$$

As we can see, these two indicators match closely away from V_F and there is a slight discrepancy when the solution looses its Gaussian shape. Figure 2D shows the evolution in time of the logarithmic relative entropy, which is given by (5.2) with $G(x) = x\log(x)$. The quadratic relative entropy was used to prove convergence towards the stationary state in [8, 14, 10] and it is a good indicator of how the stationary state is either stable or unstable. When $b < 0$ is small enough, it is proved that there exist some constants $A_0 > 0$, $\mu > 0$ and $t_0 > 0$ such that[3]

$$\forall t > t_0, \qquad \mathcal{S}(p|p_\infty)(t) \leqslant A_0 e^{-\mu t} \mathcal{S}(p|p_\infty)(t_0).$$

[3] In the delayed case $d > 0$ the precise result is a bit more complex because of the initial condition for $N(t)$ between $-d$ and 0, but the spirit is the same: exponential convergence to 0 of the quadratic relative entropy. See [10, Th. 5.3].

However, we plotted the logarithmic relative entropy rather than the quadratic one for the sake of readability. We approximate the stationary state (p_∞, N_∞) by running for a large enough time Equation (1.4) without delay ($d = 0$). Note that in each period the periodic solution passes twice close to the stationary state before moving away. This resembles an elliptic movement of the solution around the stationary state, consistently with the idea that the periodic solution arises from a Hopf bifurcation.

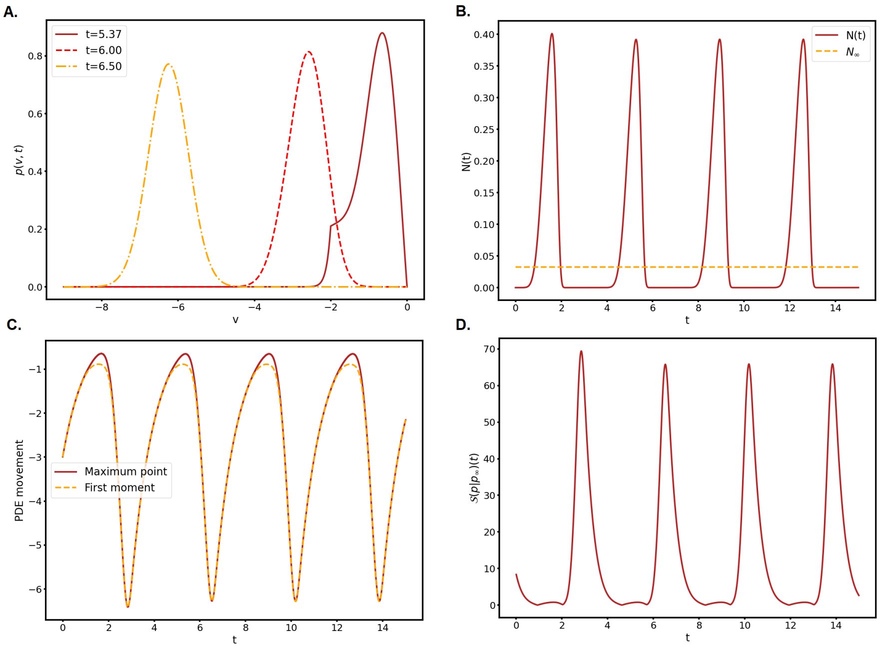

Figure 4: Simulation of the PDE (1.4) in Setup 5.2 with $b = -35$; (A.) solution at three different times; (B.) evolution in time of the firing rate $N(t)$; (C.) evolution in time of the first moment (average) and the maximum point of the solution; (D.) evolution in time of the logarithmic relative entropy $\mathcal{S}(p|p_\infty)(t)$.

In Figure 3 we validate numerically Conjecture 2.3 in Setup 5.1. We compute the solution of Equation (1.4) for different values of b ranging from -1000 to -50. Figure 3A shows the evolution in b of the L^∞ and L^1 norms of $t \mapsto N(t)$ over a period. The L^1 norm is computed by averaging over multiple periods. Figure 3B shows what the firing rate $N(t)$ looks like in the extreme case $b = -1000$. It was not possible to explore larger order of magnitude for $|b|$ because the numerical solution blows-up in finite time when the nonlinearity becomes too large. It is not possible to perform a similar study in Setup 5.2 because of higher numerical instability.

In Figure 4 we simulate the PDE (1.4) in Setup 5.2 with $b = -35$. We display the same information as in the previous setting and we observe that the solution still converges towards a periodic solution. However, this case is numerically unstable for two main reasons:

- the solution comes very close to V_F and the firing rate can change abruptly from near 0 to comparatively high values, which requires a very small time step Δt;
- the solution is pushed far away from V_F at high speed and it is thus necessary to have a large numerical domain, which in turn implies a prohibitive number of space grid points.

For these reasons, increasing more the value $|b|$ makes the numerical computation intractable and the simulation becomes unreliable. The development of a more efficient scheme based upon the logarithmic rather than quadratic relative entropy will be the subject of future research. From a biological point of view, taking $V_F = 0$ means that there is a strong external excitatory input favouring discharges of the neurons at lower potentials. It can be seen by applying to the equation the rescaling

$$\bar{p}(v,t) = p(\eta v + \zeta, t),$$

with $\eta, \zeta > 0$, which allows to choose the values of V_F and V_R at the price of replacing the flux function $-v + bN(t-d)$ by $-v + \nu_{ext} + bN(t-d)$ for some $\nu_{ext} \in (-\infty, +\infty)$.

5.2 Comparison with the ODE $c' + c = bN(t-d)$

We now investigate the decomposition $p(v,t) = \phi(v - c(t)) + R(v,t)$, where the function $c(t)$ is the solution to the Ordinary Differential Equation

$$c'(t) + c(t) = bN(t-d), \tag{5.3}$$

where $N(t)$ is the firing rate of the PDE (1.4).

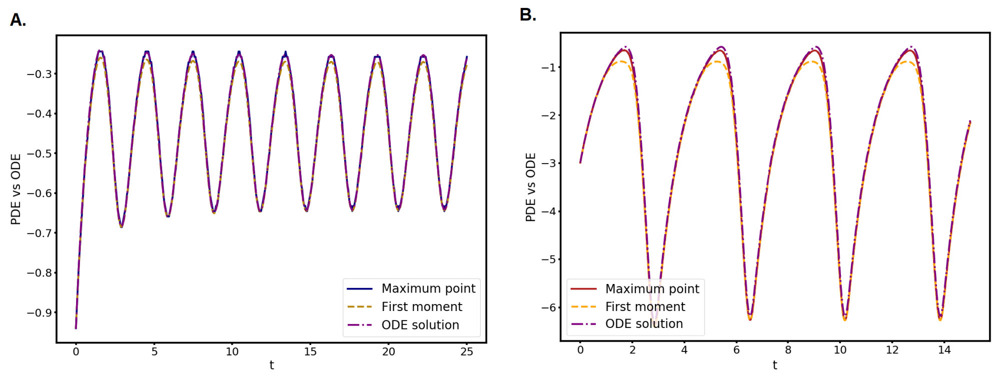

Figure 5: Comparison between the maximum point and first moment of the PDE (1.4) and the ODE (5.3); (A.) Setup 5.1 with $b = -45$; (B.) Setup 5.2 with $b = -35$.

In Figure 5 we compare our two indicators of the movement of the PDE solution (the first moment and the maximum point) to the solution of this ODE. In Figure 5A we do so in Setup 5.1 with $b = -45$ and in Figure 5B we do so in Setup 5.2 with $b = -35$. In both cases, the ODE solution matches the two indicators, but when the first moment and the maximum point are distinct (closer to V_F), the ODE matches best the maximum point of the solution. For this reason, we will henceforth focus on comparisons between the maximum point of the PDE solution $p(v,t)$ and the value $c(t)$ of the ODE and later of the DDE.

In Figure 6, we look at the shape of the remainder term $R(v,t)$ for different values of t. In Setup 5.1 and with $b = -45$, the remainder term $R(v,t)$ is very small compared to the size of the solution, as can be seen in Figure 6A. We can also see in Figure 6B that the Gaussian wave

$$\phi(v - c(t)) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{(v-c(t))^2}{2a}}$$

is a very good approximation of the solution $p(v,t)$. In Setup 5.2 with $b = -35$, the approximation is less precise, as can be seen in Figures 6C and 6D. Note that the

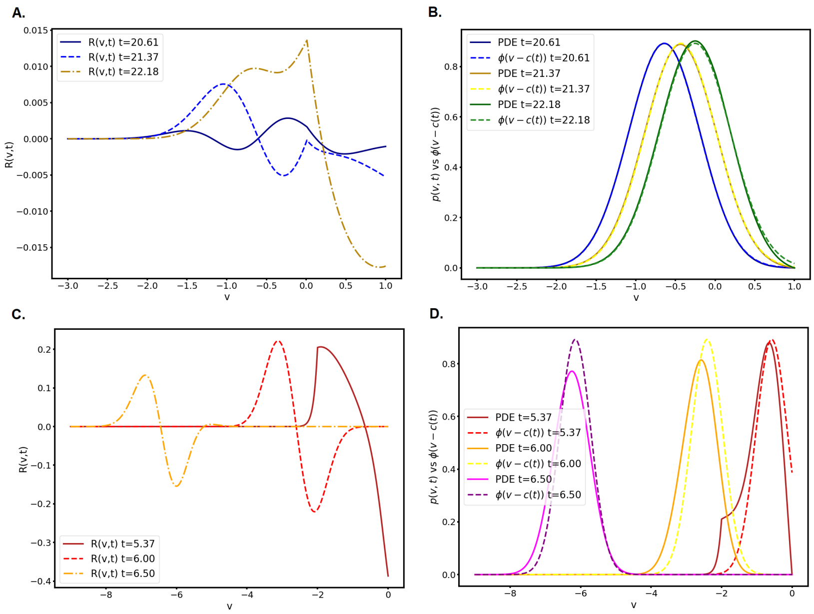

Figure 6: Comparison between the PDE (1.4) and the ODE (5.3); (A.) The remainder term $R(v,t) = p(v,t) - \phi(v - c(t))$ at different times in Setup 5.1 with $b = -45$; (B.) comparison of $p(v,t)$ and $\phi(v - c(t))$ at different times in Setup 5.1 with $b = -45$; (C.) The remainder term $R(v,t)$ at different times in Setup 5.2 with $b = -35$ (D.) comparison of $p(v,t)$ and $\phi(v - c(t))$ at different times in Setup 5.2 with $b = -35$.

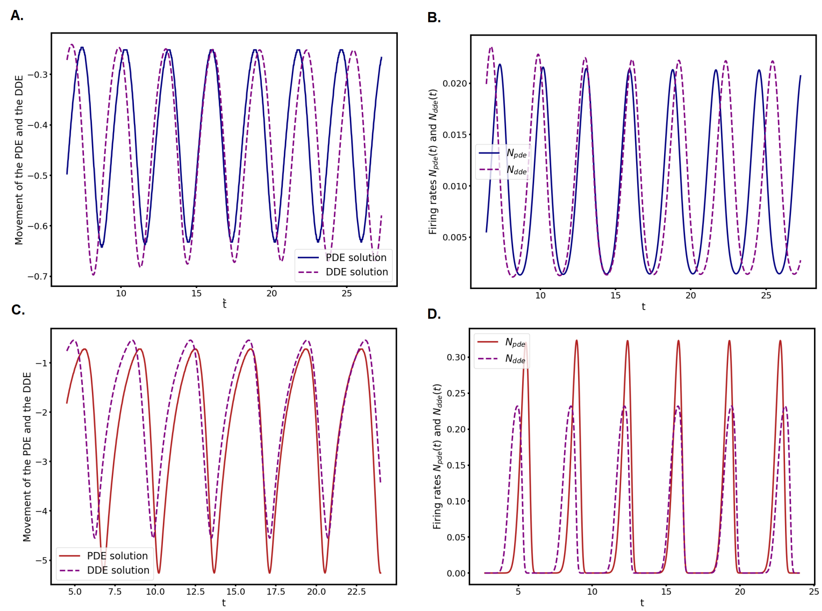

Figure 7: Comparison between the maximum point of the PDE (1.4) and the DDE (5.4); (A.) Setup 5.1 with $b = -45$; (B.) comparison of the firing rates $N(t)$ and $\mathcal{N}(c(t))$ in Setup 5.1 with $b = -45$; (C.) Setup 5.2 with $b = -35$; (D.) comparison of the firing rates $N(t)$ and $\mathcal{N}(c(t))$ in Setup 5.2 with $b = -35$.

variance of the solution is less stable in this case than it was in Setup 5.1. This can be explained by the fact that the maximal value of the firing rate and the range of motions of the solution are higher. Both of them play a role in Equation (2.12) which describes the evolution in time of the variance.

5.3 Comparison with the DDE $c' + c = b\mathcal{N}(c(t-d))$

Let us now consider the solution $c(t)$ of the Delay Differential Equation

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \qquad \mathcal{N}(c) = \frac{1}{\sqrt{2\pi a}}(V_F - c)e^{-\frac{(V_F-c)^2}{2a}}. \tag{5.4}$$

We approximate the solution by treating the ODE part with a Cranck-Nickolson scheme. This scheme is robust and efficient for linear ODEs and it allows us to compute the solution on the same time grid as the PDE without relying upon a black-box DDE solver. If j_d is such that $d \simeq t^{j_d}$, and if we denote c^j the approximation of t^j, we compute

$$c^{j+1} = \frac{1-\frac{\Delta t}{2}}{1+\frac{\Delta t}{2}}\, c^j + \frac{\Delta t}{1+\frac{\Delta t}{2}}\, b\mathcal{N}(c^{j-j_d}),$$

with $\Delta t = t^{j+1} - t^j$. We take Δt constant.

In Figure 7, we compare the maximum value of the solution $p(v,t)$ of the PDE to the solution $c(t)$ of the DDE. Figure 7A is in Setup 5.1 with $b = -45$ and Figure 7C in Setup 5.2 with $b = -35$. Figure 7B and Figure 7D show the comparison between the PDE firing rate $N(t)$ and the DDE firing rate $\mathcal{N}(c(t))$. We can see that the PDE and the DDE are close, but a small difference in the value of the time period makes the curves separate after a short time. The ratio between the periods for the DDE and the PDE can be computed numerically. In Setup 5.1 with $b = -45$, the ratio is

$$\frac{T_{DDE}}{T_{PDE}} \simeq \frac{3.16}{3.09} \simeq 1.023,$$

which is a 2.3% difference. In Setup 5.2 with $b = -35$, the ratio is

$$\frac{T_{DDE}}{T_{PDE}} \simeq \frac{3.61}{3.3.45} \simeq 1.046$$

which is about 4.6%.

5.4 Numerical exploration of the DDE

The advantage of the DDE over the PDE is that we can simulate it without any trouble for very large values of $|b|$. The numerical solutions seem to be reliable even when $b < -10^6$.

In Figure 8 we explore the DDE solutions for different orders of magnitude of b in Setup 5.1. We can see that the period of the solution increases slowly (logarithmically) with b. The range of the motion (L^∞ norm of c over a period) also increases slowly when b tends to $-\infty$, but faster than the period.

In Figure 9 we perform the same exploration for different values of b in Setup 5.2. We can observe again that the period of the oscillations grows logarithmically with respect to $|b|$. For this setting, it matches the theoretical information given by Theorem 4.1. Indeed, this result allows us to predict that there exists $C_M > 0$ such that the period of the oscillations evolves like

$$T_b \simeq C_M \log\left(-\frac{b}{\sqrt{2\pi a}}\right)$$

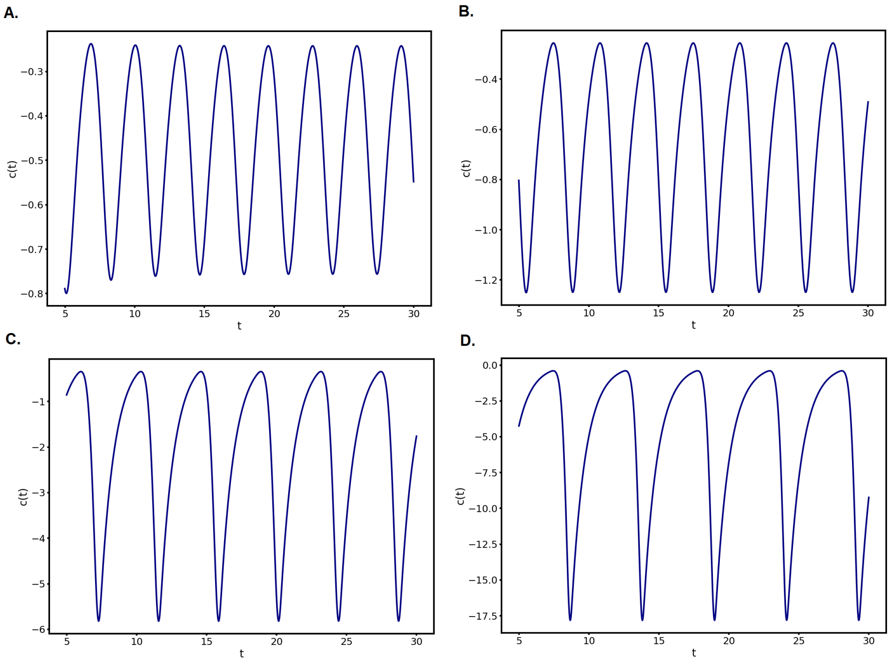

Figure 8: Simulation of the DDE (5.4) in Setup 5.1 for different values of b; (A.) $b = -50$; (B.) $b = -100$; (C.) $b = -1000$; (D.) $b = -5000$.

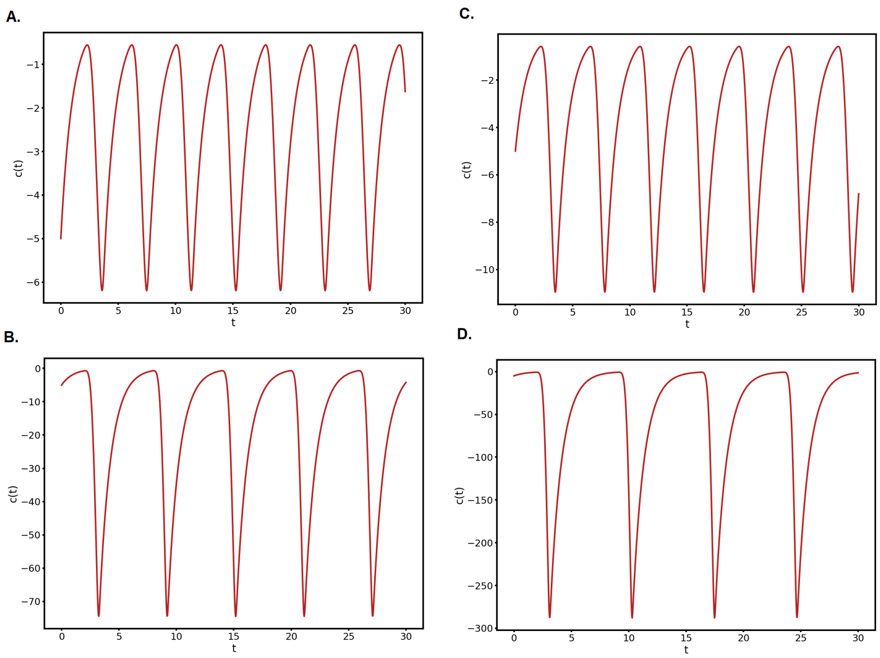

Figure 9: Simulation of the DDE (5.4) in Setup 5.2 for different values of b; (A.) $b = -50$; (B.) $b = -100$; (C.) $b = -1000$; (D.) $b = -5000$.

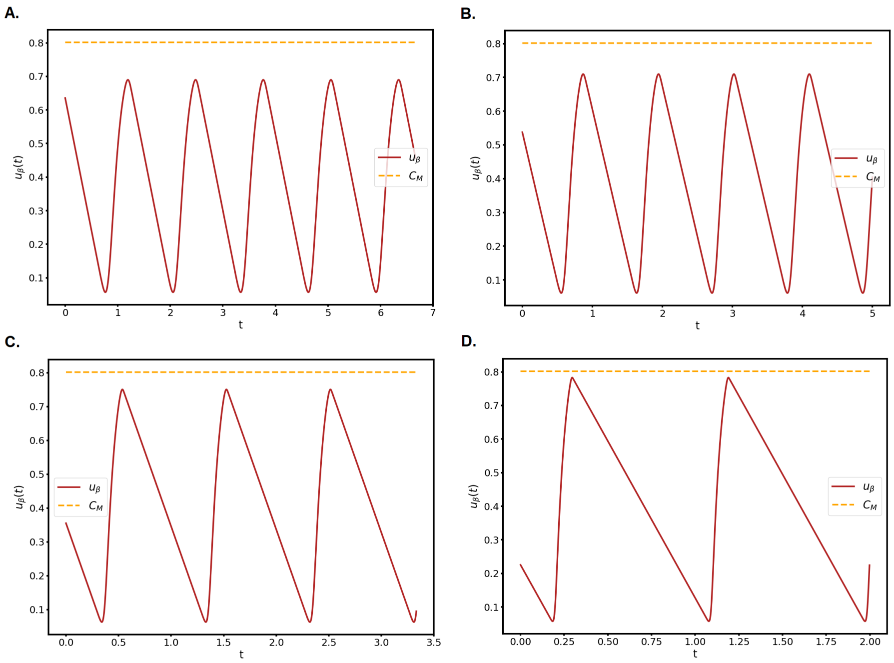

Figure 10: The solution $u_\beta(t)$ of the DDE (5.4) in Setup 5.2 in rescaled variables for different values of β; (A.) $\beta = 0.26331$, which is $b = -50$; (B.) $\beta = 0.22267$, which is $b = -100$; (C.) $\beta = 0.14719$, which is $b = -1000$; (D.) $\beta = 0.09340$, which is $b = -50000$.

and the range of the motion evolves like

$$\sqrt{a}\left(-\frac{b}{\sqrt{2\pi a}}\right)^{C_M}.$$

Notice that when we derived these expressions in the previous section we didn't have the $\sqrt{2\pi a}$ because we rescaled b linearly in the theoretical results to make the proofs easier to read.

In order to estimate the value of C_M, we apply the rescaling (4.2) to the numerical solutions of the DDE in Setup 5.2 and we set an immense value for $|b|$; past 10^6, the value stabilises around $C_M \simeq 0.8015$. In Figure 10, we display the solution of the DDE in rescaled variables for different values of $\beta = 1/\log(\frac{-b}{\sqrt{2\pi a}})$. We can see the convergence of the solution u_β towards the theoretical explicit profile of Theorem 4.1.

Given this approximate value of C_M, we compare in Figure 11 the range of the motion of the solution $c(t)$ of the DDE with the theoretical asymptote $\sqrt{a}\left(-\frac{b}{\sqrt{2\pi a}}\right)^{C_M}$.

6 Conclusion

In this article, we studied the periodic solutions of the delayed NNLIF model. Previous numerical simulations had indicated that the delayed NNLIF model can give rise to periodic solutions in the inhibitory case ($b < 0$), which is a key for the understanding of fast global oscillations in networks of weakly firing inhibitory neurons. There wasn't yet any analytical insight on this topic. In the simulations, these periodic solutions exhibit a Gaussian shape.

Based upon heuristic arguments, partial results and numerics, we introduced an associate delay differential equation which depicts the periodic movement of the centre

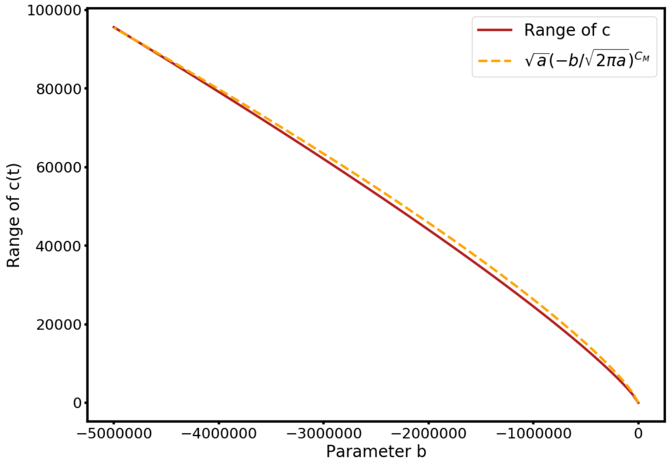

Figure 11: Range of $c(t)$ in Setup 5.2 depending on the value of b; comparison with the asymptote $\sqrt{a}\left(-\frac{b}{\sqrt{2\pi a}}\right)^{C_M}$ with the numerical value $C_M = 0.8015$

of the Gaussian wave. We proved rigorously that there exist periodic solutions to this associate equation.

Since our partial results indicate that the associate equation is valid asymptotically in $b \to -\infty$, we provided a rigorous result on the asymptotic behaviour of the solutions of this DDE. To do so, we used an appropriate change of variable and decomposed the asymptotic dynamics into distinct parts.

In this work, we didn't address two difficult open questions. First, in order to complete our partial results on the asymptotic convergence of the periodic solutions of the complete NNLIF system towards our approximate Gaussian wave, it remains to prove Conjecture 2.3. This will be the subject of further research and it will require the development of new techniques. Then, we only portrayed the inhibitory case $b < 0$. The results of [16] in the excitatory random discharge model indicate that our method could apply when $b > 0$, but in this case the shape of the solutions could be more complex than a Gaussian wave and there is no possibility to proceed asymptotically.

Note also that we investigated a Hopf bifurcation in terms of the parameter d. Previous heuristic studies (*e.g.* [5]) indicate that there is also a Hopf bifurcation in terms of the parameter b. Another subject for future investigation is the search for codimension 2 bifurcations in the parameters (b, d) for this delayed differential equation. The literature about DDEs contains a lot of tools to study bifurcations (see namely [38]) and a full rigorous study of the bifurcations in Equation (5.4) should be possible.

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