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# Theoretical study of the emergence of periodic solutions for the inhibitory NNLIFF neuron model with synaptic delay.

Kota Ikeda\*      Pierre Roux†      Delphine Salort ‡      Didier Smets §

March 3, 2021

## Abstract

Among other models aimed at understanding self-sustained oscillations in neural networks, the NNLIFF model with synaptic delay was developed twenty years ago to model fast global oscillations in networks of weakly firing inhibitory neurons. Periodic solutions were numerically observed in this model, but despite intensive study of this model by researchers in PDEs and probability, there was up to date no analytical result on this topic. In this article, we propose to approximate formally these solutions by a Gaussian wave whose periodic movement is described by an associate difference-differential equation. We prove the existence of a periodic solution for the position in time of the centre of the Gaussian wave and we give a rigorous asymptotic result on these solutions when the connectivity parameter  $b$  goes to  $-\infty$ . Last, we provide heuristic and numerical evidence of the validity of our approximation.

**Key-words:** Leaky integrate and fire models, noise, blow-up, neural networks, delay, global existence.  
**AMS Class. No:** 35K60, 82C31, 92B20, 35Q84

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# 1 Introduction

Self-sustained oscillations in neural networks are key processes in the brain and several studies proved their ubiquity ([2], [36], [20] [6] among other reviews). These spontaneous (not elicited by external inputs), stable and periodic collective behaviours play a pivotal role in vital processes like respiratory rhythmogenesis ([1]). In many cases, the activity arises from intrinsically oscillating neurons, but periodic spontaneous activity can occur in networks where individual noisy excitable neurons fire sporadically. This kind of collective behaviour is difficult to grasp without a self-contained mathematical model and many PDE models were studied numerically or analytically: the time-elapsd model [30], [31], [28], the kinetic Fitzhugh-Nagumo model [27], conductance models [32], [33]...

One of the most striking cases of spontaneous periodic activity is fast global oscillations (gamma frequency range, see [7]) in networks of slowly firing inhibitory neurons. A milestone was reached on this topic when Brunel and Hakim proposed in [5] an approach to simulate numerically and understand qualitatively those fast global oscillations: the Nonlinear Noisy Leaky Integrate & Fire (NNLIF<sup>1</sup>) neuron model.

They start from the classical Lapique Integrate & Fire model ([25]):

$$C_m \frac{dV}{dt} = -g_L(V - V_L) + I(t), \quad (1.1)$$

where  $C_m$  is the capacitance of the membrane,  $g_L$  the leak conductance and  $V_L \simeq -70mV$  the leak potential. The synaptic current is a stochastic process of the form

$$I(t) = J_E \sum_{i=1}^{C_E} \sum_{j \in \mathbb{N}} \delta(t - t_{E_j}^i) + J_I \sum_{i=1}^{C_I} \sum_{j \in \mathbb{N}} \delta(t - t_{I_j}^i), \quad (1.2)$$

where  $\delta$  is the Dirac measure,  $J_E$  and  $J_I$  the strengths of excitatory and inhibitory synapses,  $C_E$  and  $C_I$  the numbers of excitatory and inhibitory pre-synaptic neurons and  $t_{E_j}^i$ ,  $t_{I_j}^i$  the random times of the  $j^{\text{th}}$  discharge from the  $i^{\text{th}}$  pre-synaptic excitatory or inhibitory neuron.

When a neuron reach the discharge potential  $V_F \simeq -50mV$ , it emits an action potential and then returns to the reset potential  $V_R \simeq -60mV$ . This model is hard to study in this form. Thus, many authors assume that discharges follow a Poisson law and do a diffusive approximation for a great number of neurons ([5], [4],...). Denoting  $b = C_E J_E - C_I J_I$  and  $\sigma_C^2 = (C_E J_E^2 + C_I J_I^2)$  and rescaling so to have  $C_m = g_L = 1$ , it yields the stochastic differential equation

$$dV = (-V + V_L + b\nu)dt + \sigma_C dB_t, \quad V < V_F, \quad (1.3)$$

where  $B_t$  is a standard Brownian motion, with the jump process and discharge intensity

$$\limsup_{t \rightarrow t_0^-} V(t) = V_F \implies \liminf_{t \rightarrow t_0^+} V(t) = V_R \quad \text{and} \quad \nu = \nu_{ext} + N(t).$$

The quantity  $N(t)$  is the flux of neurons crossing the firing potential  $V_F$ .

The so-called NNLIF model is associated to the probability density of (1.3) and it writes:

$$\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} [(-v + bN(t-d))p(v, t)] - a \frac{\partial^2 p}{\partial v^2}(v, t) = N(t)\delta(v - V_R), \quad v \leq V_F, \quad (1.4)$$

with firing rate

$$N(t) = -a \frac{\partial p}{\partial v}(V_F, t) \geq 0, \quad (1.5)$$

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<sup>1</sup>In some articles, the first N stands for Network.

and boundary conditions

$$p(v, 0) = p^0(v) \geq 0, \quad \int_{-\infty}^{V_F} p^0(v) dv = 1 \quad \text{and} \quad p(V_F, t) = p(-\infty, t) = 0. \quad (1.6)$$

The function  $p(\cdot, t)$  represents the probability density of the electric potential of a randomly chosen neuron at time  $t$ . The parameter  $a = \frac{\sigma_c^2}{2} > 0$  is the diffusion coefficient and  $b$  is the *connectivity parameter*. If  $b$  is positive, the neural network is said to be average-excitatory; if  $b$  is negative the network is said to be average-inhibitory. Last, the parameter  $d \geq 0$  is the synaptic delay: the mean time it takes for a spike to go from one neuron to another in the network.

Both problems (1.3) and (1.4) were intensively studied from a mathematical perspective ([34, 18, 17, 23, 22, 29, 16, 15] for (1.3) and variants; [8], [9],[13],[14],[35],[10],[11],[12] for (1.4) and variants) and the recent article [26] eventually made a rigorous justification of the link between these systems in the strong sense.

The emergence of periodic solutions in NNLIF models is a crucial question and although it was numerically investigated there are up to our knowledge very few theoretical results on this topic. Indeed, these complex dynamics can solely occur when the strength  $|b|$  of the nonlinearity is sufficient. Such a strong nonlinearity is hard to tackle mathematically. Within the scope of this article, we propose to bring new theoretical insights to this question in the case of very inhibitory networks, *i.e.*  $b \ll 0$ . Before getting into the details of our methods and results, let us summarise what is already known on these periodic solutions.

Several numerical simulations allowed previous studies to identify which conditions can elicit periodic solutions in NNLIF-type models:

- in the classical NNLIF system without delay ( $d = 0$ ), no periodic solution was ever observed. Adding a refractory period ([9]) or coupling excitatory-inhibitory systems ([11, 12]) doesn't suffice to make them appear;
- in the high connectivity regime ( $b > 0$  large), it is proved in [10][Theorem 5.4] that there is no periodic solutions neither with delay ( $d > 0$ ) nor without delay ( $d = 0$ );
- in the excitatory case  $b > 0$ , periodic solutions were not observed when there is a delay  $d > 0$ , but if there is a delay and a refractory period they appear ([12, 24]). They were also observed when a random discharge mechanism (without delay) is taken into account ([9, 16, 15]). Periodic solutions for the random discharge model (without delay) were very recently constructed analytically in [15].
- in the inhibitory case  $b < 0$  with delay  $d > 0$ , periodic solutions are observed as soon as  $|b|, d$  are large enough (regardless of the presence of a refractory period), [5, 12, 24].

In this article, we put ourselves in the context of a very inhibitory network:  $b \ll 0$  with a positive delay  $d > 0$ . Our strategy is based upon the following conjecture: in the limit  $b \rightarrow -\infty$  the delayed NNLIF system can be approximated by the sum of a periodic wave  $\varphi(v, t) = \phi(v - c(t))$  plus a remainder term  $R(v, t)$ :

$$p(v, t) = \phi(v - c(t)) + R(v, t), \quad (1.7)$$

with

$$c'(t) + c(t) = bN(t - d). \quad (1.8)$$

The remainder term  $R(v, t)$  is expected to vanish in some sense when  $b \rightarrow -\infty$ . However, this problem is still strongly coupled since the periodic movement  $c(t)$  depends upon the firing rate

$$N(t) = -a \frac{\partial p}{\partial v}(V_F, t) = -a \frac{\partial \varphi}{\partial v}(V_F, t) - a \frac{\partial R}{\partial v}(V_F, t).$$

In order to make the problem tractable, we do a second approximation, which is more involved: we assume that in the limit  $b \rightarrow -\infty$  we have  $bN(t) \simeq -ba\partial_v\varphi(V_F, t)$ . Hence, we obtain an autonomous difference-differential equation:

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \quad (1.9)$$

where this new firing rate can be computed in terms of  $c$ :

$$\mathcal{N}(c) = \frac{1}{\sqrt{2\pi a}}(V_F - c)e^{-\frac{(V_F - c)^2}{2a}}. \quad (1.10)$$

Implementing this strategy requires to answer two separate questions.

- The first one, which is the hardest and which we leave open, is to justify rigorously this approximations in the limit  $b \rightarrow -\infty$ . Concerning the first approximation (1.7), we provide a partial answer to this question by proving that if we assume  $N(t) \rightarrow 0$ , then the remainder  $R$  goes to 0 in some sense. The conjecture  $N(t) \rightarrow 0$  in the sense that we need is observed numerically. Our second approximation (1.9) is much more involved: since  $bN(t) = -ba\partial_v\phi(V_F, t) - ba\partial_v R(V_F, t)$ , proving our method to be valid requires to prove that both  $R$  and  $b\partial_v R(V_F, t)$  converge appropriately to 0 when  $b \rightarrow -\infty$  (*i.e.*  $R(v, t) = o(1)$  and  $\partial_v R(V_F, t) = o(\frac{1}{b})$ ). This difficult question is beyond the scope of our paper.
- The second one is to study theoretically the solution  $c(t)$  of the associate difference-differential equation (1.9). We provide here a thorough study of this associate equation. On the one hand, we prove first a general existence result of periodic solutions to this equation in the case  $V_F^2 \geq a$  and  $b < b^*, d > d^*$  where  $|b^*|, d^*$  are large enough (see Theorem 3.1). On the other hand, we looked further into the dynamics of the solutions of (1.9) when  $b \rightarrow -\infty$ ; setting  $V_F = 0, d = 1$  for the sake of clearness. We construct in rescaled variables an explicit periodic asymptotic profile  $P(t)$  towards which the solutions converge when  $b \rightarrow -\infty$  (see Theorem 4.1) This profile gives us a precise idea of the behaviour of the solutions, in accordance with numerical simulations: the periodic solutions are projected very fast away from  $V_F$  (at a distance  $\mathcal{O}((-b)^{C_M})$ ,  $C_M \in (0, 1)$ ), then they come back at an exponential speed over a period of time in  $\mathcal{O}(\log(-b))$ , and so on back and forth.

Note that our explicit periodic asymptotic profile  $c(t) \rightarrow P(t)$  is in the case  $V_F = 0$  and our existence result for a periodic solution  $c(t)$  is in the case  $V_F > 0$ . Numerical simulations indicate that both these results are valid in a larger range of parameters: periodic solutions arise when  $V_F = 0$  and  $d = 1$  and the asymptotic profile is satisfied when  $V_F > 0$ .

A method was proposed in [21] for proving the existence of periodic solutions of difference-differential equations: it consists of studying the correspondent eigenvalues and applying Browder's fixed-point theorem on an *ad hoc* functional. We use a similar method in order to prove analytically that there are periodic solutions to equation (1.9). Note that, as Brunel and Hakim guessed in their original article [5], the apparition of fast global oscillations in an inhibitory network is likely to be a Hopf bifurcation. However, [5] suggests a bifurcation in  $b$  and our result indicates a bifurcation in  $d$  when  $b$  is negative and large enough. From a mathematical point of view: two opposite complex eigenvalues cross the imaginary axis. Note that the periodic solutions of the random discharge NNLIF model constructed in [15] appear through a Hopf bifurcation in  $b$  (and not in  $d$ ).

In order to carry out our study of the asymptotic profile  $P$  when the connectivity  $b$  goes to  $-\infty$ , we make a handy rescaling and we decompose the evolution of such solutions in different phases (growth phase, decay phase, etc). Then, we use  $b$ -dependent estimates to prove convergence towards an explicit profile. Our numerical simulations indicate that this asymptotic profile is a good approximation even for not so large values of  $|b|$ .

This article is organised as follows. In Section 2 we derive (heuristically) the associate difference-differential equation (1.9) and we justify its relevance in a heuristic and numerical way. In Section 3, we use the method of [21] to prove the existence of periodic solutions for equation (1.9). Last, in Section 4, we prove a result on the asymptotic profile of solutions of (1.9) when the parameter  $b$  goes to  $-\infty$ .

## 2 An associate difference-differential equation

### 2.1 Formal derivation of the wave-type solution

Note first that we can rewrite (1.4) on the whole real line as follows:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial v} [(-v + bN(t-d))p] - a \frac{\partial^2 p}{\partial v^2} = \delta_{V_R} N(t) - \delta_{V_F} N(t), \quad v \in \mathbb{R}, t > 0,$$

$$p(-\infty, t) = p(+\infty, t) = 0, \quad N(t) = -a \frac{\partial p}{\partial v}(V_F, t), \quad t > 0,$$

$$p(v, 0) = p^0(v) \geq 0, \quad v \in \mathbb{R}, \quad \int_{-\infty}^{+\infty} p^0(v) dv = 1.$$

If  $p^0 \equiv 0$  on  $[V_F, +\infty)$  then for all  $t > 0$ ,  $p(\cdot, t) \equiv 0$  on  $[V_F, +\infty)$ .

As Lemma 2.3 below and numerical simulations in the literature indicate (see *e.g.* [5]), when  $b \ll 0$  the firing rate  $N$  tends to be low. As a consequence, the term  $(\delta_{V_R} - \delta_{V_F})N(t)$  is of lesser importance in the equation. Hence, we are looking for a solution composed of a periodic wave of unit mass  $\varphi(v, t) = \phi(v - c(t))$  defined on  $\mathbb{R}$  plus a corrective term  $R(v, t)$  needed to account for the boundary and reset conditions:

$$p(v, t) = \phi(v - c(t)) + R(v, t), \tag{2.1}$$

and it is sound to look for the wave-type periodic solution  $\varphi : (v, t) \mapsto \phi(v - c(t))$  as a solution of equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial v} [(-v + bN(t-d))\varphi] - a \frac{\partial^2 \varphi}{\partial v^2} = 0, & v \in \mathbb{R}, t > 0 \\ \varphi(-\infty, t) = \varphi(+\infty, t) = 0, & \int_{-\infty}^{+\infty} \varphi(v, t) dv = 1. \end{cases} \tag{2.2}$$

Therefore, the remainder term  $R : (v, t) \mapsto R(v, t)$  must be a solution of

$$\begin{cases} \frac{\partial R}{\partial t} + \frac{\partial}{\partial v} [(-v + bN(t-d))R] - a \frac{\partial^2 R}{\partial v^2} = \delta_{V_R} N(t) - \delta_{V_F} N(t), & v \in \mathbb{R}, t > 0 \\ R(-\infty, t) = R(+\infty, t) = 0, & R(v, 0) = p^0(v) - \varphi(v, 0), \quad v \in \mathbb{R}, t > 0. \end{cases} \tag{2.3}$$

Unfortunately, systems (2.2) and (2.3) are strongly coupled through the firing rate associated to  $p(v, t)$ :

$$N(t) = -a \partial_v \varphi(V_F, t) - a \partial_v R(V_F, t).$$

However, we have the following result about wave-type solutions of (2.2):

**Proposition 2.1** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$\phi(v) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{v^2}{2a}},$$

Let  $c$  be a solution of

$$c'(t) + c(t) = bN(t - d)$$

Then the function  $\varphi$  defined by  $\varphi(v, t) = \phi(v - c(t))$  is a solution of (2.2).

**Proof.** Replacing  $\varphi(v, t) = \phi(v - c(t))$  in (2.2), we get for all  $v \in \mathbb{R}$ ,

$$-\phi(v - c(t)) - (v + c(t) + c'(t) - bN(t - d))\phi'(v - c(t)) - a\phi''(v - c(t)) = 0,$$

By change of variables, it yields

$$-\phi(v) - (v + c'(t) + c(t) - bN(t - d))\phi'(v) - a\phi''(v) = 0.$$

Assume that the function  $c$  satisfies  $c'(t) + c(t) = bN(t - d)$ . Then,  $-\phi(v) - v\phi'(v) - a\phi''(v) = 0$ , that is to say  $-(v\phi(v))' = a\phi''(v)$ . We integrate and apply boundary conditions:

$$\phi'(v) = -\frac{v}{a}\phi(v).$$

The positive solutions of this equation are of the form  $\phi(v) = C_g e^{-\frac{v^2}{2a}}$ , where  $C_g > 0$  is any given positive constant. Since

$$\int_{-\infty}^{+\infty} C_g e^{-\frac{v^2}{2a}} dv = \int_{-\infty}^{+\infty} \varphi(v, t) dv = 1,$$

we have  $C_g = (2\pi a)^{-\frac{1}{2}}$ . □

In order to make the problem autonomous and thus theoretically tractable, we make a last and more involved assumption: that  $b\partial_v R(V_F, t) \simeq 0$  in an appropriate sense when  $b \rightarrow -\infty$ . Hence, we write

$$p(v, t) \simeq \tilde{\varphi}(v, t) + \tilde{R}(v, t), \quad (2.4)$$

and we replace  $N(t)$  by a simpler firing rate

$$N(t) = -a \frac{\partial \tilde{\varphi}}{\partial v}(V_F, t). \quad (2.5)$$

We can now consider the simpler autonomous equation

$$\begin{cases} \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial}{\partial v} \left[ (-v + bN(t - d)) \tilde{\varphi} \right] - a \frac{\partial^2 \tilde{\varphi}}{\partial v^2} = 0, & v \in \mathbb{R}, t > 0 \\ N(t) = -a \frac{\partial \tilde{\varphi}}{\partial v}(V_F, t) \\ \tilde{\varphi}(-\infty, t) = \tilde{\varphi}(+\infty, t) = 0, & \int_{-\infty}^{+\infty} \tilde{\varphi}(v, t) dv = 1. \end{cases} \quad (2.6)$$

Proceeding as in the proof of Theorem 2.1, we obtain:

**Proposition 2.2** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$\phi(v) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{v^2}{2a}},$$

Let  $c$  be a solution of

$$c'(t) + c(t) = \frac{b}{\sqrt{2\pi a}} (V_F - c(t - d)) \exp\left(-\frac{(V_F - c(t - d))^2}{2a}\right)$$

Then the function  $\tilde{\varphi}$  defined by  $\tilde{\varphi}(v, t) = \phi(v - c(t))$  is a solution of (2.6).

Note that  $\tilde{R}(v, t) \neq R(v, t)$  and that we have absolutely no information on the behaviour of  $\tilde{R}$ . Proving that it converges to 0 might be very difficult and it is beyond the scope of the paper.

We are now reduced to prove the existence of a periodic solution of the difference-differential equation

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \quad (2.7)$$

where the function  $\mathcal{N}$  is given by

$$\mathcal{N}(c) = \frac{1}{\sqrt{2\pi a}}(V_F - c) \exp\left(-\frac{(V_F - c)^2}{2a}\right).$$

In Section 3 we prove the following existence result for this difference-differential equation and in Section 4 we analyse rigorously its asymptotic behaviour when  $b \rightarrow -\infty$ .

## 2.2 Partial results on the remainder term $R$

We will give here some partial results on the remainder term  $R$  in the first approximation (2.1). Of course, it tells us nothing about the remainder  $\tilde{R}$  in the second approximation (2.4). Note first that as the following lemma indicates, the stationary firing rate vanishes when  $b$  tends to  $-\infty$ .

**Lemma 2.3** *For all  $b \leq 0$ , denote  $N_\infty^b$  the firing rate of the unique stationary state of (1.4) with connectivity parameter  $b$ . Then, the function  $b \mapsto N_\infty^b$  is increasing on  $(-\infty, 0]$  and*

$$\lim_{b \rightarrow -\infty} N_\infty^b = 0.$$

**Proof.** We know from [8] that when  $b \leq 0$  there exists a unique stationary state of (1.4) and that it satisfies

$$\frac{1}{N_\infty^b} = \int_0^{+\infty} \frac{\varepsilon^{-\frac{s^2}{2}}}{s} \varepsilon^{-\frac{bN_\infty^b}{\sqrt{a}}s} \left( \varepsilon^{s\frac{V_F}{\sqrt{a}}} - \varepsilon^{s\frac{V_R}{\sqrt{a}}} \right) ds.$$

Let us define two functions:

$$I : (b, N) \mapsto \int_0^{+\infty} \frac{\varepsilon^{-\frac{s^2}{2}}}{s} \varepsilon^{-\frac{bN}{\sqrt{a}}s} \left( \varepsilon^{s\frac{V_F}{\sqrt{a}}} - \varepsilon^{s\frac{V_R}{\sqrt{a}}} \right) ds \quad \text{and} \quad J : (b, N) \mapsto I(b, N) - \frac{1}{N}.$$

For all  $b < 0$ , equation  $J(b, N) = 0$  has a unique solution ([8]). The function  $J$  is smooth on  $(-\infty, 0) \times (0, +\infty)$  and for all  $(b, N) \in (-\infty, 0) \times (0, +\infty)$ ,

$$\frac{\partial J}{\partial N}(b, N) = -\frac{b}{\sqrt{a}}I(b, N) + \frac{1}{N^2} > 0 \quad \text{and} \quad \frac{\partial J}{\partial b}(b, N) = -\frac{N}{\sqrt{a}}I(b, N) < 0.$$

Hence, by the implicit functions theorem applied to equation  $J(b, N) = 0$ , the function  $b \mapsto N_\infty^b$  has the following derivative:

$$b \mapsto -\frac{\frac{\partial J}{\partial b}(b, N)}{\frac{\partial J}{\partial N}(b, N)} > 0.$$

Function  $b \mapsto N_\infty^b$  is thus increasing on  $(-\infty, 0)$ .

Then, note that for all  $N > 0$ , the function  $b \mapsto I(b, N)$  is non-increasing and

$$\lim_{b \rightarrow -\infty} I_N(b) = +\infty.$$

Function  $b \mapsto N_\infty^b$  being continuous, bounded by 0 and increasing, it has a limit  $N_\infty^* \geq 0$  when  $b$  goes



to  $-\infty$ . If we assume  $N_\infty^* > 0$ , then

$$0 = \lim_{b \rightarrow -\infty} J(b, N_\infty^*) = \lim_{b \rightarrow -\infty} I(b, N_\infty^*) - \frac{1}{N_\infty^*} = +\infty,$$

and we reached a contradiction. Thus, we must have  $\lim_{b \rightarrow -\infty} N_\infty^b = 0$ .  $\square$

We now make the following guess:

**Conjecture 2.4** *Let  $(p^0, N^0)$  be an initial condition. For all  $\varepsilon > 0$ , there exists some time  $T > 0$  and some connectivity parameter  $b < 0$  such that any solution  $(p, N)$  of (1.4) with parameter  $b$  satisfies*

$$\forall t > T, \quad \int_0^t e^{s-t} N(s) ds < \varepsilon.$$

With the previous lemma, this conjecture means that as  $b$  goes to  $-\infty$ ,  $N$  converges towards 0 in some sense. If we assume this conjecture holds, then we might expect that the remainder term  $R$  is thus asymptotically small when  $b$  goes to  $-\infty$ . More precisely, this conjecture implies:

**Proposition 2.5** *Assume that Conjecture 2.4 holds. Then, for all  $\varepsilon > 0$ , there exists  $b < 0$  and  $T > 0$  such that for all  $t > T$ ,*

$$\int_0^t e^{s-t} \|R(s)\|_{L^2}^2 ds < \varepsilon.$$

**Proof.** First, since the mass of  $p$  is conserved, we have

$$\int_{-\infty}^{+\infty} R(v, t) dv = \int_{-\infty}^{+\infty} p(v, t) dv - \int_{-\infty}^{+\infty} \varphi(v, t) dv = 1 - 1 = 0.$$

We know that  $R$  is solution of

$$\partial_t R + \partial_v((-v + bN(t-d))R) - a\partial_{vv}R = \delta_{v=V_R}N(t) - \delta_{v=V_F}N(t). \quad (2.8)$$

Define

$$U(t, v) = \int_{-\infty}^v R(t, w) dw.$$

We have  $U(t, \pm\infty) = 0$  and the equation satisfied by  $U$  is given by

$$\partial_t U + (-v + bN(t-d))\partial_v U - a\partial_{vv}U = \mathbf{1}_{(V_R, V_F)}N(t).$$

Multiplying the above equation by  $U$  and integrating, we find that

$$\frac{d}{dt} \|U\|_{L^2}^2 \leq -\|U\|_{L^2}^2 - 2a\|R\|_{L^2}^2 + 2N(t) \|U\|_{L^1(V_R, V_F)}.$$

We have

$$\begin{aligned} \|U\|_{L^1(V_R, V_F)} &= \int_{V_R}^{V_F} \left| \int_{-\infty}^v R(w, t) dw \right| dv \leq \int_{V_R}^{V_F} \int_{-\infty}^v |R(w, t)| dw dv \\ &\leq \int_{V_R}^{V_F} \int_{-\infty}^{+\infty} (p(w, t) + \phi(w, t)) dw dv \leq 2(V_F - V_R). \end{aligned} \quad (2.9)$$

This implies that

$$\frac{d}{dt} \|U\|_{L^2}^2 \leq -\|U\|_{L^2}^2 - 2a\|R\|_{L^2}^2 + 4(V_F - V_R)N(t).$$

Hence, we obtain that

$$2ae^{-t} \int_0^t e^s \|R(s)\|_{L^2}^2 ds \leq \|U(0)\|_{L^2}^2 e^{-t} + 4(V_F - V_R)e^{-t} \int_0^t e^s N(s) ds - \|U(t)\|_{L^2}^2.$$

If we choose  $T$  large enough and  $|b|$  large enough so to have for all  $t > T$ , applying 2.4 yields

$$\|U(0)\|_{L^2}^2 e^{-t} \leq a\varepsilon \quad \text{et} \quad 4(V_F - V_R)e^{-t} \int_0^t e^s N(s) ds \leq a\varepsilon,$$

and thus,

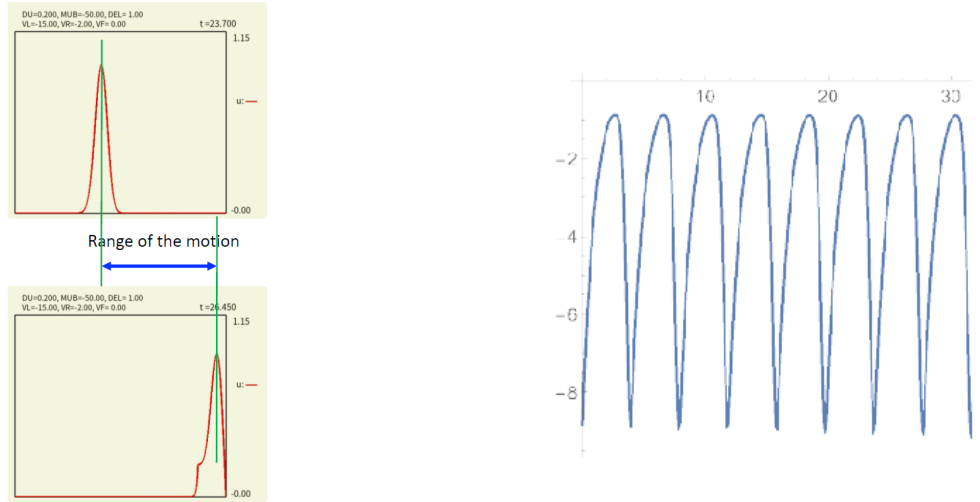
$$\int_0^t e^{s-t} \|R(s)\|_{L^2}^2 ds < \varepsilon.$$

□

### 2.3 Numerical evidence

We performed numerical simulations in order to compare the periodic Gaussian wave obtained as a solution of (2.6) and the solution of the original PDE (1.4). Complex and very effective numerical methods were designed to handle this equation ([8, 12, 24]) in the excitatory case. In the inhibitory delayed setting where blow-up cannot happen, a finite difference scheme provides a robust approximation. The diffusion part is treated implicitly and the drift part *via* a finite volume method.

First, we confirm previous numerical observations of periodic solutions ([12], [24]). These periodic solutions seem globally asymptotically stable: after a transitory period, any solution converges towards them. Moreover, they exhibit the expected Gaussian shape (Figure 1) as long as the centre of the Gaussian wave is not too close from  $V_R$  or  $V_F$ . As  $|b|$  increases the wave remains farther from  $V_F$ .



(a) Shape of the periodic solution. When the solution is far from  $-2$  it has a Gaussian shape. As we chose  $V_F = 0$ , it gets closer and loses its Gaussian shape until it moves back again.

(b) Moment of the solution in function of time. It provides an approximation for the movement of the center of the Gaussian wave.

Figure 1: Simulation with parameters  $V_F = 0$ ,  $a = 0.2$ ,  $b = -50$ ,  $d = 1$ ,  $V_R = -2$  and grid  $d_v = 0.015$ ,  $d_t = 0.0005$ .

In order to compare our approximate equation to the complete problem (1.4), we plotted the first

moment of the solution of the full problem and the solution of the difference-differential equation (2.7) on the same graph (Figure 2). As we can see, the period of the oscillations and the overall shape are the same; the height of these curves are different though.

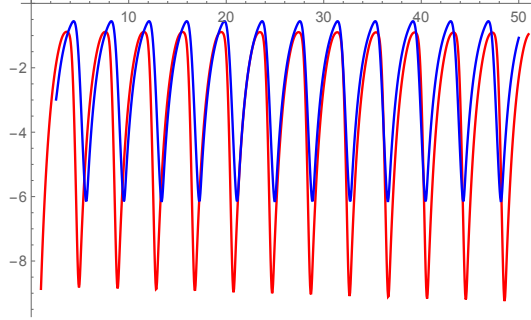


Figure 2: *In blue: moment of the solution of (1.4) with same parameters as previous graph. In red: corresponding solution of (2.7). Same parameters as Figure 1*

Last, we compared the numerical solution of the difference-differential equation (2.7) with the asymptotic profile obtained in Theorem 4.1 (Figure 3). For a reasonable value of  $b$  ( $b = -50$ ) the two curves match well.

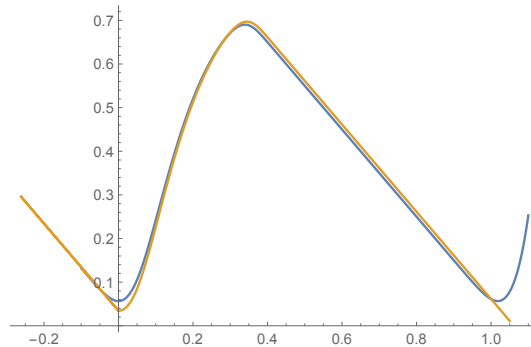


Figure 3: *Comparison between the numerical solution of equation (2.7) (blue curve) and the profile obtained in theorem 4.1 (orange curve) for  $V_F = 0$  and  $b = -50$ , that is to say  $\beta = 0.2556$*

### 3 Periodic solutions of the difference-differential equation

Here we prove the existence of a periodic solution of

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \quad (3.1)$$

where the function  $\mathcal{N}(c)$  is given by

$$\mathcal{N}(c) = (V_F - c) \exp\left(-\frac{(V_F - c)^2}{2a}\right),$$

and  $b < 0, d \geq 0, V_F \geq 0, a > 0$  are parameters.

We removed in this section the constant  $C_g = (2\pi a)^{-\frac{1}{2}}$  without loss of generality since the rescaling

$\bar{b} = b(2\pi a)^{-\frac{1}{2}}$  allows us to come back to the general case.

**Theorem 3.1** *Let  $c_* < 0$  be a stationary solution of (3.1). If  $V_F^2 \geq a$ , then there exist  $b_* < 0$  and  $d_* > 0$  such that (3.1) has a periodic solution  $c(t)$  for any  $b < b_*$  and  $d > d_*$ . Moreover, there is a negative constant  $c_0$  independent of  $d > d_*$  such that for all  $t > 0$ ,  $c_0 \leq c(t) < 0$ .*

Note that the form of this result is indicative of a Hopf-type bifurcation as suggested by Brunel and Hakim in their seminal article [5]: if we assume that  $|b|$  is large enough, that is to say larger than  $|b^*|$ , then when  $d$  crosses  $d^*$  two conjugate complex eigenvalues cross the imaginary axis and the stable stationary state is destabilised; a stable periodic cycle then appears which is a periodic solution. However, the Hopf bifurcation proposed by Brunel and Hakim was in term of the parameter  $b$ . A more complex dynamic of Hopf-Hopf type could happen when  $b$  and  $d$  evolve simultaneously, but this is beyond the scope of our paper. Note also that the Hopf bifurcation studied in [15] in the excitatory random discharge NNLIF model was along the parameter  $b$ .

We shall build upon the method of Hadeler and Tomiuk for a similar difference-differential equation ([21]). The main idea is to find the periodic solution as a *non-ejective fixed-point* of a functional  $\mathcal{F}$  on an appropriate function space and to prove that uninteresting fixed-point are *ejective*. More precisely, we recall the following definition and result.

**Definition 3.2 (Ejective fixed-point, [3])** *Let  $E$  a Banach space and  $D$  a closed subset of  $E$ . Let  $\mathcal{F} : D \rightarrow D$  a continuous map. A fixed-point  $\bar{x}$  of  $\mathcal{F}$  is said to be ejective if there exists a neighbourhood  $U \subset D$  of  $\bar{x}$  such that*

$$\forall x \in U \setminus \{\bar{x}\}, \exists n \in \mathbb{N}, \mathcal{F}^n(x) \notin U.$$

**Theorem 3.3 (Browder's fixed-point theorem, [3])** *Let  $D$  be a closed, bounded and convex subset of an infinite dimensional Banach space and let  $\mathcal{F} : D \rightarrow D$  a continuous and compact map. Then  $\mathcal{F}$  admits at least one fixed-point which is not ejective.*

Let us describe the strategy of the proof of Theorem 3.1. Be careful that we go through three different forms of the problem: original equation (3.1), equivalent equation (3.4) and the slightly modified equation (3.10).

- First we study the stationary solutions  $c_*$  of (3.1).
- We change variables and recast (3.1) into (3.4). Then we study the correspondent characteristic equation (whose solutions are called *eigenvalues*):

$$\lambda + d = bd\mathcal{N}'(c_*)e^{-\lambda}.$$

We find that  $b\mathcal{N}'(c_*) < -1$  is a necessary condition to have *unstable eigenvalues*.

- We define a functional  $\mathcal{F}$  whose solutions are either stationary or periodic solutions of problem (3.10) which is very close to our renormalised problem (3.4).
- We prove that stationary solutions of (3.10) are *ejective*; hence, applying Browder's fixed-point theorem,  $\mathcal{F}$  possess another fixed-point which happens to be a periodic solution of (3.10). Last, we prove that this solution yields a non-trivial periodic solution of (3.4) and equivalently of (3.1).

### 3.1 Stationary problem

Let us first state the existence of a stationary solution. In the following we consider two cases

$$V_F > 0 \quad (\text{Case 1}) \quad \text{and} \quad V_F = 0, \quad b < -1 \quad (\text{Case 2}).$$

Since (3.1) describes the solution from the PDE system (1.4) on  $(-\infty, V_F]$ , the solution of (3.1) must be contained in  $(-\infty, V_F)$ . Thus, we are only looking for solutions  $c(t) < V_F$ .

**Theorem 3.4** *There is a unique stationary solution  $c_* < 0$  of (3.1) in  $c < V_F$  in Cases 1 and 2. On the other hand, there is no stationary solution of (3.1) in  $c < 0$  in the case of  $V_F = 0$  and  $-1 \leq b < 0$ .*

**Proof.** Define  $F(c, b) = c - b\mathcal{N}(c)$ , which is a smooth function of  $(c, b)$  in  $c < V_F$  and  $b < 0$ . It is easy to see  $\lim_{c \rightarrow -\infty} F(c, b) = -\infty$ . Moreover, we have  $F(0, b) = -b\mathcal{N}(0) > 0$  in Case 1 while  $F(0, b) = 0$  and  $(\partial F/\partial c)(0, b) = 1 + b < 0$  in Case 2. Hence there exists a zero of  $F(c, b)$ , equivalently, a stationary solution, denoted by  $c = c_* < 0$ . On the other hand, let us consider the case  $V_F = 0$  and  $-1 \leq b \leq 0$  and assume that there is a stationary solution  $c < 0$  of (3.1). Then we have a contradiction because

$$1 = -b \exp\left(-\frac{c^2}{2a}\right) < -b \leq 1.$$

Next, uniqueness of stationary solutions  $c^*$  of (3.1) follows from the following computation:

$$\frac{\partial F}{\partial c}(c_*, b) = 1 - b\mathcal{N}'(c_*) = \frac{aV_F - c_*(V_F - c_*)^2}{a(V_F - c_*)} > 0, \quad (3.2)$$

where we calculate  $\mathcal{N}'(c_*)$  as

$$\mathcal{N}'(c_*) = \frac{(V_F - c_*)^2 - a}{a} \exp\left(-\frac{(V_F - c_*)^2}{2a}\right) = \frac{c_*}{b} \left(\frac{V_F - c_*}{a} - \frac{1}{V_F - c_*}\right). \quad (3.3)$$

□

The stationary solution  $c_*$  depends on the parameter  $b$  so that we may represent  $c_* = c_*(b)$ . According to the implicit function theorem and due to (3.2), the proof above implies that  $c_*(b)$  is a smooth function. Note that  $c_*(b)$  can be defined for  $b < 0$  in Case 1, and  $b < -1$  in Case 2, respectively. Next, we study some properties of  $c_*(b)$ .

**Lemma 3.5** *The function  $c_*(b)$  satisfies  $c'_*(b) > 0$ ,  $\lim_{b \rightarrow -\infty} c_*(b) = -\infty$ , and  $\lim_{b \rightarrow -\infty} c_*(b)/b = 0$ . Moreover, there exists  $b_* < 0$  such that  $b\mathcal{N}'(c_*(b)) < -1$  for any  $b < b_*$ .*

**Proof.** In the proof we only consider Case 1 because we can prove the statement for Case 2 by the same argument. It is obvious that  $\lim_{b \rightarrow 0} c_*(b) = 0$ . Assume that then there is a subsequence  $b_i$  ( $i = 1, 2, \dots$ ) and  $c_\infty \leq 0$  such that  $b_i \rightarrow -\infty$  and  $c_*(b_i) \rightarrow c_\infty$  as  $i \rightarrow \infty$ . We have  $\mathcal{N}(c_\infty) = 0$ , and then  $c_\infty = V_F > 0$ , which is a contradiction. Hence we see  $\lim_{b \rightarrow -\infty} c_*(b) = -\infty$ . This fact and (3.3) imply  $c_*(b)/b = \mathcal{N}(c_*(b)) \rightarrow 0$  and  $b\mathcal{N}'(c_*(b)) \rightarrow -\infty$  as  $b \rightarrow -\infty$ .

Let  $\mathcal{F}$  be the same function as in the previous proof. Since  $(\partial F/\partial b)(c, b) = -\mathcal{N}(c) < 0$  in  $c < V_F$  and  $b < 0$ , we differentiate both sides of  $F(c_*(b), b) = 0$  with respect to  $b$  and obtain  $c'_*(b) = -(\partial F/\partial b)(c_*(b), b)/(\partial F/\partial c)(c_*(b), b) > 0$  because of (3.2). □

Set  $c(t) = c_* + x(\frac{t}{d})$  and change  $\frac{t}{d}$  into  $t$ . Then we derive a difference-differential equation for  $x$  from (3.1) which writes

$$x'(t) + dx(t) = bd(\mathcal{N}(c_* + x(t-1)) - \mathcal{N}(c_*)). \quad (3.4)$$

### 3.2 Study of the characteristic equation

It is well-known that the linearised problem of (3.4) around  $x = 0$  has the characteristic equation

$$\lambda + d = bd\mathcal{N}'(c_*)e^{-\lambda}. \quad (3.5)$$

We refer to a solution  $\lambda \in \mathbb{C}$  of (3.5) as an *eigenvalue*. Particularly, the solution  $\lambda$  is called an *unstable eigenvalue* if  $\operatorname{Re}\lambda > 0$ . Note that if  $\lambda$  is an eigenvalue, then so is  $\bar{\lambda}$ , where  $\bar{\lambda}$  is the complex conjugate of  $\lambda$ . Hence we can assume  $\operatorname{Im}\lambda \geq 0$  without loss of generality. We investigate the conditions for parameters such that (3.5) has an unstable eigenvalue, and finally show the existence of an unstable eigenvalue  $\lambda$  with  $\pi/2 < \operatorname{Im}\lambda < \pi$ , where “Re” and “Im” represent real and imaginary parts of a complex value.

We first show that there is no real unstable eigenvalue in (3.5).

**Lemma 3.6** (3.5) has no real non-negative solutions.

**Proof.** We have  $1 - b\mathcal{N}'(c_*) > 0$  by (3.2), then it is easy to complete the proof of the lemma.  $\square$

Next we study complex eigenvalues in (3.5).

**Lemma 3.7** Suppose that there exists an eigenvalue  $\lambda$  in (3.5) with  $\operatorname{Im}\lambda \neq 0$ . Let  $r$  be the remainder of the division of  $\operatorname{Im}\lambda$  by  $2\pi$ . If  $\operatorname{Re}\lambda \geq 0$ , then it must hold true that  $b\mathcal{N}'(c_*) \leq -1$  and  $r \in (\pi/2, \pi)$ . Similarly, if  $\lambda$  is an unstable eigenvalue, then one has  $b\mathcal{N}'(c_*) < -1$ .

**Proof.** We follow the argument in the proof of Lemma 3 in [21]. Set  $\lambda = \mu + i\gamma$  and assume  $\mu, \gamma \geq 0$ . It follows from (3.5) that

$$\mu + d = bd\mathcal{N}'(c_*)e^{-\mu} \cos \gamma, \quad \gamma = -bd\mathcal{N}'(c_*)e^{-\mu} \sin \gamma. \quad (3.6)$$

Then we have

$$(bd\mathcal{N}'(c_*)e^{-\mu})^2 = (\mu + d)^2 + \gamma^2 \geq d^2.$$

Since  $\mu$  is non-negative, we see  $(bd\mathcal{N}'(c_*))^2 \geq e^{2\mu} \geq 1$ . From (3.2), we have  $b\mathcal{N}'(c_*) \leq -1$ . This also implies  $r \in (\pi/2, \pi)$  because both the right-hand sides of (3.6) must be positive. The remainder of the lemma can be shown similarly.  $\square$

Combining Lemmas 3.6 and 3.7, we know that the stationary solution  $c_*$  may be destabilised only in the case of  $b\mathcal{N}'(c_*) < -1$ . We focus on this case hereafter and prove that there is an unstable eigenvalue in (3.5) if  $d > 0$  is sufficiently large. We follow the argument in the proof of Lemma 3 in [21] and state the existence of an unstable eigenvalue in (3.5).

We first show these non-degeneracy conditions for eigenvalues of (3.5) with respect to  $d$  in the following two lemmas.

**Lemma 3.8** Assume that there are  $d_0 > 0$  and an eigenvalue  $\lambda_0$  of (3.5) for  $d = d_0$  with  $\operatorname{Im}\lambda_0 \neq 0$ . Then there exists a smooth function  $\lambda(d)$  defined on  $d > 0$  satisfying the following properties;

- (i)  $\lambda(d)$  is an eigenvalue of (3.5) for  $d > 0$  with  $\operatorname{Im}\lambda(d) \neq 0$ .
- (ii)  $\lambda(d_0) = \lambda_0$ .
- (iii) If  $\operatorname{Re}\lambda(d_0) = 0$ , then  $\operatorname{Re}\lambda'(d_0) > 0$

(iv) If  $\operatorname{Re}\lambda(d_0) > 0$ , then there exists  $\tilde{d}_0 \in (0, d_0)$  such that  $\operatorname{Re}\lambda(\tilde{d}_0) = 0$ .

**Lemma 3.9** Let  $\lambda_1(d)$  and  $\lambda_2(d)$  be eigenvalues of (3.5) with  $\operatorname{Im}\lambda_i(d) \neq 0$  for  $i = 1, 2$ . If there is  $d^* > 0$  such that  $\lambda_1(d^*) \neq \lambda_2(d^*)$ , then it holds true that  $\lambda_1(d) \neq \lambda_2(d)$  in all  $d > 0$ .

**Proof of Lemmas 3.8 and 3.9.** We define  $F(\lambda, d) = \lambda + d - b\mathcal{N}'(c_*)e^{-\lambda}$ . From the assumption,  $F(\lambda_0, d_0) = 0$ . Moreover it is easy to see

$$\begin{aligned}\frac{\partial F}{\partial \lambda}(\lambda_0, d_0) &= 1 + b d_0 \mathcal{N}'(c_*) e^{-\lambda_0} = 1 + \lambda_0 + d_0 \neq 0, \\ \frac{\partial F}{\partial d}(\lambda_0, d_0) &= 1 - b \mathcal{N}'(c_*) e^{-\lambda_0} = -\frac{\lambda_0}{d_0}.\end{aligned}$$

From the implicit function theorem, we can construct a smooth function  $\lambda(d)$  in a neighborhood of  $d_0$  such as  $F(\lambda(d), d) = 0$ . Denote  $\lambda(d) = \mu(d) + i\gamma(d)$  for real  $(\mu, \gamma) = (\mu(d), \gamma(d))$ . Differentiating  $F(\lambda(d), d) = 0$  by  $d$ , we have

$$\lambda'(d) = -\frac{\frac{\partial F}{\partial d}}{\frac{\partial F}{\partial \lambda}} = \frac{1}{d} \frac{\lambda}{1 + \lambda + d}, \quad \mu'(d) = \frac{1}{d} \frac{\mu(1 + \mu + d) + \gamma^2}{(1 + \mu + d)^2 + \gamma^2}, \quad \gamma'(d) = \frac{1}{d} \frac{\gamma(1 + d)}{(1 + \mu + d)^2 + \gamma^2}. \quad (3.7)$$

From Lemma 3.7,  $|\gamma|$  must be larger than  $\pi/2$ . Hence  $\mu(d)$  and  $\gamma(d)$  can be defined in  $0 < d < \infty$  by solving the differential equations (3.7). This conclusion also implies (i)–(iii) of Lemma 3.8 and Lemma 3.9.

Suppose that  $\mu(d_0) > 0$  and  $0 \leq \mu(d)$  in  $d \in (0, d_0)$ . We can make an additional assumption on  $\gamma(d)$  such as  $\pi/2 + 2\pi k < \gamma(d) < \pi + 2\pi k$  for some integer  $k \geq 0$  without loss of generality by Lemma 3.7. We have  $\mu(d) < \mu(d_0)$  because of  $\mu'(d) > 0$ . On the other hand, it follows from (3.7) that

$$\begin{aligned}\mu(d_0) &= \mu(d) + \int_d^{d_0} \frac{1}{s} \frac{\mu(s)(1 + \mu(s) + s) + \gamma(s)^2}{(1 + \mu(s) + s)^2 + \gamma(s)^2} ds \\ &\geq \frac{1}{(1 + \mu(d_0) + d_0)^2 + (2k + 1)^2 \pi^2} \left(2k + \frac{1}{2}\right)^2 \pi^2 (\log d_0 - \log d) \rightarrow \infty\end{aligned}$$

when  $d \rightarrow 0$ , which constitutes a contradiction.  $\square$

Lemma 3.8 implies that any unstable eigenvalue of (3.5) must be connected to an eigenvalue on the imaginary axis. Then we study the conditions for parameters that (3.5) has a solution  $\lambda$  with  $\operatorname{Re}\lambda = 0$  and  $\operatorname{Im}\lambda > 0$ . Putting  $\mu = 0$  in the first equation of (3.6), we have

$$1 = b\mathcal{N}'(c_*) \cos \gamma. \quad (3.8)$$

(3.8) has denumerably infinite many solutions  $\gamma$  because of  $b\mathcal{N}'(c_*) < -1$ . We denote the minimal positive solution of (3.8) by  $\gamma_1$ , where  $\gamma_1$  can be estimated as  $\pi/2 < \gamma_1 < \pi$ . Moreover we readily see that  $\gamma_k = \gamma_1 + 2\pi(k - 1)$  ( $k = 2, 3, \dots$ ) are also solutions of (3.8). Then we define  $d_k$  by

$$d_k \equiv -\frac{\gamma_k}{b\mathcal{N}'(c_*) \sin \gamma_k}.$$

We can show the following lemma by the same argument as in [21].

**Lemma 3.10** Suppose that  $b\mathcal{N}'(c_*) < -1$ . Let  $k \geq 1$  be an integer. If  $d_k < d < d_{k+1}$ , then there are exactly  $2k$  unstable eigenvalues  $\lambda_i$  and  $\overline{\lambda_i}$  ( $i = 1, \dots, k$ ) of (3.5). On the other hand, if  $d = d_k$ , (3.5) has exactly  $2k - 2$  unstable eigenvalues  $\lambda_i$  and  $\overline{\lambda_i}$  ( $i = 1, \dots, k - 1$ ) and eigenvalues  $\lambda_k$  and  $\overline{\lambda_k}$  with  $\operatorname{Re}\lambda_k = 0$ . Moreover,  $\lambda_i$  satisfies  $\pi/2 < \operatorname{Im}\lambda_i - 2\pi(i - 1) < \pi$ .

**Proof.** Suppose that there is an eigenvalue  $\lambda$  in (3.5) with  $\operatorname{Re}\lambda \geq 0$  and  $\operatorname{Im}\lambda \neq 0$  for  $d^* \in (0, d_1)$ . Then it follows from Lemma 3.8 that there are  $\lambda(d)$  defined in  $d \in (0, \infty)$  and  $\tilde{d}^* \leq d^*$  such that  $\lambda(d^*) = \lambda$  and  $\operatorname{Re}\lambda(\tilde{d}^*) = 0$ . However,  $\tilde{d}^*$  must be equal to  $d_k$  for some  $k \in \mathbb{N}$ , which is a contradiction because  $d_1 \leq \tilde{d}^* \leq d^* < d_1$ .

As seen above, (3.5) has an eigenvalue  $\lambda = i\gamma_1$  for  $d = d_1$ . Thanks to Lemma 3.8, there exists  $\lambda_1(d) = \mu_1(d) + i\gamma_1(d)$  of an eigenvalue in (3.5) for  $d$  such that  $\gamma_1(d_1) = \gamma_1$  and  $\mu_1(d) > 0$  in  $d > d_1$ . Moreover, we see  $\pi/2 < \gamma_1(d) < \pi$  in  $d > d_1$  because  $\gamma_1(d)$  is continuous and (3.6) holds true. By the same argument as in the proof of Lemma 3.9, it can be shown that there is no eigenvalue in (3.5) in  $d \in [d_1, d_2)$  except for  $\lambda_1(d)$  and  $\overline{\lambda_1(d)}$ .

By the induction, we can complete the proof of Lemma 3.10.  $\square$

From Lemma 3.10, we have the following corollary.

**Corollary 3.11** *If  $b\mathcal{N}'(c_*) < -1$ , then (3.5) has an unstable eigenvalue  $\lambda$  with  $\pi/2 < \operatorname{Im}\lambda < \pi$ . if and only if  $d_1 < d$ .*

### 3.3 Properties of the non-linear term

Here we summarise the properties of the nonlinear term  $f_0(x) \equiv -b(\mathcal{N}(c_* + x) - \mathcal{N}(c_*))$ , which appears in the right-hand side of (3.4). It is easy to see that the following lemma holds true so that we omit the details of the proof.

**Lemma 3.12** *Suppose that  $b\mathcal{N}'(c_*) \leq -1$ . Then the smooth function  $f_0(x)$  satisfies the following properties;*

- (i)  $f_0(0) = 0$ .
- (ii)  $\lim_{x \rightarrow -\infty} f_0(x) = b\mathcal{N}(c_*) = c_* < 0$ .
- (iii)  $f'_0(x) > 0$  in  $x \leq 0$  while there are exactly two zeros of  $f'_0(x) = 0$  in  $x > 0$ , denoted by  $x_1 < x_2$ . In particular,  $f'_0(0) = -b\mathcal{N}'(c_*) \geq 1$ .
- (iv)  $f_0(x)$  is a Lipschitz function on  $\mathbb{R}$ , that is, there is  $L_0 > 0$  such that  $|f_0(x) - f_0(y)| \leq L_0|x - y|$  in any  $x, y \in \mathbb{R}$ .

We can give  $x_1$  explicitly such as  $x_1 = V_F - c_* - \sqrt{a}$  by a direct calculation and see  $x_1 > 0$  because of (3.3) and the assumption. Lemma 3.12 implies that the function  $f_0(x)$  is monotonically increasing and bounded in  $(-\infty, x_1]$ . Thus we define a function  $f(x)$  by

$$f(x) = \begin{cases} f_0(x), & x \leq x_1, \\ f_0(x_1), & x > x_1. \end{cases}$$

Under the assumption  $b\mathcal{N}'(c_*) \leq -1$ ,  $f \in C^1(\mathbb{R})$  satisfies the following properties;

- (f1)  $f(x)x > 0$  in  $x \neq 0$ .
- (f2)  $f(x) \geq c_*$  in  $x \in \mathbb{R}$ .
- (f3)  $f'(0) = -b\mathcal{N}'(c_*) \geq 1$ .
- (f4) There is  $L > 0$  such that  $|f(x) - f(y)| \leq L|x - y|$  in any  $x, y \in \mathbb{R}$ .



From the condition (f3), it is easy to show that there exist  $\alpha > 0$  and  $\delta > 0$  such that

$$|f(x)| \geq \alpha|x| \quad (3.9)$$

in  $|x| < \delta$ , where  $\delta$  is supposed to be less than  $-c_*$  without loss of generality.

### 3.4 The Browder fixed-point method

We replace  $f_0$  in the right-hand side of (3.4) into  $f$  and consider

$$x'(t) + dx(t) = -df(x(t-1)). \quad (3.10)$$

From the conditions (f1)–(f4), we can prove the existence of a periodic solution in (3.4) if  $d$  is sufficiently large. Actually, our difference-differential equation is slightly different from that in the previous work [21], and a parameter  $d$  appears not only in the coefficient of  $x(t)$  of the left-hand side of (3.4) but also in that of  $f(x(t-1))$ . Here, we adapt the argument in [21], and prove that (3.10) has a periodic solution.

The functional space  $C[-1, 0]$  denotes the set of all continuous functions on  $[-1, 0]$  equipped with the usual sup norm  $\|\cdot\|$ . We define  $K$  by the set of all functions  $\varphi \in C[-1, 0]$  such that  $\varphi(-1) = 0$ , and  $e^{dt}\varphi(t)$  is monotonically increasing on  $0 \leq t \leq 1$ . Moreover, we set  $K_0 = K \setminus \{0\}$  and  $B_M = \{\varphi \in K \mid \|\varphi\| \leq M\}$  for a constant  $M > 0$ . The initial function in (3.10) is assumed to be in  $C[-1, 0]$ . Then  $x(t; \varphi)$  denotes the solution of (3.10) with an initial function  $\varphi \in C[-1, 0]$ . We first show the global existence of a solution in (3.10).

**Lemma 3.13** *Set  $\varphi \in C[-1, 0]$  arbitrarily. Then there exists a solution  $x(t; \varphi)$  of (3.10) globally in time. The solution  $x(t; \varphi)$  is Lipschitz continuous with respect to  $\varphi \in C[-1, 0]$ , that is, for any  $T > 0$ , there are  $L_x > 0$  depending on only  $T$  such that*

$$\sup_{t \in [0, T]} |x(t; \varphi_1) - x(t; \varphi_2)| \leq L_x \|\varphi_1 - \varphi_2\|. \quad (3.11)$$

**Proof.** We rewrite (3.10) into an integral form

$$x(t; \varphi) = e^{-d(t-t_0)}x(t_0; \varphi) - \int_{t_0}^t e^{-d(t-s)}df(x(s-1; \varphi))ds \quad (3.12)$$

for any  $0 \leq t_0 \leq t$ . Let  $t_0 = 0$  and  $t \in [0, 1]$ . It is easy to see that the solution of (3.10) can be defined by the equation above because of  $x(0; \varphi) = \varphi(0)$  and  $x(t-1; \varphi) = \varphi(t-1)$ . In addition, we have

$$|x(t; \varphi_1) - x(t; \varphi_2)| \leq |\varphi_1(0) - \varphi_2(0)| + d \int_0^t e^{-d(t-s)}|f(\varphi_1(s-1)) - f(\varphi_2(s-1))|ds \leq (1+L)\|\varphi_1 - \varphi_2\|,$$

which implies that if  $T \in [0, 1]$ , (3.11) holds true.

By an induction argument, we complete the proof of the lemma.  $\square$

Lemma 3.13 implies that for  $T > 0$ , the solution  $x(t; \varphi)$  is estimated such as

$$|x(t; \varphi)| \leq L_x \|\varphi\| \quad (3.13)$$

on  $t \in [0, T]$ , which can be shown directly from (3.11) because  $x(t; 0) \equiv 0$ .

Next we prove that  $x(t; \varphi)$  has a zero at some  $t$ .

**Lemma 3.14** *Assume  $(1 + \alpha)/\alpha \leq e^d$ . For  $M > \delta$  given arbitrarily, let  $\varphi \in K_0$  with  $\|\varphi\| \leq M$ . Then there exists a zero of  $x(t; \varphi)$ , denoted by  $t = z_1$ , such that  $z_1 > 0$  and  $x'(z_1; \varphi) < 0$ . Moreover,  $e^{dt}x(t; \varphi)$  is monotonically decreasing on  $[z_1, z_1 + 1]$ .*

**Proof.** Denote  $x(t) = x(t; \varphi)$  for simplicity. Since  $\varphi \in K_0$ , we see  $\varphi(t) \geq 0$  on  $-1 \leq t \leq 0$  and  $x(0) = \varphi(0) > 0$ . Let  $z_1 = \inf\{t \geq 0 \mid x(t) \leq 0\}$ . If  $z_1 \leq 1$  and  $x'(z_1) = 0$ , it follows from (3.10) that  $f(\varphi(z_1 - 1)) = 0$  so that  $\varphi(t) = 0$  in  $-1 \leq t \leq z_1 - 1$ . Then we substitute  $t_0 = 0$  and  $t = z_1$  into (3.12) and then have  $x(0) = 0$ , which is a contradiction. Hence we obtain  $x'(z_1) < 0$  if  $z_1 \leq 1$ .

Next we assume  $z_1 > 1$ . Then  $x(t)$  decreases monotonically because  $x'(t) = -dx(t) - df(x(t-1)) \leq 0$  on  $t \in [0, z_1]$ , from which we have  $x(t) \leq x(0) \leq M$  on  $t \in [0, z_1]$ . Define  $t_1 = \inf\{t \geq 1 \mid x(t) \leq \delta\}$ . We prove that  $t_1$  is finite. If  $x(1) > \delta$ , it follows from (f1) that

$$x'(t) = -dx(t) - df(x(t-1)) \leq -\delta d$$

on  $t \in [1, t_1]$ . Integrating the both sides above on  $[1, t_1]$ , we obtain  $t_1 \leq 1 + M/(\delta d)$ . Then we assume that  $z_1 > t_1 + 1$ . Setting  $t_0 = t_1 + 1$  and  $t \in [t_1 + 1, t_1 + 2]$  in (3.12), we have

$$x(t) = e^{-d(t-(t_1+1))}x(t_1+1) - \int_{t_1+1}^t e^{-d(t-s)}df(x(s-1))ds \leq ((1+\alpha)e^{-d(t-(t_1+1))} - \alpha)x(t_1+1).$$

The assumption implies  $x(t_1 + 2) \leq 0$ , from which we have

$$z_1 \leq t_1 + 2 \leq 3 + \frac{M}{\delta d}. \quad (3.14)$$

In addition, we see  $x'(z_1) = -df(x(z_1 - 1)) < 0$ .

Finally we see that  $(e^{dt}x(t))' = -e^{dt}df(x(t-1)) \leq 0$  on  $t \in [z_1, z_1 + 1]$ , which completes the proof of the lemma.  $\square$

For  $\varphi \in K_0$ , there is another zero of  $x(t; \varphi)$ .

**Lemma 3.15** *Assume the same conditions as in Lemma 3.14. Then there exists a zero of  $x(t; \varphi)$ , denoted by  $t = z_2$ , such that  $z_2 > z_1 + 1$  and  $x'(z_2; \varphi) > 0$ . Moreover,  $e^{dt}x(t; \varphi)$  is monotonically increasing and  $0 \leq x(t; \varphi) \leq -c_*$  on  $t \in [z_2, z_2 + 1]$ .*

**Proof.** Denote  $x(t) = x(t; \varphi)$  for simplicity. Define  $z_2 = \inf\{t > z_1 \mid x(t) > 0\}$ . Substituting  $t_0 = z_1$  in (3.12), we see  $x(t) < 0$  in  $t \in (z_1, z_1 + 1]$  and

$$x(z_1 + 1) = - \int_{z_1}^{z_1+1} e^{-d(z_1+1-s)}df(x(s-1))ds \geq -LM. \quad (3.15)$$

Hence we have  $z_1 + 1 < z_2$ . Then  $x(t)$  increases monotonically on  $t \in [z_1 + 1, z_2]$  because

$$x'(t) = -dx(t) - df(x(t-1)) > 0.$$

Define  $t_2 = \inf\{t \geq z_1 + 1 \mid x(t) > -\delta\}$ . In the case of  $x(z_1 + 1) \geq -\delta$ , we set  $t_2 = z_1 + 1$ . We assume  $t_2 > z_1 + 1$  and show that  $t_2$  is finite. It follows from (f1) that

$$x'(t) = -dx(t) - df(x(t-1)) \geq \delta d$$

on  $z_1 + 1 \leq t \leq t_2$ . Integrating the inequality above over  $(z_1 + 1, t_2)$  and owing to (3.14), we have

$$t_2 \leq z_1 + 1 + \frac{LM}{\delta d} \leq 4 + \frac{M}{\delta d}(1 + L),$$

which implies that  $t_2$  is finite.

Suppose that  $z_2 > t_2 + 1$ . Setting  $t_0 = t_2 + 1$  and  $t \in [t_2 + 1, t_2 + 2]$  in (3.12), we see that

$$x(t) = e^{-d(t-(t_2+1))}x(t_2+1) - \int_{t_2+1}^t e^{-d(t-s)}df(x(s-1))ds \geq ((1+\alpha)e^{-d(t-(t_2+1))} - \alpha)x(t_2+1).$$

Therefore it follows from the assumption that  $x(t_2 + 2)$  is nonnegative, which implies  $z_2 \leq t_2 + 2$ .

Finally we see that  $(e^{dt}x(t))' = -e^{dt}df(x(t-1)) \geq 0$  from (3.10) on  $z_2 \leq t \leq z_2 + 1$ . By setting  $t_0 = z_2$  and  $t \in [z_2, z_2 + 1]$ , it follows from (f2) that

$$0 \leq x(t) = - \int_{z_2}^t e^{-d(t-s)} df(x(s-1)) ds \leq -c_*,$$

which completes the proof of the lemma.  $\square$

We may emphasise the  $\varphi$ -dependency of  $z_2$  and denote  $z_2 = z_2(\varphi)$ . Fix  $M > 0$  arbitrarily. Lemmas 3.13–3.15 imply that  $z_2(\varphi)$  is well-defined, continuous and uniformly bounded in  $B_M \cap K_0$ . In particular, we readily see that  $z_2(\varphi) > z_1 + 1$  and

$$z_2(\varphi) \leq t_2 + 2 \leq 6 + \frac{M}{\delta d}(1 + L). \quad (3.16)$$

We define the functional  $\mathcal{F} : K \rightarrow C[-1, 0]$  by  $[\mathcal{F}\varphi](t) = x(z_2(\varphi) + 1 + t; \varphi)$  in  $t \in [0, 1]$  for  $\varphi \neq 0$  while  $[\mathcal{F}\varphi](t) = 0$  for  $\varphi \equiv 0$ . Then  $\mathcal{F}$  satisfies the following lemma.

**Lemma 3.16** *Assume the same conditions as in Lemma 3.14. Then the following two conditions hold true;*

(i)  $\mathcal{F}(B_{-c_*}) \subset B_{-c_*}$

(ii)  $\mathcal{F}$  is continuous and compact.

**Proof.** The condition (i) can be verified by Lemma 3.15. The continuity of  $\mathcal{F}$  can be proved by Lemma 3.13. Set  $M = -c_*$  and give  $\varphi_n \in B_M$  ( $n = 1, 2, \dots$ ). We can assume that  $\varphi_n \neq 0$  in all  $n$  without loss of generality. Denote  $x_n(t) \equiv x(z_2(\varphi_n) + 1 + t; \varphi_n)$ . We see  $0 \leq x_n(t) \leq M$  and

$$|x_n(t_1) - x_n(t_2)| \leq \int_{t_2}^{t_1} |x_n'(s)| ds \leq dM(1 + L)(t_1 - t_2)$$

uniformly in  $n \geq 1$  and  $t, t_1, t_2 \in [-1, 0]$  with  $t_2 \leq t_1$  due to the assumption (f4). Since  $\{x_n(t)\}$  is uniformly bounded and uniformly equicontinuous, a subsequence  $x_{n_k}(t)$  ( $k = 1, 2, \dots$ ) converges in  $C[-1, 0]$  as  $k \rightarrow \infty$ , which implies that  $\mathcal{F}$  is compact.  $\square$

According to the Schauder's fixed point theorem ([19]), Lemma 3.16 implies the existence of a fixed point  $\varphi_*$  of  $\mathcal{F}$ . However, since  $\mathcal{F}(0) = 0$ ,  $\varphi_*$  may be identically equal to 0. Actually, we can exclude this possibility by the Browder's fixed point theorem (Theorem 1 in [3]). We first prove the next lemma.

**Lemma 3.17** *Assume  $b\mathcal{N}'(c_*) < -1$  and  $d_1 < d$ . Then there is a positive constant  $A > 0$  such that  $\sup_{t \geq z} |x(t; \varphi)| \geq A$  for  $\varphi \in K_0$  and any zero  $z$  of  $x(t; \varphi)$ .*

**Proof.** From Corollary 3.11, (3.5) has an eigenvalue  $\lambda = \mu + i\gamma$  with  $\mu > 0$  and  $\pi/2 < \gamma < \pi$ . Let  $\varepsilon$  be a positive constant satisfying

$$\varepsilon < \frac{\mu}{d} e^{-d} \frac{1}{2} \cos \frac{\gamma}{2}. \quad (3.17)$$

Define  $h(x) = f'(0)x - f(x)$ . Since  $f$  belongs to a class of  $C^1(\mathbb{R})$  and  $f(0) = 0$ , there is a positive constant  $A$  such that

$$|h(x)| \leq \int_0^{|x|} |f'(0) - f'(y)| dy \leq \varepsilon|x| \quad (3.18)$$

uniformly in  $|x| < A$ . Assume that there are  $\varphi \in K_0$  and a zero  $z$  of  $x(t) \equiv x(t; \varphi)$  such that  $\kappa \equiv \sup_{t \geq z} |x(t; \varphi)| < A$ . Define  $y(t) = e^{dt}x(t)$ . From Lemmas 3.14 and 3.15, we can make an additional assumption such that there is an extremum  $m \in (z, z+1)$  such that  $|x(m)| \geq \kappa/2$ . Moreover, we assume that  $x'(z) > 0$  and  $y'(t) > 0$  in  $(z, z+1)$  because we can analogously consider another case that  $x'(z) < 0$  and  $y'(t) < 0$  in  $(z, z+1)$ .

From (3.10), we have

$$x'(t) = -dx(t) - df'(0)x(t-1) + dh(x(t-1)).$$

Set  $T = z + 1$ . Multiplying  $e^{-\lambda t}$  to the both sides above and integrating over  $(T, \infty)$ , we have

$$\int_T^\infty x'(t)e^{-\lambda t} dt = d \int_{T-1}^T x(t)e^{-\lambda t} dt + \lambda \int_{T-1}^\infty x(t)e^{-\lambda t} dt + d \int_T^\infty h(x(t-1))e^{-\lambda t} dt.$$

Integrating by parts, we obtain

$$\int_T^\infty x'(t)e^{-\lambda t} dt = -x(T)e^{-\lambda T} + \lambda \int_T^\infty x(t)e^{-\lambda t} dt.$$

Then it follows that

$$-x(T)e^{-\lambda T} = (d + \lambda) \int_{T-1}^T x(t)e^{-\lambda t} dt + d \int_T^\infty h(x(t-1))e^{-\lambda t} dt.$$

Multiplying  $e^{\lambda(T-1/2)}$  to the both sides, we have

$$- \int_{T-1}^T y'(t)e^{-dt-\lambda(t-T+1/2)} dt = d \int_T^\infty h(x(t-1))e^{-\lambda(t-T+1/2)} dt. \quad (3.19)$$

It is easy to see that

$$\begin{aligned} \left| \int_{T-1}^T y'(t)e^{-dt-\lambda(t-T+1/2)} dt \right| &\geq \int_{T-1}^T y'(t)e^{-dt-\mu(t-T+1/2)} \cos \gamma \left( t - T + \frac{1}{2} \right) dt \\ &\geq y(T)e^{-dT-\mu/2} \cos \frac{\gamma}{2} = x(T)e^{-\mu/2} \cos \frac{\gamma}{2} \end{aligned} \quad (3.20)$$

owing to  $y'(t) > 0$  in  $(T-1, T)$ . On the other hand, we estimate the right-hand side of (3.19) as

$$\left| \int_T^\infty h(x(t-1))e^{-\lambda(t-T+1/2)} dt \right| \leq \varepsilon \kappa \frac{1}{\mu} e^{-\mu/2}. \quad (3.21)$$

Moreover, we put  $t = T$  and  $t_0 = m$  in (3.12) and then have

$$x(T) = e^{-d(T-m)}x(m) - \int_m^T e^{-d(t-s)} df(x(s-1); \varphi) ds \geq e^{-d\frac{\kappa}{2}}.$$

From this inequality and (3.19)–(3.21), we see

$$e^{-\mu/2} \cos \frac{\gamma}{2} e^{-d\frac{\kappa}{2}} \leq d\varepsilon \kappa \frac{1}{\mu} e^{-\mu/2},$$

which contradicts to (3.17). Hence we complete the proof of Lemma 3.17.  $\square$

From Lemma 3.17, there are infinitely many zeroes and extrema of  $x(t; \varphi)$  for  $\varphi \in K_0$ , denoted by  $z_n$  and  $m_n$  respectively, such that  $m_n \in (z_n, z_n + 1)$  and  $|x(m_n)| \geq A/2$ . This result yields the following lemma.

**Lemma 3.18** *Assume  $b\mathcal{N}'(c_*) < -1$  and  $d_1 < d$ . Let  $A$  be a positive constant given in Lemma 3.17. For all  $\varphi \in K_0$ , there are an even integer  $n$  and a positive constant  $A_1 > 0$  such that  $x(m_n; \varphi) \geq A_1$ .*

**Proof.** Denote  $x(t) = x(t; \varphi)$ . Set  $A_1 = \min\{A/(2L), A/2\}$  and suppose that  $x(m_{2n}) < A_1$  for all

$n \geq 1$ . By putting  $t_0 = z_{2n+1}$  and  $t = m_{2n+1}$  in (3.12), it follows from (f4) that

$$x(m_{2n+1}) = - \int_{z_{2n+1}}^{m_{2n+1}} e^{-d(m_{2n+1}-s)} df(x(s-1)) ds > -\frac{A}{2},$$

which is in contrast to Lemma 3.17.  $\square$

### Proof of Theorem 3.1.

From Lemma 3.18, we see that for any  $\varphi \in K_0$ , there is a sufficiently large integer  $n$  such that  $\|F^{(n)}(\varphi)\| \geq A_1$ , which implies that  $\varphi \equiv 0$  is an *ejective point* (see Definition 2 in [3]). Therefore we obtain a nonzero fixed point  $\varphi_* \neq 0$  of  $F$ . From the assumption  $V_F^2 \geq a$  in Theorem 3.1, we see that  $x_1 \geq -c_*$  and then  $c(t) \equiv c_* + x(t/d; \varphi_*)$  is a periodic solution of (3.1).

In order to complete the proof of Theorem 3.1, we estimate  $x(t; \varphi_*)$ . From the proof of Lemma 3.14,  $x(t; \varphi_*) \leq -c_*$  on  $t \in [0, z_1]$ . By the same argument as in the proof of Lemma 3.15,  $x(t; \varphi_*)$  attains a local minimum at  $t = m \in (z_1, z_1 + 1)$ . Putting  $t_0 = z_1$  and  $t = m$  in (3.12), we have

$$x(m; \varphi_*) = - \int_{z_1}^m e^{-d(m-s)} df(x(s-1; \varphi_*)) ds \geq Lc_*.$$

Then it holds that  $(L+1)c_* \leq c(t) \leq 0$ , which also implies that  $c(t)$  must be negative because  $b\mathcal{N}(c(t-d))$  is negative. Since  $L$  and  $c_*$  are independent of  $d, t$ , we have proved Theorem 3.1.  $\square$

## 4 Asymptotic description of the periodic solution

We now prove an asymptotic result on the shape of solutions of equation

$$c'(t) + c(t) = b\mathcal{N}(c(t-d)), \quad \mathcal{N}(c) = (V_F - c) \exp\left(-\frac{(V_F - c)^2}{2a}\right) \quad (4.1)$$

In order to do so, we assume for the sake of clarity that  $V_F = 0$ ,  $d = 1$  and we make the change of variables

$$\beta = \frac{1}{\log(-b)}, \quad u(s) = \beta \log\left(\frac{1}{\sqrt{a}} c\left(\frac{s}{\beta}\right)\right). \quad (4.2)$$

This change of variable is meaningful only when  $b < -1$ , which reminds us of Case 2 ( $V_F = 0$ ,  $b < -1$ ) in the previous section. We come to the following equivalent system:

$$u'_\beta(t) + 1 = \exp\left(\frac{1}{\beta}(1 - u_\beta(t)) + f(u_\beta(t - \beta))\right), \quad 1 \geq u_\beta([- \beta, 0]) > 0 \quad (4.3)$$

with

$$f(x) = \frac{1}{\beta}x - \frac{1}{2}e^{\frac{2}{\beta}x}. \quad (4.4)$$

Equation (4.3) has a unique constant positive stationary state given by

$$\bar{u}_\beta = \frac{\beta}{2} \log\left(\frac{2}{\beta}\right), \quad (4.5)$$

and we readily observe that

$$\lim_{\beta \rightarrow 0} \bar{u}_\beta = 0.$$

Note that when  $b$  goes to  $-\infty$ ,  $\beta$  tends to 0. If  $b \ll 0$ , then  $0 < \beta \ll 1$ .

**Theorem 4.1** Assume that  $u_\beta(-\beta, 0) \equiv 1$  and let  $T > 2$ . Then, there exists a constant  $C_M \in (0, 1]$  such that

$$\lim_{\beta \rightarrow 0} u_\beta = P, \quad \text{in } L^1(0, T),$$

where  $P$  has the following form:

- $P(t) = 1 - t$  on  $(0, 1)$ ;
- $P(t) = C_M + 1 - t$  on  $(1, 1 + C_M)$ ;
- $P$  is  $C_M$  periodic on  $(1, T)$ .

We summarise our approach in Figure 4.

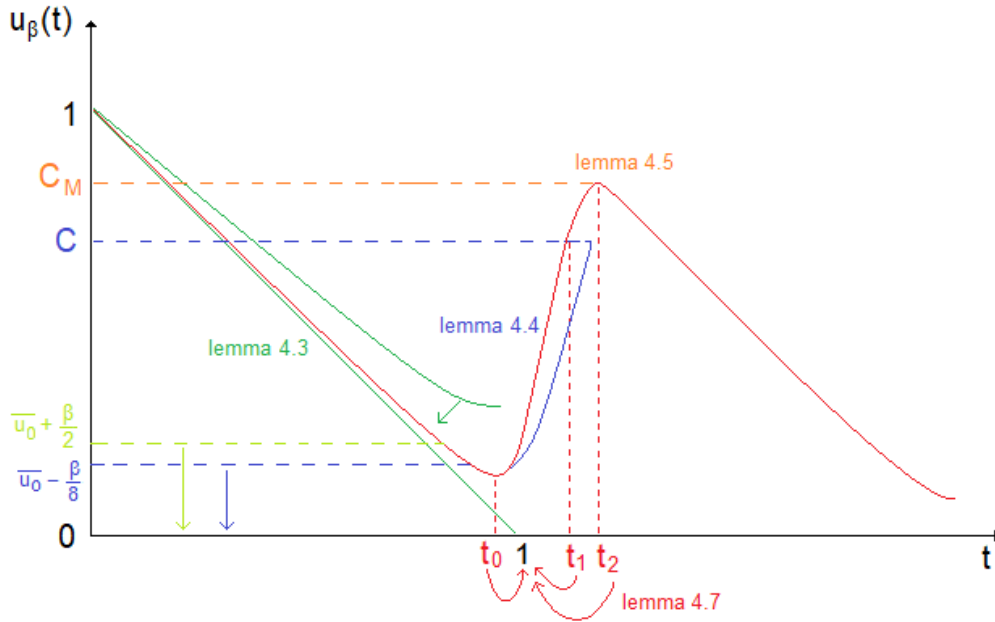


Figure 4: Strategy of the proof : Lemma 4.3 is used to prove convergence to a straight line, Lemma 4.4 gives rapid growth to a  $\beta$ -independent constant  $C$ , Lemma 4.7 gives  $t_0, t_1, t_2 \rightarrow 1$ ; Lemma 4.5 helps proving periodicity of  $P$ .

Before proving this result, let us make some comment on it's implications.

- First, we make the assumption  $u_\beta(-\beta, 0) \equiv 1$  for practicality and clearness reasons. However, a key mechanism being the unstable nature of the stationary state  $u_\beta$ , we expect that it is enough to take an initial condition which is not uniformly  $u_\beta$  to get the same convergence result after a transitory period. Aforementioned numerical simulations confirm the fact that the hypothesis is purely technical.
- The form of our rescaling and Theorem 4.1 indicate that the period  $T_b$  of the numerically observed periodic solutions  $c$  of (2.7) evolves in  $\mathcal{O}(\log(-b))$  when  $b \rightarrow -\infty$ .

- More precisely, we have in the original variable

$$c(t) \simeq \sqrt{a} \exp \left( \log(-b) P \left( \frac{t}{\log(-b)} \right) \right),$$

In decay phases, this approximation means that  $c$  behaves like

$$c(t) \simeq \sqrt{a} e^{C_M \log(-b) - t} = S_b e^{-t},$$

which constitutes an exponential decay over a spatial area of length  $S_b = \sqrt{a}(-b)^{C_M}$  during a time  $T_b = C_M \log(-b)$ . Remember that  $C_M \in (0, 1)$ .

- The approximation of  $c(t)$  by the discontinuous profile hides a very fast growth phase which makes the exponential decay appear slow in comparison. Lemma 4.4 gives an idea of this growth phase in rescaled variables but it does not allow us to provide a precise quantitative insight in original variables.

We are going to prove the convergence of  $u_\beta$  towards this periodic asymptotic profile  $P$  when  $\beta$  goes to 0. To do so, we rely on the instability of the stationary state  $\bar{u}_\beta$ . Our strategy is to study independently different phases of the solution:

1. a decay phase with asymptotic slope  $-1$  (Lemma 4.3);
2. a rapid growth phase up to some uniform constant  $C$  (Lemma 4.4);
3. a maximal value which presents some handy stability properties from which we will derive the periodicity of the final profile (Lemma 4.5, Lemma 4.7).

#### 4.1 Convergence of $u_\beta$ when $\beta \rightarrow 0$ to a profile $P$ .

Let us first observe that the function (4.4) is strictly decreasing on  $[0, +\infty)$  with  $f(0) = -\frac{1}{2}$  and  $f(+\infty) = -\infty$ . Indeed, we have

$$f'(x) = \frac{1}{\beta} (1 - e^{\frac{2}{\beta}x}) < 0 \quad \text{on} \quad (0, +\infty).$$

Here, we ensure that a limit profile does exist, without describing it yet. The following lemma holds

**Lemma 4.2** *There exists  $\beta_0 > 0$  such that for all  $\beta \in (0, \beta_0)$ , the solution of (4.3) satisfies : for all  $t > 0$ ,*

$$0 < u_\beta(t) < 1$$

and for all  $T > 0$ ,

$$\int_0^T |u'_\beta(t)| dt \leq 2T + 1.$$

Moreover, there exist a sequence  $\beta_n$  which converges to 0 and a function  $P$  such that

- $0 \leq P \leq 1$
- $P \in BV$ , with  $BV$  the set of functions with bounded variations
- $\lim_{n \rightarrow +\infty} u_{\beta_n} = P$  in  $L^1(0, T)$ .

**Proof.** Let us first prove that  $u_\beta > 0$  for all  $t > 0$ . To this, assume that there exists a first time  $t_0$  such that  $u(t_0) = 0$ , this means in particular that for all  $t < t_0$ ,  $u(t) > 0$ . Note that, as  $t_0$  is the first time the function reach 0 from above, we must have by continuity of the derivative:  $u'_\beta(t_0) \leq 0$ . At this step, there are two possibilities :

- either  $u_\beta(t_0 - \beta) > \beta$ , and in this case, as for all  $t \geq 0$ ,  $u'_\beta(t) > -1$ , this implies that  $u_\beta(t_0) > 0$  which is incompatible with the fact that  $u(t_0) = 0$ ;
- either  $u_\beta(t_0 - \beta) \leq \beta$ , and in this case, because  $f$  is decreasing, we obtain that

$$u'_\beta(t_0) \geq -1 + e^{\frac{1}{\beta} + f(\beta)},$$

but  $f(\beta) = 1 - \frac{1}{2}e^2$  and so, for  $\beta$  small enough,

$$u'_\beta(t_0) \geq e^{\frac{1}{2\beta}}$$

which is in contradiction with  $u'_\beta(t_0) \leq 0$ .

Hence, for all  $t > 0$ , we have  $u_\beta(t) > 0$ .

Let us now prove that for all  $t > 0$ ,  $u_\beta(t) < 1$ . To check this, it is enough to remark that if there exists  $t_0 > 0$  such that  $u_\beta(t_0) = 1$ , then because  $f < 0$ , we have  $u'_\beta(t_0) < 0$ , hence there exists  $t_1 > 0$  such that for all  $t \in (t_0, t_1)$ ,  $u_\beta(t) < 1$ . Therefore, for all  $t > 0$ ,  $u_\beta(t) < 1$ .

To prove uniform estimates on the derivative of  $u_\beta$ , we first integrate Equation (4.3) between 0 and  $T$ , and obtain that

$$u_\beta(T) - u_\beta(0) = -T + \int_0^T e^{\frac{1}{\beta}(1-u_\beta(t)) + f(u_\beta(t-\beta))} dt.$$

Hence,

$$\int_0^T e^{\frac{1}{\beta}(1-u_\beta(t)) + f(u_\beta(t-\beta))} dt \leq T + 1$$

and so

$$\int_0^T |u'_\beta(t)| dt \leq 2T + 1$$

Which ends the proof of Lemma 4.2. □

## 4.2 Description of the qualitative properties of $u_\beta$

To describe the profile  $P$ , let us come back to a more precise description of the dynamic related to the delay equation associated to  $u_\beta$ .

When the solution starts at the value 1, the solution first decays down to a local minimum which is smaller than the stationary state. The following Lemma describe the asymptotic pace of such decay with respect to  $\beta$ .

**Lemma 4.3** *Let  $I$  be an interval of  $\mathbb{R}_+$ . Assume that for all  $t \in I$ ,*

$$u_\beta(t - \beta) \geq \bar{u}_\beta + \frac{\beta}{2},$$

*then, for all  $t \in I$ , the following estimate holds*

$$-1 \leq u'_\beta(t) \leq -1 + e^{\frac{1}{\beta}(1-e)+1}. \tag{4.6}$$



**Proof.** Let us first remark that, because we always have  $u' \geq -1$ , this implies that

$$u_\beta(t) \geq u_\beta(t - \beta) - \beta,$$

hence, this implies that

$$u'_\beta(t) \leq -1 + e^{\frac{1}{\beta}(1-u_\beta(t-\beta))+1+f(u_\beta(t-\beta))}.$$

Now, if we consider the function  $g$  given by

$$g(x) = \frac{1}{\beta}(1-x) + 1 + f(x) = \frac{1}{\beta} + 1 - \frac{1}{2}e^{\frac{2}{\beta}x},$$

then  $g'(x) < 0$ , hence  $g$  is strictly decreasing. We deduce that while  $u(t - \beta) \geq \bar{u}_\beta + \frac{\beta}{2}$ , we have

$$u'_\beta(t) \leq -1 + e^{g(\bar{u}_\beta + \frac{\beta}{2})}.$$

But,

$$g(\bar{u}_\beta + \frac{\beta}{2}) = \frac{1}{\beta}(1-e) + 1$$

which proves Lemma 4.3 □

Then, the solution grow rapidly. The following Lemma provides a quantitative grasp on the phenomenon.

**Lemma 4.4** *Let  $\varepsilon > 0$ . Let  $I$  be an interval of  $\mathbb{R}_+$ . Assume that for all  $t \in I$ ,*

$$u_\beta(t - \beta) \leq \bar{u}_\beta - \frac{\beta}{8} \text{ and } u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon,$$

*then, for all  $t \in I$ , the following estimate holds*

$$u'_\beta(t) \geq -1 + e^{-\frac{1}{8}} \left(\frac{2}{\beta}\right)^{\frac{1}{2}} e^{\frac{\varepsilon}{\beta}}.$$

**Proof.** Because

$$u_\beta(t - \beta) \leq \bar{u}_\beta - \frac{\beta}{8}$$

and  $f$  decreasing, we have

$$u'_\beta(t) \geq -1 + e^{\frac{1}{\beta}(1-u_\beta(t))+f(\bar{u}_\beta - \frac{\beta}{8})}.$$

But

$$f(\bar{u}_\beta - \frac{\beta}{8}) = \frac{1}{2} \log\left(\frac{2}{\beta}\right) - \frac{1}{8} - \frac{1}{\beta}e^{-\frac{1}{4}}.$$

Hence, while  $u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon$ , we have

$$u'_\beta(t) \geq -1 + e^{-\frac{1}{8}} \left(\frac{2}{\beta}\right)^{\frac{1}{2}} e^{\frac{\varepsilon}{\beta}}$$

which proves Lemma 4.4. □

Now, we are going to prove a stability estimate on Equation (4.3). As the solution goes down below the fading  $u_\beta$ , we want to show that small initial conditions give stable patterns. The goal is to prove later than the rapid growth phase gives a stable asymptotic maximum value. More precisely, the following result holds.

**Lemma 4.5** *Let  $u_1, u_2$  be two solutions of Equation (4.3) and let a time  $t$  such that there exists two constants  $C$  and  $\alpha_1 > 0$  independent of  $\beta$  and a value  $\beta_0 > 0$  small enough such that for all  $\beta \in (0, \beta_0)$ ,*

$$|u_1(s) - u_2(s)| \leq C e^{-\alpha_1 \beta^{-1}} \quad \forall s \in (t - \beta, t), \quad (4.7)$$

*Assume that there exists  $k \in \mathbb{N}$  such that for all  $s \in (t - \beta, t + k\beta)$*

$$|u_1(s)| \leq \bar{u}_\beta + C\beta \quad \text{or} \quad |u_2(s)| \leq \bar{u}_\beta + C\beta. \quad (4.8)$$

*Then, there exists  $\alpha_2 > 0$  independent of  $\beta$  such that for all  $\beta \in (0, \beta_0)$ , for all  $s \in (t, t + (k + 1)\beta)$ , the following estimate holds*

$$|u_1 - u_2(s)| \leq e^{-\alpha_2 \beta^{-1}}. \quad (4.9)$$

**Proof.** Without loss of generality, with assumption (4.8), we can assume that for all  $s \in (t - \beta, t)$

$$|u_1(s)| \leq \bar{u}_\beta + C\beta.$$

We have for all  $s \geq t$

$$(u_1 - u_2)'(s) = e^{\frac{1}{\beta}(1-u_1(s))} e^{f(u_1(s-\beta))} - e^{\frac{1}{\beta}(1-u_2(s))} e^{f(u_2(s-\beta))}$$

and so

$$(u_1 - u_2)'(s) = (u_1'(t) + 1)(1 - e^{(f(u_2) - f(u_1))(s-\beta)}) + (u_2'(t) + 1)(e^{\frac{1}{\beta}(u_2 - u_1)(s)} - 1).$$

Multiplying the above equation by  $u_1 - u_2$ , we obtain that

$$\frac{1}{2} ((u_1 - u_2)^2)'(s) = (u_1'(s) + 1)(1 - e^{(f(u_2) - f(u_1))(s-\beta)})(u_1 - u_2) + (u_2'(s) + 1)(e^{\beta^{-1}(u_2 - u_1)(s)} - 1)(u_1 - u_2)(s).$$

As

$$(u_2'(s) + 1)(e^{\beta^{-1}(u_2 - u_1)(s)} - 1)(u_1 - u_2)(s) \leq 0,$$

we deduce that

$$((u_1 - u_2)^2)'(s) \leq (u_1'(s) + 1) \left( \left| 1 - e^{f(u_2(s-\beta)) - f(u_1(s-\beta))} \right|^2 + (u_1 - u_2)^2 \right).$$

But, with Lemma 4.2, we know that there exists a constant  $C_1 > 0$  such that for all  $s \in (t, t + (k + 1)\beta)$

$$e^{\int_t^s 1 + u_1'(w) dw} \leq C_1.$$

By Gronwall inequality, we then obtain that for all  $s \in (t, t + (k + 1)\beta)$ , the following estimate holds

$$(u_1 - u_2)^2(s) \leq C_1 \left( (u_1 - u_2)^2(t) + \|1 - e^{f(u_2) - f(u_1)}\|_{L^\infty(t-\beta, s-\beta)}^2 \right).$$

and so there exists a constant  $C_2 > 0$  such that for all  $s \in (t, t + (k + 1)\beta)$

$$(u_1 - u_2)^2(s) \leq C_2 \left( e^{-2\alpha_1 \beta^{-1}} + \|1 - e^{f(u_2) - f(u_1)}\|_{L^\infty(t-\beta, s-\beta)}^2 \right). \quad (4.10)$$

Let us deal with the term

$$\|1 - e^{f(u_1) - f(u_2)}\|_{L^\infty(t-\beta, s-\beta)}.$$

As with assumption (4.8),

$$|u_1(s)| \leq \bar{u}_\beta + C\beta,$$

we obtain that for all  $s \in (t - \beta, t)$

$$|f(u_1) - f(u_2)|(s) \leq C\beta^{-1} e^{-\alpha_1 \beta^{-1}} + \frac{1}{2} e^{2\beta^{-1}(\bar{u}_\beta + C\beta)} (1 - e^{2\beta^{-1}(u_2 - u_1)}).$$

But

$$e^{2\beta^{-1}(\bar{u}_\beta + C\beta)} \leq e^{2C\frac{2}{\beta}}.$$

Hence, for all  $s \in (t - \beta, t)$ , there exist constants  $C_3, C_4 > 0$  such that

$$\begin{aligned} \|1 - e^{f(u_1) - f(u_2)}\|_{L^\infty(t-\beta, s-\beta)} &\leq C_3 \|f(u_1) - f(u_2)\|_{L^\infty(t-\beta, s-\beta)} \\ &\leq C_4 \frac{1}{\beta^2} e^{-\alpha_1 \beta^{-1}} \end{aligned}$$

For  $\beta$  small enough we have

$$e^{-2\alpha_1 \beta^{-1}} = e^{-\alpha_1 \beta^{-1}} e^{-\alpha_1 \beta^{-1}} \leq \frac{1}{\beta} e^{-\alpha_1 \beta^{-1}}.$$

Hence, coming back to estimate (4.10), we obtain that for all  $s \in (t, t + \beta)$ , there exists a constant  $C_5$  such that

$$|u_1 - u_2|(s) \leq C_5 \frac{1}{\beta^2} e^{-\alpha_1 \beta^{-1}}.$$

By induction, we then obtain that there exists a constant  $C_6$  such that for all  $s \in (t - \beta, t + k\beta)$

$$|f(u_1) - f(u_2)|(s) \leq C_6 \left(\frac{1}{\beta}\right)^{2k+2} e^{-\alpha_1 \beta^{-1}}$$

and so there exists a constant  $C_7 > 0$  such that for all  $s \in (t - \beta, t + (k + 1)\beta)$

$$|u_1 - u_2|(s) \leq C_7 \left(\frac{1}{\beta}\right)^{2k+2} e^{-\alpha_1 \beta^{-1}}.$$

The constant  $C_7$  being independent of  $\beta$ , we can choose  $\beta_0$  so to have, for all  $\beta \in (0, \beta_0)$ ,

$$C_7 \left(\frac{1}{\beta}\right)^{2k+2} \leq 1,$$

which conclude the proof of Lemma 4.5. □

### 4.3 Description of the profile $P$ .

The following Theorem holds.

**Theorem 4.6** *Assume that  $u_\beta(-\beta, 0) \equiv 1$  and let  $T > 2$ . Then, there exists a constant  $C_M > 0$  such that the profile  $P$  has the following form*

$$P(t) = 1 - t \text{ on } (0, 1)$$

$$P(t) = C_M + 1 - t \text{ on } (1, 1 + C_M)$$

and  $P$  is  $C_M$  periodic on  $(1, T)$ .

**Proof.**

*Step 1: proof of the shape of the profile on  $(0, 1)$ .* To this, we are going to prove that  $u'(\beta)$  converges uniformly to  $-1$  on the interval  $[0, u_\beta(0) - \varepsilon]$ , for all  $\varepsilon > 0$ .

First, note that  $\bar{u}_\beta$  tends to 0 when  $\beta$  goes to 0. Let  $\varepsilon > 0$ . Since  $u'_\beta \geq -1$ , we can choose  $\beta$  small enough in order to have for all  $t \in [-\beta, u_\beta(0) - \varepsilon]$ ,

$$u_\beta(t) \geq \bar{u}_\beta + \frac{\beta}{2}.$$

Then, using Lemma 4.3, we deduce that for  $\beta$  small enough, for all  $t \in [0, 1 - \varepsilon]$

$$-1 \leq u'_\beta(t) \leq -1 + e^{\frac{1}{\beta}(1-e)+1},$$

which proves the result.

*Step 2: Description of the discontinuity of  $P$  at  $t = 1$  via  $u_\beta$ .* To understand what happens at the point 1 at the limit when  $\beta$  goes to 0, let us describe more precisely some qualitative properties of the function  $u_\beta$ . More precisely, the following Lemma holds

**Lemma 4.7** *Assume that  $u_\beta(-\beta, 0) \equiv 1$ . The following properties hold*

- *Let  $t_0(\beta)$  be the first time such that  $u'_\beta(t_0(\beta)) = 0$ . Then,  $u_\beta(t_0(\beta)) \leq \bar{u}_\beta - \frac{\beta}{4}$  is a local minimum. Moreover, we have*

$$1 - \bar{u}_\beta - 2\beta \leq t_0(\beta) \leq 1 - \bar{u}_\beta + 2\beta.$$

- *There exists a constant  $C > 0$  independent of  $\beta$  and a minimal time  $t_1(\beta)$  with  $t_0(\beta) + \beta > t_1(\beta) > t_0(\beta)$  such that*

$$u_\beta(t_1(\beta)) \geq C.$$

- *There exists  $t_2(\beta) \in [t_1, t_1 + \beta)$  such that*

$$C_1(\beta) := u_\beta(t_2(\beta)) = \sup_{t \in (t_0, t_1 + \beta)} u_\beta(t). \quad (4.11)$$

Moreover  $u'_\beta$  converges uniformly to  $-1$  on  $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon]$ , for all  $\varepsilon > 0$ .

**Proof of Lemma 4.7.** Let us prove the first property of Lemma 4.7. Let us first remark that with estimate (4.6), if we consider the first time  $t$  such that  $u(t - \beta) = \bar{u}_\beta + \frac{\beta}{2}$ , then as

$$u'_\beta(s) \leq -1 + e^{\frac{1}{\beta}(1-e)+1}$$

for all  $s \in (0, t]$ , this implies in particular that if  $\beta$  is small enough the derivative is close enough to  $-1$  so to have  $u_\beta(t) \leq \bar{u}_\beta - \frac{\beta}{4}$  and so at the first time  $t_0(\beta) > t$  such that  $u'_\beta(t_0) = 0$ , we have  $u_\beta(t_0) < \bar{u}_\beta - \frac{\beta}{4}$ . Let us now prove that  $t_0(\beta)$  is a local minimum. To this, let us derivate Equation (4.3), we find that

$$u''_\beta(t_0) = u'_\beta(t_0 - \beta) f'(u(t_0 - \beta)) e^{\frac{1}{\beta}(1-u_\beta(t_0)) + f(u_\beta(t_0 - \beta))}.$$

As  $f' < 0$  and  $u'_\beta(t_0 - \beta) < 0$ , we deduce that  $u''_\beta(t_0) > 0$  and hence  $t_0(\beta)$  is a local minimum. The proof of the fact that

$$1 - \bar{u}_\beta - 2\beta \leq t_0(\beta) \leq 1 - \bar{u}_\beta + 2\beta$$

is a direct consequence of Lemmas 4.3 and 4.4.

Let us now prove the second property of Lemma 4.7. To this, we remark that as  $u_\beta(t_0) < \bar{u}_\beta - \frac{\beta}{4}$ , and  $u' \geq -1$ , then, for all  $t \in (t_0 - \frac{\beta}{8}, t_0)$ , we have  $u_\beta(t) \leq \bar{u}_\beta - \frac{\beta}{8}$ . this implies that on  $(t_0 + \frac{7\beta}{8}, t_0 + \beta)$ , we can apply Lemma 4.4. Let  $\varepsilon > 0$  small enough. Hence, while  $u_\beta(t) \leq 1 - e^{-\frac{1}{4}} - \varepsilon$ , we have

$$u'_\beta(t) \geq -1 + Ce^{\frac{\varepsilon}{\beta}}.$$

Moreover, integrating the above estimate on an interval of size  $\beta$ , we conclude that there exists  $t_1 \in (t_0, t_0 + \beta)$  such that  $u_\beta(t) \geq 1 - e^{-\frac{1}{4}} - \varepsilon$ .

To prove the third property of Lemma 4.7, we observe that as  $u'_\beta \geq -1$ , on  $(t_1, t_1 + \beta)$ ,

$$u_\beta(t) \geq 1 - e^{-\frac{1}{4}} - \beta - \varepsilon.$$

Hence, using estimate (4.6), we deduce that while  $u(t - \beta) \geq \bar{u}_\beta + \frac{\beta}{2}$ ,  $u_\beta$  is strictly decreasing. We deduce that there exists a time  $t_2(\beta) \in [t_1, t_1 + \beta)$  such that

$$C_1 := u_\beta(t_2(\beta)) = \sup_{t \in (t_0, t_1 + \beta)} u_\beta(t),$$

with  $u_\beta$  satisfying estimate (4.6) on  $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon(\beta)]$  with  $\varepsilon(\beta) \rightarrow 0$  when  $\beta$  goes to 0. We then deduce that  $u'_\beta$  converges uniformly to  $-1$  on  $[t_1 + \beta, t_1 + \beta + C_1(\beta) - \varepsilon]$ , for all  $\varepsilon > 0$  which ends the proof of Lemma 4.7.  $\square$

*Step 3: Proof of the  $C_M$  periodicity of  $P$ .* With Lemma 4.7, we know that the shape of  $P$  is a function piecewise linear with a tangent equal to  $-1$ , that the function decrease until reaching the value 0 and that the function has a jump just after reaching the value 0. We have to prove that this jump is always the same at each step to obtain the periodicity property.

To do this, we first observe that, on the one hand, up to a subsequence,  $C_1(\beta)$  defined in (4.11) converges to a value  $C_M > 0$  uniformly bounded above. To prove that  $C_M$  is exactly the jump of  $P$  at the value 1, we observe that, as  $u'_\beta \geq -1$ , we know that for all  $t \in (t_2(\beta), t_1(\beta) + \beta)$ ,

$$C_1(\beta) - \beta \leq u_\beta(t) \leq C_1(\beta).$$

On the other hand, combining Lemmas 4.7 and 4.3, we know that there exists a first time  $t \in [1, 2]$  such that we have for all  $s \in (t, t + \beta)$

$$u_\beta(s) = \bar{u}_\beta + \frac{\beta}{2} - s + \mathcal{O}(e^{(1-\varepsilon)\beta^{-1}}).$$

Hence, to prove that  $P$  is periodic, we have to prove that by taking an initial data such that for  $s \in (-\beta, 0)$ ,

$$u_1(s) = \bar{u}_\beta + \frac{\beta}{2} - s$$

and another initial data such that for  $s \in (-\beta, 0)$ ,

$$u_2(s) = \bar{u}_\beta + \frac{\beta}{2} - s + \mathcal{O}(e^{(1-\varepsilon)\beta^{-1}}),$$

then the maximal values  $C_1^1(\beta)$  and  $C_1^2(\beta)$  defined in (4.11), associated to  $u_1$  and  $u_2$  given by the third step of Lemma 4.7 are such that

$$\lim_{\beta \rightarrow 0} C_1^1(\beta) = \lim_{\beta \rightarrow 0} C_1^2(\beta).$$

To prove this, we first use Lemma 4.7 which implies that there exists  $t_2^1(\beta)$  and  $t_2^2(\beta)$  such that

$$u_1(t_2^1(\beta)) = C_1^1(\beta) \quad \text{and} \quad u_1(t_2^2(\beta)) = C_1^2(\beta).$$

Moreover, we know that there exists a constant  $C$  independent of  $\beta$  such that

$$|t_2^1(\beta)| + |t_2^2(\beta)| \leq C\beta.$$

We can assume without loss of generality that  $t_2^2(\beta) \geq t_2^1(\beta)$ . By the contraposition of Lemma 4.3, we know that

$$u_2(t_2^2 - \beta) \leq \bar{u}_\beta + \frac{\beta}{2}.$$

Hence, again, because  $u_2' \geq -1$ , this implies that for all  $s \in (t_2^2 - 2\beta, t_2^2 - \beta)$ ,

$$u_2(s) \leq \bar{u}_\beta + \frac{3}{2}\beta.$$

Hence, we are in the setting of Lemma 4.5 as soon as  $t \leq t_2^2 - \beta$ . Moreover, using that

$$|t_2^1(\beta)| + |t_2^2(\beta)| \leq C\beta,$$

we obtain that there exists a constant  $\alpha > 0$  such that for all  $s \in (-\beta, t_2^2)$

$$|u_1(s) - u_2(s)| \leq e^{-\alpha\beta^{-1}},$$

which implies that necessarily

$$\lim_{\beta \rightarrow 0} C_1^1(\beta) = \lim_{\beta \rightarrow 0} C_1^2(\beta).$$

This ends the proof of Theorem 4.6. □

**Proof of Theorem 4.1.** This result is a direct consequence of Theorem 4.2 and Lemma 4.6. □

## 5 Conclusion

In this article, we studied the periodic solutions of the delayed>NNLIF model. Previous numerical simulations had indicated that the delayed>NNLIF model can give rise to periodic solutions in the inhibitory case ( $b < 0$ ), which is of tremendous importance for the understanding of fast global oscillations in networks of weakly firing inhibitory neurons. There wasn't yet any analytical insight on this topic. In the simulations, these periodic solutions exhibit a Gaussian shape.

Based upon heuristic arguments, partial results and numerics, we introduced an associate difference-differential equation which depicts the periodic movement of the center of the Gaussian wave. We prove rigorously that there exist periodic solutions to this associate equation.

Since our partial results indicate that the associate equation is valid asymptotically in  $b \rightarrow -\infty$ , we provide a rigorous results on the asymptotic behaviour of these approximate periodic solutions. The asymptotic profile is fully characterised.

We didn't address two difficult open questions. First, in order to complete our partial results on the asymptotic convergence of the periodic solutions of the complete>NNLIF system towards our approximate Gaussian wave, it remains to prove Conjecture (2.4). This will be the subject of further research and it will require the development of new techniques. Then, we only portrayed the inhibitory case  $b < 0$ . The results of [15] in the excitatory random discharge model indicate that our method could apply when  $b > 0$ , but in this case the shape of the solutions could be more complex than a Gaussian wave and there is no possibility to proceed asymptotically.

Note also that we investigated a bifurcation of Hopf time in term of the parameter  $d$ . Previous heuristic studies (*e.g.* [5]) indicate that there is also a Hopf bifurcation in term of the parameter  $b$ . Another subject for future investigation is to properly characterise the  $(b, d)$  two-parameters bifurcation of our associate difference-differential equation which we expect to be of Hopf-Hopf type.

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