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KURANISHI AND TEICHMÜLLER

LAURENT MEERSSEMAN

ABSTRACT. The goal of this short article is to describe the local structure of the Teichmüller stack of [8] in the neighborhood of a Kähler point. In particular we show that at a generic Kähler point X , Catanese Kur=Teich question, when interpreted at the level of stacks, has an affirmative answer. The situation may be much more complicated if X is non-Kähler suggesting that Teichmüller spaces/stacks of non-Kähler manifold has a much richer geometry.

1. INTRODUCTION.

Let X_0 be a compact complex manifold with underlying C^∞ manifold denoted by M . There are traditionnally two ways of describing the complex structures *near* X_0 . From the one hand, one may (try to) construct a moduli space of complex structures on M and look at a neighborhood of the class of X_0 in this moduli space. From the other hand, one may focus on small deformations of X_0 and look for a deformation from which all other deformations can be obtained by pull-back, after restriction to an adequate neighborhood of the base point.

Teichmüller refers to the Teichmüller space, that is the set of classes of complex manifolds diffeomorphic to M up to biholomorphism smoothly isotopic to the identity; hence to the first setting. Kuranishi refers to the Kuranishi semi-universal deformation; hence to the second setting.

In complex dimension one, the Teichmüller space has a natural structure of a complex manifold and the base of the Kuranishi deformation of X_0 is a (germ of) neighborhood of the class of X_0 in the Teichmüller space. So the link between the two notions is direct.

In complex dimension strictly greater than one, the Teichmüller space is just a topological space, usually non Hausdorff and non locally Hausdorff (see [8], Examples 13.3 and 13.6); and there exists a surjective continuous mapping from the (\mathbb{C} -analytic) base of the Kuranishi family onto a neighborhood of the class of X_0 in the Teichmüller space. Catanese asked in [1], see also [2], for conditions under which these two spaces are locally homeomorphic and proved it is the case in several cases including the case with trivial canonical bundle. However, due to usual non-local Hausdorffness, the existence of such a local homeomorphism is a rather restrictive property. And a useful criterion is missing. We use the slogan Kur=Teich to refer to this question and to a positive answer to it.

In [8], we replace the Teichmüller space with the Teichmüller stack, an analytic Artin stack over the category of \mathbb{C} -analytic spaces (see also [9] for

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a comprehensive presentation). On the way of exhibiting an atlas for this stack, we also construct Kuranishi stacks which are, roughly speaking, the quotient of Kuranishi base spaces by the automorphism group of the base points.

Catanese's question $\text{Kur}=\text{Teich}$ becomes in this new framework to find conditions for these two stacks to be locally isomorphic. It is now more natural since we stick to the analytic world.

The goal of this short article is to show that, in the Kähler setting, we have $\text{Kur}=\text{Teich}$ at a generic point of the Teichmüller space. We obtain this result as a consequence of a much more detailed statement. Indeed, we give a complete description of the Teichmüller stack around a point encoding a Kähler structure by showing in Theorem 6.2 that the natural inclusion of stacks $\text{Kur} \subset \text{Teich}$ is a finite analytic morphism. We then prove that $\text{Kur}=\text{Teich}$ outside a strict analytic substack of Teich that we characterize in Theorem 6.5. We point out that such a result does not hold at the level of the Riemann moduli stack since the mapping class group of a, say projective, manifold can act on the Teichmüller stack with dense orbits (this is the case for 2-dimensional tori [5] or for Hyperkähler manifolds [14]), hence the inclusion of Kur in this moduli stack may be far from being a finite morphism.

Moreover, at a non-Kähler point, the situation may be much more complicated and the finiteness property is also lost. This is only a theoretical statement and we unfortunately lack of examples. Indeed, we do not know of a single example with infinite fibers. However, our results and methods strongly suggest that they should exist and point towards a dichotomy between points with the above mentioned inclusion being a finite morphism (including but not equal to Kähler ones) and points with infinite fibers (which have to be non-Kähler). As a consequence, the local structure of the Teichmüller stack is much more singular in a sense at a non-Kähler point. If correct, this would really be surprising since, at the level of the Kuranishi space (and Kuranishi stack), there is no difference between Kähler and non-Kähler manifolds: the Kuranishi space of a Kähler, even of a projective, manifold can exhibit all the pathologies (for example not irreducible [4], not reduced [11], arbitrary singularities [13]) the Kuranishi space of a non-Kähler one can have. In the same way, as noted above, there is no difference between them at the level of the Riemann moduli stack. *This difference only appear when considering the Teichmüller stack* and suggests that the full complexity of the Teichmüller stack is only seen at non-Kähler points *hence that its geometry cannot be fully understood without dealing with non-Kähler manifolds.*

We also prove two related results in the paper. Firstly, we show in Theorem 3.2 that the germ of Kuranishi stack at a point has a universal property. This generalizes the semi-universality property of the Kuranishi space. At a rough level, this is folklore (see for example [15]), but we never saw a precise statement of such a property, probably because the stack setting developed in [8] is necessary to a clear formulation. Secondly, we investigate when the local Teichmüller stack is an orbifold. We solve this question in Theorem 6.10 in the Kähler setting.

2. THE TEICHMÜLLER STACK: BASIC FACTS

We recollect some facts about the Teichmüller stack of a connected, compact oriented C^∞ manifold M admitting complex structures. We refer to [8] for more details.

Let \mathfrak{S} be the category of analytic spaces and morphisms endowed with the transcendental topology. Given $S \in \mathfrak{S}$, we call M -*deformation over* S a proper and smooth morphism $\mathcal{X} \rightarrow S$ whose fibers are compact complex manifolds diffeomorphic to M . As C^∞ -object, such a deformation is a bundle over S with fiber M and structural group $\text{Diff}^+(M)$ (diffeomorphisms of M that preserve its orientation). It is called *reduced* if the structural group is reduced to $\text{Diff}^0(M)$. In the same way, a morphism of reduced M -deformations \mathcal{X} and \mathcal{X}' over an analytic morphism $f : S \rightarrow S'$ is a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' \end{array}$$

such that \mathcal{X} and $f^*\mathcal{X}'$ are isomorphic as $\text{Diff}^0(M)$ -bundles over S .

The Teichmüller stack $\mathcal{T}(M)$ is defined as the stack over the site \mathfrak{S} such that

- i) $\mathcal{T}(M)(S)$ is the groupoid of isomorphism classes of reduced M -deformations over S .
- ii) $\mathcal{T}(M)(f)$ is the pull-back morphism f^* from $\mathcal{T}(M)(S')$ to $\mathcal{T}(M)(S)$.

A point $X_0 := (M, J_0)$ is an object of $\mathcal{T}(M)(pt)$ that is a complex structure on M up to biholomorphisms smoothly isotopic to the identity.

Alternatively, $\mathcal{T}(M)$ can be considered as an analytic version of the quotient $\mathcal{I}(M)/\text{Diff}^0(M)$. Here, $\mathcal{I}(M)$ is the set of integrable complex operators on M compatible with its orientation (o.c.), that is

$$(2.1) \quad \mathcal{I}(M) = \{J : TX \rightarrow TX \mid J^2 \equiv -Id, J \text{ o.c.}, [T^{1,0}, T^{1,0}] \subset T^{1,0}\}$$

for

$$T^{1,0} = \{v - iJv \mid v \in TX\}.$$

and $\text{Diff}^0(M)$ is the group of diffeomorphisms of M which are C^∞ -isotopic to the identity.

Two points have to be emphasized here. Firstly, as stack, $\mathcal{T}(M)$ also encodes the isotropy groups of the action. Recall that the isotropy group at X_0 is the group

$$(2.2) \quad \text{Aut}^1(X_0) := \text{Aut}(X_0) \cap \text{Diff}^0(M).$$

which may be different from $\text{Aut}^0(X_0)$, the connected component of the identity of the automorphism group $\text{Aut}(X_0)$, see [10]. Secondly, the action of $\text{Diff}^0(M)$ onto $\mathcal{I}(M)$ is not a holomorphic action but an action by biholomorphisms. Indeed $\text{Diff}^0(M)$ can be endowed with a complex structure as an open set of the complex Fréchet space of C^∞ -maps from M to X_0 , but this complex structure depends on X_0 , that is depends on the choice of a complex structure on M . Taking this into account means putting as complex structure on $\text{Diff}^0(M) \times \mathcal{I}(M)$ not a product structure but the structure

such that $\text{Diff}^0(M) \times \{J\}$ is an open set of the complex Fréchet space of C^∞ -maps from M to X_J . In other words, $\text{Diff}^0(M) \times \mathcal{I}(M)$ is endowed with a complex structure as an open set of the complex Fréchet space of C^∞ -maps from $M \times \mathcal{I}(M)$ to $\mathcal{X}_{\mathcal{I}(M)}$ the tautological family over $\mathcal{I}(M)$.

Since $\text{Diff}^0(M)$ acts on the (infinite-dimensional) analytic space $\mathcal{I}(M)$ preserving its connected components and its irreducible components, we may speak in this way of connected components and irreducible components of $\mathcal{S}(M)$. Indeed Kuranishi's Theorem tells us that there exist local analytic sections K_0 of finite dimension at each point $X_0 = (M, J_0)$ of $\mathcal{I}(M)$. Hence, locally, the irreducible components of $\mathcal{I}(M)$ at J_0 are those of the finite-dimensional space K_0 .

In [8], a finite-dimensional atlas of (a connected component of) $\mathcal{S}(M)$ is described under the hypothesis that the dimension of the automorphism group of the complex manifolds encoded in $\mathcal{S}(M)$ is bounded. The rough idea is that the $\text{Diff}^0(M)$ -action on $\mathcal{I}(M)$, though not locally free when the complex structures admit holomorphic vector fields, defines a sort of foliation that we call a TG foliation. A holonomy groupoid can be defined for this sort of foliation and gives the desired atlas.

Such a groupoid is obtained by taking a complete set of local transversals to the foliation and considering its quotient through the holonomy morphisms. In our situation, the transversal at a point X_0 of $\mathcal{S}(M)$ is the Kuranishi stack. It is build from K_0 , see [8, §2.3] and Section 3.3.

3. THE KURANISHI STACK

In the first two sections, we review the construction of the Kuranishi family first from classical deformation theory point of view, then from Kuranishi-Douady's point of view.

We then review the construction of the Kuranishi stack(s) introduced in [8] and finally describe some important properties of them.

It is worth pointing out that the classical point of view (which presents Kuranishi family from a formal/algebraic point of view leaving aside the analytic details of the construction) is not enough for our purposes. This is indeed an infinitesimal point of view and even if it gives complete equations for the Kuranishi space, it fails in describing the properties of the structures close to the base complex structure. Kuranishi-Douady's point of view allows to pass from the infinitesimal point of view to a local one.

3.1. The Kuranishi family. The Kuranishi family $\pi : \mathcal{K}_0 \rightarrow K_0$ of X_0 is a semi-universal deformation of X_0 . It comes with a choice of a marking, that is of an isomorphism i between X_0 and the fiber $\pi^{-1}(0)$ over the base point 0 of K_0 . The semi-universal property means that

- i) Every marked deformation $\mathcal{X} \rightarrow B$ of X_0 is locally isomorphic to the pull-back of the Kuranishi family by a pointed holomorphic map defined in a small neighborhood of the base point of B with values in a neighborhood of 0 in K_0 .
- ii) Neither the mapping f nor its germ at the base point are unique; but its differential at the base point is.

Two such semi-universal deformations of X_0 are isomorphic up to restriction to a smaller neighborhood of their base points. Hence the germ of deformation $(\mathcal{X}_0, \pi^{-1}(0)) \rightarrow (K_0, 0)$ is unique. This explains why we talk of the Kuranishi family, even if, in many cases, we work with a representant of the germ rather than with the germ itself.

The Zariski tangent space to the Kuranishi space K_0 at 0 identifies naturally with $H^1(X_0, \Theta_0)$. Indeed, K_0 is locally isomorphic to an analytic subspace of $H^1(X_0, \Theta_0)$ whose equations coincide at order 2 with the vanishing of the Schouten bracket.

The groups $\text{Aut}(X_0)$, $\text{Aut}^0(X_0)$ and $\text{Aut}^1(X_0)$ act on this tangent space.

However, this infinitesimal action cannot always be integrated in an action of the automorphism groups of X_0 onto K_0 . Still there exists an action of each 1-parameter subgroup and all these actions can be encoded in an analytic groupoid and thus in a stack. To do this, we need to know more about the complex properties of the structures encoded in a neighborhood of 0 in K_0 .

3.2. Kuranishi-Douady's presentation and automorphisms. Let V be an open neighborhood of J_0 in $\mathcal{I}(M)$. Complex structures close to J_0 can be encoded as $(0, 1)$ -forms ω with values in $T^{1,0}$ which satisfy the equation

$$(3.1) \quad \bar{\partial}\omega + \frac{1}{2}[\omega, \omega] = 0$$

Choose an hermitian metric and let $\bar{\partial}^*$ be the L^2 -adjoint of $\bar{\partial}$ with respect to this metric. Let U be a neighborhood of 0 in $(T^{0,1})^* \otimes T^{1,0}$. Set

$$(3.2) \quad K_0 := \{\omega \in U \mid \bar{\partial}\omega + \frac{1}{2}[\omega, \omega] = \bar{\partial}^*\omega = 0\}$$

Let W an open neighborhood of 0 in the vector space of vector fields L^2 -orthogonal to the vector space of holomorphic vector fields $H^0(X_0, \Theta)$. In Douady's setting [3], Kuranishi's Theorem states a local isomorphism between $\mathcal{I}(M)$ at J_0 and the product of K_0 with W such that every plaque $\{pt\} \times L$ is sent through the inverse of this isomorphism into a single local $\text{Diff}^0(M)$ -orbit. To be more precise, up to restricting U , V and W , the Kuranishi mapping

$$(3.3) \quad (\xi, J) \in W \times K_0 \longmapsto J \cdot e(\xi) \in V$$

is an isomorphism of infinite-dimensional analytic spaces. As usual, we use the exponential map associated to a riemannian metric on M in order to define the map e which gives a local chart of $\text{Diff}^0(M)$ at Id . And \cdot denotes the natural right action of $\text{Diff}^0(M)$ onto $\mathcal{I}(M)$ given by

$$(3.4) \quad J \cdot f := df^{-1} \circ J \circ df$$

Composing the inverse of (3.3) with the projection onto K_0 gives a retraction map $\Xi : V \rightarrow K_0$. Let now f be an element of $\text{Aut}(X_0)$. There exists some maximal open set $U_f \subset K_0$ such that

$$(3.5) \quad \text{Hol}_f : J \in U_f \subset K_0 \longmapsto \Xi(J \cdot f) \in K_0$$

is a well defined analytic map. Observe that Hol_f fixes J_0 and that it fixes each leaf of the foliation¹ of K_0 described in [7, §3].

Of course, in (3.5), we may restrict ourselves to elements of $\text{Aut}^0(X_0)$ or $\text{Aut}^1(X_0)$.

3.3. The Kuranishi stacks. The Kuranishi stacks encode the maps (3.5) in an analytic groupoid. The first step to do this consists in proving that there is an isomorphism

$$(3.6) \quad (\xi, g) \in W \times \text{Aut}^0(X_0) \longmapsto g \circ e(\xi) \in \mathcal{D}_0$$

with values in a neighborhood \mathcal{D}_0 of $\text{Aut}^0(X_0)$ in $\text{Diff}^0(M)$, see [8, Lemma 4.2].

Let now $\text{Diff}^0(M, \mathcal{K}_0)$ denote the set of C^∞ diffeomorphisms from M to a fiber of the Kuranishi family $\mathcal{K}_0 \rightarrow K_0$. This is an infinite-dimensional analytic space², see [3]. Here by $(J, F) \in \text{Diff}^0(M, \mathcal{K}_0)$, we mean that we consider F as a diffeomorphism from M to the complex manifold X_J .

Given (J, F) an element of $\text{Diff}^0(M, \mathcal{K}_0)$, we say it is (V, \mathcal{D}_0) -admissible if there exist a finite sequence (J_i, F_i) (for $0 \leq i \leq p$) of $\text{Diff}^0(M, \mathcal{K}_0)$ such that

- i) $J_0 = J$ and each J_i belongs to K_0 .
- ii) Each F_i belongs to \mathcal{D}_0 .
- iii) $F = F_0 \circ \dots \circ F_p$
- iv) $J_{i+1} = J_i \cdot F_i$

We set then

$$(3.7) \quad \mathcal{A}_0 = \{(J, F) \in \text{Diff}^0(M, \mathcal{K}_0) \mid (J, F) \text{ is } (V, \mathcal{D}_0)\text{-admissible}\}$$

We also consider the two maps from \mathcal{A}_0 to K_0

$$(3.8) \quad s(J, F) = J \quad \text{and} \quad t(J, F) = J \cdot F$$

and the composition and inverse maps

$$(3.9) \quad m((J, F), (J \cdot F, F')) = (J, F \circ F'), \quad i(J, F) = (J \cdot F, F^{-1})$$

With these structure maps, the groupoid $\mathcal{A}_0 \rightrightarrows K_0$ is an analytic groupoid [8, Prop. 4.6] whose stackification over \mathfrak{S} is called *the Kuranishi stack* of X_0 . We denote it by \mathcal{A}_0 . Note that it depends indeed of the particular choice of V .

As a category, its objects are still reduced M -deformations over bases belonging to \mathfrak{S} . However, the allowed complex structures are those encoded in V ; and the allowed families are those obtained by gluing pull-back families of $\mathcal{K}_0 \rightarrow K_0$ with respect to (V, \mathcal{D}_0) -admissible diffeomorphisms. In the same way, morphisms are those induced by (V, \mathcal{D}_0) -admissible diffeomorphisms. Hence, not only the complex fibers of the families have to be isomorphic to those of \mathcal{K}_0 , but gluings and morphisms of families are restricted.

Of course, the same construction can be carried out for the automorphism groups $\text{Aut}^1(X_0)$, resp. $\text{Aut}(X_0)$, with the obvious modifications.

¹The leaves correspond to the connected components in K_0 of the following equivalence relation: $J \equiv J'$ iff both operators belong to the same $\text{Diff}^0(M)$ -orbit.

²Strictly speaking, we have to pass to Sobolev L_l^2 -structures for a big l to have an analytic space, and $\text{Diff}^0(M, \mathcal{K}_0)$ is the subset of C^∞ points of this analytic set. In the sequel, we automatically make this slight abuse of terminology, cf. Convention 3.2 in [8].

In (3.6), $\text{Aut}^0(X_0)$ is replaced with $\text{Aut}^1(X_0)$, resp. $\text{Aut}(X_0)$, defining a neighborhood \mathcal{D}_1 of $\text{Aut}^1(X_0)$ in $\text{Diff}^0(M)$, resp. \mathcal{D} of $\text{Aut}(X_0)$ in $\text{Diff}(M)$. This allows to speak of (V, \mathcal{D}_1) -admissible, resp. (V, \mathcal{D}) -admissible diffeomorphisms. Then, replacing \mathcal{D}_0 with \mathcal{D}_1 , resp. \mathcal{D} in (3.7) we obtain the analytic groupoid $\mathcal{A}_1 \rightrightarrows K_0$, resp. $\mathcal{A} \rightrightarrows K_0$. Its stackification over \mathfrak{S} gives a stack \mathcal{A}_1 , resp. \mathcal{A} . The previous description of \mathcal{A}_0 as a category applies to \mathcal{A} , resp. \mathcal{A} with the obvious changes. We also call them *Kuranishi stacks*.

3.4. Universality and the Kuranishi stacks. Recall that Kuranishi's Theorem asserts the existence of a semi-universal deformation for any compact complex manifold. This is not however a universal deformation when the dimension of the automorphism group varies in the fibers of the Kuranishi family, i.e. in the setting of section 3.1, the germ of mapping f is not unique. Replacing the Kuranishi space with the Kuranishi stack allows to recover a universality property.

To do that, we need to germify the Kuranishi stacks. We replace our base category \mathfrak{S} with the base category \mathfrak{G} of germs of analytic spaces. We turn \mathfrak{G} into a site by considering the trivial coverings. Hence each object of \mathfrak{G} has a unique covering and there is no non trivial descent data.

We then germify the groupoids. Starting with $\mathcal{A} \rightrightarrows K_0$, resp. $\mathcal{A}_0 \rightrightarrows K_0$ and $\mathcal{A}_1 \rightrightarrows K_0$, and using s and t as defined in (3.8), we germify K_0 at 0, \mathcal{A} , resp. \mathcal{A}_0 and \mathcal{A}_1 , at the fiber $(s \times t)^{-1}(0)$ and germify consequently all the structure maps. We thus obtain the groupoids $(\mathcal{A}, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, resp. $(\mathcal{A}_0, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$ and $(\mathcal{A}_1, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$.

Finally, we stackify $(\mathcal{A}, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, resp. $(\mathcal{A}_0, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$ and $(\mathcal{A}_1, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, over \mathfrak{G} . We denote the corresponding stacks by $(\mathcal{A}, 0)$, resp. $(\mathcal{A}_0, 0)$ and $(\mathcal{A}_1, 0)$.

The objects of $(\mathcal{A}, 0)$ over a germ of analytic space $(S, 0)$ are germs of M -deformations $p : \mathcal{X} \rightarrow S$ with fiber at the point 0 of S isomorphic to X_0 . We denote them by $(\mathcal{X}, p^{-1}(0)) \rightarrow (S, 0)$. The morphisms over some analytic mapping $f : S \rightarrow S'$ are germs of morphisms between M deformations $(\mathcal{X}, p^{-1}(0)) \rightarrow (S, 0)$ and $(\mathcal{X}', p'^{-1}(0)) \rightarrow (S', 0')$ over f . Note that $f(0) = 0'$.

Remark 3.1. It is crucial to notice that we deal with germs of *unmarked* deformations. There is obviously a distinguished point (since we deal with germs), but there is no marking of the distinguished fiber.

The following theorem shows that $(\mathcal{A}, 0)$ contains indeed *all* such germs of M -deformations and of morphisms between M -deformations. It is folklore although we never saw a paper stating this in a precise way.

Theorem 3.2. *The stack $(\mathcal{A}, 0)$ is the stack over \mathfrak{G} whose objects are the germs of M -deformations of X_0 and whose morphisms are the germs of morphisms between M -deformations.*

Proof. Since the site \mathfrak{G} does not contain any non-trivial covering, there is no gluings of families, and the torsors associated to $(\mathcal{A}, 0)$ are just given by the pull-backs of the germ of Kuranishi family $(\mathcal{K}_0, \pi^{-1}(0)) \rightarrow (K_0, 0)$. Kuranishi's Theorem implies that the natural inclusion of $(\mathcal{A}, 0)$ in the stack of germs of M -deformations over \mathfrak{G} is essentially surjective.

Morphisms over the identity of some germ $(S, 0)$ of analytic space are thus given by morphisms F between two germs of families $(f^* \mathcal{K}_0, \pi^{-1}(0)) \rightarrow (B, 0)$ and $(g^* \mathcal{K}_0, \pi^{-1}(0)) \rightarrow (B, 0)$ for f and g germs of analytic mappings from $(B, 0)$ to $(K_0, 0)$. Hence F restricted to the central fiber $X_0 \simeq \pi^{-1}(0)$ is an automorphism of the central fiber that is an element of $\text{Aut}(X_0)$. But $(s \times t)^{-1}(0)$ is isomorphic to $\text{Aut}(X_0)$ so such a morphism F is induced by an analytic mapping from $(B, 0)$ to $(\mathcal{A}, (s \times t)^{-1}(0))$ that we still denote by F which satisfies $s \circ F = f$ and $t \circ F = g$. This shows that the natural inclusion of $(\mathcal{A}, 0)$ in the stack of germs of M -deformations over \mathfrak{G} is fully faithful. \square

This must be thought of as a property of universality. Indeed, the failure of universality in Kuranishi's theorem comes from the existence of automorphisms of the Kuranishi family fixing the central fiber but not all the fibers. Imposing a marking is an artificial and incomplete solution to this problem because this automorphism group is not in general isomorphic to $\text{Aut}(X_0)$ which is killed by the marking.

In the same way, we have

Corollary 3.3. *The stack $(\mathcal{A}_1, 0)$ is the stack over \mathfrak{G} whose objects are the germs of reduced M -deformations and whose morphisms are the germs of morphisms between reduced M -deformations.*

Here C^∞ -markings of the M -families, that is the choice of a C^∞ diffeomorphism from M to the central fiber, can be used to characterize reduced families. Morphisms are required to induce on M a diffeomorphism isotopic to the identity through the markings.

We also have

Corollary 3.4. *The stack $(\mathcal{A}_0, 0)$ is the stack over \mathfrak{G} whose objects are the germs of 0-reduced M -deformations and whose morphisms are the germs of morphisms between 0-reduced M -deformations.*

A 0-reduced M -deformation is just a marked family. We use a different terminology because morphisms are different. A morphism of marked families is required to induce on X_0 the identity through the markings, whereas a morphism of 0-reduced M -deformation is required to induce on X_0 an element of $\text{Aut}^0(X_0)$ through the markings.

3.5. Kuranishi stack as an orbifold. We now investigate when the Kuranishi stack(s) is (are) an orbifold. Here by orbifold, we mean the stack given by the global quotient of an analytic space by an holomorphic action of a finite group fixing exactly one point. We have

Theorem 3.5. *The Kuranishi stack \mathcal{A} is an orbifold if and only if $\text{Aut}(X_0)$ is finite.*

Remark 3.6. To be more precise, the statement "the Kuranishi stack \mathcal{A} is an orbifold" must be understood as: "for a good choice of V and K_0 , the Kuranishi stack \mathcal{A} is an orbifold". This should appear clearly in the proof.

Proof. Since the isotropy group of X_0 is $\text{Aut}(X_0)$, the condition is obviously necessary. So let us assume that $\text{Aut}(X_0)$ is finite. We show that we may

choose V and K_0 so that the corresponding atlas $\mathcal{A} \rightrightarrows K_0$ of \mathcal{A} is Morita equivalent to the translation groupoid $\text{Aut}(X_0) \times K_0 \rightrightarrows K_0$.

We start with an arbitrary atlas $\mathcal{A} \rightrightarrows K_0$. We assume that the $\text{Aut}(X_0)$ version of (3.6) is valid. Given $f \in \text{Aut}(X_0)$, define Hol_f as in (3.5) and set

$$(3.10) \quad \sigma_f : J \in U_f \subset K_0 \mapsto (J, f \circ e(\chi_J)) \in \mathcal{A}$$

where χ is an analytic mapping from $U_f \subset K_0$ to W with $\chi(0) = 0$ defined by

$$(3.11) \quad \Xi(J \cdot f) = (J \cdot f) \cdot e(\chi(J))$$

Observe that

$$(3.12) \quad \text{Hol}_f(J) = f \circ e(\chi(J))$$

The map σ_f is a local analytic section of the source map $s : \mathcal{A} \rightarrow K_0$ defined on U_f . It satisfies

$$(3.13) \quad t \circ \sigma = \text{Hol}_f$$

The proof of Theorem 3.5 consists in the following two lemmas.

Lemma 3.7. *For all $f \in \text{Aut}(X_0)$, the map $\sigma_f : U_f \subset K_0 \rightarrow \mathcal{A}$ is the unique (up to restriction) extension of f .*

By extension of f , we mean a section F of s defined in a neighborhood of 0 and such that $F(0) = f$.

Proof of Lemma 3.7. The map σ_f is obviously an extension of f as desired.

Let now G be another extension of f . Then, for all $J \in K_0$ close to 0, we have a decomposition

$$(3.14) \quad G(J) = f \circ e(\eta(J))$$

using (3.6). Here the factor in $\text{Aut}(X_0)$ is constant equal to f since $\text{Aut}(X_0)$ is discrete.

The mapping η also satisfies (3.11). But since (3.3) is an isomorphism, (3.11) is uniquely verified and $\eta = \chi$. Thus $G = \sigma_f$ on a neighborhood of J_0 in K_0 . \square

Redefine K_0 as the intersection of all U_f for f in the finite group $\text{Aut}(X_0)$. Then all extensions σ_f are defined on K_0 . Set

$$(3.15) \quad \mathcal{E}xt = \{\sigma_f : K_0 \rightarrow \mathcal{A} \mid f \in \text{Aut}(X_0)\}$$

We have

Lemma 3.8. *$(\mathcal{E}xt, \circ)$ is a group isomorphic to $\text{Aut}(X_0)$.*

Proof of Lemma 3.8. Define

$$(3.16) \quad \sigma_g \circ \sigma_f : J \in K_0 \mapsto \sigma_g(J \cdot \sigma_f(J)) \circ \sigma_f(J)$$

This is an extension of $g \circ f$, and thus by Lemma 3.7 is equal to $\sigma_{g \circ f}$ on a neighborhood of J_0 , hence on K_0 by analyticity.

Still by Lemma 3.7, if $g_1 \circ \dots \circ g_k = \text{Id}$, then the same relation holds for the σ_{g_i} 's. \square

The space K_0 is invariant by the group $(\mathcal{E}xt, \circ)$. We may thus take as atlas for \mathcal{A} the translation groupoid $\mathcal{E}xt \times K_0 \rightrightarrows K_0$, or, equivalently the translation groupoid $\text{Aut}(X_0) \times K_0 \rightrightarrows K_0$. \square

Replacing \mathcal{A} with \mathcal{A}_1 , resp. \mathcal{A}_0 and $\text{Aut}(X_0)$ with $\text{Aut}^1(X_0)$, resp. $\text{Aut}^0(X_0)$ yields the following immediate corollaries.

Corollary 3.9. *The Kuranishi stack \mathcal{A}_1 is an orbifold if and only if the group $\text{Aut}^1(X_0)$ is finite.*

and

Corollary 3.10. *The Kuranishi stack \mathcal{A}_0 is an orbifold if and only if $\text{Aut}^0(X_0)$ is zero.*

4. THE NEIGHBORHOOD OF A POINT IN THE TEICHMÜLLER STACK

A neighborhood of X_0 in $\mathcal{T}(M)$ consists of M -deformations all of whose fibers are close to X_0 , that is can be encoded by structures J living in a neighborhood V of J_0 in $\mathcal{I}(M)$. As in [8], we shall denote it by $\mathcal{T}(M, V)$. From now on, we assume that V is open, connected and small enough to come equipped with a Kuranishi mapping (3.3).

4.1. Atlas. The main difficulty to construct an atlas in [8] was to describe all the morphisms between the different Kuranishi spaces involved. Here, in the local case, we just need to use one Kuranishi space and it is straightforward to give an atlas for $\mathcal{T}(M, V)$. Just consider

$$(4.1) \quad \mathcal{T}_V := \{(J, f) \in \text{Diff}^0(M, \mathcal{X}_0) \mid J \cdot f \in K_0\}$$

with structure maps as in (3.8) and (3.9). This gives an atlas $\mathcal{T}_V \rightrightarrows K_0$ for $\mathcal{T}(M, V)$. Observe that it is very close to that of the Kuranishi stacks. Indeed the points are exactly the same than those of \mathcal{A}_1 , but \mathcal{A}_1 has less morphisms, hence also less descent data and thus less objects. To understand how to pass from \mathcal{A}_1 to $\mathcal{T}(M, V)$, we need to understand and encode the "missing" morphisms.

4.2. Target Germification. To compare the local Teichmüller stack with the Kuranishi stacks, we need to germify V and consider only complex structures belonging to the germ of some point J_0 in V . This process is different from the germification process of section 3.4 which was about germifying the base category and thus the base of M -deformations. Here we still want to consider M -deformations over any analytic bases, but need to germify the set of possible fibers. Hence we need a target germification process, as opposed to the source germification process used in Section 3.4.

To do that, we look at sequences of stacks $\mathcal{T}(M, V_n)$ for (V_n) an inclusion decreasing sequence of neighborhoods of a fixed point J_0 with $V_0 = V$. Corresponding to a nesting sequence

$$(4.2) \quad \dots \subset V_n \subset \dots \subset V \subset \mathcal{I}$$

we obtain the sequence

$$(4.3) \quad \dots \longleftarrow \mathcal{T}(M, V_n) \longleftarrow \dots \longleftarrow \mathcal{T}(M, V)$$

As in the standard case of germ of topological spaces, we may endow the set of stacks $\mathcal{T}(M, V)$ for $V \subset \mathcal{S}$ with the following equivalence relation

$$(4.4) \quad \mathcal{T}(M, V) \sim \mathcal{T}(M, W) \iff V \cap V' = W \cap V'$$

for some neighborhood V' of J_0 in \mathcal{S} .

We call the resulting equivalence class of $\mathcal{T}(M)$ the *target germination* of $\mathcal{T}(M)$ at J_0 and denote it by $(\mathcal{T}(M), J_0)$.

To understand $(\mathcal{T}(M), J_0)$, we need to consider sequences of M -deformations over the same base $(\mathcal{X}_n \rightarrow B)$ such that \mathcal{X}_n is an object of $\mathcal{T}(M, V_n)$ for some decreasing sequence (4.2). We identify two such sequences $(\mathcal{X}'_n \rightarrow B)$ and $(\mathcal{X}_n \rightarrow B)$ if the families $\mathcal{X}'_n \rightarrow B$ and $\mathcal{X}_n \rightarrow B$ are isomorphic for every large n . Here are some examples of such sequences

- i) Start with a M -deformation $\mathcal{X} \rightarrow \mathbb{D}$ over the disk with central fiber isomorphic to X_0 and thus can be encoded in J_0 . Then consider the pull-back sequence $(\lambda_n^* \mathcal{X} \rightarrow \mathbb{D})$ where (λ_n) is a sequence of homotheties with ratio decreasing from 1 to 0.
- ii) Start with a fiber bundle $E \rightarrow B$ with fiber X_0 and structural group $\text{Aut}^1(X_0)$ and a M -deformation $\pi : \mathcal{X} \rightarrow B \times \mathbb{D}$ which coincides with the bundle E over $B \times \{0\}$. Then pick up some sequence (x_n) in the disk which converges to 0. Then consider the sequence of families $(\pi^{-1}(B \times \{x_n\}) \rightarrow B)$.

Morphisms from $(\mathcal{X}'_n \rightarrow B)$ to $(\mathcal{X}_n \rightarrow B)$ are sequences (f_n) with f_n a family morphism over B from \mathcal{X}'_n to \mathcal{X}_n for every n . Once again, we identify two such sequences (f_n) and (g_n) if there exists some integer k such that $f_n = g_n$ for $n \geq k$.

4.3. Sequences of isomorphic structures in the Kuranishi space.

Let (f_n) be a morphism from $(\mathcal{X}'_n \rightarrow B)$ to $(\mathcal{X}_n \rightarrow B)$, two sequences of $(\mathcal{T}(M), J_0)$, as explained above in Section 4.2. Since $\mathcal{T}_{V_n} \rightrightarrows K_0$ is an atlas for $\mathcal{T}(M, V_n)$, each f_n is obtained by gluing a cocycle of morphisms in \mathcal{T}_{V_n} over an open cover of V_n . Such morphisms are local morphisms of the Kuranishi family. Hence the existence of a morphism f_n acting non-trivially on the base is subject to the existence of two isomorphic distinct fibers of the Kuranishi family, that is to the existence of two distinct points in K_0 encoding the same complex manifold up to isomorphism. In the same way, the existence of sequences of morphisms (f_n) acting non-trivially on the base is subject to the existence of sequences (x_n) and (y_n) of points in K_0 such that

- i) Both sequences (x_n) and (y_n) converge to the base point of K_0 .
- ii) For every n , the fibers of the Kuranishi family above x_n and y_n are isomorphic.

In particular, there exists a sequence (ϕ_n) of $\text{Diff}^0(M)$ such that

$$(4.5) \quad \forall n, \quad x_n \cdot \phi_n = y_n$$

Now, we are looking for *missing* morphisms in the Kuranishi stacks. In other words, we are looking for such sequences (x_n) and (y_n) with the additional property that (x_n, ϕ_n) does not belong to \mathcal{A}_1 , that is (x_n, ϕ_n) is not (V_n, \mathcal{D}_0) -admissible.

From (4.5) and the convergence of (x_n) and (y_n) , we see that the sequence (ϕ_n) may exhibit three different types of behaviour.

- i) It converges uniformly on compact sets to a diffeomorphism g . This g fixes the base point of K_0 , hence is an automorphism of X_0 and (ϕ_n) is already encoded in the Kuranishi stacks.
- ii) It does not converge uniformly but the associated sequence of graphs converges in the cycle space \mathcal{C} .
- iii) Neither the sequence of diffeomorphisms nor the sequence of graphs converge.

Here \mathcal{C} denotes the Barlet space of (relative) n -cycles of $\mathcal{K}_0^{red} \times \mathcal{K}_0^{red}$ for \mathcal{K}_0^{red} the reduction of the Kuranishi family. Let also \mathcal{C}_0 be the Barlet space of n -cycles of $X_0 \times X_0$.

Each ϕ_n is an isomorphism between the fiber $\pi^{-1}(x_n)$ and the fiber $\pi^{-1}(y_n)$ of the Kuranishi family, hence defines an element γ_n of \mathcal{C} . When these cycles converge, the limit belongs to \mathcal{C}_0 . So we may reformulate the three previous cases as follows.

- i) The cycles γ_n converge to the graph of an automorphism of X_0 .
- ii) The cycles γ_n converge to a cycle in \mathcal{C}_0 which does not belong to the irreducible component(s) formed by $\text{Aut}^1(X_0)$.
- iii) The cycles γ_n do not converge.

5. FINITENESS PROPERTIES IN THE KÄHLER SETTING

In this section, we recall and apply a basic result on cycle spaces in the Kähler case, which is due to Lieberman [6]. We state the relative version, which is adapted to our purposes.

Proposition 5.1. *Let $\pi_i : \mathcal{X}_i \rightarrow B_i$ be smooth morphisms with compact Kähler fibers over reduced analytic spaces B_i for $i = 0, 1$. Let $E \subset B_0 \times B_1$ be a subset and let $\mathcal{Z} \rightarrow E$ be a family of relative cycles of $\mathcal{X}_0 \times \mathcal{X}_1$. Assume that the projection of E is included in a compact of $B_0 \times B_1$. Assume also that any cycle Z is the graph of a biholomorphism from some fiber $(X_0)_t$ onto some fiber $(X_1)_{t'}$. Then,*

- i) \mathcal{Z} only meets a finite number of irreducible components of the cycle space of $\mathcal{X}_0 \times \mathcal{X}_1$.
- ii) Let \mathcal{C} be such a component. Then \mathcal{C} contains a Zariski open subset \mathcal{C}_0 all of whose members are graphs of a biholomorphism between a fiber of \mathcal{X}_0 and a fiber of \mathcal{X}_1 .

Proof. i) Let $(\omega_t^i)_{t \in B_i}$ be a continuous family of Kähler forms on the π_i -fibers ($i = 0, 1$). Let M be the smooth model of X_0 and let $(J_t^i)_{t \in B_i}$ be a continuous family of integrable almost complex operators on M such that $(X_i)_t = (M, J_t^i)$. For every $e \in E$, call $f_e : M \rightarrow M$ the biholomorphism from some fiber $(X_0)_t$ onto some fiber $(X_1)_{t'}$ corresponding to the cycle Z_e . We compute the volume of these cycles using the ω_t . We have

$$\text{Vol}(Z_e) = \int_M (\omega_t^0 + f_e^* \omega_{t'}^1)^n = \int_M (\omega_t^0 + \omega_{t'}^1)^n$$

since f_e is isotopic to the identity hence $f_e^* \omega_{t'}^1$ and $\omega_{t'}^1$ differs from an exact form. Since the projection of E is included in a compact of $B_0 \times B_1$, we

obtain that the volume of the Z_e is uniformly bounded. It follows from [6, Theorem 1] that \mathcal{Z} has compact closure in the cycle space of $\mathcal{X}_0 \times \mathcal{X}_1$. Hence \mathcal{Z} only meets a finite number of irreducible components of this cycle space.

ii) Consider the family of cycles $\tilde{\mathcal{C}} \subset \mathcal{X}_0 \times \mathcal{X}_1 \rightarrow \mathcal{C}$. Since this map is proper and surjective, it is smooth on a Zariski open subset. Since some fibers are non singular, the generic fiber is non singular. The cycles above E are submanifolds of some $(X_0)_t \times (X_1)_{t'}$ with projections pr_i being bijective onto both factors. Hence, on a Zariski open subset of \mathcal{C} , every cycle enjoys such properties. So is the graph of a biholomorphism between a fiber of \mathcal{X}_0 and a fiber of \mathcal{X}_1 . \square

Setting $X_0 = (M, J_0)$, considering the orbit \mathcal{O} of J_0 in $\mathcal{I}(M)$ and viewing K_0 as a local transverse section, we obtain a first interesting Corollary.

Corollary 5.2. *If K_0 is small enough then K_0 intersects \mathcal{O} only at J_0 .*

Proof. We assume that K_0 is small enough so that it only contains Kähler points. We also assume that K_0 is reduced, replacing it with its reduction if necessary. Take $\mathcal{X}_1 = \mathcal{X}_0$ and $\mathcal{X}_0 = X_0$ seen as a family over the point $\{J_0\}$. Let E' be the subset of K_0 corresponding to complex structures J in the same $\text{Diff}^0(M)$ -orbit that J_0 . Now, E' is a subset of K_0 which does not contain any continuous path by Fischer-Grauert Theorem, see [7, Lemma 5] for the convenient geometric reformulation. So it contains at most a countable number of points. Since we are only interested in what happens close to J_0 , we may replace E' with its intersection with a compact neighborhood of J_0 in K_0 . Then for each $J \in E'$, choose some element f_J of $\text{Diff}^0(M)$ mapping J_0 onto it. Set $E = \{J_0\} \times E'$ and let \mathcal{Z} be the cycles corresponding to the graphs of the f_J . Apply Proposition 5.1. We conclude that \mathcal{Z} meets a finite number of irreducible components of the cycle space of $X_0 \times \mathcal{X}_0$, say $\mathcal{C}_1, \dots, \mathcal{C}_p$.

Still by Proposition 5.1, it follows that a Zariski open subset of each \mathcal{C}_i only contains graph of biholomorphisms between X_0 and some X_J with $J \in E'$. Hence each of these components only contains cycles in a fixed product $X_0 \times X_J$ and E' is a finite subset. Reducing K_0 if necessary, we may assume that E' is just $\{J_0\}$ as wanted. \square

Let Γ be the union of the irreducible components of \mathcal{C} containing a sequence (4.5). Let Γ_0 be the intersection of Γ with \mathcal{C}_0 . Observe that every cycle in Γ_0 projects onto each factor of $X_0 \times X_0$ through a map of degree one. In the same way, every cycle in Γ projects onto each factor of some $X_t \times X_{t'}$ through a map of degree one. We may state our second Corollary

Corollary 5.3. *Assume X_0 is Kähler. Then,*

- i) *Each irreducible component of Γ contains at least one cycle of $X_0 \times X_0$.*
- ii) *Γ has a finite number of components.*
- iii) *Take a component \mathcal{C} of Γ . Every irreducible component of the intersection $\Gamma_0 \cap \mathcal{C}$ is either the closure of a connected component of $\text{Aut}^1(X_0)$ or a component all of whose members are singular cycles.*

Definition 5.4. An exceptional component of Γ_0 is an irreducible component of some $\Gamma_0 \cap \mathcal{C}$, all of whose members are singular cycles. In this case, we also say that \mathcal{C} is exceptional above J_0 .

Exceptional components may be isolated singular cycles or components of positive dimension with a reducible generic member.

Notice that the irreducible components of an intersection $\Gamma_0 \cap \mathcal{C}$ may not be irreducible components of Γ_0 . Two distinct components of Γ may intersect in Γ_0 .

Proof. As above in the proof of Corollary 5.2, assume that K_0 is reduced and all the fibers of the Kuranishi family are Kähler. We may thus apply Proposition 5.1 to $\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}_0$ and to Γ . This proves that Γ has a finite number of components. It also proves that every sequence of Γ converges up to passing to a subsequence, and so case iii) of the list at the end of Section 4 is not possible. Moreover, if a component of some $\Gamma_0 \cap \mathcal{C}$ contains a (irreducible) manifold, this has to be the graph of an automorphism, since it projects with degree one onto each factor of $X_0 \times X_0$. By Proposition 5.1 applied to $\mathcal{X}_0 = \mathcal{X}_1 = X_0$ and to Γ_0 , this is a component of the closure of $\text{Aut}^1(X_0)$. \square

More precisely, let $\text{Irr } \Gamma$ be the set of irreducible components of Γ ; that is, each point of $\text{Irr } \Gamma$, encodes an irreducible component of Γ .

Corollary 5.5. *Assume X_0 is Kähler. Then,*

- i) *The number of irreducible components of the reduction of \mathcal{T}_V is finite.*
- ii) *If V is a sufficiently small neighborhood of J_0 in $\mathcal{I}(M)$, then there exists a natural bijection between the set of irreducible components of the reduction of \mathcal{T}_V and $\text{Irr } \Gamma$.*
- iii) *If V is a sufficiently small neighborhood of J_0 in $\mathcal{I}(M)$, and $V' \subset V$ contains also J_0 , then the natural inclusion of $\mathcal{T}_{V'}$ in \mathcal{T}_V is a bijection between the corresponding sets of irreducible components.*

Proof. Assume \mathcal{T}_V is reduced. Every irreducible component of \mathcal{T}_V injects in an irreducible component of \mathcal{C} . Just send (J, f) to its graph as a cycle of $\mathcal{X}_0 \times \mathcal{X}_0$. Since \mathcal{C} has only a finite number of components by Kählerianity, this proves i). Moreover, by Proposition 5.1, a component of \mathcal{T}_V forms a Zariski open subset of the corresponding component of \mathcal{C} . Then, using Corollary 5.3, for a sufficiently small neighborhood V of J_0 , the point J_0 is adherent to the s -image of every such component. Hence they contain a sequence (4.5). So the irreducible components of \mathcal{T}_V are in fact in 1:1 correspondence with the irreducible components of Γ , proving ii). Then iii) follows from ii). \square

As a consequence of Corollary 5.5, we do not need to consider the full target germification in the Kähler case. It is enough to look at $\mathcal{S}(M, V)$ for a fixed small enough V since restricting to smaller neighborhoods of 0 will not change the number of components of \mathcal{T}_V .

In other words, there is no wandering sequence (4.5) with each ϕ_n belonging to a different component of \mathcal{T}_V .

6. LOCAL STRUCTURE OF THE TEICHMÜLLER STACK IN THE KÄHLER SETTING

We want to analyse the structure of the analytic space \mathcal{T}_V defined in (4.1) and compare it with \mathcal{A}_1 in the Kähler setting.

We already observed in Section 4 that there is a natural inclusion of groupoids of \mathcal{A}_1 into \mathcal{T}_V . It comes from the fact that \mathcal{T}_V encodes every morphism between fibers of the Kuranishi family, whereas \mathcal{A}_1 encodes some morphisms between fibers of the Kuranishi family. This inclusion is just the description at the level of atlases of the natural inclusion of \mathcal{A}_1 into $\mathcal{T}(M, V)$: \mathcal{A}_1 -objects, resp. \mathcal{A}_1 -morphisms, inject in $\mathcal{T}(M, V)$ -objects, resp. $\mathcal{T}(M, V)$ -morphisms. So our final goal here is to give the structure of this inclusion.

There exists also a natural inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$. We first relate the morphisms encoded in \mathcal{T}_V to those encoded in \mathcal{A}_0 .

Lemma 6.1. *Let $(J, f) \in \mathcal{T}_V$ and let $(J, g) \in \mathcal{T}_V$. Then these two elements belong to the same connected component of \mathcal{T}_V if and only if $(J \cdot f, f^{-1} \circ g)$ belongs to \mathcal{A}_0 .*

Proof. Assume $(J \cdot f, f^{-1} \circ g)$ belongs to \mathcal{A}_0 , that is $(J \cdot f, f^{-1} \circ g)$ is (V, \mathcal{D}_0) -admissible. Then we may decompose it as

$$(J_1, f^{-1} \circ g) = (J_1, h_1) \circ (J_2, h_2) \circ \dots \circ (J_k, h_k)$$

where $J_1 := J \cdot f$. Now each $h_i \in \mathcal{D}_0$ can be decomposed as $k_i \circ e(\chi_i)$ through (3.6) and is thus isotopic to the identity inside \mathcal{D}_0 . Hence (J_1, Id) and (J_1, h_1) stay in the same connected component of \mathcal{T}_V , and so do (J, f) and $(J, f \circ h_1)$.

Conversely, let (J, f) and (J, g) belong to the same connected component of \mathcal{T}_V . Then, there exists an isotopy (J_t, f_t) joining these two points in \mathcal{T}_V . But then we may find $0 < t_1 < \dots < t_k < 1$ such that

$$(J_1, h_1) := (J \cdot f, f^{-1} \circ f_{t_1})$$

satisfies $h_1 \in U$, as well as

$$(J_2, h_2) := (J_1 \cdot h_1, f_{t_1}^{-1} \circ f_{t_2})$$

satisfies $h_2 \in U$ and so on. \square

We are now in position to state and prove our first main result.

Theorem 6.2. *The natural inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$, resp. of \mathcal{A}_1 into $\mathcal{T}(M, V)$, is a finite morphism of analytic stacks.*

By finite morphism of analytic stacks, we mean that, given any $B \in \mathfrak{S}$ and any morphism u from B to $\mathcal{T}(M, V)$, the fiber product

$$(6.1) \quad \begin{array}{ccc} B \times_u \mathcal{A}_0 & \xrightarrow{f_2} & \mathcal{A}_0 \\ f_1 \downarrow \square & & \downarrow \text{inclusion} \\ B & \xrightarrow{u} & \mathcal{T}(M, V) \end{array}, \quad \text{resp.} \quad \begin{array}{ccc} B \times_u \mathcal{A}_1 & \xrightarrow{f_2} & \mathcal{A}_1 \\ f_1 \downarrow \square & & \downarrow \text{inclusion} \\ B & \xrightarrow{u} & \mathcal{T}(M, V) \end{array}$$

satisfies

- i) $B \times_u \mathcal{A}_0$, resp. $B \times_u \mathcal{A}_1$, is a \mathbb{C} -analytic space.
- ii) The morphism f_1 is a finite morphism between \mathbb{C} -analytic spaces.

Remark 6.3. We emphasize that in the definition of analytic stack used in [8], we did not impose that the diagonal is representable, see the discussion in §2.4 of [8]. As a consequence, point (i) above does not follow from the fact that $\mathcal{T}(M, V)$ is an analytic stack but shall be proved by hands.

Proof. By Yoneda's lemma, a morphism $u : B \rightarrow \mathcal{T}(M, V)$ corresponds to a family $\mathcal{X} \rightarrow B$. The fiber product $B \times_u \mathcal{A}_0$ encodes the isomorphisms of $\mathcal{T}(M, V)$

$$(6.2) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\alpha} & \mathcal{X}' \\ & \searrow & \swarrow \\ & B & \end{array}$$

between families $\mathcal{X} \rightarrow \mathcal{X}'$ over B modulo isomorphisms β over B

$$(6.3) \quad \begin{array}{ccc} & & \mathcal{X}' \\ & \nearrow \alpha & \downarrow \beta \\ \mathcal{X} & & \mathcal{X}'' \\ & \searrow \alpha' & \end{array}$$

belonging to \mathcal{A}_0 .

Decompose B as a union of connected open sets $B_1 \cup \dots \cup B_k$ in such a way that the family \mathcal{X} , resp. \mathcal{X}' , is locally isomorphic above B_i to $u_i^* \mathcal{K}_0$ for some $u_i : B_i \rightarrow K_0$, resp. to $(u'_i)^* \mathcal{K}_0$ for some $u'_i : B_i \rightarrow K_0$. These local models are glued through a cocycle $u_{ij} : B_i \cap B_j \rightarrow \mathcal{T}_V$, resp. $u'_{ij} : B_i \cap B_j \rightarrow \mathcal{A}_0$, satisfying $s(u_{ij}) = u_i$ and $t(u_{ij}) = u_j$, resp. $s(u'_{ij}) = u'_i$ and $t(u'_{ij}) = u'_j$, to obtain a family isomorphic to \mathcal{X} , resp. \mathcal{X}' .

In these models, up to passing to a finer covering, an isomorphism (6.2) corresponds to a collection $F_i : B_i \rightarrow \mathcal{T}_V$ fulfilling

- i) $s \circ F_i = u_i$ and $t \circ F_i = u'_i$
- ii) $F_i \circ u_{ij} = u'_{ij} \circ F_j$

Then \mathcal{X}'' corresponds to a cocycle $u''_i : B_i \rightarrow K_0$ and α' to a collection $F'_i : B_i \rightarrow \mathcal{T}_V$ satisfying similar relations.

Denoting by α_i the isomorphism between $u_i^* \mathcal{K}_0$ and $(u'_i)^* \mathcal{K}_0$, and setting

$$(6.4) \quad F_i(b) = (u_i(b), F_{i,b}) \quad \text{and} \quad F'_i(b) = (u'_i(b), F'_{i,b})$$

we have

$$(6.5) \quad \alpha_i(b, v) = (u_i(b), F_{i,b}(v)) \quad \text{and} \quad \alpha'_i(b, v) = (u'_i(b), F'_{i,b}(v))$$

and

$$(6.6) \quad u_i(b) \cdot F_{i,b} = u'_i(b) \quad \text{and} \quad u_i(b) \cdot F'_{i,b} = u''_i(b)$$

Let β be $\alpha' \circ \alpha^{-1}$. This is a morphism of $\mathcal{T}(M, V)$ which is given in our localisation by

$$(6.7) \quad \beta_i(b, v) = (u'_i(b), G_{i,b}(v) := F_{i,b}^{-1} \circ F'_{i,b}(v))$$

We want to know when β is a morphism of \mathcal{A}_0 .

Since the B_i are connected, the image of each map F_i , F'_i and G_i , is included in a single connected component of \mathcal{T}_V . By Lemma 6.1, F_i and F'_i land in the same connected component of \mathcal{T}_V if and only if G_i lands in \mathcal{A}_0 . Choose a point b_i in each B_i . Then α and α' are equivalent through (6.3) if and only if $(b_i, F_i(b_i))$ and $(b_i, F'_i(b_i))$ belong to the same connected component of \mathcal{T}_V for all i .

Now, assume that $(b_1, F_1(b_1))$ and $(b_1, F'_1(b_1))$ belong to the same connected component of \mathcal{T}_V . Given $i \neq 1$ and taking $c \in B_1 \cap B_i$, it follows from the compatibility relations that

$$(6.8) \quad F_i(c) = u'_{i1} \circ F_1 \circ u_{1i}(c) \quad \text{and} \quad F'_i(c) = u''_{i1} \circ F'_1 \circ u_{1i}(c)$$

But u'_{i1} and u''_{i1} are mappings with values in \mathcal{A}_0 , hence $(c, F_i(c))$ and $(c, F'_i(c))$ belong to the same connected component of \mathcal{T}_V . And so do $(b_i, F_i(b_i))$ and $(b_i, F'_i(b_i))$.

Therefore, the fiber product $B \times_u \mathcal{A}_0$ identifies with the disjoint union of g copies of B , where g is at most the number of connected components of \mathcal{T}_V , through the map

$$(6.9) \quad (b, (u(b), F_b)) \in (B \times_u \mathcal{A}_0)_{\{b\}} \longmapsto (b, \sharp(u(b), F_b)) \in B \times \sharp\mathcal{T}_V$$

Here $(B \times_u \mathcal{A}_0)_{\{b\}}$ denotes the set of objects above $\{b\}$, the set $\sharp\mathcal{T}_V$ is the set of connected components of \mathcal{T}_V and the \sharp application maps an element of \mathcal{T}_V to the connected component of \mathcal{T}_V which contains it.

Finally f_1 can be rewritten as the natural projection map

$$(6.10) \quad B \times \sharp_g \mathcal{T}_V \longrightarrow B$$

for $\sharp_g \mathcal{T}_V$ the connected components of \mathcal{T}_V that can be attained through (6.9). This proves that the inclusion of \mathcal{A}_0 in $\mathcal{S}(M, V)$ is a finite morphism.

Recall that any element f of $\text{Aut}^1(X_0)$ admits a natural extension σ_f - see (3.10) - in \mathcal{T}_V whose s - and t -projections cover a neighborhood of J_0 in K_0 . Recall also that the s -projection of any component is at least adherent to J_0 . Hence, given an element (J, g) of some connected component \mathcal{C} of \mathcal{T}_V and an element f of some connected component A' of $\text{Aut}^1(X_0)$, we may compose (J, g) with $\sigma_f(J \cdot g)$ as soon as J is sufficiently close to J_0 . Moreover this composition lands in some component \mathcal{C}' of \mathcal{T}_V which depends only on \mathcal{C} and on A' . Hence there is an action of $\sharp\text{Aut}^1(X_0)$ onto $\sharp\mathcal{T}_V$.

One eventually finds that the fiber product $B \times_u \mathcal{A}_1$ identifies with the disjoint union of a finite number of copies of B and f_1 with the natural projection map

$$(6.11) \quad B \times \sharp_g \mathcal{T}_V / (\sharp\text{Aut}^1(X_0) \cap \sharp_g \mathcal{T}_V) \longrightarrow B$$

□

So, going back to the setting "Kur/Teich", we obtain that, in the Kähler case, there is a finite morphism from Kur to Teich. Our next step is to characterize the Kur=Teich case.

The fiber of this finite morphism is $\sharp\mathcal{T}_V / \sharp\text{Aut}^1(X_0)$, hence we obtain Kur=Teich if and only these two finite groups have same cardinal. Now, this occurs if and only if, for some \mathcal{C} in $\text{Irr } \Gamma$, the intersection $\Gamma_0 \cap \mathcal{C}$ contains a component with only singular cycles. This motivates the following definition.

Definition 6.4. A point X_0 of $\mathcal{S}(M)$ is called *exceptional* if Γ_0 has at least one exceptional component in the sense of Definition 5.4.

By the mere definition, X_0 satisfies Kur=Teich if and only if it is not an exceptional point of the Teichmüller stack. The atlas (4.1) being an

atlas of a neighborhood of X_0 in the Teichmüller stack, it contains all the information we need to decide which points close to X_0 are exceptional.

Indeed, pick a point X_J in K_0 . Assume it is exceptional. If V is sufficiently small, every irreducible component of \mathcal{C} contains a cycle of $X_0 \times X_0$, hence the set Γ_J of cycles of $X_J \times X_J$ which are limits of cycles of \mathcal{C} is included in Γ . As a consequence, J is exceptional if and only if there are only singular cycles in a component of Γ above J . Let S be the analytic set of singular cycles of \mathcal{C} . Let p denote the projection of Γ to $K_0 \times K_0$. The set of exceptional points is thus equal to

$$(6.12) \quad E = \bigcup_{\mathcal{C} \in \text{Irr } \Gamma} \{J \in K_0 \mid p^{-1}(J, J) \cap \mathcal{C} \neq \emptyset \text{ and } (p^{-1}(J, J) \cap \mathcal{C})_0 \subset S\}$$

where $(p^{-1}(J, J) \cap \mathcal{C})_0 \subset S$ means that some irreducible component of $p^{-1}(J, J) \cap \mathcal{C}$ is included in S . This is a constructible set.

Let E^c be the closure of (6.12). Now, E^c is a closed constructible set, hence an analytic set. We claim that E^c is a *strict* analytic subspace of K_0 ³. Assume the contrary. Then every cycle of a component \mathcal{C} above a Zariski open set of the diagonal in $K_0 \times K_0$ is singular. As we already argued several times, each component \mathcal{C} contains a Zariski open subset of graphs of biholomorphisms between fibers of the Kuranishi family. Moreover, if p_1 denotes the projection of $K_0 \times K_0$ onto the first factor composed with p , then the image of this Zariski open set by p_1 is an open set of K_0 . Hence, $p(\mathcal{C})$ is an analytic set of $K_0 \times K_0$ strictly containing the diagonal; and a Zariski open subset of it encodes graphs of biholomorphisms. We may thus find for a well chosen J close to 0 in K_0 , a path of biholomorphisms between X_J and some $X_{J'_t}$ with J'_t distinct from J . Hence K_0 has a non trivial foliated structure in the sense of [7]. But this implies that the dimension of $\text{Aut}^0(X_0)$ jumps at 0, that is is not constant in a neighborhood of 0 in K_0 . Since $\mathcal{K}_0 \rightarrow K_0$ is complete at every point J of K_0 , denoting its Kuranishi space K_J , then the set of exceptional points in K_J is also the full K_J . Hence the same argument tells that the dimension of the automorphism group also jumps at X_J . But it cannot jump at every point of K_0 . Contradiction. The set E^c is a strict analytic subspace of K_0 .

So we may define a strict analytic substack of $\mathcal{T}(M, V)$ as the stackification of the full subgroupoid of $\mathcal{T}_V \rightrightarrows K_0$ above $E^c \subset K_0$. Since the notion of exceptional point is an intrinsic notion, this substack is just a neighborhood of X_0 of an analytic substack of $\mathcal{T}(M, V_K)$. Here V_K is the open⁴ set of Kähler points of $\mathcal{I}(M)$. So we have proved our second main Theorem

Theorem 6.5. *The closure of exceptional points of the Teichmüller stack $\mathcal{T}(M, V_K)$ of Kählerian structures form a strict analytic substack $\mathcal{E}(M)$ of $\mathcal{T}(M, V_K)$.*

and its immediate Corollary

³As above in Section 5, we replace K_0 with its reduction if necessary, so strict means that E^c is not the whole K_0 .

⁴By a classical result of Kodaira, Kählerianity is a stable property through small deformations.

Corollary 6.6. *A compact complex Kähler manifold X satisfies Kur=Teich as well as the structures sufficiently close to it if and only if it belongs to $\mathcal{T}(M, V_K) \setminus \mathcal{E}(M)$.*

Hence Kähler points such that Kur=Teich as well as the structures sufficiently close to it fill a Zariski open substack of the Teichmüller stack.

Remark 6.7. The closure of exceptional points also form a strict analytic subspace of V_K .

Remark 6.8. In particular, the set of points satisfying Kur=Teich is dense in the Kählerian Teichmüller space $V_K/\text{Diff}^0(M)$ and contains an open set. However, due to the non-Hausdorff topology this space may have, this may be a misleading statement. For example, if M is $\mathbb{S}^2 \times \mathbb{S}^2$, then the (Kählerian) Teichmüller space of M , as a set, is \mathbb{Z} , a point $a \in \mathbb{Z}$ encoding the Hirzebruch surface \mathbb{F}_{2a} ⁵. Now, the topology to put on \mathbb{Z} has for (non trivial) open sets $\{0\}$, $\{0, 1\}$, $\{-1, 0\}$ and so on, cf. [8], Examples 5.14 and 12.6. Hence 0 is an open and dense subset of the Teichmüller space.

Remark 6.9. If the intersection of an exceptional component of X_0 and Γ_J is non-empty but contains regular cycles, then the corresponding morphisms

- i) either form a component of $\text{Aut}^1(X_J)$ which is not induced by $\text{Aut}^1(X_0)$.
- ii) or send J to points that are distinct from J and not adherent to it.

Hence, when restricting to a sufficiently small neighborhood W of X_J , this component disappears from $\mathcal{T}(M, W)$.

Finally, we deal with the orbifold case.

Theorem 6.10. *Assume X_0 Kähler. Then, the following statements are equivalent*

- i) $\mathcal{T}(M, V)$ is an orbifold.
- ii) \mathcal{A}_1 is an orbifold and X_0 is not exceptional.
- iii) $\text{Aut}^1(X_0)$ is finite and X_0 is not exceptional.
- iv) $\text{Aut}^0(X_0)$ is trivial and X_0 is not exceptional.

Remark 3.6 applies also here.

Proof. By Theorem 6.2, $\mathcal{T}(M, V)$ has finite isotropy groups if and only if \mathcal{A}_1 has finite isotropy groups. Now if X_0 is exceptional the finite group $\text{Irr } \Gamma/\text{Aut}^1(X_0)$ is not the stabilizer of the base structure X_0 , since the exceptional components do not yield automorphisms at 0. Also, if X_0 is not exceptional, then Kur=Teich by Theorem 6.5. This proves that i) and ii) are equivalent. Then, ii) and iii) are equivalent by Corollary 3.3. Finally, iii) and iv) are equivalent because of the fact that $\text{Aut}^0(X_0)$ has finite index in $\text{Aut}^1(X_0)$ in the Kähler case. \square

Remark 6.11. Consider the case of compact complex tori. Then $\text{Aut}^0(X_0)$ is not trivial, since it contains the translations. So neither $\mathcal{T}(M, V)$ nor \mathcal{A}_1 is an orbifold, since their isotropy groups are not finite. However, if we forget about the stack structure, the Teichmüller space is naturally a complex manifold. Indeed, roughly speaking, the stack is obtained from

⁵The surfaces \mathbb{F}_{2a} and \mathbb{F}_{-2a} are isomorphic, but not through a biholomorphism isotopic to the identity.

this complex manifold by attaching a group of translations to each point. More precisely, it is the universal family of tori, see [8], Example 13.1.

7. THE NON-KÄHLER CASE

In this section, we briefly investigate the non-Kähler case.

The atlases of the Kuranishi stacks and of $\mathcal{T}(M, V)$ in Sections 3.3 and 4.1 are still valid. However, Proposition 5.1 and Corollaries 5.2, 5.3, 5.5 are no more correct. As a consequence, there may be a countable number of connected components in \mathcal{T}_V ; and this number is not stable when restricting to smaller neighborhoods V . There may be wandering components. Indeed, taking $V' \subset V$, there is a natural injection of the set of connected components of \mathcal{T}_V into the set of connected components of $\mathcal{T}_{V'}$, but it is possible that for a basis of neighborhoods $V \supset V_1 \supset V_2 \dots$, all the injection maps are strict. So sequences (4.5) may jump from one component to another and exceptional components do not capture all sequences (4.5).

For this reason, we have to deal with the target germification of $\mathcal{T}(M, V)$ and it is not even clear that there is a reasonable sense to the assertion: the inclusion of \mathcal{A}_0 and \mathcal{A}_1 in $\mathcal{T}(M, V)$ is an analytic morphism of stacks with countable fiber.

Also recall that cycles do not always converge in the non-Kähler setting, hence case iii) at the end of Section 4 really occurs. So we add the definitions

Definition 7.1. We say that a component of Γ is *adherent* to X_0 if its projection onto $K_0 \times K_0$ does not contain $(0, 0)$ but is adherent to it. We say that a sequence of components of \mathcal{C} is *wandering* at X_0 if none of them is adherent to $(0, 0)$, but their union is.

and

Definition 7.2. A point X_0 of $\mathcal{T}(M)$ is called *adherent* if Γ has a component adherent to it. It is *wandering* if \mathcal{C} has a wandering sequence at X_0 .

Theorem 6.5 is no more correct but may now be replaced with

Theorem 7.3. *The closure of exceptional, adherent and wandering points of the Teichmüller stack form a countable union of strict analytic substacks of $\mathcal{T}(M)$.*

Proof. Remark that every exceptional component, every adherent component and every component of wandering points forms a constructible set of K_0 . Follow the proof of Theorem 6.5 to show that their closure is a strict analytic substack of $\mathcal{T}(M, V)$ and thus of $\mathcal{T}(M, V_K)$. Since there may exist a countable number of such components, the statement follows. \square

Corollary 7.4. *A compact complex Kähler manifold X satisfies Kur=Teich as well as the structures sufficiently close to it if and only if it belongs to $\mathcal{T}(M, V_K)$ minus an at most countable union of strict analytic substacks.*

Still due to the wandering components, Theorem 6.10 is no more correct in the non-Kähler setting and we do not have any equivalent.

8. OPEN PROBLEMS

There are many open problems around this. We list some of them in this section. Firstly, at the level of the Teichmüller stack/Kuranishi space,

Problem 8.1. Find exceptional points.

and, in the non-Kähler setting,

Problem 8.2. Find exceptional, adherent and wandering points.

Remark 8.3. Consider the second Hirzebruch surface \mathbb{F}_2 . It deforms onto $\mathbb{P}^1 \times \mathbb{P}^1$. The automorphism g that exchanges the two components of the product $\mathbb{P}^1 \times \mathbb{P}^1$ does not extend as an automorphism of \mathbb{F}_2 . So the corresponding graphs of biholomorphisms extend as singular cycles in $\mathbb{F}_2 \times \mathbb{F}_2$. Hence there is a component of singular cycles in the cycle space of $\mathbb{F}_2 \times \mathbb{F}_2$ that deform as automorphisms of the nearby structures. This really looks like an exceptional component. However, it is not, for g is clearly not isotopic to the identity.

Then, at the level of compact complex manifolds,

Problem 8.4. Find a compact complex manifold X_0 with $\text{Aut}^1(X_0)$ having an infinite number of connected components.

Such an example would automatically have $\sharp\mathcal{T}_V$ infinite. But there is no reason for them to be equal. Hence we ask

Problem 8.5. Find a compact complex manifold X_0 with infinite $\sharp\mathcal{T}_V$ but $\text{Aut}^1(X_0)$ having a finite number of connected components.

Both problems concern non-Kähler manifolds but in the Kähler setting, the difference between $\sharp\mathcal{T}_V$ and $\text{Aut}^1(X_0)/\text{Aut}^0(X_0)$ is also unknown. So we state our last problem

Problem 8.6. Find a compact Kähler manifold with $\sharp\mathcal{T}_V$ different from $\text{Aut}^1(X_0)/\text{Aut}^0(X_0)$.

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