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Shape optimization of a Dirichlet type energy for semilinear elliptic partial differential equations

Antoine Henrot* Idriss Mazari† Yannick Privat‡

Abstract

Minimizing the so-called “Dirichlet energy” with respect to the domain under a volume constraint is a standard problem in shape optimization which is now well understood. This article is devoted to a prototypical non-linear version of the problem, where one aims at minimizing a Dirichlet-type energy involving the solution to a semilinear elliptic PDE with respect to the domain, under a volume constraint. One of the main differences with the standard version of this problem rests upon the fact that the criterion to minimize does not write as the minimum of an energy, and thus most of the usual tools to analyze this problem cannot be used. By using a relaxed version of this problem, we first prove the existence of optimal shapes under several assumptions on the problem parameters. We then analyze the stability of the ball, expected to be a good candidate for solving the shape optimization problem, when the coefficients of the involved PDE are radially symmetric.

Keywords: shape optimization, Dirichlet energy, existence/stability of optimal shapes.

AMS classification: 49J45, 49K20.

1 Introduction

1.1 Motivations and state of the art

Existence and characterization of domains minimizing or maximizing a given shape functional under constraint is a long story. Such issues have been much studied over the last decades (see e.g. [1, 6, 9, 15, 12]). Recent progress has been made in understanding such issues for problems involving for instance spectral functionals (see e.g. [11]).

The issue of minimizing the Dirichlet energy (in the linear case) with respect to the domain is a basic and academical shape optimization problem under PDE constraint, which is by now well understood. This problem reads:

Let $d \in \mathbb{N}^$ and D be a smooth compact set of \mathbb{R}^d . Given $g \in H^{-1}(D)$ and $m \leq |D|$, minimize the Dirichlet energy*

$$J(\Omega) = \frac{1}{2} \int_{\Omega} |\nabla u_{\Omega}|^2 - \langle g, u_{\Omega} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)},$$

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where u_Ω is the unique solution of the Dirichlet problem¹ on Ω associated to g , among all open bounded sets $\Omega \subset D$ of Lebesgue measure $|\Omega| \leq m$.

As such, this problem is not well-posed and it has been shown (see e.g. [8] or [12, Chap. 4] for a survey of results about this problem) that optimal sets only exist within the class

$$\mathcal{O}_m = \{\Omega \in \mathcal{A}(D), |\Omega| \leq m\}, \quad (1)$$

where $\mathcal{A}(D)$ denotes the class of quasi-open sets² of D .

This article is motivated by the observation that, in general, the techniques used to prove existence, regularity and even characterization of optimal shapes for this problem rely on the fact that the functional is "energetic", in other words that the PDE constraint can be handled by noting that the full shape optimization problem rewrites

$$\min_{\substack{\Omega \in \mathcal{A}(D) \\ |\Omega| \leq m}} \min_{u \in W_0^{1,2}(D)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \langle g, u \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)} \right\}.$$

In this article, we introduce and investigate a prototypical problem close to the standard "Dirichlet energy shape minimization", involving a nonlinear differential operator. The questions we wish to study here concern existence of optimal shapes and stability issues for "non energetic" models. We note that the literature regarding existence and qualitative properties for non-energetic, non-linear optimization problems is scarce. We nevertheless mention [16], where existence results are established in certain asymptotic regimes for a shape optimization problem arising in population dynamics.

Since our aim is to investigate the optimization problems in the broadest classes of measurable domains, a volume constraint, known to lead to potential difficulties due to lack of compactness for standard topologies, is considered.

In the close version of the Dirichlet problem we will deal with, the linear PDE solved by u_Ω is changed into a nonlinear one but the functional to minimize remains the same. Since, in such a case, the problem is not "energetic" anymore (in the sense made precise above), the PDE constraint cannot be incorporated into the shape functional. This calls for new tools to be developed in order to overcome this difficulty. Among others, we are interested in the following issues:

- **Existence:** is the resulting shape optimization problem well-posed?
- **Stability of optimal sets:** given a minimizer Ω_0^* for the Dirichlet energy in the linear case, is Ω_0^* still a minimizer when considering a "small enough" non-linear perturbation of the problem?

This article is organized as follows: the main results, related to the existence of optimal shapes for Problem (3) and the criticality/stability of the ball are gathered in Section 2. Section 3 is dedicated to the proofs of the existence results whereas Section 4 is dedicated to the proofs of the stability results.

¹in other words

$$u_\Omega = \operatorname{argmax}_{u \in W_0^{1,2}(D)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \langle g, u \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)} \right\}.$$

²Recall that $\Omega \subset D$ is said quasi-open whenever there exists a non-increasing sequence $(\omega_n)_{n \in \mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \Omega \cup \omega_n \text{ is open} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \operatorname{cap}(\omega_n) = 0$$

1.2 The shape optimization problem

In what follows, we consider a modified version of the problem described above, where the involved PDE constraint is now nonlinear.

Let $d \in \mathbb{N}^*$, D a compact set of \mathbb{R}^d , $g \in L^2(D)$ and $f \in W^{1,\infty}(\mathbb{R})$. For a small enough positive parameter ρ , let $u_\Omega \in W_0^{1,2}(\Omega)$ be the unique solution of the problem

$$\begin{cases} -\Delta u_{\rho,\Omega} + \rho f(u_{\rho,\Omega}) = g & \text{in } \Omega \\ u_{\rho,\Omega} \in W_0^{1,2}(\Omega). \end{cases} \quad (2)$$

For $m \leq |D|$, solve the problem:

$$\inf_{\Omega \in \mathcal{O}_m} J_\rho(\Omega) \quad \text{where } J_\rho(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_{\rho,\Omega}|^2 - \int_\Omega g u_{\rho,\Omega}, \quad (3)$$

where \mathcal{O}_m is defined in (1).

In this problem, the smallness assumption on the parameter ρ guarantees the well-posedness of the PDE problem (2) for generic choices of nonlinearities f .

Lemma 1. *There exists $\underline{\rho} > 0$ such that, for any $\Omega \in \mathcal{O}_m$, for any $\rho \in [0, \underline{\rho})$, Equation (2), understood through its variational formulation, has a unique solution in $W_0^{1,2}(\Omega)$.*

This follows from a simple fixed-point argument: let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet Laplacian on Ω . We note that the operator

$$\begin{aligned} T : W_0^{1,2}(\Omega) &\longrightarrow W_0^{1,2}(\Omega) \\ u &\longmapsto w_\Omega, \end{aligned}$$

where w_Ω is the unique solution of

$$\begin{cases} -\Delta w - g = -\rho f(u) & \text{in } \Omega \\ w \in W_0^{1,2}(\Omega), \end{cases}$$

is Lipschitz with Lipschitz constant $C_T(\Omega)$ such that $C_T(\Omega) \leq \rho \frac{1}{\lambda_1(\Omega)} \|f\|_{W^{1,\infty}}$. By the monotony of λ_1 with respect to domain inclusion (see [10]), we have, for every $\Omega \in \mathcal{O}_m$, $\lambda_1(D) \leq \lambda_1(\Omega)$, so that $C_T(\Omega) \leq \frac{\rho \|f\|_{W^{1,\infty}}}{\lambda_1(D)}$.

2 Main results of the paper

2.1 Existence results

We state hereafter a partial existence result inherited from the linear case. Indeed, we will exploit a monotonicity property of the shape functional J_ρ together with its lower-semi continuity for the γ -convergence to apply the classical theorem by Buttazzo-DalMaso (see Subsection 3.1). Our approach takes advantage of the analysis of a relaxed formulation of Problem (3). To introduce it, let us first consider a given box $D \subset \mathbb{R}^n$ (i.e a smooth, compact subset of \mathbb{R}^n) such that $|D| > V_0$.

In the minimization problem (3), let us identify a shape Ω with its characteristic function $\mathbb{1}_\Omega$. This leads to introducing the “relaxation” set

$$\widehat{\mathcal{O}}_m = \left\{ a \in L^\infty(D, [0, 1]) \text{ such that } \int_D a \leq m \right\}$$

For a given positive relaxation parameter M , we define the (relaxed) functional $\hat{J}_{M,\rho}$ by

$$\hat{J}_{M,\rho}(a) = \frac{1}{2} \int_D |\nabla u_{M,\rho,a}|^2 + \frac{M}{2} \int_\Omega (1-a) u_{M,\rho,a}^2 - \int_\Omega g u_{\rho,\Omega}, \quad (4)$$

for every $a \in \widehat{\mathcal{O}}_m$, where $u_{M,\rho,a} \in W_0^{1,2}(D)$ denotes the unique solution of the non-linear problem

$$\begin{cases} -\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) = g & \text{in } D \\ u_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (5)$$

Our existence result involves a careful asymptotic analysis of $u_{M,\rho,a}$ as $\rho \rightarrow 0$ to derive a monotonicity property.

Standard elliptic estimates entail that, for every $M > 0$ and $a \in \widehat{\mathcal{O}}_m$, one has $u_{M,\rho,a} \in \mathcal{C}^0(\overline{\Omega})$.

Remark 1. Such an approximation of $u_{\rho,\Omega}$ is rather standard in the framework of fictitious domains. The introduction of the term $M(1-a)$ in the PDE has an interpretation in terms of porous materials (see e.g. [7]) and it may be expected that $u_{M,\rho,a}$ converges in some sense to $u_{\rho,\Omega}$ as $M \rightarrow +\infty$ and whenever $a = \mathbb{1}_\Omega$. This will be confirmed in the analysis to follow.

Roughly speaking, the existence result stated in what follows requires the right-hand side of equation (2) to have a constant sign. To write the hypothesis down, we need a few notations related to the relaxed problem (5), which is the purpose of the next lemma.

Lemma 2. *Let $m \in [0, |D|]$, $a \in \widehat{\mathcal{O}}_m$ and $g \in L^2(D)$ be nonnegative. There exists a positive constant $N_{m,g}$ such that*

$$\forall a \in \widehat{\mathcal{O}}_m, \forall M > 0, \forall \rho \in [0, \underline{\rho}], \quad \|u_{M,\rho,a}\|_\infty \leq N_{m,g}, \quad (6)$$

where $\underline{\rho}$ is defined in Lemma 1, $u_{M,\rho,a}$ denotes the unique solution to (5). In what follows, $N_{m,g}$ will denote the optimal constant in the inequality above, namely

$$N_{m,g} = \sup\{\|u_{M,\rho,a}\|_{L^\infty(\Omega)}, a \in \widehat{\mathcal{O}}_m, M > 0, \rho \in [0, \underline{\rho}]\}.$$

This follows from standard arguments postponed to Section A.

We now state the main results of this section. Let us introduce the assumptions we will consider hereafter:

- (H₁) There exist two positive numbers g_0, g_1 such that $g_0 < g_1$ and $g_0 \leq g(\cdot) \leq g_1$ a.e. in D .
- (H₂) One has $f \in W^{1,\infty}(\mathbb{R}) \cap D^2$, where D^2 is the set of twice differentiable functions (with second derivatives not necessarily continuous). Moreover, $f(0) \leq 0$ and there exists $\delta > 0$ such that the mapping $x \mapsto xf(x)$ is non-decreasing on $[0, N_{m,g} + \delta]$ where $N_{m,g}$ is given by Lemma 2.

Theorem 1. *Let us assume that one of the following assumptions holds true:*

- g or $-g$ satisfies the assumption (H₁);
- g is non-negative and the function f satisfies the assumption (H₂) or g is non-positive and the function $-f$ satisfies the assumption (H₂);

Then, there exists a positive constant $\rho_0 = \rho_0(D, f(0), \|f\|_{W^{1,\infty}})$ such that the shape optimization problem (3) has a solution Ω^ for every $\rho \in (0, \rho_0)$.*

Remark 2. The proof of Theorem 1 rests upon a monotonicity property of the relaxed functional $\hat{J}_{M,\rho}$ given by (4). This is the first ingredient that subsequently allows the well-known existence result of Buttazzo and Dal-Maso to be applied.

It is natural to wonder whether or not it would be possible to obtain this result in a more direct way, for instance by using shape derivatives. We claim that such an approach would require to consider domains Ω having a smooth boundary so that the shape derivative (in the sense of Hadamard) of J_ρ at Ω in direction V , where V denotes an adequate vector field, makes sense. This relaxed version enables us to encompass less regular domains.

2.2 Stability results

In what follows, we will work in \mathbb{R}^2 . \mathbb{B}^* denotes the centered ball with radius $R > 0$ such that $\mathbb{B}^* \in \mathcal{O}_m$ and we introduce $\mathbb{S}^* = \partial\mathbb{B}^*$. The notation ν stands for the outward unit vector on \mathbb{S}^* , in other words $\nu(x) = x/|x|$ for all $x \in \mathbb{S}^*$.

In this section, we will discuss the local optimality of the ball for small nonlinearities. We will in particular highlight that the local optimality of the ball can be either preserved or lost depending on the choice of the right-hand side g . Indeed, if $\rho = 0$ and if g is radially symmetric and non-increasing, the Schwarz rearrangement³ ensures that, for any $\Omega \in \mathcal{O}_m$, $J_0(\Omega) \geq J_0(\mathbb{B}^*)$. Without such assumptions, not much is known about the qualitative properties of the optimizers.

According to the considerations above, we will assume in the whole section that

(H₃) g is a non-increasing, radially symmetric and non-negative function in $L^2(\mathbb{B}^*)$ and f is a given \mathcal{C}^2 non-linearity.

Notice that the analysis to follow can be generalized to sign-changing g . Here, this assumption allows us to avoid distinguishing between the cases where the signs of normal derivatives on \mathbb{S}^* are positive or negative. For the sake of simplicity, for every $\rho \geq 0$, we will call u_ρ the solution of the PDE

$$\begin{cases} -\Delta u_\rho + \rho f(u_\rho) = g & \text{in } \mathbb{B}^* \\ u_\rho \in W_0^{1,2}(\mathbb{B}^*) & \text{on } \partial\mathbb{B}^* = \mathbb{S}^*. \end{cases} \quad (7)$$

Proving a full stationarity result⁴ is too intricate to tackle, since we do not know the minimizers topology. Hereafter, we investigate the local stability of the ball \mathbb{B}^* : we will prove that the ball is always a critical point, and show that we obtain different stability results, related to the non-negativity of the second shape derivative of the Lagrangian, depending on f and g .

To compute the first and second order shape derivatives, it is convenient to consider vector fields $V \in W^{3,\infty}(\mathbb{R}^2)$ and to introduce, for a given admissible vector field V (i.e such that, for t small enough, $(\text{Id} + tV)\mathbb{B}^* \in \mathcal{O}_m$), the mapping

$$f_V : t \mapsto J_\rho((\text{Id} + tV)\mathbb{B}^*).$$

The first (resp. second) order shape derivative of J_ρ in the direction V is defined as

$$J'_\rho(\mathbb{B}^*)[V] := f'_V(0), \text{ (resp. } J''_\rho(\mathbb{B}^*)[V, V] := f''_V(0)).$$

To enforce the volume constraint $|\Omega| = m$, we work with the unconstrained functional

$$\mathcal{L}_{\Lambda_\rho} : \Omega \mapsto J_\rho(\Omega) - \Lambda_\rho(\text{Vol}(\Omega) - m),$$

³see e.g. [14] for an introduction to the Schwarz rearrangement.

⁴ in other words, proving that for any $\rho \leq \rho^*$ \mathbb{B}^* is the unique minimizer of J_ρ in \mathcal{O}_m

where Vol denotes the Lebesgue measure in \mathbb{R}^2 and Λ_ρ denotes a Lagrange multiplier associated with the volume constraint. Recall that, for every domain Ω with a \mathcal{C}^2 boundary and every vector field in $W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, we have

$$\text{Vol}'(\Omega)[V] = \int_{\partial\Omega} V \cdot \nu \quad \text{and} \quad \text{Vol}''(\Omega)[V, V] = \int_{\partial\Omega} H(V \cdot \nu)^2,$$

where H stands for the mean curvature of $\partial\Omega$. The local first and second order optimality conditions for Problem (3) read as follow:

$$\left. \begin{array}{l} \mathcal{L}'_{\Lambda_\rho}(\Omega)[V] = 0 \\ \mathcal{L}''_{\Lambda_\rho}(\Omega)[V, V] \geq 0 \end{array} \right\} \text{ for every } V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2) \text{ such that } \int_{\mathbb{S}^*} V \cdot \nu = 0.$$

For further informations about shape derivatives, we refer for instance to [12, Chapitre 5]. Let us state the main result of this section. In what follows, ρ is chosen small enough so that Equation (2) has a unique solution.

Theorem 2. *Let f and g satisfying the assumption (\mathbf{H}_3) . Let $V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ denote a vector field such that $\int_{\mathbb{S}^*} V \cdot \nu = 0$.*

1. *(Shape criticality) \mathbb{B}^* is a critical shape, in other words $J'_\rho(\mathbb{B}^*)[V] = 0$.*
2. *(Shape stability) Assume that*

$$\pi R^2 g(R) \leq \int_{\mathbb{B}^*} g \quad \text{and} \quad 0 < \int_{\mathbb{B}^*} g, \quad (8)$$

where R denotes the radius of the ball \mathbb{B}^* . Let Λ_ρ be the Lagrange multiplier associated with the volume constraint. There exists $\bar{\rho} > 0$ and $C > 0$ such that, for any $\rho \leq \bar{\rho}$,

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V] \geq C \|V \cdot \nu\|_{L^2(\Omega)}^2. \quad (9)$$

3. *(Shape instability) Assume that g is the constant function equal to 1 and that f is a \mathcal{C}^1 non-negative function such that $f' < -1$ on $[0, 2\|u_0\|_{L^\infty})$, where u_0 is the solution of (2) with $\rho = 0$ and $\Omega = \mathbb{B}^*$. Then, the second order optimality conditions are not fulfilled on \mathbb{B}^* : there exists $\bar{\rho} > 0$ and $\hat{V} \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\int_{\mathbb{S}^*} \hat{V} \cdot \nu = 0$ and, for any $\rho \leq \bar{\rho}$,*

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[\hat{V}, \hat{V}] < 0.$$

Remark 3. Let us comment on the strategy of proof. It is known that estimates of the kind (9) can lead to local quantitative inequalities [4]. We first establish (9) in the case $\rho = 0$, and then extend it to small parameters ρ with the help of a perturbation argument. Assumptions of the type (8) are fairly well-known, and amount to requiring that \mathbb{B}^* is a stable shape minimizer [5, 13]. Finally, the instability result rests upon the following observation: if $g = 1$ and if V_1 is the vector field given by $V(r \cos(\theta), r \sin(\theta)) = \cos(\theta)(r \cos(\theta), r \sin(\theta))$, then one has

$$(J'_0 - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] = 0$$

while higher order modes are stable [5, 13]. It therefore seems natural to consider such perturbations when dealing with for small parameters ρ .

It should also be noted that our proof uses a comparison principle, which shortens many otherwise lengthy computations.

3 Proof of Theorem 1

3.1 General outline of the proof

The proof of Theorem 1 rests upon an adaptation of the standard existence result by Buttazzo-DalMaso (see either the original article [2] or [12, Thm 4.7.6] for a proof), based on the notion of γ -convergence, that we recall below.

Definition 1. For any quasi-open set $\Omega \in \mathcal{O}_m$, let R_Ω be the resolvent of the Laplace operator on Ω . We say that a sequence of quasi-open sets $(\Omega_k)_{k \in \mathbb{N}}$ in \mathcal{O}_m γ -converges to $\Omega \in \mathcal{O}_m$ if, for any $\ell \in H^{-1}(D)$, $(R_{\Omega_k}(\ell))_{k \in \mathbb{N}}$ converges in $W_0^{1,2}(D)$ to $R_\Omega(\ell)$.

The aforementioned existence theorem reads as follows.

Theorem (Buttazzo-DalMaso). Let $J : \mathcal{O}_m \rightarrow \mathbb{R}$ be a shape functional satisfying the two following assumptions:

1. (monotonicity) For every $\Omega_1, \Omega_2 \in \mathcal{O}_m$, $\Omega_1 \subseteq \Omega_2 \Rightarrow J(\Omega_2) \leq J(\Omega_1)$.
2. (γ -continuity) J is lower semi-continuous for the γ -convergence.

Then the shape optimization problem

$$\inf_{\Omega \in \mathcal{O}_m} J(\Omega)$$

has a solution.

As is customary when using this result, the lower semi-continuity for the γ -convergence is valid regardless of any sign assumptions on g or of any additional hypothesis on f . This is the content of the next result, whose proof is standard and thus, postponed to Appendix B.

Proposition 1. Let $f \in W^{1,\infty}(\mathbb{R})$ and $\rho \geq 0$. The functional J_ρ is lower semi-continuous for the γ -convergence.

It remains hence to investigate the monotonicity of J_ρ . Our approach uses a relaxed version of J_ρ , namely the functional $\hat{J}_{M,\rho}$ defined by (4). More precisely, we will prove under suitable assumptions that

$$\forall M \geq 0, \forall a_1, a_2 \in \widehat{\mathcal{O}}_m, a_1 \leq a_2 \implies \hat{J}_{M,\rho}(a_1) \geq \hat{J}_{M,\rho}(a_2). \quad (10)$$

The following result, whose proof is postponed to Appendix C for the sake of clarity, allows us to make the link between $\hat{J}_{M,\rho}$ and J_ρ .

Lemma 3. Let $\Omega \in \mathcal{O}_m$. There exists $(M_n)_{n \in \mathbb{N}}$ such that

$$M_n \rightarrow +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \hat{J}_{M_n,\rho}(\mathbb{1}_\Omega) = J_\rho(\Omega).$$

Setting then $a_1 = \mathbb{1}_{\Omega_1}$, $a_2 = \mathbb{1}_{\Omega_2}$, $M = M_n$, where M_n is chosen as in the statement of Lemma 3, and passing to the limit in (10) as $n \rightarrow \infty$ gives the monotonicity of J_ρ .

In the next sections, we will concentrate on showing the monotonicity property (10). To this aim, we will carefully analyze the so-called ‘‘switching function’’ (representing the gradient of the functional $\hat{J}_{M,\rho}$) as the parameter M is large enough.

3.2 Structure of the switching function

It is notable that, in this section, we will not make any assumption on g or f . Let $M > 0$. Considering the following relaxed version of Problem (3)

$$\inf_{a \in \widehat{\mathcal{O}}_m} \hat{J}_{M,\rho}(a), \quad (11)$$

it is convenient to introduce the set of *admissible perturbations* in view of deriving first order optimality conditions.

Definition 2 (tangent cone, see e.g. [3]). *Let $a^* \in \mathcal{M}$ and \mathcal{T}_{a^*} be the tangent cone to the set $\widehat{\mathcal{O}}_m$ at a^* . The cone \mathcal{T}_{a^*} is the set of functions $h \in L^\infty(D)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(D)$ converging to h as $n \rightarrow +\infty$, and $a^* + \varepsilon_n h_n \in \mathcal{O}_m$ for every $n \in \mathbb{N}$.*

In what follows, for any $a \in \widehat{\mathcal{O}}_m$, any element h of the tangent cone \mathcal{T}_a will be called an *admissible direction*.

Lemma 4 (Differential of $\hat{J}_{M,\rho}$). *Let $a \in \widehat{\mathcal{O}}_m$ and $h \in \mathcal{T}_a$. Let $v_{M,\rho,a}$ be the unique solution of*

$$\begin{cases} -\Delta v_{M,\rho,a} + M(1-a)v_{M,\rho,a} + \rho f'(u_{M,\rho,a})v_{M,\rho,a} = \rho f(u_{M,\rho,a}) & \text{in } D \\ v_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (12)$$

Then, $\hat{J}_{M,\rho}$ is differentiable in the sense of Fréchet at a in the direction h and its differential reads $\langle d\hat{J}_{M,\rho}(a), h \rangle = \int_D h \Psi_a$, where Ψ_a is the so-called “switching function” defined by

$$\Psi_a = -M \left(v_{M,\rho,a} + \frac{u_{M,\rho,a}}{2} \right) u_{M,\rho,a}.$$

Proof of Lemma 4. The Fréchet-differentiability of $\hat{J}_{M,\rho}$ and of the mapping $\mathcal{O}_m \ni a \mapsto u_{M,\rho,a} \in W_0^{1,2}(D)$ at m^* is standard (see e.g. [12, Chap. 8]). Let us consider an admissible perturbation h of a and let $\dot{u}_{M,\rho,a}$ be the differential of $u_{M,\rho,a}$ at a in direction h . One has

$$\begin{aligned} \langle d\hat{J}_{M,\rho}(a), h \rangle &= \int_D \nabla u_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a)u_{M,\rho,a} \dot{u}_{M,\rho,a} - \frac{M}{2} \int_D h u_{M,\rho,a}^2 \\ &\quad - \langle g, \dot{u}_{M,\rho,a} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)}, \end{aligned}$$

where $\dot{u}_{M,\rho,a}$ solves the system

$$\begin{cases} -\Delta \dot{u}_{M,\rho,a} + M(1-a)\dot{u}_{M,\rho,a} + \rho f'(u_{M,\rho,a})\dot{u}_{M,\rho,a} = M h u_{M,\rho,a} & \text{in } D \\ \dot{u}_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (13)$$

Let us multiply the main equation of (5) by $\dot{u}_{M,\rho,a}$ and then integrate by parts. We get

$$\int_D \nabla u_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a)u_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a} = \langle g, \dot{u}_{M,\rho,a} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)}$$

and therefore,

$$\langle d\hat{J}_{M,\rho}(a), h \rangle = -\frac{M}{2} \int_D h u_{M,\rho,a}^2 - \rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a}$$

Let us multiply the main equation of (13) by $v_{M,\rho,a}$ and then integrate by parts. We get

$$\int_D \nabla v_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a)v_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f'(u_{M,\rho,a}) \dot{u}_{M,\rho,a} v_{M,\rho,a} = M \int_D h u_{M,\rho,a} v_{M,\rho,a}.$$

Similarly, multiplying the main equation of (12) by $\dot{u}_{M,\rho,a}$ and then integrating by parts yields

$$\int_D \nabla v_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a) v_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f'(u_{M,\rho,a}) \dot{u}_{M,\rho,a} v_{M,\rho,a} = \rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a}.$$

Combining the two relations above leads to

$$\rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a} = M \int_D h u_{M,\rho,a} v_{M,\rho,a}.$$

Plugging this relation into the expression of $\langle d\hat{J}_{M,\rho}(a), h \rangle$ above yields the expected conclusion. \square

3.3 Proof that (10) holds true whenever ρ is small enough

Let us consider each set of assumptions separately.

Existence under the first assumption: g or $-g$ satisfies the assumption (\mathbf{H}_1) .

According to the discussion carried out in Section 3.1, proving Theorem 1 boils down to proving monotonicity properties for the functional $\hat{J}_{M,\rho}$ whenever ρ is small enough, which is the purpose of the next result.

Lemma 5. *Let a_1 and a_2 be two elements of $\hat{\mathcal{O}}_m$ such that $a_1 \leq a_2$ a.e. in D . If g or $-g$ satisfies the assumption (\mathbf{H}_1) , then there exists $\rho_1 = \rho_1(D, g_0, g_1, \|f\|_{W^{1,\infty}}) > 0$ such that*

$$\rho \in (0, \rho_1) \Rightarrow \hat{J}_{M,\rho}(a_1) \geq \hat{J}_{M,\rho}(a_2).$$

Proof of Lemma 5. Assume without loss of generality that $g_0 > 0$, the case $g_1 < 0$ being easily inferred by modifying all the signs in the proof below. Then, one has

$$-\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} = g - \rho f(u_{M,\rho,a}) \geq 0 \quad \text{in } D,$$

whenever $\rho \in (0, g_0/\|f\|_\infty)$, and therefore, one has $u_{M,\rho,a} \geq 0$ by the comparison principle.

Similarly, notice that

$$-\Delta u_{M,\rho,a} \leq g_1 + \rho \|f\|_\infty \quad \text{in } D,$$

which implies that $u_{M,\rho,a} \leq (g_1 + \rho \|f\|_\infty) w_D$ where w_D is the torsion function of D . By the classical Talenti's estimate of the torsion function [17], we have $\|w_D\|_\infty \leq \frac{1}{2d} \left(\frac{|D|}{\omega_d} \right)^{2/d}$ (where ω_d is the volume of the unit ball). Thus

$$\|u_{M,\rho,a}\|_\infty \leq (g_1 + \rho \|f\|_\infty) \frac{1}{2d} \left(\frac{|D|}{\omega_d} \right)^{2/d} := C(g_0, \rho, \|f\|_\infty, D).$$

Setting $U_{M,\rho,a} = \frac{1}{2} u_{M,\rho,a} + v_{M,\rho,a}$, elementary computations show that $U_{M,\rho,a}$ solves the problem

$$\begin{cases} -\Delta U_{M,\rho,a} + (M(1-a) + \rho f'(u_{M,\rho,a})) U_{M,\rho,a} = \frac{\rho}{2} (f(u_{M,\rho,a}) + u_{M,\rho,a} f'(u_{M,\rho,a})) + \frac{g}{2} & \text{in } D, \\ U_{M,\rho,a} = 0 & \text{on } \partial D. \end{cases} \quad (14)$$

Lemma 6. *Let us choose ρ_1 in such a way that*

$$\rho_1 (\|f\|_\infty + C(g_0, \frac{1}{2}, \|f\|_\infty, D) \|f'\|_\infty) < g_0, \quad \text{and} \quad \rho_1 \|f'\|_\infty \leq \frac{\lambda_1(D)}{2}, \quad (15)$$

where $\lambda_1(D)$ denotes the first eigenvalue of the Dirichlet-Laplacian operator on D . For every $\rho \in [0, \rho_1)$, $U_{M,\rho,a}$ is non-negative in D .

Proof of Lemma 6. The result follows immediately from the generalized maximum principle which claims that if a function v satisfies

$$-\Delta v + a(\cdot)v \geq 0 \quad \text{with } a(\cdot) > -\lambda_1(D) \quad (16)$$

and $v = 0$ on ∂D , then $v \geq 0$ a.e. in D . Here we have chosen ρ_1 in such a way that

$$M(1 - a) + \rho f'(u_{M,\rho,a}) \geq -\lambda_1(D)$$

and the right-hand side of (14) is non-negative which yields the result. \square

Coming back to the proof of Lemma 5, consider $h = a_2 - a_1$. According to the mean value theorem, there exists $\varepsilon \in (0, 1)$ such that

$$\hat{J}_{M,\rho}(a_2) - \hat{J}_{M,\rho}(a_1) = \langle d\hat{J}_{M,\rho}(a_1 + \varepsilon h), h \rangle = -M \int_D h u_{M,a_1+\varepsilon h} U_{M,a_1+\varepsilon h} \leq 0,$$

according to the combination of the analysis above with Lemma 4. The expected conclusion follows. \square

Existence under the second assumption: g is non-negative and the function f satisfies the assumption **(H₂)** or g is non-positive and the function $-f$ satisfies the assumption **(H₂)**.

The main difference with the previous case is that g might possibly be zero. Deriving the conclusion is therefore trickier and relies on a careful asymptotic analysis of the solution $u_{M,\rho,a}$ as $\rho \rightarrow 0$.

Proposition 2. *There exists $C = C(D, \|f\|_{L^\infty(\Omega)}) > 0$ such that, for any $M \in \mathbb{R}_+$, any $a \in \hat{\mathcal{O}}_m$ such that $a \geq 0$ a.e. in D , there holds*

$$\|u_{M,\rho,a} - u_{M,0,a}\|_{L^\infty(D)} \leq C\rho. \quad (17)$$

Proof. Let us set $z_\rho = u_{M,\rho,a} - u_{M,0,a}$ for any $\rho > 0$. A direct computation yields that z_ρ satisfies

$$-\Delta z_\rho + M(1 - a)z_\rho = -\rho f(u_{M,\rho,a}).$$

By comparison with the torsion function w_D of D , this implies

$$\|z_\rho\|_\infty \leq \rho \|f\|_\infty \|w_D\|_\infty$$

and the result follows, with a constant C explicit by Talenti's Theorem like in the proof of Lemma 5. \square

Let us consider the switching function $\Psi = -M U_{M,\rho,a} u_{M,\rho,a}$ where $u_{M,\rho,a}$ and $U_{M,\rho,a}$ respectively solve (5) and (14), and we will prove that both $u_{M,\rho,a}$ and $U_{M,\rho,a}$ are non-negative, so that one can conclude similarly to the previous case.

Lemma 7. *The functions $u_{M,\rho,a}$ and $U_{M,\rho,a}$ non-negative whenever ρ is small enough.*

Proof. Let us choose ρ such that $\rho\|f\|_\infty < \lambda_1(D)$. Since $u_{M,\rho,a}$ satisfies

$$-\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) \geq 0$$

the non-negativity of $u_{M,\rho,a}$ is a consequence of the generalized maximum principle (16). Indeed, for ρ small enough, we have

$$M(1-a) + \rho f(u_{M,\rho,a}) > -\lambda_1(D).$$

Since $U_{M,\rho,a}$ satisfies (14), the proof follows the same lines assuming the $\rho\|f'\|_\infty < \lambda_1(D)$ and using the assumption (\mathbf{H}_2) to get non-negativity of the right-hand side. By mimicking the reasoning done at the end of the first case, one gets that (10) is true if ρ is small enough. \square

Thus, in both cases, the monotonicity of the functional is established, so that the theorem of Buttazzo and Dal Maso applies.

4 Proof of Theorem 2

Note first that the functional J_ρ is shape differentiable, which follows from standard arguments, see e.g. [12, Chapitre 5].

Our proof of Theorem 2 is divided into two steps: after proving the criticality of \mathbb{B}^* for ρ small enough, we compute the second order shape derivative of the Lagrangian associated with the problem at the ball. Next, we establish that, under Assumption (8), there exists a positive constant C_0 such that, for any admissible V , one has

$$(J_0 - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] \geq C_0 \|V \cdot \nu\|_{L^2(\Omega)}^2. \quad (18)$$

Finally, we prove that, for any radially symmetric, non-increasing non-negative g , there exists $M \in \mathbb{R}$ such that, for any admissible V , one has

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V] \geq (J_0 - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] - M\rho \|V \cdot \nu\|_{L^2(\Omega)}^2. \quad (19)$$

Local shape minimality of \mathbb{B}^* for ρ small enough can then be inferred in a straightforward way.

If V is an admissible vector field, we will denote by $u'_{\rho,V}$ and $u''_{\rho,V}$ the first and second order (eulerian) shape derivatives of u_ρ at \mathbb{B}^* with respect to V .

4.1 Preliminary material

Lemma 8. *Under the assumptions of Theorem 2, i.e when g is radially symmetric and non-increasing function, for ρ small enough, the function u_ρ is radially symmetric nonincreasing. We write it $u_\rho = \varphi_\rho(|\cdot|)$. Furthermore, if $\rho = 0$, one has*

$$-\frac{\partial u_0}{\partial \nu} \geq \frac{R}{2}g(R).$$

Proof of Lemma 8. The fact that u_ρ is a radially symmetric nonincreasing function follows from a simple application of the Schwarz rearrangement. Integrating the equation on the ball \mathbb{B}^* yields

$$-\int_{\mathbb{B}^*} \Delta u_0 = -\int_{\partial \mathbb{B}^*} \frac{\partial u_0}{\partial \nu} = -2\pi \frac{\partial u_0}{\partial \nu}$$

on the one-hand, while using the fact that g is decreasing:

$$-\int_{\mathbb{B}^*} \Delta u_0 = \int_{\mathbb{B}^*} g \geq 2\pi g(R) \int_0^R r dr = \pi R g(R)$$

\square

By differentiating the main equation and the boundary conditions (see e.g. [12, Chapitre 5]), we get that the functions $u'_{\rho,V}$ and $u''_{\rho,V}$ satisfy

$$\begin{cases} -\Delta u'_{\rho,V} + \rho f'(u_\rho) u'_{\rho,V} = 0 & \text{in } \mathbb{B}^* \\ u'_{\rho,V} = -\frac{\partial u_\rho}{\partial \nu} V \cdot \nu & \text{on } \partial \mathbb{B}^* \end{cases} \quad (20)$$

and

$$\begin{cases} -\Delta u''_{\rho,V} + \rho f'(u_\rho) u''_{\rho,V} + \rho f''(u_\rho) (u'_{\rho,V})^2 = 0 & \text{in } \mathbb{B}^* \\ u''_{\rho,V} = -2\frac{\partial u'_{\rho,V}}{\partial \nu} V \cdot \nu - (V \cdot \nu)^2 \frac{\partial^2 u_\rho}{\partial \nu^2} & \text{on } \partial \mathbb{B}^*. \end{cases} \quad (21)$$

4.2 Proof of the shape criticality of the ball

Proving the shape criticality of the ball boils down to showing the existence of a Lagrange multiplier $\Lambda_\rho \in \mathbb{R}$ such that for every admissible vector field $V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, one has

$$(J_\rho - \Lambda_\rho \text{Vol})'(\mathbb{B}^*)[V] = 0 \quad (22)$$

Standard computations (see e.g. [12, chapitre 5]) yield

$$\begin{aligned} J'_\rho(\mathbb{B}^*)[V] &= \int_{\mathbb{B}^*} \langle \nabla u_\rho, \nabla u'_{\rho,V} \rangle - \int_{\mathbb{B}^*} g u'_{\rho,V} + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu \\ &= \int_{\mathbb{S}^*} u'_{\rho,V} \frac{\partial u_\rho}{\partial \nu} - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu \\ &= - \int_{\mathbb{S}^*} \left(\frac{\partial u_\rho}{\partial \nu} \right)^2 V \cdot \nu + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) \\ &= -\frac{1}{2} \int_{\mathbb{S}^*} |\nabla u_\rho|^2 V \cdot \nu - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho). \end{aligned}$$

We introduce the adjoint state p_ρ as the unique solution of

$$\begin{cases} -\Delta p_\rho + \rho p_\rho f'(u_\rho) + \rho f(u_\rho) = 0 & \text{in } \mathbb{B}^* \\ p_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (23)$$

Since u_ρ is radially symmetric, so is p_ρ . Multiplying the main equation of (23) by $u'_{\rho,V}$ and integrating by parts yields

$$-\rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) = \int_{\mathbb{S}^*} \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} V \cdot \nu,$$

and finally

$$J'_\rho(\mathbb{B}^*)[V] = \int_{\mathbb{S}^*} \left(\frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left(\frac{\partial u_\rho}{\partial \nu} \right)^2 \right) V \cdot \nu.$$

Observe that $\frac{\partial p_\rho}{\partial \nu}$ and $\frac{\partial u_\rho}{\partial \nu}$ are constant on \mathbb{S}^* since u_ρ and p_ρ are radially symmetric. Introduce the real number Λ_ρ given by

$$\Lambda_\rho = \left. \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left(\frac{\partial u_\rho}{\partial \nu} \right)^2 \right|_{\mathbb{S}^*}, \quad (24)$$

we get that (22) is satisfied, whence the result.

In what follows, we will exploit the fact that the adjoint state is radially symmetric. In the following definition, we sum-up the notations we will use in what follows.

Definition 3. Recall that φ_ρ (defined in Lemma 8) is such that

$$u_\rho(x) = \varphi_\rho(|x|), \quad \forall x \in \mathbb{B}^*.$$

Since p_ρ is also radially symmetric, introduce ϕ_ρ such that

$$p_\rho(x) = \phi_\rho(|x|), \quad \forall x \in \mathbb{B}^*.$$

4.3 Second order optimality conditions

Let us focus on the second and third points of Theorem 2, especially on (9). Since \mathbb{B}^* is a critical shape, it is enough to work with normal vector fields, in other words vector fields V such that $V = (V \cdot \nu)\nu$ on \mathbb{S}^* . Consider such a vector field V . For the sake of notational simplicity, let us set $J_\rho'' = J_\rho''(\mathbb{B}^*)[V, V]$, $\mathcal{L}_{\Lambda_\rho}'' = (J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V]$, $u = u_\rho$, $u' = u'_{\rho, V}$ and $u'' = u''_{\rho, V}$.

4.3.1 Computation of the second order derivative

To compute the second order derivative, we use the Hadamard second order formula [12, Chap. 5, p. 227] for normal vector fields, namely

$$\frac{d^2}{dt^2} \Big|_{t=0} \int_{(\text{Id} + tV)\mathbb{B}^*} f(t) = \int_{\mathbb{B}^*} f''(0) + 2 \int_{\mathbb{S}^*} f'(0)V \cdot \nu + \int_{\mathbb{S}^*} \left(\frac{1}{R}f(0) + \frac{\partial f(0)}{\partial \nu} \right) (V \cdot \nu)^2,$$

applied to $f(t) = \frac{1}{2}|\nabla u_t|^2 - gu_t$, where u_t denotes the solution of (2) on $(\text{Id} + tV)\mathbb{B}^*$.

The Hadamard formula along with the weak formulation of Equations (20)-(21) yields

$$\begin{aligned} J_\rho'' &= \int_{\mathbb{B}^*} \langle \nabla u, \nabla u'' \rangle - \int_{\mathbb{B}^*} gu'' + \int_{\mathbb{B}^*} |\nabla u'|^2 + 2 \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu - 2 \int_{\mathbb{S}^*} gu'V \cdot \nu \\ &+ \int_{\mathbb{S}^*} \left(\frac{1}{2R}|\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u)u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} u'' \frac{\partial u}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial u'}{\partial \nu} \\ &+ 2 \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu - 2 \int_{\mathbb{S}^*} gu'V \cdot \nu + \int_{\mathbb{S}^*} \left(\frac{1}{2R}|\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u)u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} \left(-2 \frac{\partial u'}{\partial \nu} V \cdot \nu - \frac{\partial^2 u}{\partial \nu^2} (V \cdot \nu)^2 \right) \frac{\partial u}{\partial \nu} \\ &- \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu + 2 \int_{\mathbb{S}^*} \left(\frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} - gu' \right) V \cdot \nu + \int_{\mathbb{S}^*} \left(\frac{1}{2R}|\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u)u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} (V \cdot \nu)^2 - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} (V \cdot \nu) \\ &+ 2 \int_{\mathbb{S}^*} g \frac{\partial u}{\partial \nu} (V \cdot \nu)^2 + \int_{\mathbb{S}^*} \left(\frac{1}{2R}|\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u)u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} \left(\frac{1}{2R} \left(\frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu. \end{aligned}$$

As such, the two first terms of the sum in the expression above are not tractable. Let us rewrite them. Multiplying the main equation of (23) by u'' and integrating two times by parts yields

$$-\rho \int_{\mathbb{B}^*} f(u)u'' = \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} - \rho \int_{\mathbb{B}^*} (u')^2 p_\rho f''(u).$$

To handle the last term of the right-hand side, let us introduce the function λ_ρ defined as the solution of

$$\begin{cases} -\Delta\lambda_\rho + \rho\lambda_\rho f'(u) + \rho u' p_\rho f''(u) = 0 & \text{in } \mathbb{B}^* \\ \lambda_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (25)$$

Multiplying this equation by u' and integrating by parts gives

$$-\rho \int_{\mathbb{B}^*} f''(u)(u')^2 = \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu}.$$

To handle the term $-\rho \int_{\mathbb{B}^*} (u')^2 f'(u)$ of J''_ρ , we introduce the function η_ρ , defined as the only solution to

$$\begin{cases} -\Delta\eta_\rho + \rho\eta_\rho f'(u) + \rho u' f'(u) = 0 & \text{in } \mathbb{B}^* \\ \eta_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (26)$$

Multiplying this equation by u' and integrating by parts gives

$$-\rho \int_{\mathbb{B}^*} (u')^2 f'(u) = \int_{\mathbb{S}^*} u' \frac{\partial \eta_\rho}{\partial \nu} = - \int_{\mathbb{S}^*} V \cdot \nu \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu}.$$

Gathering these terms, we have

$$\begin{aligned} J''_\rho &= \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\ &\quad + \int_{\mathbb{S}^*} \left(\frac{1}{2R} \left(\frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2. \end{aligned}$$

Using that

$$\Lambda_\rho = \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left(\frac{\partial u_\rho}{\partial \nu} \right)^2 \Big|_{\mathbb{S}^*} \quad \text{and} \quad \text{Vol}''(\mathbb{B}^*) = \int_{\mathbb{S}^*} \frac{1}{R} (V \cdot \nu)^2,$$

one computes

$$\boxed{\begin{aligned} \mathcal{L}''_{\Lambda_\rho} &= \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\ &\quad + \int_{\mathbb{S}^*} \left(-\frac{\Lambda_\rho}{R} + \frac{1}{2R} \left(\frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \end{aligned}} \quad (27)$$

4.3.2 Expansion in Fourier Series

In this section, we recast the expression of $\mathcal{L}''_{\Lambda_\rho}$ in a more tractable form, by using the method introduced by Lord Rayleigh: since we are dealing with vector fields normal to \mathbb{S}^* , we expand $V \cdot \nu$ as a Fourier series. This leads to introduce the sequences of Fourier coefficients $(\alpha_k)_{k \in \mathbb{N}^*}$ and $(\beta_k)_{k \in \mathbb{N}^*}$ defined by:

$$V \cdot \nu = \sum_{k \in \mathbb{N}^*} \left(\alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot) \right),$$

the equality above being understood in a $L^2(\mathbb{S}^*)$ sense.

Let $v_{k,\rho}$ (resp. $w_{k,\rho}$) denote the function u' associated to the perturbation choice V_k given by $V_k = V_k^c := \cos(k \cdot) \nu$ (resp. $V_k = V_k^s := \sin(k \cdot) \nu$), in other words, $v_{k,\rho} = u'_{\rho, V_k^c}$ (resp. $w_{k,\rho} = u'_{\rho, V_k^s}$). Then, one shows easily (by uniqueness of the solutions of the considered PDEs) that for every $k \in \mathbb{N}$, there holds

$$v_{k,\rho}(r, \theta) = \psi_{k,\rho}(r) \cos(k\theta) \quad (\text{resp. } w_{k,\rho}(r, \theta) = \psi_{k,\rho}(r) \sin(k\theta)),$$

where (r, θ) denote the polar coordinates in \mathbb{R}^2 , where $\psi_{k,\rho}$ solves

$$\begin{cases} -\frac{1}{r}(r\psi'_{k,\rho})' = -\left(\frac{k^2}{r^2} + \rho f'(u)\right)\psi_{k,\rho} & \text{in } (0, R) \\ \psi_{k,\rho}(R) = -\varphi'_\rho(R). \end{cases} \quad (28)$$

By linearity, we infer that

$$u' = \sum_{k \in \mathbb{N}^*} \alpha_k v_{k,\rho} + \beta_k w_{k,\rho}.$$

For every $k \in \mathbb{N}^*$, let us introduce $\eta_{k,\rho}$ as the solution of (26) associated with $v_{k,\rho}$. One shows that $\eta_{k,\rho}$ satisfies

$$\begin{cases} -\Delta \eta_{k,\rho} + \rho f'(u)\eta_{k,\rho} + \rho f'(u)v_{k,\rho} = 0 & \text{in } \mathbb{B}^* \\ \eta_{k,\rho} = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (29)$$

Similarly, one shows easily that

$$\eta_{k,\rho}(r, \theta) = \xi_{k,\rho}(r) \cos(k\theta),$$

where $\xi_{k,\rho}$ satisfies

$$\begin{cases} -\frac{1}{r}(r\xi'_{k,\rho})' = -\left(\frac{k^2}{r^2} + \rho f'(u)\right)\xi_{k,\rho} - \rho\psi_{k,\rho} & \text{in } (0, R) \\ \xi_{k,\rho}(R) = 0. \end{cases} \quad (30)$$

Notice that one has $\xi_{k,\rho} = 0$ whenever $\rho = 0$, which can be derived obviously from (26).

Since u_ρ is radially symmetric we denote by $r \mapsto \varphi_\rho(r)$ this radial function.

Finally, we introduce a last set of equations related to λ_ρ . Let us define $\zeta_{k,\rho}$ as the solution of

$$\begin{cases} -(r\zeta'_{k,\rho})' = -\frac{k^2}{r^2}\zeta_{k,\rho} - r\rho\zeta_{k,\rho}f'(u) - \rho r\psi_{k,\rho}\phi_\rho f''(u) & \text{in } (0, R) \\ \zeta_{k,\rho}(R) = 0. \end{cases} \quad (31)$$

and verify that $\lambda_\rho = \zeta_{k,\rho}(r) \cos(k\theta)$ whenever $V = V_k$.

Proposition 3. *The quadratic form $\mathcal{L}''_{\Lambda_\rho}$ expands as*

$$\mathcal{L}''_{\Lambda_\rho} = \sum_{k=1}^{\infty} \omega_{k,\rho} (\alpha_k^2 + \beta_k^2), \quad (32)$$

where, for any $k \in \mathbb{N}^*$,

$$\begin{aligned} \omega_{k,\rho} = \pi R \left(-2\psi'_{k,\rho}(R)\phi'_\rho(R) - \varphi'_\rho(R)\zeta'_{k,\rho}(R) - \varphi''_\rho(R)\phi'_\rho(R) \right. \\ \left. - \xi'_{k,\rho}(R)\varphi'_\rho(R) - \frac{\Lambda_\rho}{R} + \frac{1}{2R}(\varphi'_\rho)^2 + g(R)\varphi'_\rho(R) - \varphi'_\rho(R)\psi'_{k,\rho}(R) \right), \end{aligned} \quad (33)$$

the functions $\psi_{k,\rho}$, $\xi_{k,\rho}$, $\zeta_{k,\rho}$ being respectively defined by (28), (30), (31), and Λ_ρ is given by (24).

Proof of Proposition 3. Let us first deal with the particular case $V \cdot \nu = \cos(k \cdot)$. According to (20), (21) and (27), one has

$$\begin{aligned} \mathcal{L}''_{\Lambda_\rho} &= \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\ &+ \int_{\mathbb{S}^*} \left(-\frac{\Lambda_\rho}{R} + \frac{1}{2R} \left(\frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \end{aligned}$$

$$\begin{aligned}
&= R \int_0^{2\pi} (-2 \cos(k\theta)^2 \psi'_{k,\rho}(R) - \cos(k\theta)^2 \varphi''_\rho(R)) \phi'_\rho(R) d\theta - R \int_0^{2\pi} \cos(k\theta)^2 \varphi'_\rho(R) \zeta'_{k,\rho}(R) d\theta \\
&\quad - R \int_0^{2\pi} \cos(k\theta)^2 \xi'_{k,\rho}(R) \varphi'_\rho(R) d\theta + R \int_0^{2\pi} \cos(k\theta)^2 \left(-\frac{\Lambda_\rho}{R} + \frac{1}{2R} (\varphi'_\rho)^2 + g(R) \varphi'_\rho(R) \right) d\theta \\
&\quad - R \int_0^{2\pi} \cos(k\theta)^2 \varphi'_\rho(R) \psi'_{k,\rho}(R) d\theta
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\mathcal{L}''_{\Lambda_\rho}}{\pi R} &= -2\psi'_{k,\rho}(R)\phi'_\rho(R) - \varphi'_\rho(R)\zeta'_{k,\rho}(R) - \varphi''_\rho(R)\phi'_\rho(R) - \xi'_{k,\rho}(R)\varphi'_\rho(R) \\
&\quad - \frac{\Lambda_\rho}{R} + \frac{1}{2R}(\varphi'_\rho)^2 + g(R)\varphi'_\rho(R) - \varphi'_\rho(R)\psi'_{k,\rho}(R)
\end{aligned}$$

We have then obtained the expected expression for this particular choice of vector field V . Similar computations enable us to recover the formula when dealing with the vector field V given by $V \cdot \nu = \sin(k \cdot)$. Finally, for general V , one has to expand the square $(V \cdot \nu)^2$, and the computation follows exactly the same lines as before. Note that all the crossed terms of the sum (i.e. the term that do not write as squares of real numbers) vanish, by using the $L^2(\mathbb{S})$ orthogonality properties of the families $(\cos(k \cdot), \sin(k \cdot))_{k \in \mathbb{N}^*}$. \square

4.3.3 Comparison principle on the family $\{\omega_{k,\rho}\}_{k \in \mathbb{N}^*}$

The next result allows us to recast the ball stability issue in terms of the sign of $\omega_{1,\rho}$.

Proposition 4. *There exists $M > 0$ such that, for any ρ small enough,*

$$\forall k \in \mathbb{N}^*, \quad \omega_{k,\rho} - \omega_{1,\rho} \geq -M\rho \quad \text{and} \quad |\omega_{1,\rho} - \omega_{1,0}| \leq M\rho.$$

Proof of Proposition 4. Fix $k \in \mathbb{N}$ and introduce $\tilde{\omega}_{k,\rho} = \omega_{k,\rho}/(\pi R)$. Using (33), one computes

$$\begin{aligned}
\tilde{\omega}_{k,\rho} - \tilde{\omega}_{1,\rho} &= (-\varphi'_\rho(R) - 2\phi'_\rho(R)) (\psi'_{k,\rho}(R) - \psi'_{1,\rho}(R)) \\
&\quad - \varphi'_\rho(R) (\xi'_{k,\rho}(R) - \xi'_{1,\rho}(R) + \zeta'_{k,\rho}(R) - \zeta'_{1,\rho}(R)).
\end{aligned}$$

We need to control each term of the expression above, which is the goal of the next results, whose proofs are postponed at the end of this section.

Lemma 9. *There exists $M > 0$ and $\bar{\rho} > 0$ such that for $\rho \in [0, \bar{\rho}]$, one has*

$$\max \{ \|\varphi'_\rho - \varphi'_0\|_{L^\infty(0,R)}, \|\phi'_\rho\|_{L^\infty(0,R)}, \|\xi'_{k,\rho}\|_{L^\infty} \} \leq M\rho \quad \text{and} \quad \|\zeta'_{k,\rho}\|_{L^\infty} \leq M\rho^2.$$

According to Lemma 8, one has in particular $\varphi'_0(R) < 0$. We thus infer from Lemma 9 the existence of $\delta > 0$ such that

$$\min \{ -\varphi'_\rho(R) - 2\phi'_\rho(R), -\varphi'_\rho(R) \} \geq \delta > 0.$$

for ρ small enough. Furthermore, Lemma 9 also yields easily the estimate

$$|\zeta'_{k,\rho}(R) - \zeta'_{1,\rho}(R)| \leq M\rho^2$$

Hence, we are done by applying the following result.

Lemma 10. *There exists $\hat{M} > 0$ and $\bar{\rho} > 0$ such that for $\rho \in [0, \bar{\rho}]$, one has*

$$\psi'_{k,\rho}(R) - \psi'_{1,\rho}(R) \geq 0 \quad \text{and} \quad |\xi'_{k,\rho}(R) - \xi'_{1,\rho}(R)| \leq \hat{M}\rho. \quad (34)$$

Indeed, the results above lead to

$$\omega_{k,\rho} - \omega_{1,\rho} \geq \delta(\psi'_k(R) - \psi'_{k,\rho}(R) + \xi'_{k,\rho}(R) - \xi'_{k,\rho}(R)) \geq 0$$

for every $k \geq 1$ and ρ small enough.

Finally, the proof of the second inequality follows the same lines and are left to the reader. \square

Proof of Lemma 9. These convergence rates are simple consequences of elliptic regularity theory. Since the reasonings for each terms are similar, we only focus on the estimate of $\|\phi'_\rho\|_{L^\infty(\Omega)}$. Recall that p_ρ solves the equation (23). Multiplying this equation by p_ρ , integrating by parts and using the Poincaré inequality yield the existence of $C > 0$ such that

$$(1 - \rho C \|f'\|_{L^\infty(\mathbb{B}^*)}) \|\nabla p_\rho\|_{L^2(\mathbb{B}^*)}^2 \leq \rho \|f\|_{L^\infty(\mathbb{B}^*)} \|p_\rho\|_{L^2(\mathbb{B}^*)},$$

so that $\|p_\rho\|_{W_0^{1,2}(\mathbb{B}^*)}$ is uniformly bounded for ρ small enough. Hence, the elliptic regularity theory yields that p_ρ is in fact uniformly bounded in $W^{2,2}(\mathbb{B}^*)$, and there exists $\hat{M} > 0$ such that $\|p_\rho\|_{W_0^{2,2}(\mathbb{B}^*)} \leq \hat{M}\rho$ and, since $\mathbb{B}^* \subset \mathbb{R}^2$, we get

$$\|p_\rho\|_{L^\infty(\mathbb{B}^*)} \leq M\rho.$$

Since $\Delta p_\rho = \rho p_\rho f'(u_\rho) + \rho f(u_\rho)$ and the right-hand side belongs to $L_0^p(\mathbb{B}^*)$ for all $p \geq 1$, the elliptic regularity theory yields the existence of $C > 0$ such that

$$\|p_\rho\|_{W_0^{2,p}(\Omega)} \leq C(\rho \|p_\rho\|_{L^\infty} \|f'\|_{L^\infty} + \rho \|f\|_{L^\infty}) \leq M\rho$$

and using the embedding $W^{2,p} \hookrightarrow \mathcal{C}^{1,\alpha}$ for p large enough, one finally gets

$$\|\nabla p_\rho\|_{L^\infty(\mathbb{B}^*)} \leq M\rho.$$

\square

Proof of Lemma 10. The two estimates are proved using the maximum principle. Let us first prove that, for any k and any ρ small enough, $\psi_{k,\rho}$ is non-negative on $(0, R)$. Since, for ρ small enough, $-\varphi'_\rho(R)$ is positive, and therefore $\psi_{k,\rho}(R) > 0$. Since v_k belongs to $W_0^{1,2}$, one has necessarily $\psi_{k,\rho}(0) = 0$. Furthermore, according to (28), by considering $\rho > 0$ small enough so that

$$-\frac{1}{r^2} + \rho \|f'\|_{L^\infty} \leq -\frac{1}{2r^2}$$

it follows that

$$-\frac{1}{r}(r\psi'_{k,\rho})' = c_{k,\rho}(r)\psi_{k,\rho} \quad \text{with} \quad c_{k,\rho} = -\frac{k^2}{r^2} - \rho f'(u_0) < 0.$$

Let us argue by contradiction, assuming that $\psi_{k,\rho}$ reaches a negative minimum at a point r_1 . Because of the boundary condition, r_1 is necessarily an interior point of $(0, R)$. Then, from the equation,

$$0 \geq -\psi''_{k,\rho}(r_1) = c_{k,\rho}(r_1)\psi_{k,\rho}(r_1) > 0,$$

which is a contradiction. Thus there exists $\bar{\rho} > 0$ small enough such that, for any $\rho \leq \bar{\rho}$ and every $k \in \mathbb{N}^*$, $\psi_{k,\rho}$ is non-negative on $(0, R)$.

Now, introduce $z_k = \psi_{k,\rho} - \psi_{1,\rho}$ for every $k \geq 1$ and notice that it satisfies

$$-\frac{1}{r}(rz'_k)' = \frac{1}{r^2}\psi_{1,\rho} - \frac{k^2}{r^2}\psi_{k,\rho} - \rho f'(u_0)z_k.$$

Since $\psi_{k,\rho}$ is non-negative, it implies

$$-\frac{1}{r}(rz'_k)' \leq \left(-\frac{k^2}{r^2} - \rho f'(u_0)\right) z_k, \quad \text{and} \quad z_k(R) = z_k(0) = 0.$$

Up to decreasing $\bar{\rho}$, one may assume that for $\rho \leq \bar{\rho}$, $-\frac{k^2}{r^2} - \rho f'(u_0) < 0$ in $(0, R)$. If z_k reached a positive maximum, it would be at an interior point r_1 , but we would have

$$0 \leq -z''_k(r_1) < \left(-\frac{k^2}{r^2} - \rho f'(u_0)\right) z_k(r_1) < 0.$$

Hence, one has necessarily $z_k \leq 0$ in $(0, R)$ and z_k reaches a maximum at R , which means in particular that $z'_k(R) = \psi'_{k,\rho}(R) - \psi'_{1,\rho}(R) \geq 0$. \square

4.4 Shape (in)stability of \mathbb{B}^*

4.4.1 Under Assumption (8)

Stability under Assumption (8) is well known (see [5]) in the case where $\rho = 0$. Hereafter, we recall the proof, showing by the same a stability result for $\rho > 0$.

Lemma 11. *Under assumption (8), one has $\omega_{1,0} > 0$.*

This Lemma concludes the proof of the second part of Theorem 2. Indeed, according to Propositions 3 and 4, there holds

$$\begin{aligned} \mathcal{L}''_{\Lambda_\rho}(\mathbb{B}^*)[V, V] &\geq \left(\frac{\omega_{1,0}}{2} + O(\rho)\right) \sum_{k=1}^{\infty} (\alpha_k^2 + \beta_k^2) \\ &= \left(\frac{\omega_{1,0}}{2} + O(\rho)\right) \|V \cdot \nu\|_{L^2}^2 \end{aligned}$$

for ρ small enough.

Proof of Lemma 11. To compute $\omega_{1,0}$, recall that, for $\rho = 0$, the function $\psi_{1,0}$ solves

$$-\frac{1}{r}(r\psi'_{1,0})' = -\frac{1}{r^2}\psi_{1,0} \quad \text{and} \quad \psi_{1,0}(R) = -\varphi'_0(R),$$

and therefore, $\psi_{1,0}(r) = -\frac{r}{R}\varphi'_0(R)$ for all $r \in [0, R]$, so that

$$\begin{aligned} \frac{\omega_{1,0}}{\pi R} &= -\frac{\Lambda_0}{R} + \frac{1}{2R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) - \varphi'_0(R)\psi'_{1,0}(R) \\ &= \frac{1}{R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) + \frac{1}{R}(\varphi'_0(R))^2 \\ &= \frac{2}{R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) \\ &= -\varphi'_0(R) \left(-\frac{2}{R}\varphi'_0(R) - g(R)\right) \end{aligned}$$

where the expression of Λ_0 is given by (24). Since $-R\varphi'_0(R) = \int_0^R tg(t)dt = \frac{1}{2\pi} \int_{\mathbb{B}^*} g$, and $\varphi'_0(R) < 0$, we infer that the sign of $\omega_{1,0}$ is the sign of

$$-\frac{2}{R}\varphi'_0(R) - g(R) = \frac{1}{\pi R^2} \int_{\mathbb{B}^*} g - g(R),$$

and the positivity of this last quantity is exactly Assumption (8). The conclusion follows.

4.4.2 An example of instability

In this part, we will assume that g is the constant function equal to 1, i.e. $g = 1$. Even if the ball \mathbb{B}^* is known to be a minimizer in the case $\rho = 0$, it is a degenerate one in the sense that $\omega_{1,0} = 0$ coming from the invariance by translations of the problem. In what follows, we exploit this fact and will construct a suitable nonlinearity f such that \mathbb{B}^* is not a local minimizer for ρ small enough, in other words such that $\omega_{1,\rho} < 0$.

We assume without loss of generality that $R = 1$ for the sake of simplicity.

Lemma 12. *There holds*

$$\omega_{1,\rho} = \frac{\rho}{4}(w_1 + w'_1)(1) + O(\rho^2)$$

where w_1 solves

$$\begin{cases} -(rw'_1)' = -\frac{1}{r}w_1 - \frac{r^2}{2}f'(\varphi_0) - \frac{r^2}{2} & \text{in } (0,1) \\ w_1(1) = -\int_0^1 tf(\varphi_0) dt. \end{cases} \quad (35)$$

Proof of Lemma 12. The techniques to derive estimates follow exactly the same lines as in Lemma 9. First, we claim that

$$\varphi_\rho = \varphi_0 + \rho\varphi_1 + O(\rho^2) \quad \text{in } \mathcal{C}^1, \quad (36)$$

where φ_1 satisfies

$$\begin{cases} -\frac{1}{r}(r\varphi'_1)' = -f(\varphi_0) & \text{in } (0,1) \\ \varphi_1(1) = 0. \end{cases} \quad (37)$$

Indeed, considering the function $\delta = \varphi_\rho - \varphi_0 - \rho\varphi_1$, one shows easily that it satisfies

$$\begin{cases} -\frac{1}{r}(r\delta')' = \rho(f(\varphi_0) - f(\varphi_\rho)) & \text{in } (0,1) \\ \delta(1) = 0. \end{cases}$$

Therefore, by mimicking the reasonings done in the proof of Lemma 9, involving the elliptic regularity theory, and the fact that $\|\varphi_\rho - \varphi_0\|_{W^{1,\infty}} = O(\rho)$, we infer that $\|\delta\|_{\mathcal{C}^1} = O(\rho^2)$, whence the result.

Using that φ_ρ satisfies $-\frac{1}{r}(r\varphi'_\rho)' + \rho f(\varphi_\rho) = g$ and integrating this equation yields

$$-\varphi'_\rho(1) = \frac{1}{2} - \rho \int_0^1 tf(\varphi_\rho) dt = \frac{1}{2} - \rho \int_0^1 tf(\varphi_0(t)) dt + O(\rho^2). \quad (38)$$

The Equation on ϕ_ρ reads

$$\begin{cases} -(r\phi'_\rho)' = r\left(-\rho\phi_\rho f'(\varphi_\rho) - \rho f(\varphi_\rho)\right) & \text{in } (0,1) \\ \phi_\rho(0) = 0. \end{cases}$$

and according to Lemma 9, there holds $\|\phi_\rho\|_{L^\infty} = O(\rho)$. We thus infer that

$$-\phi'_\rho(1) = -\rho \int_0^1 tf(\varphi_0) dt + O(\rho^2). \quad (39)$$

From (38) and (39), we infer that

$$\begin{aligned} \Lambda_\rho &= \frac{1}{2}((\varphi'_\rho(1))^2 - \phi'_\rho(1)\varphi'_\rho(1)) = \frac{1}{2}\varphi'_0(1)^2 - \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\ &= \frac{1}{2}\varphi'_0(1)^2 + O(\rho^2). \end{aligned} \quad (40)$$

Regarding $\psi_{1,\rho}$ and using that it satisfies (28), we get

$$\psi_{1,\rho}(1) = -\varphi'_\rho(1) = \frac{1}{2} - \rho \int_0^1 tf(\varphi_0) dt.$$

We then infer that $\|\psi_{1,\rho} + r\varphi'_{0,\rho}(1)\|_{\mathcal{C}^1} = O(\rho)$. Plugging this estimate in (28) allows us to show that

$$\psi_{1,\rho}(r) = -\varphi'_0(1)r + \rho y_1(r) + O(\rho^2) \quad \text{in } \mathcal{C}^1(0,1), \quad (41)$$

where y_1 solves

$$\begin{cases} -(ry'_1)' = -\frac{1}{r}y_1 + r^2\varphi'_0(1)f'(\varphi_0) & \text{in } (0,1) \\ y_1(1) = -\int_0^1 tf(\varphi_0) dt. \end{cases} \quad (42)$$

Regarding $\xi_{1,\rho}$ and using that it satisfies (30), we easily get that $\|\xi_{1,\rho}\|_{W^{1,\infty}} = O(\rho)$, according to Lemma 9. This allows us to write

$$\xi_{1,\rho} = \rho z_1 + O(\rho^2) \quad \text{in } \mathcal{C}^1(0,1) \quad (43)$$

where z_1 satisfies

$$\begin{cases} -(rz'_1)' = -\frac{1}{r}z_1 + r^2\varphi'_0(1) & \text{in } (0,1) \\ z_1(1) = 0. \end{cases} \quad (44)$$

Let us now expand $\omega_{1,\rho}$ with respect to the parameter ρ . Recall that

$$\begin{aligned} \omega_{1,\rho} = \frac{1}{2} & \left(-2\psi'_{1,\rho}(1)\phi'_{1,\rho}(1) - \varphi''_\rho(1)\phi'_{1,\rho}(1) - \varphi'_\rho(R)\zeta'_{1,\rho}(R) \right. \\ & \left. - \xi'_{1,\rho}(1)\varphi'_\rho(1) + \Lambda_\rho + \frac{1}{2}(\varphi'_\rho)^2 + \varphi'_\rho(1) - \varphi'_\rho(1)\psi'_{1,\rho}(1) \right). \end{aligned}$$

Regarding the term $\varphi'_{0,\rho}(R)\zeta'_{1,\rho}(R)$, we know from Lemma 9 that $\|\zeta'_{1,\rho}(R)\|_{L^\infty} = O(\rho^2)$.

Using this estimate and plugging the expansions (36)-(40)-(41)-(43) in the expression above yields successively

$$\begin{aligned} -2\psi'_{1,\rho}(1)\phi'_{1,\rho}(1) &= 2\rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) = -\rho \int_0^1 tf(\varphi_0) dt + O(\rho^2). \\ -\varphi''_\rho(1)\phi'_{1,\rho}(1) &= -\varphi''_0(1)\phi'_\rho(1) + O(\rho^2) = \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2). \\ -\xi'_{1,\rho}(1)\varphi'_\rho(1) &= -\varphi'_0(1)\xi'_{1,\rho}(1) + O(\rho^2) = \frac{\rho}{2}z'_1(1) \\ \Lambda_\rho + \frac{1}{2}(\varphi'_\rho)^2 &= \varphi'_0(1)^2 - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2) = \frac{1}{4} - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\ \varphi'_\rho(1) &= -\frac{1}{2} + \rho \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\ -\varphi'_\rho(1)\psi'_{1,\rho}(1) &= \varphi'_0(1)^2 - \rho\varphi'_0(1)y'_1(1) + \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\ &= \frac{1}{4} + \frac{\rho}{2}y'_1(1) - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt, \end{aligned}$$

by using that $\|\phi_\rho\|_{W^{1,\infty}} = O(\rho)$ and $\|\xi_{1,\rho}\|_{W^{1,\infty}} = O(\rho)$. This gives

$$\omega_{1,\rho} = -\rho \int_0^1 tf(\varphi_0) dt + \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + \frac{\rho}{2}z'_1(1)$$

$$+ \frac{1}{4} - \frac{\rho}{2} \int_0^1 t f(\varphi_0) dt - \frac{1}{2} + \rho \int_0^1 t f(\varphi_0) dt + \frac{1}{4} + \frac{\rho}{2} y_1'(1) - \frac{\rho}{2} \int_0^1 t f(\varphi_0) dt + O(\rho^2).$$

As expected, the zero order terms cancel each other out and we get

$$\omega_{1,\rho} = -\frac{\rho}{2} \int_0^1 t f(\varphi_0) dt + \frac{\rho}{2} z_1'(1) + \frac{\rho}{2} y_1'(1) + O(\rho^2),$$

which concludes the proof by setting $w_1 = y_1 + z_1$. □

Construction of the non-linearity. Recall that we are looking for a non-linearity f such that $\omega_{1,\rho} < 0$, in other words such that $(w_1 + w_1')(1) < 0$ according to Lemma 12. To this aim, let us consider the function w_1 solving (35). Let us consider a non-negative function f such that

$$f'(\cdot) < -1 \quad \text{on } [0, \|\varphi_0\|_{L^\infty}]. \quad (45)$$

It follows that

$$w_1(1) = -\int_0^1 t f(\varphi_0) dt < 0.$$

Besides,

$$-(r w_1')' = -\frac{1}{r} w_1 - \frac{r^2}{2} (f'(\varphi_0) + 1) \geq -\frac{1}{r} w_1$$

by using (45). Thus w_1 cannot reach a local negative minimum in $(0, 1)$. Moreover, by using that w_1 is regular (w_1 is the sum of two functions at least \mathcal{C}^1 according to the proof of Lemma 12) and integrating the equation above yields

$$-r w_1'(r) + \frac{1}{2} \int_0^r s^2 (f'(\varphi_0(s)) + 1) ds = -\int_0^r \frac{w_1(s)}{s} ds$$

for $r > 0$. The left-hand side is well-defined and it follows that so is the right-hand side, which implies that necessarily $w_1(0) = 0$ (else, we would immediately reach a contradiction).

Since w_1 cannot reach a local minimum on $(0, 1)$ and since $0 = w_1(0) > w_1(1)$, we get that w_1 is decreasing on $(1 - \delta, 1)$ for some $\delta > 0$, ensuring that $w_1'(1) < 0$. The conclusion follows. □

Appendix

A Proof of Lemma 2

Recall that we want to establish a uniform (with respect to a and M) L^∞ bound on the solutions of

$$\begin{cases} -\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) = g, & \text{in } D, \\ u_{M,\rho,a} \in W_0^{1,2}(\Omega). \end{cases} \quad (46)$$

Here, it is assumed that g is non-negative.

Define ϕ_g as the solution of

$$\begin{cases} -\Delta \phi_g + \rho f(\phi_g) = g, & \text{in } D, \\ \phi_g \in W_0^{1,2}(\Omega). \end{cases}$$

Standard L^p estimates show that ϕ_g is continuous and that

$$\|\phi_g\|_{L^\infty(\Omega)} < +\infty.$$

Define $z := \phi_g - u_{M,\rho,a} \in W_0^{1,2}(\Omega)$. We can write

$$-\Delta z + \rho \frac{f(\phi_g) - f(u_{M,\rho,a})}{\phi_g - u_{M,\rho,a}} z = M(1-a)u_{M,\rho,a} \geq 0$$

The generalized maximum principle, and the fact that f is Lipschitz entails that z reaches its minimum on the boundary ∂D , so that z is non-negative. Thus

$$0 \leq u_{M,\rho,a} \leq \phi_g \leq \|\phi_g\|_{L^\infty(\Omega)} < +\infty$$

and we conclude by noting that the quantity in the right-hand side is uniformly bounded with respect to $\rho \in [0, \underline{\rho}]$.

B Proof of Proposition 1

We recall that we want to establish that if $(\Omega_k)_{k \in \mathbb{N}} \in \mathcal{O}_m^{\mathbb{N}}$ γ -converges to Ω , then

$$J_\rho(\Omega) \leq \liminf_{k \rightarrow \infty} J_\rho(\Omega_k).$$

Fix such a sequence $(\Omega_k)_{k \in \mathbb{N}}$ that γ -converges to Ω . For the sake of clarity, we drop the subscript ρ, f and g and define, for every $k \in \mathbb{N}$, $u_k \in W_0^{1,2}(D)$ the unique solution to

$$\begin{cases} -\Delta u_k + \rho f(u_k) = g & \text{in } \Omega_k, \\ u_k \in W_0^{1,2}(\Omega_k), \\ u_k \text{ is extended by continuity as a function in } W_0^{1,2}(D). \end{cases}$$

First note that, for any $k \in \mathbb{N}$, multiplying the equation by u_k and integrating by parts immediately yields

$$\begin{aligned} \lambda_1(D) \int_D u_k^2 &= \lambda_1(D) \int_{\Omega_k} u_k^2 \leq \lambda_1(\Omega_k) \int_{\Omega_k} u_k^2 \leq \int_{\Omega_k} |\nabla u_k|^2 \\ &\leq \|g\|_{L^2(\Omega_k)} \|u_k\|_{L^2(\Omega_k)} + \rho \|f\|_{L^\infty(\mathbb{R})} |\Omega_k|^{\frac{1}{2}} \|u_k\|_{L^2(\Omega_k)}. \end{aligned}$$

The sequence $(u_k)_{k \in \mathbb{N}}$ is thus uniformly bounded in $W_0^{1,2}(D)$. By the Rellich-Kondrachov Theorem, $(u_k)_{k \in \mathbb{N}}$ converges (up to a subsequence, strongly in $L^2(D)$ and weakly in $W_0^{1,2}(D)$) to a function $u \in W_0^{1,2}(D)$.

The dominated convergence theorem then yields that the sequence $(f(u_k))_{k \in \mathbb{N}}$ converges strongly in $W_0^{-1,2}(D)$, to $f(u)$. Thus, the sequence $(g - f(u_k))_{k \in \mathbb{N}}$ converges strongly in $W_0^{-1,2}(D)$ to $g - f(u)$. Since by assumption $(\Omega_k)_{k \in \mathbb{N}}$ γ -converges to Ω and since the right hand term converges strongly to $g - \rho f(u)$ in $W_0^{-1,2}(D)$, it follows that $(u_k)_{k \in \mathbb{N}}$ converges strongly in $W_0^{1,2}(D)$ to u and that u solves

$$\begin{cases} -\Delta u + \rho f(u) = g & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

This strong convergence immediately implies that

$$J(\Omega) \leq \liminf_{k \rightarrow \infty} J(\Omega_k),$$

thus concluding the proof of Proposition 1.

C Proof of Lemma 3

Let us first prove that $(u_{M,\rho,a})_{M \geq 0}$ is uniformly bounded in $W_0^{1,2}(D)$ with respect to M and ρ . To this aim, let us multiply (2) by $u_{M,\rho,a}$ and integrate by parts. One gets

$$\begin{aligned} \int_D |\nabla u_{M,\rho,a}|^2 &\leq \int_D |\nabla u_{M,\rho,a}|^2 + M(1-a)u_{M,\rho,a}^2 \\ &\leq \|g\|_{H^{-1}(D)} \|u_{M,\rho,a}\|_{L^2(D)} + \rho(f(0) + \|f\|_{W^{1,\infty}}) \|u_{M,\rho,a}\|_{L^2(D)}. \end{aligned}$$

By using the Poincaré inequality, we infer an uniform estimate of $u_{M,\rho,a}$ in $W_0^{1,2}(D)$. According to the Rellich-Kondrachov Theorem, there exists $u^* \in W_0^{1,2}(D)$ such that, up to a subfamily, $(u_{M,\rho,a})_{M \geq 0}$ converges to u^* weakly in $H^1(D)$ and strongly in $L^2(D)$. As a consequence, up to a subsequence, $(f(u_{M,\rho,a}))_{M \geq 0}$ converges to $f(u^*)$ in $L^2(D)$ by using that f is Lipschitz and $(\langle g, u_{M,\rho,a} \rangle_{H^{-1}, H_0^1})_{M \geq 0}$ converges to $\langle g, u^* \rangle_{H^{-1}, H_0^1}$. By rewriting (5) under variational form with $u = u_{M,\rho,a}$, and passing to the limit as $M \rightarrow +\infty$ after having adequately extracted subsequences, we infer that u^* is the unique solution of (5). By using the previous convergence results and the fact that

$$\hat{J}_{M,\rho}(a) = -\frac{\rho}{2} \int_D u_{M,\rho,a} f(u_{M,\rho,a}) - \frac{1}{2} \langle g, u_{M,\rho,a} \rangle_{H^{-1}(D), W_0^{1,2}(D)}$$

we have

$$\hat{J}_{M,\rho}(a) \rightarrow -\frac{\rho}{2} \int_D u^* f(u^*) - \frac{1}{2} \langle g, u^* \rangle_{H^{-1}(D), W_0^{1,2}(D)} \quad \text{as } M \rightarrow +\infty.$$

Finally, if $a = \mathbb{1}_\Omega$, by multiplying (5) by $u_{M,\rho,a}$ and integrating by parts, one gets

$$\int_D |\nabla u_{M,\rho,a}|^2 + M \int_{D \setminus \Omega} u_{M,\rho,a}^2 = \int_D (g - \rho f(u_{M,\rho,a})) u_{M,\rho,a},$$

and since the right-hand side is uniformly bounded with respect to M , we infer that $\sqrt{M}u_{M,\rho,a}$ is bounded in $L^2(D \setminus \Omega)$ so that $u^* \in W_0^{1,2}(\Omega)$.

The conclusion follows.

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