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Georges Gras. On the  $\lambda$ -stability of  $p$ -class groups along cyclic  $p$ -towers of a number field. 2021.  
hal-03155190v5

**HAL Id: hal-03155190**

**<https://hal.science/hal-03155190v5>**

Preprint submitted on 12 Aug 2021

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# ON THE $\lambda$ -STABILITY OF $p$ -CLASS GROUPS ALONG CYCLIC $p$ -TOWERS OF A NUMBER FIELD

GEORGES GRAS

ABSTRACT. Let  $k$  be a number field,  $p \geq 2$  a prime and  $S$  a set of tame or wild finite places of  $k$ . We call  $K/k$  a totally  $S$ -ramified cyclic  $p$ -tower if  $\text{Gal}(K/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$  and if  $S \neq \emptyset$  is totally ramified. Using analogues of Chevalley's formula (Gras, Proc. Math. Sci. **127**(1) (2017)), we give an elementary proof of a stability theorem (Theorem 3.1) for generalized  $p$ -class groups  $\mathcal{X}_n$  of the layers  $k_n \subseteq K$ : let  $\lambda = \max(0, \#S - 1 - \rho)$  given in Definition 1.1; then  $\#\mathcal{X}_n = \#\mathcal{X}_0 \cdot p^{\lambda \cdot n}$  for all  $n \in [0, N]$ , if and only if  $\#\mathcal{X}_1 = \#\mathcal{X}_0 \cdot p^\lambda$ . This improves the case  $\lambda = 0$  of Fukuda (1994), Li–Ouyang–Xu–Zhang (2020), Mizusawa–Yamamoto (2020), whose techniques are based on Iwasawa's theory or Galois theory of pro- $p$ -groups. We deduce capitulation properties of  $\mathcal{X}_0$  in the tower. Finally we apply our principles to the torsion groups  $\mathcal{T}_n$  of abelian  $p$ -ramification theory. Numerical examples are given.

## CONTENTS

1. Introduction	1
2. Around Chevalley's ambiguous class number formula	4
3. Introduction of Hasse's norm symbols – Main theorem	5
4. On the existence of capitulation fields for real class groups	7
4.1. Monogenic class groups and $K \subset k(\mu_\ell)$	7
4.2. Non-monogenic class groups and $K \subset k(\mu_\ell)$	8
5. Examples and counterexamples, with modulus, of $\lambda$ -stabilities	10
5.1. Cyclic $p$ -towers over $k = \mathbb{Q}(\sqrt{\pm m})$	10
5.2. Cyclic $p$ -towers over $\mathbb{Q}$ with modulus	11
5.3. Cyclic $p$ -towers over a cyclic cubic field $k$	11
6. Other examples of applications and Remarks	12
7. Behavior of other invariants in cyclic $p$ -towers	13
7.1. Case of the cyclotomic $\mathbb{Z}_p$ -extensions $k_\infty/k$ of $k$ real	15
7.2. Case of real cyclic $p$ -towers with tame ramification	16
References	17

## 1. INTRODUCTION

The behavior of  $p$ -class groups in a  $\mathbb{Z}_p$ -extension gives rise to many theoretical and computational results; a main observation is the unpredictability of these

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*Date:* August 12, 2021.

*2020 Mathematics Subject Classification.* Primary 11R23; 11R29; 11R37.

*Key words and phrases.*  $p$ -class groups, class field theory, Chevalley's formula, cyclic  $p$ -extensions.

groups in the first layers of the  $\mathbb{Z}_p$ -extension, then their regularization from some not effective level, Iwasawa's theory giving the famous formula of the orders by means of the invariants  $\lambda, \mu, \nu$ ; nevertheless, this algebraic context is not sufficient to estimate these parameters (e.g., Greenberg's conjecture [13]). Our purpose is to see, more generally, what is the behavior of the  $p$ -class groups (and some other arithmetic invariants) in cyclic  $p$ -extensions in which one allows tame ramification.

Let  $k$  be a number field,  $p \geq 2$  a prime number and  $K$  a cyclic extension of degree  $p^N$  of  $k$ ,  $N \geq 1$ , and let  $k_n$ ,  $n \in [0, N]$ , be the degree  $p^n$  extension of  $k$  in  $K$ . We speak of cyclic  $p$ -towers  $K/k$  and put  $G = \text{Gal}(K/k)$ . Let  $S$  be the set of places of  $k$ , ramified in  $K/k$ . We will assume  $S \neq \emptyset$  and  $S$  totally ramified in  $K/k$ ; in general,  $S$  contains tame places, except if  $K$  is contained in a  $\mathbb{Z}_p$ -extension of  $k$ , in which case  $S$  is a set of  $p$ -places of  $k$ . Due to the very nature of a tower, there will never be "complexification=ramification" of infinite real places.

Let  $\mathfrak{m} \neq 0$  be an integer ideal of  $k$ , of support  $T$  disjoint from  $S$ ; let  $\mathcal{C}_{k, \mathfrak{m}}$  and  $\mathcal{C}_{k_n, \mathfrak{m}}$ , denoted simply  $\mathcal{C}_{0, \mathfrak{m}}$  and  $\mathcal{C}_{n, \mathfrak{m}}$ , be the  $p$ -Sylow subgroups of the ray class groups modulo  $\mathfrak{m}$ , of  $k$  and  $k_n$ , respectively (for  $n > 0$ ,  $\mathfrak{m}$  is seen as extended ideal in  $k_n$ ). The class of an ideal  $\mathfrak{a}$  of  $k_n$  is denoted  $\mathcal{C}_{n, \mathfrak{m}}(\mathfrak{a})$ .

Let  $N_{K/k_n}$  be the arithmetic norm in  $K/k_n$ , defined on class groups from norms of ideals. Since  $\mathfrak{m}$  is prime to  $S \neq \emptyset$  totally ramified, the corresponding  $p$ -Hilbert's ray class fields  $H_{n, \mathfrak{m}}$ , of the  $k_n$ 's are linearly disjoint from the relative  $p$ -towers  $K/k_n$ ; since, by class field theory,  $N_{K/k_n}$  corresponds to the restriction of automorphisms  $\text{Gal}(H_{N, \mathfrak{m}}/K) \rightarrow \text{Gal}(H_{n, \mathfrak{m}}/k_n)$ , we get  $N_{K/k_n}(\mathcal{C}_{N, \mathfrak{m}}) = \mathcal{C}_{n, \mathfrak{m}}$ , for all  $n \in [0, N]$ .

Let  $\mathcal{H}_N$  be a sub- $\mathbb{Z}_p[G]$ -module of  $\mathcal{C}_{N, \mathfrak{m}}$ ; we may represent a minimal set of  $\mathbb{Z}_p[G]$ -generators of  $\mathcal{H}_N$  with prime ideals  $\mathfrak{Q}_1, \dots, \mathfrak{Q}_t \notin S \cup T$ ,  $\mathfrak{Q}_i \mid \mathfrak{q}_i$  in  $K/k$ . Let  $\mathcal{I}_N$  be the  $\mathbb{Z}[G]$ -module generated by these ideals; thus  $\mathcal{I}_0 := N_{K/k}(\mathcal{I}_N)$  is of minimal  $\mathbb{Z}$ -rank  $t$  since  $N_{K/k}(\mathcal{I}_N) \otimes \mathbb{Q} = \langle \mathfrak{q}_1, \dots, \mathfrak{q}_t \rangle_{\mathbb{Z}} \otimes \mathbb{Q}$ . We set, for all  $n \in [0, N]$ :

$$(1.1) \quad \mathcal{I}_n := N_{K/k_n}(\mathcal{I}_N), \quad \mathcal{H}_n := N_{K/k_n}(\mathcal{H}_N), \quad \mathcal{X}_{n, \mathfrak{m}} := \mathcal{C}_{n, \mathfrak{m}}/\mathcal{H}_n.$$

This defines the family  $\{\mathcal{X}_{n, \mathfrak{m}}\}_{n \in [0, N]}$  ("generalized  $p$ -class groups") such that:

$$N_{K/k_n}(\mathcal{X}_{N, \mathfrak{m}}) = \mathcal{X}_{n, \mathfrak{m}}, \text{ for all } n \in [0, N].$$

Any place  $\mathfrak{Q}$  of  $K$ , such that  $\mathcal{C}_{N, \mathfrak{m}}(\mathfrak{Q}) \in \mathcal{H}_N$ , totally splits in the subfield of  $H_{N, \mathfrak{m}}$  fixed by the image of  $\mathcal{H}_N$  in  $\text{Gal}(H_{N, \mathfrak{m}}/K)$ ; so this general definition allows to enforce decomposition conditions in the ray class fields.

To simplify, we shall remove the indices  $\mathfrak{m}$ , except necessity.

One speaks of " $\lambda$ -stability" of these  $\mathcal{X}_n$  along the cyclic  $p$ -tower  $K/k$  when there exists  $\lambda \geq 0$  such that  $\#\mathcal{X}_n = \#\mathcal{X}_0 \cdot p^{\lambda \cdot n}$ , for all  $n \in [0, N]$ . Of course, this is not a workable definition, both theoretically (classical context of  $\mathbb{Z}_p$ -extensions or less familiar case of  $p$ -towers with tame ramification) and computationally (inability to use PARI/GP [30] beyond some level  $n = 3$  or  $4$ , and almost nothing for  $p > 5$ ). So we intend to get an accessible criterion likely to give information in all the tower.

**Definition 1.1.** Let  $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{I}_N)\}$ . For instance,  $\mathcal{I}_N = 1$  yields  $\{\mathcal{X}_n\} = \{\mathcal{C}_n\}$  and  $\Lambda = E$ , the group of units  $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$  of  $k$ . We have the exact sequence  $1 \rightarrow \Lambda/E \rightarrow \mathcal{I}_0 \rightarrow \mathcal{H}_0 \rightarrow 1$ . Since  $\Lambda/E$  is  $\mathbb{Z}$ -free (of  $\mathbb{Z}$ -rank  $t$  because  $\mathcal{H}_0$  is finite), one can write, with representatives  $\alpha_j$  of  $\Lambda/E$ :

$$\Lambda = \text{tor}_{\mathbb{Z}}(E) \oplus \langle \varepsilon_1, \dots, \varepsilon_r; \alpha_1, \dots, \alpha_t \rangle_{\mathbb{Z}} =: \text{tor}_{\mathbb{Z}}(E) \oplus X, \quad \varepsilon_i \in E,$$

$r := r_1 + r_2 - 1$ , where  $(r_1, r_2)$  is the signature of  $k$ , and where  $X$  is  $\mathbb{Z}$ -free of  $\mathbb{Z}$ -rank  $\rho := r + t$ ; we define the parameter  $\lambda := \max(0, \#S - 1 - \rho)$ .

We shall see that the norm properties, in  $K/k$ , of  $\text{tor}_{\mathbb{Z}}(E) \otimes \mathbb{Z}_p$  are specific and may give some obstructions that are elucidated by the following Lemma.

**Lemma 1.2.** (Albert's Theorem for  $K/k$ ; for a proof, see [8, Exercise I.6.2.3]). *If  $k$  contains the group  $\mu_{p^e}$  of  $p^e$ th roots of unity, then  $\mu_{p^e} \subset N_{K/k}(K^\times)$  if and only if there exists a degree  $p^e$  cyclic extension  $L/K$  such that  $L/k$  is cyclic.*

**Remark 1.3.** *We shall restrict ourselves to the case  $\text{tor}_{\mathbb{Z}}(E) \otimes \mathbb{Z}_p \subset N_{K/k}(K^\times)$ . If necessary, it suffices to restrict the tower  $K/k$  to the sub-tower  $K'/k$  such that  $[K : K']$  be the order of the group of  $p$ th roots of unity of  $k$ , or to notice that  $L/K/k$  does exist (recall that no arithmetic condition is required on  $L/K$ ).*

*Thus, the norm properties of  $\Lambda$  are assigned to  $X$ .*

We put to simplify  $\mathcal{C}_0 =: \mathcal{C}$ ,  $\mathcal{H}_0 =: \mathcal{H}$ ,  $\mathcal{I}_0 =: \mathcal{I}$ ,  $\mathcal{X}_0 =: \mathcal{X}$ , and so on,  $\mathfrak{m}$  being implied; let  $p^e$  be the exponent of  $\mathcal{X}$ . Under the assumptions on  $S$ ,  $T$ , we will prove the following result for the cyclic  $p$ -tower  $K/k$  of degree  $p^N$  (see Theorem 3.1 and Corollary 3.2 for more complete statements):

**Main Result.** *Assume that  $\text{tor}_{\mathbb{Z}}(E) \subset N_{K/k}(K^\times)$  (Remark 1.3). Let  $\rho \geq 0$  be the  $\mathbb{Z}$ -rank of the  $\mathbb{Z}$ -module  $X$  and let  $\lambda := \max(0, \#S - 1 - \rho)$  (Definition 1.1). Then  $\#\mathcal{X}'_n = \#\mathcal{X} \cdot p^{\lambda n}$  for all  $n \in [0, N]$ , if and only if  $\#\mathcal{X}'_1 = \#\mathcal{X} \cdot p^\lambda$ . If the criterion applies with  $\lambda = 0$  (i.e.,  $\#\mathcal{X}'_1 = \#\mathcal{X}$ ) and if  $e \leq N$ , then  $\mathcal{X}$  capitulates in  $K$ .*

**Remark 1.4.** (a) *If  $\lambda = \max(0, \#S - 1 - \rho)$  fulfills the criterion of  $\lambda$ -stability from the base field  $k$ , it fulfills the same  $\lambda$ -stability criterion in any relative  $p$ -tower  $K/k'$ ,  $k' := k_{n_0}$ ,  $n_0 \in [0, N[$ , since (with  $\mathcal{X}'_n := \mathcal{X}'_{n_0+n}$ ,  $n \geq 0$ ) we get  $\#\mathcal{X}'_n = \#\mathcal{X}' \cdot p^{\lambda n}$  since  $\#\mathcal{X}'_n = \#\mathcal{X}'_{n_0+n} = \#\mathcal{X} \cdot p^{\lambda(n_0+n)} = (\#\mathcal{X} \cdot p^{\lambda n_0}) \cdot p^{\lambda n} = \#\mathcal{X}' \cdot p^{\lambda n}$ .*

(b) *If  $\lambda = \max(0, \#S - 1 - \rho)$  is not suitable for the criterion in  $k_1/k$  (which shall be equivalent to  $\#\mathcal{X}'_1 > \#\mathcal{X} \cdot p^\lambda$  due to Chevalley's formula), we can consider a relative tower  $K/k'$ ; then, in  $k'$ ,  $\#S' = \#S$ ,  $t' = t$ , and  $\lambda' := \max(0, \#S - 1 - \rho')$  is a strictly decreasing function of  $[k' : k]$  (indeed, from Definition 1.1, we compute that  $\rho' - \rho = r' - r = (r_1 + r_2) \cdot ([k' : k] - 1)$ ), so that two cases may arise:*

(i) *For some  $k' \subset K$ , we have  $\#\mathcal{X}'_1 = \#\mathcal{X}' \cdot p^{\lambda'}$  giving the  $\lambda'$ -stability in  $K/k'$  with regular linear increasing orders from  $k'$ .*

(ii) *Whatever  $k' \subset K$ ,  $\lambda'$  is not suitable (i.e.,  $\#\mathcal{X}'_1 > \#\mathcal{X}' \cdot p^{\lambda'}$ , even if  $\lambda' = 0$ , which occurs rapidly for  $k'$  high enough in the tower), which means that  $\#\mathcal{X}'_n$  is strictly increasing from some level  $n_0$ , which is illustrated by numerical examples and can define a non-linear increasing (in a  $\mathbb{Z}_p$ -extension  $\tilde{k}$  this means " $\tilde{\mu} \neq 0$ "). It is important to note that a  $\tilde{\lambda}$ -stability may exist from some level, with  $\tilde{\lambda} > \lambda'$ ; this is the case in a  $\mathbb{Z}_p$ -extension  $\tilde{k}$  with Iwasawa's invariants  $\tilde{\lambda} \neq 0$  and  $\tilde{\mu} = 0$ .*

Numerical experiments are out of reach as soon as  $n > 3$  or 4, but a reasonable heuristic is a tendency to stabilization in totally real  $p$ -towers, by analogy with the study of Greenberg's conjecture carried out in [11]. In a different framework, let  $k_n$  be the  $n$ th layer of the cyclotomic  $\mathbb{Z}_p$ -extension  $k_\infty$  of  $k$  and let  $C_n$  be the whole class group of  $k_n$ ; then, for  $\ell \neq p$ ,  $\#(C_n \otimes \mathbb{Z}_\ell)$  stabilizes in  $k_\infty$  (a deep result in the abelian case [33], extended in [4] to Iwasawa context in the  $\mathbb{Z}_p \times \mathbb{Z}_\ell$ -extension of  $k$ ).

(c) *For  $k$  fixed,  $\lambda$  is unbounded (which only depends on the tower via  $S$ ) and the condition  $\#\mathcal{C}'_1 = \#\mathcal{C}$  of the literature ( $\lambda = t = 0$ , thus  $\#S \leq r_1 + r_2$ ) is empty as*

soon as  $\lambda \geq 1$ , whence the interest of the factor  $p^\lambda$  to get examples whatever  $S$ . For known results (all relative to  $\lambda = 0$ ), one may cite Fukuda [5] using Iwasawa's theory, Li–Ouyang–Xu–Zhang [25, §3] working in a non-abelian Galois context, in Kummer towers, via the use of the fixed points formulas [6, 9], then Mizusawa [27] above  $\mathbb{Z}_2$ -extensions, and Mizusawa–Yamamoto [28] for generalizations, including ramification and splitting conditions, via the Galois theory of pro- $p$ -groups.

(d) If Lemma 1.2 does not apply, some counterexamples can arise. For instance, let  $k = \mathbb{Q}(\sqrt{-55})$ ,  $p = 2$ ,  $K = k(\mu_{17})$  of degree  $2^4$  over  $k$  ( $-1 \notin N_{K/k}(K^\times)$ ,  $\#S = 2$ ,  $\lambda = 1$ ); the successive 2-structures are [4], [8], [16], [32], [32], for which  $\#\mathcal{C}_n = \#\mathcal{C} \cdot 2^n$  holds for  $n \in [0, 3]$ , since  $-1$  is norm in  $k_3/k$ , but not for  $n = 4$ .

To conclude, we will show that the behavior, in  $K/k$ , of the finite torsion groups  $\mathcal{T}_n$  of  $p$ -ramification theory (as “dual” invariants of  $p$ -class groups) gives much strongly increasing groups with possibly non-zero  $\tilde{\mu}$ -invariants in  $\mathbb{Z}_p$ -extensions  $\tilde{k}/k$ .

## 2. AROUND CHEVALLEY'S AMBIGUOUS CLASS NUMBER FORMULA

The well-known Chevalley's formula [2, p. 402] is the pivotal element for a great lot of “fixed points formulas”. For the ordinary sense, it takes into account the complexification of real infinite places, which does not occur for us as explained in the Introduction. Let  $L/F$  be a cyclic  $p$ -extension of Galois group  $G$ ,  $S$ -ramified, non-complexified if  $p = 2$ , and let  $e_v(L/F)$  be the ramification index of  $v \in S$ ; we have  $\#\mathcal{C}_L^G = \frac{\#\mathcal{C}_F \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot (E_F : E_F \cap N_{L/F}(L^\times))}$ , where  $\mathcal{C}_F$ ,  $\mathcal{C}_L$  denote the  $p$ -class groups of  $F$ ,  $L$ , respectively, and  $E_F$  the group of units of  $F$ .

Various generalizations of this formula were given (Gras [6, Théorème 2.7, Corollaire 2.8], [9, Theorem 3.6, Corollaries 3.7, 3.9], Jaulent [16, Chap. II, §3]); mention their recent idelic proof by Li–Yu [26, Theorem 2.1 & §2.3, Examples]. We shall use the following one for the layers of a  $S$ -ramified cyclic  $p$ -tower  $K/k$ :

**Proposition 2.1.** *Assume  $S \neq \emptyset$  totally ramified in  $L/F$ . Let  $\mathfrak{m}$ , of support  $T$  disjoint from  $S$ , be a modulus of  $F$  and let  $\mathcal{C}_F$ ,  $\mathcal{C}_L$ , be the  $p$ -Sylow subgroups of the ray class groups modulo  $\mathfrak{m}$ , of  $F$  and  $L$ , respectively. For any  $\mathbb{Z}[G]$ -module  $\mathcal{I}_L$ , of prime-to- $T \cup S$  ideals of  $L$ , defining  $\mathcal{H}_L := \mathcal{C}_L(\mathcal{I}_L)$ ,  $\mathcal{H}_F := N_{L/F}(\mathcal{H}_L) = \mathcal{C}_F(N_{L/F}(\mathcal{I}_L))$ , we have, with  $\Lambda_{L/F} := \{x \in F^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{L/F}(\mathcal{I}_L)\}$ :*

$$(2.1) \quad \#(\mathcal{C}_L/\mathcal{H}_L)^G = \#(\mathcal{C}_F/\mathcal{H}_F) \times \frac{(\#G)^{\#S-1}}{(\Lambda_{L/F} : \Lambda_{L/F} \cap N_{L/F}(L^\times))}.$$

**Lemma 2.2.** *Let  $K/k$  be a cyclic  $p$ -tower of degree  $p^N$ , of Galois group  $G = \langle \sigma \rangle$ . Let  $\mathcal{H}_N := \mathcal{C}_N(\mathcal{I}_N) \subseteq \mathcal{C}_N$  and  $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{I}_N)\}$  (see Definition 1.1):*

- (i)  $\Lambda = \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{k_n/k}(\mathcal{I}_n)\}$  (i.e.,  $\Lambda_{K/k} = \Lambda_{k_n/k}$ ),
- (ii)  $\Lambda \subseteq \Lambda_1 := \{x_1 \in k_1^\times, x_1 \equiv 1 \pmod{\mathfrak{m}}, (x_1) \in N_{K/k_1}(\mathcal{I}_N)\}$ .

*Proof.* Point (i) comes from  $\mathcal{I}_n = N_{K/k_n}(\mathcal{I}_N)$ . Let  $x \in \Lambda$  seen in  $k_1$ ; then  $(x) = N_{K/k}(\mathfrak{A}) = N_{K/k_1}(\mathfrak{A}^\Theta)$ , where  $\Theta = 1 + \sigma + \dots + \sigma^{p-1}$ , and  $\mathfrak{A}^\Theta \in \mathcal{I}_N$  since  $\mathcal{I}_N$  is a  $\mathbb{Z}[G]$ -module, whence (ii). Notice that  $(x) \in N_{K/k}(\mathcal{I}_N)$  expresses that  $x$  is local norm in  $K/k$  at every  $v \notin S$ ; if moreover  $x$  is local norm at  $S$  it is a global one (Hasse norm theorem).  $\square$

In what follows, throughout the article, only the invariant  $\rho = \text{rk}_{\mathbb{Z}}(X)$  is needed (Definition 1.1 giving  $\lambda := \max(0, \#S - 1 - \rho)$ ) and never  $\mathcal{S}_N, \Lambda, \dots$ . However,  $\rho = r + t$  may be unknown because of the minimal number  $t$  of generators of the  $\mathbb{Z}[G]$ -module  $\mathcal{S}_N$  (or the  $\mathbb{Z}$ -module  $\mathcal{S}$ ) if it is too general; but, in practice,  $\rho$  is known (e.g.,  $\rho = r_1 + r_2 - 1$  for ray class groups modulo  $\mathfrak{m}$ ) or  $\rho$  is large enough to get  $\lambda = 0$ .

**Lemma 2.3.** *Let  $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$  be a free  $\mathbb{Z}$ -module of  $\mathbb{Z}$ -rank  $\rho \geq 0$ , let  $\Omega := (\mathbb{Z}/p^n\mathbb{Z})^\delta$ ,  $n \geq 1$ ,  $\delta \geq 0$ , and let  $f : X \rightarrow \Omega$  be an homomorphism such that the image of  $X \rightarrow \Omega/\Omega^p$  is of  $\mathbb{F}_p$ -dimension  $\min(\rho, \delta)$ . Then  $\#f(X) = p^{\min(\rho, \delta) \cdot n}$  and for all  $m \in [0, n]$ ,  $\#f^{p^m}(X) = p^{\min(\rho, \delta) \cdot (n-m)}$  (where  $f^{p^m}(x) := f(x)^{p^m}$ ).*

*Proof.* From  $f : X \rightarrow \Omega$ , let  $\bar{f} : X/X^p \rightarrow \Omega/\Omega^p$ ; by assumption,  $\dim_{\mathbb{F}_p}(\text{Im}(\bar{f})) = \min(\rho, \delta)$ . Let  $M = \langle e_1, \dots, e_\delta \rangle_{\mathbb{Z}_p}$  be a free  $\mathbb{Z}_p$ -module of  $\mathbb{Z}_p$ -rank  $\delta$ , and replace  $\Omega$  by  $M/M^{p^n}$ . Let  $q : X \rightarrow X/X^p$ ,  $\pi_n : M \rightarrow M/M^{p^n}$ ,  $\pi : M/M^{p^n} \rightarrow M/M^p$ , and  $\pi_1 = \pi \circ \pi_n : M \rightarrow M/M^p$ , be the canonical maps. Then, let  $F : X \rightarrow M$  be any map such that  $\pi_n \circ F = f$ . We have, since  $n \geq 1$ , the commutative diagram:

$$\begin{array}{ccc}
 X \simeq \mathbb{Z}^\rho & \xrightarrow{F} & M \simeq \mathbb{Z}_p^\delta \\
 \downarrow q & \searrow f & \downarrow \pi_n \\
 & & \Omega \simeq M/M^{p^n} \\
 & & \downarrow \pi \\
 X/X^p & \xrightarrow{\bar{f}} & \Omega/\Omega^p \simeq M/M^p
 \end{array}
 \quad \begin{array}{l} \\ \\ \\ \pi_1 \end{array}$$

(i) Case  $\rho \geq \delta$  (surjectivity of  $\bar{f}$ ). Let  $y \in M$ ; there exists  $x \in X$  such that  $\pi_1(y) = \bar{f}(q(x)) = \pi_1(F(x))$ , whence  $y = F(x) \cdot y'^p$ ,  $y' \in M$ ; so, by a finite induction,  $M = F(X) \cdot M^{p^n}$  and  $\pi_n(M) = \Omega = f(X)$  is of order  $p^{\delta \cdot n}$ .

(ii) Case  $\rho < \delta$  (injectivity of  $\bar{f}$ ). Let  $y \in M$  such that  $y^p = F(x)$ ,  $x \in X$ ; then  $1 = \pi_1(y^p) = \pi_1(F(x)) = \bar{f}(q(x))$ , whence  $q(x) = 1$  and  $x = x'^p$ ,  $x' \in X$ , giving  $y^p = F(x')^p$  in  $M$  thus  $y = F(x')$ , proving that  $F(X)$  is direct factor in  $M$ ; thus  $f(X) = \pi_n(F(X))$  is a direct factor, in  $\Omega$ , isomorphic to  $(\mathbb{Z}/p^n\mathbb{Z})^\rho$ .

Since  $f^{p^m}(X) = f(X)^{p^m}$ , this gives  $f(X)^{p^m} = \Omega^{p^m} \simeq ((\mathbb{Z}/p^n\mathbb{Z})^\delta)^{p^m}$  in case (i) and  $((\mathbb{Z}/p^n\mathbb{Z})^\rho)^{p^m}$  in case (ii). Whence the orders.  $\square$

### 3. INTRODUCTION OF HASSE'S NORM SYMBOLS – MAIN THEOREM

Let  $K/k$  be a  $S$ -ramified cyclic  $p$ -tower of degree  $p^N$ ,  $N \geq 1$ . Let  $\omega_{k_n/k}$  be the map which associates with  $x \in \Lambda$  the family of Hasse's norm symbols  $(\frac{x, k_n/k}{v}) \in I_v(k_n/k)$  (inertia groups of  $v \in S$ ), where  $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{v \mathfrak{m}}, (x) \in N_{K/k}(\mathcal{S}_N)\}$ . Since  $x$  is local norm at the places  $v \notin S$ , the image of  $\omega_{k_n/k}$  is contained in  $\Omega_{k_n/k} := \{(\tau_v)_{v \in S} \in \bigoplus_{v \in S} I_v(k_n/k), \prod_{v \in S} \tau_v = 1\}$  (from the product formula); then  $\text{Ker}(\omega_{k_n/k}) = \Lambda \cap N_{k_n/k}(k_n^\times)$ .

Assume  $S \neq \emptyset$  totally ramified; so, fixing any  $v_0 \in S$ ,  $\Omega_{k_n/k} \simeq \bigoplus_{v \neq v_0} G_n$ , with  $G_n := \text{Gal}(k_n/k)$ . We consider formula (2.1) in  $k_n/k$ , using Lemma 2.2 (i):

$$(3.1) \quad \#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot \frac{p^{n(\#S-1)}}{(\Lambda : \Lambda \cap N_{k_n/k}(k_n^\times))} = \#\mathcal{X} \cdot \frac{p^{n(\#S-1)}}{\#\omega_{k_n/k}(\Lambda)},$$

where  $\mathcal{X}_n := \mathcal{C}_n/\mathcal{H}_n$  for the family  $\{\mathcal{H}_n\}$  (cf. (1.1) with a prime-to- $S$  modulus  $\mathfrak{m}$ ).

**Theorem 3.1.** *Set  $\Lambda =: \text{tor}_{\mathbb{Z}}(E) \oplus X$ , with  $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$  free of  $\mathbb{Z}$ -rank  $\rho$ , and let  $\lambda := \max(0, \#S - 1 - \rho)$ . Assume that  $\text{tor}_{\mathbb{Z}}(E) \subset N_{K/k}(K^\times)$  (see Lemma 1.2 and Remark 1.3). We then have the following properties of  $\lambda$ -stability:*

(i) *If  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ , then  $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda \cdot n}$  and  $\mathcal{X}_n^{G_n} = \mathcal{X}_n$ , for all  $n \in [0, N]$ .*

(ii) *If  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ , then  $J_{k_n/k}(\mathcal{X}) = \mathcal{X}_n^{p^n}$  and  $\text{Ker}(J_{k_n/k}) = N_{k_n/k}(\mathcal{X}_n[p^n])$ , for all  $n \in [0, N]$ , where the  $J_{k_n/k}$ 's are the transfer maps in  $k_n/k$  and  $\mathcal{X}_n[p^n] := \{x \in \mathcal{X}_n, x^{p^n} = 1\}$ . If moreover,  $\lambda = 0$ , the norm maps  $N_{k_n/k} : \mathcal{X}_n \twoheadrightarrow \mathcal{X}$  are isomorphisms.*

*Proof.* Put  $N_n := N_{k_n/k}$ ,  $J_n := J_{k_n/k}$ ,  $G_n =: \langle \sigma_n \rangle$ ,  $\Omega_n := \Omega_{k_n/k}$ ,  $\omega_n := \omega_{k_n/k}$ ,  $s := \#S$ . By assumption,  $\omega_n(\Lambda) = \omega_n(X)$  for all  $n \in [1, N]$ . From (3.1),  $\#\mathcal{X}_1^{G_1} = \#\mathcal{X} \cdot \frac{p^{s-1}}{\#\omega_1(X)} \geq \#\mathcal{X} \cdot p^\lambda$  since  $\#\omega_1(X) \leq p^{\min(\rho, s-1)}$ ; thus, with  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ , we obtain  $\mathcal{X}_1^{G_1} = \mathcal{X}_1$ , whence  $\#\mathcal{X}_1^{G_1} = \#\mathcal{X} \cdot p^\lambda$  and  $\#\omega_1(X) = p^{\min(\rho, s-1)}$ .

By restriction of Hasse's symbols,  $\omega_1 = \pi \circ \omega_n$ , with  $\pi : \Omega_n \rightarrow \Omega_n/\Omega_n^p \simeq G_1^{s-1}$ . Since we have proven that  $\dim_{\mathbb{F}_p}(\omega_1(X)) = \min(\rho, s-1)$ , Lemma 2.3 applies to  $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$ ,  $\delta = s-1$ ,  $f = \omega_n$ ; so,  $\#\omega_n(X) = p^{\min(\rho, s-1)n}$  and, from (3.1):

$$(3.2) \quad \#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda \cdot n}, \text{ for all } n \in [0, N].$$

Consider the extension  $k_n/k_1$ , of Galois group  $G_n^p = \langle \sigma_n^p \rangle$ , and the corresponding map  $\omega'_n$  on  $k_1^\times$  with values in  $\Omega_{k_n/k_1} \simeq \Omega_n^p$ ; then  $\#\mathcal{X}_n^{G_n^p} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda_1)}$ ,  $\Lambda_1 := \{x \in k_1^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k_1}(\mathcal{I}_N)\}$ . Since  $\Lambda \subseteq \Lambda_1$  (Lemma 2.2 (ii)), the functorial properties of Hasse's symbols on  $X$  imply (where  $v_1 \mid v$  in  $k_1$ ):

$$\omega'_n(x_i) = \left( \left( \frac{x_i, k_n/k_1}{v_1} \right) \right)_v = \left( \left( \frac{N_1(x_i), k_n/k}{v} \right) \right)_v = \left( \left( \frac{x_i^p, k_n/k}{v} \right) \right)_v = \left( \left( \frac{x_i, k_n/k}{v} \right)^p \right)_v,$$

giving  $\omega'_n = \omega_n^p$  on  $X$ . Then,  $\omega'_n(X) \subseteq \omega'_n(\Lambda_1)$  yields:

$$\#\mathcal{X}_n^{G_n^p} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda_1)} \leq \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(X)} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega_n^p(X)};$$

Lemma 2.3, for  $\omega_n^p$ , gives  $\#\omega_n^p(X) = p^{(n-1)\min(\rho, s-1)}$ , then, with  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ :

$$\#\mathcal{X}_n^{G_n^p} \leq \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{p^{(n-1)\min(\rho, s-1)}} = \#\mathcal{X}_1 \cdot p^{\lambda \cdot (n-1)} = \#\mathcal{X} \cdot p^{\lambda \cdot n}.$$

Since  $\mathcal{X}_n^{G_n^p} \supseteq \mathcal{X}_n^{G_n}$  and  $\#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda \cdot n}$  from (3.2), we get  $\mathcal{X}_n^{G_n} = \mathcal{X}_n^{G_n^p}$ , equivalent to  $\mathcal{X}_n^{1-\sigma_n} = \mathcal{X}_n^{1-\sigma_n^p} = \mathcal{X}_n^{(1-\sigma_n) \cdot \theta}$ , where  $\theta = 1 + \sigma_n + \dots + \sigma_n^{p-1} \in (p, 1-\sigma_n)$ , a maximal ideal of  $\mathbb{Z}_p[G_n]$  since  $\mathbb{Z}_p[G_n]/(p, 1-\sigma_n) \simeq \mathbb{F}_p$ ; so  $\mathcal{X}_n^{1-\sigma_n} = 1$ , thus  $\mathcal{X}_n = \mathcal{X}_n^{G_n}$ . Whence (i).

From  $N_n(\mathcal{X}_n) = \mathcal{X}$ ,  $\mathcal{X}_n = \mathcal{X}_n^{G_n}$  (from (i)) and  $J_n \circ N_n = \nu_n$  (the algebraic norm), one obtains  $J_n(\mathcal{X}) = J_n(N_n(\mathcal{X}_n)) = \mathcal{X}_n^{\nu_n} = \mathcal{X}_n^{p^n}$ . Let  $x \in \text{Ker}(J_n)$  and put  $x = N_n(y)$ ,  $y \in \mathcal{X}_n$ ; then  $1 = J_n(x) = J_n(N_n(y)) = y^{p^n}$ , so  $\text{Ker}(J_n) \subseteq N_n(\mathcal{X}_n[p^n])$  and if  $x = N_n(y)$ ,  $y^{p^n} = 1$ , then  $J_n(x) = J_n(N_n(y)) = y^{p^n} = 1$ . Whence (ii).  $\square$

**Corollary 3.2.** *Let  $p^e$  be the exponent of  $\mathcal{X}$  in  $k$  fixed, and let  $K$  be a totally  $S$ -ramified cyclic  $p$ -tower of degree  $p^N$ , with  $N \geq e$ , such that  $\lambda = 0$  (namely  $1 \leq \#S \leq r_1 + r_2$  for the family  $\{\mathcal{C}_n\}$  of  $p$ -class groups). If  $\#\mathcal{X}_1 = \#\mathcal{X}$ , then*



$\mathcal{X}$  capitulates in  $k_e$ . In particular, this applies in a  $\mathbb{Z}_p$ -extension  $\tilde{k}/k$  with  $\lambda = 0$ ,  $\#\mathcal{X}_1 = \#\mathcal{X}$  and  $S$  (set of  $p$ -places) totally ramified.

*Proof.* Since  $\lambda = 0$ ,  $\mathcal{X}_e \simeq \mathcal{X}$  from the isomorphism induced by  $N_e : \mathcal{X}_e \rightarrow \mathcal{X}$ , whence  $\text{Ker}(J_e) = N_e(\mathcal{X}_e[p^e]) \simeq \mathcal{X}[p^e] = \mathcal{X}$ .  $\square$

**Example 3.3.** Put  $p^e := \#\text{tor}_{\mathbb{Z}_p}(E)$ . For a prime  $\ell \equiv 1 \pmod{p^{N+e}}$ , let  $S_\ell$  be the set of prime ideals  $\mathfrak{l}$  of  $k$  dividing  $\ell$  and let  $K \subseteq k(\mu_\ell)$  be the subfield of degree  $p^N$ . Sections 4 and 5 will give many examples of such capitulations of  $p$ -class groups of real fields  $k$  in  $K \subseteq k(\mu_\ell)$  (totally  $S$ -ramified with  $S = S_\ell$ ); for instance, for  $k = \mathbb{Q}(\sqrt{4409})$ , the capitulation of the 3-class group  $\mathcal{C} \simeq \mathbb{Z}/9\mathbb{Z}$  occurs in  $k_2$  for  $\ell \in \{19, 37, 73, 109, 127, 271, 307, 379, 397, 523, 541, 577, 739, 883, \dots\}$ . The 3-class group  $\mathcal{C} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  of the cyclic cubic field of conductor 5383, defined by  $P = x^3 + x^2 - 1794x + 17744$ , capitulates in  $k_2$  for  $\ell = 109, 163, 919, \dots$ . The 2-class group  $\mathcal{C} \simeq \mathbb{Z}/16\mathbb{Z}$  of  $\mathbb{Q}(\sqrt{2305})$  capitulates in  $k_4$  for  $\ell = 97, 193, 353, 449, 929, \dots$ . The 5-class group  $\mathcal{C} \simeq \mathbb{Z}/25\mathbb{Z}$  of  $\mathbb{Q}(\sqrt{24859})$  capitulates in  $k_2$  for  $\ell = 101, 151, 251, 401, \dots$

Totally real fields  $k$  give  $\lambda = 0$  whatever  $S = S_\ell$ , but some non-totally real fields may give  $\lambda = 0$ ; for instance, let  $P = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 2x + 7$  defining  $k$ , of signature  $(0, 3)$ , of Galois group  $S_6$  and discriminant  $-11 \cdot 31 \cdot 971 \cdot 2801$ ; so  $\lambda = \max(0, \#S - 1 - (r_1 + r_2 - 1)) = \max(0, \#S - 3)$ . For  $p = 3$ ,  $\mathcal{C} \simeq \mathbb{Z}/3\mathbb{Z}$  capitulates in  $k_1$  for  $\ell = 37, 61, 67, 73, 97, 103, 109, \dots$  for which  $\#S_\ell \in \{1, 2, 3\}$ , giving  $\lambda = 0$  as expected. We have verified the capitulation of  $\mathcal{C}$  in  $k_1$  for  $\ell = 37$  (for this and other complements see [12]).

#### 4. ON THE EXISTENCE OF CAPITULATION FIELDS FOR REAL CLASS GROUPS

A totally real field  $k$  been given, a problem is *the existence* of cyclic  $p$ -towers  $K/k$  such that  $\mathcal{C}$ , of exponent  $p^e$ , capitulates in  $K$ . We do not intend to establish again the ‘‘abelian capitulation’’ proved in the literature (Gras [7] (1997), Kurihara [24] (1999), Bosca [1] (2009), Jaulent [18, 20] (2019, 2020)), but we will examine this possibility in a simpler way using Theorem 3.1. We may assume that  $k$  is Galois real and we shall make some heuristics using only cyclic  $p$ -towers totally  $S$ -ramified with  $S = S_\ell$ , the set of places of  $k$  above  $\ell$ ; this is an important difference compared to the previous references.

**4.1. Monogenic class groups and  $K \subset k(\mu_\ell)$ .** We assume, at first, that  $\mathcal{C}$  is monogenic, that is to say that there exists a prime ideal  $\mathfrak{q}$  such that  $\mathcal{C}$  is generated by the classes of the conjugates of  $\mathfrak{q}$ . Let  $K \subseteq k(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ ,  $N \gg 0$ , be a cyclic  $p$ -tower of degree  $p^N$  and let  $S = S_\ell$ . We assume  $\ell$  totally split in  $k$ , whence  $\#S = [k : \mathbb{Q}]$ . We have, for  $k_e/k$  of Galois group  $G_e$ , the exact sequence:  $1 \rightarrow \mathcal{C}_e^{G_e} \rightarrow E \cap N_e(k_e^\times)/N_e(E_e) \rightarrow 1$ , where the group  $\mathcal{C}_e^{G_e}$ , of invariant ideals of  $k_e$ , is of the form  $\langle S_e \rangle \cdot J_e(\mathcal{C})$ , where  $S_e$  is the set of prime ideals of  $k_e$  above  $S$  and  $J_e(\mathcal{C})$  the extension to  $k_e$  of the group of ideals of  $k$ . From Theorem 3.1, necessary conditions, to get the criterion of stability  $\#\mathcal{C}_1 = \#\mathcal{C}$ , are  $\mathcal{C}_e = \mathcal{C}_e^{G_e}$  and  $(E : E \cap N_e(k_e^\times)) = p^{e(\#S-1)} = p^{e([k:\mathbb{Q}]-1)}$ , whence  $E \cap N_e(k_e^\times) = E^{p^e} \subseteq N_e(E_e)$  giving  $\mathcal{C}_e^{G_e} = \mathcal{C}_e \cdot J_e(\mathcal{C})$ . But the condition  $\mathcal{C}_e = \mathcal{C}_e^{G_e}$  is equivalent to

$$(\mathcal{C}_e/\mathcal{C}_e^{G_e})^{G_e} = 1, \text{ whence (from (2.1)), to } \frac{\#\mathcal{C}}{\#N_e(\mathcal{C}_e^{G_e})} \times \frac{p^{e([k:\mathbb{Q}]-1)}}{(\Lambda' : \Lambda' \cap N_e(k_e^\times))} = 1, \text{ for}$$



some  $\Lambda' \supseteq E$  so that the second factor is trivial. Since  $N_e(\langle S_e \rangle) = \langle S \rangle$ ,  $\frac{\#\mathcal{C}}{\#\mathcal{N}_e(\mathcal{C}_e^{G_e})} = \frac{\#\mathcal{C}}{\#\mathcal{d}(\langle S \rangle) \cdot \#\mathcal{C}^{p^e}} = \frac{\#\mathcal{C}}{\#\mathcal{d}(\langle S \rangle)}$  if and only if  $S$  generates  $\mathcal{C}$ . Under the assumption of monogenicity, from Chebotarev density theorem, there exist infinitely many such sets and an obvious heuristic is that among these sets, infinitely many ones give the criterion  $\#\mathcal{C}_1 = \#\mathcal{C}$ , hence capitulation of  $\mathcal{C}$ . But we can go further for non-monogenic  $\mathcal{C}$ 's.

**4.2. Non-monogenic class groups and  $K \subset k(\mu_\ell)$ .** Using only single primes  $\ell$ , the condition  $\#\mathcal{C}_1 = \#\mathcal{C}$  does not hold in general in the first layer  $k_1$ . Nevertheless, many examples lead to a stabilization from some  $n_0 > 1$ , hence capitulation of  $\mathcal{C}$  from  $k'_e = k_{n_0+e}$ , so that we must have  $N > n_0 + e$ , where  $e$  is a constant, which suggests the existence of such towers. Unfortunately, any verification becomes out of reach if  $n_0$  is too large. For instance, using the program of §5.1 or the program below, for  $k = \mathbb{Q}(\sqrt{130})$ , the  $p$ -class group  $\mathcal{C} \simeq (\mathbb{Z}/2\mathbb{Z})^2$  capitulates in  $k_2$  for  $\ell = 1697, 2017, 5153, 5857, \dots$ . For  $k = \mathbb{Q}(\sqrt{2310})$ , where  $\mathcal{C} \simeq (\mathbb{Z}/2\mathbb{Z})^3$ , we have the following results:

e11=593	e11=1217	e11=4289
v0=3 C0=[2,2,2]	v0=3 C0=[2,2,2]	v0=3 C0=[2,2,2]
v1=6 C1=[2,4,8]	v1=6 C1=[2,4,8]	v1=7 C1=[2,2,2,16]
v2=7 C2=[2,4,16]	v2=7 C2=[2,4,16]	v2=9 C2=[2,2,4,32]
v3=7 C3=[2,4,16]	v3=7 C3=[2,4,16]	v3=10 C3=[2,2,4,64]

For  $\ell = 593$  (resp.  $\ell = 1217$ ) the stability holds from  $n_0 = 2$  with  $N = 3$  (resp.  $N = 5$ ); thus  $\mathcal{C}_2[2]$  capitulates in  $k'_1 = k_3$ , whence  $\mathcal{C}[2] = \mathcal{C}$  capitulates in  $k_3$ . For  $\ell = 4289$ , a stabilization seems possible from  $n_0 = 3$ , but  $\mathcal{C}_4$  is not computable.

Concerning capitulation of  $\mathcal{C}$  in towers  $K \subset k(\mu_\ell)$ , we have the following experiments, showing that it can happen even if no stabilization is obtained; the notation  $\mathbf{Cn} = [A, \dots, Z]$  means  $\mathcal{C}_n \simeq \mathbb{Z}/AZ \times \dots \times \mathbb{Z}/ZZ$  in  $k_n$  and a box  $[a, \dots, z]$  means that the generators of  $\mathcal{C}$  extended in  $k_n$  are, respectively, the  $a$ th,  $\dots$ ,  $z$ th powers of suitable generators of  $\mathcal{C}_n$  computed by PARI/GP ( $a, \dots, z$ , being integers modulo  $A, \dots, Z$ , respectively); so that  $[0, \dots, 0]$  is equivalent to the capitulation of  $\mathcal{C}$ . For the program, one must precise  $\mathbf{p}$  the minimal  $p$ -rank of  $\mathcal{C}$  required  $\mathbf{rmin}$ , the length  $\mathbf{N}$  of the tower and the interval for  $\mathbf{m}$ :

```
{p=2;rmin=3;N=3;bm=2;Bm=10^6;for(m=bm,Bm,if(core(m)!=m,next);
P=x^2-m;k=bnfinit(P,1);Ck=k.clgp;r=matsize(Ck[2])[2];\Ck=class group of k
L=List;for(i=1,r,listput(L,0,i));rp=0;for(i=1,r,ei=Ck[2][i];v=valuation(ei,p);
if(v>0,rp=rp+1));if(rp<rmin,next);\computation and test of the p-rank rp
h=k.no;u=h/p^valuation(h,p);Ckp=List;for(i=1,r,ai=idealpov(k,Ck[3][i],u);
listput(Ckp,ai,i));\representatives ai of the p-class group Ckp
forprime(e11=5,200,if(Mod(e11-1,2*p^N)!=0||Mod(m,e11)==0,next);Lq=List;
\the program computes prime representatives qi, inert in K/k, split in k:
for(i=1,r,ai=Ckp[i];forprime(q=2,10^4,if(q==e11 || kronecker(m,q)!=1,next);
o=znorder(Mod(q,e11));if(valuation(o,p)!=valuation(e11-1,p),next);\inertia
f=idealfactor(k,q);qi=component(f,1)[1];cij=ai;for(j=1,p-1,cij=idealmul(k,cij,qi);
if(List(bnfisprincipal(k,cij)[1])=L,listput(Lq,q,i);break(2)))));
print("m=",m," Lq=",Lq);\Lq = list of primes qi generating the p-class group
for(n=0,N,R=polcompositum(P,polsubcyclo(e11,p^n))[1];K=bnfinit(R,1);H=K.no;
U=H/p^valuation(H,p);print();print("e11=",e11," n=",n," CK",n,"=",K.cyc);
for(i=1,r,Fi=idealfactor(K,Lq[i]);Qi=idealpov(K,component(Fi,1)[1],U);
print(bnfisprincipal(K,Qi)[1])))}
p = 2
```

m=1155 e11=193					
C0=[2,2,2]	C1=[2,2,2,2,2]	C2=[4,4,2,2,2,2,2]	C3=[8,8,4,2,2,2,2]		
[1,0,0]	[1,0,1,0,0]	[0,0,0,0,0,0,0]	[0,0,0,0,0,0,0]		
[0,1,0]	[1,1,1,0,0]	[2,2,1,1,1,0,0]	[0,0,0,0,0,0,0]		
[0,0,1]	[0,1,0,0,0]	[2,2,1,1,1,0,0]	[0,0,0,0,0,0,0]		
m=1155 e11=337					
C0=[2,2,2]	C1=[2,2,2,2,2]	C2=[20,4,4,2,2]	C3=[40,4,4,2,2]		
[1,0,0]	[0,1,1,1,1]	[0,0,0,0,0]	[0,0,0,0,0]		
[0,1,0]	[1,0,1,1,0]	[10,2,0,0,0]	[0,0,0,0,0]		
[0,0,1]	[1,1,0,0,1]	[10,2,0,0,0]	[0,0,0,0,0]		
m=1995 e11=17					
C0=[2,2,2]	C1=[4,2,2,2]	C2=[8,4,2,2]	C3=[8,4,2,2]		
[1,0,0]	[2,1,0,0]	[0,0,0,0]	[0,0,0,0]		
[0,1,0]	[2,1,0,0]	[0,0,0,0]	[0,0,0,0]		
[0,0,1]	[0,1,0,0]	[4,0,0,0]	[0,0,0,0]		
m=1995 e11=113					
C0=[2,2,2]	C1=[12,2,2,2]	C2=[12,2,2,2,2,2]	C3=[24,4,2,2,2,2]		
[1,0,0]	[6,1,1,1]	[0,0,1,1,1,0]	[0,0,0,0,0,0]		
[0,1,0]	[6,0,0,0]	[0,0,0,0,0,0]	[0,0,0,0,0,0]		
[0,0,1]	[0,0,0,0]	[0,0,0,0,0,0]	[0,0,0,0,0,0]		
m=1995 e11=193					
C0=[2,2,2]	C1=[2,2,2,2,2]	C2=[12,6,2,2,2,2,2]	C3=[24,12,4,4,2,2,2]		
[1,0,0]	[1,0,1,1,0]	[6,3,1,0,0,0,0]	[0,0,0,0,0,0,0]		
[0,1,0]	[0,1,1,1,0]	[6,3,1,0,0,0,0]	[0,0,0,0,0,0,0]		
[0,0,1]	[0,0,0,0,0]	[0,0,0,0,0,0,0]	[0,0,0,0,0,0,0]		
m=2210 e11=97					
C0=[2,2,2]	C1=[12,4,2]	C2=[24,4,2]	C3=[24,4,2]		
[1,0,0]	[0,0,0]	[0,0,0]	[0,0,0]		
[0,1,0]	[6,0,0]	[12,0,0]	[0,0,0]		
[0,0,1]	[6,2,0]	[12,0,0]	[0,0,0]		
m=2210 e11=113					
C0=[2,2,2]	C1=[12,2,2,2]	C2=[24,4,2,2,2]	C3=[48,12,12,4,2]		
[1,0,0]	[0,1,1,0]	[0,2,0,0,0]	[0,0,0,0,0]		
[0,1,0]	[6,0,0,0]	[12,0,0,0,0]	[24,0,0,0,0]		
[0,0,1]	[6,1,1,0]	[12,2,0,0,0]	[24,0,0,0,0]		
$p = 3$					
m=23659 e11=19		m=23659 e11=37		m=32009 e11=19	
C0=[6,3]	C1=[18,3]	C2=[18,3]	C0=[6,3]	C1=[18,3,3]	C2=[18,3,3]
[0,1]	[0,0]	[0,0]	[0,2]	[12,0,0]	[0,0,0]
[2,0]	[6,0]	[0,0]	[2,0]	[6,0,0]	[0,0,0]
m=32009 e11=37		m=42817 e11=19			
C0=[3,3]	C1=[9,3]	C2=[9,3]	C0=[3,3]	C1=[9,3]	C2=[27,3]
[0,1]	[6,0]	[0,0]	[2,1]	[6,0]	[9,0]
[1,2]	[3,0]	[0,0]	[1,0]	[3,0]	[18,0]

In the non-monogenic case, we observe many capitulations (or partial ones), in cyclic  $p$ -tower contained in  $k(\mu_\ell)$  (possibly, the capitulation holds in larger layers but this would require several days of computer); so we propose the following conjecture:

**Conjecture 4.1.** *Let  $k$  be a totally real number field with generalized  $p$ -class group  $\mathcal{X}_{k,m} = \mathcal{C}_{k,m}/\mathcal{H}_k$  (cf. (1.1)). There are infinitely many primes  $\ell$ ,  $\ell \equiv 1 \pmod{2p^N}$ ,  $N \gg 0$ , such that  $\mathcal{X}_{k,m}$  capitulates in  $k(\mu_\ell)$ .*

Examples of non-totally real fields (case of the degree 6 example and non-Galois cubic fields, as illustrated in [12]) show that the property can be enlarged to some non-totally real fields, with some conditions on the infinite places since it is obvious

that capitulation in  $k(\mu_\ell)$  does not hold for an imaginary quadratic field. Since the structures of these  $p$ -class groups depend on norm properties, the conjecture can probably be also completed by densities results thanks to the techniques used in the classical approaches [1, 7, 20, 24] and especially those of [22, 31].

## 5. EXAMPLES AND COUNTEREXAMPLES, WITH MODULUS, OF $\lambda$ -STABILITIES

5.1. **Cyclic  $p$ -towers over  $k = \mathbb{Q}(\sqrt{\pm m})$ .** Consider, for  $\ell \equiv 1 \pmod{p^N}$  (in `ell`), the cyclic  $p$ -tower contained in  $k(\mu_\ell)/k$ , for  $k = \mathbb{Q}(\sqrt{sm})$ ,  $m > 0$  square free (in `m`),  $s = \pm 1$ . The following PARI/GP program computes  $\mathcal{C}_{k_n, m} =: \mathcal{C}_n$  for any prime-to- $\ell$  modulus `m` (in `mod`). The polynomial  $P_n$  (in `Pn`) defines the layer  $k_n$ . If  $s = -1$  (resp.  $s = 1$ ),  $\lambda = \#S - 1 - \rho \in \{0, 1\}$  (resp.  $\lambda = 0$ ) in  $k$  but decreases in the tower (Remark 1.4). Then `vn` denotes the  $p$ -valuation of the order of  $\mathcal{C}_n$  whose structure `Cn` is given by the list `C`.

The computations up to  $N = 3$  or  $4$  ( $n \in [0, N]$ ) are only for verification when a  $\lambda$ -stability exists; otherwise some examples do not  $\lambda$ -stabilize in the interval considered and no conclusion is possible. For the program, one must precise `p`, `ell`, the modulus `mod`,  $s = \pm 1$  defining real or imaginary quadratic fields  $k$ , the length `N` of the tower and the interval for `m`:

```
{p=2;ell=257;mod=5;s=-1;N=3;bm=2;Bm=10^3;
for(m=bm,Bm,if(core(m)!=m || Mod(m,ell)==0,next);
P=x^2-s*m;lambda=0;if(s=-1&kroncker(s*m,ell)==1,lambda=1);\\lambda
print();print("p=",p," mod=",mod," ell=",ell," sm=",s*m," lambda=",lambda);
for(n=0,N,Pn=polcompositum(polsubcyclo(ell,p^n),P)[1];kn=bnfinit(Pn,1);\\layer kn
knmod=bnrinit(kn,mod);v=valuation(knmod.no,p);Cn=knmod.cyc;\\ray class group of kn
C>List;d=matsize(Cn)[2];for(j=1,d,c=Cn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(C,p^w,1));\\end of computation of the p-ray class group of kn
print("v",n,"=",v," p-ray class group=",C))}

IMAGINARY QUADRATIC FIELDS, p=2, ell=257, mod=1:
m=-2,lambda=1      m=-11,lambda=1      m=-14,lambda=0      m=-17,lambda=1      m=-15,lambda=1
v0=0 []            v0=0 []            v0=2 [4]            v0=2 [4]            v0=1 [2]
v1=3 [8]           v1=2 [4]           v1=4 [4,4]          v1=5 [8,2,2]        v1=2 [4]
v2=7 [16,4,2]      v2=7 [8,8,2]       v2=8 [4,4,4,4]      v2=10[16,2,2,2,2,2,2] v2=3 [8]
v3=13 [32,8,2,2]   v3=10 [16,16,4]    v3=16 [8,8,8,4,4]   v3=19[32,2,2,2,2,2,2,2] v3=4 [16]
                2,2,2,2]                4,2,2,2]            2,2,2,2,2,2,2,2,2]

m=-782,lambda=1    m=-858,lambda=0
v0=3 [4,2]         v0=4 [4,2,2]
v1=8 [4,4,4,2,2]  v1=12 [128,2,2,2,2,2]
v2=22[32,32,4,4,4,2,2,2,2,2,2] v2=22 [256,16,8,2,2,2,2,2,2,2]
v3=35[64,64,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2] v3=32 [512,32,16,4,4,4,2,2,2,2,2,2,2,2,2]

m=-15,mod=5,lambda=1 m=-35,mod=5,lambda=1 m=-253,mod=5,lambda=1
v0=2 [4]            v0=2 [4]            v0=4 [4,2,2]
v1=3 [4,2]          v1=3 [4,2]          v1=12 [64,4,2,2,2,2]
v2=4 [8,2]          v2=4 [8,2]          v2=22 [128,8,4,4,2,2,2,2,2,2,2,2]
                v3=34 [256,16,8,4,4,4,4,4,2,2,2,2,2,2,2,2]

IMAGINARY QUADRATIC FIELDS, p=3, ell=163, mod=1:
m=-2,lambda=1      m=-293,lambda=1      m=-983,lambda=1      m=-3671,lambda=0
v0=0 []            v0=2 [9]             v0=3 [27]            v0=4 [81]
v1=1 [3]           v1=3 [27]            v1=4 [81]            v1=7 [243,3,3]
v2=2 [9]           v2=4 [81]            v2=5 [243]           v2=11 [729,9,9,3]

REAL QUADRATIC FIELDS, p=2, lambda=0:
mod=7,ell=12289:
m=2                m=19                 m=89                 m=15                 m=39
v0=0 []            v0=1 [2]             v0=0 []              v0=2 [2,2]          v0=2 [2,2]
```

```

v1=1 [2]      v1=2 [2,2]      v1=0 []      v1=5 [2,2,2,2,2]      v1=5 [2,2,2,2,2]
v2=1 [2]      v2=2 [2,2]      v2=0 []      v2=9 [8,4,4,2,2]      v2=10 [8,4,2,2,2,2,2]
mod=17,e11=7340033
m=17          m=19          m=21          m=38          m=26          m=226
v0=2 [4]      v0=6 [16,4]      v0=4 [16]     v0=5 [16,2]   v0=6 [16,2,2] v0=5 [8,8]
v1=2 [4]      v1=6 [16,4]      v1=4 [16]     v1=5 [16,2]   v1=8 [16,4,2,2] v1=10 [32,8,8]
REAL QUADRATIC FIELDS, p=3, lambda=0:
mod=1,e11=109      mod=1,e11=109      mod=7,e11=163      mod=7,e11=163
m=326              m=4409              m=5                 m=6
v0=1 [3]           v0=2 [9]           v0=1 [3]           v0=1 [3]
v1=1 [3]           v1=2 [9]           v1=1 [3]           v1=2 [3,3]
v2=1 [3]           v2=2 [9]           v2=1 [3]           v2=2 [3,3]
mod=7,e11=163      mod=7,e11=163      mod=19,e11=163     mod=19,e11=163
m=22              m=15                m=2                 m=11
v0=2 [3,3]         v0=1 [3]           v0=2 [9]           v0=3 [9,3]
v1=2 [3,3]         v1=3 [3,3,3]       v1=3 [27]          v1=4 [27,3]
v2=2 [3,3]         v2=5 [9,9,3]       v2=3 [27]          v2=4 [27,3]

```

One finds examples of  $\lambda$ -stabilities taking for instance  $\#S = 6$  with three primes split in  $k = \mathbb{Q}(\sqrt{\pm m})$  and any of the 1024 totally  $S$ -ramified cyclic 2-towers  $K/k$  of degree  $2^6$  contained in  $k(\mu_{257 \cdot 449 \cdot 577})$  but, unfortunately, only computing in  $k$  and  $k_1 = k(\sqrt{257 \cdot 449 \cdot 577})$  with  $\lambda = 4$  (resp. 5) for  $k$  real (resp. imaginary):

```

m=193          m=1591          m=2669          m=-527          m=-1739
v0=0 []        v0=1 [2]        v0=2 [4]        v0=1 [2]        v0=2 [4]
v1=4 [2,2,2,2] v1=5 [2,2,2,2,2] v1=6 [4,2,2,2,2] v1=6 [4,2,2,2,2] v1=7 [8,2,2,2,2]

```

**5.2. Cyclic  $p$ -towers over  $\mathbb{Q}$  with modulus.** In a cyclic  $p$ -tower  $K/\mathbb{Q}$  for  $S = S_\ell$ ,  $m = 1$ , (2.1) shows that  $\lambda = 0$  and  $\#\mathcal{C}_n = 1$  for all  $n$ . For  $m \neq 1$ , a stability may occur (possibly from  $n_0 > 0$ ) as shown by the following examples ( $p = 2, 3$ ) analogous to [28, Example 4.1]:

```

{p=2;e11=257;N=5;print("p=",p," e11=",e11," mod=",mod);
for(n=0,N,Pn=polsubcyclo(e11,p^n);kn=bnfinit(Pn,1);\\layer kn
knmod=bnrinit(kn,mod);v=valuation(knmod.no,p);Cn=knmod.cyc;\\ray class group
C=List;d=matsize(Cn)[2];for(j=1,d,c=Cn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(C,p^w,1)));print("v",n,"=",v," C",n,"=",C)}
p=2,e11=257,mod=17      e11=257,mod=4*17      p=3,e11=163,mod=19      e11=163,mod=9*19
v0=3 [8]               v0=4 [16]            v0=2 [9]               v0=3 [9,3]
v1=4 [16]              v1=5 [16,2]         v1=3 [27]              v1=5 [27,3,3]
v2=4 [16]              v2=7 [16,2,2,2]     v2=3 [27]              v2=6 [81,3,3]
v3=4 [16]              v3=11 [16,2,2,2,2,2,2] v3=3 [27]              v3=6 [81,3,3]
v4=4 [16]              v4=12 [32,2,2,2,2,2,2]
v5=4 [16]              v5=12 [32,2,2,2,2,2,2]

```

**5.3. Cyclic  $p$ -towers over a cyclic cubic field  $k$ .** Let  $K \subset k(\mu_\ell)$  defining a cyclic  $p$ -tower of  $k$ . The program gives the complete list of cyclic cubic fields of conductor  $f \in [\text{bf}, \text{Bf}]$ . An interesting fact is that, in most cases, the stability occurs (with  $\lambda = 0$  in the case  $S = S_\ell$ ); we give an excerpt with the defining polynomial  $P$  of  $k$  of conductor  $f$ :

```

{p=2;e11=257;N=3;bf=7;Bf=10^3;for(f=bf,Bf,if(Mod(f,e11)==0,next);
h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;if(core(F)!=F,next);F=factor(F);
Div=component(F,1);d=matsize(F)[1];for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,\\computation of a and b such that f=(a^2+27b^2)/4
if(h==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);print();print("f=",f," P=",P);

```

```

\\end of computation of P defining k of conductor f
for(n=0,N,Pn=polcompositum(polsubcyclo(e11,p^n),P)[1];kn=bnfinit(Pn,1);
v=valuation(kn.no,p);Cn=kn.cyc;C=List;d=matsize(Cn)[2];for(j=1,d,c=Cn[d-j+1];
w=valuation(c,p);if(w>0,listinsert(C,p^w,1)));print("v",n,"=",v," C",n,"=",C))))}

p=2,e11=257,lambda=0
f=63          f=163          f=277          f=279
P=x^3-21*x-35  P=x^3+x^2-54*x-169  P=x^3+x^2-92*x+236  P=x^3-93*x+217
v0=0 []        v0=2 [2,2]       v0=2 [2,2]       v0=0 []
v1=2 [2,2]     v1=4 [4,4]       v1=4 [2,2,2,2]   v1=2 [2,2]
v2=2 [2,2]     v2=4 [4,4]       v2=6 [2,2,2,2,2] v2=4 [4,4]
v3=2 [2,2]     v3=4 [4,4]       v3=6 [2,2,2,2,2] ?

f=333          f=349          f=397          f=547
P=x^3-111*x+37 P=x^3+x^2-116*x-517 P=x^3+x^2-132*x-544 P=x^3+x^2-182*x-81
v0=0 []        v0=2 [2,2]       v0=2 [2,2]       v0=2 [2,2]
v1=4 [4,4]     v1=4 [2,2,2,2]  v1=2 [2,2]       v1=4 [2,2,2,2]
v2=4 [4,4]     v2=4 [2,2,2,2]  v2=2 [2,2]       v2=6 [4,4,2,2]

p=3,e11=109,lambda=0
f=1687          f=1897          f=2359          f=5383
x^3+x^2-562*x-4936 x^3+x^2-632*x-4075 x^3+x^2-786*x+7776 x^3+x^2-1794*x+17744
v0=1 [3]        v0=1 [3]        v0=2 [3,3]       v0=3 [9,3]
v1=1 [3]        v1=1 [3]        v1=2 [3,3]       v1=3 [9,3]

```

As for the quadratic case, we may use  $k_1 = k(\sqrt{257 \cdot 449 \cdot 577})$  for  $p = 2$  with  $\#S = 9$ ,  $\rho = 2$ ,  $\lambda = 6$ . The contribution of the 2-tower over  $\mathbb{Q}$  is given by the 2-stability from the structures  $[], [2, 2]$ . We find a great lot of  $\lambda$ -stabilities with the structures  $[], [2, 2, 2, 2, 2, 2]$  or  $[2, 2], [4, 4, 2, 2, 2, 2]$  and very few exceptions for which one does not know if a  $\lambda$ -stabilization does exist for  $n > 1$  (case of  $f = 1531$  below):

```

f=171,P=x^3-57*x-152  f=349,P=x^3+x^2-116*x-517  f=1531,P=x^3+x^2-510*x-567
C0=[]                  C0=[2,2]                     C0=[]
C1=[2,2,2,2,2,2,2]   C1=[4,4,2,2,2,2,2]          C1=[4,4,2,2,2,2,2]

```

## 6. OTHER EXAMPLES OF APPLICATIONS AND REMARKS

Assume the total ramification of  $S \neq \emptyset$  in the cyclic  $p$ -tower  $K/k$  and the main condition  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$  with  $\lambda = \max(0, \#S - 1 - \rho)$  (Definition 1.1); remember that it is easy to ensure that  $\text{tor}_{\mathbb{Z}}(E) \subset N_{K/k}(K^\times)$  (Remark 1.3):

(a) Let  $\tilde{k}/k$  be a  $\mathbb{Z}_p$ -extension with Iwasawa's invariants  $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$ . We then have a set of  $p$ -places  $S$  with  $1 \leq \#S \leq [k : \mathbb{Q}] = r_1 + 2r_2$  and  $\lambda = \max(0, \#S - r_1 - r_2) \in [0, r_2]$ ; one gets the formula  $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda \cdot n}$  for all  $n$ , as soon as  $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$  holds, and in that case,  $\tilde{\lambda}$  is equal to  $\lambda$  and  $\tilde{\mu} = 0$ ; the Iwasawa formula being  $\#\mathcal{X}_n = p^{\lambda \cdot n + \nu}$  with  $p^\nu := \#\mathcal{X}$  (recall that this may apply only from a larger layer  $k'$ ). For CM-fields  $k$  with  $p$  totally split in  $k$ ,  $\lambda = \frac{1}{2}[k : \mathbb{Q}]$ .

When  $k$  is totally real, Greenberg's conjecture [13] for  $p$ -class groups ( $\tilde{\lambda} = \tilde{\mu} = 0$ ) holds if and only if  $\#\mathcal{C}_{n_0+1} = \#\mathcal{C}_{n_0}$  is fulfilled for some  $n_0 \geq 0$  ( $\lambda'$ -stabilization from  $k' = k_{n_0}$  with  $\lambda' = \lambda = 0$ ), which may be checked if  $n_0$  is not too large by chance (as in [23] for real quadratic fields,  $p = 3$  non-split, or in various works of Taya [32], Fukuda, Komatsu).

(b) Taking  $\mathcal{H}_N = \mathcal{C}_N^{p^r}$  may give the  $\lambda(r)$ -stability of the  $\mathcal{C}_n/\mathcal{C}_n^{p^r}$ 's, for some  $\lambda(r)$ , whence  $\text{rk}_p(\mathcal{C}_n) = \text{rk}_p(\mathcal{C}) + \lambda(1) \cdot n$  for all  $n$ , as soon as this equality holds for  $n = 1$  (where  $\text{rk}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$ ). For instance, let  $k = \mathbb{Q}(\sqrt{-m})$  and

$K \subset k(\mu_{257})$  and consider the stability of the 2-ranks when  $\text{rk}_2(\mathcal{C}) \geq 1$  ( $\lambda(1) = 0$  since  $\#S \leq 2$  and  $\rho = t \geq 1$  when  $\mathcal{C} \neq 1$ ); we find many examples:

For  $m \in \{-15, -35, -95, -111, -123, -159, -215, -235, -259, -287, -291, -303, -327, -355, -371, -415, -447, \dots\}$  the 2-rank is 1 and stable from  $n = 0$ .

For  $m \in \{-39, -87, -91, -155, -183, -195, -203, -247, -267, -295, -339, -395, -399, -403, -427, -435, -455, -471, \dots\}$  the 2-rank (equal to 2 or 3) is stable from  $n = 1$ .

For  $m = -55$  (successive structures [4], [4, 4], [8, 4, 4, 2], [16, 8, 8, 4]),  $m = -115$  (structures [2], [2, 2], [2, 2, 2, 2], [8, 8, 8, 8]), we have two examples of stabilization from  $n = 2$  (rank 4).

We see that the orders are not necessarily  $\lambda$ -stable from  $n = 0$  ( $\lambda = \#S - 1$ ):

m=-15	m=-403	m=-259	m=-95	m=-195	m=-895
v0=1 [2]	v0=1 [2]	v0=2 [4]	v0=3 [8]	v0=2 [2, 2]	v0=4 [16]
v1=2 [4]	v1=5 [8, 2, 2]	v1=3 [8]	v1=4 [16]	v1=4 [4, 2, 2]	v1=5 [32]
v2=3 [8]	v2=8 [16, 4, 4]	v2=4 [16]	v2=5 [32]	v2=9 [8, 8, 8]	v2=6 [64]
v3=4 [16]	v3=11 [32, 8, 8]	v3=5 [32]	v3=6 [64]	v3=12 [16, 16, 16]	v3=7 [128]

If  $k \subseteq K \subseteq \tilde{k}$ , for a  $\mathbb{Z}_p$ -extension  $\tilde{k}$  of  $k$ , and if  $\lambda(1) = 0$  (e.g.,  $k$  is any quadratic field and  $\mathcal{C} \neq 1$ ) then  $\text{rk}_p(\mathcal{C}_n) = \text{rk}_p(\mathcal{C})$  as soon as this is true for  $n = 1$  (this may hold from some layer). See [12, Section 7] for many examples. The following ones (among others) show a great tendency to stabilization of the rank:

p=2 m=-93 [2, 2], [2, 2, 2], [2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 4, 4, 4, 4, 4, 4, 8, 8]  
p=3 m=6559 [9], [3, 27], [9, 27]; m=-362 [9], [3, 27], [9, 81]; m=-929 [9], [3, 9, 27], [9, 27, 81]

(c) If  $K/\mathbb{Q}$  is abelian with  $p \nmid [k : \mathbb{Q}]$ ,  $\lambda$ -stabilities may hold for the isotopic  $\mathbb{Z}_p[\text{Gal}(k/\mathbb{Q})]$ -components of the  $\mathcal{X}_n$ 's, using fixed points formulas [16, 17].

(d) Remarks. Let's consider, for an arbitrary  $k$ , the family  $\{\mathcal{C}_n\}$  assuming the context of Lemma 1.2 and Remark 1.3; so  $\lambda = \max(0, \#S - 1 - \rho)$  with  $\rho = r_1 + r_2 - 1$ .

(i) The condition of  $\lambda$ -stability,  $\#\mathcal{C}_1 = \#\mathcal{C} \cdot p^\lambda$ , implies  $\mathcal{C}_n^{G_n} = \mathcal{C}_n$  for all  $n \geq 1$ , but is not equivalent. Indeed, let  $k = \mathbb{Q}(\sqrt{m})$ ,  $m > 0$  such that 17 splits in  $k$ , and let  $K = kL$ , where  $L$  is the subfield of  $\mathbb{Q}(\mu_{17})$  of degree 8; whence  $\rho = 1$ ,  $\lambda = 0$ . Assume that  $\omega_1(\varepsilon) = 1$  ( $\varepsilon$  norm in  $k_1/k$ ); then, from Chevalley's formula,  $\#\mathcal{C}_1^{G_1} = \#\mathcal{C} \cdot p \neq \#\mathcal{C}$ . For  $m = 26$ ,  $\varepsilon = 3 + \sqrt{26} \equiv 5^2 \pmod{17}$  in the completion  $\mathbb{Q}_{17}$ ; one obtains the structures [2], [2, 2], [4, 2] giving  $\#\mathcal{C} = 2$  and  $\#\mathcal{C}_1^{G_1} = 4$ .

(ii) When  $\#\mathcal{C}_1 = \#\mathcal{C} \cdot p^\lambda$  does not hold (whence  $\#\mathcal{C}_1 > \#\mathcal{C} \cdot p^\lambda$ ), we observe that orders and  $p$ -ranks are unpredictable and possibly unlimited in the tower; indeed, put  $\mathcal{C}_n^i := \{c \in \mathcal{C}_n, c^{(1-\sigma_n)^i} = 1\} =: \mathcal{C}_n(\mathcal{J}_n^i)$ ,  $i \geq 1$ ; we have, from (2.1),  $\#(\mathcal{C}_n^{i+1}/\mathcal{C}_n^i) = \frac{\#\mathcal{C}_n}{\#\mathbb{N}_n(\mathcal{C}_n^i)} \times \frac{p^n(\#\mathcal{S}-1)}{\#\omega_n(\Lambda^i)}$ , where  $\Lambda^i := \{x \in k^\times, (x) \in \mathbb{N}_n(\mathcal{J}_n^i)\}$ ; in general these two factors are non-trivial giving  $\#\mathcal{C}_n^{i+1} > \#\mathcal{C}_n^i$  and the filtration increases.

## 7. BEHAVIOR OF OTHER INVARIANTS IN CYCLIC $p$ -TOWERS

Let  $\Sigma_p$  be the set of  $p$ -places of  $k$ . In relation with class field theory and cohomology of the Galois group  $\mathcal{G}_{k, \Sigma_p}$  of the maximal  $\Sigma_p$ -ramified (=  $p$ -ramified) pro- $p$ -extension of  $k$  and of its local analogues  $\mathcal{G}_{k_p}$ , the fundamental Tate–Chafarevich groups [21, Theorem 3.74], [29, § 1]:

$$\text{III}_k^i(\Sigma_p) := \text{Ker}[\text{H}^i(\mathcal{G}_{k, \Sigma_p}, \mathbb{F}_p) \rightarrow \bigoplus_{p \in \Sigma_p} \text{H}^i(\mathcal{G}_{k_p}, \mathbb{F}_p)], \quad i = 1, 2,$$

can be examined, in the same way in towers, as for generalized class groups. This is possible for  $\text{III}_k^1(\Sigma_p) \simeq (\mathcal{C}_k/\mathcal{C}_k(\Sigma_p)) \otimes \mathbb{F}_p$ , but  $\text{III}_k^2(\Sigma_p)$  does not appear immediately as a generalized class group and is not of easy computational access (see [15] for

a thorough study of  $\text{III}_k^2(S)$  when  $\Sigma_p \not\subseteq S$ ; the case  $\Sigma_p \subseteq S$  easily reduces to  $S = \Sigma_p$  under Leopoldt's conjecture [8, Theorem III.4.1.5]). Nevertheless, we have an arithmetic interpretation, much more easily computable and having well-known properties, with the key invariant, closely related to  $\text{III}_k^2(\Sigma_p)$ ,

$$\mathcal{T}_k := \text{tor}_{\mathbb{Z}_p}(\mathcal{G}_{k, \Sigma_p}^{\text{ab}}) \simeq \text{H}^2(\mathcal{G}_{k, \Sigma_p}, \mathbb{Z}_p)^* \quad [29, \text{Théorème 1.1}]$$

(see general rank computations in [8, Theorem III.4.2 and Corollaries]). Nevertheless,  $\mathcal{T}_k$ , linked to the residue at  $s = 1$  of  $p$ -adic  $\zeta$ -functions [3], gives rise to some analogies with generalized class groups (due to reflection theorems [8, Theorem II.5.4.5]). See in [10] the interpretation of  $\mathcal{T}_k$  with the  $p$ -class group  $\mathcal{C}_k$  and the normalized  $p$ -adic regulator  $\mathcal{R}_k$ . Since the deep aspects of these cohomology groups are due to the  $p$ -adic properties of global units (i.e.,  $\mathcal{R}_k$ ) we shall restrict ourselves to the totally real case under Leopoldt's conjecture. For any field  $\kappa$ , let  $\kappa_\infty$  be its cyclotomic  $\mathbb{Z}_p$ -extension; then  $\mathcal{T}_\kappa = \text{Gal}((H_\kappa^{\text{pr}})^{\text{ab}}/\kappa_\infty)$  where  $H_\kappa^{\text{pr}}$  is the maximal abelian  $\Sigma_p$ -ramified pro- $p$ -extension of  $\kappa$ .

Let  $K/k$  be a totally real,  $S$ -ramified, cyclic  $p$ -tower; denote by  $S^{\text{ta}} \subseteq S$  the subset of tame places (when non-empty,  $S^{\text{ta}}$  is assumed totally ramified, but the subset of  $p$ -places may be arbitrary). For all extensions  $L/F$  contained in  $K/k$ , the transfer maps  $J_{L/F} : \mathcal{T}_F \rightarrow \mathcal{T}_L$  are injective [8, Theorem IV.2.1] and, since  $L_\infty/F_\infty$  is totally  $S^{\text{ta}}$ -ramified, the norm maps  $N_{L/F} : \mathcal{T}_L \rightarrow \mathcal{T}_F$  are surjective. In the particular case  $L/F \subset k_\infty/k$ ,  $S^{\text{ta}} = \emptyset$ , but  $N_{L/F}$  is still surjective (indeed,  $F_\infty = L_\infty = k_\infty$  and  $(H_k^{\text{pr}})^{\text{ab}} \subseteq (H_F^{\text{pr}})^{\text{ab}} \subseteq (H_L^{\text{pr}})^{\text{ab}}$ ).

**Lemma 7.1.** *Let  $K/k$  be a totally real cyclic  $p$ -tower of degree  $qp^N$ ,  $p$ -ramified or else totally ramified at  $S^{\text{ta}} \neq \emptyset$ , where  $q = p$  or  $4$ ; assume that  $k \cap \mathbb{Q}_\infty = \mathbb{Q}$ . Then, for any  $0 \leq n \leq n+m \leq N$ , and  $g_n^m := \text{Gal}(k_{n+m}/k_n)$ ,  $\#\mathcal{T}_{n+m}^{g_n^m} = \#\mathcal{T}_n \cdot p^{m \cdot \#S^{\text{ta}}}$ .*

*Proof.* We have (with our notations)  $\#\mathcal{T}_{n+m}^{g_n^m} = \#\mathcal{T}_n \cdot p^{m \cdot \#S^{\text{ta}}} \cdot p^{R-m}$  [8, Theorem IV.3.3, Exercise 3.3.1],  $R = \min_{l \in S^{\text{ta}}}(m; \dots, \nu_l + \varphi_l + \gamma_l, \dots)$  (with  $p^{\nu_l} \sim q^{-1} \log(\ell)$  where  $l \cap \mathbb{Z} = \ell \mathbb{Z}$ ,  $p^{\varphi_l} \sim$  residue degree of  $l$  in  $k/\mathbb{Q}$ ,  $\gamma_l = 0$  since  $S^{\text{ta}}$  does not split in  $K/k$ ). When  $S^{\text{ta}} \neq \emptyset$ , the existence (see [8, V, Theorem 2.9] for a characterization) of a cyclic  $p$ -tower  $K/k$  implies  $\mathcal{N}(l) \equiv 1 \pmod{qp^N}$ , for all  $l \in S^{\text{ta}}$ , where  $\mathcal{N}$  is the absolute norm; indeed, the inertia group of the local extension  $K_{\mathfrak{L}}/k_l$ ,  $\mathfrak{L} | l$ , is isomorphic to a subgroup of  $\mathbb{F}_{\mathcal{N}(l)}^\times$ . So, we have  $\nu_l + \varphi_l \geq N$  and  $R = m$ ; this explains the limitation of the level  $n$  to get the given fixed points formula in any case. If  $k \cap \mathbb{Q}_\infty = \mathbb{Q}_{n_0}$ , one must take  $\ell \equiv 1 \pmod{qp^{n_0} \cdot p^N}$   $\square$

If a  $\lambda$ -stability does exist with  $\mathcal{T}_n = \mathcal{T}_n^{G_n}$  of order  $\#\mathcal{T} \cdot p^{\lambda \cdot n}$ , then  $\lambda = \#S^{\text{ta}}$  (from the relation  $\#\mathcal{T}_1 = \#\mathcal{T} p^{\#S^{\text{ta}}}$  giving  $\mathcal{T}_n^{G_n} = \mathcal{T}_n^{G_n^p}$ ). However, the properties of the groups  $\mathcal{T}_n$  will give sometimes non-trivial Iwasawa's  $\tilde{\mu}$ -invariants in  $k_\infty/k$  and very large ranks in cyclic  $p$ -towers with tame ramification; moreover  $\tilde{\lambda}$  may be much larger than  $\lambda = \#S^{\text{ta}}$  which explains that the  $\lambda$ -stability problem is less pertinent in this framework (few examples). A reason (based on the properties, in  $k_{n+m}/k_n$ , of the norms  $N_n^m$  (surjectivity) and transfer maps  $J_n^m$  (injectivity)) is the following, where  $g_n^m := \text{Gal}(k_{n+m}/k_n)$ :

From  $1 \rightarrow J_n^m \mathcal{T}_n \rightarrow \mathcal{T}_{n+m} \rightarrow \mathcal{T}_{n+m}/J_n^m \mathcal{T}_n \rightarrow 1$ , we get:

$$1 \rightarrow \mathcal{T}_{n+m}^{g_n^m}/J_n^m \mathcal{T}_n \rightarrow (\mathcal{T}_{n+m}/J_n^m \mathcal{T}_n)^{g_n^m} \rightarrow \text{H}^1(g_n^m, J_n^m \mathcal{T}_n) \rightarrow \text{H}^1(g_n^m, \mathcal{T}_{n+m}),$$

where  $\text{H}^1(g_n^m, J_n^m \mathcal{T}_n) = (J_n^m \mathcal{T}_n)[p^m] \simeq \mathcal{T}_n[p^m]$ , and:



$$\#H^1(g_n^m, \mathcal{T}_{n+m}) = \#H^2(g_n^m, \mathcal{T}_{n+m}) = \#(\mathcal{T}_{n+m}^{g_n^m}/J_n^m \mathcal{T}_n) = p^{m \cdot \#S^{\text{ta}}},$$

giving an exact sequence of the form  $1 \rightarrow A \rightarrow (\mathcal{T}_{n+m}/J_n^m \mathcal{T}_n)^{g_n^m} \rightarrow \mathcal{T}_n[p^m] \rightarrow A'$ , with  $\#A' = \#A = p^{m \cdot \#S^{\text{ta}}}$ . We then obtain the following inequality:

$$(7.1) \quad \#\mathcal{T}_{n+m} \geq \#\mathcal{T}_n \cdot \#\mathcal{T}_n[p^m],$$

more precise than  $\#\mathcal{T}_{n+m} \geq \#\mathcal{T}_{n+m}^{g_n^m} = \#\mathcal{T}_n \cdot p^{m \cdot \#S^{\text{ta}}}$ , when the  $p$ -rank of  $\mathcal{T}_n$  increases, since  $\#S^{\text{ta}}$  is a constant.

For  $m = 1$  one gets  $\#\mathcal{T}_{n+1} \geq \#\mathcal{T}_n \cdot p^{\text{rk}_p(\mathcal{T}_n)}$ . This tendency to give large orders and  $p$ -ranks is enforced by the presence of tame ramification (see §7.2 for numerical illustrations) and leads to the vast theory of pro- $p$ -extensions and Galois cohomology of pro- $p$ -groups in a broader context than  $p$ -ramification theory, which is not our purpose (see [14] for more information). This explains that, for  $\lambda = \#S^{\text{ta}} = 0$ , Corollary 3.2 never applies for  $\mathcal{T}$  (except  $\mathcal{T} = 1$  giving  $\mathcal{T}_n = 1$  for all  $n$ ).

It would be interesting to give an analogous study for the Jaulent logarithmic class group [19] since its capitulation in the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$  (real) characterizes Greenberg's conjecture.

**Remark 7.2.** For  $p$ -class groups  $\mathcal{C}_n$ , in totally real cyclic  $p$ -towers, one obtains an inequality similar to (7.1) provided one replaces  $\mathcal{C}_n$  by  $J_n^m \mathcal{C}_n$ , where  $J_n^m$  is in general non-injective, a crucial point in this comparison. But for CM-fields  $k$  and  $p \neq 2$ ,  $J_n$  is injective on the “minus parts”, giving  $\#\mathcal{C}_{n+m}^- \geq \#\mathcal{C}_n^- \cdot \#\mathcal{C}_n^-[p^m]$ , which explains the results for imaginary quadratic fields in §5.1, especially for  $m = -782$  and  $m = -858$ , for which we can predict  $\#\mathcal{C}_4$  larger than  $2^{54}$  and  $2^{46}$ , respectively.

**Remark 7.3.** One can wonder about the contribution of the  $p$ -towers contained in  $\mathbb{Q}(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{qp^N}$ , used in the compositum with  $k$ . We have the following examples of  $\mathcal{T}_n$  for  $p = 2, 3$  and  $S = S_\ell$ ; let  $\text{vn}, \text{rkn}$  denote the  $p$ -valuation, the  $p$ -rank, of  $\mathcal{T}_n$ , respectively; the parameter  $E$  must be chosen such that  $E > e + 1$ , where  $p^e$  is the exponent of  $\mathcal{T}_n$ .

```
{p=3;ell=17497;E=8;print("p=",p," ell=",ell);for(n=0,4,Qn=polsubcyclo(ell,p^n);
kn=bnfinit(Qn,1);knmod=bnrinit(kn,p^E);Tn=knmod.cyc;\computation of Tn
T=List;dn=size(Tn)[2];rkn=0;vn=0;for(j=1,d-1,c=Tn[d-j+1];w=valuation(c,p);
if(w>0,vn=vn+w;rkn=rkn+1;listinsert(T,p^w,1));\computation of vn and rkn
print("v",n,"=",vn," rk",n,"=",rkn," ",T))}

p=2,ell=17      p=2,ell=7681      p=2,ell=257
v0=0 rk0=0 []   v0=0 rk0=0 []       v0=0 rk0=0 []
v1=1 rk1=1 [2]  v1=1 rk1=1 [2]     v1=3 rk1=1 [8]
v2=2 rk2=1 [4]  v2=2 rk2=1 [4]     v2=7 rk2=3 [16,4,2]
v3=3 rk3=1 [8]  v3=3 rk3=1 [8]     v3=13 rk3=7 [32,8,2,2,2,2]
v4=4 rk4=1 [16] v4=4 rk4=1 [16]    v4=23 rk4=15 [64,16,2,2,2,2,2,2,2,2,2,2,2,2]
v5=38 rk5=15 [128,32,4,4,4,4,4,4,4,4,4,4,4,4,4]

p=3,ell=109     p=3,ell=163         p=3,ell=487         p=3,ell=17497
v0=0 rk0=0 []   v0=0 rk0=0 []       v0=0 rk0=0 []       v0=0 rk0=0 []
v1=1 rk1=1 [3]  v1=1 rk1=1 [3]     v1=2 rk1=2 [3,3]    v1=3 rk1=2 [9,3]
v2=2 rk2=1 [9]  v2=2 rk2=1 [9]     v2=4 rk2=2 [9,9]    v2=6 rk2=3 [27,9,3]
v3=3 rk3=1 [27] v3=3 rk3=1 [27]
```

**7.1. Case of the cyclotomic  $\mathbb{Z}_p$ -extensions  $k_\infty/k$  of  $k$  real.** The Iwasawa invariants  $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$ , attached to  $\lim_{\leftarrow} \mathcal{T}_n$  satisfy  $\tilde{\lambda} = \tilde{\mu} = 0$  if and only if  $\mathcal{T} = 1$  (in which case  $\tilde{\nu} = 0$ ). Indeed, if  $\tilde{\lambda} = \tilde{\mu} = 0$ ,  $\#\mathcal{T}_n = p^{\tilde{\nu}}$  for all  $n \gg 0$ ; but (7.1), with

$m = 1$ , gives  $\tilde{\nu} \geq \tilde{\nu} + r_n$ , where  $r_n := \text{rk}_p(\mathcal{T}_n)$ , whence  $r_n = 0$ ; but  $r_n \geq r$ , so  $r = 0$  ( $J_n$  is injective). The reciprocal comes from  $\mathcal{T}_n^{G_n} \simeq \mathcal{T}$  since  $S^{\text{ta}} = \emptyset$ . Thus a  $\lambda$ -stability does exist only for  $\lambda = 0$ , which is excluded if  $\mathcal{T} \neq 1$ . Computations suggest that the true Iwasawa parameters  $\tilde{\lambda}$ ,  $\tilde{\mu}$  may be often non-zero, even if few levels are accessible with PARI/GP; often there is stability for the  $p$ -class groups  $\text{Cn}$  ( $\#\text{Sp}$  is the number of  $p$ -places of  $k = \mathbb{Q}(\sqrt{m})$ ):

```
{p=3;E=8;N=4;bm=2;Bm=10^3;for(m=bm,Bm,if(core(m)!=m,next);P=x^2-m;
Sp=1;if(kronecker(m,p)==1,Sp=2);print();print("p=",p," m=",m," #Sp=",Sp);
for(n=0,N,if(p==2,Qn=x;for(k=1,n,Qn=Qn^2-2));if(p!=2,Qn=polsubcyclo(p^(n+1),p^n));
Pn=polcompositum(Qn,P)[1];kn=bnfinit(Pn,1);Cn=kn.cyc;C=List;d=matsize(Cn)[2];
for(j=1,d,c=Cn[j];w=valuation(c,p);if(w>0,listinsert(C,p^w,1)))\\p-class group Cn
knmod=bnrinit(kn,p^E);rkn=0;vn=0;Tn=knmod.cyc;\\computation of Tn
T=List;d=matsize(Tn)[2];for(j=1,d-1,c=Tn[d-j+1];w=valuation(c,p);
if(w>0,vn=vn+w;rkn=rkn+1;listinsert(T,p^w,1)))\\computation of vn and rkn
print("v",n,"=",vn," rk",n,"=",rkn," T",n,"=",T," C",n,"=",C))}

p=2
m=33,#Sp=2          m=41,#Sp=2          m=217,#Sp=2
v0=1 rk0=1 [2] C0=[] v0=4 rk0=1 [16] C0=[] v0=2 rk0=2 [2,2] C0=[]
v1=4 rk1=1 [16] C1=[] v1=5 rk1=1 [32] C1=[2] v1=4 rk1=3 [4,2,2] C1=[2]
v2=5 rk2=1 [32] C2=[] v2=6 rk2=1 [64] C2=[4] v2=8 rk2=5 [4,4,4,2,2] C2=[2]
v3=6 rk3=1 [64] C3=[] v3=7 rk3=1 [128] C3=[8] v3=15 rk3=9 [8,8,8,2,2,2,2,2,2] C3=[2]
v4=7 rk4=1 [128] C4=[] v4=8 rk4=1 [256] C4=[8] v4=24 rk4=9 [16,16,16,4,4,4,4,4,4] C4=[2]

m=193,#Sp=2          m=69,#Sp=1
v0=1 rk0=1 [2] C0=[] v0=2 rk0=1 [4] C0=[]
v1=2 rk1=2 [2,2] C1=[] v1=4 rk1=2 [4,4] C1=[]
v2=4 rk2=4 [2,2,2,2] C2=[] v2=6 rk2=2 [8,8] C2=[]
v3=8 rk3=8 [2,2,2,2,2,2,2,2] C3=[] v3=8 rk3=2 [16,16] C3=[]
v4=16 rk4=15 [4,2,2,2,2,2,2,2,2,2,2,2,2,2,2] C4=[]

p=3
m=62,#Sp=1          m=103,#Sp=2          m=1714,#Sp=2
v0=1 rk0=[3] C0=[] v0=1 rk0=1 [3] C0=[] v0=2 rk0=2 [3,3] C0=[3]
v1=3 rk1=[9,3] C1=[] v1=3 rk1=2 [9,3] C1=[3,3] v1=5 rk1=4 [9,3,3,3] C1=[3,3]
v2=5 rk2=[27,9] C2=[] v2=5 rk2=2 [27,9] C2=[3,3] v2=9 rk2=4 [27,9,9,9] C2=[9,3]
```

Formula (7.1) predicts  $\#\mathcal{T}_5 \geq 2^{16+15} = 2^{31}$  for  $m = 193$ , giving possibly  $\tilde{\mu} = 1$ .

**7.2. Case of real cyclic  $p$ -towers with tame ramification.** In the case of  $S_\ell$ -ramified cyclic  $p$ -towers, one gets few examples of  $\lambda$ -stability (for  $\lambda = \#S_\ell$ ) and mainly spectacular groups  $\mathcal{T}_n$ ; for example let  $k = \mathbb{Q}(\sqrt{m})$  and  $K = k(\mu_{257})$  with  $p = 2$ , then  $K = k(\mu_{109})$  and  $K = k(\mu_{163})$  with  $p = 3$ :

(i) Case  $p = 2$ .

```
{p=2;ell=257;N=3;E=8;bm=2;Bm=10^3;for(m=bm,Bm,if(core(m)!=m|Mod(m,ell)==0,next);
P=x^2-m;S=1;if(kronecker(m,ell)==1,S=2);print("p=",p," ell=",ell," m=",m," #S=",S);
for(n=0,N,Pn=polcompositum(polsubcyclo(ell,p^n),P)[1];kn=bnfinit(Pn,1);\\layer kn
Cn=kn.cyc;C=List;d=matsize(Cn)[2];for(j=1,d,c=Cn[j];w=valuation(c,p);
if(w>0,listinsert(C,p^w,1)))\\p-class group Cn
knmod=bnrinit(kn,p^E);Tn=knmod.cyc;T=List;d=matsize(Tn)[2];\\computation of Tn
rkn=0;vn=0;for(j=1,d-1,c=Tn[d-j+1];w=valuation(c,p);if(w>0,rkn=rkn+1;vn=vn+w;
listinsert(T,p^w,1)));print("v",n,"=",vn," rk",n,"=",rkn," T",n,"=",T," C",n,"=",C))}

p=2,ell=257,m=17,#S=2
v0=1 rk0=1 [2] C0=[]
v1=7 rk1=3 [16,4,2] C1=[]
v2=16 rk2=7 [32,8,8,4,2,2,2] C2=[]
v3=31 rk3=15 [64,16,16,4,4,4,4,4,2,2,2,2,2,2,2] C3=[]
v4=58 rk4=31 [128,32,32,8,8,8,8,4,4,4,4,4,4,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2] C4=[]
```

$p=2, e_{11}=257, m=34, \#S=2$   
 $v_0=1 \text{ rk}_0=1 \quad [2] \quad C_0=[2]$   
 $v_1=8 \text{ rk}_1=3 \quad [16, 8, 2] \quad C_1=[4, 2]$   
 $v_2=20 \text{ rk}_2=7 \quad [32, 16, 8, 8, 4, 4, 2] \quad C_2=[4, 2, 2, 2]$   
 $v_3=37 \text{ rk}_3=15 \quad [64, 32, 16, 16, 8, 4, 4, 4, 4, 2, 2, 2, 2, 2] \quad C_3=[4, 2, 2, 2, 2, 2, 2, 2]$   
 $v_4=65 \text{ rk}_4=27 \quad [128, 64, 32, 32, 16, 8, 8, 8, 8, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] \quad C_4=[8, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2]$

Strikingly, for  $m = 17$ , one verifies the formula  $\#\mathcal{T}_n = 2^{3 \cdot n + 3 \cdot 2^n - 2}$  for  $n \in [0, 4]$  and (7.1) predicts  $\#\mathcal{T}_5 \geq 2^{89}$ ; for  $m = 34$ , (7.1) predicts  $\#\mathcal{T}_5 \geq 2^{92}$  (or much more).

(ii) Case  $p = 3$ .

$p=3, e_{11}=163, m=2, \#S=1$	$p=3, e_{11}=163, m=10, \#S=2$	$p=3, e_{11}=109, m=29, \#S=2$
$v_0=0 \text{ rk}_0=1 \quad [] \quad C_0=[]$	$v_0=0 \text{ rk}_0=0 \quad [] \quad C_0=[]$	$v_0=2 \text{ rk}_0=1 \quad [9] \quad C_0=[]$
$v_1=1 \text{ rk}_1=1 \quad [3] \quad C_1=[]$	$v_1=2 \text{ rk}_1=2 \quad [3, 3] \quad C_1=[]$	$v_1=4 \text{ rk}_1=2 \quad [27, 3] \quad C_1=[]$
$v_2=2 \text{ rk}_2=1 \quad [9] \quad C_2=[]$	$v_2=4 \text{ rk}_2=2 \quad [9, 9] \quad C_2=[]$	$v_2=6 \text{ rk}_2=2 \quad [81, 9] \quad C_2=[]$
$p=3, e_{11}=109, m=15, \#S=2$	$p=3, e_{11}=109, m=145, \#S=2$	
$v_0=1 \text{ rk}_0=1 \quad [3] \quad C_0=[]$	$v_0=0 \text{ rk}_0=0 \quad [] \quad C_0=[]$	$C_0=[]$
$v_1=6 \text{ rk}_1=4 \quad [9, 9, 3, 3] \quad C_1=[]$	$v_1=8 \text{ rk}_1=3 \quad [81, 27, 3] \quad C_1=[]$	$C_1=[]$
$v_2=10 \text{ rk}_2=4 \quad [27, 27, 9, 9] \quad C_2=[]$	$v_2=15 \text{ rk}_2=7 \quad [243, 81, 9, 3, 3, 3, 3] \quad C_2=[]$	
$p=3, e_{11}=163, m=15, \#S=2$	$p=3, e_{11}=163, m=79, \#S=1$	$p=3, e_{11}=163, m=118, \#S=2$
$v_0=1 \text{ rk}_0=1 \quad [3] \quad C_0=[]$	$v_0=2 \text{ rk}_0=1 \quad [9] \quad C_0=[3]$	$v_0=0 \text{ rk}_0=0 \quad [] \quad C_0=[]$
$v_1=5 \text{ rk}_1=4 \quad [9, 3, 3, 3] \quad C_1=[3]$	$v_1=7 \text{ rk}_1=4 \quad [27, 9, 3, 3] \quad C_1=[9]$	$v_1=4 \text{ rk}_1=3 \quad [9, 3, 3] \quad C_1=[]$
$v_2=9 \text{ rk}_2=4 \quad [27, 9, 9, 9] \quad C_2=[9]$	$v_2=11 \text{ rk}_2=4 \quad [81, 27, 9, 9] \quad C_2=[9]$	$v_2=8 \text{ rk}_2=4 \quad [27, 9, 9, 3] \quad C_2=[]$

See Remark 7.3 to compute the contributions of the towers over  $\mathbb{Q}$ .

**Remark 7.4.** Let  $K/k$  be a totally real,  $p$ -ramified (whence  $S \subseteq \Sigma_p$ , arbitrary), cyclic  $p$ -tower (i.e.,  $k \subseteq K \subseteq (H_k^{\text{Pr}})^{\text{ab}}$ , where  $\text{Gal}((H_k^{\text{Pr}})^{\text{ab}}/k) \simeq \mathbb{Z}_p \oplus \mathcal{T}_k$ ). Then if the  $p$ -ranks of  $k$  and  $k_1$  coincide, the  $p$ -rank is constant in the tower (indeed, this corresponds to the stability of  $\mathcal{T}_n[p]$ ,  $n \in [0, N]$ , since  $\mathcal{T}_n[p]^{G_n} = \mathcal{T}[p]$  and  $\mathcal{T}_n[p]^{G_n^p} = \mathcal{T}_1[p]$  which leads to  $\mathcal{T}_n[p] \simeq \mathcal{T}[p]$  for all  $n$ ). Of course, this may occur only from some layer  $k'$  and the numerical data of § 7.1 gives many examples.

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