



HAL
open science

ON THE p -CLASS GROUP STABILITY ALONG CYCLIC p -TOWERS OF A NUMBER FIELD

Georges Gras

► **To cite this version:**

Georges Gras. ON THE p -CLASS GROUP STABILITY ALONG CYCLIC p -TOWERS OF A NUMBER FIELD. 2021. hal-03155190v3

HAL Id: hal-03155190

<https://hal.science/hal-03155190v3>

Preprint submitted on 31 Mar 2021 (v3), last revised 12 Aug 2021 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE p -CLASS GROUP STABILITY ALONG CYCLIC p -TOWERS OF A NUMBER FIELD III

GEORGES GRAS

ABSTRACT. Let k be a number field and let $p \geq 2$ be prime. We call K/k a cyclic p -tower if $\text{Gal}(K/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$, $N \geq 2$. We give an elementary proof of a stability theorem for generalized p -class groups \mathcal{X}_n of the layers k_n (Theorem 3.1). Using generalizations of Chevalley's formula (Gras, J. Math. Soc. Japan **46**(3) (1994), Proc. Math. Sci. **127**(1) (2017)), we improve results by Fukuda (1994), Li–Ouyang–Xu–Zhang (2020), Mizusawa–Yamamoto (2020) and others, whose techniques are based on Iwasawa's theory, Galois theory of pro- p -groups or specific methods. The main difference, compared to these works, is that we introduce an explicit parameter $\lambda \geq 0$ giving formulas of the form $\#\mathcal{X}_n = \#\mathcal{X}_0 \cdot p^{\lambda \cdot n}$, for all $n \in [0, N]$, as soon as this equality holds for $n = 1$.

CONTENTS

1. Introduction	1
2. Around Chevalley's ambiguous class number formula	3
3. Introduction of Hasse's norm symbols	4
3.1. Main theorem	4
3.2. Remarks	5
4. Numerical examples	6
4.1. Cyclic p -towers over imaginary quadratic fields	6
4.2. Cyclic p -towers over real fields with $p = 3$ and $\mathcal{X} = \mathcal{C}$	6
4.3. Cyclic p -towers over \mathbb{Q} , with $p = 3$ and non-trivial modulus \mathfrak{m}	7
5. Other applications	7
References	8

1. INTRODUCTION

Let k be a number field, let $p \geq 2$ be a prime number. Let K be a cyclic extension of degree p^N of k and let k_n , $n \in [0, N]$, be the degree p^n extension of k in K . We speak of cyclic p -towers K/k and put $G = \text{Gal}(K/k)$. Let S be the set of finite places of k , ramified in K/k . In the sequel we will assume $S \neq \emptyset$ and S totally ramified in K/k ; in general, S contains tame places, except if K is a subfield of a \mathbb{Z}_p -extension of k , in which case S is necessarily a set of p -places of k . Due to

Date: 31 March 2021.

2020 Mathematics Subject Classification. Primary 11R23; 11R29; 11R37.

Key words and phrases. p -class groups, class field theory, Chevalley's formula, cyclic p -extensions.

the very nature of a tower, there will never be “complexification=ramification” of infinite real places (such places cannot be *totally ramified* since $N \geq 2$ in practice).

Let \mathfrak{m} be an integer ideal of k , *prime to* S ; let $\mathcal{C}_{k,\mathfrak{m}}$ and $\mathcal{C}_{k_n,\mathfrak{m}}$, denoted simply $\mathcal{C}_{0,\mathfrak{m}}$ and $\mathcal{C}_{n,\mathfrak{m}}$, be the p -Sylow subgroups of the ray class groups modulo \mathfrak{m} , of k and k_n , respectively (for $n > 0$, \mathfrak{m} is seen as extended ideal in k_n). The class of an ideal \mathfrak{a} of k_n is denoted $\mathcal{C}_{n,\mathfrak{m}}(\mathfrak{a})$. Let N_{k_m/k_n} be the arithmetic norm, in k_m/k_n , defined on class groups from norms of ideals.

Let $(\mathcal{H}_n)_{n \in [0,N]}$ be a family of sub- G -modules of the class groups $\mathcal{C}_{n,\mathfrak{m}}$ such that $N_{k_n/k_{n-1}}(\mathcal{H}_n) = \mathcal{H}_{n-1}$, for all $n \in [1, N]$, and let \mathcal{I}_n be any G -module of prime to S ideals of k_n such that $\mathcal{H}_n = \mathcal{C}_{n,\mathfrak{m}}(\mathcal{I}_n)$ (one can take $\mathcal{I}_n = N_{K/k_n}(\mathcal{I}_N)$).

We consider the following “generalized p -class groups”:

$$\mathcal{X}_{n,\mathfrak{m}} := \mathcal{C}_{n,\mathfrak{m}}/\mathcal{H}_n, \quad n \in [0, N].$$

Since \mathfrak{m} is prime to $S \neq \emptyset$, the corresponding generalized p -Hilbert’s ray class fields of the layers k_n are linearly disjoint from the relative towers K/k_n ; thus $N_{k_n/k_{n-1}}(\mathcal{X}_{n,\mathfrak{m}}) = \mathcal{X}_{n-1,\mathfrak{m}}$ for all $n \in [1, N]$ by class field theory. To simplify, we shall remove the indices \mathfrak{m} in all the forthcoming notations, except necessity.

One speaks of “ λ -stability” of these generalized p -class groups along the tower K/k when there exists $\lambda \geq 0$ such that $\#\mathcal{X}_n \simeq \#\mathcal{X}_0 \cdot p^{\lambda \cdot n}$, for all $n \in [0, N]$.

Definition 1.1. Let $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{I}_N)\}$. For instance, when $\mathcal{X} = \mathcal{C}$ (i.e., $\mathcal{I}_n = 1$ for all n), $\Lambda = E$, the group of global units $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$ of k . We shall see, Lemma 2.2, that *in some sense* Λ is an invariant of k replacing E in the more general context defined by the \mathcal{H}_n ’s. Ideal groups \mathcal{I}_n may be finitely generated and, due to the inclusion $E \subseteq \Lambda$, one can write:

$$\Lambda = \langle \zeta \rangle_{\mathbb{Z}} \oplus \langle \varepsilon_1, \dots, \varepsilon_r; \alpha_1, \dots, \alpha_t \rangle_{\mathbb{Z}},$$

where $\langle \zeta \rangle_{\mathbb{Z}} = \text{tor}_{\mathbb{Z}}(E)$, $r = r_1 + r_2 - 1$ as usual, and $t = \dim_{\mathbb{F}_p}(\mathcal{H}_0/\mathcal{H}_0^p)$.

Denote by $\mu_p(k)$ the group of roots of unity of p -power order contained in k . We shall see that the norm properties of $\langle \zeta \rangle_{\mathbb{Z}} \otimes \mathbb{Z}_p \subseteq \mu_p(k)$ are specific and may give some obstructions that are elucidated by the following Lemma.

Lemma 1.2. (*Albert’s Theorem for K/k*). *If k contains the group μ_{p^e} of p^e th roots of unity, then $\mu_{p^e} \subset N_{K/k}(K^\times)$ if and only if there exists a degree p^e cyclic extension L/K such that L/k is cyclic.*

Remark 1.3. *We shall restrict ourselves to the case $\zeta \in N_{K/k}(K^\times)$ (e.g., $p = 2$, $\mu_2(k) = \langle -1 \rangle_{\mathbb{Z}}$, $L = k\mathbb{Q}(\mu_{17})$, $K \subset L$ with $[L : K] = 2$; thus $-1 \in N_{K/k}(K^\times)$). If necessary, it suffices to restrict the tower to a sub-tower K'/k or to notice that $L/K/k$ does exist (even if the ramification in L/K is unknown). Then we can write $\Lambda N_{K/k}(K^\times) =: X N_{K/k}(K^\times)$ where X is a free \mathbb{Z} -module of \mathbb{Z} -rank $\rho = r + t$.*

Under the assumptions about S , \mathfrak{m} , and the family $(\mathcal{H}_n)_{n \in [0,N]}$, we will prove the following result, where we put $\mathcal{C}_0 =: \mathcal{C}$, $\mathcal{H}_0 =: \mathcal{H}$, $\mathcal{I}_0 =: \mathcal{I}$, $\mathcal{X}_0 =: \mathcal{X}$, and so on in the sequel (see Theorem 3.1 for a more complete statement):

Main Theorem. *Assume that $\Lambda N_{K/k}(K^\times) =: X N_{K/k}(K^\times)$, with X free of \mathbb{Z} -rank $\rho \geq 0$, and put $\lambda := \max(0, \#S - 1 - \rho)$. The λ -stability of the \mathcal{X}_n ’s along K/k (i.e., $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda \cdot n}$ for all $n \in [0, N]$), is equivalent to $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$.*

For known results (all relative to $\lambda = 0$), one may cite Fukuda [2] using Iwasawa’s theory, Li–Ouyang–Xu–Zhang [10, §3] working in a non-abelian Galois context, in

Kummer towers, via the use of the fixed points formulas [3, 4], then Mizusawa [12] above \mathbb{Z}_2 -extensions, and Mizusawa–Yamamoto [13] for generalizations, including ramification and splitting conditions, via the Galois theory of pro- p -groups.

The present work gives, in an elementary way, the “ λ -generalization” of these works, using only classical fixed points formulas issued from Chevalley’s one.

2. AROUND CHEVALLEY’S AMBIGUOUS CLASS NUMBER FORMULA

The well-known Chevalley’s formula [1, p. 402] (also known as the Herbrand–Chevalley formula) is the pivotal element for a great lot of “fixed point formulas”. For the ordinary sense, it takes into account the “complexification” of real infinite places which does not occur for us as explained in the Introduction.

Let L/F be a cyclic p -extension of Galois group G , S -ramified, non-complexified at infinity if $p = 2$, and let $e_v(L/F)$ be the ramification index of $v \in S$; so:

$$\#\mathcal{C}_L^G = \frac{\#\mathcal{C}_F \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot (E_F : E_F \cap N_{L/F}(L^\times))}.$$

Various generalizations of this formula were given (Gras [3, Théorème 2.7, Corollaire 2.8], [4, Theorem 3.6, Corollaries 3.7, 3.9], Jaulent [8, Chap. II, § 3]); mention the recent idelic proof by Li–Yu [11, Theorem 2.1 & § 2.3, Examples] of such formulas. We shall use the following one in L/F :

Proposition 2.1. *Assume $S \neq \emptyset$ totally ramified in L/F . Let \mathfrak{m} be a modulus of F , prime to S , and let $\mathcal{C}_F, \mathcal{C}_L$, be the p -Sylow subgroups of the ray class groups modulo \mathfrak{m} , of F and L , respectively. For any sub- G -module \mathcal{H}_L of \mathcal{C}_L and any G -module \mathcal{I}_L , of prime to S ideals of L , such that $\mathcal{H}_L = \alpha_L(\mathcal{I}_L)$, we have, with $\mathcal{H}_F := N_{L/F}(\mathcal{H}_L)$:*

$$(2.1) \quad \#(\mathcal{C}_L/\mathcal{H}_L)^G = \#(\mathcal{C}_F/\mathcal{H}_F) \times \frac{(\#G)^{\#S-1}}{(\Lambda_{L/F} : \Lambda_{L/F} \cap N_{L/F}(L^\times))},$$

where $\Lambda_{L/F} := \{x \in F^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{L/F}(\mathcal{I}_L)\}$.

Lemma 2.2. *Let K/k be a cyclic p -tower of degree p^N , of Galois group $G = \langle \sigma \rangle$. Let $\mathcal{H}_n =: \alpha_n(\mathcal{I}_n) \subseteq \mathcal{C}_n$ be the given family of sub- G -modules, $n \in [0, N]$. Let $\Lambda_{k_n/k} := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{k_n/k}(\mathcal{I}_n)\}$ and let $\Lambda := \Lambda_{K/k}$. Then:*

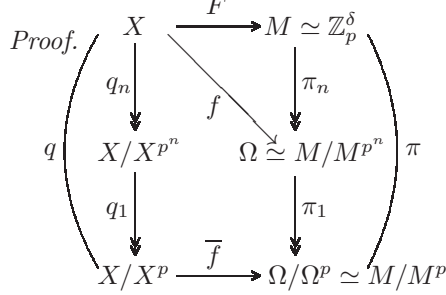
(i) $\Lambda_{k_n/k} \cdot N_{k_n/k}(k_n^\times) = \Lambda \cdot N_{k_n/k}(k_n^\times)$, whence $(\Lambda_{k_n/k} : \Lambda_{k_n/k} \cap N_{k_n/k}(k_n^\times)) = (\Lambda : \Lambda \cap N_{k_n/k}(k_n^\times))$;

(ii) $\Lambda \subseteq \Lambda_1 := \{x \in k_1^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k_1}(\mathcal{I}_N)\}$.

Proof. (i) Let $x \in \Lambda_{k_n/k}$. Thus $(x) = N_{k_n/k}(\mathfrak{a})$, $\mathfrak{a} \in \mathcal{I}_n$; since $\mathcal{H}_n = N_{K/k_n}(\mathcal{H}_N)$, there exists $\mathfrak{A} \in \mathcal{I}_N$ such that $\mathfrak{a} = (\alpha) \cdot N_{K/k_n}(\mathfrak{A})$, with $\alpha \in k_n^\times$, $\alpha \equiv 1 \pmod{\mathfrak{m}}$, whence $(x) = (N_{k_n/k}(\alpha)) \cdot N_{K/k}(\mathfrak{A})$, proving the non-trivial inclusion.

(ii) Let $x \in \Lambda$; then $(x) = N_{K/k}(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{I}_N$. Let $J_{K/k}, J_{k_1/k}$ be the transfer maps (extension of ideals). Then $J_{K/k} \circ N_{K/k} = \nu_{K/k} = \sum_{i=0}^{N-1} \sigma^i = \Theta \cdot \nu_{K/k_1}$, with $\Theta = \sum_{i=0}^{p-1} \sigma^i$; thus, $J_{K/k}((x)) = \mathfrak{A}^{\Theta \cdot \nu_{K/k_1}} =: \mathfrak{B}^{\nu_{K/k_1}} = J_{K/k_1}(N_{K/k_1}(\mathfrak{B}))$, $\mathfrak{B} = \mathfrak{A}^\Theta \in \mathcal{I}_N$. Since the map J_{K/k_1} is injective on ideal groups, from $J_{K/k}((x)) = J_{K/k_1}(J_{k_1/k}((x)))$, we get $(x) = N_{K/k_1}(\mathfrak{B})$ considering $x \in k_1$, whence the claim. \square

Lemma 2.3. *Let $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$ be a free \mathbb{Z} -module of \mathbb{Z} -rank $\rho \geq 0$, let $\Omega := (\mathbb{Z}/p^n\mathbb{Z})^\delta$, $n \geq 1$, $\delta \geq 0$, and let $f : X \rightarrow \Omega$ be an homomorphism such that the image of $X \rightarrow \Omega/\Omega^p$ be of \mathbb{F}_p -dimension $\min(\rho, \delta)$. Then $\#f(X) = p^{\min(\rho, \delta) \cdot n}$ and for all $m \in [0, n]$, $\#f^{p^m}(X) = p^{\min(\rho, \delta) \cdot (n-m)}$ (where $f^{p^m}(x) := f(x)^{p^m}$).*



Let $M := \langle e_1, \dots, e_\delta \rangle_{\mathbb{Z}_p}$ be a free \mathbb{Z}_p -module of \mathbb{Z}_p -rank δ , and replace Ω by M/M^{p^n} . From the map $f : X \rightarrow \Omega$, let $\bar{f} : X/X^p \rightarrow \Omega/\Omega^p$ deduced from f . By assumption, $\dim_{\mathbb{F}_p}(\text{Im}(\bar{f})) = \min(\rho, \delta)$. Let $q : X \rightarrow X/X^p$ and $\pi : M \rightarrow M/M^p$ be the canonical maps. Let $F : X \rightarrow M$ be any map such that $\pi \circ F = \bar{f} \circ q$.

(i) Case $\rho \geq \delta$ (surjectivity of \bar{f}). Let $y \in M$; there exists $x \in X$ such that $\pi(y) = \bar{f}(q(x)) = \pi(F(x))$, whence $y = F(x) \cdot y'$, $y' \in M$, and $M = F(X)$ (Hensel Lemma).

(ii) Case $\rho < \delta$ (injectivity of \bar{f}). Let $y \in M$ such that $y^p = F(x)$, $x \in X$; then $1 = \pi(y^p) = \pi(F(x)) = \bar{f}(q(x))$, whence $q(x) = 1$ and $x = x'^p$, $x' \in X$, giving $y^p = F(x')^p$, thus $y = F(x')$ proving that $F(X)$ is direct factor in M .

(iii) In case (i), $f(X) = \Omega$ is of order $p^{\delta \cdot n}$. In case (ii), $f(X) = \pi_n(F(X))$ is direct factor in Ω isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\rho$ and f^{p^m} applies X onto $(\mathbb{Z}/p^{n-m}\mathbb{Z})^\rho$. \square

3. INTRODUCTION OF HASSE'S NORM SYMBOLS

3.1. Main theorem. Let K/k be a cyclic p -tower of degree p^N , $N \geq 2$. Let $\omega_{k_n/k}$ be the map which associates with $x \in \Lambda$ the family of Hasse's norm symbols $(\frac{x, k_n/k}{v}) \in I_v(k_n/k)$ (inertia groups of $v \in S$). Being norm of an ideal, x is a local norm at the places $v \notin S$ and the image of $\omega_{k_n/k}$ is contained in

$$\Omega_{k_n/k} := \left\{ (\tau_v)_{v \in S} \in \bigoplus_{v \in S} I_v(k_n/k), \prod_{v \in S} \tau_v = 1 \right\};$$

then $\text{Ker}(\omega_{k_n/k}) = \Lambda \cap N_{k_n/k}(k_n^\times)$.

Assume $S \neq \emptyset$ totally ramified; so, fixing any $v_0 \in S$, $\Omega_{k_n/k} \simeq \bigoplus_{v \neq v_0} G_n$, with $G_n := \text{Gal}(k_n/k)$. We consider formulas (2.1) in k_n/k , using Lemma 2.2 (i):

$$(3.1) \quad \#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot \frac{p^{n(\#S-1)}}{(\Lambda : \Lambda \cap N_{k_n/k}(k_n^\times))} = \#\mathcal{X} \cdot \frac{p^{n(\#S-1)}}{\#\omega_1(\Lambda)},$$

where $\mathcal{X}_n := \mathcal{C}_n/\mathcal{H}_n$ for the given $\mathcal{H}_n = \mathcal{d}_n(\mathcal{J}_n)$, the modulus \mathfrak{m} being prime to S , and where $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{J}_N)\}$.

Theorem 3.1. *Set $\Lambda := \text{tor}_{\mathbb{Z}}(\Lambda) \oplus X$ with $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$, free of \mathbb{Z} -rank ρ and let $\lambda := \max(0, \#S - 1 - \rho)$. Assume that $\text{tor}_{\mathbb{Z}}(\Lambda) \subset N_{K/k}(K^\times)$ (see Lemma 1.2 and Remark 1.3). We then have the following properties of λ -stability:*

(i) *If $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$, then for all $n \in [0, N]$, $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda \cdot n}$ and $\mathcal{X}_n^{G_n} = \mathcal{X}_n$.*

(ii) *Under the above condition $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$, we have $J_{k_n/k}(\mathcal{X}) = \mathcal{X}_n^{p^n}$ and $\text{Ker}(J_{k_n/k}) = \mathcal{X}[p^n]$, for all $n \in [0, N]$, where the $J_{k_n/k}$'s are the transfer maps. If \tilde{k}/k is a \mathbb{Z}_p -extension, then \mathcal{X} capitulates in some $K \subset \tilde{k}$.*

Proof. Put $N_n := N_{k_n/k}(k_n^\times)$, $G_n := \langle \sigma_n \rangle$, $\Omega_n := \Omega_{k_n/k}$, $\omega_n := \omega_{k_n/k}$, $s := \#S$. By assumption, $\omega_n(\Lambda) = \omega_n(X)$ for all $n \in [1, N]$. We have (from (3.1)) $\#\mathcal{X}_1^{G_1} = \#\mathcal{X} \cdot \frac{p^{s-1}}{\#\omega_1(X)} \geq \#\mathcal{X} \cdot p^\lambda$ since $\#\omega_1(X) \leq p^{\min(\rho, s-1)}$; thus, with $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$, we obtain $\mathcal{X}_1^{G_1} = \mathcal{X}_1$, whence $\#\mathcal{X}_1^{G_1} = \#\mathcal{X} \cdot p^\lambda$ and $\#\omega_1(X) = p^{\min(\rho, s-1)}$.

By restriction of Hasse's symbols, $\omega_1 = q \circ \omega_n$, with $q : \Omega_n \rightarrow \Omega_n/\Omega_n^p \simeq G_1^{s-1}$; since we have proven that $\dim_{\mathbb{F}_p}(\omega_1(X)) = \min(\rho, s-1)$, Lemma 2.3 applies to $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$, $\delta = s-1$, $f = \omega_n$; so, $\#\omega_n(X) = p^{\min(\rho, s-1)n}$ and, from (3.1):

$$\#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda n}, \text{ for all } n \in [0, N], \lambda = \max(0, s-1-\rho).$$

Consider the extension k_n/k_1 of Galois group $G_n^p = \langle \sigma_n^p \rangle$, and the corresponding map ω'_n on k_1^\times with values in $\Omega_{k_n/k_1} \simeq (G_n^p)^{s-1}$; then $\#\mathcal{X}_n^{G_n^p} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda_1)}$, $\Lambda_1 := \{x \in k_1^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k_1}(\mathcal{I}_N)\}$. Since $X \subseteq \Lambda \subseteq \Lambda_1$ (Lemma 2.2 (ii)), the functorial properties of Hasse's symbols imply (where $v_1 \mid v$ in k_1):

$$\left(\frac{x_i, k_n/k_1}{v_1} \right) = \left(\frac{N_1(x_i), k_n/k}{v} \right) = \left(\frac{x_i^p, k_n/k}{v} \right) = \left(\frac{x_i, k_n/k}{v} \right)^p = \sigma_n^{pa_i, v},$$

giving $\omega'_n = \omega_n^p$ on X . Then, $\omega'_n(X) \subseteq \omega'_n(\Lambda_1)$ yields:

$$\#\mathcal{X}_n^{G_n^p} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda_1)} \leq \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(X)} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega_n^p(X)};$$

Lemma 2.3, for $f^p = \omega_n^p$, gives $\#\omega_n^p(X) = p^{(n-1)\min(\rho, s-1)}$, then:

$$\#\mathcal{X}_n^{G_n^p} \leq \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{p^{(n-1)\min(\rho, s-1)}} = \#\mathcal{X}_1 \cdot p^{\lambda \cdot (n-1)} = \#\mathcal{X} \cdot p^{\lambda n}.$$

Since $\mathcal{X}_n^{G_n^p} \supseteq \mathcal{X}_n^{G_n}$ and $\#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda n}$ we get $\mathcal{X}_n^{G_n^p} = \mathcal{X}_n^{G_n}$ equivalent to $\mathcal{X}_n^{1-\sigma_n} = \mathcal{X}_n^{1-\sigma_n^p}$, whence $\mathcal{X}_n^{1-\sigma_n} = \mathcal{X}_n^{(1-\sigma_n) \cdot \theta}$, where $\theta = 1 + \sigma_n + \dots + \sigma_n^{p-1} \in (p, 1 - \sigma_n)$, a maximal ideal since $\mathbb{Z}_p[G_n]/(p, 1 - \sigma_n) \simeq \mathbb{F}_p$; so $\mathcal{X}_n^{1-\sigma_n} = 1$, thus $\mathcal{X}_n = \mathcal{X}_n^{G_n}$. Whence (i), then (ii) from $N_n(\mathcal{X}_n) = \mathcal{X}$ and $J_n \circ N_n = \nu_n$. \square

3.2. Remarks. To simplify the comments, let's consider $\mathcal{X} = \mathcal{C}$, so that $\lambda = \max(0, \#S - 1 - \rho)$ with $\rho = r_1 + r_2 - 1$ assuming the context of Lemma 1.2 and Remark 1.3, where $\omega_N(\Lambda) = \omega_N(X)$:

(a) The parameter λ may be arbitrarily large (which only depends on S) and the condition $\#\mathcal{C}_1 = \#\mathcal{C}$ of the literature (which implies $\lambda = 0$, thus $\#S \leq r_1 + r_2$) is empty as soon as $\lambda \geq 1$, whence the interest of the factor p^λ to get examples whatever S . Moreover, the condition $\#\mathcal{C}_1 = \#\mathcal{C} \cdot p^\lambda$ implies $\mathcal{C}_1^{G_1} = \mathcal{C}_1$, which is very restrictive. Indeed, put $\mathcal{C}_1^i := \{c \in \mathcal{C}_1, c^{(1-\sigma_1)^i} = 1\}$, $i \geq 1$; we have, from the general formula, $\#(\mathcal{C}_1^2/\mathcal{C}_1^1) = \frac{\#\mathcal{C}}{\#N_1(\mathcal{C}_1^1)} \times \frac{p^{\#S-1}}{\#\omega_1(\Lambda^1)}$, where Λ^1 is defined from \mathcal{C}_1^1 ; in general this order is non-trivial and the filtration increases to get $\#\mathcal{C}_1 > \#\mathcal{C}_1^1$.

(b) The condition $\mathcal{C}_1^{G_1} = \mathcal{C}_1$ is not equivalent to $\#\mathcal{C}_1 = \#\mathcal{C} \cdot p^\lambda$. Let $k = \mathbb{Q}(\sqrt{m})$ be a real quadratic field and $\#S = 2$ and assume that $\omega_1(\varepsilon) = 1$ (ε is norm in k_1/k); indeed, in this case $\rho = 1$, $\lambda = 0$, then $\#\mathcal{C}_1 = \#\mathcal{C} \cdot p \neq \#\mathcal{C} \cdot p^\lambda = \#\mathcal{C}$. Consider the towers $K = kL$, where L is the subfield of $\mathbb{Q}(\mu_{17})$ with $p = 2$ (resp. $\mathbb{Q}(\mu_{109})$ with $p = 3$), of degree 8 (resp. 27); ν_n denotes the p -valuation of $\#\mathcal{C}_n$ and $[a, b, \dots]$ gives the structure $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \cdots$ of the whole class group:

p=2	q=17	m=26	p=3	q=109	m=223	p=5	q=251	m=727
n=0	vn=1	Cn=[2]	n=0	vn=1	Cn=[3]	n=0	vn=1	Cn=[5]
n=1	vn=2	Cn=[2,2]	n=1	vn=2	Cn=[9]	n=1	vn=2	Cn=[25]
n=2	vn=3	Cn=[4,2]	n=2	vn=2	Cn=[9]			

(c) If $\lambda = 0$ for the tower K/k (i.e., $\rho \geq \#S - 1$) and if we consider a relative tower K/k' , with $k' = k_{n_0}$, the corresponding invariant λ' is also trivial since $\Lambda' \supseteq \Lambda$ giving $\rho' \geq \rho$. So, as soon as $\#\mathcal{C}_{n_0+1} = \#\mathcal{C}_{n_0}$, the equality propagates in the relative tower. Otherwise, one must compare $\#S - 1$ with $\rho' = r'_1 + r'_2 - 1$ in k_{n_0} .

(d) If Lemma 1.2 does not apply, some counterexamples can arise as for $k = \mathbb{Q}(\sqrt{-55})$, $p = 2$, $K = k\mathbb{Q}(\mu_{17})$ of degree 2^4 over k ($-1 \notin N_{K/k}(K^\times)$) and the structures [4], [56], [560], [126560], [126560], for which $\#\mathcal{C}_n = \#\mathcal{C} \cdot 2^n$ holds for $n \in [0, 3]$ since -1 is norm in k_3/k , but not for $n = 4$.

(e) For the construction of more general cyclic p -towers, see [6, V, Theorem 2.9] using explicit governing fields.

4. NUMERICAL EXAMPLES

4.1. Cyclic p -towers over imaginary quadratic fields. The following program PARI/GP [14] deals with $\#S = 4$ and $\lambda = 3$ in a real cyclic 2-tower of degree 4 in the compositum $k\mathbb{Q}(\mu_{q_1q_2})$, for primes $q_i \equiv 1 \pmod{8}$; the polynomial P is obtained via `polsubcyclo(q1*q2,4)` to give total ramification of S and $G := \text{Gal}(K/k) \simeq \mathbb{Z}/4\mathbb{Z}$. The field k_1 is the compositum $k\mathbb{Q}(\sqrt{q_1q_2})$. The computation for the layer k_2 is only for verification (the execution time becomes unaffordable for higher degrees). One obtains examples with non-trivial modulus, as shown by the excerpt hereafter ($\mathbf{m} = (5)$ and $\mathbf{m} = (13)$ giving -1 norm); in fact, the modulus $\mathbf{m} = 1$ is suitable from Lemma 1.2 and the PARI/GP verification $-1 = N_{K/k}(5717/36x^3 + 18397/9x^2 - 150695/12x - 3658)$:

```
{p=2;q1=41;q2=17;P=x^4-x^3-261*x^2+1089*x+324;mod=5;for(m=2,10^4,
if(core(m)!=m || kronecker(-m,q1)!=1 || kronecker(-m,q2)!=1,next);Q=x^2+m;
k0=bnfinit(Q);k0mod=bnrinit(k0,mod);v0=valuation(k0mod.no,p);Ck0=k0mod.cyc;
R1=polcompositum(x^2-q1*q2,Q)[1];k1=bnfinit(R1,1);k1mod=bnrinit(k1,mod);
v1=valuation(k1mod.no,p);Ck1=k1mod.cyc;if(v1!=v0+3,next);print(-m);
R2=polcompositum(P,Q)[1];k2=bnfinit(R2,1);k2mod=bnrinit(k2,mod);
v2=valuation(k2mod.no,p);Ck2=k2mod.cyc;print("v0=",v0," Ck0=",Ck0);
print("v1=",v1," Ck1=",Ck1);print("v2=",v2," Ck2=",Ck2)}
```

m=-59 mod=(1)	m=-115 mod=(1)	m=-6995 mod=(1)
v0=0 [3]	v0=1 [2]	v0=5 [32]
v1=3 [18,6,6]	v1=4 [84,2,2]	v1=8 [17088,2,2]
v2=6 [7956,12,12]	v2=7 [23016,4,4]	v2=11 [1187445120,4,4]
m=-115 mod=(5)	m=-155 mod=(5)	m=-695 mod=(5)
v0=2 [20]	v0=3 [20,2]	v0=4 [120, 2]
v1=5 [1260,10,2,2]	v1=6 [360,90,2,2]	v1=7 [69840,30,2,2]
v2=8 [345240,20,20,10]	v2=9 [867600,180,20,10]	v2=10 [4968836640,60,20,10]
m=-247 mod=(13)	m=-2327 mod=(13)	m=-2483 mod=(13)
v0=2 [156,3]	v0=5 [624,6]	v0=3 [780,2]
v1=5 [3588,78,6,2]	v1=8 [116064,78,6,6]	v1=6 [7800,390,6,2]
v2=8 [27850056,156,156,26]	v2=11 [17083924416,156,156,78]	v2=9 [19172400,780,780,26]

4.2. Cyclic p -towers over real fields with $p = 3$ and $\mathcal{X} = \mathcal{C}$. (i) Let $k = \mathbb{Q}(\sqrt{m})$, $m > 0$. Then $X = \langle \varepsilon \rangle$, $\rho = 1$, $\lambda = 0$. Let $K \subset k(\mu_{109})$ of degree 27 over k

with 109 split in k ; we search equalities $\#\mathcal{X}_1 = \#\mathcal{X}$. For instance $m = 326$ gives the structures [3], [6, 2], [6, 2] and $m = 4409$ gives [9], [9], [9]; so, a consequence is that ε is non-norm in k_1/k for these quadratic fields.

(ii) Let's give examples for cyclic cubic fields k . Then $X = E$ is of \mathbb{Z} -rank $\rho = 2$, $\lambda = 0$. Let $K \subset k(\mu_{109})$ with 109 split in k . We find the following examples for which we indicate the defining polynomial of the cubic field of conductor f ; as for the real quadratic cases, the units of k are necessarily non-norms in k_1/k :

f=1687	f=1897	f=2359	f=5383
$x^3+x^2-562x-4936$	$x^3+x^2-632x-4075$	$x^3+x^2-786x+7776$	$x^3+x^2-1794x+17744$
v0=1 [39]	v0=1 [3]	v0=2 [3, 3]	v0=3 [9, 3]
v1=1 [78, 2]	v1=1 [3]	v1=2 [3, 3]	v1=3 [9, 3]

4.3. Cyclic p -towers over \mathbb{Q} , with $p = 3$ and non-trivial modulus \mathfrak{m} . Some examples in [13, §4] are cyclic p -towers K/\mathbb{Q} with $K \subseteq \mathbb{Q}(\mu_f)$, f prime. The following program, for $p = 3$, allows such illustrations with the modulus $\mathfrak{m} = (7 \cdot 13)$, $f \equiv 1 \pmod{3^4}$; the stabilization occurs at the layer $n_0 = 1$ in K/k of degree 3^4 (of course, $\#S = 1$, $\lambda = 0$ whatever the base field $k' \subset K$):

f=163 [12, 3], [36, 6, 2], [36, 6, 2], [36, 6, 2]	f=1297 [12, 3], [36, 3], [36, 3], [36, 3]
f=487 [12, 3], [36, 3], [36, 3], [36, 3]	f=1459 [12, 3], [468, 3], [8892, 3], ...

For $f = 5023$, because of the strict increasing of the $\#\mathcal{C}_n$, $n \in [0, 4]$ (structures [12, 3], [12, 12, 12, 3], [36, 12, 12, 3, 3, 3]), the stabilization is unknown (if any).

5. OTHER APPLICATIONS

Let's give examples, with $\mathfrak{m} = 1$, under the condition $\#(\mathcal{X}_1) = \#(\mathcal{X}) \cdot p^\lambda$, with the total ramification of $S \neq \emptyset$ and $\lambda = \max(0, \#S - 1 - \text{rk}_{\mathbb{Z}}(X))$ (see Remark 1.3):

(i) Let Σ be a set of primes of k , totally split in K ; $\mathcal{H}_n = \mathcal{C}_n(\Sigma_n)$, $\Sigma_n \mid \Sigma$ in k_n , gives the λ -stability of the Σ_n -class groups (then that of the Σ_n -split p -Hilbert class fields of the layers).

(ii) If \tilde{k}/k is a \mathbb{Z}_p -extension (thus $\#S \leq [k : \mathbb{Q}] = r_1 + 2r_2$), one gets $\#\mathcal{C}_n = \#\mathcal{C} \cdot p^{\lambda \cdot n}$ for all n where $\lambda = \max(0, \#S - 1 - (r_1 + r_2 - 1))$ is the corresponding Iwasawa invariant and $\mu = 0$. For CM-fields with p totally split in k , $\lambda = \frac{1}{2}[k : \mathbb{Q}]$.

(iii) Taking $\mathcal{H}_n = \mathcal{C}_n^{p^e}$ may yield the λ^e -stability of the $\mathcal{C}_n/\mathcal{C}_n^{p^e}$'s, for some λ^e 's, whence $\text{rk}_p(\mathcal{C}_n) = \text{rk}_p(\mathcal{C}) + \lambda^1 \cdot n$, for all n , under this condition for $n = 1$.

(iv) If K/\mathbb{Q} is abelian with $\Delta := \text{Gal}(k/\mathbb{Q})$ of prime to p order, similar statements hold for the isotopic $\mathbb{Z}_p[\Delta]$ -components of the \mathcal{X}_n 's, using fixed points formulas [8, 9].

(v) Put $\mu_p(k) =: \mu_{p^e}$, $e \geq 0$. Consider primes $\ell \equiv 1 \pmod{p^{N+1+e}}$, $N \gg 0$, and let L be the maximal subfield of $\mathbb{Q}(\mu_\ell)$ of degree p^N . Let's take the cyclic p -tower $K = kL$; from Lemma 1.2, $\mu_p(k) \subset N_{K/k}(K^\times)$. Assume that $\lambda = 0$ (or that it is the case for a layer, still denoted k , which makes sense since λ decreases in the tower).

So, either there exists $n_0 \geq 0$ such that $\#\mathcal{X}_{n_0+1} = \#\mathcal{X}_{n_0}$, giving the stability along K/k_{n_0} or, for all $n \in [1, N]$, $\#\mathcal{X}_{n+1} > \#\mathcal{X}_n$ giving, roughly, at least $\#\mathcal{X}_N = O(p^N)$, perhaps much more; unfortunately, numerical experiments are out of reach. However, a reasonable conjecture is a tendency to stabilization, by analogy with the study of Greenberg's conjecture (for totally real fields) [7] carried out in [5].

REFERENCES

- [1] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux* (Thèse), Jour. of the Faculty of Sciences Tokyo **2** (1933), 365–476. <http://eudml.org/doc/192833> 3
- [2] T. Fukuda, *Remarks on \mathbb{Z}_p -extensions of number fields*, Proc. Japan Acad. Ser. A **70**(8) (1994), 264–266. <https://doi.org/10.3792/pjaa.70.264> 2
- [3] G. Gras, *Classes généralisées invariantes*, J. Math. Soc. Japan **46**(3) (1994), 467–476. <http://projecteuclid.org/euclid.jmsj/1227104692> 3
- [4] G. Gras, *Invariant generalized ideal classes—Structure theorems for p -class groups in p -extensions*, Proc. Math. Sci. **127**(1) (2017), 1–34. <http://doi.org/10.1007/s12044-016-0324-1> 3
- [5] G. Gras, *Algorithmic complexity of Greenberg’s conjecture* (2021). <https://arxiv.org/abs/2004.06959v3> 7
- [6] G. Gras, *Class Field Theory: from theory to practice*, corr. 2nd ed. Springer Monographs in Mathematics, Springer, xiii+507 pages (2005). 6
- [7] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. **98**(1) (1976), 263–284. <https://doi.org/10.2307/2373625> 7
- [8] J.-F. Jaulent, *L’arithmétique des ℓ -extensions* (Thèse d’Etat), Pub. Math. Fac. Sci. Besançon, Théorie des Nombres 1985/86 (1986). https://pmb.centre-mersenne.org/item/PMB_1986__1_A1_0/ 3, 7
- [9] J.-F. Jaulent, *Théorie ℓ -adique globale du corps de classes*, J. Théorie des Nombres de Bordeaux **10**(2) (1998), 355–397. http://www.numdam.org/article/JTNB_1998__10_2_355_0.pdf 7
- [10] J. Li, Y. Ouyang, Y. Xu, S. Zhang, *ℓ -Class groups of fields in Kummer towers* (2020). <https://arxiv.org/pdf/1905.04966v2.pdf> 2
- [11] J. Li, C.F. Yu, *The Chevalley–Gras formula over global fields*, J. Théorie des Nombres de Bordeaux **32**(2) (2020), 525–543. http://jtnb.centre-mersenne.org/item?id=JTNB_2020__32_2_525_0 3
- [12] Y. Mizusawa, *Tame pro-2 Galois groups and the basic \mathbb{Z}_2 -extension*, Trans. Amer. Math. Soc. **370**(4) (2018), 2423–2461. <https://doi.org/10.1090/tran/7023> 3
- [13] Y. Mizusawa, K. Yamamoto, *On p -class groups of relative cyclic p -extensions* (2020). <https://arxiv.org/abs/2003.06546> 3, 7
- [14] The PARI Group, *PARI/GP, version 2.9.0*, Université de Bordeaux (2016). <http://pari.math.u-bordeaux.fr/> 6

VILLA LA GARDETTE, CHEMIN CHÂTEAU GAGNIÈRE, F-38520 LE BOURG D’OISANS
 Email address: g.mn.gras@wanadoo.fr