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ON THE p -CLASS GROUP STABILITY ALONG CYCLIC p -TOWERS OF A NUMBER FIELD

GEORGES GRAS

ABSTRACT. Let k be a number field and let $p \geq 2$ be a prime number. We call K/k a cyclic p -tower if $\text{Gal}(K/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$. We give an elementary proof of a stability theorem for generalized p -class groups \mathcal{X}_n along the tower (Theorem 3.1). Using generalizations of Chevalley’s formula (Gras, J. Math. Soc. Japan **46**(3) (1994), Proc. Math. Sci. **127**(1) (2017)), we improve results by Fukuda (1994), Li–Ouyang–Xu–Zhang (2020), Mizusawa–Yamamoto (2020) and others, whose techniques are based on Iwasawa’s theory, pro- p -group Galois theory or specific methods. The main difference, compared to these works, is that we introduce a parameter $\lambda \geq 0$ giving formulas of the form $\mathcal{X}_n = \mathcal{X}_0 \cdot p^{\lambda \cdot n}$, $n \in [0, N]$.

1. INTRODUCTION

Let k be a number field, let $p \geq 2$ be a prime number. Let K be a cyclic extension of degree p^N of k and let k_n , $n \in [0, N]$, be the degree p^n extension of k in K . We speak of cyclic p -towers K/k and put $G = \text{Gal}(K/k)$. Let S be the set of finite places of k , ramified in K/k . In the sequel we will assume $S \neq \emptyset$ and S *totally ramified* in K/k ; in general, S may contain tame places, except if K/k is contained in a \mathbb{Z}_p -extension of k , in which case S is necessarily a set of p -places of k . Due to the very nature of a tower, there will never be “complexification=ramification” of infinite real places (indeed, such places cannot be *totally ramified* since, in practice, $N \geq 2$); therefore, it will be easy to use both ordinary or restricted sense for $p = 2$.

Let \mathfrak{m} be an integer ideal of k , *prime to* S ; let $\mathcal{C}_{k,\mathfrak{m}}$, $\mathcal{C}_{k_n,\mathfrak{m}}$, denoted simply $\mathcal{C}_{0,\mathfrak{m}}$ and $\mathcal{C}_{n,\mathfrak{m}}$ be the p -Sylow subgroups of the ray class groups modulo \mathfrak{m} , of k and k_n , respectively (for $n > 0$, \mathfrak{m} is seen as extended ideal in k_n). The class of an ideal \mathfrak{a} is denoted $\mathcal{C}_{n,\mathfrak{m}}(\mathfrak{a})$, $n \in [0, N]$.

Let $(\mathcal{H}_n)_{n \in [0, N]}$ be a family of sub- G -modules of the class groups $\mathcal{C}_{n,\mathfrak{m}}$ such that $N_{k_n/k_{n-1}}(\mathcal{H}_n) = \mathcal{H}_{n-1}$, for all $n \in [1, N]$, and let \mathcal{I}_n be any G -module of prime to S ideals of k_n such that $\mathcal{H}_n = \mathcal{C}_{n,\mathfrak{m}}(\mathcal{I}_n)$ (one can take $\mathcal{I}_n = N_{K/k_n}(\mathcal{I}_N)$).

We consider the following “generalized p -class groups”:

$$\mathcal{X}_n := \mathcal{C}_{n,\mathfrak{m}}/\mathcal{H}_n, \quad \text{for } n \in [0, N].$$

Since \mathfrak{m} is prime to S , the corresponding p -Hilbert’s ray class fields are linearly disjoint from the layers of the tower; thus we have $N_{k_n/k_{n-1}}(\mathcal{X}_n) = \mathcal{X}_{n-1}$ for all $n \in [1, N]$. To simplify, we shall remove the indices \mathfrak{m} in all the forthcoming notations, except necessity.

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One speaks of “ λ -stability” of these generalized p -class groups along the tower K/k when there exists a constant $\lambda \geq 0$ such that:

$$\#\mathcal{X}_n \simeq \#\mathcal{X}_0 \cdot p^{\lambda n}, \text{ for all } n \in [0, N].$$

Definition 1.1. Let $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in \mathbb{N}_{K/k}(\mathcal{I}_N)\}$. For instance, when $\mathcal{X}_n := \mathcal{C}_n$ (i.e., $\mathcal{H}_n = 1$ for all n), $\Lambda = E$, the group of global units $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$ of k . We shall see, from Lemma 2.2 on the norm properties of Λ , that *in some sense* it is independent of any layer in K , thus an invariant of the base field k , replacing E in the more general context defined by the \mathcal{H}_n 's.

Since class groups are finite, the ideal groups \mathcal{I}_n may be finitely generated and, due to the inclusion $E \subseteq \Lambda$, we can write:

$$\Lambda = \langle \zeta, \varepsilon_1, \dots, \varepsilon_r; \alpha_1, \dots, \alpha_t \rangle_{\mathbb{Z}},$$

where $\text{tor}_{\mathbb{Z}_p}(E) = \langle \zeta \rangle_{\mathbb{Z}}$, $r = r_1 + r_2 - 1$ as usual, and $t \geq 0$ is the p -rank of \mathcal{H}_0 .

For $n = 0$, we write $k_0 =: k$, $\mathcal{H}_0 =: \mathcal{H}$, $\mathcal{I}_0 =: \mathcal{I}$, $\mathcal{X}_0 =: \mathcal{X}$, and so on.

Under the assumptions about S , \mathfrak{m} , and the family $(\mathcal{H}_n)_{n \in [0, N]}$, we will prove the following result (see Theorem 3.1 and Section 5):

Main Theorem. Write Λ under the form $\langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$, with $\rho \geq 0$ minimal, and put $\lambda := \max(0, \#S - 1 - \rho)$. The λ -stability of the \mathcal{X}_n 's along K/k (i.e., $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda n}$ for all $n \in [0, N]$), is equivalent to $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$.

For known results (all relative to $\lambda = 0$), one may cite Fukuda [2] using Iwasawa's theory, Li–Ouyang–Xu–Zhang [7, §3] working in a non-abelian Galois context, in Kummer towers, via the use of the fixed points formulas [3, 4], then Mizusawa [9] above \mathbb{Z}_2 -extensions, and Mizusawa–Yamamoto [10] for generalizations, including ramification and splitting conditions, via the Galois theory of pro- p -groups.

The present work gives, in an elementary way, the above “ λ -generalization”, only using classical fixed points formulas issued from Chevalley's one.

2. AROUND CHEVALLEY'S AMBIGUOUS CLASS NUMBER FORMULA

The well-known Chevalley's formula [1, p.402] (also known as the Herbrand–Chevalley formula) is the pivotal element for a great lot of “fixed point formulas”. For the ordinary sense, it takes into account the “complexification” of real infinite places which does not occur for us as explained in the Introduction.

Let L/F be cyclic of Galois group G , S -ramified, non-complexified at infinity and let $e_v(L/F)$ be the ramification index of $v \in S$:

$$\#((\mathcal{C}_L)^G) = \frac{\#\mathcal{C}_F \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot (E_F : E_F \cap \mathbb{N}_{L/F}(L^\times))}.$$

Various generalizations of this formula were given (Gras [3, Théorème 2.7, Corollaire 2.8], [4, Theorem 3.6, Corollaries 3.7, 3.9], Jaulent [5, Chap. II, §3]); mention the recent idelic proof by Li–Yu [8, Theorem 2.1 & §2.3 Examples] of such formulas.

Proposition 2.1. Let \mathfrak{m} be a modulus of F , prime to S and let $\mathcal{C}_{F, \mathfrak{m}}$, $\mathcal{C}_{L, \mathfrak{m}}$, be the ray class groups modulo \mathfrak{m} , of F and L , respectively (ordinary sense). For any sub- G -module \mathcal{H}_L of $\mathcal{C}_{L, \mathfrak{m}}$ and any G -module \mathcal{I}_L of prime to S ideals of L such that $\mathcal{H}_L = \mathfrak{d}_{L, \mathfrak{m}}(\mathcal{I}_L)$, we have (where $\mathbb{N}_{L/F}$ is the arithmetic norm):

$$\#((\mathcal{C}_{L, \mathfrak{m}}/\mathcal{H}_L)^G) = \frac{\#\mathcal{C}_{F, \mathfrak{m}} \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot \#\mathbb{N}_{L/F}(\mathcal{H}_L) \cdot (\Lambda_{L/F} : \Lambda_{L/F} \cap \mathbb{N}_{L/F}(L^\times))},$$

where $\Lambda_{L/F} := \{x \in F^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{L/F}(\mathcal{I}_L)\}$. If $S \neq \emptyset$ is totally ramified in L/F , and if $N_{L/F}(\mathcal{H}_L) =: \mathcal{H}_F$, the formula becomes:

$$(2.1) \quad \#((\mathcal{C}_{L,\mathfrak{m}}/\mathcal{H}_L)^G) = \#(\mathcal{C}_{F,\mathfrak{m}}/\mathcal{H}_F) \times \frac{(\#G)^{\#S-1}}{(\Lambda_{L/F} : \Lambda_{L/F} \cap N_{L/F}(L^\times))}.$$

Lemma 2.2. *Let K/k be a cyclic p -tower of degree p^N of Galois group $G =: \langle \sigma \rangle$. Let $\mathcal{H}_n =: \mathcal{d}_n(\mathcal{I}_n) \subseteq \mathcal{C}_n$ be a family of sub- G -modules as above.*

Let $\Lambda_{k_n/k} := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{k_n/k}(\mathcal{I}_n)\}$ and (Definition 1.1) let $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{I}_N)\}$. Then:

- (i) $\Lambda_{k_n/k} \cdot N_{k_n/k}(k_n^\times) = \Lambda \cdot N_{k_n/k}(k_n^\times)$,
- (ii) $\Lambda \subseteq \Lambda_1 := \{x \in k_1^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k_1}(\mathcal{I}_N)\}$.

Proof. (i) Let $x \in \Lambda_{k_n/k}$. Thus $(x) = N_{k_n/k}(\mathfrak{a})$, $\mathfrak{a} \in \mathcal{I}_n$; since $\mathcal{H}_n = N_{K/k_n}(\mathcal{H}_N)$, there exists $\mathfrak{A} \in \mathcal{I}_N$ such that $\mathfrak{a} = (\alpha) \cdot N_{K/k_n}(\mathfrak{A})$, with $\alpha \in k_n^\times$, $\alpha \equiv 1 \pmod{\mathfrak{m}}$, whence $(x) = (N_{k_n/k}(\alpha)) \cdot N_{K/k}(\mathfrak{A})$, proving the non-trivial inclusion.

(ii) Let $x \in \Lambda$; then $(x) = N_{K/k}(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{I}_N$. Let $J_{K/k}, J_{k_1/k}$ be the transfer maps (via extension of ideals). Then $J_{K/k}((x)) = \mathfrak{A}^{\nu_{K/k}}$, where $\nu_{K/k} = \sum_{i=0}^{N-1} \sigma^i = \Theta \cdot \nu_{K/k_1}$, with $\Theta = \sum_{i=0}^{p-1} \sigma^i$, thus:

$$J_{K/k}((x)) = \mathfrak{A}^{\Theta \cdot \nu_{K/k_1}} =: \mathfrak{B}^{\nu_{K/k_1}} = J_{K/k_1}(N_{K/k_1}(\mathfrak{B})), \quad \mathfrak{B} \in \mathcal{I}_N.$$

Since J_{K/k_1} is injective on ideal groups, from $J_{K/k}((x)) = J_{K/k_1}(J_{k_1/k}((x)))$, we get $(x)_{k_1} = N_{K/k_1}(\mathfrak{B})$ considering $x \in K_1$, whence the claim. \square

Lemma 2.3. *Let $X = \langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$ be a \mathbb{Z} -module with $\rho \geq 0$ minimal, let $\Omega := (\mathbb{Z}/p^n\mathbb{Z})^\delta$, $n \geq 1$, $\delta \geq 0$, and let $f : X \rightarrow \Omega$ be an homomorphism such that the image of $f_1 : X \xrightarrow{f} \Omega/\Omega^p$ be of \mathbb{F}_p -dimension $\min(\rho, \delta)$. Then $\#f(X) = p^{\min(\rho, \delta) \cdot n}$ and for all $m \in [0, n]$, $\#f^{p^m}(X) = p^{\min(\rho, \delta) \cdot (n-m)}$ (where $f^{p^m}(x) := f(x)^{p^m}$).*

Proof. Let $M := \langle e_1, \dots, e_\delta \rangle_{\mathbb{Z}}$, free of \mathbb{Z} -rank δ , and replace Ω by M/M^{p^n} . Put $f(x_i) = \prod_{j=1}^{\delta} q(e_j)^{a_{i,j}}$, $1 \leq i \leq \rho$, $a_{i,j} \in \mathbb{Z}$, where $q : M \rightarrow M/M^{p^n}$ is the canonical map; let $F : X \rightarrow M$ be the map defined by $F(x_i) = \prod_{j=1}^{\delta} e_j^{a_{i,j}}$, so that $f = q \circ F$, and let A be the matrix $(a_{i,j})_{1 \leq i \leq \rho, 1 \leq j \leq \delta}$ defined modulo p^n .

By assumption, the image of $F_1 : X \rightarrow M/M^p$ (which coincides with that of f_1 since $n \geq 1$) is of \mathbb{F}_p -dimension $\min(\rho, \delta)$ and defined via the $\mathbb{Z}/p\mathbb{Z}$ -matrix $\overline{A} := (\overline{a}_{i,j})_{i,j}$ of rank $\min(\rho, \delta)$; thus (for instance) the sub-determinant $\overline{D} = |\overline{a}_{i,j}|_{1 \leq i,j \leq \min(\rho, \delta)}$ is invertible and $D = |a_{i,j}|_{1 \leq i,j \leq \min(\rho, \delta)}$ is a p -adic unit.

The case $\rho \geq \delta$ being obvious (f is surjective and $\#f(X) = p^{\delta \cdot n}$), assume $\rho < \delta$. It follows that $F(X) \otimes \mathbb{Z}_p$ is a direct factor in $M \otimes \mathbb{Z}_p$ of \mathbb{Z}_p -rank ρ (indeed, if $y = \prod_{j=1}^{\delta} e_j^{u_j}$ with $y^p = \prod_{i=1}^{\rho} F(x_i)^{v_i}$, the linear system $\sum_{i=1}^{\rho} a_{i,j} v_i = pu_j$, $1 \leq j \leq \rho$, gives, by identification, $v_i \in p\mathbb{Z}$ for all i , whence $y \in M$). This defines the map $F_n : X \xrightarrow{f} M/M^{p^n}$ whose image is a direct factor of M/M^{p^n} , isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\rho$, whence $\#f(X) = p^{\rho \cdot n}$. Thus f^{p^m} applies X onto $(\mathbb{Z}/p^{n-m}\mathbb{Z})^\rho$. \square

3. MAIN THEOREM

Let K/k be a cyclic p -tower. Let $\omega_{k_n/k}$ be the map which associates with $x \in \Lambda$ the family of Hasse's norm symbols $(\frac{x, k_n/k}{v}) \in I_v(k_n/k)$ (inertia groups of $v \in S$).

Being norm of an ideal, x is a local norm at the places $v \notin S$ and the image of $\omega_{k_n/k}$ is contained in:

$$\Omega_{k_n/k} := \left\{ (\tau_v)_{v \in S} \in \bigoplus_{v \in S} I_v(k_n/k), \prod_{v \in S} \tau_v = 1 \right\}.$$

We assume $S \neq \emptyset$ totally ramified; so $\Omega_{k_n/k} \simeq \bigoplus_{v \neq v_0} G_n$, with $G_n := \text{Gal}(k_n/k)$ and for any $v_0 \in S$. We consider formulas (2.1) in k_n/k , using Lemma 2.2 (i):

$$(3.1) \quad \#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot \frac{p^{n(\#S-1)}}{(\Lambda : \Lambda \cap N_{k_n/k}(k_n^\times))}.$$

where $\mathcal{X}_n := \mathcal{C}_{n,m}/\mathcal{H}_n$ for the given \mathcal{H}_n 's, with \mathfrak{m} prime to S .

Theorem 3.1. *Let $\Lambda := \{x \in k^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k}(\mathcal{I}_N)\}$ written on the form $\langle x_1, \dots, x_\rho \rangle_{\mathbb{Z}}$, with ρ minimal, and let $\lambda := \max(0, \#S - 1 - \rho)$. We then have the following properties of λ -stability only depending on the layer k_1/k :*

(i) *If $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$, then for all $n \in [0, N]$, $\#\mathcal{X}_n = \#\mathcal{X} \cdot p^{\lambda n}$ and $\mathcal{X}_n^{G_n} = \mathcal{X}_n$.*

(ii) *Under the condition $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ with $\lambda := \max(0, \#S - 1 - \rho)$, we have $J_{k_n/k}(\mathcal{X}) = \mathcal{X}_n^{p^n}$ and $\text{Ker}(J_{k_n/k}) = \mathcal{X}[p^n]$, for all $n \in [0, N]$, where the $J_{k_n/k}$'s are the transfer maps. If \tilde{k}/k is a \mathbb{Z}_p -extension, \mathcal{X} capitulates in some $K \subset \tilde{k}$.*

Proof. Put $N_n := N_{k_n/k}(k_n^\times)$, $G_n := \langle \sigma_n \rangle$, $\Omega_n := \Omega_{k_n/k}$, $\omega_n := \omega_{k_n/k}$, and $s := \#S$.

We have (from (3.1)) $\#\mathcal{X}_1^{G_1} = \#\mathcal{X} \cdot \frac{p^{s-1}}{\#\omega_1(\Lambda)} \geq \#\mathcal{X} \cdot p^\lambda$ since $\#\omega_1(\Lambda) \leq p^{\min(\rho, s-1)}$;

thus, by assumption, we obtain $\mathcal{X}_1^{G_1} = \mathcal{X}_1$; whence $\#\mathcal{X}_1^{G_1} = \#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ and $\#\omega_1(\Lambda) = p^{\min(\rho, s-1)}$.

Put $\omega_n(x_i) := (\sigma_n^{a_{i,v}})_{v \neq v_0}$, $a_{i,v} \in \mathbb{Z}/p^n\mathbb{Z}$, $1 \leq i \leq \rho$. The restriction to k_1 yields $\omega_1(x_i) = (\bar{\sigma}_n^{a_{i,v}})_{v \neq v_0}$, $1 \leq i \leq \rho$, where $\langle \bar{\sigma}_n \rangle = G_1 \simeq G_n/G_n^p$, so the map $\omega_1 = \bar{\omega}_n$ has values in Ω/Ω^p and we have $\dim_{\mathbb{F}_p}(\omega_1(\Lambda)) = \min(\rho, s-1)$. So Lemma 2.3 applies to $X = \Lambda$, $\delta = s-1$, $f = \omega_n$; thus $\#\omega_n(\Lambda) = p^{\min(\rho, s-1)n}$ and, from (3.1):

$$\#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda \cdot n} \text{ for all } n \in [0, N], \lambda = \max(0, s-1-\rho).$$

Consider, for $n \geq 2$, the extension k_n/k_1 , of Galois group $G_n^p = \langle \sigma_n^p \rangle$, and the map ω'_n on k_1^\times with values in $(G_n^p)^{s-1}$; then $\#\mathcal{X}_n^{G_n^p} = \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{(\Lambda_1 : \Lambda_1 \cap N_{k_n/k_1}(k_n^\times))}$, where $\Lambda_1 := \{x \in k_1^\times, x \equiv 1 \pmod{\mathfrak{m}}, (x) \in N_{K/k_1}(\mathcal{I}_N)\}$. Since $\Lambda \subseteq \Lambda_1$ (Lemma 2.2 (ii)), the functorial properties of Hasse's symbols imply (where $v_1 \mid v$ in k_1):

$$\left(\frac{x_i, k_n/k_1}{v_1} \right) = \left(\frac{N_1(x_i), k_n/k}{v} \right) = \left(\frac{x_i^p, k_n/k}{v} \right) = \left(\frac{x_i, k_n/k}{v} \right)^p = \sigma_n^{pa_{i,v}},$$

giving $\omega'_n = \omega_n^p$ on Λ . Then, $\omega'_n(\Lambda) \subseteq \omega'_n(\Lambda_1)$ and Lemma 2.3 for $f^p = \omega_n^p$, give:

$$\begin{aligned} \#\mathcal{X}_n^{G_n^p} &= \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda_1)} \leq \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{\#\omega'_n(\Lambda)} \\ &= \#\mathcal{X}_1 \cdot \frac{p^{(n-1)(s-1)}}{p^{(n-1)\min(\rho, s-1)}} = \#\mathcal{X}_1 \cdot p^{\lambda \cdot (n-1)} = \#\mathcal{X} \cdot p^{\lambda \cdot n}. \end{aligned}$$

Since $\mathcal{X}_n^{G_n^p} \supseteq \mathcal{X}_n^{G_n}$ and $\#\mathcal{X}_n^{G_n} = \#\mathcal{X} \cdot p^{\lambda \cdot n}$ we get:

$$(3.2) \quad \mathcal{X}_n^{G_n^p} = \mathcal{X}_n^{G_n}.$$

From obvious exact sequences of the form $1 \rightarrow \mathcal{X}^G \rightarrow \mathcal{X}^{1-\sigma} \mathcal{X}^{1-\sigma} \rightarrow 1$, equality (3.2) is equivalent to $\mathcal{X}_n^{1-\sigma_n} = \mathcal{X}_n^{1-\sigma_n^p}$, whence $\mathcal{X}_n^{1-\sigma_n} = \mathcal{X}_n^{(1-\sigma_n) \cdot \theta}$,

where $\theta = 1 + \sigma_n + \cdots + \sigma_n^{p-1} \in (p, 1 - \sigma_n)$, a maximal ideal since $\mathbb{Z}_p[G_n]/(p, 1 - \sigma_n) \simeq \mathbb{F}_p$; so $\mathcal{X}_n^{1-\sigma_n} = 1$, thus $\mathcal{X}_n = \mathcal{X}_n^{G_n}$. Whence point (i). Point (ii) comes from the relation $J_n \circ N_n = \nu_n$. \square

Remark 3.2. *The condition $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ is, in some sense, drastic since it implies $\#\omega_1(\Lambda) = p^{\min(\rho, \#S-1)}$; if for instance $\rho \leq \#S - 1$, a necessary condition is that the x_i 's, defining Λ , be non-norm in k_1/k . If $\rho > \#S - 1$, $\lambda = 0$, but this supposes (for $\mathcal{X} = \mathcal{C}$) $\#S \leq r_1 + r_2$ if $\mu_p(k) \neq 1$. It is clear that $\lambda = \#S - 1 - \rho$ may be arbitrary large. In the literature the statements deal with $\lambda = 0$.*

One may say that, under the condition $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$, there is no “exceptional” p -classes (i.e., non-ambiguous). Recall that the second element of the filtration defined by $\mathcal{C}_1^i := \{c \in \mathcal{C}_1, c^{(1-\sigma_1)^i} = 1\}$ fulfills (with $\mathcal{H}_1^1 = \mathcal{C}_1^1$) the relation $\#(\mathcal{C}_1^2/\mathcal{C}_1^1) = \frac{\#\mathcal{C}}{\#N_1(\mathcal{C}_1^1)} \times \frac{p^{\#S-1}}{\#\omega_1(\Lambda^1)}$, where Λ^1 is defined as usual from \mathcal{H}_1^1 ; in general this order is non-trivial and the filtration increases.

4. NUMERICAL EXAMPLES

If $\#S = 1$, then $\lambda = 0$ whatever ρ , $\Lambda \subset N_{k_n/k}(k_n^\times)$ and $\#\mathcal{C}_n^{G_n} = \#\mathcal{C}$ for all n . Numerical examples show that “any result” is possible as shown in the towers $K = k\mathbb{Q}(\mu_{17})$ and $p = 2$ (resp. $K = k\mathbb{Q}(\mu_{109})$ and $p = 3$) for $k = \mathbb{Q}(\sqrt{m})$ real in which 17 (resp. 109) is inert (vn denotes the p -valuation of $\#\mathcal{C}_n$):

m=10 p=2	m=39 p=2	m=82 p=2
v0=1 Cn=[2]	v0=1 Cn=[2]	v0=2 Cn=[4]
v1=1 Cn=[2]	v1=2 Cn=[2,2]	v1=2 Cn=[4]
v2=1 Cn=[2]	v2=4 Cn=[4,2,2]	v2=2 Cn=[4]
v3=1 Cn=[2]	v3=5 Cn=[4,4,2]	v3=2 Cn=[4]
m=142 p=3	m=1129 p=3	m=5178 p=3
v0=1 Cn=[3]	v0=2 Cn=[9]	v0=1 Cn=[6]
v1=2 Cn=[18,2]	v1=3 Cn=[27]	v1=2 Cn=[6,3]
v2=3 Cn=[54,2]	v2=4 Cn=[81]	

4.1. Cyclic p -towers over imaginary quadratic fields. In the case $\mathfrak{m} = 1$, $\Lambda = \text{tor}_{\mathbb{Z}_p}(k^\times)$; so, for $p > 3$, $\Lambda = 1$, $\rho = 0$, $\lambda = \#S - 1$. For $k = \mathbb{Q}(\sqrt{-1})$ with $p = 2$ (resp. $\mathbb{Q}(\sqrt{-3})$ with $p = 3$) one gets $\Lambda = \langle \zeta \rangle$, ζ of order 4 (resp. 6) giving $\rho = 1$ and $\lambda = \#S - 2$; but one must have $\zeta \notin N_{k_1/k}(k_1^\times)$, otherwise, for $n > 2$, $\omega_{k_n/k}(\zeta)$ is of order less than p^n and examples with $\#\mathcal{X}_1 = \#\mathcal{X} \cdot p^\lambda$ do not exist. For other imaginary quadratic fields and $p = 2$, $\Lambda = \langle -1 \rangle$ and we must have $-1 \notin N_{k_1/k}(k_1^\times)$.

For $\mathfrak{m} \neq 1$, it is easy to obtain $\Lambda = 1$, $\rho = 0$, $\lambda = \#S - 1$ since in that case, roots of unity are not in $E_{\mathfrak{m}}$.

The following program PARI/GP [11] deals with $\#S = 4$ and $\lambda = 3$ in a cyclic 2-tower of degree 4 in the compositum $k\mathbb{Q}(\mu_{q_1 q_2})$, for primes $q_i \equiv 1 \pmod{8}$; the polynomial P is obtained via `polsubcyclo(q1 * q2, 4)` to give total ramification of S and $G := \text{Gal}(K/k) \simeq \mathbb{Z}/4\mathbb{Z}$. The field k_1 is the compositum $k\mathbb{Q}(\sqrt{q_1 q_2})$. The computation for the layer k_2 is only for verification (the execution time becomes unaffordable for higher degrees).

One obtains many examples with non-trivial modulus, as shown by the excerpt hereafter ($\mathfrak{m} = (5)$ and $\mathfrak{m} = (13)$):

```

{p=2;q1=41;q2=17;mod=5;P=x^4-x^3-261*x^2+1089*x+324;for(m=2,10^4,
if(core(m)!=m || kronecker(-m,q1)!=1 || kronecker(-m,q2)!=1,next);Q=x^2+m;
k0=bnfinit(Q);k0mod=bnrinit(k0,mod);v0=valuation(k0mod.no,p);Ck0=k0mod.cyc;
R1=polcompositum(x^2-q1*q2,Q)[1];k1=bnfinit(R1,1);k1mod=bnrinit(k1,mod);
v1=valuation(k1mod.no,p);Ck1=k1mod.cyc;if(v1!=v0+3,next);print(m);
R2=polcompositum(P,Q)[1];k2=bnfinit(R2,1);k2mod=bnrinit(k2,mod);
v2=valuation(k2mod.no,p);Ck2=k2mod.cyc;print("v0=",v0," Ck0=",Ck0);
print("v1=",v1," Ck1=",Ck1);print("v2=",v2," Ck2=",Ck2)}}
m=115                m=155                m=695
v0=2 [20]            v0=3 [20,2]            v0=4 [120, 2]
v1=5 [1260,10,2,2]   v1=6 [360,90,2,2]     v1=7 [69840,30,2,2]
v2=8 [345240,20,20,10] v2=9 [867600,180,20,10] v2=10 [4968836640,60,20,10]

m=247                m=2327                m=2483
v0=2 [156,3]         v0=5 [624,6]          v0=3 [780,2]
v1=5 [3588,78,6,2]   v1=8 [116064,78,6,6]  v1=6 [7800,390,6,2]
v2=8 [27850056,156,156,26] v2=11 [17083924416,156,156,78] v2=9 [19172400,780,780,26]

```

4.2. Cyclic p -towers over real quadratic fields with $p = 3$. For $m = 1$ and $p = 3$, $\Lambda = \langle \varepsilon \rangle$. Thus $\rho = 1$, $\lambda = \#S - 2$. For instance, let $K \subset k(\mu_{109})$ of degree 27, $p = 3$ and 109 split in $k = \mathbb{Q}(\sqrt{m})$; thus $\lambda = 0$ and we search equalities $\#\mathcal{X}_1 = \#\mathcal{X}$. Such examples are not always possible if ε is norm in k_1/k ; on may verify that ε is non-norm in k_1/k using congruences of the form $\varepsilon \equiv a \pmod{\mathfrak{q}}$ and $\varepsilon \equiv b \pmod{\mathfrak{q}'}$, where $S = \{\mathfrak{q}, \mathfrak{q}'\}$. For instance $m = 326$ gives the structures $[3]$, $[6, 2]$, $[6, 2]$ and $m = 4409$ gives $[9]$, $[9]$, $[9]$.

4.3. Cyclic p -towers over cyclic cubic fields with $p = 3$. Let's give examples for cyclic cubic fields k when $K = k_3 \subset \mathbb{Q}(\mu_{109})$, $p = 3$ and 109 split in k ; the λ -stability may work for the $\#\mathcal{C}$'s with $\lambda = 0$ since K is totally real and $\Lambda = E$ is of \mathbb{Z} -rank $\rho = 2$. As for the previous cases, the required equality may hold when the units of k are non-norms in k_1/k ; we find the following examples for which we indicate the defining polynomial of the cubic field of conductor f .

f=1687	f=1897	f=2359	f=5383
$x^3+x^2-562*x-4936$	$x^3+x^2-632*x-4075$	$x^3+x^2-786*x+7776$	$x^3+x^2-1794*x+17744$
v0=1 Cn=[39]	v0=1 Cn=[3]	v0=2 Cn=[3,3]	v0=3 Cn=[9,3]
v1=1 Cn=[78,2]	v1=1 Cn=[3]	v1=2 Cn=[3,3]	v1=3 Cn=[9,3]

4.4. Cyclic p -towers over \mathbb{Q} , with $p = 3$ and non-trivial modulus m . Some examples in [10, § 4] are cyclic p -towers K/\mathbb{Q} with $K \subseteq \mathbb{Q}(\mu_f)$, f prime; some tame ramification corresponds to the choice of the modulus m . The following program, for $p = 3$, allows such illustrations with the modulus $7 \cdot 43$ and the subfield $k = \mathbb{Q}_1$ (of course, $\Lambda = 1$, $\#S = 1$, $\rho = \lambda = 0$):

```

{mod=7*13;p=3;N=3;forprime(f=1,10^3,if(Mod(f,p^N)!=1,next);print();
for(n=1,N,P=polsubcyclo(f,p^n);kn=bnfinit(P,1);knm=bnrinit(kn,mod);
vn=valuation(knm.no,p);Ckm=knm.cyc;print("f=",f," n=",n," vn=",vn," Ckm=",Ckm))}
f=271  v1=3  [36,6,2]  v2=3  [36,6,2]  v3=3  [36,6,2]
f=163  v1=3  [36,6,2]  v2=3  [36,6,2]  v3=3  [36,6,2]
f=487  v1=3  [36,3]    v2=3  [36,3]    v3=3  [36,3]

```

5. OTHER APPLICATIONS

They come from the choice of the \mathcal{H}_n 's, under the usual conditions on S , \mathbf{m} , \mathcal{H} , especially $\#(\mathcal{C}_{1,\mathbf{m}}/\mathcal{H}_1) = \#(\mathcal{C}_{\mathbf{m}}/\mathcal{H}) \cdot p^\lambda$. Let's give some examples with $\mathbf{m} = 1$:

(i) Let Σ be a set of primes of k , totally split in K ; $\mathcal{H}_n = \mathcal{C}_n(\Sigma)$ gives the λ -stability of the Σ -class groups (then that of the Σ -split p -Hilbert class fields).

(ii) If \tilde{k}/k is a \mathbb{Z}_p -extension, one gets $\#\mathcal{C}_n = \#\mathcal{C} \cdot p^{\lambda \cdot n}$ for all n ; thus λ is the corresponding Iwasawa invariant and $\mu = 0$. The case where \tilde{k} is the cyclotomic \mathbb{Z}_p -extension of k totally real, being nothing but Greenberg's conjecture ($\lambda = 0$).

(iii) Taking $\mathcal{H}_n = \mathcal{C}_n^{p^e}$ yields the λ -stability of the $\mathcal{C}_n/\mathcal{C}_n^{p^e}$'s (whence that of the p -ranks, $\text{rk}_p(\mathcal{C}_n) = \text{rk}_p(\mathcal{C}) + \lambda_1 \cdot n$; one only assumes $\text{rk}_p(\mathcal{C}_1) = \text{rk}_p(\mathcal{C}) + \lambda_1$, and the ρ_e 's and λ_e 's are not ρ and λ).

(iv) Suppose that the cyclic p -tower is abelian over \mathbb{Q} and that $\Delta := \text{Gal}(k/\mathbb{Q})$ is of order prime to p ; then similar statements for the isotopic $\mathbb{Z}_p[\Delta]$ -components of the \mathcal{X}_n are immediate using corresponding fixed points formulas [5, 6].

REFERENCES

- [1] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux* (Thèse), Jour. of the Faculty of Sciences Tokyo **2** (1933), 365–476. <http://eudml.org/doc/192833> 2
- [2] T. Fukuda, *Remarks on \mathbb{Z}_p -extensions of number fields*, Proc. Japan Acad. Ser. A **70**(8) (1994), 264–266. <https://doi.org/10.3792/pjaa.70.264> 2
- [3] G. Gras, *Classes généralisées invariantes*, J. Math. Soc. Japan **46**(3) (1994), 467–476. <http://projecteuclid.org/euclid.jmsj/1227104692> 2
- [4] G. Gras, *Invariant generalized ideal classes—Structure theorems for p -class groups in p -extensions*, Proc. Math. Sci. **127**(1) (2017), 1–34. <http://doi.org/10.1007/s12044-016-0324-1> 2
- [5] J.-F. Jaulent, *L'arithmétique des ℓ -extensions* (Thèse d'Etat), Pub. Math. Fac. Sci. Besançon, Théorie des Nombres 1985/86 (1986). https://pmb.centre-mersenne.org/item/PMB_1986__1_A1_0/ 2, 7
- [6] J.-F. Jaulent, *Théorie ℓ -adique globale du corps de classes*, J. Théorie des Nombres de Bordeaux **10**(2) (1998), 355–397. http://www.numdam.org/article/JTNB_1998__10_2_355_0.pdf 7
- [7] J. Li, Y. Ouyang, Y. Xu, S. Zhang, *ℓ -Class groups of fields in Kummer towers* (2020). <https://arxiv.org/pdf/1905.04966v2.pdf> 2
- [8] J. Li, C.F. Yu, *The Chevalley–Gras formula over global fields*, J. Théorie des Nombres de Bordeaux **32**(2) (2020), 525–543. http://jtnb.centre-mersenne.org/item?id=JTNB_2020__32_2_525_0 2
- [9] Y. Mizusawa, *Tame pro-2 Galois groups and the basic \mathbb{Z}_2 -extension*, Trans. Amer. Math. Soc. **370**(4) (2018), 2423–2461. <https://doi.org/10.1090/tran/7023> 2
- [10] Y. Mizusawa, K. Yamamoto, *On p -class groups of relative cyclic p -extensions* (2020). <https://arxiv.org/abs/2003.06546> 2, 6
- [11] The PARI Group, *PARI/GP, version 2.9.0*, Université de Bordeaux (2016). <http://pari.math.u-bordeaux.fr/> 5

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