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# ON THE $p$ -CLASS GROUP STABILITY ALONG CYCLIC $p$ -TOWERS OF A NUMBER FIELD

GEORGES GRAS

ABSTRACT. Let  $k$  be a number field and let  $p \geq 2$  be a prime number. We call  $K/k$  a (pro)-cyclic  $p$ -tower if  $\text{Gal}(K/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$  or  $\mathbb{Z}_p$ . We give an elementary proof of a stability theorem about generalized  $p$ -class groups along the tower (Main Theorem 3.1). This result, using generalizations of Chevalley’s formula (Gras, J. Math. Soc. Japan **46**(3) (1994), Proc. Math. Sci. **127**(1) (2017)), finds again or generalizes results by Fukuda (1994), Li–Ouyang–Xu–Zhang (2020), Mizusawa–Yamamoto (2020) and many others proving particular cases from Iwasawa’s theory, pro- $p$ -group theory or specific methods.

## 1. INTRODUCTION

Let  $k$  be a number field, let  $p \geq 2$  be a prime number. Let  $K$  be a cyclic extension of degree  $p^N$  of  $k$ , or a  $\mathbb{Z}_p$ -extension, and let  $k_n$ ,  $n \geq 0$ , be the degree  $p^n$  extension of  $k$  in  $K$ . To include the two possibilities we speak of cyclic  $p$ -towers  $K/k$ . Let  $S$  be the set of finite places of  $k$ , ramified in  $K/k$ . When  $K/k$  is finite, the set  $S$  may contain tame places, otherwise  $S$  is necessary a set of  $p$ -places of  $k$ . In the sequel we will assume  $S \neq \emptyset$  and  $S$  totally ramified in  $K/k$ .

Let  $\mathfrak{m}$  be an integer ideal of  $k$ , prime to  $S$ , and let  $\mathcal{C}_{k,\mathfrak{m}}^{\text{res}}$ ,  $\mathcal{C}_{k_n,\mathfrak{m}}^{\text{res}}$ , be the  $p$ -Sylow subgroups of the ray class groups modulo  $\mathfrak{m}$ , of  $k$  and  $k_n$ , respectively, in the restricted sense. For  $\mathfrak{m} = 1$ , we obtain the  $p$ -class groups  $\mathcal{C}_k^{\text{res}}$  and  $\mathcal{C}_{k_n}^{\text{res}}$ .

Let  $(\mathcal{H}_{k_n})_{n \geq 0}$  be a family of subgroups of the  $\mathcal{C}_{k_n,\mathfrak{m}}^{\text{res}}$  such that  $N_{k_n/k_{n-1}}(\mathcal{H}_{k_n}) = \mathcal{H}_{k_{n-1}}$  for all  $n \geq 1$ . We consider the following “generalized  $p$ -class groups”:

$$\mathcal{X}_{k_n} := \mathcal{C}_{k_n,\mathfrak{m}}^{\text{res}} / \mathcal{H}_{k_n}, \text{ for all } n \geq 0.$$

One speaks of “stability” of these generalized  $p$ -class groups along the tower when  $\mathcal{X}_{k_n} \simeq \mathcal{X}_k$ , for all  $n \geq 1$ . Under the previous assumptions about  $S$ ,  $\mathfrak{m}$  and the  $\mathcal{H}_{k_n}$ , we will prove the following result (Theorem 3.1):

*The stability of the  $\mathcal{X}_{k_n}$  along the cyclic  $p$ -tower  $K/k$  is equivalent to  $\#\mathcal{X}_{k_1} = \#\mathcal{X}_k$ ; if so, the arithmetic norms  $N_{k_n/k} : \mathcal{X}_{k_n} \rightarrow \mathcal{X}_k$  are isomorphisms, for all  $n \geq 1$ .*

For known results, one may cite Fukuda [2] about  $\mathbb{Z}_p$ -extensions with application to Greenberg’s conjecture, Li–Ouyang–Xu–Zhang [7, §3], moreover computing in Kummer towers via the use of generalizations of Chevalley’s formula that we gave in [3, 4], Mizusawa [9] above  $\mathbb{Z}_2$ -extensions, Mizusawa–Yamamoto [10] for generalizations including conditions of ramification and splitting.

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The present work gives a synthetic approach, in a simpler elementary way, of this kind of results and proposes some applications.

## 2. AROUND CHEVALLEY'S AMBIGUOUS CLASS NUMBER FORMULA

The well-known Chevalley's formula [1, p.402] (also known as the Herbrand–Chevalley formula) is the pivotal element for a great lot of “fixed point formulas”. It was given in the ordinary sense, taking into account the “ramification or complexification” of real infinite places. We shall prefer, when  $p = 2$ , the restricted sense version using totally positive units and only the ramification of the finite places, giving, for  $L/F$  cyclic of Galois group  $G$ :  $\#((\mathcal{C}_L^{\text{res}})^G) = \frac{\#\mathcal{C}_F^{\text{res}} \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot (E_F^{\text{pos}} : E_F^{\text{pos}} \cap N_{L/F}(L^\times))}$ , where  $e_v(L/F)$  is the ramification index.

Various generalizations of this formula were given (Gras [3, Théorème 2.7, Corollaire 2.8], [4, Theorem 3.6, Corollary 3.7], Jaulent [5, Chap. II, §3]); mention the recent idelic proof (Li–Yu [8, Theorem 2.1]) of such formulas. More precisely:

**Theorem 2.1.** *Let  $L/F$  be a cyclic extension of Galois group  $G$ ; let  $\mathfrak{m}$  be an integer ideal of  $F$ , prime to the set  $S$  of finite places ramified in  $L/F$ , and let  $\mathcal{C}_{F,\mathfrak{m}}^{\text{res}}, \mathcal{C}_{L,\mathfrak{m}}^{\text{res}}$  be the ray class groups of modulus  $\mathfrak{m}$ , of  $F$  and  $L$ , respectively. Let  $\kappa_{F,\mathfrak{m}}^\times := \{(x), x \in F^\times, x \equiv 1 \pmod{\mathfrak{m}}, x \gg 0\}$  be the ray group modulo  $\mathfrak{m}$ . Let  $e_v(L/F)$  be the ramification index of  $v \in S$ , in  $L/F$ . For any sub- $G$ -module  $\mathcal{H}_L$  of  $\mathcal{C}_{L,\mathfrak{m}}^{\text{res}}$  and any  $G$ -module  $\mathcal{I}_L$  of prime to  $S$  ideals of  $L$  representing  $\mathcal{C}_{L,\mathfrak{m}}^{\text{res}}$ , we have:*

$$\#((\mathcal{C}_{L,\mathfrak{m}}^{\text{res}}/\mathcal{H}_L)^G) = \frac{\#\mathcal{C}_{F,\mathfrak{m}}^{\text{res}} \cdot \prod_{v \in S} e_v(L/F)}{[L:F] \cdot \#N_{L/F}(\mathcal{H}_L) \cdot (\Lambda_{L/F} : \Lambda_{L/F} \cap N_{L/F}(L^\times))},$$

where  $\Lambda_{L/F} := \{x \in \kappa_{F,\mathfrak{m}}^\times, (x) \in N_{L/F}(\mathcal{I}_L)\}$ . If  $S \neq \emptyset$  is totally ramified in  $L/F$ , and if  $N_{L/F}(\mathcal{H}_L) = \mathcal{H}_F$ , the formula becomes:

$$(2.1) \quad \#((\mathcal{C}_{L,\mathfrak{m}}^{\text{res}}/\mathcal{H}_L)^G) = \#(\mathcal{C}_{F,\mathfrak{m}}^{\text{res}}/\mathcal{H}_F) \times \frac{\#G^{\#S-1}}{(\Lambda_{L/F} : \Lambda_{L/F} \cap N_{L/F}(L^\times))}.$$

**Lemma 2.2.** *Let  $K/k$  be a cyclic  $p$ -tower of degree  $p^N$  of Galois group  $G_N = \langle \sigma_N \rangle$ , and let  $k_n$  be the layers of degrees  $p^n$ ,  $n \in [0, N]$ . Let  $\mathcal{H}_{k_n} =: \mathcal{A}_{k_n,\mathfrak{m}}^{\text{res}}(\mathcal{I}_{k_n}) \subseteq \mathcal{C}_{k_n,\mathfrak{m}}^{\text{res}}$  be a family of sub- $G$ -modules such that  $N_{k_n/k_{n-1}}(\mathcal{H}_{k_n}) = \mathcal{H}_{k_{n-1}}$  for all  $n \geq 1$ . Put  $\Lambda_{k_n/k} := \{x \in \kappa_{k,\mathfrak{m}}^\times, (x) \in N_{k_n/k}(\mathcal{I}_{k_n})\}$ .*

(i)  $\Lambda_{k_n/k} = \Lambda_{K/k} \cdot N_{k_n/k}(\kappa_{k_n,\mathfrak{m}}^\times)$ , where  $\Lambda_{K/k} := \{x \in \kappa_{k,\mathfrak{m}}^\times, (x) \in N_{K/k}(\mathcal{I}_K)\}$ .

(ii)  $\Lambda_{K/k} \subseteq \Lambda_{K/k_1}$ , where  $\Lambda_{K/k_1} := \{x \in \kappa_{k_1,\mathfrak{m}}^\times, (x) \in N_{K/k_1}(\mathcal{I}_K)\}$ .

*Proof.* (i) Let  $x \in \Lambda_{k_n/k}$ . Thus  $(x) = N_{k_n/k}(\mathfrak{a})$ ,  $\mathfrak{a} \in \mathcal{I}_{k_n}$ ; since  $\mathcal{H}_{k_n} = N_{K/k_n}(\mathcal{H}_K)$ , there exists  $\mathfrak{A} \in \mathcal{I}_K$  such that  $\mathfrak{a} = (\alpha) \cdot N_{K/k_n}(\mathfrak{A})$ , with  $\alpha \in \kappa_{k_n,\mathfrak{m}}^\times$ , whence  $(x) = (N_{k_n/k}(\alpha)) \cdot N_{K/k}(\mathfrak{A})$ ; thus,  $x \cdot N_{k_n/k}(\alpha)^{-1} \in \Lambda_{K/k}$ , proving an inclusion, the inclusion  $\Lambda_{K/k} \cdot N_{k_n/k}(\kappa_{k_n,\mathfrak{m}}^\times) \subseteq \Lambda_{k_n/k}$  being obvious.

(ii) Let  $x \in \Lambda_{K/k}$ ; then  $(x) = N_{K/k}(\mathfrak{A})$ ,  $\mathfrak{A} \in \mathcal{I}_K$ . Let  $J_{K/k}, J_{k_1/k}$  be the transfer maps. Then  $J_{K/k}((x)) = \mathfrak{A}^{\nu_{K/k}}$ , where  $\nu_{K/k} = \sum_{i=0}^{p^N-1} \sigma_N^i = \Theta \cdot \nu_{K/k_1}$ , with  $\Theta = \sum_{i=0}^{p-1} \sigma_N^i$ , thus  $J_{K/k}((x)) = \mathfrak{A}^{\Theta \cdot \nu_{K/k_1}} =: \mathfrak{B}^{\nu_{K/k_1}} = J_{K/k_1}(N_{K/k_1}(\mathfrak{B}))$ ,  $\mathfrak{B} \in \mathcal{I}_K$ . Since  $J_{K/k_1}$  is injective on ideal groups, from  $J_{K/k}((x)) = J_{K/k_1}(J_{k_1/k}((x)))$ , we get  $(x)_{k_1} = N_{K/k_1}(\mathfrak{B})$ . Whence the claim.  $\square$

## 3. THE MAIN THEOREM

Let  $K/k$  be a cyclic  $p$ -tower or a  $\mathbb{Z}_p$ -extension. Let  $\omega_{k_n/k}$  be the map which associates with  $x \in \kappa_{k,m}^\times$  the family of Hasse's norm symbols  $\left(\frac{x, k_n/k}{v}\right) \in I_v(k_n/k)$  (inertia groups of  $v \in S$  in  $k_n/k$ ). The image of  $\omega_{k_n/k}$  is contained in  $\Omega_{k_n/k} := \left\{(\tau_v)_{v \in S} \in \bigoplus_{v \in S} I_v(k_n/k), \prod_{v \in S} \tau_v = 1\right\}$ . If  $S \neq \emptyset$  is totally ramified in  $K/k$ , we get  $\Omega_{k_n/k} \simeq \bigoplus_{v \neq v_0} G_n$ , for any  $v_0 \in S$ , where  $G_n := \text{Gal}(k_n/k)$ ; we shall use formulas (2.1) in  $k_n/k$  via Lemma 2.2 (i):  $\#\mathcal{X}_{k_n}^{G_n} = \#\mathcal{X}_k \cdot \frac{p^{n(\#S-1)}}{(\Lambda_{K/k} : \Lambda_{K/k} \cap N_{k_n/k}(k_n^\times))}$ .

Note that  $\kappa_{k,m}^\times \cap N_{k_n/k}(k_n^\times) = N_{k_n/k}(\kappa_{k_n,m}^\times)$  [4, Lemma 3.4], which clarifies the use of the Hasse norm theorem when an element of  $\kappa_{k,m}^\times$  is everywhere local norm.

**Theorem 3.1.** *Let  $K/k$  be a cyclic  $p$ -tower or a  $\mathbb{Z}_p$ -extension and let  $S$  be the set of finite ramified places; one assumes  $S \neq \emptyset$  totally ramified in  $K/k$  and the modulus  $\mathfrak{m}$  prime to  $S$ .<sup>1</sup> Put  $G_n := \text{Gal}(k_n/k) =: \langle \sigma_n \rangle$ . Let  $\mathcal{H}_{k_n} \subseteq \mathcal{C}_{k_n,m}^{\text{res}}$  be sub- $G_n$ -modules such that  $N_{k_n/k_{n-1}}(\mathcal{H}_{k_n}) = \mathcal{H}_{k_{n-1}}$  for all  $n \geq 1$ . Put  $\mathcal{X}_{k_n} := \mathcal{C}_{k_n,m}^{\text{res}}/\mathcal{H}_{k_n}$ .*

*If  $\#\mathcal{X}_{k_1} = \#\mathcal{X}_k$ , then  $\mathcal{X}_{k_n}^{G_n} = \mathcal{X}_{k_n}$  and the norm maps  $N_{k_n/k} : \mathcal{X}_{k_n} \rightarrow \mathcal{X}_k$  are isomorphisms, for all  $n \geq 1$  (stability of the  $\mathcal{X}_{k_n}$  along the tower).*

*Proof.* Put  $N_n := N_{k_n/k}(k_n^\times)$ ,  $\omega_n := \omega_{k_n/k}$ ,  $s := \#S$ . From  $\#\mathcal{X}_{k_1} = \#\mathcal{X}_k$  one gets  $\#\mathcal{X}_{k_1}^{G_1} = \#\mathcal{X}_k$  and  $(\Lambda_{K/k} : \Lambda_{K/k} \cap N_1) = p^{s-1}$ , whence  $\Lambda_{K/k}N_1/N_1 = \langle x_1, \dots, x_{s-1} \rangle N_1/N_1$  of  $\mathbb{F}_p$ -dimension  $s-1$ . Put  $\omega_n(x_i) =: (\sigma_n^{a_{v,i}})_{v \neq v_0}$ ,  $a_{v,i} \in \mathbb{Z}/p^n\mathbb{Z}$ . The restriction to  $k_1$  yields  $\omega_1(x_i) = (\bar{\sigma}_n^{a_{v,i}})_{v \neq v_0}$ , where  $\langle \bar{\sigma}_n \rangle = G_1$ . By assumption,  $\omega_1(\langle x_1, \dots, x_{s-1} \rangle) = \Omega_{k_1/k}$ ; so the  $\mathbb{F}_p$ -matrix  $(a_{v,i} \pmod{p})_{v,i}$  is invertible and the determinant of  $(a_{v,i} \pmod{p^n})_{v,i}$  is thus a  $p$ -adic unit, giving  $\omega_n(\langle x_1, \dots, x_{s-1} \rangle) = \Omega_{k_n/k}$ . Whence  $(\Lambda_{K/k} : \Lambda_{K/k} \cap N_n) = p^{n(s-1)}$  and  $\#\mathcal{X}_{k_n}^{G_n} = \#\mathcal{X}_k$ .

Consider, for  $n \geq 2$ , the extension  $k_n/k_1$ , of Galois group  $G_n^p = \langle \sigma_n^p \rangle$ , for which  $\#\mathcal{X}_{k_n}^{G_n^p} = \#\mathcal{X}_{k_1} \cdot \frac{p^{(n-1)(s-1)}}{(\Lambda_{K/k_1} : \Lambda_{K/k_1} \cap N_{k_n/k_1}(k_n^\times))}$ . Since  $x_i \in \Lambda_{K/k_1}$  (Lemma 2.2 (ii)), the functorial properties of Hasse's symbols imply (where  $v_1 \mid v$  in  $k_1$ ):

$$\left(\frac{x_i, k_n/k_1}{v_1}\right) = \left(\frac{N_1(x_i), k_n/k}{v}\right) = \left(\frac{x_i^p, k_n/k}{v}\right) = \left(\frac{x_i, k_n/k}{v}\right)^p = (\sigma_n^{a_{v,i}})^p = \sigma_n^{pa_{v,i}};$$

this leads to the same matrix modulo  $p^{n-1}$  and gives  $\omega_{k_n/k_1}(\Lambda_{K/k}) = \Omega_{k_n/k_1}$ , whence  $\#\mathcal{X}_{k_n}^{G_n^p} = \#\mathcal{X}_{k_1} = \#\mathcal{X}_k$ . Since  $\mathcal{X}_{k_n}^{G_n^p} \supseteq \mathcal{X}_{k_n}^{G_n}$  and  $\#\mathcal{X}_{k_n}^{G_n} = \#\mathcal{X}_k$  we get:

$$(3.1) \quad \mathcal{X}_{k_n}^{G_n^p} = \mathcal{X}_{k_n}^{G_n}.$$

From obvious exact sequences of the form  $1 \rightarrow \mathcal{X}^G \rightarrow \mathcal{X} \xrightarrow{1-\sigma} \mathcal{X}^{1-\sigma} \rightarrow 1$ , equality (3.1) is equivalent to  $\mathcal{X}_{k_n}^{1-\sigma_n} = \mathcal{X}_{k_n}^{1-\sigma_n^p}$ , whence  $\mathcal{X}_{k_n}^{1-\sigma_n} = \mathcal{X}_{k_n}^{(1-\sigma_n) \cdot \theta}$ , where  $\theta = 1 + \sigma_n + \dots + \sigma_n^{p-1} \in (p, 1 - \sigma_n)$ , a maximal ideal since  $\mathbb{Z}_p[G_n]/(p, 1 - \sigma_n) \simeq \mathbb{F}_p$ ; so  $\mathcal{X}_{k_n}^{1-\sigma_n} = 1$ , thus  $\mathcal{X}_{k_n} = \mathcal{X}_{k_n}^{G_n}$ . Since  $\#\mathcal{X}_{k_n} = \#\mathcal{X}_k$  and  $N_{k_n/k}(\mathcal{X}_{k_n}) = \mathcal{X}_k$ , this proves  $\mathcal{X}_{k_n} = \mathcal{X}_{k_n}^{G_n} \simeq \mathcal{X}_k$ , that is to say,  $\mathcal{C}_{k_n,m}^{\text{res}}/\mathcal{H}_{k_n} = (\mathcal{C}_{k_n,m}^{\text{res}}/\mathcal{H}_{k_n})^{G_n} \simeq \mathcal{C}_{k,m}^{\text{res}}/\mathcal{H}_k$ , and that norms are isomorphisms.  $\square$

<sup>1</sup> So the corresponding  $p$ -Hilbert's ray class fields are linearly disjoint from  $K/k$ .

**Corollary 3.2.** *When Theorem 3.1 applies,  $J_{k_n/k}(\mathcal{X}_k) = \mathcal{X}_{k_n}^{p^n}$  and  $\text{Ker}(J_{k_n/k}) = \mathcal{X}_k[p^n]$ , for all  $n \in [0, N]$ . If moreover  $K/k$  is a  $\mathbb{Z}_p$ -extension,  $\mathcal{X}_k$  capitulates in  $K$ .*

#### 4. EXAMPLES OF APPLICATIONS

They come from the choice of the  $\mathcal{H}_{k_n}$  along the cyclic  $p$ -tower under the conditions:

*Total ramification in  $K/k$  of  $S \neq \emptyset$ ,  $N_{k_n/k}(\mathcal{H}_{k_n}) = \mathcal{H}_{k_{n-1}}$  for all  $n \geq 1$ , and  $\#(\mathcal{C}_{k_1, \mathfrak{m}}^{\text{res}}/\mathcal{H}_{k_1}) = \#(\mathcal{C}_{k, \mathfrak{m}}^{\text{res}}/\mathcal{H}_k)$ .*

Let's give some statements with  $\mathfrak{m} = 1$ :

(i) The case  $\mathcal{H}_{k_n} = 1$  for all  $n$  gives the stability of the  $\mathcal{C}_{k_n}^{\text{res}}$ . If, for a set  $\Sigma$  of primes of  $k$ , totally split in  $K$ ,  $\mathcal{H}_{k_n} = \mathcal{C}_{k_n}^{\text{res}}(\Sigma)$ , one gets the stability of the  $\Sigma$ -class groups corresponding to the Galois groups of the  $\Sigma$ -split  $p$ -Hilbert class fields.

(ii) If  $K/k$  is a  $\mathbb{Z}_p$ -extension, then we conclude, by induction, that the Iwasawa invariants  $\lambda$  and  $\mu$  are zero.

(iii) Taking, for  $e \geq 1$ ,  $\mathcal{H}_{k_n} = (\mathcal{C}_{k_n}^{\text{res}})^{p^e}$  yields the stability of the  $\mathcal{C}_{k_n}^{\text{res}}/(\mathcal{C}_{k_n}^{\text{res}})^{p^e}$  (whence the stability of the  $p$ -ranks; this is distinct from (i) since one only assumes  $\text{rk}_p(\mathcal{C}_{k_1}^{\text{res}}) = \text{rk}_p(\mathcal{C}_k^{\text{res}})$ ).

(iv) Suppose that the cyclic  $p$ -tower is abelian over  $\mathbb{Q}$  and that  $\Delta := \text{Gal}(k/\mathbb{Q})$  is of order prime to  $p$ ; then similar statements for the isotopic  $\mathbb{Z}_p[\Delta]$ -components of the class groups are immediate using corresponding fixed points formulas [5, 6].

(v) If  $\#S = 1$ , the stability occurs for the  $\mathcal{C}_{k_n}^{\text{res}}$  as soon as  $(\mathcal{C}_{k_1}^{\text{res}})^{G_1} = \mathcal{C}_{k_1}^{\text{res}}$ .

(vi) Let  $k_1/k$  be a cyclic extension of degree  $p$ , totally ramified at the  $p$ -places, for which  $\#\mathcal{C}_{k_1}^{\text{res}} = \#\mathcal{C}_k^{\text{res}}$ , and suppose that this extension is embeddable in a  $p$ -cyclic extension of arbitrary large degree. Then this well-known situation implies that  $k_1$  is contained in the Bertrandias–Payan field  $H_k^{\text{bp}}$ ; for the convenience of the reader, recall the definition:

Let  $\mathcal{T}_k = \text{Gal}(H_k^{\text{pr}}/\tilde{k})$  be the torsion group of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension  $H_k^{\text{pr}}$  of  $k$ , where  $\tilde{k}$  is the compositum of the  $\mathbb{Z}_p$ -extensions of  $k$ . Let  $U_k := \bigoplus_{v \in S} U_v$ , where  $U_v$  is the group of  $v$ -principal units of the  $v$ -completion  $k_v$  of  $k$ . Let  $\mu_k$  (resp.  $\mu_v$ ) be the group of  $p$ th roots of unity of  $k$  (resp.  $k_v$ ). Then  $\mathcal{W}_k := (\bigoplus_{v \in S} \mu_v)/\mu_k \subseteq \mathcal{T}_k$  fixes  $H_k^{\text{bp}}$  [6, Exemple 2.9, §2.2].

This gives unbounded cyclic  $p$ -towers in which  $\mathcal{C}_{k_n}^{\text{res}} \simeq \mathcal{C}_k^{\text{res}}$  for all  $n \geq 1$  as the following example shows:

Let  $k = \mathbb{Q}(\sqrt{1714})$ ,  $p = 3$ . We have (with PARI/GP)  $\mathcal{T}_k \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ ,  $\mathcal{C}_k \simeq \mathbb{Z}/3\mathbb{Z}$  and  $\mathcal{W}_k = 1$ . Using Kummer–Scholz reflection by means of the mirror field  $\mathbb{Q}(\sqrt{-3 \cdot 1714})$ , of 3-class group  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , we found  $\alpha = 2593 + 15 \cdot \sqrt{-5142}$ ,  $\beta = 157 + \sqrt{-5142}$  (cubes of ideals) of irreducible polynomials  $P_\alpha = x^3 - 597x - 5186$  giving the 3-Hilbert class field of  $k$ ,  $P_\beta = x^3 - 93x - 314$ , ramified at the two 3-places. The field  $k_1$  defined by  $P_\beta$  is such that  $\mathcal{C}_{k_1} \simeq \mathbb{Z}/3\mathbb{Z}$ . Thus Theorem 3.1 applies in any cyclic 3-tower built over  $k_1/k$  in  $H_k^{\text{bp}} = H_k^{\text{pr}}$ . This does not work for  $k_1 \subset \tilde{k}$  since in that case  $\mathcal{C}_{k_1} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  and furthermore  $\mathcal{C}_{k_2} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , but if Greenberg's conjecture is true ( $\lambda = \mu = 0$ ), the theorem applies, non effectively, from some layers  $n_0, n_0 + 1$ .

To illustrate the case  $p = 2$ , we have computed, for  $k = \mathbb{Q}(\sqrt{3})$  and  $K = k(\mu_{17})$ , imaginarity of degree 16 over  $k$ , that  $\mathcal{C}_{k_n}^{\text{res}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , for all  $n \in [0, 4]$ .

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VILLA LA GARDETTE, CHEMIN CHÂTEAU GAGNIÈRE, F-38520 LE BOURG D'OISANS  
Email address: g.mm.gras@wanadoo.fr