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Rate of convergence for particle approximation of PDEs in Wasserstein space ^{*}

Maximilien GERMAIN [†] Huy en PHAM [‡] Xavier WARIN [§]

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Abstract

We prove a rate of convergence for the N -particle approximation of a second-order partial differential equation in the space of probability measures, like the Master equation or Bellman equation of mean-field control problem under common noise. The rate is of order $1/N$ for the pathwise error on the solution v and of order $1/\sqrt{N}$ for the L^2 -error on its L -derivative $\partial_\mu v$. The proof relies on backward stochastic differential equations techniques.

1 Introduction

Let us consider the second-order parabolic partial differential equation (PDE) on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ of square-integrable probability measures on \mathbb{R}^d , in the form:

$$\begin{cases} \partial_t v + \mathcal{H}(t, \mu, v, \partial_\mu v, \partial_x \partial_\mu v, \partial_\mu^2 v) = 0, & (t, \mu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \\ v(T, \mu) = G(\mu), & \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{cases} \quad (1.1)$$

Here, $\partial_\mu v(t, \mu)$ is the L -derivative on $\mathcal{P}_2(\mathbb{R}^d)$ (see [CD18a]) of $\mu \mapsto v(t, \mu)$, and it is a function from \mathbb{R}^d into \mathbb{R}^d , $\partial_x \partial_\mu v(t, \mu)$ is the usual derivative on \mathbb{R}^d of $x \in \mathbb{R}^d \mapsto \partial_\mu v(t, \mu)(x) \in \mathbb{R}^d$, hence valued in $\mathbb{R}^{d \times d}$ the set of $d \times d$ -matrices with real coefficients, and $\partial_\mu^2 v(t, \mu)$ is the L -derivative of $\mu \mapsto \partial_\mu v(t, \mu)(\cdot)$, hence a function from $\mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^{d \times d}$. The terminal condition is given by a real-valued function G on $\mathcal{P}_2(\mathbb{R}^d)$, and the Hamiltonian \mathcal{H} of this PDE is assumed to be in semi-linear (non linear w.r.t. v , $\partial_\mu v$, and linear w.r.t. $\partial_x \partial_\mu v$, $\partial_\mu^2 v$) expectation form:

$$\begin{aligned} \mathcal{H}(t, \mu, y, z(\cdot), \gamma(\cdot), \gamma_0(\cdot, \cdot)) &= \int_{\mathbb{R}^d} [H(t, x, \mu, y, z(x)) + \frac{1}{2} \text{tr}((\sigma \sigma^\top + \sigma_0 \sigma_0^\top)(t, x, \mu) \gamma(x))] \mu(dx) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}(\sigma_0(t, x, \mu) \sigma_0^\top(t, x', \mu) \gamma_0(x, x')) \mu(dx) \mu(dx'), \end{aligned} \quad (1.2)$$

for some real-valued measurable function H defined on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^d$, and where σ , σ_0 are measurable functions on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, valued respectively in $\mathbb{R}^{d \times n}$, and $\mathbb{R}^{d \times m}$. Here $\text{tr}(M)$ denotes the trace of a square matrix M , while M^\top is its transpose, and \cdot is the scalar product.

PDEs in Wasserstein space have been largely studied in the literature over the last years, notably with the emergence of the mean-field game theory, and we mention among others the

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papers [BFY15], [GS15], [PW18], [Car+19], [SZ19], [Bur+20], and other references in the two-volume monographs [CD18a]-[CD18b].

An important application concerns mean-field type control problems with common noise. The controlled stochastic McKean-Vlasov dynamics is given by

$$\begin{aligned} dX_s^\alpha &= \beta(s, X_s^\alpha, \mathbb{P}_{X_s^\alpha}^0, \alpha_s)ds + \sigma(s, X_s^\alpha, \mathbb{P}_{X_s^\alpha}^0)dW_s \\ &\quad + \sigma_0(s, X_s^\alpha, \mathbb{P}_{X_s^\alpha}^0)dW_s^0, \quad t \leq s \leq T, \quad X_t^\alpha = \xi, \end{aligned} \quad (1.3)$$

where W is a n -dimensional Brownian motion, independent of a m -dimensional Brownian motion W^0 (representing the common noise) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, the control process α is \mathbb{F} -adapted valued in some Polish space A , and here \mathbb{P}^0 denotes the conditional law given W^0 . The value function defined on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ by

$$v(t, \mu) = \inf_{\alpha} \mathbb{E}_{t, \mu} \left[\int_t^T e^{-r(s-t)} f(X_s^\alpha, \mathbb{P}_{X_s^\alpha}^0, \alpha_s) ds + e^{-r(T-t)} g(X_T^\alpha, \mathbb{P}_{X_T^\alpha}^0) \right],$$

(here $\mathbb{E}_{t, \mu}[\cdot]$ is the conditional expectation given that the law at time t of X solution to (1.3) is equal to μ) is shown to satisfy the Bellman equation (1.1)-(1.2) (see [BFY13], [CP19], [DPT19]) with $G(\mu) = \int g(x, \mu)\mu(dx)$, σ , σ_0 as in (1.3) and

$$H(t, x, \mu, y, z) = -ry + \inf_{a \in A} [\beta(t, x, \mu, a).z + f(x, \mu, a)]. \quad (1.4)$$

We now consider a finite-dimensional approximation of the PDE (1.1)-(1.2) in the Wasserstein space. This can be derived formally by looking at the PDE for μ to averages of Dirac masses, and it turns out that the corresponding PDE takes the form

$$\begin{cases} \partial_t v^N + \frac{1}{N} \sum_{i=1}^N H(t, x_i, \bar{\mu}(\mathbf{x}), v^N, ND_{x_i} v^N) + \frac{1}{2} \text{tr}(\Sigma_N(t, \mathbf{x}) D_{\mathbf{x}}^2 v^N) = 0, & \text{on } [0, T] \times (\mathbb{R}^d)^N \\ v^N(T, \mathbf{x}) = G(\bar{\mu}(\mathbf{x})), & \mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket} \in (\mathbb{R}^d)^N, \end{cases} \quad (1.5)$$

where $\bar{\mu}(\cdot)$ is the empirical measure function defined by $\bar{\mu}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$, for any $\mathbf{x} = (x_1, \dots, x_N)$, $N \in \mathbb{N}^*$, and $\Sigma_N = (\Sigma_N^{ij})_{i, j \in \llbracket 1, N \rrbracket}$ is the $\mathbb{R}^{Nd \times Nd}$ -valued function with block matrices $\Sigma_N^{ij}(t, \mathbf{x}) = \sigma(t, x_i, \bar{\mu}(\mathbf{x}))\sigma^\top(t, x_j, \bar{\mu}(\mathbf{x}))\delta_{ij} + \sigma_0(t, x_i, \bar{\mu}(\mathbf{x}))\sigma_0^\top(t, x_j, \bar{\mu}(\mathbf{x})) \in \mathbb{R}^{d \times d}$. In the special case where H has the form (1.4), we notice that (1.5) is the Bellman equation for the N -cooperative problem, whose convergence to the mean-field control problem has been studied in [Lac17], [CD18b], [LT19; LT20], when $\sigma_0 \equiv 0$ (no common noise), and recently by [Dje20] in the common noise case. We point out that these works do not consider the same master equation. In particular their master equation is stated on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ and is linear in $\partial_\mu u$ whereas we allow a non-linear dependence in this derivative. Moreover our master equation is in expectation form. In [LT20] the master equation is approached by a system of N coupled PDEs on $[0, T] \times (\mathbb{R}^d)^N$ whereas we consider a single approximating PDE on $[0, T] \times (\mathbb{R}^d)^N$. For more general Hamiltonian functions H , it has been recently proved in [GMS21] that the sequence of viscosity solutions $(v^N)_N$ to (1.5) converge locally uniformly to the viscosity solution v to (1.1) when $\sigma = 0$ and σ_0 does not depend on space and measure arguments. For a detailed comparison between this work and ours, we refer to Remark 2.9.

In this paper, we adopt a probabilistic approach by considering a backward stochastic differential equation (BSDE) representation for the finite-dimensional PDE (1.5) according to the classical work [PP90]. The solution $(Y^N, \mathbf{Z}^N = (Z^{i, N})_{1 \leq i \leq N})$ to this BSDE is written with an underlying forward particle system $\mathbf{X}^N = (X^{i, N})_{1 \leq i \leq N}$ of a McKean-Vlasov SDE, and connected to the PDE (1.5) via the Feynman-Kac formula: $Y_t^N = v^N(t, \mathbf{X}_t^N)$, $Z_t^{i, N} = D_{x_i} v^N(t, \mathbf{X}_t^{i, N})$,

$0 \leq t \leq T$. By using BSDE techniques, our main contribution is to show a rate of convergence of order $1/N$ of $|Y_t^N - v(t, \bar{\mu}(\mathbf{X}_t^N))|$, and also of $|NZ_t^{i,N} - \partial_\mu v(t, \bar{\mu}(\mathbf{X}_t^N))(X_t^{i,N})|^2$, $i = 1, \dots, N$, for suitable norms, and under some regularity conditions on v (see Theorem 2.6 and Theorem 2.7). This rate of convergence on the particles approximation of v and its L -derivative is new to the best of our knowledge. We point out that classical BSDE arguments for proving the rate of convergence do not apply directly due to the presence of the factor N in front of $D_{x_i} v^N$ in the generator H , and we rather use linearization arguments and change of probability measures to overcome these issues. Another issue is due to the fact that the BSDE dimension $d \times N$ is exploding with the number of particles therefore we have to track down the influence of the dimension in the estimations, whereas classical BSDE works usually consider a fixed dimension d which is incorporated into constants.

The outline of the paper is organized as follows. In Section 2, we formulate the particle approximation of the PDE and its BSDE representation, and state the rate of convergence for v and its L -derivative. Section 3 is devoted to the proof of these results.

2 Particles approximation of Wasserstein PDEs

The formal derivation of the finite-dimensional approximation PDE is obtained as follows. We look at the PDE (1.1)-(1.2) for $\mu = \bar{\mu}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in \mathcal{P}_2(\mathbb{R}^d)$, when $\mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket}$ runs over $(\mathbb{R}^d)^N$. By setting $\tilde{v}^N(t, \mathbf{x}) = v(t, \bar{\mu}(\mathbf{x}))$, and assuming that v is smooth, we have for all $(i, j) \in \llbracket 1, N \rrbracket$ (see Proposition 5.35 and Proposition 5.91 in [CD18a]):

$$\begin{cases} D_{x_i} \tilde{v}^N(t, \mathbf{x}) = \frac{1}{N} \partial_\mu v(t, \bar{\mu}(\mathbf{x}))(x_i), \\ D_{x_i x_j}^2 \tilde{v}^N(t, \mathbf{x}) = \frac{1}{N} \partial_x \partial_\mu v(t, \bar{\mu}(\mathbf{x}))(x_i) \delta_{ij} + \frac{1}{N^2} \partial_\mu^2 v(t, \bar{\mu}(\mathbf{x}))(x_i, x_j). \end{cases} \quad (2.1)$$

By substituting into the PDE (1.1)-(1.2) for $\mu = \bar{\mu}(\mathbf{x})$, and using (2.1), we then see that \tilde{v}^N satisfies the relation:

$$\begin{aligned} & \partial_t \tilde{v}^N + \frac{1}{N} \sum_{i=1}^N H(t, x_i, \bar{\mu}(\mathbf{x}), \tilde{v}^N, ND_{x_i} \tilde{v}^N) \\ & + \frac{1}{2} \sum_{i=1}^N \text{tr}[(\sigma \sigma^\top + \sigma_0 \sigma_0^\top)(t, x_i, \bar{\mu}(\mathbf{x})) (D_{x_i}^2 \tilde{v}^N - \frac{1}{N^2} \partial_\mu^2 v(t, \bar{\mu}(\mathbf{x}))(x_i, x_i))] \\ & + \frac{1}{2} \sum_{i \neq j \in \llbracket 1, N \rrbracket} \text{tr}(\sigma_0(t, x_i, \bar{\mu}(\mathbf{x})) \sigma_0^\top(t, x_j, \bar{\mu}(\mathbf{x})) D_{x_i x_j}^2 \tilde{v}^N) \\ & + \frac{1}{2N^2} \sum_{i=1}^N \text{tr}(\sigma_0 \sigma_0^\top(t, x_i, \bar{\mu}(\mathbf{x})) \partial_\mu^2 v(t, \bar{\mu}(\mathbf{x}))(x_i, x_i)) = 0 \end{aligned} \quad (2.2)$$

for $(t, \mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket}) \in [0, T) \times (\mathbb{R}^d)^N$, together with the terminal condition $\tilde{v}^N(t, \mathbf{x}) = G(\bar{\mu}(\mathbf{x}))$. By neglecting the terms $\partial_\mu^2 v/N^2$ in the above relation, we obtain the PDE (1.5) for $v^N \simeq \tilde{v}^N$. The purpose of this section is to rigorously justify this approximation and state a rate of convergence for v^N towards v , as well as a convergence for their gradients.

2.1 Particles BSDE approximation

Let us introduce an arbitrary measurable \mathbb{R}^d -valued function b on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and set B_N the $(\mathbb{R}^d)^N$ -valued function defined on $[0, T] \times (\mathbb{R}^d)^N$ by $B_N(t, \mathbf{x}) = (b(t, x_i, \bar{\mu}(\mathbf{x}))_{i \in \llbracket 1, N \rrbracket}$ for

$(t, \mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket}) \in [0, T) \times (\mathbb{R}^d)^N$. The finite-dimensional PDE (1.5) may then be written as

$$\begin{cases} \partial_t v^N + B_N(t, \mathbf{x}) \cdot D_{\mathbf{x}} v^N + \frac{1}{2} \text{tr}(\Sigma_N(t, \mathbf{x}) D_{\mathbf{x}}^2 v^N) \\ \quad + \frac{1}{N} \sum_{i=1}^N H_b(t, x_i, \bar{\mu}(\mathbf{x}), v^N, N D_{x_i} v^N) = 0, & \text{on } [0, T) \times (\mathbb{R}^d)^N, \\ v^N(T, \mathbf{x}) = G(\bar{\mu}(\mathbf{x})), & \mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket} \in (\mathbb{R}^d)^N, \end{cases} \quad (2.3)$$

where $H_b(t, x, \mu, y, z) := H(t, x, \mu, y, z) - b(t, x, \mu) \cdot z$. For error analysis purpose, the function b can be simply taken to be zero. The introduction of the function b is actually motivated by numerical purpose. It corresponds indeed to the drift of training simulations for approximating the function v^N , notably by machine learning methods, and should be chosen for suitable exploration of the state space, see a detailed discussion in our companion paper [Ger+21]. In this paper, we fix an arbitrary function b (satisfying Lipschitz condition to be precised later).

Following [PP90], it is well-known that the semi-linear PDE (2.3) admits a probabilistic representation in terms of forward backward SDE. The forward component is defined by the process $\mathbf{X}^N = (X^{i,N})_{i \in \llbracket 1, N \rrbracket}$ valued in $(\mathbb{R}^d)^N$, solution to the SDE:

$$d\mathbf{X}_t^N = B_N(t, \mathbf{X}_t^N) dt + \sigma_N(t, \mathbf{X}_t^N) d\mathbf{W}_t + \boldsymbol{\sigma}_0(t, \mathbf{X}_t^N) dW_t^0 \quad (2.4)$$

where σ_N is the block diagonal matrix with block diagonals $\sigma_N^{ii}(t, \mathbf{x}) = \sigma(t, x_i, \bar{\mu}(\mathbf{x}))$, $\boldsymbol{\sigma}_0 = (\sigma_0^i)_{i \in \llbracket 1, N \rrbracket}$ is the $(\mathbb{R}^{d \times m})^N$ -valued function with $\sigma_0^i(t, \mathbf{x}) = \sigma_0(t, x_i, \bar{\mu}(\mathbf{x}))$, for $\mathbf{x} = (x_i)_{i \in \llbracket 1, N \rrbracket}$, $\mathbf{W} = (W^1, \dots, W^N)$ where W^i , $i = 1, \dots, N$, are independent n -dimensional Brownian motions, independent of a m -dimensional Brownian motion W^0 on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. Notice that $\Sigma_N = \sigma_N \sigma_N^\top + \boldsymbol{\sigma}_0 \boldsymbol{\sigma}_0^\top$, and \mathbf{X}^N is the particles system of the McKean-Vlasov SDE:

$$dX_t = b(t, X_t, \mathbb{P}_{X_t}) dt + \sigma(t, X_t, \mathbb{P}_{X_t}^0) dW_t + \sigma_0(t, X_t, \mathbb{P}_{X_t}^0) dW_t^0, \quad (2.5)$$

where W is an n -dimensional Brownian motion independent of W^0 . The backward component is defined by the pair process $(Y^N, \mathbf{Z}^N = (Z^{i,N})_{i \in \llbracket 1, N \rrbracket})$ valued in $\mathbb{R} \times (\mathbb{R}^d)^N$, solution to

$$\begin{aligned} Y_t^N &= G(\bar{\mu}(\mathbf{X}_T^N)) + \frac{1}{N} \sum_{i=1}^N \int_t^T H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N, N Z_s^{i,N}) ds \\ &\quad - \sum_{i=1}^N \int_t^T (Z_s^{i,N})^\top \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i, \\ &\quad - \sum_{i=1}^N \int_t^T (Z_s^{i,N})^\top \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0, \quad 0 \leq t \leq T. \end{aligned} \quad (2.6)$$

We shall assume that the measurable functions $(t, x, \mu) \mapsto b(t, x, \mu)$, $\sigma(t, x, \mu)$ satisfy a Lipschitz condition in $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ uniformly w.r.t. $t \in [0, T]$, which ensures the existence and uniqueness of a strong solution $\mathbf{X}^N \in \mathcal{S}_{\mathbb{F}}^2((\mathbb{R}^d)^N)$ to (2.4) given an initial condition. Here, $\mathcal{S}_{\mathbb{F}}^2(\mathbb{R}^q)$ is the set of \mathbb{F} -adapted process $(V_t)_t$ valued in \mathbb{R}^q s.t. $\mathbb{E}[\sup_{0 \leq t \leq T} |V_t|^2] < \infty$, ($|\cdot|$ is the Euclidian norm on \mathbb{R}^q , and for a matrix M , we choose the Frobenius norm $|M| = \sqrt{\text{tr}(MM^\top)}$) and the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ is endowed with the Wasserstein distance

$$\mathcal{W}_2(\mu, \mu') = \left(\inf \{ \mathbb{E} |\xi - \xi'|^2 : \xi \sim \mu, \xi' \sim \mu' \} \right)^{\frac{1}{2}},$$

and we set $\|\mu\|_2 := \left(\int_{\mathbb{R}^d} |x|^2 \mu(dx) \right)^{\frac{1}{2}}$ for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Assuming also that the measurable function $(t, x, \mu, y, z) \mapsto H_b(t, x, \mu, y, z)$ is Lipschitz in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ uniformly with respect to (t, x, μ)

$\in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and the measurable function G satisfies a quadratic growth condition on $\mathcal{P}_2(\mathbb{R}^d)$, we have the existence and uniqueness of a solution $(Y^N, \mathbf{Z}^N = (Z^{i,N})_{i \in [1, N]}) \in \mathcal{S}_{\mathbb{F}}^2(\mathbb{R}) \times \mathbb{H}_{\mathbb{F}}^2((\mathbb{R}^d)^N)$ to (2.6), and the connection with the PDE (2.3) (satisfied in general in the viscosity sense) via the (non linear) Feynman-Kac formula:

$$Y_t^N = v^N(t, \mathbf{X}_t^N), \quad \text{and} \quad Z_t^{i,N} = D_{x_i} v^N(t, \mathbf{X}_t^N), \quad i = 1, \dots, N, \quad 0 \leq t \leq T, \quad (2.7)$$

(when v^N is smooth for the last relation). Here, $\mathbb{H}_{\mathbb{F}}^2(\mathbb{R}^q)$ is the set of \mathbb{F} -adapted process (V_t) valued in \mathbb{R}^q s.t. $\mathbb{E}[\int_0^T |V_t|^2 dt] < \infty$.

2.2 Main results

We aim to analyze the particles approximation error on the solution v to the PDE (1.1), and its L -derivative $\partial_{\mu} v$ by considering the pathwise error on v :

$$\mathcal{E}_N^v := \sup_{0 \leq t \leq T} |Y_t^N - v(t, \bar{\mu}(\mathbf{X}_t^N))|,$$

and the L^2 -error on its L -derivative

$$\|\mathcal{E}_N^z\|_2 := \frac{1}{N} \sum_{i=1}^N \left(\int_0^T \mathbb{E} |N Z_t^{i,N} - \partial_{\mu} v(t, \bar{\mu}(\mathbf{X}_t^N))(X_t^{i,N})|^2 dt \right)^{\frac{1}{2}},$$

where the initial conditions of the particles system, $X_0^{i,N}$, $i = 1, \dots, N$, are i.i.d. with distribution μ_0 .

Here, it is assumed that we have the existence and uniqueness of a classical solution v to the PDE (1.1)-(1.2). More precisely, we make the following assumption:

Assumption 2.1 (Smooth solution to the Master Bellman PDE). *There exists a unique solution v to (1.1), which lies in $C_b^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ that is:*

- $v(\cdot, \mu) \in C^1([0, T])$, and continuous on $[0, T]$, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,
- $v(t, \cdot)$ is fully C^2 on $\mathcal{P}_2(\mathbb{R}^d)$, for any $t \in [0, T]$ in the sense that: $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_{\mu} v(t, \mu)(x) \in \mathbb{R}^d$, $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_x \partial_{\mu} v(t, \mu)(x) \in \mathbb{M}^d$, and $(x, x', \mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_{\mu}^2 v(t, \mu)(x, x') \in \mathbb{M}^d$, are well-defined and jointly continuous,
- there exists some constant $L > 0$ s.t. for all $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$$|\partial_{\mu} v(t, \mu)(x)| \leq L(1 + |x| + \|\mu\|_2), \quad |\partial_{\mu}^2 v(t, \mu)(x, x)| \leq L.$$

The existence of classical solutions to mean-field PDE in Wasserstein space is a challenging problem, and beyond the scope of this paper. We refer to [Buc+17], [CCD15], [SZ19], [WZ20] for conditions ensuring regularity results of some Master PDEs. Notice also that linear-quadratic mean-field control problems have explicit smooth solutions as in Assumption 2.1, see e.g. [PW18].

We also make some rather standard assumptions on the coefficients of the forward backward SDE:

Assumption 2.2 (Lipschitz condition on the coefficients of the forward backward SDE).

- (i) *The drift and volatility coefficients b, σ, σ_0 are Lipschitz: there exist positive constants $[b]$, $[\sigma]$, and $[\sigma_0]$ s.t. for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$\begin{aligned} |b(t, x, \mu) - b(t, x', \mu')| &\leq [b](|x - x'| + \mathcal{W}_2(\mu, \mu')) \\ |\sigma(t, x, \mu) - \sigma(t, x', \mu')| &\leq [\sigma](|x - x'| + \mathcal{W}_2(\mu, \mu')) \\ |\sigma_0(t, x, \mu) - \sigma_0(t, x', \mu')| &\leq [\sigma_0](|x - x'| + \mathcal{W}_2(\mu, \mu')). \end{aligned}$$

(ii) For all $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $\Sigma(t, x, \mu) := \sigma \sigma^\top(t, x, \mu)$ is invertible, and the function σ , and its pseudo-inverse $\sigma^+ := \sigma^\top \Sigma^{-1}$ are bounded.

(iii) $\mu_0 \in \mathcal{P}_{4q}(\mathbb{R}^d)$ for some $q > 1$, i.e., $\|\mu_0\|_{4q} := \left(\int |x|^{4q} \mu_0(dx) \right)^{\frac{1}{4q}} < \infty$, and

$$\int_0^T |b(t, 0, \delta_0)|^{4q} + |\sigma(t, 0, \delta_0)|^{4q} + |\sigma_0(t, 0, \delta_0)|^{4q} dt < \infty.$$

(iv) The driver H_b satisfies the Lipschitz condition: there exist positive constants $[H_b]_1$ and $[H_b]_2$ s.t. for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} |H_b(t, x, \mu, y, z) - H_b(t, x, \mu, y', z')| &\leq [H_b]_1 (|y - y'| + |z - z'|) \\ |H_b(t, x, \mu, y, z) - H_b(t, x', \mu', y, z)| &\leq [H_b]_2 (1 + |x| + |x'| + \|\mu\|_2 + \|\mu'\|_2) \\ &\quad (|x - x'| + \mathcal{W}_2(\mu, \mu')). \end{aligned}$$

(v) The terminal condition satisfies the (locally) Lipschitz condition: there exists some positive constant $[G]$ s.t. for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$

$$|G(\mu) - G(\mu')| \leq [G] (\|\mu\|_2 + \|\mu'\|_2) \mathcal{W}_2(\mu, \mu').$$

In order to have a convergence result for the first order Lions derivative we have to make a stronger assumption.

Assumption 2.3.

(i) The function H_b is in the form:

$$H_b(t, x, \mu, y, z) = H_1(t, x, \mu, y) + H_2(t, \mu, y) \cdot z,$$

where $H_1 : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \mapsto \mathbb{R}$ verifies for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} |H_1(t, x, \mu, y) - H_1(t, x, \mu, y')| &\leq [H_1]_1 |y - y'| \\ |H_1(t, x, \mu, y) - H_1(t, x', \mu', y)| &\leq [H_1]_2 (1 + |x| + |x'| + \|\mu\|_2 + \|\mu'\|_2) \\ &\quad (|x - x'| + \mathcal{W}_2(\mu, \mu')), \end{aligned}$$

and $H_2 : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \mapsto \mathbb{R}^d$ is bounded and verifies for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} |H_2(t, x, \mu, y) - H_2(t, x, \mu, y')| &\leq [H_2]_1 |y - y'| \\ |H_2(t, x, \mu, y) - H_2(t, x', \mu', y)| &\leq [H_2]_2 (1 + |x| + |x'| + \|\mu\|_2 + \|\mu'\|_2) \\ &\quad (|x - x'| + \mathcal{W}_2(\mu, \mu')). \end{aligned}$$

(ii) σ_0 is uniformly elliptic and does not depend on x , namely there exists $c_0 > 0$ such that for all $t \in [0, T]$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $z \in \mathbb{R}^d$

$$z^\top \sigma_0(t, \mu) \sigma_0^\top(t, \mu) z \geq c_0 |z|^2.$$

(iii) There exists some constant $L > 0$ s.t. for all $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$$|\partial_\mu v(t, \mu)(x)| \leq L.$$

Remark 2.4. The Lipschitz condition on b, σ in Assumption 2.2(i) implies that the functions $\mathbf{x} \in (\mathbb{R}^d)^N \mapsto B_N(t, \mathbf{x})$, resp. $\sigma_N(t, \mathbf{x})$ and $\boldsymbol{\sigma}_0(t, \mathbf{x})$, defined in (2.4), are Lipschitz (with Lipschitz constant $2[b]$, resp. $2[\sigma]$ and $2[\sigma_0]$). Indeed, we have

$$\begin{aligned} |B_N(t, \mathbf{x}) - B_N(t, \mathbf{x}')|^2 &= \sum_{i=1}^N |b(t, x_i, \bar{\mu}(\mathbf{x})) - b(t, x_i, \bar{\mu}(\mathbf{x}'))|^2 \\ &\leq 2[b]^2 \sum_{i=1}^N (|x_i - x'_i|^2 + \mathcal{W}_2(\bar{\mu}(\mathbf{x}), \bar{\mu}(\mathbf{x}'))^2) \\ &\leq 2[b]^2 (|\mathbf{x} - \mathbf{x}'|^2 + \sum_{i=1}^N \frac{1}{N} |\mathbf{x} - \mathbf{x}'|^2) = 4[b]^2 |\mathbf{x} - \mathbf{x}'|^2, \end{aligned}$$

for $\mathbf{x} = (x_i)_{i \in [1, N]}$, and similarly for σ_N and $\boldsymbol{\sigma}_0$.

This yields the existence and uniqueness of a solution $\mathbf{X}^N = (X^{i, N})_{i \in [1, N]}$ to (2.4) given initial conditions. Moreover, under Assumption 2.2(iii), we have the standard estimate:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\mathbf{X}_t^N|^{4q} \right] \leq C(1 + \|\mu_0\|_{4q}^{4q}) < \infty, \quad i = 1, \dots, N, \quad (2.8)$$

for some constant C (possibly depending on N). The Lipschitz condition on H_b w.r.t. (y, z) in Assumption 2.2(iv), and the quadratic growth condition on G from Assumption 2.2(v) gives the existence and uniqueness of a solution $(Y^N, \mathbf{Z}^N = (Z^{i, N})_{i \in [1, N]}) \in \mathcal{S}_{\mathbb{F}}^2(\mathbb{R}) \times \mathbb{H}_{\mathbb{F}}^2((\mathbb{R}^d)^N)$ to (2.6). Moreover, by Assumption 2.2(iv)(v), we see that

$$\begin{aligned} &\left| \frac{1}{N} \sum_{i=1}^N H_b(t, x_i, \bar{\mu}(\mathbf{x}), y, z_i) - \frac{1}{N} \sum_{i=1}^N H_b(t, x'_i, \bar{\mu}(\mathbf{x}'), y, z_i) \right| \\ &\leq [H_b]_2 \frac{1}{N} \sum_{i=1}^N (1 + |x_i| + |x'_i| + \frac{1}{\sqrt{N}} (|\mathbf{x}| + |\mathbf{x}'|)) (|x_i - x'_i| + \frac{1}{\sqrt{N}} |\mathbf{x} - \mathbf{x}'|) \\ &\leq 4[H_b]_2 (1 + |\mathbf{x}| + |\mathbf{x}'|) |\mathbf{x} - \mathbf{x}'| \\ |G(\bar{\mu}(\mathbf{x})) - G(\bar{\mu}(\mathbf{x}'))| &\leq \frac{[G]}{N} (|\mathbf{x}| + |\mathbf{x}'|) |\mathbf{x} - \mathbf{x}'|, \end{aligned}$$

for all $x, x' \in \mathbb{R}^d$, $\mathbf{x} = (x_i)_{i \in [1, N]}$, $\mathbf{x}' = (x'_i)_{i \in [1, N]} \in (\mathbb{R}^d)^N$, which yields by standard stability results for BSDE (see e.g. Theorems 4.2.1 and 5.2.1 in [Zha17]) that the function v^N in (2.7) inherits the locally Lipschitz condition:

$$|v^N(t, \mathbf{x}) - v^N(t, \mathbf{x}')| \leq C(1 + |\mathbf{x}| + |\mathbf{x}'|) |\mathbf{x} - \mathbf{x}'|, \quad \forall \mathbf{x}, \mathbf{x}' \in (\mathbb{R}^d)^N,$$

for some constant C (possibly depending on N). This implies

$$|Z_t^{i, N}| \leq C(1 + |\mathbf{X}_t^N|), \quad 0 \leq t \leq T, \quad i = 1, \dots, N, \quad (2.9)$$

(this is clear when v^N is smooth, and otherwise obtained by a mollifying argument as in Theorem 5.2.2 in [Zha17]).

Remark 2.5. Assumption 2.1 is verified in the case of linear quadratic control problems for which explicit smooth solutions are found in [PW18; PW17] respectively without and with common noise. These papers prove that the second order Lions derivative ∂_μ^2 is a continuous function of time which does not depend on the μ, x arguments hence is bounded whereas ∂_μ is affine in both the state and the first moment of the measure thus satisfies linear growth. Notice that in general,

Assumptions 2.2 and 2.3 are not satisfied due to the quadratic nature of H_b in the z . However, in the uncontrolled case

$$\begin{aligned} v(t, \mu) &= \mathbb{E}_{t, \mu} \left[\int_t^T \left(X_t^\top A(t) X_t + \mathbb{E}[X_t]^\top B(t) \mathbb{E}[X_t] + C(t) X_t + D(t) \mathbb{E}[X_t] \right) dt \right. \\ &\quad \left. + X_T^\top E X_T + \mathbb{E}[X_T]^\top F \mathbb{E}[X_T] + G X_T + H \mathbb{E}[X_T] \right] \\ dX_t &= (b_0(t) + b_1(t) X_t + b_2(t) \mathbb{E}[X_t]) dt + \sigma(t) dW_t, \end{aligned}$$

we see that v is a solution to the linear PDE

$$\begin{cases} \partial_t v + \int_{\mathbb{R}^d} [x^\top A(t)x + \bar{\mu}^\top B(t)\bar{\mu} + C(t)x + D(t)\bar{\mu} \\ + (b_0(t) + b_1(t)x + b_2(t)\bar{\mu}) \partial_\mu v(t, \mu)(x) \\ + \frac{1}{2} \text{tr}((\sigma \sigma^\top)(t) \partial_x \partial_\mu v(t, \mu)(x))] \mu(dx) = 0, & (t, \mu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \\ v(T, \mu) = E \text{Var}(\mu) + \bar{\mu}^\top (E + F) \bar{\mu} + (G + H) \bar{\mu}, & \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

where $\bar{\mu} = \int_{\mathbb{R}^d} x \mu(dx)$, $\text{Var}(\mu) = \int_{\mathbb{R}^d} x^2 \mu(dx) - \bar{\mu}^2$. In that case both assumptions 2.1 and 2.2 are satisfied.

Theorem 2.6. Under Assumptions 2.1 and 2.2, we have \mathbb{P} -almost surely

$$\mathcal{E}_N^y \leq \frac{C_y}{N},$$

where $C_y = \frac{T}{2} e^{[H_b]_1 T} L \|\sigma\|_\infty^2$, with $\|\sigma\|_\infty = \sup_{(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)} |\sigma(t, x, \mu)|$.

Theorem 2.7. Under Assumptions 2.1, 2.2 and 2.3, we have

$$\|\mathcal{E}_N^z\|_2 \leq \frac{C_z}{N^{\frac{1}{2}}},$$

where $C_z = \|\sigma^+\|_\infty \sqrt{2T([H_1]_1 + [H_2]_1 L) \bar{C}_y^2 + \bar{C}_y T \|\sigma\|_\infty^2 L + \frac{2}{c_0} \bar{C}_y^2 T \|H_2\|_\infty^2}$ and $\bar{C}_y = \frac{T}{2} e^{([H_1]_1 + [H_2]_1 L) T} L \|\sigma\|_\infty^2$.

Remark 2.8. Let us consider the global weak errors on v and its L -derivative $\partial_\mu v$ along the limiting McKean-Vlasov SDE, and defined by

$$\begin{aligned} E_N^y &:= \sup_{0 \leq t \leq T} |\mathbb{E}[Y_t^N] - \mathbb{E}[v(t, \mathbb{P}_{X_t}^0)]| \\ E_N^z &:= \frac{1}{N} \sum_{i=1}^N \left(\int_0^T \left| \mathbb{E}[N Z_t^{i, N}] - \mathbb{E}[\partial_\mu v(t, \mathbb{P}_{X_t^i}^0)(X_t^i)] \right|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

where X^i has the same law than X , and with McKean-Vlasov dynamics as in (2.5) but driven by W^i , $i = 1, \dots, N$. Then, they can be decomposed as

$$E_N^y \leq \mathbb{E}[\mathcal{E}_N^y] + \tilde{E}_N^y, \quad E_N^z \leq \|\mathcal{E}_N^z\|_2 + \tilde{E}_N^z,$$

where $\tilde{E}_N^y, \tilde{E}_N^z$ are the (weak) propagation of chaos errors defined by

$$\begin{aligned} \tilde{E}_N^y &:= \sup_{0 \leq t \leq T} |\mathbb{E}[v(t, \bar{\mu}(\mathbf{X}_t^N))] - \mathbb{E}[v(t, \mathbb{P}_{X_t}^0)]| \\ \tilde{E}_N^z &:= \frac{1}{N} \sum_{i=1}^N \left(\int_0^T \left| \mathbb{E}[\partial_\mu v(t, \bar{\mu}(\mathbf{X}_t^N))(X_t^{i, N})] - \mathbb{E}[\partial_\mu v(t, \mathbb{P}_{X_t^i}^0)(X_t^i)] \right|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

From the conditional propagation of chaos result, which states that for any fixed $k \geq 1$, the law of $(X_t^{i,N})_{t \in [0,T]}^{i \in [1,k]}$ converges toward the conditional law of $(X_t^i)_{t \in [0,T]}^{i \in [1,k]}$, as N goes to infinity, we deduce that $\tilde{E}_N^y, \tilde{E}_N^z \rightarrow 0$. Furthermore, under additional assumptions on v , we can obtain a rate of convergence. Namely, if $v(t, \cdot)$ is Lipschitz uniformly in $t \in [0, T]$, with Lipschitz constant $[v]$, we have

$$\begin{aligned} \tilde{E}_N^y &\leq [v] \sup_{0 \leq t \leq T} \left(\mathbb{E}[\mathcal{W}_2(\bar{\mu}(\mathbf{X}_t^N), \mathbb{P}_{X_t}^0)^2] \right)^{\frac{1}{2}} = O\left(N^{-\frac{1}{\max(d,4)}} \sqrt{1 + \ln(N)1_{d=4}}\right), \\ \text{hence } E_N^y &= O\left(N^{-\frac{1}{\max(d,4)}} \sqrt{1 + \ln(N)1_{d=4}}\right), \end{aligned} \quad (2.10)$$

where we use the rate of convergence of empirical measures in Wasserstein distance stated in [FG15] (see also Theorem 2.12 in [CD18b]), and since we have the standard estimate $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^{4q}] \leq C(1 + \|\mu_0\|_{2q}^{4q})$ by Assumption 2.2(iii). The rate of convergence in (2.10) is consistent with the one found in Theorem 6.17 [CD18b] for mean-field control problem. Furthermore, if the function $\partial_\mu v(t, \cdot)(\cdot)$ is Lipschitz in (x, μ) uniformly in t , then by the rate of convergence in Theorem 2.12 of [CD18b], we have

$$\tilde{E}_N^z = O\left(N^{-\frac{1}{\max(d,4)}} (1 + \ln(N)1_{d=4})\right), \text{ hence } E_N^z = O\left(N^{-\frac{1}{\max(d,4)}} (1 + \ln(N)1_{d=4})\right).$$

Remark 2.9 (Comparison with [GMS21]). In the related paper [GMS21] the authors consider a pure common noise case, that is $\sigma = 0$ and restrict themselves to $\sigma_0(t, x_i, \bar{\mu}(\mathbf{x})) = \kappa I_d$ for $\kappa \in \mathbb{R}$. If we consider these assumptions in our smooth setting, we directly see that $\Delta Y_s^N = 0$ and $\Delta Z_s^N = 0$ \mathbb{P} a.s. Indeed by (2.6) and (3.2) we notice that (Y_t^N, Z_t^N) and

$$(\tilde{Y}_t^N := v(t, \bar{\mu}(\mathbf{X}_t^N)), \{\tilde{Z}_t^{i,N} := \frac{1}{N} \partial_\mu v(t, \bar{\mu}(\mathbf{X}_t^N))(X_t^{i,N}), i = 1, \dots, N\}),$$

solve the same BSDE therefore by existence and pathwise uniqueness for Lipschitz BSDEs the result follows. Moreover, [GMS21] does not allow H to depend on y . Our approach allows to extend their findings to the case of idiosyncratic noises and in contrast to them we are able to choose a state-dependent volatility coefficient. Moreover we provide a convergence rate for the solution. However, we have to assume existence of a smooth solution for the master equation which is a restrictive assumption.

3 Proof of main results

3.1 Proof of Theorem 2.6

Step 1. Under the smoothness condition on v in Assumption 2.1, one can apply the standard Itô's formula in $(\mathbb{R}^d)^N$ to the process $\tilde{v}^N(t, \mathbf{X}_t^N) = v(t, \bar{\mu}(\mathbf{X}_t^N))$, and get

$$\begin{aligned} \tilde{v}^N(t, \mathbf{X}_t^N) &= \tilde{v}^N(T, \mathbf{X}_T^N) - \int_t^T \partial_t \tilde{v}^N(s, \mathbf{X}_s^N) ds \\ &\quad - \int_t^T B_N(s, \mathbf{X}_s^N) \cdot D_{\mathbf{x}} \tilde{v}^N(s, \mathbf{X}_s^N) + \frac{1}{2} \text{tr}(\Sigma_N(s, \mathbf{X}_s^N) D_{\mathbf{x}}^2 \tilde{v}^N(s, \mathbf{X}_s^N)) ds \\ &\quad - \sum_{i=1}^N \int_t^T (D_{x_i} \tilde{v}^N(s, \mathbf{X}_s^N))^\top \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i \\ &\quad - \sum_{i=1}^N \int_t^T (D_{x_i} \tilde{v}^N(s, \mathbf{X}_s^N))^\top \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0, \end{aligned} \quad (3.1)$$

Now, by setting (recall (2.1)):

$$\begin{aligned}\tilde{Y}_t^N &:= v(t, \bar{\mu}(\mathbf{X}_t^N)) = \tilde{v}^N(t, \mathbf{X}_t^N), \\ \tilde{Z}_t^{i,N} &:= \frac{1}{N} \partial_{\mu} v(t, \bar{\mu}(\mathbf{X}_t^N))(X_t^{i,N}) = D_{x_i} \tilde{v}^N(t, \mathbf{X}_t^N), \quad i = 1, \dots, N, \quad 0 \leq t \leq T,\end{aligned}$$

and using the relation (2.2) satisfied by \tilde{v}^N into (3.1), we have for all $0 \leq t \leq T$,

$$\begin{aligned}\tilde{Y}_t^N &= G(\bar{\mu}(\mathbf{X}_T^N)) + \frac{1}{N} \sum_{i=1}^N \int_t^T H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N, N \tilde{Z}_s^{i,N}) ds \\ &\quad - \frac{1}{2N^2} \sum_{i=1}^N \int_t^T \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_{\mu}^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds \\ &\quad - \sum_{i=1}^N \int_t^T (\tilde{Z}_s^{i,N})^{\top} \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i - \sum_{i=1}^N \int_t^T (\tilde{Z}_s^{i,N})^{\top} \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0.\end{aligned}\tag{3.2}$$

Step 2: Linearization. We set

$$\Delta Y_t^N := Y_t^N - \tilde{Y}_t^N, \quad \Delta Z_t^{i,N} := N(Z_t^{i,N} - \tilde{Z}_t^{i,N}), \quad i = 1, \dots, N, \quad 0 \leq t \leq T,$$

so that by (2.6)-(3.2),

$$\begin{aligned}\Delta Y_t^N &= \frac{1}{N} \sum_{i=1}^N \int_t^T [H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N, N Z_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N, N \tilde{Z}_s^{i,N})] ds \\ &\quad + \frac{1}{2N^2} \sum_{i=1}^N \int_t^T \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_{\mu}^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^{\top} \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^{\top} \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0, \quad 0 \leq t \leq T.\end{aligned}\tag{3.3}$$

We now use the linearization method for BSDEs and rewrite the above equation as

$$\begin{aligned}\Delta Y_t^N &= \int_t^T \alpha_s \Delta Y_s^N ds + \frac{1}{N} \sum_{i=1}^N \int_t^T \beta_s^i \Delta Z_s^{i,N} ds \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^{\top} \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^{\top} \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0 \\ &\quad + \frac{1}{2N^2} \sum_{i=1}^N \int_t^T \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_{\mu}^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds,\end{aligned}\tag{3.4}$$

with

$$\left\{ \begin{aligned}\alpha_s &= \frac{1}{N} \sum_{i=1}^N \frac{H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N, N \tilde{Z}_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N, N \tilde{Z}_s^{i,N})}{\Delta Y_s^N} \mathbf{1}_{\Delta Y_s^N \neq 0} \\ \beta_s^i &= \frac{H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N, N \tilde{Z}_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N, N \tilde{Z}_s^{i,N})}{|\Delta Z_s^{i,N}|^2} \Delta Z_s^{i,N} \mathbf{1}_{\Delta Z_s^{i,N} \neq 0}\end{aligned}\right.\tag{3.5}$$

for $i = 1, \dots, N$, and we notice by Assumption 2.2(iv) that the processes α and β^i are bounded by $[H_b]_1$. Under Assumption 2.2(ii), let us define the bounded processes $\lambda_s^i = \sigma^+(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N))\beta_s^i$, $s \in [0, T]$, $i = 1, \dots, N$, and introduce the change of probability measure \mathbb{P}^λ with Radon-Nikodym density:

$$\frac{d\mathbb{P}^\lambda}{d\mathbb{P}} = \mathcal{E}_T^\lambda := \exp\left(\sum_{i=1}^N \int_0^T \lambda_s^i dW_s^i - \sum_{i=1}^N \frac{1}{2} \int_0^T |\lambda_s^i|^2 ds\right),$$

so that by Girsanov's theorem: $\widetilde{W}_t^i = W_t^i - \int_0^t \lambda_s^i ds$, $i = 1, \dots, N$, and W^0 are independent Brownian motion under \mathbb{P}^λ . By applying Itô's Lemma to $e^{\int_0^s \alpha_u ds} \Delta Y_t^N$ under \mathbb{P}^λ , we then obtain

$$\begin{aligned} \Delta Y_t^N &= \frac{1}{2N^2} \sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u du} \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u du} (\Delta Z_s^{i,N})^\top \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) d\widetilde{W}_s^i \\ &\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u du} (\Delta Z_s^{i,N})^\top \sigma_0(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0, \quad 0 \leq t \leq T. \end{aligned} \quad (3.6)$$

Step 3. Let us check that the stochastic integrals in (3.6), namely $\int \widetilde{Z}_s^{i,N} \cdot d\widetilde{W}_s^i$, and $\int \widetilde{Z}_s^{0,i,N} \cdot dW_s^0$ are "true" martingales under \mathbb{P}^λ , where $\widetilde{Z}_s^{i,N} := e^{\int_0^s \alpha_u du} \sigma^\top(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \Delta Z_s^{i,N}$, $\widetilde{Z}_s^{0,i,N} := e^{\int_0^s \alpha_u du} \sigma_0^\top(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \Delta Z_s^{i,N}$, $i = 1, \dots, N$, $0 \leq s \leq T$.

Indeed, for fixed $i \in \llbracket 1, N \rrbracket$, recalling that α is bounded, and by the linear growth condition of σ_0 from Assumption 2.2(i), we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^\lambda} \left[\int_0^T |\widetilde{Z}_s^{0,i,N}|^2 ds \right] &\leq C \mathbb{E}^{\mathbb{P}^\lambda} \left[\int_0^T (|\sigma_0(s, 0, \delta_0)|^2 + |X_s^{i,N}|^2 + \|\bar{\mu}(\mathbf{X}_s^N)\|_2^2) \right. \\ &\quad \left. (|N \Delta Z_s^{i,N}|^2 + |\partial_\mu v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N})|^2) ds \right] \\ &\leq C \mathbb{E} \left[\mathcal{E}_T^\lambda \int_0^T N^2 (|\sigma_0(s, 0, \delta_0)|^4 + |\mathbf{X}_s^N|^4) ds \right] \end{aligned}$$

where we use Bayes formula, the estimation (2.9), the growth condition on $\partial_\mu v(\cdot)(\cdot)$ in Assumption 2.1, and noting that $|X_s^{i,N}| \leq |\mathbf{X}_s^N|$, $\|\bar{\mu}(\mathbf{X}_s^N)\|_2^2 = |\mathbf{X}_s^N|^2/N$. By Hölder inequality with q as in Assumption 2.2(iii), and $\frac{1}{p} + \frac{1}{q} = 1$, the above inequality yields

$$\mathbb{E}^{\mathbb{P}^\lambda} \left[\int_0^T |\widetilde{Z}_s^{0,i,N}|^2 ds \right] \leq CN^2 \left(\mathbb{E} [|\mathcal{E}_T^\lambda|^p] \right)^{\frac{1}{p}} \left(\mathbb{E} \left[\int_0^T (|\sigma_0(s, 0, \delta_0)|^{4q} + |\mathbf{X}_s^N|^{4q}) ds \right] \right)^{\frac{1}{q}},$$

which is finite by (2.8), and since λ is bounded. This shows the square-integrable martingale property of $\int \widetilde{Z}_s^{0,i,N} \cdot dW_s^0$ under \mathbb{P}^λ . By the same arguments, we get the square-integrable martingale property of $\int \widetilde{Z}_s^{i,N} \cdot d\widetilde{W}_s^i$ under \mathbb{P}^λ .

Step 4: Estimation of \mathcal{E}_N^Y . By taking the \mathbb{P}^λ conditional expectation in (3.6), we obtain

$$\Delta Y_t^N = \frac{1}{2N^2} \sum_{i=1}^N \mathbb{E}^{\mathbb{P}^\lambda} \left[\int_t^T e^{\int_t^s \alpha_u du} \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds \middle| \mathcal{F}_t \right],$$

for all $t \in [0, T]$. Under the boundedness condition on $\Sigma = \sigma\sigma^\top$ in Assumption 2.2(ii), and on $\partial_\mu^2 v$ in Assumption 2.1, it follows immediately that

$$\mathcal{E}_N^Y = \sup_{0 \leq t \leq T} |\Delta Y_t^N| \leq \frac{C_y}{N}, \quad a.s.$$

where $C_y = \frac{T}{2} e^{[H_b]_1 T} L \|\sigma\|_\infty^2$, with $\|\sigma\|_\infty = \sup_{(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)} |\sigma(t, x, \mu)|$.

3.2 Proof of Theorem 2.7

From (3.5), and under Assumption 2.3(i) and (iii), we see that

$$\begin{aligned}
\alpha_s &= \frac{1}{N} \sum_{i=1}^N \frac{H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N, N\tilde{Z}_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N, N\tilde{Z}_s^{i,N})}{\Delta Y_s^N} \mathbb{1}_{\Delta Y_s^N \neq 0} \\
&= \frac{1}{N} \sum_{i=1}^N \frac{H_1(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) - H_1(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N)}{\Delta Y_s^N} \mathbb{1}_{\Delta Y_s^N \neq 0} \\
&\quad + \frac{1}{N} \sum_{i=1}^N N\tilde{Z}_s^{i,N} \cdot \frac{H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) - H_2(s, \bar{\mu}(\mathbf{X}_s^N), \tilde{Y}_s^N)}{\Delta Y_s^N} \mathbb{1}_{\Delta Y_s^N \neq 0}, \tag{3.7}
\end{aligned}$$

is bounded by $[H_1]_1 + [H_2]_1 L$, recalling $N\tilde{Z}_s^{i,N} = \partial_\mu v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N})$. As a consequence, the proof of Theorem 2.6 still applies. Then by (3.3)

$$\begin{aligned}
\Delta Y_t^N &= \int_t^T \alpha_s \Delta Y_s^N ds + \frac{1}{N} \sum_{i=1}^N \int_t^T H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) \cdot \Delta Z_s^{i,N} ds \\
&\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\top \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i \\
&\quad - \frac{1}{N} \sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\top \sigma_0(s, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0 \\
&\quad + \frac{1}{2N^2} \sum_{i=1}^N \int_t^T \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds.
\end{aligned}$$

By applying Itô's formula to $|\Delta Y_t^N|^2$ in (3.4) under \mathbb{P}

$$\begin{aligned}
|\Delta Y_0^N|^2 &+ \frac{1}{N^2} \int_0^T \sum_{i=1}^N |\sigma^\top(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \Delta Z_s^{i,N}|^2 ds + |\sigma_0^\top(s, \bar{\mu}(\mathbf{X}_s^N))|^2 \sum_{j=1}^N |\Delta Z_s^{j,N}|^2 ds \\
&= 2 \int_0^T \alpha_s |\Delta Y_s^N|^2 ds + \frac{2}{N} \sum_{i=1}^N \int_0^T \Delta Y_s^N H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) \cdot \Delta Z_s^{i,N} ds \\
&\quad - \frac{2}{N} \sum_{i=1}^N \int_0^T \Delta Y_s^N (\Delta Z_s^{i,N})^\top \sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) dW_s^i \\
&\quad - \frac{2}{N} \sum_{i=1}^N \int_0^T \Delta Y_s^N (\Delta Z_s^{i,N})^\top \sigma_0(s, \bar{\mu}(\mathbf{X}_s^N)) dW_s^0 \\
&\quad + \frac{1}{N^2} \sum_{i=1}^N \int_0^T \Delta Y_s^N \text{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N})) ds,
\end{aligned}$$

so by taking expectation under \mathbb{P} , and using the Cauchy-Schwarz inequality in \mathbb{R}^d

$$\begin{aligned}
& \frac{1}{N^2} \int_0^T \sum_{i=1}^N \mathbb{E} \left[|\sigma^\top(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \Delta Z_s^{i,N}|^2 + |\sigma_0^\top(s, \bar{\mu}(\mathbf{X}_s^N)) \sum_{j=1}^N \Delta Z_s^{j,N}|^2 \right] ds \\
& \leq \mathbb{E} \left[2 \int_0^T |\alpha_s| |\Delta Y_s^N|^2 ds + 2 \int_0^T \left| \Delta Y_s^N H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) \cdot \sum_{i=1}^N \frac{\Delta Z_s^{i,N}}{N} \right|^2 ds \right] \\
& \quad + \frac{1}{N^2} \sum_{i=1}^N \int_0^T \mathbb{E} |\Delta Y_s^N \operatorname{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N}))| ds, \\
& \leq \mathbb{E} \left[2 \int_0^T |\alpha_s| |\Delta Y_s^N|^2 ds + 2 \int_0^T \left| \Delta Y_s^N H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N) \right|^2 \sum_{i=1}^N \frac{\Delta Z_s^{i,N}}{N} ds \right] \\
& \quad + \frac{1}{N^2} \sum_{i=1}^N \int_0^T \mathbb{E} |\Delta Y_s^N \operatorname{tr}(\Sigma(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \partial_\mu^2 v(s, \bar{\mu}(\mathbf{X}_s^N))(X_s^{i,N}, X_s^{i,N}))| ds, \\
& \leq \mathbb{E} \left[\vartheta \int_0^T |\Delta Y_s^N H_2(s, \bar{\mu}(\mathbf{X}_s^N), Y_s^N)|^2 ds + \frac{1}{\vartheta N^2} \int_0^T \left| \sum_{i=1}^N \Delta Z_s^{i,N} \right|^2 ds \right] + \frac{\tilde{C}_z}{N^2},
\end{aligned}$$

by Young inequality for any $\vartheta > 0$, boundedness of α (see (3.7)), Σ , $\partial_\mu^2 v$ and Theorem 2.6, where $\tilde{C}_z = 2T([H_1]_1 + [H_2]_1 L) \bar{C}_y^2 + \bar{C}_y T \|\sigma\|_\infty^2 L$ and $\bar{C}_y = \frac{T}{2} e^{([H_1]_1 + [H_2]_1 L)T} L \|\sigma\|_\infty^2$. Thus, by Assumption 2.3(ii), and by choosing $\vartheta = \frac{2}{c_0}$, it follows from the boundedness of H_2 , and Theorem 2.6 that

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \int_0^T |\sigma^\top(s, X_s^{i,N}, \bar{\mu}(\mathbf{X}_s^N)) \Delta Z_s^{i,N}|^2 ds \right] \leq \frac{\tilde{C}_z + \frac{2}{c_0} \bar{C}_y^2 T \|H_2\|_\infty^2}{N},$$

which ends the proof by recalling that $\|\sigma^+\|_\infty < +\infty$, using Cauchy-Schwarz inequality in \mathbb{R}^N (in the form $\frac{1}{N} \sum_i \sqrt{|a_i|} \leq \sqrt{\frac{1}{N} \sum_i |a_i|}$) and Jensen inequality (in the form $\mathbb{E}[\sqrt{|X|}] \leq \sqrt{\mathbb{E}[|X|]}$).

References

- [BFY13] A. Bensoussan, J. Frehse, and P. Yam. *Mean field games and mean field type control theory*. Springer Briefs in Mathematics. Springer, 2013.
- [BFY15] A. Bensoussan, J. Frehse, and P. Yam. “The Master equation in mean-field theory”. In: *J. de Math. Pures et Appliquées* 103.6 (2015), pp. 1441–1474.
- [Buc+17] R. Buckdahn et al. “Mean-field stochastic differential equations and associated PDEs”. In: *Ann. Probab.* 45.2 (Mar. 2017), pp. 824–878.
- [Bur+20] M. Burzoni et al. “Viscosity solutions for controlled McKean-Vlasov jump-diffusions”. In: *SIAM Journal on Control and Optimization* 58.3 (2020), pp. 1676–1699.
- [Car+19] P. Cardaliaguet et al. *The Master equation and the convergence problem in mean-field games*. Vol. 201. Annals of mathematics studies. Princeton University Press, 2019.
- [CCD15] J.-F. Chassagneux, D. Crisan, and F. Delarue. “A probabilistic approach to classical solutions of the master equation for large population equilibria”. In: *to appear in Memoirs of the AMS* (2015).
- [CD18a] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games: vol. I, Mean Field FBSDEs, Control, and Games*, Springer, 2018.
- [CD18b] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games: vol. II, Mean Field FBSDEs, Control, and Games*, Springer, 2018.

- [CP19] A. Cosso and H. Pham. “Zero-sum stochastic differential games of generalized McKean-Vlasov type”. In: *Journal de Mathématiques Pures et Appliquées* 129 (2019), pp. 180–212.
- [Dje20] F Djete. “Extended mean-field control problem: a propagation of chaos result”. In: *arXiv:2006.12996* (2020).
- [DPT19] F. Djete, D. Possamai, and X. Tan. “McKean-Vlasov optimal control: the dynamic programming principle”. In: *arXiv:1907.08860* (2019).
- [FG15] N. Fournier and A. Guillin. “On the rate of convergence in the Wasserstein distance of the empirical measure”. In: *Probability Theory and Related Fields* 162 (2015), pp. 707–738.
- [Ger+21] M. Germain et al. “DeepSets and derivative networks for solving symmetric PDEs”. In: *arXiv: 2103.00838* (2021).
- [GMS21] W. Gangbo, S. Mayorga, and A. Swiech. “Finite Dimensional Approximations of Hamilton–Jacobi–Bellman Equations in Spaces of Probability Measures”. In: *SIAM Journal on Mathematical Analysis* 53.2 (2021), pp. 1320–1356.
- [GS15] W. Gangbo and A. Swiech. “Existence of a solution to an equation arising from the theory of mean-field games”. In: *Journal of Differential equations* 259.11 (2015), pp. 6573–6643.
- [Lac17] D. Lacker. “Limit theory for controlled McKean-Vlasov dynamics”. In: *SIAM Journal on Control and Optimization* 55.3 (2017), pp. 1641–1672.
- [LT19] M. Laurière and L. Tangpi. “Backward propagation of chaos”. In: *arXiv:1911.06835* (2019).
- [LT20] M. Laurière and L. Tangpi. “Convergence of large population games to mean field games with interaction through the controls”. In: *arXiv:2004.08351* (2020).
- [PP90] E. Pardoux and S. Peng. “Adapted solution of a backward stochastic differential equation”. In: *Systems & Control Letters* 14.1 (1990), pp. 55–61.
- [PW17] H. Pham and X. Wei. “Dynamic programming for optimal control of stochastic McKean-Vlasov dynamics”. In: *SIAM Journal on Control and Optimization* 55.2 (2017), pp. 1069–1101.
- [PW18] H. Pham and X. Wei. “Bellman equation and viscosity solutions for mean-field stochastic control problem”. In: *ESAIM: COCV* 24.1 (2018), pp. 437–461.
- [SZ19] Y. F. Saporito and J. Zhang. “Stochastic Control with Delayed Information and Related Nonlinear Master Equation”. In: *SIAM Journal on Control and Optimization* 57.1 (2019), pp. 693–717.
- [WZ20] C. Wu and J. Zhang. “Viscosity solutions to parabolic master equations and McKean–Vlasov SDEs with closed-loop controls”. In: *Ann. Appl. Probab.* 30.2 (2020), pp. 936–986.
- [Zha17] J. Zhang. *Backward stochastic differential equations: from linear to fully nonlinear theory*. Vol. 86. Probability theory and stochastic modelling. Springer, 2017.