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Rate of convergence for particles approximation of PDEs in Wasserstein space *

Maximilien Germain † Huyên Pham ‡ Xavier Warin §

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Abstract

We prove a rate of convergence of order $1/N$ for the N-particle approximation of a second-order partial differential equation in the space of probability measures, like the Master equation or Bellman equation of mean-field control problem under common noise. The proof relies on backward stochastic differential equations techniques.

1 Introduction

Let us consider the second-order parabolic partial differential equation (PDE) on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ of square-integrable probability measures on $\mathbb{R}^d$, in the form:

$$\left\{ \begin{array}{rcll} \partial_t v + \mathcal{H}(t,\mu,v,\partial_\mu v,\partial_x\partial_\mu v,\partial^2_\mu v) & = & 0, & (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ v(T,\mu) & = & G(\mu), & \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{array} \right. \tag{1.1}$$

Here, $\partial_\mu v(t,\mu)$ is the L-derivative on $\mathcal{P}_2(\mathbb{R}^d)$ (see [CD18a]) of $\mu\mapsto v(t,\mu)$, and it is a function from $\mathbb{R}^d$ into $\mathbb{R}^d$, $\partial_x\partial_\mu v(t,\mu)$ is the usual derivative on $\mathbb{R}^d$ of $x\in\mathbb{R}^d\mapsto\partial_\mu v(t,\mu)(x)\in\mathbb{R}^d$, hence valued in $\mathbb{R}^{d\times d}$ the set of $d\times d$-matrices with real coefficients, and $\partial^2_\mu v(t,\mu)$ is the L-derivative of $\mu\mapsto\partial_\mu v(t,\mu)(.)$, hence a function from $\mathbb{R}^d\times\mathbb{R}^d$ into $\mathbb{R}^{d\times d}$. The terminal condition is given by a real-valued function G on $\mathcal{P}_2(\mathbb{R}^d)$, and the Hamiltonian $\mathcal{H}$ of this PDE is assumed to be in semi-linear (non linear w.r.t. v, $\partial_\mu v$, and linear w.r.t. $\partial_x\partial_\mu$, $\partial^2_\mu v$) expectation form:

$$\begin{aligned} \mathcal{H}(t,\mu,y,\mathrm{z}(.),\gamma(.),\gamma_0(.,.)) = & \int_{\mathbb{R}^d}\big[H(t,x,\mu,y,\mathrm{z}(x)) + \frac{1}{2}\mathrm{tr}\big((\sigma\sigma^\intercal+\sigma_0\sigma_0^\intercal)(t,x,\mu)\gamma(x)\big)\big]\mu(\mathrm{d}x) \\ & + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{tr}\big(\sigma_0(t,x,\mu)\sigma_0^\intercal(t,x',\mu)\gamma_0(x,x')\big)\mu(\mathrm{d}x)\mu(\mathrm{d}x'), \end{aligned} \tag{1.2}$$

for some real-valued measurable function H defined on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d$, and where σ, σ_0 are measurable functions on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, valued respectively in $\mathbb{R}^{d\times n}$,

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and $\mathbb{R}^{d\times m}$. Here $\mathrm{tr}(M)$ denotes the trace of a square matrix M, while $M^\intercal$ is its transpose, and . is the scalar product.

PDEs in Wasserstein space have been largely studied in the literature over the last years, notably with the emergence of the mean-field game theory, and we mention among others the papers [BFY15], [GS15], [PW18], [Car+19], [SZ19], [Bur+20], and other references in the two-volume monographs [CD18a]-[CD18b]. An important application concerns mean-field type control problems with common noise. The controlled stochastic McKean-Vlasov dynamics is given by

$$\begin{aligned} \mathrm{d}X_s^\alpha = \ & \beta(s, X_s^\alpha, \mathbb{P}^0_{X_s^\alpha}, \alpha_s)\mathrm{d}s + \sigma(s, X_s^\alpha, \mathbb{P}^0_{X_s^\alpha})\mathrm{d}W_s \\ & + \ \sigma_0(s, X_s^\alpha, \mathbb{P}^0_{X_s^\alpha})\mathrm{d}W_s^0, \quad t \le s \le T, \ X_t^\alpha = \xi, \end{aligned} \tag{1.3}$$

where W is a n-dimensional Brownian motion, independent of a m-dimensional Brownian motion W^0 (representing the common noise) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, the control process α is $\mathbb{F}$-adapted valued in some Polish space A, and here $\mathbb{P}^0$ denotes the conditional law given W^0. The value function defined on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ by

$$v(t,\mu) \ = \ \inf_\alpha \mathbb{E}_{t,\mu}\Big[\int_t^T e^{-r(s-t)} f(X_s^\alpha, \mathbb{P}^0_{X_s^\alpha}, \alpha_s)\mathrm{d}s + e^{-r(T-t)} g(X_T^\alpha, \mathbb{P}^0_{X_T^\alpha})\Big],$$

(here $\mathbb{E}_{t,\mu}[\cdot]$ is the conditional expectation given that the law at time t of X solution to (1.3) is equal to μ) is shown to satisfy the Bellman equation (1.1)-(1.2) (see [BFY13], [CP19], [DPT19]) with $G(\mu) = \int g(x,\mu)\mu(dx)$, σ, σ_0 as in (1.3) and

$$H(t,x,\mu,y,z) = \ -ry + \inf_{a\in A}\big[\beta(t,x,\mu,a).z + f(x,\mu,a)\big]. \tag{1.4}$$

We now consider a finite-dimensional approximation of the PDE (1.1)-(1.2) in the Wasserstein space. This can be derived formally by looking at the PDE for μ to averages of Dirac masses, and it turns out that the corresponding PDE takes the form

$$\begin{cases} \partial_t v^N + \frac{1}{N}\displaystyle\sum_{i=1}^N H\big(t, x_i, \bar\mu(\boldsymbol{x}), v^N, ND_{x_i}v^N\big) + \frac{1}{2}\mathrm{tr}(\Sigma_N(t,\boldsymbol{x})D^2_{\boldsymbol{x}}v^N) \ = \ 0, \ \text{on } [0,T)\times(\mathbb{R}^d)^N \\ v^N(T,\boldsymbol{x}) \ = \ G\big(\bar\mu(\boldsymbol{x})\big), \quad \boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]} \in (\mathbb{R}^d)^N, \end{cases} \tag{1.5}$$

where $\bar\mu(.)$ is the empirical measure function defined by $\bar\mu(\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, for any $\boldsymbol{x} = (x_1,\ldots,x_N)$, $N\in\mathbb{N}^*$, and $\Sigma_N = (\Sigma_N^{ij})_{i,j\in[\![1,N]\!]}$ is the $\mathbb{R}^{Nd\times Nd}$-valued function with block matrices $\Sigma_N^{ij}(t,\boldsymbol{x}) = \sigma(t,x_i,\bar\mu(\boldsymbol{x}))\sigma^\intercal(t,x_j,\bar\mu(\boldsymbol{x}))\delta_{ij} + \sigma_0(t,x_i,\bar\mu(\boldsymbol{x}))\sigma_0^\intercal(t,x_j,\bar\mu(\boldsymbol{x})) \in \mathbb{R}^{d\times d}$. In the special case where H has the form (1.4), we notice that (1.5) is the Bellman equation for the N-cooperative problem, whose convergence to the mean-field control problem has been studied in [Lac17], [CD18b], [LT20], when $\sigma_0 \equiv 0$ (no common noise), and recently by [Dje20] in the common noise case. For more general Hamiltonian functions H, it has been recently proved in [GMS20] that the sequence of viscosity solutions $(v^N)_N$ to (1.5) converge locally uniformly to the viscosity solution v to (1.1).

In this paper, we adopt a probabilistic approach by considering a backward stochastic differential equation (BSDE) representation for the finite-dimensional PDE (1.5) according to the classical work [PP90]. The solution $(Y^N, \boldsymbol{Z}^N = (Z^{i,N})_{1\le i\le N})$ to this BSDE is written with an underlying forward particle system $\boldsymbol{X}^N = (X^{i,N})_{1\le i\le N}$ of a McKean-Vlasov

SDE, and connected to the PDE (1.5) via the Feynman-Kac formula: $Y_t^N = v^N(t, \boldsymbol{X}_t^N)$, $Z_t^{i,N} = D_{x_i} v^N(t, \boldsymbol{X}_t^{i,N})$, $0 \leq t \leq T$. By using BSDE techniques, our main contribution is to show a rate of convergence of order $1/N$ of $|Y_t^N - v(t, \bar{\mu}(\boldsymbol{X}_t^N))|$, and also of $|NZ_t^{i,N} - \partial_\mu v(t, \bar{\mu}(\boldsymbol{X}_t^N))(X_t^{i,N})|^2$, $i = 1, \ldots, N$, for suitable norms, and under some regularity conditions on v (see Theorem 2.4). This rate of convergence on the particles approximation of v and its L-derivative is new to the best of our knowledge. We point out that classical BSDE arguments for proving the rate of convergence do not apply directly due to the presence of the factor N in front of $D_{x_i} v^N$ in the generator H, and we rather use linearization arguments and change of probability measures to overcome these issues.

The outline of the paper is organized as follows. In Section 2, we formulate the particle approximation of the PDE and its BSDE representation, and state the rate of convergence. Section 3 is devoted to the proof of this result.

2 Particles approximation of Wasserstein PDEs

The formal derivation of the finite-dimensional approximation PDE is obtained as follows. We look at the PDE (1.1)-(1.2) for $\mu = \bar{\mu}(\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \in \mathcal{P}_2(\mathbb{R}^d)$, when $\boldsymbol{x} = (x_i)_{i \in [\![1,N]\!]}$ runs over $(\mathbb{R}^d)^N$. By setting $\tilde{v}^N(t, \boldsymbol{x}) = v(t, \bar{\mu}(\boldsymbol{x}))$, and assuming that v is smooth, we have for all $(i,j) \in [\![1,N]\!]$ (see Proposition 5.35 and Proposition 5.91 in [CD18a]):

$$\left\{ \begin{array}{rcl} D_{x_i}\tilde{v}^N(t,\boldsymbol{x}) &=& \frac{1}{N}\partial_\mu v(t,\bar{\mu}(\boldsymbol{x}))(x_i), \\ D^2_{x_i x_j}\tilde{v}^N(t,\boldsymbol{x}) &=& \frac{1}{N}\partial_x\partial_\mu v(t,\bar{\mu}(\boldsymbol{x}))(x_i)\delta_{ij} + \frac{1}{N^2}\partial^2_\mu v(t,\bar{\mu}(\boldsymbol{x}))(x_i,x_j). \end{array} \right. \tag{2.1}$$

By substituting into the PDE (1.1)-(1.2) for $\mu = \bar{\mu}(\boldsymbol{x})$, and using (2.1), we then see that $\tilde{v}^N$ satisfies the relation:

$$\begin{aligned} &\partial_t \tilde{v}^N + \frac{1}{N}\sum_{i=1}^N H\big(t, x_i, \bar{\mu}(\boldsymbol{x}), \tilde{v}^N, ND_{x_i}\tilde{v}^N\big) \\ &+ \frac{1}{2}\sum_{i=1}^N \mathrm{tr}\big[\big(\sigma\sigma^\intercal + \sigma_0\sigma_0^\intercal\big)(t,x_i,\bar{\mu}(\boldsymbol{x}))\big(D^2_{x_i}\tilde{v}^N - \frac{1}{N^2}\partial^2_\mu v(t,\bar{\mu}(\boldsymbol{x}))(x_i,x_i)\big)\big] \\ &+ \frac{1}{2}\sum_{i\neq j\in[\![1,N]\!]} \mathrm{tr}\big(\sigma_0(t,x_i,\bar{\mu}(\boldsymbol{x}))\sigma_0^\intercal(t,x_j,\bar{\mu}(\boldsymbol{x}))D^2_{x_i x_j}\tilde{v}^N\big) \\ &+ \frac{1}{2N^2}\sum_{i=1}^N \mathrm{tr}\big(\sigma_0\sigma_0^\intercal(t,x_i,\bar{\mu}(\boldsymbol{x}))\partial^2_\mu v(t,\bar{\mu}(\boldsymbol{x}))(x_i,x_i)\big) = 0 \end{aligned} \tag{2.2}$$

for $(t, \boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]}) \in [0,T) \times (\mathbb{R}^d)^N$, together with the terminal condition $\tilde{v}^N(t,\boldsymbol{x}) = G(\bar{\mu}(\boldsymbol{x}))$. By neglecting the terms $\partial^2_\mu v/N^2$ in the above relation, we obtain the PDE (1.5) for $v^N \simeq \tilde{v}^N$. The purpose of this section is to rigorously justify this approximation and state a rate of convergence for v^N towards v, as well as a convergence for their gradients.

2.1 Particles BSDE approximation

Let us introduce an arbitrary measurable $\mathbb{R}^d$-valued function b on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and set B_N the $(\mathbb{R}^d)^N$-valued function defined on $[0,T]\times(\mathbb{R}^d)^N$ by $B_N(t,\boldsymbol{x}) = (b(t,x_i,\bar{\mu}(\boldsymbol{x}))_{i\in[\![1,N]\!]}$

for $(t, \boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]}) \in [0,T) \times (\mathbb{R}^d)^N$. The finite-dimensional PDE (1.5) may then be written as

$$\left\{\begin{array}{l} \partial_t v^N + B_N(t,\boldsymbol{x}).D_{\boldsymbol{x}} v^N + \frac{1}{2}\mathrm{tr}\big(\Sigma_N(t,\boldsymbol{x})D^2_{\boldsymbol{x}} v^N\big) \\ \qquad + \frac{1}{N}\sum_{i=1}^N H_b\big(t, x_i, \bar{\mu}(\boldsymbol{x}), v^N, ND_{x_i} v^N\big) \;=\; 0, \quad \text{on } [0,T)\times(\mathbb{R}^d)^N, \\ v^N(T,\boldsymbol{x}) \;=\; G\big(\bar{\mu}(\boldsymbol{x})\big), \quad \boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]} \in (\mathbb{R}^d)^N, \end{array}\right. \tag{2.3}$$

where $H_b(t,x,\mu,y,z) := H(t,x,\mu,y,z) - b(t,x,\mu).z$. For error analysis purpose, the function b can be simply taken to be zero. The introduction of the function b is actually motivated by numerical purpose. It corresponds indeed to the drift of training simulations for approximating the function v^N, notably by machine learning methods, and should be chosen for suitable exploration of the state space, see a detailed discussion in our companion paper [Ger+21]. In this paper, we fix an arbitrary function b (satisfying Lipschitz condition to be precised later).

Following [PP90], it is well-known that the semi-linear PDE (2.3) admits a probabilistic representation in terms of forward backward SDE. The forward component is defined by the process $\boldsymbol{X}^N = (X^{i,N})_{i\in[\![1,N]\!]}$ valued in $(\mathbb{R}^d)^N$, solution to the SDE:

$$\mathrm{d}\boldsymbol{X}^N_t = \; B_N(t,\boldsymbol{X}^N_t)\mathrm{d}t + \sigma_N(t,\boldsymbol{X}^N_t)\mathrm{d}\boldsymbol{W}_t + \boldsymbol{\sigma}_0(t,\boldsymbol{X}^N_t)\mathrm{d}W^0_t \tag{2.4}$$

where σ_N is the block diagonal matrix with block diagonals $\sigma^{ii}_N(t,\boldsymbol{x}) = \sigma(t,x_i,\bar{\mu}(\boldsymbol{x}))$, $\boldsymbol{\sigma}_0 = (\sigma^i_0)_{i\in[\![1,N]\!]}$ is the $(\mathbb{R}^{d\times m})^N$-valued function with $\boldsymbol{\sigma}^i_0(t,\boldsymbol{x}) = \sigma_0(t,x_i,\bar{\mu}(\boldsymbol{x}))$, for $\boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]}$, $\boldsymbol{W} = (W^1,\ldots,W^N)$ where W^i, $i = 1,\ldots,N$, are independent n-dimensional Brownian motions, independent of a m-dimensional Brownian motion W^0 on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$. Notice that $\Sigma_N = \sigma_N\sigma_N^\intercal + \boldsymbol{\sigma}_0\boldsymbol{\sigma}_0^\intercal$, and $\boldsymbol{X}^N$ is the particles system of the McKean-Vlasov SDE:

$$\mathrm{d}X_t = \; b(t,X_t,\mathbb{P}_{X_t})\mathrm{d}t + \sigma(t,X_t,\mathbb{P}^0_{X_t})\mathrm{d}W_t + \sigma_0(t,X_t,\mathbb{P}^0_{X_t})\mathrm{d}W^0_t, \tag{2.5}$$

where W is an n-dimensional Brownian motion independent of W^0. The backward component is defined by the pair process $(Y^N, \boldsymbol{Z}^N = (Z^{i,N})_{i\in[\![1,N]\!]})$ valued in $\mathbb{R}\times(\mathbb{R}^d)^N$, solution to

$$\begin{aligned} Y^N_t =\; & G\big(\bar{\mu}(\boldsymbol{X}^N_T)\big) + \frac{1}{N}\sum_{i=1}^N\int_t^T H_b(s,X^{i,N}_s,\bar{\mu}(\boldsymbol{X}^N_s),Y^N_s,NZ^{i,N}_s)\mathrm{d}s \\ & - \sum_{i=1}^N\int_t^T (Z^{i,N}_s)^\intercal\sigma\big(s,X^{i,N}_s,\bar{\mu}(\boldsymbol{X}^N_s)\big)\mathrm{d}W^i_s, \\ & - \sum_{i=1}^N\int_t^T (Z^{i,N}_s)^\intercal\sigma_0\big(s,X^{i,N}_s,\bar{\mu}(\boldsymbol{X}^N_s)\big)\mathrm{d}W^0_s, \qquad 0\le t\le T. \end{aligned} \tag{2.6}$$

We shall assume that the measurable functions $(t,x,\mu)\mapsto b(t,x,\mu)$, $\sigma(t,x,\mu)$ satisfy a Lipschitz condition in $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ uniformly w.r.t. $t\in[0,T]$, which ensures the existence and uniqueness of a strong solution $\boldsymbol{X}^N\in\mathcal{S}^2_{\mathbb{F}}((\mathbb{R}^d)^N)$ to (2.4) given an initial condition. Here, $\mathcal{S}^2_{\mathbb{F}}(\mathbb{R}^q)$ is the set of $\mathbb{F}$-adapted process $(V_t)_t$ valued in $\mathbb{R}^q$ s.t. $\mathbb{E}\big[\sup_{0\le t\le T}|V_t|^2\big] < \infty$, ($|.|$ is the Euclidian norm on $\mathbb{R}^q$, and for a matrix M, we choose the Frobenius norm

$|M| = \sqrt{\mathrm{tr}(MM^\intercal)}$), and the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ is endowed with the Wasserstein distance

$$\mathcal{W}_2(\mu,\mu') = \Big(\inf\big\{\mathbb{E}|\xi-\xi'|^2 : \xi\sim\mu, \xi'\sim\mu'\big\}\Big)^{\frac{1}{2}},$$

and we set $\|\mu\|_2 := \big(\int_{\mathbb{R}^d}|x|^2\ \mu(\mathrm{d}x)\big)^{\frac{1}{2}}$ for $\mu\in\mathcal{P}_2(\mathbb{R}^d)$. Assuming also that the measurable function $(t,x,\mu,y,z)\mapsto H_b(t,x,\mu,y,z)$ is Lipschitz in $(y,z)\in\mathbb{R}\times\mathbb{R}^d$ uniformly with respect to $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, and the measurable function G satisfied a quadratic growth condition on $\mathcal{P}_2(\mathbb{R}^d)$, we have the existence and uniqueness of a solution $(Y^N, \boldsymbol{Z}^N = (Z^{i,N})_{i\in[\![1,N]\!]})\in\mathcal{S}^2_{\mathbb{F}}(\mathbb{R})\times\mathbb{H}^2_{\mathbb{F}}((\mathbb{R}^d)^N)$ to (2.6), and the connection with the PDE (2.3) (satisfied in general in the viscosity sense) via the (non linear) Feynman-Kac formula:

$$Y^N_t = v^N(t,\boldsymbol{X}^N_t), \quad \text{and} \quad Z^{i,N}_t = D_{x_i}v^N(t,\boldsymbol{X}^N_t),\ \ i=1,\dots,N,\ \ 0\le t\le T, \tag{2.7}$$

(when v^N is smooth for the last relation). Here, $\mathbb{H}^2_{\mathbb{F}}(\mathbb{R}^q)$ is the set of $\mathbb{F}$-adapted process $(V_t)_t$ valued in $\mathbb{R}^q$ s.t. $\mathbb{E}\big[\int_0^T|V_t|^2\mathrm{d}t\big]<\infty$.

2.2 Main result

We aim to analyze the particles approximation error on the solution v to the PDE (1.1), and its L-derivative $\partial_\mu v$ by considering the following pathwise quantities:

$$\mathcal{E}^{\mathrm{y}}_N := \sup_{0\le t\le T}\big|Y^N_t - v(t,\bar\mu(\boldsymbol{X}^N_t))\big|,$$

$$\mathcal{E}^{\mathrm{z}}_N := \frac{1}{N}\sum_{i=1}^N\int_0^T\big|NZ^{i,N}_t - \partial_\mu v(t,\bar\mu(\boldsymbol{X}^N_t))(X^{i,N}_t)\big|^2\mathrm{d}t,$$

where the initial conditions of the particles system, $X^{i,N}_0$, $i=1,\dots,N$, are i.i.d. with distribution μ_0.

Here, it is assumed that we have the existence and uniqueness of a classical solution v to the PDE (1.1)-(1.2). More precisely, we make the following assumption:

Assumption 2.1 (Smooth solution to the Master Bellman PDE)**.** *There exists a unique solution v to to* (1.1)*, which lies in $C^{1,2}_b([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ that is:*

- *$v(.,\mu)\in C^1([0,T))$, and continuous on $[0,T]$, for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$,*
- *$v(t,.)$ is fully C^2 on $\mathcal{P}_2(\mathbb{R}^d)$, for any $t\in[0,T]$ in the sense that: $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\mapsto\partial_\mu v(t,\mu)(x)\in\mathbb{R}^d$, $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\mapsto\partial_x\partial_\mu v(t,\mu)(x)\in\mathbb{M}^d$, and $(x,x',\mu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\mapsto\partial^2_\mu v(t,\mu)(x,x')\in\mathbb{M}^d$, are well-defined and jointly continuous,*
- *there exists some constant $L>0$ s.t. for all $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$*

$$\big|\partial_\mu v(t,\mu)(x)\big| \le L\big(1+|x|+\|\mu\|_2\big), \quad \big|\partial^2_\mu v(t,\mu)(x,x)\big| \le L.$$

The existence of classical solutions to mean-field PDE in Wasserstein space is a challenging problem, and beyond the scope of this paper. We refer to [Buc+17], [CCD15], [SZ19], [WZ20] for conditions ensuring regularity results of some Master PDEs. Notice also that linear-quadratic mean-field control problems have explicit smooth solutions as in Assumption 2.1, see e.g. [PW18].

We also make some rather standard assumptions on the coefficients of the forward backward SDE:

Assumption 2.2 (Lipschitz condition on the coefficients of the forward backward SDE)**.**

(i) The drift and volatility coefficients b, σ, σ_0 are Lipschitz: there exist positive constants $[b]$, $[\sigma]$, and $[\sigma_0]$ s.t. for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}
|b(t,x,\mu) - b(t,x',\mu')| &\leq [b]\big(|x-x'| + \mathcal{W}_2(\mu,\mu')\big)\\
|\sigma(t,x,\mu) - \sigma(t,x',\mu')| &\leq [\sigma]\big(|x-x'| + \mathcal{W}_2(\mu,\mu')\big)\\
|\sigma_0(t,x,\mu) - \sigma_0(t,x',\mu')| &\leq [\sigma_0]\big(|x-x'| + \mathcal{W}_2(\mu,\mu')\big).
\end{aligned}$$

(ii) For all $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $\Sigma(t,x,\mu) := \sigma\sigma^\intercal(t,x,\mu)$ is invertible, and the function σ, and its pseudo-inverse $\sigma^+ := \sigma^\intercal \Sigma^{-1}$ are bounded.

(iii) $\mu_0 \in \mathcal{P}_{4q}(\mathbb{R}^d)$ for some $q > 1$, i.e., $\|\mu_0\|_{4q} := \big(\int |x|^{4q}\mu_0(\mathrm{d}x)\big)^{\frac{1}{4q}} < \infty$, and

$$\int_0^T |b(t,0,\delta_0)|^{4q} + |\sigma(t,0,\delta_0)|^{4q} + |\sigma_0(t,0,\delta_0)|^{4q}\,\mathrm{d}t < \infty.$$

(iv) The driver H_b satisfies the Lipschitz condition: there exist positive constants $[H_b]_1$ and $[H_b]_2$ s.t. for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned}
|H_b(t,x,\mu,y,z) - H_b(t,x,\mu,y',z')| &\leq [H_b]_1(|y-y'| + |z-z'|)\\
|H_b(t,x,\mu,y,z) - H_b(t,x',\mu',y,z)| &\leq [H_b]_2\big(1 + |x| + |x'| + \|\mu\|_2 + \|\mu'\|_2\big)\\
&\qquad \big(|x-x'| + \mathcal{W}_2(\mu,\mu')\big).
\end{aligned}$$

(v) The terminal condition satisfies the (locally) Lipschitz condition: there exists some positive constant $[G]$ s.t. for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$

$$|G(\mu) - G(\mu')| \leq [G]\big(\|\mu\|_2 + \|\mu'\|_2\big)\mathcal{W}_2(\mu,\mu').$$

Remark 2.3. *The Lipschitz condition on b, σ in Assumption 2.2(i) implies that the functions $\boldsymbol{x} \in (\mathbb{R}^d)^N \mapsto B_N(t,\boldsymbol{x})$, resp. $\sigma_N(t,\boldsymbol{x})$ and $\boldsymbol{\sigma}_0(t,\boldsymbol{x})$, defined in (2.4), are Lipschitz (with Lipschitz constant $2[b]$, resp. $2[\sigma]$ and $2[\sigma_0]$). Indeed, we have*

$$\begin{aligned}
|B_N(t,\boldsymbol{x}) - B_N(t,\boldsymbol{x}')|^2 &= \sum_{i=1}^N |b(t,x_i,\bar{\mu}(\boldsymbol{x})) - b(t,x_i,\bar{\mu}(\boldsymbol{x}'))|^2\\
&\leq 2[b]^2 \sum_{i=1}^N (|x_i - x_i'|^2 + \mathcal{W}_2(\bar{\mu}(\boldsymbol{x}),\bar{\mu}(\boldsymbol{x}'))^2)\\
&\leq 2[b]^2(|\boldsymbol{x}-\boldsymbol{x}'|^2 + \sum_{i=1}^N \frac{1}{N}|\boldsymbol{x}-\boldsymbol{x}'|^2) \;=\; 4[b]^2|\boldsymbol{x}-\boldsymbol{x}'|^2,
\end{aligned}$$

for $\boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]}$, and similarly for σ_N and $\boldsymbol{\sigma}_0$. This yields the existence and uniqueness of a solution $\boldsymbol{X}^N = (X^{i,N})_{i\in[\![1,N]\!]}$ to (2.4) given initial conditions. Moreover, under Assumption 2.2(iii), we have the standard estimate:

$$\mathbb{E}\big[\sup_{0\leq t\leq T} |\boldsymbol{X}_t^N|^{4q}\big] \leq C\big(1 + \|\mu_0\|_{4q}^{4q}\big) < \infty, \quad i = 1,\dots,N, \tag{2.8}$$

for some constant C (possibly depending on N). The Lipschitz condition on H_b w.r.t. (y,z) in Assumption 2.2(iv), and the quadratic growth condition on G from Assumption 2.2(v) gives the existence and uniqueness of a solution $(Y^N, \boldsymbol{Z}^N = (Z^{i,N})_{i\in[\![1,N]\!]}) \in \mathcal{S}^2_{\mathbb{F}}(\mathbb{R}) \times \mathbb{H}^2_{\mathbb{F}}((\mathbb{R}^d)^N)$ to (2.6). Moreover, by Assumption 2.2(iv)(v), we see that

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{i=1}^N H_b(t,x_i,\bar\mu(\boldsymbol{x}),y,z_i) - \frac{1}{N}\sum_{i=1}^N H_b(t,x'_i,\bar\mu(\boldsymbol{x}'),y,z_i)\Big| \\
&\le [H_b]_2 \frac{1}{N}\sum_{i=1}^N \big(1+|x_i|+|x'_i|+\frac{1}{\sqrt{N}}(|\boldsymbol{x}|+|\boldsymbol{x}'|)\big)\big(|x_i-x'_i|+\frac{1}{\sqrt{N}}|\boldsymbol{x}-\boldsymbol{x}'|\big) \\
&\le 4[H_b]_2\big(1+|\boldsymbol{x}|+|\boldsymbol{x}'|\big)|\boldsymbol{x}-\boldsymbol{x}'| \\
\big|G(\bar\mu(\boldsymbol{x})) - G(\bar\mu(\boldsymbol{x}'))\big| &\le \frac{[G]}{N}(|\boldsymbol{x}|+|\boldsymbol{x}'|)|\boldsymbol{x}-\boldsymbol{x}'|,
\end{aligned}$$

for all $x, x' \in \mathbb{R}^d$, $\boldsymbol{x} = (x_i)_{i\in[\![1,N]\!]}$, $\boldsymbol{x}' = (x'_i)_{i\in[\![1,N]\!]} \in (\mathbb{R}^d)^N$, which yields by standard stability results for BSDE (see e.g. Theorems 4.2.1 and 5.2.1 in [Zha17]) that the function v^N in (2.7) inherits the locally Lipschitz condition:

$$\big|v^N(t,\boldsymbol{x}) - v^N(t,\boldsymbol{x}')\big| \le C(1+|\boldsymbol{x}|+|\boldsymbol{x}'|)|\boldsymbol{x}-\boldsymbol{x}'|, \quad \forall \boldsymbol{x},\boldsymbol{x}' \in (\mathbb{R}^d)^N.$$

for some constant C (possibly depending on N). This implies

$$|Z^{i,N}_t| \le C\big(1+|\boldsymbol{X}^N_t|\big), \quad 0\le t\le T, \quad i=1,\dots,N, \tag{2.9}$$

(this is clear when v^N is smooth, and otherwise obtained by a mollifying argument as in Theorem 5.2.2 in [Zha17]).

Theorem 2.4. *Under Assumptions 2.1 and 2.2, we have $\mathbb{P}$-almost surely*

$$\mathcal{E}^{\mathrm{y}}_N + \mathcal{E}^{\mathrm{z}}_N = O\big(N^{-1}\big),$$

where the constant in the big Bachmann-Landau term $O(.)$ depends only on the data of the PDE-BSDE problem.

Remark 2.5. *Let us consider the global weak errors on v and its L-derivative $\partial_\mu v$ along the limiting McKean-Vlasov SDE, and defined by*

$$\begin{aligned}
E^{\mathrm{y}}_N &:= \sup_{0\le t\le T}\big|\mathbb{E}[Y^N_t] - \mathbb{E}[v(t,\mathbb{P}^0_{X_t})]\big| \\
E^{\mathrm{z}}_N &:= \frac{1}{N}\sum_{i=1}^N \int_0^T \Big|\mathbb{E}\big[NZ^{i,N}_t\big] - \mathbb{E}\big[\partial_\mu v(t,\mathbb{P}^0_{X^i_t})(X^i_t)\big]\Big|^2 dt,
\end{aligned}$$

where X^i has the same law than X, and with McKean-Vlasov dynamics as in (2.5) but driven by W^i, $i=1,\dots,N$. Then, they can be decomposed as

$$E^{\mathrm{y}}_N \le \mathbb{E}\big[\mathcal{E}^{\mathrm{y}}_N\big] + \tilde E^{\mathrm{y}}_N, \quad E^{\mathrm{z}}_N \le 2\big(\mathbb{E}\big[\mathcal{E}^{\mathrm{z}}_N\big] + \tilde E^{\mathrm{z}}_N\big),$$

where $\tilde{E}^{\mathrm{y}}_N$, $\tilde{E}^{\mathrm{z}}_N$ are the (weak) propagation of chaos errors defined by

$$\tilde{E}^{\mathrm{y}}_N := \sup_{0\le t\le T} \big|\mathbb{E}[v(t,\bar{\mu}(\boldsymbol{X}^N_t))] - \mathbb{E}[v(t,\mathbb{P}^0_{X_t})]\big|$$
$$\tilde{E}^{\mathrm{z}}_N := \frac{1}{N}\sum_{i=1}^N \int_0^T \Big|\mathbb{E}\big[\partial_\mu v(t,\bar{\mu}(\boldsymbol{X}^N_t))(X^{i,N}_t)\big] - \mathbb{E}\big[\partial_\mu v(t,\mathbb{P}^0_{X^i_t})(X^i_t)\big]\Big|^2 \mathrm{d}t,$$

From the conditional propagation of chaos result, which states that for any fixed $k\ge 1$, the law of $(X^{i,N}_t)^{i\in[\![1,k]\!]}_{t\in[0,T]}$ converges toward the conditional law of $(X^i_t)^{i\in[\![1,k]\!]}_{t\in[0,T]}$, as N goes to infinity, we deduce that $\tilde{E}^{\mathrm{y}}_N, \tilde{E}^{\mathrm{z}}_N \to 0$. Furthermore, under additional assumptions on v, we can obtain a rate of convergence. Namely, if $v(t,.)$ is Lipschitz uniformly in $t\in[0,T]$, with Lipschitz constant $[v]$, we have

$$\tilde{E}^{\mathrm{y}}_N \le [v] \sup_{0\le t\le T}\Big(\mathbb{E}[\mathcal{W}_2(\bar{\mu}(\boldsymbol{X}^N_t),\mathbb{P}^0_{X_t})^2]\Big)^{\frac{1}{2}} = O\Big(N^{-\frac{1}{\max(d,4)}}\sqrt{1+\ln(N)1_{d=4}}\Big),$$
$$\textit{hence}\quad E^{\mathrm{y}}_N = O\Big(N^{-\frac{1}{\max(d,4)}}\sqrt{1+\ln(N)1_{d=4}}\Big), \tag{2.10}$$

where we use the rate of convergence of empirical measures in Wasserstein distance stated in [FG15], and since we have the standard estimate $\mathbb{E}[\sup_{0\le t\le T}|X_t|^{4q}] \le C(1+\|\mu_0\|^{4q}_{2q})$ by Assumption 2.2(iii). The rate of convergence in (2.10) is consistent with the one found in Theorem 6.17 [CD18b] for mean-field control problem. Furthermore, if the function $\partial_\mu v(t,.)(.)$ is Lipschitz in (x,μ) uniformly in t, then by the rate of convergence in Theorem 2.12 of [CD18b], we have

$$\tilde{E}^{\mathrm{z}}_N = O\big(N^{-\frac{2}{\max(d,4)}}\big(1+\ln(N)1_{d=4}\big)\big),\quad \textit{hence}\quad E^{\mathrm{z}}_N = O\big(N^{-\frac{2}{\max(d,4)}}\big(1+\ln(N)1_{d=4}\big)\big).$$

3 Proof of Theorem 2.4

<u>*Step 1.*</u> Under the smoothness condition on v in Assumption 2.1, one can apply the standard Itô's formula in $(\mathbb{R}^d)^N$ to the process $\tilde{v}^N(t,\boldsymbol{X}^N_t) = v(t,\bar{\mu}(\boldsymbol{X}^N_t))$, and get

$$\begin{aligned}\tilde{v}^N(t,\boldsymbol{X}^N_t) =\ & \tilde{v}^N(T,\boldsymbol{X}^N_T) - \int_t^T \partial_t\tilde{v}^N(s,\boldsymbol{X}^N_s)\mathrm{d}s \\ & - \int_t^T B_N(s,\boldsymbol{X}^N_s).D_{\boldsymbol{x}}\tilde{v}^N(s,\boldsymbol{X}^N_s) + \frac{1}{2}\mathrm{tr}\big(\Sigma_N(s,\boldsymbol{X}^N_s)D^2_{\boldsymbol{x}}\tilde{v}^N(s,\boldsymbol{X}^N_s)\big)\mathrm{d}s \\ & - \sum_{i=1}^N\int_t^T \big(D_{x_i}\tilde{v}^N(s,\boldsymbol{X}^N_s)\big)^\intercal\sigma(s,X^{i,N}_s,\bar{\mu}(\boldsymbol{X}^N_s))\mathrm{d}W^i_s \\ & - \sum_{i=1}^N\int_t^T \big(D_{x_i}\tilde{v}^N(s,\boldsymbol{X}^N_s)\big)^\intercal\sigma_0(s,X^{i,N}_s,\bar{\mu}(\boldsymbol{X}^N_s))\mathrm{d}W^0_s,\end{aligned} \tag{3.1}$$

Now, by setting (recall (2.1)):

$$\tilde{Y}^N_t := v(t,\bar{\mu}(\boldsymbol{X}^N_t)) = \tilde{v}^N(t,\boldsymbol{X}^N_t),$$
$$\tilde{Z}^{i,N}_t := \frac{1}{N}\partial_\mu v(t,\bar{\mu}(\boldsymbol{X}^N_t))(X^{i,N}_t) = D_{x_i}\tilde{v}^N(t,\boldsymbol{X}^N_t),\quad i=1,\dots,N,\quad 0\le t\le T,$$

and using the relation (2.2) satisfied by $\tilde{v}^N$ into (3.1), we have for all $0 \le t \le T$,

$$\begin{aligned}\tilde{Y}_t^N =\ & G\big(\bar{\mu}(\boldsymbol{X}_T^N)\big) + \frac{1}{N}\sum_{i=1}^N \int_t^T H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), \tilde{Y}_s^N, N\tilde{Z}_s^{i,N})\mathrm{d}s \\ & - \frac{1}{2N^2}\sum_{i=1}^N \int_t^T \mathrm{tr}\big(\Sigma(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N))\partial_\mu^2 v(s, \bar{\mu}(\boldsymbol{X}_s^N))(X_s^{i,N}, X_s^{i,N})\big)\mathrm{d}s \\ & - \sum_{i=1}^N \int_t^T (\tilde{Z}_s^{i,N})^\intercal \sigma\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^i - \sum_{i=1}^N \int_t^T (\tilde{Z}_s^{i,N})^\intercal \sigma_0\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^0.\end{aligned} \tag{3.2}$$

<u>*Step 2*</u>: *Linearization.* We set

$$\Delta Y_t^N := Y_t^N - \tilde{Y}_t^N, \quad \Delta Z_t^{i,N} := N(Z_t^{i,N} - \tilde{Z}_t^{i,N}), \quad i = 1, \dots, N, \quad 0 \le t \le T,$$

so that by (2.6)-(3.2),

$$\begin{aligned}\Delta Y_t^N =\ & \frac{1}{N}\sum_{i=1}^N \int_t^T \big[H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), Y_s^N, NZ_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), \tilde{Y}_s^N, N\tilde{Z}_s^{i,N})\big]\mathrm{d}s \\ & + \frac{1}{2N^2}\sum_{i=1}^N \int_t^T \mathrm{tr}\big(\Sigma(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N))\partial_\mu^2 v(s, \bar{\mu}(\boldsymbol{X}_s^N))(X_s^{i,N}, X_s^{i,N})\big)\mathrm{d}s \\ & - \frac{1}{N}\sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\intercal \sigma\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^i \\ & - \frac{1}{N}\sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\intercal \sigma_0\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^0, \qquad 0 \le t \le T.\end{aligned}$$

We now use the linearization method for BSDEs and rewrite the above equation as

$$\begin{aligned}\Delta Y_t^N =\ & \int_t^T \alpha_s \Delta Y_s^N \mathrm{d}s + \frac{1}{N}\sum_{i=1}^N \int_t^T \beta_s^i . \Delta Z_s^{i,N}\mathrm{d}s \\ & - \frac{1}{N}\sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\intercal \sigma\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^i \\ & - \frac{1}{N}\sum_{i=1}^N \int_t^T (\Delta Z_s^{i,N})^\intercal \sigma_0\big(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^0 \\ & + \frac{1}{2N^2}\sum_{i=1}^N \int_t^T \mathrm{tr}\big(\Sigma(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N))\partial_\mu^2 v(s, \bar{\mu}(\boldsymbol{X}_s^N))(X_s^{i,N}, X_s^{i,N})\big)\mathrm{d}s,\end{aligned}$$

with

$$\begin{cases} \alpha_s = \frac{1}{N}\sum_{i=1}^N \dfrac{H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), Y_s^N, NZ_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), \tilde{Y}_s^N, NZ_s^{i,N})}{\Delta Y_s^N}\mathbb{1}_{\Delta Y_s^N \neq 0} \\ \beta_s^i = \frac{H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), \tilde{Y}_s^N, NZ_s^{i,N}) - H_b(s, X_s^{i,N}, \bar{\mu}(\boldsymbol{X}_s^N), \tilde{Y}_s^N, N\tilde{Z}_s^{i,N})}{|\Delta Z_s^{i,N}|^2}\Delta Z_s^{i,N}\mathbb{1}_{\Delta Z_s^{i,N} \neq 0} \end{cases}$$

for $i = 1, \ldots, N$, and we notice by Assumption 2.2(iv) that the processes α and β^i are bounded by $[H_b]_1$. Under Assumption 2.2(ii), let us define the bounded processes $\lambda^i_s = \sigma^+(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s))\beta^i_s$, $s \in [0,T]$, $i = 1, \ldots, N$, and introduce the change of probability measure $\mathbb{P}^\lambda$ with Radon-Nikodym density:

$$\frac{\mathrm{d}\mathbb{P}^\lambda}{\mathrm{d}\mathbb{P}} = \mathcal{E}^\lambda_T := \exp\Big(\sum_{i=1}^N \int_0^T \lambda^i_s \mathrm{d}W^i_s - \sum_{i=1}^N \frac{1}{2}\int_0^T |\lambda^i_s|^2 \mathrm{d}s\Big),$$

so that by Girsanov's theorem: $\widetilde{W}^i_t = W^i_t - \int_0^t \lambda^i_s \mathrm{d}s$, $i = 1, \ldots, N$, and W^0 are independant Brownian motion under $\mathbb{P}^\lambda$. By applying Itô's Lemma to $e^{\int_0^t \alpha_s \mathrm{d}s}\Delta Y^N_t$ under $\mathbb{P}^\lambda$, we then obtain

$$\begin{aligned}
\Delta Y^N_t = & \frac{1}{2N^2}\sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u \mathrm{d}u} \mathrm{tr}\big(\Sigma(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s))\partial^2_\mu v(s, \bar{\mu}(\boldsymbol{X}^N_s))(X^{i,N}_s, X^{i,N}_s)\big)\mathrm{d}s \\
& - \frac{1}{N}\sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u \mathrm{d}u}(\Delta Z^{i,N}_s)^\intercal \sigma\big(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s)\big)\mathrm{d}\widetilde{W}^i_s \\
& - \frac{1}{N}\sum_{i=1}^N \int_t^T e^{\int_t^s \alpha_u \mathrm{d}u}(\Delta Z^{i,N}_s)^\intercal \sigma_0\big(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s)\big)\mathrm{d}W^0_s, \quad 0 \le t \le T. \qquad (3.3)
\end{aligned}$$

<u>*Step 3.*</u> Let us check that the stochastic integrals in (3.3), namely $\int \tilde{\mathcal{Z}}^{i,N}_s.\mathrm{d}\widetilde{W}^i_s$, and $\int \tilde{\mathcal{Z}}^{0,i,N}_s.\mathrm{d}W^0_s$ are "true" martingales under $\mathbb{P}^\lambda$, where $\tilde{Z}^{i,N}_s := e^{\int_0^s \alpha_u \mathrm{d}u}\sigma^\intercal\big(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s)\big)\Delta Z^{i,N}_s$, $\tilde{Z}^{0,i,N}_s := e^{\int_0^s \alpha_u \mathrm{d}u}\sigma_0^\intercal\big(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s)\big)\Delta Z^{i,N}_s$, $i = 1, \ldots, N$, $0 \le s \le T$.

Indeed, for fixed $i \in [\![1, N]\!]$, recalling that α is bounded, and by the linear growth condition of σ_0 from Assumption 2.2(i), we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^\lambda}\Big[\int_0^T |\tilde{Z}^{0,i,N}_s|^2 \mathrm{d}s\Big] \le & \; C\mathbb{E}^{\mathbb{P}^\lambda}\Big[\int_0^T \big(|\sigma_0(s,0,\delta_0)|^2 + |X^{i,N}_s|^2 + \|\bar{\mu}(\boldsymbol{X}^N_s)\|_2^2\big) \\
& \qquad \big(|NZ^{i,N}_s|^2 + |\partial_\mu v(s, \bar{\mu}(\boldsymbol{X}^N_s))(X^{i,N}_s)|^2\big)\mathrm{d}s \\
\le & \; C\mathbb{E}\Big[\mathcal{E}^\lambda_T \int_0^T N^2(|\sigma_0(s,0,\delta_0)|^4 + |\boldsymbol{X}^N_s|^4)\mathrm{d}s\Big]
\end{aligned}$$

where we use Bayes formula, the estimation (2.9), the growth condition on $\partial_\mu v(.)(.)$ in Assumption 2.1, and noting that $|X^{i,N}_s| \le |\boldsymbol{X}^N_s|$, $\|\bar{\mu}(\boldsymbol{X}^N_s)\|_2^2 = |\boldsymbol{X}^N_s|^2/N$. By Hölder inequality with q as in Assumption 2.2(iii), and $\frac{1}{p} + \frac{1}{q} = 1$, the above inequality yields

$$\mathbb{E}^{\mathbb{P}^\lambda}\Big[\int_0^T |\tilde{Z}^{0,i,N}_s|^2 \mathrm{d}s\Big] \le CN^2\Big(\mathbb{E}\big[|\mathcal{E}^\lambda_T|^p\big]\Big)^{\frac{1}{p}}\Big(\mathbb{E}\Big[\int_0^T (|\sigma_0(s,0,\delta_0)|^{4q} + |\boldsymbol{X}^N_s|^{4q})\mathrm{d}s\Big]\Big)^{\frac{1}{q}},$$

which is finite by (2.8), and since λ is bounded. This shows the square-integrable martingale property of $\int \tilde{\mathcal{Z}}^{0,i,N}_s.\mathrm{d}W^0_s$ under $\mathbb{P}^\lambda$. By the same arguments, we get the square-integrable martingale property of $\int \tilde{\mathcal{Z}}^{i,N}_s.\mathrm{d}\tilde{W}^i_s$ under $\mathbb{P}^\lambda$

<u>*Step 4*</u>: *Estimation of* $\mathcal{E}^{\mathrm{y}}_N$. By taking the $\mathbb{P}^\lambda$ conditional expectation in (3.3), we obtain

$$\Delta Y^N_t = \frac{1}{2N^2}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}^\lambda}\Big[\int_t^T e^{\int_t^s \alpha_u \mathrm{d}u}\mathrm{tr}\big(\Sigma(s, X^{i,N}_s, \bar{\mu}(\boldsymbol{X}^N_s))\partial^2_\mu v(s, \bar{\mu}(\boldsymbol{X}^N_s))(X^{i,N}_s, X^{i,N}_s)\big)\mathrm{d}s\big|\mathcal{F}_t\Big],$$

for all $t \in [0,T]$. Under the boundedness condition on $\Sigma = \sigma\sigma^\intercal$ in Assumption 2.2(ii), and on $\partial_\mu^2 v$ in Assumption 2.1, it follows immediately that

$$\mathcal{E}_N^{\mathrm{y}} \;=\; \sup_{0\le t\le T} |\Delta Y_t^N| \;=\; O(N^{-1}), \quad a.s. \tag{3.4}$$

<u>*Step 5:*</u> *Estimation of $\mathcal{E}_N^{\mathrm{z}}$.* By applying Itô's formula to $|\Delta Y_t^N|^2$ in (3.3) under $\mathbb{P}^\lambda$, and integrating from 0 to T, we have

$$\begin{aligned}
&|\Delta Y_0^N|^2 + \frac{1}{N^2}\sum_{i=1}^N \int_0^T \big|\sigma^\intercal\big(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N)\big)\Delta Z_s^{i,N}\big|^2 + \big|\sigma_0^\intercal\big(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N)\big)\Delta Z_s^{i,N}\big|^2 \mathrm{d}s \\
&= 2\int_0^T \alpha_s |\Delta Y_s^N|^2 + \frac{1}{N^2}\sum_{i=1}^N \Delta Y_s^N \mathrm{tr}\big(\Sigma(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N))\partial_\mu^2 v(s, \bar\mu(\boldsymbol{X}_s^N))(X_s^{i,N}, X_s^{i,N})\big)\mathrm{d}s \\
&\quad - \frac{2}{N}\sum_{i=1}^N \int_0^T \Delta Y_s^N (\Delta Z_s^{i,N})^\intercal \sigma\big(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N)\big)\mathrm{d}\widetilde{W}_s^i \\
&\quad - \frac{2}{N}\sum_{i=1}^N \int_0^T \Delta Y_s^N (\Delta Z_s^{i,N})^\intercal \sigma_0\big(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N)\big)\mathrm{d}W_s^0.
\end{aligned}$$

As σ^+ is bounded by Assumption 2.2(ii), and taking expectation under $\mathbb{P}^\lambda$, we then get

$$\begin{aligned}
\mathcal{E}_N^{\mathrm{z}} \le\; & \frac{\|\sigma^+\|_\infty^2}{N}\sum_{i=1}^N \int_0^T \big|\sigma^\intercal\big(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N)\big)\Delta Z_s^{i,N}\big|^2 \mathrm{d}s \\
\le\; & \|\sigma^+\|_\infty^2 \mathbb{E}^{\mathbb{P}^\lambda}\Big[2\int_0^T \alpha_s N |\Delta Y_s^N|^2 \\
& + \frac{1}{N}\sum_{i=1}^N \Delta Y_s^N \mathrm{tr}\big(\Sigma(s, X_s^{i,N}, \bar\mu(\boldsymbol{X}_s^N))\partial_\mu^2 v(s, \bar\mu(\boldsymbol{X}_s^N))(X_s^{i,N}, X_s^{i,N})\big)\mathrm{d}s\Big].
\end{aligned}$$

Recalling that α, Σ and $\partial_\mu v^2$ are bounded, and using the estimate (3.4), we conclude that: $\mathcal{E}_N^{\mathrm{z}} = O(N^{-1})$, a.s. □

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