# Colored operads, series on colored operads, and combinatorial generating systems <br> Samuele Giraudo 

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# COLORED OPERADS, SERIES ON COLORED OPERADS, AND COMBINATORIAL GENERATING SYSTEMS 

SAMUELE GIRAUDO


#### Abstract

We introduce bud generating systems, which are used for combinatorial generation. They specify sets of various kinds of combinatorial objects, called languages. They can emulate context-free grammars, regular tree grammars, and synchronous grammars, allowing us to work with all these generating systems in a unified way. The theory of bud generating systems uses colored operads. Indeed, an object is generated by a bud generating system if it satisfies a certain equation in a colored operad. To compute the generating series of the languages of bud generating systems, we introduce formal power series on colored operads and several operations on these. Series on colored operads are crucial to express the languages specified by bud generating systems and allow us to enumerate combinatorial objects with respect to some statistics. Some examples of bud generating systems are constructed; in particular to specify some sorts of balanced trees and to obtain recursive formulas enumerating these.


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## Introduction

Coming from theoretical computer science and formal language theory, formal grammars [Har78, HMU06] are powerful tools having many applications in several fields of mathematics. A formal grammar is a device which describes-more or less concisely and with more or less restrictions-a set of words, called a language. There are several variations in the definitions of formal grammars and some types of these are classified by the ChomskySchützenberger hierarchy [Cho59, CS63] according to four different categories, taking into account their expressive power. In an increasing order of power, there are the classes of Type-3 to Type-0 grammars, known respectively as regular grammars, context-free grammars, context-sensitive grammars, and unrestricted grammars. One of the most striking similarities between all these variations of formal grammars is that they work by constructing words by applying rewrite rules [BN98]. Indeed, a word of the language described by a formal grammar is obtained by considering a starting word and by iteratively altering it in accordance with the production rules of the grammar.

Similar mechanisms and ideas are translatable into the world of trees, instead only those of words. Grammars of trees [CDG $\left.{ }^{+} 07\right]$ are thus the natural counterpart of formal grammars to describe sets of trees, and here also, there exist several very different types of grammars. One can cite for instance tree grammars, regular tree grammars [GS84], and synchronous grammars [Gir12], which describe sets of various kinds of treelike structures. These grammars, like the previous ones, work by applying rewrite rules on trees. In this framework, trees are constructed by growing from the root to the leaves by replacing some subtrees by other ones.

In contrast, the theory of operads seems to have no link with formal grammars. Operads are algebraic structures introduced in the context of algebraic topology [May72, BV73] (see also [Mar08, LV12, Mén15] for a modern conspectus of the theory). Recently, many links between the theory of operads and combinatorics have been developed (see, for instance [CL01, Cha08, CG14]]. Many operads involving various sets of combinatorial objects have been defined, so that almost every classical object can be seen as an element of at least one operad (see the previous references and for instance [Zin12, Gir15, Gir16a,FFM18]). From an intuitive point of view, an operad is a set of abstract operators with several inputs and one output. More precisely, if $x$ is an operator with $n$ inputs and $y$ is an operator with $m$ inputs, $x \circ_{i} y$ denotes the operator with $n+m-1$ inputs obtained by gluing the output of $y$ to the $i$-th input of $x$. Operads are algebraic structures related to trees in the same way that monoids are algebraic structures related to words. For this reason, the study of operads has many connections with combinatorial properties of trees.

The initial spark of this work was the following simple observation. The partial composition $x \circ_{i} y$ of two elements $x$ and $y$ of an operad $\theta$ can be regarded as the application of a rewrite rule on $x$ to obtain a new element of $\theta$-the rewrite rule being encoded essentially by $y$. This leads to the idea of using an operad $\theta$ to define grammars generating some subsets of $\theta$. In this way, according to the nature of the elements of $\mathcal{O}$, this provides a way to define grammars which generate objects different to words and trees. We use colored operads [BV73, Yau16], a generalization of operads. In a colored operad $\mathcal{G}$, every input and every output for the
elements of $\mathfrak{G}$ has a color. These colors create constraints for the partial compositions of two elements. Indeed, $x \circ_{i} y$ is defined only if the color of the output of $y$ is the same as the color of the $i$-th input of $x$. Colored operads enable us to define a new kind of grammar, since the restrictions provided by the colors allow precise control on how the rewrite rules can be applied.

We introduce a new kind of grammar, the bud generating system. This is defined mainly from a ground operad $\mathcal{O}$, a set $\mathscr{C}$ of colors, and a set $\mathfrak{R}$ of production rules. A bud generating system describes a subset of $\operatorname{Bud}_{\mathscr{C}}(\theta)$-the colored operad obtained by augmenting the elements of $\mathcal{O}$ with input and output colors taken from $\mathscr{C}$. An element is generated by iteratively altering an element $x$ of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ by composing it with an element $y$ of $\mathfrak{R}$. In this context, the colors play the role analogous to nonterminal symbols in formal grammars and in grammars of trees. Any bud generating system $\mathscr{3}$ specifies two sets of objects: its language $L(\mathscr{B})$ and its synchronous language $L_{S}(\mathscr{B})$. For instance, they can be used to describe sets of Motkzin paths with some constraints, the set of \{2,3\}-perfect trees [MPRS79, CLRS09] and some of their generalizations, and the set of balanced binary trees [AVL62]. Indeed, bud generating systems can emulate both context-free grammars and regular tree grammars, and allow us to see these in a unified manner.

This paper is organized as follows. Section 1 is devoted to set our notations and definitions about operads and colored operads, and to introduce the construction Bud $\mathscr{C}^{( }(\theta)$ producing a colored operad from a noncolored one $\mathcal{O}$ and a set $\mathscr{C}$ of colors. Section 2 contains the main definition: bud generating systems. We establish some properties of these. Next, we introduce formal power series on colored operads in Section 3, define several products on these, and explain how these series can be used to obtain enumerative results from bud generating systems. This article ends with Section 4 , which contains a collection of examples for most of the notions introduced by this work. We have taken the freedom to put all the examples in this section. For this reason, the reader is encouraged to consult this section whilst reading the first ones, by following the references.

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General notations and conventions. We denote by $\delta_{x, y}$ the Kronecker delta function (that is, for any elements $x$ and $y$ of a same set, $\delta_{x, y}=1$ if $x=y$ and $\delta_{x, y}=0$ otherwise). For any integers $a$ and $c,[a, c]$ denotes the set $\{b \in \mathbb{N}: a \leqslant b \leqslant c\}$ and [n], the set $[1, n]$. The cardinality of a finite set $S$ is denoted by $\# S$. For any finite multiset $S:=\left\{s_{1}, \ldots, s_{n}\right\}$ of nonnegative integers, we denote by $\sum S$ the sum

$$
\begin{equation*}
\sum S:=s_{1}+\cdots+s_{n} \tag{0.0.1}
\end{equation*}
$$

of its elements and by $S$ ! the multinomial coefficient

$$
\begin{equation*}
S!:=\binom{\sum S}{s_{1}, \ldots, s_{n}} \tag{0.0.2}
\end{equation*}
$$

For any set $A, A^{*}$ denotes the set of all finite sequences, called words, of elements of $A$. We denote by $A^{+}$the subset of $A^{*}$ consisting in nonempty words. For any $n \geqslant 0, A^{n}$ is the set of
all words on $A$ of length $n$. If $u$ is a word, its letters are indexed from left to right from 1 to its length $|u|$. For any $i \in[|u|], u_{i}$ is the letter of $u$ at position $i$. If $a$ is a letter and $n$ is a nonnegative integer, $a^{n}$ denotes the word consisting in $n$ occurrences of $a$. Notice that $a^{0}$ is the empty word $\epsilon$.

In our graphical representations of trees, the uppermost nodes are always roots. Moreover, internal nodes are represented by circles $O$, leaves by squares $\square$, and edges by segments |. To distinguish trees and syntax trees, we shall draw the latter without circles for internal nodes and without squares for leaves (only the labels of the nodes are depicted).

In graphical representations of multigraphs, labels of edges denote their multiplicities. All unlabeled edges have 1 as multiplicity.

## 1. COlORED operads and bud operads

The aim of this section is to set our notations about operads, colored operads, and colored syntax trees. We also establish some properties of treelike expressions in colored operads and present a construction producing colored operads from operads.
1.1. Colored operads. Let us recall here the definitions of colored graded collections and colored operads.
1.1.1. Colored graded collections. Let $\mathscr{C}$ be a finite set, called set of colors. A $\mathscr{C}$-colored graded collection is a graded set

$$
\begin{equation*}
C:=\bigsqcup_{n \geqslant 1} C(n) \tag{1.1.1}
\end{equation*}
$$

together with two maps out :C $\rightarrow \mathscr{C}$ and in : $C(n) \rightarrow \mathscr{C}^{n}, n \geqslant 1$, respectively sending any $x \in C(n)$ to its output color out $(x)$ and to its word of input colors $\mathrm{in}(x)$. The $i$-th input color of $x$ is the $i$-th letter of $\operatorname{in}(x)$, denoted by $\operatorname{in}_{i}(x)$. For any $n \geqslant 1$ and $x \in C(n)$, the arity $|x|$ of $x$ is $n$. We say that $C$ is locally finite if for all $n \geqslant 1$, the $C(n)$ are finite sets. A monochrome graded collection is a $\mathscr{C}$-colored graded collection where $\mathscr{C}$ is a singleton. If $C_{1}$ and $C_{2}$ are two $\mathscr{C}$-colored graded collections, a map $\phi: C_{1} \rightarrow C_{2}$ is a $\mathscr{C}$-colored graded collection morphism if it preserves arities. Besides, $C_{2}$ is a $\mathscr{C}$-colored graded subcollection of $C_{1}$ if for all $n \geqslant 1$, $C_{2}(n) \subseteq C_{1}(n)$, and $C_{1}$ and $C_{2}$ have the same maps out and in.
1.1.2. Hilbert series. In all this work, we consider that $\mathscr{C}$ has cardinal $k$ and that the colors of $\mathscr{C}$ are arbitrarily indexed so that $\mathscr{C}=\left\{c_{1}, \ldots, c_{k}\right\}$. Let $\mathbb{X}_{\mathscr{C}}:=\left\{\mathrm{x}_{\mathrm{c}_{1}}, \ldots, \mathrm{x}_{c_{k}}\right\}$ and $\mathbb{Y}_{\mathscr{C}}:=$ $\left\{y_{c_{1}}, \ldots, y_{c_{k}}\right\}$ be two alphabets of mutually commutative parameters and $\mathbb{N}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right]$ be the set of commutative multivariate series on $\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}$ with nonnegative integer coefficients. As usual, if $\mathbf{s}$ is a series of $\mathbb{N}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right],\langle m, \mathbf{s}\rangle$ denotes the coefficient of the monomial $m$ in $\mathbf{s}$.

For any $\mathscr{C}$-colored graded collection $C$, the Hilbert series $\mathbf{h}_{C}$ of $C$ is the series of $\mathbb{N}\left[\left[\mathbb{X}_{\mathscr{C}} U\right.\right.$ $\mathbb{Y}_{\mathscr{C}}$ ]] defined by

$$
\begin{equation*}
\mathbf{h}_{C}:=\sum_{x \in C}\left(\mathrm{x}_{\mathrm{out}(x)} \prod_{i \in[|x|]} \mathrm{y}_{\mathrm{in} i}(x)\right) . \tag{1.1.2}
\end{equation*}
$$

The coefficient of $x_{a} y_{c_{1}}^{\alpha_{1}} \ldots y_{c_{k}}^{\alpha_{k}}$ in $\mathbf{h}_{C}$ thus counts the elements of $C$ having $a$ as output color and $\alpha_{j}$ inputs of color $c_{j}$ for any $\boldsymbol{j} \in[k]$. Note that (1.1.2) is defined only if there are only finitely many such elements for any $a \in \mathscr{C}$ and any $\alpha_{j} \geqslant 0, j \in[k]$. This is the case when $C$ is locally finite.

Besides, the generating series of $C$ is the series $\mathbf{s}_{C}$ of $\mathbb{N}[[t]]$ defined as the specialization of $\mathbf{h}_{C}$ at $x_{a}:=1$ and $y_{a}:=t$ for all $a \in \mathscr{C}$. Therefore, for any $n \geqslant 1$ the coefficient $\left\langle t^{n}, \mathbf{s}_{C}\right\rangle$ counts the elements of arity $n$ in $C$.
1.1.3. Colored operads. A nonsymmetric colored set-operad on $\mathscr{C}$, or a $\mathscr{C}$-colored operad for short, is a colored graded collection $\mathfrak{G}$ together with partially defined maps

$$
\begin{equation*}
\circ_{i}: \mathscr{G}(n) \times \mathfrak{G}(m) \rightarrow \mathfrak{G}(n+m-1), \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant m \tag{1.1.3}
\end{equation*}
$$

called partial compositions, and a subset $\left\{\mathbb{1}_{a}: a \in \mathscr{C}\right\}$ of $\mathscr{G}(1)$ such that any $\mathbb{1}_{a}, a \in \mathscr{C}$, is called unit of color $a$ and satisfies out $\left(\mathbb{1}_{a}\right)=\operatorname{in}\left(\mathbb{1}_{a}\right)=a$. This data has to satisfy, for any $x, y, z \in \mathscr{G}$, the following constraints. First, for any $i \in[|x|], x \circ_{i} y$ is defined if and only if out $(y)=\operatorname{in}_{i}(x)$. Moreover, the relations

$$
\begin{gather*}
\left(x \circ_{i} y\right) \circ_{i+j-1} z=x \circ_{i}\left(y \circ_{j} z\right), \quad 1 \leqslant i \leqslant|x|, 1 \leqslant j \leqslant|y|,  \tag{1.1.4a}\\
\left(x \circ_{i} y\right) \circ_{j+|y|-1} z=\left(x \circ_{j} z\right) \circ_{i} y, \quad 1 \leqslant i<j \leqslant|x|,  \tag{1.1.4b}\\
\mathbb{1}_{a} \circ_{1} x=x=x \circ_{i} \mathbb{1}_{b}, \quad 1 \leqslant i \leqslant|x|, \quad a, b \in \mathscr{C}, \tag{1.1.4c}
\end{gather*}
$$

have to hold when they are well-defined.
The complete composition map of $\mathfrak{G}$ is the partially defined map

$$
\begin{equation*}
\circ: \mathfrak{G}(n) \times \mathfrak{G}\left(m_{1}\right) \times \cdots \times \mathfrak{G}\left(m_{n}\right) \rightarrow \mathfrak{G}\left(m_{1}+\cdots+m_{n}\right), \tag{1.1.5}
\end{equation*}
$$

defined from the partial composition maps in the following way. For any $x \in \mathscr{G}(n)$ and $y_{1}, \ldots, y_{n} \in \mathscr{G}$ such that out $\left(y_{i}\right)=\operatorname{in}_{i}(y)$ for all $i \in[n]$, we set

$$
\begin{equation*}
x \circ\left[y_{1}, \ldots, y_{n}\right]:=\left(\ldots\left(\left(x \circ_{n} y_{n}\right) \circ_{n-1} y_{n-1}\right) \ldots\right) \circ_{1} y_{1} . \tag{1.1.6}
\end{equation*}
$$

Let $\mathscr{G}_{1}$ and $\mathscr{G}_{2}$ are two $\mathscr{C}$-colored operads. A $\mathscr{C}$-colored graded collection morphism $\phi$ : $\mathfrak{G}_{1} \rightarrow \mathfrak{G}_{2}$ is a $\mathscr{C}$-colored operad morphism if it sends any unit of color $a \in \mathscr{C}$ of $\mathfrak{G}_{1}$ to the unit of color $a$ of $\mathfrak{G}_{2}$, if it commutes with partial composition maps and, if for any $x, y \in \mathfrak{G}_{1}$ and $i \in[|x|]$, if $x \circ_{i} y$ is defined in $\mathfrak{G}_{1}$, then $\phi(x) \circ_{i} \phi(y)$ is defined in $\mathfrak{G}_{2}$. Besides, $\mathfrak{G}_{2}$ is a colored suboperad of $\mathscr{G}_{1}$ if $\mathscr{G}_{2}$ is a $\mathscr{C}$-colored graded subcollection of $\mathscr{G}_{1}$ and $\mathfrak{G}_{1}$ and $\mathfrak{G}_{2}$ have the same colored units and the same partial composition maps. If $G$ is a $\mathscr{C}$-colored graded subcollection of $\mathscr{G}$, we denote by $\mathscr{G}^{G}$ the $\mathscr{C}$-colored operad generated by $G$, that is the smallest $\mathscr{C}$-colored suboperad of $\mathfrak{G}$ containing $G$. When the $\mathscr{C}$-colored operad generated by $G$ is $\mathscr{G}$ itself, $G$ is a generating $\mathscr{C}$-colored graded collection of $\mathscr{G}$. Moreover, when $G$ is minimal with respect to inclusion among the $\mathscr{C}$-colored graded subcollections of $\mathscr{G}$ satisfying this property, $G$ is a minimal generating $\mathscr{C}$-colored graded collection of $\mathfrak{G}$. We say that $\mathfrak{G}$ is locally finite if, as a colored graded collection, $\mathfrak{G}$ is locally finite.

A monochrome operad (or an operad for short) $\mathcal{O}$ is a $\mathscr{C}$-colored operad with a monochrome graded collection as underlying set. In this case, $\mathscr{C}$ is a singleton $\left\{\mathrm{c}_{1}\right\}$ and, since for all $x \in \mathcal{O}(n)$, we necessarily have out $(x)=c_{1}$ and $\operatorname{in}(x)=c_{1}^{n}$, for all $x, y \in \mathcal{O}$ and $i \in[|x|]$, all partial compositions $x o_{i} y$ are defined. In this case, $\mathscr{C}$ and its single element $c_{1}$ do not play any role. For this reason, in the future definitions of monochrome operads, we shall not define their set of colors $\mathscr{C}$.
1.2. Free colored operads. Free colored operads and more particularly colored syntax trees play an important role in this work. We recall here the definitions of these two notions and establish some of their properties.
1.2.1. Colored syntax trees. Unless otherwise specified, we use in the sequel the standard terminology (i.e., node, edge, root, parent, child, path, etc.) about planar rooted trees [Knu97]. Let $\mathscr{C}$ be a set of colors and $C$ be a $\mathscr{C}$-colored graded collection. A $\mathscr{C}$-colored $C$-syntax tree is a planar rooted tree $t$ such that, for any $n \geqslant 1$, any internal node of $\mathfrak{t}$ having $n$ children is labeled by an element of arity $n$ of $C$ and, for any internal nodes $u$ and $v$ of $\mathfrak{t}$ such that $v$ is the $i$-th child of $u, \operatorname{out}(y)=\operatorname{in}_{i}(x)$ where $x$ (resp. $y$ ) is the label of $u$ (resp. $v$ ). In our graphical representations of a $\mathscr{C}$-colored $C$-syntax tree $\mathfrak{t}$, we write the colors of the leaves of $\mathfrak{t}$ below them and the color of the edge exiting the root of $\mathfrak{t}$ above it (see Figure 1).

(A) The degree of this $\mathscr{C}$-colored $C$ syntax tree is 5 , its arity is 8 , and its height is 3

(B) A perfect $\mathscr{C}$-colored $C$-syntax tree. The degree of this colored syntax tree is 8 , its arity is 11 , and its height is 3 .

Figure 1. Two $\mathscr{C}$-colored $C$-syntax trees, where $\mathscr{C}$ is the set of colors $\{1,2\}$ and $C$ is the $\mathscr{C}$ graded colored collection defined by $C:=C(2) \sqcup C(3)$ with $C(2):=\{a, b\}, C(3):=\{c\}$, out $(a):=1$, $\operatorname{out}(\mathrm{b}):=2, \operatorname{out}(\mathrm{c}):=1, \operatorname{in}(\mathrm{a}):=11, \operatorname{in}(\mathrm{~b}):=21$, and in(c) $:=221$.

Let $\mathfrak{t}$ be a $\mathscr{C}$-colored $C$-syntax tree. The arity of an internal node $v$ of $\mathfrak{t}$ is its number $|\nu|$ of children and its label is the element of $C$ labeling it and denoted by $\mathfrak{t}(v)$. The degree $\operatorname{deg}(t)$ (resp. arity $|\mathfrak{t}|$ ) of $\mathfrak{t}$ is its number of internal nodes (resp. leaves). We say that $\mathfrak{t}$ is a corolla if $\operatorname{deg}(t)=1$. The height of $\mathfrak{t}$ is the length $h t(t)$ of a longest path connecting the root of $\mathfrak{t}$ to one of its leaves. For instance, the height of a colored syntax tree of degree 0 is 0 and the one of a corolla is 1 . The set of all internal nodes of $\mathfrak{t}$ is denoted by $\mathcal{N}(\mathfrak{t})$. For any $v \in \mathcal{N}(\mathfrak{t})$, $\mathfrak{t}_{v}$ is the subtree of $\mathfrak{t}$ rooted at the node $v$. We say that $\mathfrak{t}$ is perfect if all paths connecting the root of $\mathfrak{t}$ to its leaves have the same length. Finally, $\mathfrak{t}$ is a monochrome $C$-syntax tree if $C$ is a monochrome graded collection.
1.2.2. Free colored operads. The free $\mathscr{C}$-colored operad over $C$ is the operad Free $(C)$ wherein for any $n \geqslant 1$, Free $(C)(n)$ is the set of all $\mathscr{C}$-colored $C$-syntax trees of arity $n$. For any $\mathfrak{t} \in \operatorname{Free}(C)$, out $(\mathfrak{t})$ is the output color of the label of the root of $\mathfrak{t}$ and $\operatorname{in}(\mathfrak{t})$ is the word obtained by reading, from left to right, the input colors of the leaves of $\mathfrak{t}$. For any $\mathfrak{s}, \mathfrak{t} \in \operatorname{Free}(C)$, the partial composition $\mathfrak{s} \circ_{i} \mathfrak{t}$, defined if and only if the output color of $\mathfrak{t}$ is the input color of the $i$-th leaf of $\mathfrak{s}$, is the tree obtained by grafting the root of $\mathfrak{t}$ to the $i$-th leaf of $\mathfrak{s}$. For instance, with the $\mathscr{C}$-colored graded collection $C$ defined in Figure 1, one has in Free $(C)$,

1.2.3. Treelike expressions and finitely factorizing sets. For any $\mathscr{C}$-colored operad $\mathfrak{G}$, the evaluation map of $\mathscr{G}$ is the map $\mathrm{ev}_{\mathcal{G}}:$ Free $(\mathscr{G}) \rightarrow \mathfrak{G}$, defined as the unique surjective morphism of colored operads satisfying $\mathrm{ev}_{\mathcal{G}}(\mathrm{t})=x$ where $\mathfrak{t}$ is a tree of degree 1 having its root labeled by $x$. If $S$ is a colored graded subcollection of $\mathfrak{G}$, an S-treelike expression of $x \in \mathscr{G}$ is a tree $\mathfrak{t}$ of Free(G) such that $\mathrm{ev}_{\mathscr{G}}(\mathfrak{t})=x$ and all internal nodes of $\mathfrak{t}$ are labeled on $S$.

Besides, when $S$ is such that any $x \in \mathscr{G}$ admits finitely many $S$-treelike expressions, we say that $S$ finitely factorizes G. This notion is important in the sequel and is used as sufficient condition for the well-definition of some formal power series on colored operads.

Lemma 1.2.1. Let $\mathfrak{G}$ be a locally finite $\mathscr{C}$-colored operad and $S$ be a $\mathscr{C}$-colored graded subcollection of $\mathfrak{G}$ such that $S(1)$ finitely factorizes $\mathfrak{G}$. Then, $S$ finitely factorizes $\mathfrak{G}$.

Proof. Since $\mathfrak{G}$ is locally finite and $S(1)$ finitely factorizes $\mathfrak{G}$, there is a nonnegative integer $k$ such that $k$ is the degree of a $\mathscr{C}$-colored $S(1)$-syntax tree with a maximal number of internal nodes. Let $x$ be an element of $\mathfrak{G}$ of arity $n$ admitting an $S$-treelike expression $\mathfrak{t}$. Observe first that $\mathfrak{t}$ has at most $n-1$ non-unary internal nodes and at most $2 n-1$ edges. Moreover, by the pigeonhole principle, if $t$ would have more than $(2 n-1) k$ unary internal nodes, there would be a chain made of more than $k$ unary internal nodes in $\mathfrak{t}$. This cannot happen since, by hypothesis, it is not possible to form any $\mathscr{C}$-colored $S(1)$-syntax tree with more than $k$ nodes. Therefore, we have shown that all $S$-treelike expressions of $x$ are of degrees at most $n-1+(2 n-1) k$. Moreover, since $\mathfrak{G}$ is locally finite and $S$ is a $\mathscr{C}$-colored graded subcollection of $\mathfrak{G}$, all $S(m)$ are finite for all $m \geqslant 1$. Therefore, there are finitely many $S$-treelike expressions of $x$.

Assume now that $\mathscr{G}$ is locally finite and that $S$ is a $\mathscr{C}$-colored graded subcollection of $\mathscr{G}$ such that $S(1)$ finitely factorizes $\mathfrak{G}$. For any element $x$ of $\mathscr{G}^{S}$, the colored suboperad of $\mathfrak{G}$ generated by $S$, the $S$-degree of $x$ is defined by

$$
\begin{equation*}
\operatorname{deg}_{S}(x):=\max \left\{\operatorname{deg}(\mathfrak{t}): \mathfrak{t} \in \operatorname{Free}(S) \text { and } \mathrm{ev}_{G}(\mathrm{t})=x\right\} . \tag{1.2.2}
\end{equation*}
$$

Thanks to the fact that, by hypothesis, $x$ admits at least one $S$-treelike expression and, by Lemma 1.2.1, the fact that $x$ admits finitely many $S$-treelike expressions, $\operatorname{deg}_{S}(x)$ is welldefined.
1.2.4. Left expressions and hook-length formula. Let $S$ be a $\mathscr{C}$-colored graded subcollection of $\mathfrak{G}$ and $x \in \mathfrak{G}$. An S-left expression of $x$ is an expression for $x$ in $\mathfrak{G}$ of the form

$$
\begin{equation*}
x=\left(\ldots\left(\left(\mathbb{1}_{\mathrm{out}(x)} \circ_{1} s_{1}\right) \circ_{i_{1}} s_{2}\right) \circ_{i_{2}} \ldots\right) \circ_{i_{\ell-1}} s_{\ell} \tag{1.2.3}
\end{equation*}
$$

where $s_{1}, \ldots, s_{\ell} \in S$ and $i_{1}, \ldots, \boldsymbol{i}_{\ell-1} \in \mathbb{N}$. Besides, if $\mathfrak{t}$ is an $S$-treelike expression of $x$ such that $\mathcal{N}(\mathfrak{t})=\left\{\mathrm{e}_{1}, \ldots, \mathrm{e}_{\ell}\right\}$, a sequence $\left(\mathrm{e}_{1}, \ldots, \mathrm{e}_{\ell}\right)$ is a linear extension of $\mathfrak{t}$ if the sequence is a linear extension of the poset induced by $\mathfrak{t}$ seen as an Hasse diagram where the root of $\mathfrak{t}$ is the smallest element.

Lemma 1.2.2. Let $\mathscr{G}$ be a locally finite $\mathscr{C}$-colored operad and $S$ be a $\mathscr{C}$-colored graded subcollection of $\mathfrak{G}$. Then, for any $x \in \mathfrak{G}$, the set of all S-left expressions of $x$ is in one-to-one correspondence with the set of all pairs ( $\mathfrak{t}, \mathrm{e}$ ) where $\mathfrak{t}$ is an S-treelike expression of $x$ and e is a linear extension of $\mathfrak{t}$.

Proof. Let $\phi_{x}$ be the map sending any $S$-left expression of the form (1.2.3) of $x \in \mathcal{G}$ to the pair $(t, e)$ where $t$ is the colored syntax tree of Free $(S)$ obtained by interpreting (1.2.3) in Free $(S)$, i.e., by replacing any $s_{j}, \boldsymbol{j} \in[\ell]$, in (1.2.3) by a corolla $\mathfrak{s}_{j}$ of Free $(S)$ labeled by $s_{j}$, and where $e$ is the sequence $\left(e_{1}, \ldots, e_{\ell}\right)$ of the internal nodes of $\mathfrak{t}$, where any $e_{j}, j \in[\ell]$, is the node of $\mathfrak{t}$ coming from $\mathfrak{s}_{j}$. We then have

$$
\begin{equation*}
\mathfrak{t}=\left(\ldots\left(\left(\mathbb{1}_{\text {out }(x)} \circ_{1} \mathfrak{s}_{1}\right) \circ_{i_{1}} \mathfrak{s}_{2}\right) \circ_{i_{2}} \ldots\right) \circ_{i_{\ell-1}} \mathfrak{s}_{\ell} \tag{1.2.4}
\end{equation*}
$$

and by construction, $\mathfrak{t}$ is an $S$-treelike expression of $x$. Moreover, immediately from the definition of the partial composition in free $\mathscr{C}$-colored operads, ( $\mathrm{e}_{1}, \ldots, \mathrm{e}_{\ell}$ ) is a linear extension of $\mathfrak{t}$. Therefore, we have shown that $\phi_{x}$ sends any S-left expression of $x$ to a pair ( $\mathfrak{t}, \mathrm{e}$ ) where $\mathfrak{t}$ is an $S$-treelike expression of $x$ and e is a linear extension of $\mathfrak{t}$.

Let $\mathfrak{t}$ be an $S$-treelike expression of $x \in \mathcal{G}$ and e be a linear extension ( $\mathrm{e}_{1}, \ldots, \mathrm{e}_{\ell}$ ) of $\mathfrak{t}$. It follows by induction on the degree $\ell$ of $\mathfrak{t}$ that $\mathfrak{t}$ can be expressed by an expression of the form (1.2.4) where any $e_{j}, j \in[\ell]$, is the node of $\mathfrak{t}$ coming from $\mathfrak{s}_{j}$. Now, the interpretation of (1.2.4) in $\mathcal{G}$, i.e., by replacing any corolla $\mathfrak{s}_{j}, \boldsymbol{j} \in[\ell]$, in (1.2.4) by its label $s_{j}$, is an $S$-left expression of the form (1.2.3) for $x$. Since (1.2.3) is the only antecedent of ( $\mathfrak{t}, \mathrm{e}$ ) by $\phi_{x}$, it follows that $\phi_{x}$, with domain the set of all $S$-left expressions of $x$ and with codomain the set of all pairs ( $\mathfrak{t}, \mathrm{e}$ ) where $\mathfrak{t}$ is an $S$-treelike expression of $x$ and $e$ is a linear extension of $\mathfrak{t}$, is a bijection.

A famous result of Knuth [Knu98], known as the hook-length formula for trees, stated here in our setting, says that given a $\mathscr{C}$-colored syntax tree $\mathfrak{t}$, the number of linear extensions of $\mathfrak{t}$ is

$$
\begin{equation*}
\frac{\operatorname{deg}(t)!}{\prod_{v \in \mathcal{N}(t)} \operatorname{deg}\left(t_{v}\right)} \tag{1.2.5}
\end{equation*}
$$

When $S(1)$ finitely factorizes $\mathcal{G}$, by Lemma 1.2.1, the number of $S$-treelike expressions for any $x \in \mathscr{G}$ is finite. Hence, in this case, we deduce from Lemma 1.2.2 and (1.2.5) that the number of $S$-left expressions of $x$ is

$$
\begin{equation*}
\sum_{\substack{t \in \operatorname{Free}(S) \\ \operatorname{ev}_{6}(t)=x}} \frac{\operatorname{deg}(\mathfrak{t})!}{\prod_{v \in \mathcal{N}(t)} \operatorname{deg}\left(\mathfrak{t}_{v}\right)} \tag{1.2.6}
\end{equation*}
$$

1.3. Bud operads. Let us now present a simple construction producing colored operads from operads.
1.3.1. From monochrome operads to colored operads. If $\mathcal{O}$ is a monochrome operad and $\mathscr{C}$ is a finite set of colors, we denote by $\operatorname{Bud}_{\mathscr{C}}(\theta)$ the $\mathscr{C}$-colored graded collection defined by

$$
\begin{equation*}
\operatorname{Bud}_{\mathscr{C}}(\theta)(n):=\mathscr{C} \times \mathscr{O}(n) \times \mathscr{C}^{n}, \quad n \geqslant 1 \tag{1.3.1}
\end{equation*}
$$

and for all $(a, x, u) \in \operatorname{Bud}_{\mathscr{C}}(\theta), \operatorname{out}((a, x, u)):=a$ and $\operatorname{in}((a, x, u)):=u$. We endow $\operatorname{Bud}_{\mathscr{C}}(\theta)$ with the partially defined partial composition $\circ_{i}$ satisfying, for all triples $(a, x, u)$ and $(b, y, v)$ of $\operatorname{Bud}_{\mathscr{C}}(\theta)$, and $i \in[|x|]$ such that out $((b, y, v))=\operatorname{in}_{i}((a, x, u))$,

$$
\begin{equation*}
(a, x, u) \circ_{i}(b, y, v):=\left(a, x o_{i} y, u \longleftarrow_{i} v\right), \tag{1.3.2}
\end{equation*}
$$

where $u \leftarrow_{i} v$ is the word obtained by replacing the $i$-th letter of $u$ by $v$. Besides, if $\mathcal{O}_{1}$ and $\mathcal{O}_{2}$ are two operads and $\phi: \mathcal{O}_{1} \rightarrow \mathcal{O}_{2}$ is an operad morphism, we denote by $\operatorname{Bud}_{\mathscr{C}}(\phi)$ the map

$$
\begin{equation*}
\operatorname{Bud}_{\mathscr{C}}(\phi): \operatorname{Bud}_{\mathscr{C}}\left(\Theta_{1}\right) \rightarrow \operatorname{Bud}_{\mathscr{C}}\left(\Theta_{2}\right) \tag{1.3.3}
\end{equation*}
$$

defined by

$$
\begin{equation*}
\operatorname{Bud}_{\mathscr{C}}(\phi)((a, x, u)):=(a, \phi(x), u) \tag{1.3.4}
\end{equation*}
$$

Proposition 1.3.1. For any set of colors $\mathscr{C}$, the construction $(\theta, \phi) \mapsto\left(\operatorname{Bud}_{\mathscr{C}}(\theta), \operatorname{Bud}_{\mathscr{C}}(\phi)\right)$ is a functor from the category of monochrome operads to the category of $\mathscr{C}$-colored operads.

We omit the proof of Proposition 1.3.1 since it is very straightforward. This result shows that $\mathrm{Bud}_{\mathscr{C}}$ is a functorial construction producing colored operads from monochrome ones. We call Bud $\mathscr{C}(\theta)$ the $\mathscr{C}$-bud operad of $\mathcal{O}^{\text {a }}$. When $\mathscr{C}$ is a singleton, Bud $\mathscr{C}(\theta)$ is by definition a monochrome operad isomorphic to $\theta$. For this reason, in this case, we identify Bud $\mathscr{C}^{( }(\theta)$ with $\mathcal{O}$.

As a side observation, remark that in general, the bud operad $\mathrm{Bud}_{\mathscr{C}}(\theta)$ of a free operad $\mathcal{O}$ is not a free $\mathscr{C}$-colored operad. Indeed, consider for instance the bud operad Bud $\{1,2\}(\theta)$, where $\mathcal{O}:=\operatorname{Free}(C)$ and $C$ is the monochrome graded collection defined by $C:=C(1):=\{\mathrm{a}\}$. Then, a minimal generating set of $\operatorname{Bud}_{\{1,2\}}(\theta)$ is

$$
\left\{(1, \quad \stackrel{a}{a}, 1),\left(\begin{array}{ll}
1, & \dot{a}, 2  \tag{1.3.5}\\
& \vdots
\end{array}\right),\left(\begin{array}{cc}
2, & \dot{a}, 1 \\
1
\end{array}\right),\left(\begin{array}{cc}
2, & \dot{a} \\
1 & , 2
\end{array}\right)\right\} .
$$

These elements are subjected to the nontrivial relations

[^0]where $d, \mathrm{e} \in\{1,2\}$, implying that $\operatorname{Bud}_{\{1,2\}}(\theta)$ is not free.
1.3.2. The associative operad. The associative operad As is the monochrome operad defined by $\boldsymbol{A s}(n):=\left\{\star_{n}\right\}, n \geqslant 1$, and wherein partial composition maps are defined by
\[

$$
\begin{equation*}
\star_{n} \circ_{i} \star_{m}:=\star_{n+m-1}, \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant m . \tag{1.3.7}
\end{equation*}
$$

\]

For any set of colors $\mathscr{C}$, the bud operad $\operatorname{Bud}_{\mathscr{C}}($ As $)$ is the set of all triples

$$
\begin{equation*}
\left(a, \star_{n}, u_{1} \ldots u_{n}\right) \tag{1.3.8}
\end{equation*}
$$

where $a \in \mathscr{C}$ and $u_{1}, \ldots, u_{n} \in \mathscr{C}$. For $\mathscr{C}:=\{1,2,3\}$, one has for instance the partial composition

$$
\begin{equation*}
\left(2, \star_{4}, 3112\right) o_{2}\left(1, \star_{3}, 233\right)=\left(2, \star_{6}, 323312\right) . \tag{1.3.9}
\end{equation*}
$$

The associative operad and its bud operads will play an important role in the sequel. For this reason, to gain readability, we shall simply denote by ( $a, u$ ) any element ( $a, \star_{|u|}, u$ ) of $\operatorname{Bud}_{\mathscr{G}}(\mathbf{A s})$ without any loss of information.
1.3.3. Pruning map. Here, we use the fact that any monochrome operad $\mathcal{\theta}$ can be seen as a $\mathscr{C}$-colored operad where all output and input colors of its elements are equal to $c_{1}$, where $c_{1}$ is the first color of $\mathscr{C}$ (see Section 1.1.3). Let

$$
\begin{equation*}
\mathfrak{p r u}: \operatorname{Bud}_{\mathscr{C}}(\theta) \rightarrow \theta \tag{1.3.10}
\end{equation*}
$$

be the morphism of $\mathscr{C}$-colored operads defined, for any $(a, x, u) \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, by

$$
\begin{equation*}
\mathfrak{p r u}((a, x, u)):=x . \tag{1.3.11}
\end{equation*}
$$

We call pru the pruning map on $\operatorname{Bud}_{\mathscr{C}}(\theta)$.

## 2. Bud generating systems and combinatorial generation

In this section, we introduce bud generating systems. A bud generating system relies on an operad $\theta$, a set of colors $\mathscr{C}$, and the bud operad $\operatorname{Bud}_{\mathscr{C}}(\theta)$. The principal interest of these objects is that they allow us to specify sets of objects of $\operatorname{Bud}_{\mathscr{C}}(\theta)$. We shall also establish some first properties of bud generating systems by showing that they can emulate context-free grammars, regular tree grammars, and synchronous grammars.
2.1. Bud generating systems. We introduce here the main definitions and the main tools about bud generating systems.
2.1.1. Bud generating systems. A bud generating system is a tuple $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ where $\mathcal{O}$ is an operad called ground operad, $\mathscr{C}$ is a finite set of colors, $\mathfrak{R}$ is a finite $\mathscr{C}$-colored graded subcollection of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ called set of rules, $I$ is a subset of $\mathscr{C}$ called set of initial colors, and $T$ is a subset of $\mathscr{C}$ called set of terminal colors.

A monochrome bud generating system is a bud generating system whose set $\mathscr{C}$ of colors contains a single color, and whose sets of initial and terminal colors are equal to $\mathscr{C}$. In this case, as explained in Section 1.3.1, $\operatorname{Bud}_{\mathscr{C}}(\theta)$ and $\Theta$ are identified. These particular generating systems are thus simply denoted by pairs $(\theta, \mathfrak{R})$.

Let us explain how bud generating systems specify, in two different ways, two $\mathscr{C}$-colored graded subcollections of $\operatorname{Bud}_{\mathscr{C}}(O)$. In what follows, $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ is a bud generating system.
2.1.2. Generation. We say that $x_{2} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ is derivable in one step from $x_{1} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ if there is a rule $r \in \mathfrak{R}$ and an integer $i$ such that $x_{2}=x_{1} \circ_{i} r$. We denote this property by $x_{1} \rightarrow x_{2}$. When $x_{1}, x_{2} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ are such that $x_{1}=x_{2}$ or there are $y_{1}, \ldots, y_{\ell-1} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, $\ell \geqslant 1$, satisfying

$$
\begin{equation*}
x_{1} \rightarrow y_{1} \rightarrow \cdots \rightarrow y_{\ell-1} \rightarrow x_{2}, \tag{2.1.1}
\end{equation*}
$$

we say that $x_{2}$ is derivable from $x_{1}$. Moreover, $\mathscr{B}$ generates $x \in \operatorname{Bud} \mathscr{C}(\theta)$ if there is a color $a$ of $I$ such that $x$ is derivable from $\mathbb{1}_{a}$ and all colors of in $(x)$ are in $T$. The language $L(\mathscr{B})$ of $\mathscr{B}$ is the set of all the elements of Bud $_{\mathscr{C}}(\theta)$ generated by $\mathscr{B}$.

The derivation graph of $\mathscr{B}$ is the oriented multigraph $G(\mathscr{B})$ with the set of elements derivable from $\mathbb{1}_{a}, a \in I$, as set of vertices. In $G(\mathscr{B})$, for any $x_{1}, x_{2} \in L(\mathscr{B})$ such that $x_{1} \rightarrow x_{2}$, there are $\ell$ edges from $x_{1}$ to $x_{2}$, where $\ell$ is the number of pairs $(i, r) \in \mathbb{N} \times \Re$ such that $x_{2}=x_{1} \circ_{i} r^{b}$.
2.1.3. Synchronous generation. We say that $x_{2} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ is synchronously derivable in one step from $x_{1} \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ if there are rules $r_{1}, \ldots, r_{\left|x_{1}\right|}$ of $\mathfrak{R}$ such that $x_{2}=x_{1} \circ\left[r_{1}, \ldots, r_{\left|x_{1}\right|}\right]$. We denote this property by $x_{1} \leadsto x_{2}$. When $x_{1}, x_{2} \in \operatorname{Bud} \mathscr{C}^{\prime}(\theta)$ are such that $x_{1}=x_{2}$ or there are $y_{1}, \ldots, y_{\ell-1} \in \operatorname{Bud}_{\mathscr{C}}(\theta), \ell \geqslant 1$, satisfying

$$
\begin{equation*}
x_{1} \leadsto y_{1} \leadsto \cdots \sim y_{\ell-1} \leadsto x_{2} \tag{2.1.2}
\end{equation*}
$$

we say that $x_{2}$ is synchronously derivable from $x_{1}$. Moreover, $\mathscr{B}$ synchronously generates $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ if there is a color $a$ of $I$ such that $x$ is synchronously derivable from $\mathbb{1}_{a}$ and all colors of in $(x)$ are in $T$. The synchronous language $L_{S}(\mathscr{B})$ of $\mathscr{G}$ is the set of all the elements of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ synchronously generated by $\mathscr{B}$.

The synchronous derivation graph of $\mathscr{B}$ is the oriented multigraph $G_{S}(\mathscr{B})$ with the set of elements synchronously derivable from $\mathbb{1}_{a}, a \in I$, as set of vertices. In $G_{S}(\mathscr{B})$, for any $x_{1}, x_{2} \in \mathrm{~L}_{S}(\mathscr{B})$ such that $x_{1} \sim x_{2}$, there are $\ell$ edges from $x_{1}$ to $x_{2}$, where $\ell$ is the number of tuples $\left(r_{1}, \ldots, r_{\left|x_{1}\right|}\right) \in \mathfrak{R}^{\left|x_{1}\right|}$ such that $x_{2}=x_{1} \circ\left[r_{1}, \ldots, r_{\left|x_{1}\right|}\right] .^{c}$.

[^1]2.2. First properties. We state now two properties about the languages and the synchronous languages of bud generating systems.

Lemma 2.2.1. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system. Then, for any $x \in$ $\operatorname{Bud}_{\mathscr{C}}(\Theta), x$ belongs to $\mathrm{L}(\mathscr{3})$ if and only if $x$ admits an $\mathfrak{R}$-treelike expression with output color in I and all input colors in $T$.

Proof. Assume that $x$ belongs to $\mathrm{L}(\mathscr{3})$. Then, by definition of the derivation relation $\rightarrow, x$ admits an $\mathfrak{R}$-left expression. Lemma 1.2.2 implies in particular that $x$ admits an $\mathfrak{R}$-treelike expression $\mathfrak{t}$. Moreover, since $\mathfrak{t}$ is a treelike expression for $x, \mathfrak{t}$ has the same output and input colors as those of $x$. Hence, because $x$ belongs to $L(\mathscr{B})$, its output color is in $I$ and all its input colors are in $T$. Thus, $\mathfrak{t}$ satisfies the required properties.

Conversely, assume that $x$ is an element of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ admitting an $\mathfrak{R}$-treelike expression $\mathfrak{t}$ with output color in $I$ and all input colors in $T$. Lemma 1.2.2 implies in particular that $x$ admits an $\mathfrak{R}$-left expression. Hence, by definition of the derivation relation $\rightarrow, x$ is derivable from $\mathbb{1}_{\mathrm{out}(x)}$ and all its input colors are in $T$. Therefore, $x$ belongs to $L(\mathscr{G})$.

Lemma 2.2.2. Let $\mathscr{G}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system. Then, for any $x \in$ Bud $_{\mathscr{C}}(\Theta), x$ belongs to $\mathrm{L}_{S}(\mathscr{H})$ if and only if $x$ admits an $\mathfrak{R}$-treelike expression with output color in I and all input colors in $T$ and which is a perfect tree.

Proof. The proof of the statement of the lemma is very similar to the one of Lemma 2.2.1. The only difference lies on the fact that the definition of synchronous languages uses the complete composition map o instead of partial composition maps $\circ_{i}$, intervening in the definition of languages. Hence, in this context, $\mathfrak{R}$-treelike expressions are perfect trees.

Proposition 2.2.3. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system. Then, the language of $\mathfrak{G}$ satisfies

$$
\begin{equation*}
\mathrm{L}(\mathscr{B})=\left\{x \in \operatorname{Bud}_{\mathscr{C}}(\theta)^{\mathfrak{R}}: \operatorname{out}(x) \in I \text { and } \operatorname{in}(x) \in T^{+}\right\}, \tag{2.2.1}
\end{equation*}
$$

where $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$ is the colored suboperad of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ generated by $\Re$.
Proof. By definition of suboperads generated by a set, as a $\mathscr{C}$-colored graded collection, $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$ consists in all the elements obtained by evaluating in $\operatorname{Bud}_{\mathscr{C}}(\theta)$ all $\mathscr{C}$-colored $\mathfrak{R}$ syntax trees. Therefore, the statement of the proposition is a consequence of Lemma 2.2.1.

Proposition 2.2.4. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system. Then, the synchronous language of $\mathscr{B}$ is a subset of the language of $\mathscr{A}$. Moreover, when $\mathfrak{R}$ contains all the colored units of $\operatorname{Bud}_{\mathscr{C}}(\Theta)$, these two languages are equal.

Proof. By Lemma 2.2.1 the language of $\mathscr{B}$ is the set of the elements obtained by evaluating in $\operatorname{Bud}_{\mathscr{C}}(\theta)$ all $\mathscr{C}$-colored $\mathfrak{R}$-syntax trees satisfying some conditions for their output and input colors. Lemma 2.2.2 says that the synchronous language of $\mathscr{B}$ is the set of the elements obtained by evaluating in $\operatorname{Bud}_{\mathscr{C}}(\Theta)$ some $\mathscr{C}$-colored $\mathfrak{R}$-syntax trees satisfying at least the previous conditions. Hence, this implies the statement of the proposition.

The second part of the proposition follows from the fact that, if $x_{1} \rightarrow x_{2}$ for two elements $x_{1}$ and $x_{2}$ of $\operatorname{Bud}_{\mathscr{C}}(\theta)$, there is by definition $r \in \mathfrak{R}$ and an integer $i$ such that $x_{2}=x_{1} \circ_{i} r$. Then, one has

$$
\begin{equation*}
x_{2}=x_{1} \circ\left[\mathbb{1}_{\mathrm{in}_{1}\left(x_{1}\right)}, \ldots, \mathbb{1}_{\mathrm{in}_{i-1}\left(x_{1}\right)}, r, \mathbb{1}_{\mathrm{in}_{i+1}\left(x_{1}\right)}, \ldots, \mathbb{1}_{\mathrm{in}_{\left|x_{1}\right|}\left(x_{1}\right)}\right] . \tag{2.2.2}
\end{equation*}
$$

Since by hypothesis, all the colored units of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ are in $\Re$, this implies $x_{1} \sim x_{2}$. Hence, as binary relations, $\rightarrow$ and $\leadsto$ are equal, establishing the second part of the statement of the proposition.
2.3. Links with other generating systems. Context-free grammars, regular tree grammars, and synchronous grammars are already existing generating systems describing sets of words for the first, and sets of trees for the last two. We show here that any of these grammars can be emulated by bud generating systems. In the first case, context-free grammars are emulated by bud generating systems with the associative operad As as ground operad, and in the second and third cases, regular tree grammars and synchronous grammars are emulated by bud generating systems with free operads Free $(C)$ as ground operads, where $C$ are suitable set of generators.
2.3.1. Context-free grammars. A context-free grammar [Har78, HMU06] is a tuple $\mathcal{G}:=$ ( $V, T, P, s$ ) where $V$ is a finite alphabet of variables, $T$ is a finite alphabet of terminal symbols, $P$ is a finite subset of $V \times(V \sqcup T)^{*}$ called set of productions, and $s$ is a variable of $V$ called start symbol. If $x_{1}$ and $x_{2}$ are two words of $(V \sqcup T)^{*}, x_{2}$ is derivable in one step from $x_{1}$ if $x_{1}$ is of the form $x_{1}=u a v$ and $x_{2}$ is of the form $x_{2}=u w v$ where $u, v \in(V \sqcup T)^{*}$ and $(a, w)$ is a production of $P$. This property is denoted by $x_{1} \rightarrow x_{2}$, so that $\rightarrow$ is a binary relation on $(V \sqcup T)^{*}$. The reflexive and transitive closure of $\rightarrow$ is the derivation relation. A word $x \in T^{*}$ is generated by $\mathcal{G}$ if $x$ is derivable from the word $s$. The language of $\mathcal{G}$ is the set of all words generated by $\mathcal{G}$. We say that $\mathcal{G}$ is proper if, for any $(a, w) \in P$, w is not the empty word.

If $\mathcal{G}:=(V, T, P, s)$ is a proper context-free grammar, we denote by $\mathrm{CFG}(\mathcal{G})$ the bud generating system

$$
\begin{equation*}
\operatorname{CFG}(\mathcal{G}):=(\mathbf{A s}, V \sqcup T, \mathfrak{R},\{s\}, T) \tag{2.3.1}
\end{equation*}
$$

wherein $\mathfrak{R}$ is the set of rules

$$
\begin{equation*}
\mathfrak{R}:=\left\{(a, u) \in \operatorname{Bud}_{V \sqcup T}(\mathbf{A s}):(a, u) \in P\right\} . \tag{2.3.2}
\end{equation*}
$$

Proposition 2.3.1. Let $\mathcal{G}$ be a proper context-free grammar. Then, the restriction of the map in, sending any $(a, u) \in \operatorname{Bud}_{V \sqcup T}(\mathbf{A s})$ to $u$, on the domain $\mathrm{L}(\mathrm{CFG}(\mathcal{G})$ ) is a bijection between $\mathrm{L}(\mathrm{CFG}(\mathcal{G}))$ and the language of $\mathcal{G}$.

Proof. Let us denote by $V$ the set of variables, by $T$ the set of terminal symbols, by $P$ the set of productions, and by $s$ the start symbol of $\mathcal{G}$.

Let $(a, x) \in \operatorname{Bud}_{V \sqcup T}(\mathbf{A s}), \ell \geqslant 1$, and $y_{1}, \ldots, y_{\ell-1} \in(V \sqcup T)^{*}$. Then, by definition of CFG, there are in $\operatorname{CFG}(G)$ the derivations

$$
\begin{equation*}
\mathbb{1}_{s} \rightarrow\left(s, y_{1}\right) \rightarrow \cdots \rightarrow\left(s, y_{\ell-1}\right) \rightarrow(a, x) \tag{2.3.3}
\end{equation*}
$$

if and only if $a=s$ and there are in $\mathcal{G}$ the derivations

$$
\begin{equation*}
s \rightarrow y_{1} \rightarrow \cdots \rightarrow y_{\ell-1} \rightarrow x . \tag{2.3.4}
\end{equation*}
$$

Then, $(a, x)$ belongs to $\mathrm{L}(\operatorname{CFG}(\mathcal{G}))$ if and only if $a=s$ and $x$ belongs to the language of $\mathcal{G}$. The fact that in $((s, x))=x$ completes the proof.
2.3.2. Regular tree grammars. Let $V:=V(0)$ be a finite graded alphabet of variables and $T:=\bigsqcup_{n \geqslant 0} T(n)$ be a finite graded alphabet of terminal symbols. For any $n \geqslant 0$ and $a \in T(n)$, the arity $|a|$ of $a$ is $n$. The tuple ( $V, T)$ is called a signature.

A $(V, T)$-tree is an element of $\operatorname{Bud}_{V \sqcup T(0)}(\operatorname{Free}(T \backslash T(0)))$, where $T \backslash T(0)$ is seen as a monochrome graded collection. In other words, a $(V, T)$-tree is a planar rooted tree $\mathfrak{t}$ such that, for any $n \geqslant 1$, any internal node of $\mathfrak{t}$ having $n$ children is labeled by an element of arity $n$ of $T$, and the output and all leaves of $\mathfrak{t}$ are labeled on $V \sqcup T(0)$.

A regular tree grammar [GS84, $\left.\mathrm{CDG}^{+} 07\right]$ is a tuple $\mathcal{G}:=(V, T, P, s)$ where $(V, T)$ is a signature, $P$ is a set of pairs of the form $(v, \mathfrak{s})$ called productions where $v \in V$ and $\mathfrak{s}$ is a $(V, T)$-tree, and $s$ is a variable of $V$ called start symbol. If $\mathfrak{t}_{1}$ and $\mathfrak{t}_{2}$ are two $(V, T)$-trees, $\mathfrak{t}_{2}$ is derivable in one step from $\mathfrak{t}_{1}$ if $\mathfrak{t}_{1}$ has a leaf $y$ labeled by $a$ and the tree obtained by replacing $y$ by the root of $\mathfrak{s}$ in $\mathfrak{t}_{1}$ is $\mathfrak{t}_{2}$, provided that $(a, \mathfrak{s})$ is a production of $P$. This property is denoted by $\mathfrak{t}_{1} \rightarrow \mathfrak{t}_{2}$, so that $\rightarrow$ is a binary relation on the set of all $(V, T)$-trees. The reflexive and transitive closure of $\rightarrow$ is the derivation relation. A $(V, T)$-tree $\mathfrak{t}$ is generated by $\mathcal{G}$ if $\mathfrak{t}$ is derivable from the tree $\mathbb{1}_{s}$ consisting in one leaf labeled by $s$ and all leaves of $\mathfrak{t}$ are labeled on $T(0)$. The language of $\mathcal{G}$ is the set of all $(V, T)$-trees generated by $\mathcal{G}$.

If $\mathcal{G}:=(V, T, P, s)$ is a regular tree grammar, we denote by $\operatorname{RTG}(\mathcal{G})$ the bud generating system

$$
\begin{equation*}
\operatorname{RTG}(\mathcal{G}):=(\operatorname{Free}(T \backslash T(0)), V \sqcup T(0), \mathfrak{R},\{s\}, T(0)) \tag{2.3.5}
\end{equation*}
$$

wherein $\mathfrak{R}$ is the set of rules

$$
\begin{equation*}
\mathfrak{R}:=\left\{(a, \mathfrak{t}, u) \in \operatorname{Bud}_{V \sqcup T(0)}(\operatorname{Free}(T \backslash T(0))):\left(a, \mathfrak{t}_{a, u}\right) \in P\right\} \tag{2.3.6}
\end{equation*}
$$

where, for any $\mathfrak{t} \in \operatorname{Free}(T \backslash T(0))$, $a \in V \sqcup T(0)$, and $u \in\left(\left.V \sqcup T(0)\right|^{|\mathfrak{t}|}, \mathfrak{t}_{a, u}\right.$ is the $(V, T)$-tree obtained by labeling the output of $\mathfrak{t}$ by $a$ and by labeling from left to right the leaves of $\mathfrak{t}$ by the letters of $u$.

Proposition 2.3.2. Let $\mathcal{G}$ be a regular tree grammar. Then, the map $\phi: \mathrm{L}(\mathrm{RTG}(\mathcal{G})) \rightarrow L$ defined by $\phi((a, \mathfrak{t}, u)):=\mathfrak{t}_{a, u}$ is a bijection between the language of $\operatorname{RTG}(\mathcal{G})$ and the language $L$ of $\mathcal{G}$.

Proof. Let us denote by $(V, T)$ the underlying signature and by $s$ the start symbol of $\mathcal{G}$.
Let $(a, \mathfrak{t}, u) \in \operatorname{Bud}_{V \sqcup T(0)}(\operatorname{Free}(T \backslash T(0))), \ell \geqslant 1$, and $\mathfrak{s}^{(1)}, \ldots, \mathfrak{s}^{(\ell-1)} \in \operatorname{Free}(T \backslash T(0))$, and $v_{1}, \ldots, v_{\ell-1} \in(V \sqcup T(0))^{+}$. Then, by definition of RTG, there are in RTG $(\mathcal{G})$ the derivations

$$
\begin{equation*}
\mathbb{1}_{s} \rightarrow\left(s, \mathfrak{s}^{(1)}, v_{1}\right) \rightarrow \cdots \rightarrow\left(s, \mathfrak{s}^{(\ell-1)}, v_{\ell-1}\right) \rightarrow(a, \mathfrak{t}, u) \tag{2.3.7}
\end{equation*}
$$

if and only if $a=s$ and there are in $\mathcal{G}$ the derivations

$$
\begin{equation*}
\mathbb{1}_{s} \rightarrow \mathfrak{s}_{s, v_{1}}^{(1)} \rightarrow \cdots \rightarrow \mathfrak{s}^{(\ell-1)_{s, v_{\ell-1}}} \rightarrow \mathfrak{t}_{a, u} \tag{2.3.8}
\end{equation*}
$$

Then, $(a, t, u)$ belongs to $\mathrm{L}(\operatorname{RTG}(\mathcal{G}))$ if and only if $a=s$ and $\mathfrak{t}_{a, u}$ belongs to the language of $\mathcal{G}$. The fact that $\phi((a, t, u))=\mathfrak{t}_{a, u}$ completes the proof.
2.3.3. Synchronous grammars. In this section, we shall denote by Tree the monochrome operad defined as the free operad generated by one operation $a_{n}$ of arity $n$ for all $n \geqslant 1$. The elements of this operad are planar rooted trees where internal nodes have an arbitrary arity. Observe by the way that Tree is not locally finite.

Let $B$ be a finite alphabet. A $B$-bud tree is an element of Bud $_{B}$ (Tree). In other words, a $B$-bud tree is a planar rooted tree $\mathfrak{t}$ such that the output and all leaves of $\mathfrak{t}$ are labeled on $B$. The leaves of a $B$-bud tree are indexed from 1 from left to right.

A synchronous grammar [Gir12] is a tuple $\mathcal{G}:=(B, a, R)$ where $B$ is a finite alphabet of bud labels, $a$ is an element of $B$ called axiom, and $R$ is a finite set of pairs of the form $(b, \mathfrak{s})$ called substitution rules where $b \in B$ and $\mathfrak{s}$ is a $B$-bud tree. If $\mathfrak{t}_{1}$ and $\mathfrak{t}_{2}$ are two $B$-bud trees such that $\mathfrak{t}_{1}$ is of arity $n, \mathfrak{t}_{2}$ is derivable in one step from $\mathfrak{t}_{1}$ if there are substitution rules $\left(b_{1}, \mathfrak{s}_{1}\right), \ldots,\left(b_{n}, \mathfrak{s}_{n}\right)$ of $R$ such that for all $i \in[n]$, the $i$-th leaf of $\mathfrak{t}_{1}$ is labeled by $b_{i}$ and $\mathfrak{t}_{2}$ is obtained by replacing the $i$-th leaf of $\mathfrak{t}_{1}$ by $\mathfrak{s}_{i}$ for all $i \in[n]$. This property is denoted by $\mathfrak{t}_{1} \sim \mathfrak{t}_{2}$, so that $\leadsto$ is a binary relation on the set of all $B$-bud trees. The reflexive and transitive closure of $\leadsto$ is the derivation relation. A $B$-bud tree $\mathfrak{t}$ is generated by $\mathcal{G}$ if $\mathfrak{t}$ is derivable from the tree $\mathbb{1}_{a}$ consisting is one leaf labeled by $a$. The language of $\mathcal{G}$ is the set of all $B$-bud trees generated by $\mathcal{G}$.

If $\mathcal{G}:=(B, a, R)$ is a synchronous grammar, we denote by $\mathrm{SG}(\mathcal{G})$ the bud generating system

$$
\begin{equation*}
\mathrm{SG}(\mathcal{G}):=(\text { Tree }, B, \mathfrak{R},\{a\}, B) \tag{2.3.9}
\end{equation*}
$$

wherein $\mathfrak{R}$ is the set of rules

$$
\begin{equation*}
\mathfrak{R}:=\left\{(b, \mathfrak{t}, u) \in \operatorname{Bud}_{B}(\text { Tree }):\left(b, t_{b}, u\right) \in R\right\}, \tag{2.3.10}
\end{equation*}
$$

where, for any $\mathfrak{t} \in \operatorname{Bud}_{B}$ (Tree), $b \in B$, and $u \in B^{+}, \mathfrak{t}_{b, u}$ is the $B$-bud tree obtained by labeling the output of $t$ by $b$ and by labeling from left to right the leaves of $t$ by the letters of $u$.

Proposition 2.3.3. Let $\mathcal{G}$ be a synchronous grammar. Then, the map $\phi: \mathrm{L}_{\mathrm{S}}(\mathrm{SG}(\mathcal{G})) \rightarrow L$ defined by $\phi((b, \mathfrak{t}, u)):=\mathfrak{t}_{b, u}$ is a bijection between the synchronous language of $\mathrm{SG}(\mathcal{G})$ and the language $L$ of $\mathcal{G}$.

Proof. Let us denote by $B$ the set of bud labels and by $a$ the axiom of $\mathcal{G}$.
Let $(b, t, u) \in \operatorname{Bud}_{B}($ Tree $), \ell \geqslant 1$, and $\mathfrak{s}^{(1)}, \ldots, \mathfrak{s}^{(\ell-1)} \in$ Tree, and $v_{1}, \ldots, v_{\ell-1} \in B^{+}$. Then, by definition of SG, there are in $\operatorname{SG}(\mathscr{G})$ the synchronous derivations

$$
\begin{equation*}
\mathbb{1}_{a} \sim\left(a, \mathfrak{s}^{(1)}, v_{1}\right) \sim \cdots \sim\left(a, \mathfrak{s}^{(\ell-1)}, v_{\ell-1}\right) \sim(b, \mathfrak{t}, u) \tag{2.3.11}
\end{equation*}
$$

if and only if $b=a$ and there are in $\mathcal{G}$ the derivations

$$
\begin{equation*}
\mathbb{1}_{a} \leadsto \mathfrak{s}^{(1)}{ }_{a, v_{1}} \sim \cdots \sim \mathfrak{s}^{(\ell-1)}{ }_{a, v_{\ell-1}} \sim \mathfrak{t}_{b, u} . \tag{2.3.12}
\end{equation*}
$$

Then, $(a, t, u)$ belongs to $L_{S}(S G(\mathcal{G}))$ if and only if $b=a$ and $\mathfrak{t}_{b, u}$ belongs to the language of $\mathcal{G}$. The fact that $\phi((b, t, u))=\mathfrak{t}_{b, u}$ completes the proof.

## 3. Series on colored operads and bud generating systems

A very normal combinatorial question consists, given a bud generating system $\mathscr{B}$, in computing the generating series $\mathbf{s}_{\mathrm{L}(\mathscr{G})}(t)$ and $\mathbf{s}_{\mathrm{LS}(\mathscr{B})}(t)$, respectively counting the elements of the language and of the synchronous language of $\mathscr{B}$ with respect to the arity of the elements. To achieve this objective, we develop a new generalization of formal power series, namely series on colored operads, and define several operation on these. Any bud generating system $\mathscr{B}$ leads to the definition of three series on colored operads: its hook generating series hook( $\mathscr{A}$ ), its syntactic generating series synt(G)), and its synchronous generating series sync(G)). The hook generating series allows us to define analogues of the hook-length statistic of binary trees for objects belonging to the language of $\mathscr{B}$, possibly different than trees. The syntactic (resp. synchronous) generating series leads to obtain functional equations and recurrence formulas to compute the coefficients of $\mathbf{s}_{\mathrm{L}(\mathscr{G})}(t)$ and $\mathbf{s}_{\mathrm{LS}}(\mathscr{B})(t)$.

From now, $\mathbb{K}$ is a field of characteristic zero. Moreover, in all this section, $\mathfrak{G}$ is a $\mathscr{C}$-colored operad. Recall that the set $\mathscr{C}$ of colors is always considered on the form $\mathscr{C}=\left\{c_{1}, \ldots, c_{k}\right\}$. Besides, $\mathscr{G}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ is a bud generating system.
3.1. Series on colored operads. We introduce here the main definitions about series on colored operads. We also introduce specific notions about series on bud operads.
3.1.1. Series on colored operads. The linear span of the underlying set of $\mathfrak{G}$ is denoted by $\mathbb{K}\langle\mathscr{G}\rangle$. Let $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ be the dual space of $\mathbb{K}\langle\mathscr{G}\rangle$. By definition, the elements of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ are maps $\mathbf{f}: \mathfrak{G} \rightarrow \mathbb{K}$, called $\mathfrak{G}$-formal power series (or $\mathfrak{G}$-series for short). Let $\mathbf{f} \in \mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$. The coefficient of any $x \in \mathcal{G}$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f}\rangle$. The support of $\mathbf{f}$ is the set $\operatorname{Supp}(\mathbf{f}):=\{x \in \mathfrak{G}:\langle x, \mathbf{f}\rangle \neq 0\}$. For any $\mathscr{C}$-colored graded subcollection $S$ of $\mathfrak{G}$, the characteristic series of $S$ is the $\mathfrak{G}$-series $\bar{S}$ defined for any $x \in \mathscr{G}$ by $\langle x, \bar{S}\rangle:=1$ if $x \in S$, and by $\langle x, \bar{S}\rangle:=0$ otherwise. The series of colored units of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ is the series $\mathbf{u}$ defined as the characteristic of $\left\{\mathbb{1}_{a}: a \in \mathscr{C}\right\}$. This series will play a special role in the sequel. Since $\mathscr{C}$ is finite, $\mathbf{u}$ is a polynomial. By exploiting the vector space structure of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$, any $\mathfrak{G}$-series $\mathbf{f}$ expresses as

$$
\begin{equation*}
\mathbf{f}=\sum_{x \in G}\langle x, \mathbf{f}\rangle x . \tag{3.1.1}
\end{equation*}
$$

This notation using potentially infinite sums of elements of $\mathcal{G}$ accompanied with coefficients of $\mathbb{K}$ is common in the context of formal power series. In the sequel, we shall define and handle some $\mathfrak{G}$-series using the notation (3.1.1).

Let us provide here some bibliographical information about various types of series. Since the introduction of formal power series, a lot of generalizations were proposed in order to extend the range of problems they can help to solve. The most obvious ones are multivariate series allowing us to count objects not only with respect to their sizes but also with respect to various other statistics. Another one consists in considering noncommutative series on words [Eil74, SS78, BR10], or even, pushing the generalization one step further, on elements of a monoid [Sak09]. Besides, as another generalization, series on trees have been considered [BR82, Boz01]. Series on (noncolored) operads increase the list of these generalizations.

Chapoton is the first to have considered such series on operads [Cha02, Cha08, Cha09]. Several authors have contributed to this field by considering slight variations in the definitions of these series. Among these, one can cite van der Laan [vdL04], Frabetti [Fra08], and Loday and Nikolov [LN13]. Our notion of series on colored operads developed here is a natural generalization of series on operads.

Observe that $\mathfrak{G}$-series are defined here on fields $\mathbb{K}$ instead on the much more general structures of semirings, as it is the case for series on monoids. We choose to tolerate this loss of generality because this considerably simplifies the theory. Furthermore, we shall use in the sequel $\mathfrak{G}$-series as devices for combinatorial enumeration, so that it is sufficient to pick $\mathbb{K}$ as the field $\mathbb{Q}\left(q_{0}, q_{1}, q_{2}, \ldots\right)$ of rational functions in an infinite number of commuting parameters with rational coefficients. The parameters $q_{0}, q_{1}, q_{2}, \ldots$ intervene in the enumeration of colored graded subcollections of $\mathcal{G}$ with respect to several statistics ${ }^{d}$.
3.1.2. Colored operad morphisms and series. If $\mathscr{G}_{1}$ and $\mathscr{G}_{2}$ are two $\mathscr{C}$-colored operads and $\phi: \mathfrak{G}_{1} \rightarrow \mathfrak{G}_{2}$ is a morphism of colored operads, $\bar{\phi}$ is the map

$$
\begin{equation*}
\bar{\phi}: \mathbb{K}\left\langle\left\langle\mathcal{G}_{1}\right\rangle\right\rangle \rightarrow \mathbb{K}\left\langle\left\langle\mathfrak{G}_{2}\right\rangle\right\rangle \tag{3.1.2}
\end{equation*}
$$

defined, for any $\mathbf{f} \in \mathbb{K}\left\langle\left\langle\mathfrak{G}_{1}\right\rangle\right\rangle$ and $y \in \mathfrak{G}_{2}$, by

$$
\begin{equation*}
\langle y, \bar{\phi}(\mathbf{f})\rangle:=\sum_{\substack{x \in G_{1} \\ \phi(x)=y}}\langle x, \mathbf{f}\rangle . \tag{3.1.3}
\end{equation*}
$$

Observe first that $\bar{\phi}$ is a linear map. Moreover, notice that (3.1.3) could be undefined for arbitrary colored operads $\mathfrak{G}_{1}$ and $\mathfrak{G}_{2}$, and an arbitrary morphism of colored operads $\phi$. However, when all fibers of $\phi$ are finite, for any $y \in \mathcal{G}_{2}$, the right member of (3.1.3) is welldefined since the sum has a finite number of terms.
3.1.3. Pruned series and faithfulness. Let $\mathbf{f}$ be a $\operatorname{Bud}_{\mathscr{C}}(\theta)$-series. By a slight abuse of notation, we denote by

$$
\begin{equation*}
\mathfrak{p r u}: \mathbb{K}\left\langle\left\langle\mathbf{B u d}_{\mathscr{C}}(\Theta)\right\rangle\right\rangle \rightarrow \mathbb{K}\langle\langle\Theta\rangle\rangle \tag{3.1.4}
\end{equation*}
$$

the map $\mathfrak{p r u}$. Since $\mathscr{C}$ is finite, the series $\mathfrak{p r u}(\mathbf{f})$ is well-defined and is called pruned series of $\mathbf{f}$. Intuitively, the series $\mathfrak{p r u}(\mathbf{f})$ can be seen as a version of $\mathbf{f}$ wherein the colors of the elements of its support are forgotten ${ }^{\text {e }}$. Besides, $\mathbf{f}$ is said faithful if all coefficients of $\mathfrak{p r u}(\mathbf{f})$ are equal to 0 or to 1.

We say that $\mathscr{B}$ is faithful (resp. synchronously faithful) if the characteristic series of L( $\mathscr{A})$ (resp. $\mathrm{L}_{S}(\mathscr{\oiint})$ ) is faithful. Observe that all monochrome bud generating systems are faithful (resp. synchronously faithful). One of the reasons for requiring faithfulness (resp. synchronous faithfulness) for bud generating systems appears when $\mathscr{B}$ is utilized for specifying objects of $\mathcal{O}$ by pruning the objects of $\mathrm{L}(\mathscr{B})$ (resp. $\mathrm{L}_{S}(\mathscr{B})$ ). In this case, if $\mathscr{G}$ is not faithful (resp. synchronously faithful), there would be several distinct elements ( $a, x, u$ ) of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ generated (resp. synchronously generated) by $\mathscr{B}$ whose image by $\mathfrak{p r u}$ is $x$. This could make very hard the enumeration of the pruned version of the language (resp. synchronous language) of $\mathscr{\mathscr { B }}$.

[^2]3.1.4. Series of colors. Let
\[

$$
\begin{equation*}
\mathfrak{c o l}: \mathscr{G} \rightarrow \operatorname{Bud}_{\mathscr{C}}(\mathbf{A s}) \tag{3.1.5}
\end{equation*}
$$

\]

be the morphism of colored operads defined for any $x \in \mathscr{G}$ by

$$
\begin{equation*}
\mathfrak{c o l}(x):=(\operatorname{out}(x), \operatorname{in}(x)) . \tag{3.1.6}
\end{equation*}
$$

By a slight abuse of notation, we denote by

$$
\begin{equation*}
\mathfrak{c o l}: \mathbb{K}\langle\langle\mathscr{C}\rangle\rangle \rightarrow \mathbb{K}\left\langle\left\langle\mathbf{B u d}_{\mathscr{C}}(\mathbf{A s})\right\rangle\right\rangle \tag{3.1.7}
\end{equation*}
$$

the map $\mathfrak{c o l}$. If $\mathbf{f}$ is a $\mathfrak{G}$-series, we call $\mathfrak{c o l}(\mathbf{f})$ the series of colors of $\mathbf{f}$. Intuitively, the series $\mathfrak{c o l}(\mathbf{f})$ can be seen as a version of $\mathbf{f}$ wherein only the colors of the elements of its support are taken into account ${ }^{\mathrm{f}}$.
3.1.5. Series of color types. The $\mathscr{C}$-type of a word $u \in \mathscr{C}^{+}$is the word type $(u)$ of $\mathbb{N}^{k}$ defined by

$$
\begin{equation*}
\operatorname{type}(u):=|u|_{c_{1}} \ldots|u|_{c_{k}} \tag{3.1.8}
\end{equation*}
$$

where for any $a \in \mathscr{C},|u|_{a}$ is the number of occurrences of $a$ in $u$. By extension, we shall call $\mathscr{C}$-type any word of $\mathbb{N}^{k}$ with at least a nonzero letter and we denote by $\mathscr{T}_{\mathscr{C}}$ the set of all $\mathscr{C}$-types. The degree $\operatorname{deg}(\alpha)$ of $\alpha \in \mathscr{T}_{\mathscr{C}}$ is the sum of the letters of $\alpha$. We denote by $\mathscr{C}^{\alpha}$ the word $c_{1}^{\alpha_{1}} \ldots c_{k}^{\alpha_{k}}$.

Assume that $\mathbb{Z}_{\mathscr{C}}:=\left\{\mathrm{z}_{\mathrm{c}_{1}}, \ldots, \mathrm{z}_{\mathrm{c}_{k}}\right\}$ is any alphabet of commutative letters. For any type $\alpha$, we denote by $\mathbb{Z}_{\mathscr{C}}^{\alpha}$ the monomial $z_{c_{1}}^{\alpha_{1}} \ldots \mathrm{z}_{\mathrm{c}_{k}}^{\alpha_{k}}$ of $\mathbb{K}\left[\mathbb{Z}_{\mathscr{C}}\right]$. Moreover, for any two types $\alpha$ and $\beta$, the sum $\alpha \dot{+} \beta$ of $\alpha$ and $\beta$ is the type satisfying $(\alpha \dot{+} \beta)_{i}:=\alpha_{i}+\beta_{i}$ for all $i \in[k]$. Observe that with this notation, $\mathbb{Z}_{\mathscr{C}}^{\alpha} \mathbb{Z}_{\mathscr{C}}^{\beta}=\mathbb{Z}_{\mathscr{C}}^{\alpha \dot{+} \beta}$.

Consider now the map

$$
\begin{equation*}
\mathfrak{c o l t}: \mathbb{K}\langle\langle\mathscr{G}\rangle\rangle \rightarrow \mathbb{K}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right] \tag{3.1.9}
\end{equation*}
$$

defined for all $\alpha, \beta \in \mathscr{T}_{\mathscr{C}}$ by

$$
\begin{equation*}
\left\langle\mathbb{X}_{\mathscr{C}}^{\alpha} \mathbb{Y}_{\mathscr{C}}^{\beta}, \operatorname{colt}(\mathbf{f})\right\rangle:=\sum_{\substack{(a, u) \in \operatorname{Bud} \mathscr{G}(\mathbf{A s}) \\ \text { type}(a)=\alpha \\ \text { type }(u)=\beta}}\langle(a, u), \operatorname{col}(\mathbf{f})\rangle . \tag{3.1.10}
\end{equation*}
$$

By the definition of the map col,

$$
\begin{equation*}
\mathfrak{c o l t}(\mathbf{f})=\sum_{x \in \mathscr{G}}\langle x, \mathbf{f}\rangle \mathbb{X}_{\mathscr{C}}^{\text {type }(\operatorname{out}(x))} \mathbb{Y}_{\mathscr{C}}^{\operatorname{type}(\operatorname{in}(x))} . \tag{3.1.11}
\end{equation*}
$$

Observe that for all $\alpha, \beta \in \mathscr{T}_{\mathscr{C}}$ such that $\operatorname{deg}(\alpha) \neq 1$, the coefficients of $\mathbb{X}_{\mathscr{C}}^{\alpha} \mathbb{Y}_{\mathscr{C}}^{\beta}$ in $\mathfrak{c o l t}(\mathbf{f})$ are zero. In intuitive terms, the series $\mathfrak{c o l t}(\mathbf{f})$, called series of color types of $\mathbf{f}$, can be seen as a version of $\operatorname{col}(\mathbf{f})$ wherein only the output colors and the types of the input colors of the elements of its support are taken into account, the variables of $\mathbb{X}_{\mathscr{C}}$ encoding output colors and the variables of $\mathbb{Y}_{\mathscr{C}}$ encoding input colors ${ }^{9}$. In the sequel, we shall be concerned by the computation of the coefficients of $\mathfrak{c o l t}(\mathbf{f})$ for some $\mathfrak{G}$-series $\mathbf{f}$.

[^3]3.1.6. Elementary series of bud generating systems. We assume here that $\mathcal{O}$ is a locally finite monochrome operad. We shall denote by $\mathbf{r}$ the characteristic series of $\mathfrak{R}$, by $\mathbf{i}$ the characteristic series of $\left\{\mathbb{1}_{a}: a \in I\right\}$, and by $\mathbf{t}$ the characteristic series of $\left\{\mathbb{1}_{a}: a \in T\right\}$. For all colors $a \in \mathscr{C}$ and types $\alpha \in \mathscr{T}_{\mathscr{C}}$, let
\[

$$
\begin{equation*}
\chi_{a, \alpha}:=\#\{r \in \mathfrak{R}:(\text { out }(r), \text { type }(\operatorname{in}(r)))=(a, \alpha)\} . \tag{3.1.12}
\end{equation*}
$$

\]

For any $a \in \mathscr{C}$, let $\mathbf{g}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ be the series of $\mathbb{K}\left[\left[Y_{\mathscr{C}}\right]\right]$ defined by

$$
\begin{equation*}
\mathbf{g}_{a}\left(\mathrm{y}_{\mathrm{c}_{1}}, \ldots, \mathrm{y}_{c_{k}}\right):=\sum_{\gamma \in \mathcal{T}_{\mathscr{C}}} \chi_{a, \gamma} \mathbb{Y}_{\mathscr{C}}^{\gamma}=\sum_{\substack{r \in \mathfrak{R} \\ \operatorname{out}(r)=a}} \mathbb{Y}_{\mathscr{C}}^{\text {type(in }(r))} \tag{3.1.13}
\end{equation*}
$$

Since $\Re$ is finite, this series is a polynomial ${ }^{h}$.
In the sequel, we shall use maps $\phi: \mathscr{C} \times \mathscr{T}_{\mathscr{C}} \rightarrow \mathbb{N}$ such that $\phi(a, \gamma) \neq 0$ for a finite number of pairs $(a, \gamma) \in \mathscr{C} \times \mathscr{T}_{\mathscr{C}}$, to express in a concise manner some recurrence relations for the coefficients of series on colored operads. We shall consider the two following notations. If $\phi$ is such a map and $a \in \mathscr{C}$, we define $\phi^{(a)}$ as the natural number

$$
\begin{equation*}
\phi^{(a)}:=\sum_{\substack{b \in \mathscr{\mathscr { C }} \\ \gamma \in \mathscr{T}_{\mathscr{C}}}} \phi(b, \gamma) \gamma_{a} \tag{3.1.14}
\end{equation*}
$$

and $\phi_{a}$ as the finite multiset

$$
\begin{equation*}
\phi_{a}:=\left\{\phi(a, \gamma): \gamma \in \mathscr{T}_{\mathscr{C}} \int\right. \tag{3.1.15}
\end{equation*}
$$

3.2. Products on series. Two binary products $\curvearrowleft$ and $\odot$ on the space of $\mathfrak{G}$-series are presented. The product $\curvearrowleft$ is a generalization to series and to colored operads of a known product on monochrome operads, and $\odot$ is a generalization to colored operads of a known product on series on monochrome operads.
3.2.1. Pre-Lie product. Given two $\mathfrak{G}$-series $\mathbf{f}, \mathbf{g} \in \mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$, the pre-Lie product of $\mathbf{f}$ and $\mathbf{g}$ is the $\mathfrak{G}$-series $\mathbf{f} \curvearrowleft \mathbf{g}$ defined, for any $x \in \mathfrak{G}$, by

$$
\begin{equation*}
\langle x, \mathbf{f} \curvearrowleft \mathbf{g}\rangle:=\sum_{\substack{y, z \in G \\ i \in[\| y] \\ x=y \circ_{i} z}}\langle y, \mathbf{f}\rangle\langle z, \mathbf{g}\rangle . \tag{3.2.1}
\end{equation*}
$$

Observe that $\mathbf{f} \curvearrowleft \mathbf{g}$ could be undefined for arbitrary $\mathfrak{G}$-series $\mathbf{f}$ and $\mathbf{g}$ on an arbitrary colored operad $\mathfrak{G}$. Besides, notice from (3.2.1) that $\curvearrowleft$ is bilinear and that $\mathbf{u}$ is a left unit of $\curvearrowleft$. However, since

$$
\begin{equation*}
\mathbf{f} \curvearrowleft \mathbf{u}=\sum_{x \in G}|x|\langle x, \mathbf{f}\rangle x \tag{3.2.2}
\end{equation*}
$$

the $\mathfrak{G}$-series $\mathbf{u}$ is not a right unit of $\curvearrowleft$. This product is also nonassociative in the general case since we have, for instance in $\mathbb{K}\langle\langle\mathbf{A s}\rangle\rangle$,

$$
\begin{equation*}
\left(\star_{2} \curvearrowleft \star_{2}\right) \curvearrowleft \star_{2}=6 \star_{4} \neq 4 \star_{4}=\star_{2} \curvearrowleft\left(\star_{2} \curvearrowleft \star_{2}\right) . \tag{3.2.3}
\end{equation*}
$$

[^4]Recall that a $\mathbb{K}$-pre-Lie algebra [Vin63, Ger63] (see also [CL01,Man11]) is a $\mathbb{K}$-vector space $V$ endowed with a bilinear product $\curvearrowleft$ satisfying, for all $x, y, z \in V$, the relation

$$
\begin{equation*}
(x \curvearrowleft y) \curvearrowleft z-x \curvearrowleft(y \curvearrowleft z)=(x \curvearrowleft z) \curvearrowleft y-x \curvearrowleft(z \curvearrowleft y) . \tag{3.2.4}
\end{equation*}
$$

In this case, we say that $\curvearrowleft$ is a pre-Lie product. Observe that any associative product satisfies (3.2.4), so that associative algebras are pre-Lie algebras.

Proposition 3.2.1. For any locally finite colored operad $\mathfrak{G}$, the space $\mathbb{K}\langle\langle\mathcal{G}\rangle\rangle$ endowed with the binary product $\curvearrowleft$ is a pre-Lie algebra.

This product $\curvearrowleft$ is a generalization of a pre-Lie product defined in [Ger63] (see also [vdL04, Cha08]), endowing the $\mathbb{K}$-linear span of the underlying monochrome graded collection of a monochrome operad with a pre-Lie algebra structure. Proposition 3.2.1 is based on similar arguments as the ones contained in the previous references.
3.2.2. Composition product. Given two $\mathfrak{G}$-series $\mathbf{f}, \mathbf{g} \in \mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$, the composition product of $\mathbf{f}$ and $\mathbf{g}$ is the $\mathfrak{G}$-series $\mathbf{f} \odot \mathbf{g}$ defined, for any $x \in \mathfrak{G}$, by

$$
\begin{equation*}
\langle x, \mathbf{f} \odot \mathbf{g}\rangle:=\sum_{\substack{y, z_{1}, \ldots, z_{| | y} \mid \in \mathcal{G} \\ x=y \circ\left[z_{1}, \ldots, z_{\mid y}\right]}}\langle y, \mathbf{f}\rangle \prod_{i \in[|y|]}\left\langle z_{i}, \mathbf{g}\right\rangle . \tag{3.2.5}
\end{equation*}
$$

Observe that $\mathbf{f} \odot \mathbf{g}$ could be undefined for arbitrary $\mathcal{G}$-series $\mathbf{f}$ and $\mathbf{g}$ on an arbitrary colored operad G. Besides, notice from (3.2.5) that $\odot$ is linear on the left and that the series $\mathbf{u}$ is the left and right unit of $\odot$. However, this product is not linear on the right since we have, for instance in $\mathbb{K}\langle\langle\mathbf{A s}\rangle\rangle$,

$$
\begin{equation*}
\star_{2} \odot\left(\star_{2}+\star_{3}\right)=\star_{4}+2 \star_{5}+\star_{6} \neq \star_{4}+\star_{6}=\star_{2} \odot \star_{2}+\star_{2} \odot \star_{3} . \tag{3.2.6}
\end{equation*}
$$

Proposition 3.2.2. For any locally finite colored operad $\mathfrak{G}$, the space $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ endowed with the binary product $\odot$ and the unit $\mathbf{u}$ is a monoid.

This product $\odot$ is a generalization of the composition product of series on operads of [Cha02, Cha09] (see also [vdL04, Fra08, Cha08, LV12, LN13]). In the case where $\mathfrak{G}$ is a monochrome operad concentrated in arity 1, $\odot$ coincides with the Cauchy product on series of monoids considered in [Sak09]. Proposition 3.2.2 is based on similar arguments as the ones contained in the previous references.

Lemma 3.2.3. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \Re, I, T)$ be $a$ bud generating system and $\mathbf{f}$ be $a \mathbf{B u d}_{\mathscr{C}}(\Theta)$-series. Then, $\mathbf{i} \odot \mathbf{f} \odot \mathbf{t}$ is the $\operatorname{Bud}_{\mathscr{C}}(\theta)$-series satisfying, for all $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$,

$$
\langle x, \mathbf{i} \odot \mathbf{f} \odot \mathbf{t}\rangle= \begin{cases}\langle x, \mathbf{f}\rangle & \text { if } \operatorname{out}(x) \in I \text { and } \operatorname{in}(x) \in T^{+}  \tag{3.2.7}\\ 0 & \text { otherwise } .\end{cases}
$$

Proof. By definition of the operation $\odot$, composing $\mathbf{f}$ with $\mathbf{i}$ to the left and with $\mathbf{t}$ to the right with respect to $\odot$ amounts to annihilate the coefficients of the terms of $\mathbf{f}$ that have an output color which is not in $I$ or an input color which is not in $T$. This implies the statement of the lemma.
3.3. Series and languages. We introduce the Kleene star operation of the pre-Lie product $\curvearrowleft$ in order to define the hook generating series of a bud generating system $\mathscr{B}$. We also study the inverse of the composition product $\odot$ in order to define the syntactic generating series of $\mathscr{G}$. We relate both of these series with the language of $\mathscr{B}$ and provide ways to compute its coefficients.
3.3.1. Pre-Lie star product. For any $\mathfrak{G}$-series $\mathbf{f} \in \mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ and any $\ell \geqslant 0$, let $\mathbf{f}^{\curvearrowleft \ell}$ be the $\mathfrak{G}$-series recursively defined by

$$
\mathbf{f}^{\curvearrowleft \ell}:= \begin{cases}\mathbf{u} & \text { if } \ell=0,  \tag{3.3.1}\\ \mathbf{f}^{\curvearrowleft \ell-1} \curvearrowleft \mathbf{f} & \text { otherwise } .\end{cases}
$$

Immediately from this definition and the definition of the pre-Lie product $\curvearrowleft$, the coefficients of $\mathbf{f}^{\curvearrowleft}, \ell \geqslant 0$, satisfy for any $x \in \mathcal{G}$,

$$
\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle= \begin{cases}\delta_{x, \mathbb{1}_{\text {out }}(x)} & \text { if } \ell=0,  \tag{3.3.2}\\ \sum_{\substack{y, z \in G \\ i \in[|y|] \\ x=y \circ_{i} Z}}\left\langle y, \mathbf{f}^{\curvearrowleft-1}\right\rangle\langle z, \mathbf{f}\rangle & \text { otherwise. } \\ & \end{cases}
$$

Lemma 3.3.1. Let $\mathscr{G}$ be a locally finite $\mathscr{C}$-colored operad and $\mathbf{f}$ be a series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$. Then, the coefficients of $\mathbf{f}^{\curvearrowleft}{ }_{\ell+1}, \ell \geqslant 0$, satisfy for any $x \in \mathfrak{G}$,

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\curvearrowleft} \cap_{\ell+1}\right\rangle=\sum_{\substack{y_{1}, \ldots, y_{\ell+1} \in \mathcal{G} \\ i_{1}, \ldots, i_{i} \in \mathbb{N} \\ x=\left(\ldots\left(y_{1} \dot{i}_{1} y_{2}\right) \circ_{i_{2}} \cdots\right) \circ_{i_{\ell}} y_{\ell+1}}} \prod_{j \in[\ell+1]}\left\langle y_{j}, \mathbf{f}\right\rangle . \tag{3.3.3}
\end{equation*}
$$

Proof. By Proposition 3.2.1, since $\mathfrak{G}$ is locally finite, $\mathbf{f}^{\curvearrowleft_{\ell+1}}$ is a well-defined $\mathfrak{G}$-series. The statement of the lemma follows by induction on $\ell$ and by using (3.3.2).

The $\curvearrowleft$-star of $\mathbf{f}$ is the series

$$
\begin{equation*}
\mathbf{f}^{\curvearrowleft}:=\sum_{\ell \geqslant 0} \mathbf{f}^{\curvearrowleft}=\mathbf{u}+\mathbf{f}+\mathbf{f} \curvearrowleft \mathbf{f}+(\mathbf{f} \curvearrowleft \mathbf{f}) \curvearrowleft \mathbf{f}+((\mathbf{f} \curvearrowleft \mathbf{f}) \curvearrowleft \mathbf{f}) \curvearrowleft \mathbf{f}+\cdots \tag{3.3.4}
\end{equation*}
$$



Proposition 3.3.2. Let $\mathfrak{G}$ be a locally finite $\mathscr{C}$-colored operad and $\mathbf{f}$ be a series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ such that $\operatorname{Supp}(\mathbf{f})(1)$ finitely factorizes $\mathfrak{G}$. Then,
(i) the series $\mathbf{f}^{\curvearrowleft^{*}}$ is well-defined;
(ii) for any $x \in \mathcal{G}$, the coefficient of $x$ in $\mathbf{f}^{\curvearrowleft *}$ satisfies

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\curvearrowleft *}\right\rangle=\delta_{x, \mathbb{1}_{\text {out }}(x)}+\sum_{\substack{y, z \in G \\ i \in[\| y] \\ x=y \circ_{i} z}}\left\langle y, \mathbf{f}^{\curvearrowleft *}\right\rangle\langle z, \mathbf{f}\rangle ; \tag{3.3.5}
\end{equation*}
$$

(iii) the equation

$$
\begin{equation*}
\mathbf{x}-\mathbf{x} \curvearrowleft \mathbf{f}=\mathbf{u} \tag{3.3.6}
\end{equation*}
$$

admits $\mathbf{x}=\mathbf{f}^{\curvearrowleft *}$ as unique solution.

Proof. Let $x \in \mathscr{G}$ and let us show that the coefficient $\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle$ is well-defined. Since $\mathfrak{G}$ is locally finite and $\operatorname{Supp}(\mathbf{f})(1)$ finitely factorizes $\mathcal{G}$, by Lemma 1.2.1, there are finitely many $\operatorname{Supp}(\mathbf{f})$-treelike expressions for $x$. Thus, for all $\ell \geqslant \operatorname{deg}_{\text {Supp }(\mathbf{f})}(x)+1$ there is in particular no expression for $x$ of the form $x=\left(\ldots\left(y_{1} \circ_{i_{1}} y_{2}\right) \circ_{i_{2}} \ldots\right) \circ_{i_{\ell-1}} y_{\ell}$ where $y_{1}, \ldots, y_{\ell} \in \operatorname{Supp}(\mathbf{f})$ and $\boldsymbol{i}_{1}, \ldots, \boldsymbol{i}_{\ell-1} \in \mathbb{N}$. This implies, together with Lemma 3.3.1, that $\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle=0$. Therefore, by virtue of this observation and by definition of the $\curvearrowleft$-star operation, the coefficient of $x$ in $\mathbf{f}^{\curvearrowleft}$ * is

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\curvearrowleft *}\right\rangle=\sum_{\ell \geqslant 0}\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle=\sum_{0 \leqslant \ell \leqslant \operatorname{deg}_{\text {supp }(f)}(x)}\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle, \tag{3.3.7}
\end{equation*}
$$

showing that $\left\langle x, \mathbf{f}^{\curvearrowleft}\right\rangle$ is a sum of a finite number of terms, all well-defined by Lemma 3.3.1. Thus, $\mathbf{f}^{\curvearrowleft}{ }^{*}$ is well-defined, so that (i) holds.

Point (ii) follows straightforwardly from the definition of the $\curvearrowleft$-star operation and (3.3.2).
By (3.3.6), we have $\mathbf{x}=\mathbf{u}+\mathbf{x} \curvearrowleft \mathbf{f}$ so that the coefficients of $\mathbf{x}$ satisfy, for any $x \in \mathfrak{G}$,

$$
\begin{equation*}
\langle x, \mathbf{x}\rangle=\langle x, \mathbf{u}\rangle+\langle x, \mathbf{x} \curvearrowleft \mathbf{f}\rangle=\delta_{x, \mathbb{1}_{\text {out }}(x)}+\sum_{\substack{y, z \in G \\ i \in[|y|] \\ x=y \circ_{i} z}}\langle y, \mathbf{x}\rangle\langle z, \mathbf{f}\rangle . \tag{3.3.8}
\end{equation*}
$$

By (ii), this implies $\mathbf{x}=\mathbf{f}^{\curvearrowleft^{*}}$ and the uniqueness of this solution, so that (iii) is established.
In particular, Point (ii) of Proposition 3.3.2 gives a way, given a G-series fatisfying the stated constraints, to compute recursively the coefficients of its $\curvearrowleft$-star $\mathbf{f}^{\curvearrowleft}$.
3.3.2. Hook generating series. The hook generating series of $\mathscr{B}$ the $\mathbf{B u d}_{\mathscr{C}}(\theta)$-series hook( $\left.\mathscr{B}\right)$ defined by

$$
\begin{equation*}
\operatorname{hook}(\mathscr{B}):=\mathbf{i} \odot \mathbf{r}^{\curvearrowleft} \odot \mathbf{t} . \tag{3.3.9}
\end{equation*}
$$

Observe that (3.3.9) could be undefined for an arbitrary set of rules $\mathfrak{R}$ of $\mathscr{B}$. Nevertheless, when r satisfies the conditions of Proposition 3.3.2, that is, when $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, $\operatorname{hook}(\mathscr{3})$ is well-defined.

The aim of the following is to provide an expression to compute the coefficients of hook(6)).

Lemma 3.3.3. Let $\mathscr{B}:=(O, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$,

Proof. Since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point (i) of Proposition 3.3.2, $\mathbf{r}^{\curvearrowleft}$ * is a welldefined series. Now, (3.3.10) is a consequence of Point (ii) of Proposition 3.3.2 together with the fact that all coefficients of $\mathbf{r}$ are equal to 0 or to 1 .

Proposition 3.3.4. Let $\mathscr{B}:=(O, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, for any $x \in \operatorname{Bud}_{\mathscr{C}}(0)$ such that $\operatorname{out}(x) \in I$, the coefficient $\left\langle x, \mathbf{r}^{\curvearrowleft}\right\rangle$ is the number of multipaths from $\mathbb{1}_{\text {out }(x)}$ to $x$ in the derivation graph of $\mathscr{G}$.

Proof. First, since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point $(i)$ of Proposition 3.3.2, $\mathbf{r}^{\curvearrowleft}$ is a well-defined series. If $x=\mathbb{1}_{a}$ for an $a \in I$, since $\left\langle\mathbb{1}_{a}, \mathbf{r}^{\curvearrowleft}\right\rangle=1$, the statement of the proposition holds. Let us now assume that $x$ is different from a colored unit and let us denote by $\lambda_{x}$ the number of multipaths from $\mathbb{1}_{\text {out }(x)}$ to $x$ in the derivation graph $G(\mathscr{B})$ of $\mathscr{B}$. By definition of $\mathrm{G}(\mathscr{3})$, by denoting by $\mu_{y, x}$ the number of edges from $y \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ to $x$ in $\mathrm{G}(\mathscr{B})$, we have

$$
\begin{equation*}
\lambda_{x}=\sum_{y \in \operatorname{Bud}_{\mathscr{C}}(\theta)} \mu_{y, x} \lambda_{y}=\sum_{y \in \operatorname{Bud}_{\mathscr{C}}(\theta)} \#\left\{(i, r) \in \mathbb{N} \times \Re: x=y \circ_{i} r\right\} \lambda_{y}=\sum_{\substack{y \in \operatorname{Bud}_{\mathscr{C}}(\theta) \\ i \in[|y|] \\ r \in \mathcal{R} \\ x=y \circ_{i} r}} \lambda_{y} . \tag{3.3.11}
\end{equation*}
$$

We observe that Relation (3.3.11) satisfied by the $\lambda_{x}$ is the same as Relation (3.3.10) of Lemma 3.3.3 satisfied by the $\left\langle x, \mathbf{r}^{\curvearrowleft}\right\rangle$. This implies the statement of the proposition.

Theorem 3.3.5. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes Bud $_{\mathscr{C}}(\theta)$. Then, the hook generating series of $\mathscr{B}$ satisfies

Proof. By definition of $\mathrm{L}(\mathscr{B})$ and $\mathrm{G}(\mathscr{3})$, any $x \in \mathrm{~L}(\mathscr{B})$ can be reached from $\mathbb{1}_{\text {out }(x)}$ by a multipath

$$
\begin{equation*}
\mathbb{1}_{\mathrm{out}(x)} \rightarrow y_{1} \rightarrow y_{2} \rightarrow \cdots \rightarrow y_{\ell-1} \rightarrow x \tag{3.3.13}
\end{equation*}
$$

in $\mathrm{G}(\mathscr{3})$, where $y_{1}, \ldots, y_{\ell-1}$ are elements of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ and $\mathbb{1}_{\text {out }(x)} \in I$. Hence, by definition of $\rightarrow$, $x$ admits an $\mathfrak{R}$-left expression

$$
\begin{equation*}
x=\left(\ldots\left(\left(\mathbb{1}_{\text {out }(x)} \circ_{1} r_{1}\right) \circ_{i_{1}} r_{2}\right) \circ_{i_{2}} \ldots\right) \circ_{i_{\ell-1}} r_{\ell} \tag{3.3.14}
\end{equation*}
$$

where for any $j \in[\ell], r_{j} \in \mathfrak{R}$, and for any $j \in[\ell-1]$,

$$
\begin{equation*}
y_{j}=\left(\ldots\left(\left(\mathbb{1}_{\text {out }(x)} \circ_{1} r_{1}\right) \circ_{i_{1}} r_{2}\right) \circ_{i_{2}} \ldots\right) \circ_{i_{j-1}} r_{j} \tag{3.3.15}
\end{equation*}
$$

and $i_{j} \in\left[\left|y_{j}\right|\right]$. This shows that the set of all multipaths from $\mathbb{1}_{\text {out }(x)}$ to $x$ in $G(\mathscr{3})$ is in one-to-one correspondence with the set of all $\mathfrak{R}$-left expressions for $x$. Now, observe that since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point $(i)$ of Proposition 3.3.2, $\mathbf{r}^{\curvearrowleft}$ is a well-defined series. If $x=\mathbb{1}_{a}$ for an $a \in I$, since $\left\langle\mathbb{1}_{a}, \mathbf{r}^{\curvearrowleft}\right\rangle=1$, the statement of the proposition holds. Let us now assume that $x$ is different from a colored unit and let us denote by $\lambda_{x}$ the number of multipaths from $\mathbb{1}_{\text {out }(x)}$ to $x$ in the derivation graph $\mathrm{G}(\mathscr{B})$ of $\mathscr{G}$. By $\mathrm{d}, \mathbf{r}^{\curvearrowleft}{ }^{*}$ is a well-defined series. By Proposition 3.3.4, Lemmas 1.2.1 and 1.2.2, and (1.2.6), we obtain that

$$
\begin{equation*}
\left\langle x, \mathbf{r}^{\curvearrowleft}\right\rangle=\sum_{\substack{t \in \operatorname{Free}(\mathfrak{R}) \\ \operatorname{eV_{Bud}}(\theta)(t)=x}} \frac{\operatorname{deg}(t)!}{\prod_{v \in \mathcal{N}(t)} \operatorname{deg}\left(t_{v}\right)} . \tag{3.3.16}
\end{equation*}
$$

Finally, by Lemma 3.2.3, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ such that out $(x) \in I$ and $\operatorname{in}(x) \in T^{+}$, we have $\langle x, \operatorname{hook}(\mathscr{G})\rangle=\left\langle x, \mathbf{r}^{\curvearrowleft} \curvearrowleft^{*}\right\rangle$. This shows that the right member of (3.3.12) is equal to hook( $(\mathscr{3})$.

An alternative way to understand $\operatorname{hook}(\mathscr{B})$ thus offered by Theorem 3.3.5 consists is seeing the coefficient $\langle x, \operatorname{hook}(\mathscr{B})\rangle, x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, as the number of $\mathfrak{R}$-left expressions of $x$.

The following result establishes a link between the hook generating series of $\mathscr{G}$ and its language.

Proposition 3.3.6. Let $\mathscr{G}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, the support of the hook generating series of $\mathscr{B}$ is the language of $\mathscr{B}$.

Proof. This is an immediate consequence of Theorem 3.3.5 and Lemma 2.2.1.
Bud generating systems lead to the definition of analogues of the hook-length statistic [Knu98] for combinatorial objects possibly different than trees in the following way. Let $\mathcal{O}$ be a monochrome operad, $G$ be a generating set of $\mathcal{O}$, and $\mathrm{HS}_{\Theta, G}:=(\theta, G)$ be a monochrome bud generating system depending on $\Theta$ and $G$, called hook bud generating system. Since $G$ is a generating set of $\mathcal{O}$, by Propositions 2.2.3 and 3.3.6, the support of hook $\left(\mathrm{HS}_{\Theta, G}\right)$ is equal to $\mathrm{L}\left(\mathrm{HS}_{\mathcal{Q}, G}\right)$. We define the hook-length coefficient of any element $x$ of $\mathcal{O}$ as the coefficient $\left\langle x, \operatorname{hook}\left(\operatorname{HS}_{\Theta, G}\right)\right\rangle^{\mathrm{i}}$.
3.3.3. Invertible elements for the composition product. Since by Proposition 3.2.2, $\odot$ is an associative product and $\mathbf{u}$ is its unit, the $\odot$-inverse of a $\mathfrak{G}$-series $\mathbf{f}$ is defined as the unique $\mathfrak{G}$-series $\mathbf{x}$ satisfying

$$
\begin{equation*}
\mathbf{f} \odot \mathbf{x}=\mathbf{u}=\mathbf{x} \odot \mathbf{f} \tag{3.3.17}
\end{equation*}
$$

This series $\mathbf{x}$ could be undefined for an arbitrary $\mathfrak{G}$-series $\mathbf{f}$. The $\odot$-inverse of $\mathbf{f}$ is denoted by $\mathbf{f}^{\odot-1}$ when it is well-defined.

Immediately from this definition and the definition of the composition product $\odot$, the coefficients of $\mathbf{f}^{\odot^{-1}}$ satisfy for any $x \in \mathfrak{G}$,

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\odot-1}\right\rangle=\frac{\delta_{x, \mathbb{1}_{\text {out }}(x)}}{\left\langle\mathbb{1}_{\text {out }}(x), \mathbf{f}\right\rangle}-\frac{1}{\left\langle\mathbb{1}_{\text {out }(x)}, \mathbf{f}\right\rangle} \sum_{\substack{y, z_{1}, \ldots, z_{\mid y,} \in G \\ y \neq 1 \mathbb{1}_{\text {out }}(x) \\ x=y \circ\left[z_{1}, \ldots, z_{\mid y]}\right]}}\langle y, \mathbf{f}\rangle \prod_{i \in[|y|]}\left\langle z_{i}, \mathbf{f}^{\odot-1}\right\rangle . \tag{3.3.18}
\end{equation*}
$$

Proposition 3.3.7. Let $\mathfrak{G}$ be a locally finite colored $\mathscr{C}$-operad and $\mathbf{f}$ be a series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ such that $\operatorname{Supp}(\mathbf{f})=\left\{\mathbb{1}_{a}: a \in \mathscr{C}\right\} \sqcup S$ where $S$ is a $\mathscr{C}$-colored graded subcollection of $\mathfrak{G}$ such that $S(1)$ is finitely factorizes $\mathfrak{G}$. Then,
(i) the series $\mathbf{f}^{\odot-1}$ is well-defined;
(ii) for any $x \in \mathfrak{G}$, the coefficient of $x$ in $\mathbf{f}^{\odot^{-1}}$ satisfies

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\odot-1}\right\rangle=\frac{1}{\left\langle\mathbb{1}_{\text {out }(x)}, \mathbf{f}\right\rangle} \sum_{\substack{\mathfrak{t} \in \operatorname{Free}(S) \\ \operatorname{ev} v_{G}(t)=x}}(-1)^{\operatorname{deg}(t)} \prod_{v \in \mathcal{N}(t)} \frac{\langle\mathfrak{t}(v), \mathbf{f}\rangle}{\prod_{j \in[|v|]}\left\langle\mathbb{1}_{\mathrm{in}_{j}(v)}, \mathbf{f}\right\rangle} . \tag{3.3.19}
\end{equation*}
$$

Proof. Let us first assume that $x$ does not belong to $\mathscr{G}^{S}$, the colored suboperad of $\mathfrak{G}$ generated by $S$. Hence, since there is no $\mathfrak{t} \in \operatorname{Free}(S)$ such that $\mathrm{ev}_{\mathfrak{G}}(\mathfrak{t})=x$, the right member of (3.3.19) is equal to zero. Moreover, since $x$ does not belong to $\mathscr{G}^{S}$, for any $y \in \mathscr{G}$ and $z_{1}, \ldots, z_{|y|} \in \mathfrak{G}$ such that $y \neq \mathbb{1}_{\text {out }(x)}$ and $x=y \circ\left[z_{1}, \ldots, z_{|y|}\right]$, we have necessarily $y \notin S$ or $z_{i} \notin \mathcal{G}^{S}$ for at least

[^5]one $i \in[|y|]$. By (3.3.18), this implies that $\left\langle x, \mathbf{f}^{\odot-1}\right\rangle=0$. This hence shows that (3.3.19) holds when $x \notin \mathcal{G}^{S}$.

Let us assume that $x$ belongs to $\mathscr{G}^{S}$. By Lemma 1.2.1, since $\mathscr{G}$ is locally finite and $S(1)$ finitely factorizes $\mathfrak{G}$, the $S$-degree $\operatorname{deg}_{S}(x)$ of $x$ is well-defined. To prove (3.3.19), we proceed by induction on $\operatorname{deg}_{S}(x)$ and by using (3.3.18). A straightforward computation shows that (3.3.19) holds so that (ii) checks out.

Finally, by Lemma 1.2.1, there is a finite number of $S$-treelike expressions of $x$. This shows that (3.3.19) is well-defined and then $\mathbf{f}^{\odot-1}$ also is. Hence, $(i)$ holds.

Given a G-series $\mathbf{f}$ such that $\mathbf{f}^{\odot-1}$ is well-defined, Equation (3.3.18) (resp. Proposition 3.3.7) provides a recursive (resp. direct) way to compute the coefficients of $\mathbf{f}^{\odot-1}$.

Besides, the set of all the $\mathfrak{G}$-series satisfying the conditions of Proposition 3.3.7 forms a submonoid of $\mathbb{K}\langle\langle C\rangle\rangle$ for the composition product which is also a group. This group is a generalization of the groups constructed from operads of [Cha02, Cha09] (see also [vdL04, Fra08, Cha08, LV12, LN13]).
3.3.4. Syntactic generating series. The syntactic generating series of $\mathscr{G}$ the $\mathbf{B u d}_{\mathscr{C}}(\theta)$-series $\operatorname{synt}(\mathscr{3})$ defined by

$$
\begin{equation*}
\operatorname{synt}(\mathscr{B}):=\mathbf{i} \odot(\mathbf{u}-\mathbf{r})^{\odot-1} \odot \mathbf{t} . \tag{3.3.20}
\end{equation*}
$$

Observe that (3.3.20) could be undefined for an arbitrary set of rules $\mathfrak{R}$ of $\mathscr{B}$. Nevertheless, when $\mathbf{u}$ - $\mathbf{r}$ satisfies the conditions of Proposition 3.3.7, synt $(\mathscr{B})$ is well-defined. Remark that this condition is satisfied whenever $\mathcal{O}$ is locally finite and $\mathfrak{R}(1)$ factorizes finitely $\operatorname{Bud}_{\mathscr{C}}(\theta)$.

The aim of this section is to provide an expression to compute the coefficients of synt $(\mathscr{3})$.

Lemma 3.3.8. Let $\mathscr{G}:=(\theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$,

$$
\begin{equation*}
\left\langle x,(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle=\delta_{x, \mathbb{1}_{\text {out }}(x)}+\sum_{\substack{y \in \mathfrak{R} \\ z_{1}, \ldots, z_{|v|}\left[\operatorname{Bud}_{\mathscr{C}}(\theta) \\ x=y \circ\left[z_{1}, \ldots, z_{|v|}\right]\right.}} \prod_{i \in[|y|]}\left\langle z_{i},(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle . \tag{3.3.21}
\end{equation*}
$$

Proof. Since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point $(i)$ of Proposition 3.3.7, $(\mathbf{u}-\mathbf{r})^{\odot_{-1}}$ is a well-defined series. Now, (3.3.21) is a consequence of Point (ii) of Proposition 3.3.7 and Equation (3.3.18) for the $\odot$-inverse, together with the fact that all coefficients of $\mathbf{r}$ are equal to 0 or to 1.

Theorem 3.3.9. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \Re, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, the syntactic generating series of $\mathscr{G}$ satisfies

$$
\begin{equation*}
\operatorname{synt}(\mathscr{A})=\sum_{\substack{t \in \operatorname{Free}(\mathfrak{R}) \\ \text { out }(t) \in I \\ \text { in }(t) \in T^{+}}} \operatorname{ev}_{\operatorname{Bud}_{\mathscr{C}}(\mathcal{O})}(\mathrm{t}) . \tag{3.3.22}
\end{equation*}
$$

Proof. Let, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, $\lambda_{x}$ be the number of $\mathfrak{R}$-treelike expressions for $x$. Since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Lemma 1.2.1, all $\lambda_{x}$ are well-defined integers. Moreover, since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point $(i)$ of Proposition 3.3.7, (u $\left.-\mathbf{r}\right)^{\odot-1}$ is a well-defined series. Let us show that $\left\langle x,(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle=\lambda_{x}$. First, when $x$ does not belong to $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$, by Point (ii) of Proposition 3.3.7, $\left\langle x,(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle=0$. Since, in this case $\lambda_{x}=0$, the property holds. Let us now assume that $x$ belongs to $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$. Again by Lemma 1.2.1, the $\mathfrak{R}$-degree of $x$ is well-defined. Therefore, we proceed by induction on $\operatorname{deg}_{\mathfrak{R}}(x)$. By Lemma 3.3.8, when $x$ is a colored unit $\mathbb{1}_{a}, a \in \mathscr{C}$, one has $\left\langle x,(\mathbf{u}-\mathbf{r})^{-1}\right\rangle=1$. Since there is exactly one $\mathfrak{R}$-treelike expression for $\mathbb{1}_{a}$, namely the syntax tree consisting in one leaf of output and input color $a$, $\lambda_{\mathbb{1}_{a}}=1$ so that the base case holds. Otherwise, again by Lemma 3.3.8, we have, by using induction hypothesis,

$$
\begin{equation*}
\left\langle x,(\mathbf{u}-\mathbf{r})^{\odot_{-1}}\right\rangle=\sum_{\substack{y \in \mathfrak{R} \\ z_{1}, \ldots, z_{|y|} \in \operatorname{Bud}_{\mathscr{C}}(\theta) \\ x=y \circ\left[z_{1}, \ldots, z_{|y|}\right]}} \prod_{i \in[|y|]} \lambda_{z_{i}}=\lambda_{x} \tag{3.3.23}
\end{equation*}
$$

Notice that one can apply the induction hypothesis to state (3.3.23) since one has $\operatorname{deg}_{\mathfrak{R}}(x) \geqslant$ $1+\operatorname{deg}_{\mathfrak{R}}\left(z_{i}\right)$ for all $i \in[|y|]$.

Now, from (3.3.23) and by using Lemma 3.2.3, we obtain that for all $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ such that out $(x) \in I$ and $\operatorname{in}(x) \in T^{+},\langle x, \operatorname{synt}(\mathscr{3})\rangle=\lambda_{x}$. Denoting by $\mathbf{f}$ the series of the right member of (3.3.22), we have $\langle x, \mathbf{f}\rangle=\lambda_{x}$ if out $(x) \in I$ and $\operatorname{in}(x) \in T^{+}$, and $\langle x, \mathbf{f}\rangle=0$ otherwise. This shows that this expression is equal to $\operatorname{synt}(\mathscr{3})$.

Theorem 3.3.9 explains the name of syntactic generating series for synt( $\mathscr{B}$ ) because this series can be expressed following (3.3.22) as a sum of evaluations of syntax trees. An alternative way to see $\operatorname{synt}(\mathscr{3})$ is that for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, the coefficient $\langle x, \operatorname{synt}(\mathscr{3})\rangle$ is the number of $\Re$-treelike expressions for $x$.

The following result establishes a link between the syntactic generating series of $\mathscr{B}$ and its language.

Proposition 3.3.10. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, the support of the syntactic generating series of $\mathscr{G}$ is the language of $\mathscr{G}$.

Proof. This is an immediate consequence of Theorem 3.3.9 and Lemma 2.2.1.
By Propositions 3.3.6 and 3.3.10, the series hook( $\mathscr{3})$ and synt $(\mathscr{3})$ have the same support. The main difference between these two series is that the coefficient of an $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ in $\operatorname{synt}(\mathscr{B})$ is the number of $\mathfrak{R}$-treelike expressions for $x$, while in hook $(\mathscr{B})$ this coefficient is the number of ways to generate $x$ in $\mathscr{B}$.

We say that $\mathscr{B}$ is unambiguous if all coefficients of synt $(\mathscr{B})$ are equal to 0 or to 1 . This property is important from a combinatorial and enumerative point of view. Indeed, when $\mathscr{G}$ is unambiguous, its syntactic generating series is the characteristic series of its language. As a consequence, by definition of the series of colors $\mathfrak{c o l}$ (see Section 3.1.4) and Proposition 3.3.10, the coefficient of $(a, u) \in \operatorname{Bud}_{\mathscr{C}}(\mathbf{A s})$ in the series $\operatorname{col}(\operatorname{synt}(\mathscr{B}))$ is the number of elements $x$ of $L(\mathscr{3})$ such that $(\operatorname{out}(x), \operatorname{in}(x))=(a, u)$.

As a side remark, observe that Theorem 3.3.9 implies in particular that for any bud generating system of the form $\mathscr{B}:=(\theta, \mathscr{C}, \mathfrak{R}, \mathscr{C}, \mathscr{C})$, if $\operatorname{synt}(\mathscr{B})$ is unambiguous, then the colored suboperad of $\operatorname{Bud}_{\mathscr{C}}(\theta)$ generated by $\mathfrak{R}$ is free. The converse property does not hold.

Let us now describe the coefficients of $\mathfrak{c o l t}(\operatorname{synt}(\mathscr{B}))$, the series of color types of the syntactic series of $\mathscr{B}$, in the particular case when $\mathscr{B}$ is unambiguous. We shall give two descriptions: a first one involving a system of equations of series of $\mathbb{K}\left[\left[\mathbb{Y}_{\mathscr{C}}\right]\right]$, and a second one involving a recurrence relation on the coefficients of a series of $\mathbb{K}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right]$.

Lemma 3.3.11. Let $\mathscr{B}:=(\theta, \mathscr{C}, \mathfrak{R}, I, T)$ be an unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes Bud $\mathscr{C}^{( }(\theta)$. Then, for all colors $a \in I$ and all types $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{synt}(\mathscr{B}))\right\rangle$ count the number of elements $x$ of $\mathrm{L}(\mathscr{G})$ such that (out $(x)$, type $(\operatorname{in}(x)))=(a, \alpha)$.

Proof. By Proposition 3.3.10 and since $\mathscr{B}$ is unambiguous, synt $(\mathscr{3})$ is the characteristic series of $\mathrm{L}(\mathscr{3})$. The statement of the lemma follows immediately from the definition (3.1.10) of $\mathfrak{c o l t}$.

Proposition 3.3.12. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \Re, I, T)$ be an unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes Bud $_{\mathscr{C}}(\Theta)$. For all $a \in \mathscr{C}$, let $\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ be the series of $\mathbb{K}\left[\left[\mathbb{Y}_{\mathscr{C}}\right]\right]$ satisfying

$$
\begin{equation*}
\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)=y_{a}+\mathbf{g}_{a}\left(\mathbf{f}_{c_{1}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right), \ldots, \mathbf{f}_{c_{k}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)\right) . \tag{3.3.24}
\end{equation*}
$$

Then, for any color $a \in I$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{synt}(\mathscr{3}))\right\rangle$ and $\left\langle\mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}_{a}\right\rangle$ are equal.

Proof. Let us set $\mathbf{h}:=(\mathbf{u}-\mathbf{r})^{\odot_{-1}}$ and, for all $a \in \mathscr{C}, \mathbf{h}_{a}:=\mathbb{1}_{a} \odot \mathbf{h}$. Since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Point ( $i$ ) of Proposition 3.3.7, $\mathbf{h}$ and $\mathbf{h}_{a}$ are well-defined series. Equation (3.3.17) implies that any $\mathbf{h}_{a}, a \in \mathscr{C}$, satisfies the relation

$$
\begin{equation*}
\mathbf{h}_{a}=\mathbb{1}_{a}+\mathbf{r}_{a} \odot \mathbf{h} \tag{3.3.25}
\end{equation*}
$$

where $\mathbf{r}_{a}:=\mathbb{1}_{a} \odot \mathbf{r}$. Observe that for any $a \in \mathscr{C}, \mathfrak{c o l t}\left(\mathbf{r}_{a}\right)=\mathbf{g}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$. Moreover, from the definitions of $\mathfrak{c o l t}$ and the operation $\odot$, we obtain that $\mathfrak{c o l t}\left(\mathbf{r}_{a} \odot \mathbf{h}\right)$ can be computed by a functional composition of the series $\mathbf{g}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ with $f_{c_{1}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right), \ldots, \mathbf{f}_{c_{k}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$. Hence, Relation (3.3.25) leads to

$$
\begin{align*}
\mathfrak{c o l t}\left(\mathbf{h}_{a}\right) & =\mathfrak{c o l t}\left(\mathbb{1}_{a}\right)+\mathfrak{c o l t}\left(\mathbf{r}_{a} \odot \mathbf{h}\right) \\
& =y_{a}+\mathbf{g}_{a}\left(\mathbf{f}_{c_{1}}\left(y_{c_{1}}, \ldots, \mathrm{y}_{c_{k}}\right), \ldots, \mathbf{f}_{c_{k}}\left(y_{c_{1}}, \ldots, \mathrm{y}_{c_{k}}\right)\right)  \tag{3.3.26}\\
& =\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right) .
\end{align*}
$$

Finally, Lemma 3.2.3 implies that, when $a \in I$ and $\mathscr{C}^{\alpha} \in T^{+},\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{synt}(\mathscr{3}))\right\rangle$ and $\left\langle\mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}_{a}\right\rangle$ are equal.

When $\mathscr{B}$ is a bud generating system satisfying the conditions of Proposition 3.3.12, the generating series of the language of $\mathscr{G}$ satisfies

$$
\begin{equation*}
\mathbf{s}_{\mathrm{L}(\mathscr{B})}=\sum_{a \in I} \mathbf{f}_{a}^{T} \tag{3.3.27}
\end{equation*}
$$

where $\mathbf{f}_{a}^{T}$ is the specialization of the series $\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ at $y_{b}:=t$ for all $b \in T$ and at $y_{c}:=0$ for all $c \in \mathscr{C} \backslash T$. Therefore, the resolution of the system of equations given by Proposition 3.3.12 provides a way to compute the coefficients of $\mathbf{s}_{\mathrm{L}(\mathscr{G})}$.

Theorem 3.3.13. Let $\mathscr{B}:=(\theta, \mathscr{C}, \mathfrak{R}, I, T)$ be an unambiguous bud generating system such that $\Theta$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\mathbf{B u d}_{\mathscr{C}}(\theta)$. Then, the generating series $\mathbf{s}_{\mathrm{L}(\mathscr{G})}$ of the language of $\mathscr{G}$ is algebraic.

Proof. Proposition 3.3.12 shows that each series $\mathbf{f}_{a}$ satisfies an algebraic equation involving variables of $\mathbb{Y}_{\mathscr{C}}$ and series $\mathbf{f}_{b}, b \in \mathscr{C}$. Hence, $\mathbf{f}_{a}$ is algebraic. Moreover, the fact that, by (3.3.27), $\mathbf{s}_{\mathrm{L}(\mathscr{B})}$ is a specialized sum of some $\mathbf{f}_{a}$ implies the statement of the theorem.

Theorem 3.3.14. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be an unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\mathbf{B u d}_{\mathscr{C}}(\theta)$ Let $\mathbf{f}$ be the series of $\mathbb{K}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right]$ satisfying, for any $a \in \mathscr{C}$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$,

$$
\begin{equation*}
\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle=\delta_{\alpha, \text { type }(a)}+\sum_{\substack{\phi: \mathscr{C} \times \mathcal{F}_{\mathscr{C}} \rightarrow \mathbb{N} \\ \alpha=\phi^{\left(c_{1}\right)} \ldots \phi^{\left(c_{k}\right)}}} \chi_{a, \sum \phi_{c_{1}} \cdots \sum \phi_{c_{k}}}\left(\prod_{b \in \mathscr{C}} \phi_{b}!\right)\left(\prod_{\substack{b \in \mathscr{C} \\ \gamma \in \mathscr{T}_{\mathscr{C}}}}\left\langle\mathrm{x}_{b} \mathbb{Y}_{\mathscr{C}}^{\gamma}, \mathbf{f}\right\rangle^{\phi(b, \gamma)}\right) \tag{3.3.28}
\end{equation*}
$$

Then, for any color $a \in I$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{synt}(\mathscr{H}))\right\rangle$ and $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle$ are equal.

Proof. First, since $\mathfrak{R}(1)$ finitely factorizes Bud $_{\mathscr{C}}(\Theta)$, by Point $(i)$ of Proposition 3.3.7, $(\mathbf{u}-\mathbf{r})^{\odot-1}$ is a well-defined series. Moreover, by (3.3.17), $(\mathbf{u}-\mathbf{r})^{\odot_{-1}}$ satisfies the identity of series

$$
\begin{equation*}
(\mathbf{u}-\mathbf{r}) \odot(\mathbf{u}-\mathbf{r})^{\odot-1}=\mathbf{u} \tag{3.3.29}
\end{equation*}
$$

Since the map col commutes with the addition of series, with the composition product $\odot$, and with the inverse with respect to $\odot,(3.3 .29)$ leads to the equation

$$
\begin{equation*}
\mathfrak{c o l}(\mathbf{u}-\mathbf{r}) \odot \operatorname{col}(\mathbf{u}-\mathbf{r})^{\odot_{-1}}=\operatorname{col}(\mathbf{u}) . \tag{3.3.30}
\end{equation*}
$$

By Point (ii) of Proposition 3.3.7, by (3.3.18), and by definition of the composition map of $\operatorname{Bud}_{\mathscr{C}}(\mathbf{A s})$, the coefficients of $\operatorname{col}^{( }(\mathbf{u}-\mathbf{r})^{\odot-1}$ satisfy, for all $(a, u) \in \operatorname{Bud}_{\mathscr{C}}(\mathbf{A s})$, the recurrence relation

$$
\begin{equation*}
\left\langle(a, u), \operatorname{col}(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle=\delta_{u, a}+\sum_{\substack{w \in \mathscr{C}^{+} \\ w \neq a}} \lambda_{a, w} \sum_{\substack{v^{(1)}, \ldots, v^{(|w|)} \in \mathscr{C}^{+} \\ u=v^{(1)} \ldots . v^{(|w|)}}} \prod_{i \in[|w|]}\left\langle\left(w_{i}, v^{(i)}\right), \operatorname{col}(\mathbf{u}-\mathbf{r})^{\odot-1}\right\rangle, \tag{3.3.31}
\end{equation*}
$$

where $\lambda_{a, w}$ denotes the number of rules $r \in \mathfrak{R}$ such that out $(r)=a$ and $\operatorname{in}(r)=w$. By definition of $\mathfrak{c o l t}$ and by (3.3.31), a straightforward computation shows that the coefficients of $\mathfrak{c o l t}\left((\mathbf{u}-\mathbf{r})^{\mathcal{Q}_{-1}}\right)$ express for any $\alpha \in \mathscr{T}_{\mathscr{C}}$, as

$$
\begin{align*}
& \left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}\left((\mathbf{u}-\mathbf{r})^{\odot_{-1}}\right)\right\rangle \\
& \left.=\delta_{\alpha, \text { type }(a)}+\sum_{\substack{\gamma \in \mathcal{T}_{\mathscr{C}} \\
\gamma \neq \operatorname{type}(a)}} \chi_{a, \gamma} \sum_{\begin{array}{c}
\beta^{(1)}, \ldots, \beta^{\left.(\operatorname{deg}(\gamma)) \in \mathcal{F}_{\mathscr{C}}\right)} \\
\alpha=\beta^{(1)}+\ldots+\beta^{(\operatorname{deg}(\gamma))}
\end{array}}\left\langle\mathrm{x}_{\mathscr{C}_{i}^{\gamma}} \mathbb{Y}_{\mathscr{C}}^{\beta^{(i)}}, \operatorname{colt}((\mathbf{d e g}(\gamma)]-\mathbf{r})^{\odot-1}\right)\right\rangle . \tag{3.3.32}
\end{align*}
$$

Therefore, (3.3.32) provides a recurrence relation for the coefficients of $\mathfrak{c o l t}\left((\mathbf{u}-\mathbf{r})^{\odot-1}\right)$. By using the notations introduced in Section 3.1.6 about mappings $\phi: \mathscr{C} \times \mathscr{T}_{\mathscr{C}} \rightarrow \mathbb{N}$, we obtain
that the coefficients of $\mathfrak{c o l t}\left((\mathbf{u}-\mathbf{r})^{\odot_{-1}}\right)$ satisfy the same recurrence relation (3.3.28) as the ones of $\mathbf{f}$. Finally, Lemma 3.2.3 implies that, when $a \in I$ and $\mathscr{C}^{\alpha} \in T^{+},\left\langle\mathbf{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{synt}(\mathscr{B}))\right\rangle$ and $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle$ are equal.

When $\mathscr{B}$ is a bud generating system satisfying the conditions of Theorem 3.3.14 (which are the same as the ones required by Proposition 3.3.12), one has for any $n \geqslant 1$,

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{L}(\mathscr{B})}\right\rangle=\sum_{a \in I} \sum_{\substack{\alpha \in \mathcal{T}_{\mathscr{C}} \\ \alpha_{i}=0, c_{i} \in \mathscr{C} \backslash T}}\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle . \tag{3.3.33}
\end{equation*}
$$

Therefore, this provides an alternative and recursive way to compute the coefficients of $\mathbf{s}_{\mathrm{L}(\mathscr{G})}$, different from the one of Proposition 3.3.12 .
3.4. Series and synchronous languages. We introduce the Kleene star operation of the composition product $\odot$ in order to define the synchronous generating series of a bud generating system $\mathscr{B}$. We relate this series with the synchronous language of $\mathscr{B}$ and provide ways to compute its coefficients.

Proofs of some results of this section are very similar to ones of Section 3.3. For this reason, some proofs are sketched here.
3.4.1. Composition star product. For any $\mathfrak{G}$-series $\mathbf{f} \in \mathbb{K}\langle\langle\mathcal{G}\rangle\rangle$ and any $\ell \geqslant 0$, let $\mathbf{f}^{\odot}$ be the series defined by

$$
\begin{equation*}
\mathbf{f}^{\odot \ell}:=\prod_{1 \leqslant i \leqslant \ell} \mathbf{f} \tag{3.4.1}
\end{equation*}
$$

where the product of (3.4.1) denotes the iterated version of the composition product $\odot$. Observe that since $\odot$ is associative (see Proposition 3.2.2), this definition is consistent. Immediately from this definition and the definition of the composition product $\odot$, the coefficient of $\mathbf{f}^{\odot}, \ell \geqslant 0$, satisfies for any $x \in \mathfrak{G}$,

$$
\left\langle x, \mathbf{f}^{\odot}\right\rangle= \begin{cases}\delta_{x, \mathbb{1}_{\text {out }}(x)} & \text { if } \ell=0,  \tag{3.4.2}\\ \sum_{\substack{y, z_{1}, \ldots, z_{|y|} \in \in \\ x=y \circ\left[z_{1}, \ldots, z_{|y|}\right]}}\left\langle y, \mathbf{f}^{\odot \prec-1}\right\rangle \prod_{i \in[|y|]}\left\langle z_{i}, \mathbf{f}\right\rangle & \text { otherwise. }\end{cases}
$$

Lemma 3.4.1. Let $\mathfrak{G}$ be a locally finite $\mathscr{C}$-colored operad and $\mathbf{f}$ be a series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$. Then, the coefficients of $\mathbf{f}^{\odot}{ }_{\ell+1}, \ell \geqslant 0$, satisfy for any $x \in \mathfrak{G}$,

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\odot+1}\right\rangle=\sum_{\substack{t \in \mathbf{F r e e}_{\text {perf }}(G) \\ \text { ht }(t)=\ell+1 \\ \operatorname{ev}(t)=x}} \prod_{v \in \mathcal{N}(t)}\langle\mathfrak{t}(v), \mathbf{f}\rangle . \tag{3.4.3}
\end{equation*}
$$

Proof. By Proposition 3.2.2, since $\mathfrak{G}$ is locally finite, $\mathbf{f}^{\odot \ell+1}$ is a well-defined $\mathfrak{G}$-series. The statement of the lemma follows by induction on $\ell$ and by using (3.4.2).

[^6]The $\odot$-star of $\mathbf{f}$ is the series

$$
\begin{equation*}
\mathbf{f}^{\odot_{*}}:=\sum_{\ell \geqslant 0} \mathbf{f}^{\odot_{\ell}}=\mathbf{u}+\mathbf{f}+\mathbf{f} \odot \mathbf{f}+\mathbf{f} \odot \mathbf{f} \odot \mathbf{f}+\mathbf{f} \odot \mathbf{f} \odot \mathbf{f} \odot \mathbf{f}+\cdots \tag{3.4.4}
\end{equation*}
$$

Observe that $\mathbf{f}^{\odot_{*}}$ could be undefined for an arbitrary $\mathfrak{G}$-series $\mathbf{f}$.
Proposition 3.4.2. Let $\mathfrak{G}$ be a locally finite $\mathscr{C}$-colored operad and $\mathbf{f}$ be a series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ such that $\operatorname{Supp}(\mathbf{f})(1)$ finitely factorizes $\mathfrak{G}$. Then,
(i) the series $\mathbf{f}^{\odot^{*}}$ is well-defined;
(ii) for any $x \in \mathcal{G}$, the coefficient of $x$ in $\mathbf{f}^{\odot^{*}}$ satisfies

$$
\begin{equation*}
\left\langle x, \mathbf{f}^{\odot_{*}}\right\rangle=\delta_{x, \mathbb{1}_{\mathrm{out}(x)}}+\sum_{\substack{y, z_{1}, \ldots, z_{|| |} \in \mathcal{C} \\ x=y \circ\left[z_{1}, \ldots, z_{|v|}\right]}}\left\langle y, \mathbf{f}^{\odot_{*}}\right\rangle \prod_{i \in[|y|]}\left\langle z_{i}, \mathbf{f}\right\rangle ; \tag{3.4.5}
\end{equation*}
$$

(iii) the equation

$$
\begin{equation*}
\mathbf{x}-\mathbf{x} \odot \mathbf{f}=\mathbf{u} \tag{3.4.6}
\end{equation*}
$$

admits $\mathbf{x}=\mathbf{f}^{\ominus_{*}}$ as unique solution.
Proof. The proof is similar to the one of Proposition 3.3.2 and uses (3.4.2) and Lemmas 1.2.1 and 3.4.1.

In particular, Point (ii) of Proposition 3.4.2 gives a way, given a G-series f satisfying the stated constraints, to compute recursively the coefficients of its $\odot$-star $\mathbf{f}^{\odot_{*}}$.
3.4.2. Synchronous generating series. The synchronous generating series of $\mathscr{B}^{\text {the }} \mathbf{B u d}_{\mathscr{C}}(0)$ series sync(63) defined by

$$
\begin{equation*}
\operatorname{sync}(\mathscr{B}):=\mathbf{i} \odot \mathbf{r}^{\odot_{*}} \odot \mathbf{t} . \tag{3.4.7}
\end{equation*}
$$

Observe that (3.4.7) could be undefined for an arbitrary set of rules $\mathfrak{R}$ of $\mathscr{B}$. Nevertheless, when r satisfies the conditions of Proposition 3.4.2, that is, when $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta), \operatorname{sync}(\mathscr{3})$ is well-defined.

The aim of the following is to provide an expression to compute the coefficients of sync $(\mathscr{A})$.

Lemma 3.4.3. Let $\mathscr{G}:=(O, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$,

Proof. The proof is similar to the one of Lemma 3.3.3 and uses Proposition 3.4.2.
Theorem 3.4.4. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\Theta$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$. Then, the synchronous generating series of $\mathscr{G}$ satisfies

$$
\begin{equation*}
\operatorname{sync}(\mathscr{B})=\sum_{\substack{\mathfrak{t} \in \operatorname{Free}_{\text {perf }}(\mathfrak{R}) \\ \text { out }(t) \in I \\ \operatorname{in}(t) \in T^{+}}} \operatorname{ev}_{\operatorname{Bud}_{\mathscr{C}}(\theta)}(\mathfrak{t}) . \tag{3.4.9}
\end{equation*}
$$

Proof. Let, for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, $\lambda_{x}$ be the number of perfect $\mathfrak{R}$-treelike expressions for $x$. Since $\mathfrak{R}(1)$ finitely factorizes $\operatorname{Bud}_{\mathscr{C}}(\theta)$, by Lemma 1.2.1, all $\lambda_{x}$ are well-defined integers. Moreover, since $\mathfrak{R}(1)$ finitely factorizes Bud $_{\mathscr{C}}(9)$, by Point $(i)$ of Proposition 3.4.2, $\mathbf{r}^{\odot^{*}}$ is a welldefined series. Let us show that $\left\langle x, \mathbf{r}^{\odot_{*}}\right\rangle=\lambda_{x}$. First, when $x$ does not belong to $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$, by Lemma 3.4.3, $\left\langle x, \mathbf{r}^{\odot_{*}}\right\rangle=0$. Since, in this case $\lambda_{x}=0$, the property holds. Let us now assume that $x$ belongs to $\operatorname{Bud}_{\mathscr{C}}(\theta)^{\Re}$. Again by Lemma 1.2.1, the $\Re$-degree of $x$ is well-defined. Therefore, we proceed by induction on $\operatorname{deg}_{\mathfrak{R}}(x)$. By Lemma 3.4.3, when $x$ is a colored unit $\mathbb{1}_{a}$, $a \in \mathscr{C}$, one has $\left\langle x, \mathbf{r}^{\odot_{*}}\right\rangle=1$. Since there is exactly one treelike expression which is a perfect tree for $\mathbb{1}_{a}$, namely the syntax tree consisting in one leaf of output and input color $a, \lambda_{1_{a}}=1$ so that the base case holds. Otherwise, again by Lemma 3.4.3, we have, by using induction hypothesis,

Notice that one can apply the induction hypothesis to state (3.4.10) since one has $\operatorname{deg}_{\mathfrak{R}}(x) \geqslant$ $1+\operatorname{deg}_{\mathfrak{R}}(y)$.

Now, from (3.4.10) and by using Lemma 3.2.3, we obtain that for all $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$ such that $\operatorname{out}(x) \in I$ and $\operatorname{in}(x) \in T^{+},\langle x, \operatorname{sync}(\mathscr{G})\rangle=\lambda_{x}$. By denoting by $\mathbf{f}$ the series of the right member of (3.4.9), we have $\langle x, \mathbf{f}\rangle=\lambda_{x}$ if out $(x) \in I$ and $\operatorname{in}(x) \in T^{*}$, and $\langle x, \mathbf{f}\rangle=0$ otherwise. This shows that this expression is equal to $\operatorname{sync}(\mathscr{3})$.

Theorem 3.4.4 implies that for any $x \in \operatorname{Bud}_{\mathscr{C}}(\theta)$, the coefficient of $\langle x, \operatorname{sync}(\mathscr{3})\rangle$ is the number of $\Re$-treelike expressions for $x$ which are perfect trees.

The following result establishes a link between the synchronous generating series of $\mathscr{G}$ and its synchronous language.

Proposition 3.4.5. Let $\mathfrak{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes Bud $_{\mathscr{C}}(\theta)$. Then, the support of the synchronous generating series of $\mathscr{B}$ is the synchronous language of $\mathscr{G}$.

Proof. This is an immediate consequence of Theorem 3.4.4 and Lemma 2.2.2.
We say that $\mathscr{G}$ is synchronously unambiguous if all coefficients of sync( $\mathscr{B}$ ) are equal to 0 or to 1. This property is important to describe the coefficients of $\operatorname{col}(\operatorname{sync}(\mathscr{3}))$ for the same reasons as the ones concerning the unambiguity property exposed in Section 3.3.4.

Let us now describe the coefficients of $\mathfrak{c o l t}(\operatorname{sync}(\mathscr{3}))$, the series of colors types of the synchronous series of $\mathscr{B}$, in the particular case when $\mathscr{B}$ is unambiguous. We shall give two descriptions: a first one involving a system of functional equations of series of $\mathbb{K}\left[\left[\mathbb{Y}_{\mathscr{C}}\right]\right]$, and a second one involving a recurrence relation on the coefficients of a series of $\mathbb{K}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right]$.

Lemma 3.4.6. Let $\mathscr{B}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a synchronously unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\mathbf{B u d}_{\mathscr{C}}(\Theta)$. Then, for all colors $a \in I$ and all types $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \operatorname{colt}(\operatorname{sync}(\mathscr{B}))\right\rangle$ count the number of elements $x$ of $\mathrm{L}_{S}(\mathscr{3})$ such that $(\operatorname{out}(x), \operatorname{type}(\operatorname{in}(x)))=(a, \alpha)$.

Proof. The proof is similar to the one of Lemma 3.3.11 and uses (3.1.10) and Proposition 3.4.5.

Proposition 3.4.7. Let $\mathscr{G}:=(\Theta, \mathscr{C}, \mathfrak{R}, I, T)$ be a synchronously unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes $\mathrm{Bud}_{\mathscr{C}}(\theta)$. For all $a \in \mathscr{C}$, let $\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ be the series of $\mathbb{K}\left[\left[\mathbb{Y}_{\mathscr{C}}\right]\right]$ satisfying

$$
\begin{equation*}
\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)=y_{a}+\mathbf{f}_{a}\left(\mathbf{g}_{c_{1}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right), \ldots, \mathbf{g}_{c_{k}}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)\right) . \tag{3.4.11}
\end{equation*}
$$

Then, for any color $a \in I$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{sync}(\mathscr{B}))\right\rangle$ and $\left\langle\mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}_{a}\right\rangle$ are equal.

Proof. The proof is similar to the one of Proposition 3.3.12 and uses Lemma 3.2.3 and Proposition 3.4.2.

When $\mathscr{B}$ is a bud generating system satisfying the conditions of Proposition 3.4.7, the generating series of the synchronous language of $\mathscr{B}$ satisfies

$$
\begin{equation*}
\mathbf{s}_{\mathrm{Ls}(\mathscr{B})}=\sum_{a \in I} \mathbf{f}_{a}^{T} \tag{3.4.12}
\end{equation*}
$$

where $\mathbf{f}_{a}^{T}$ is the specialization of the series $\mathbf{f}_{a}\left(y_{c_{1}}, \ldots, y_{c_{k}}\right)$ at $y_{b}:=t$ for all $b \in T$ and at $y_{c}:=0$ for all $c \in \mathscr{C} \backslash T$. Therefore, the resolution of the system of equations given by Proposition 3.4.7 provides a way to compute the coefficients of $\mathbf{s}_{\mathrm{L}_{\mathrm{s}}(\mathscr{B})}$. This resolution can be made in most cases by iteration [BLL97,FS09] . Moreover, when $\mathcal{G}$ is a synchronous grammar [Gir12] (see also Section 2.3.3 for a description of these grammars) and when $\mathrm{SG}(\mathscr{G})=\mathscr{B}$, the system of functional equations provided by Proposition 3.4.7 and (3.4.12) for $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}(\mathscr{B})}$ is the same as the one which can be extracted from $\mathcal{G}$.

Theorem 3.4.8. Let $\mathfrak{G}:=(O, \mathscr{C}, \mathfrak{R}, I, T)$ be a synchronously unambiguous bud generating system such that $\mathcal{O}$ is a locally finite operad and $\mathfrak{R}(1)$ finitely factorizes Bud $\mathscr{C}(0)$. Let $\mathbf{f}$ be the series of $\mathbb{K}\left[\left[\mathbb{X}_{\mathscr{C}} \sqcup \mathbb{Y}_{\mathscr{C}}\right]\right]$ satisfying, for any $a \in \mathscr{C}$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$,

$$
\begin{equation*}
\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle=\delta_{\alpha, \text { type }(a)}+\sum_{\substack{\phi: \mathscr{C} \times \mathscr{T}_{\mathscr{C}} \rightarrow \mathbb{N} \\ \alpha=\phi^{\left(\mathcal{C}_{1}\right)} \ldots \phi^{\left(c_{k}\right)}}}\left(\prod_{b \in \mathscr{C}} \phi_{b}!\right)\left(\prod_{\substack{b \in \mathscr{C} \\ \gamma \in \mathscr{T}}} \chi_{b, \gamma}^{\phi(b, \gamma)}\right)\left\langle\mathrm{x}_{a} \prod_{b \in \mathscr{C}} y_{b}^{\sum \phi_{b}}, \mathbf{f}\right\rangle . \tag{3.4.13}
\end{equation*}
$$

Then, for any color $a \in I$ and any type $\alpha \in \mathscr{T}_{\mathscr{C}}$ such that $\mathscr{C}^{\alpha} \in T^{+}$, the coefficients $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathfrak{c o l t}(\operatorname{sync}(\mathscr{G}))\right\rangle$ and $\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle$ are equal.

Proof. The proof is similar to the one of Theorem 3.3.14 and uses Lemma 3.2.3 and Proposition 3.4.2.

When $\mathscr{G}$ is a bud generating system satisfying the conditions of Theorem 3.4.8 (which are the same as the ones required by Proposition 3.4.7), one has for any $n \geqslant 1$,

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{L} S}(\mathscr{A})\right\rangle=\sum_{a \in I} \sum_{\substack{\alpha \in \mathcal{T}_{\mathscr{C}} \\ \alpha_{i}=0, c_{i} \in \mathscr{C} \backslash T}}\left\langle\mathrm{x}_{a} \mathbb{Y}_{\mathscr{C}}^{\alpha}, \mathbf{f}\right\rangle \tag{3.4.14}
\end{equation*}
$$

[^7]Therefore, this provides an alternative and recursive way to compute the coefficients of $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}(\mathscr{B})}$, different from the one of Proposition 3.4.7 ${ }^{1}$.

## 4. Examples

This final section is devoted to illustrate the notions and the results contained in the previous ones. We first define here some monochrome operads, then give examples of series on colored operads, and construct some bud generating systems. We end this section by explaining how bud generating systems can be used as tools for enumeration. For this purpose, we use the syntactic and synchronous generating series of several bud generating systems to compute the generating series of various combinatorial objects.
4.1. Monochrome operads and bud operads. Let us start by defining three monochrome operads involving some classical combinatorial objects: binary trees, some words of integers, and Motzkin paths.
4.1.1. The magmatic operad. A binary tree is a planar rooted tree $\mathfrak{t}$ such that any internal node of $\mathfrak{t}$ has two children. The magmatic operad Mag is the monochrome operad wherein $\operatorname{Mag}(n)$ is the set of all binary trees with $n$ leaves. The partial composition $\mathfrak{s} \circ_{i} t$ of two binary trees $\mathfrak{s}$ and $\mathfrak{t}$ is the binary tree obtained by grafting the root of $\mathfrak{t}$ on the $i$-th leaf of $\mathfrak{s}$. The only tree of Mag consisting in exactly one leaf is denoted by $\dot{b}$ and is the unit of Mag. Notice that Mag is isomorphic to the operad Free $(C)$ where $C$ is the monochrome graded collection defined by $C:=C(2):=\{\mathrm{a}\}$.

For any set $\mathscr{C}$ of colors, the bud operad $\operatorname{Bud}_{\mathscr{C}}(\mathbf{M a g})$ is the $\mathscr{C}$-graded colored collection of all binary trees $\mathfrak{t}$ where all leaves (inputs) and the root (output) of $\mathfrak{t}$ are labeled on $\mathscr{C}$. For instance, in Bud ${ }_{\{1,2,3\}}(\mathbf{M a g})$, one has

4.1.2. The pluriassociative operad. Let $\gamma$ be a nonnegative integer. The $\gamma$-pluriassociative operad $\operatorname{Dias}_{\gamma}$ [Gir16b] is the monochrome operad wherein $\operatorname{Dias}_{\gamma}(n)$ is the set of all words of length $n$ on the alphabet $\{0\} \cup[\gamma]$ with exactly one occurrence of 0 . The partial composition $u \circ_{i} v$ of two such words $u$ and $v$ consists in replacing the $i$-th letter of $u$ by $v^{\prime}$, where $v^{\prime}$ is the word obtained from $v$ by replacing all its letters $a$ by the greatest integer in $\left\{a, u_{i}\right\}$. For instance, in Dias $_{4}$, one has

$$
\begin{equation*}
313321 \circ_{4} 4112=313433321 \tag{4.1.2}
\end{equation*}
$$

Observe that Dias $_{0}$ is the operad As and that Dias $_{1}$ is the diassociative operad introduced by Loday [Lod01].

[^8]For any set $\mathscr{C}$ of colors, the bud operad $\operatorname{Bud}_{\mathscr{C}}\left(\operatorname{Dias}_{\gamma}\right)$ is the $\mathscr{C}$-colored graded collection of all words $u$ of $\operatorname{Dias}_{\gamma}$ where all letters of $u$ (inputs) and the whole word $u$ (output) are labeled on $\mathscr{C}$.
4.1.3. The operad of Motzkin paths. The operad of Motzkin paths Motz [Gir15] is a monochrome operad where $\operatorname{Motz}(n)$ is the set of all Motzkin paths consisting in $n-1$ steps. A Motzkin path of arity $n$ is a path in $\mathbb{N}^{2}$ connecting the points $(0,0)$ and ( $n-1,0$ ), and made of steps $(1,0),(1,1)$, and $(1,-1)$. If $\mathfrak{a}$ is a Motzkin path, the $i$-th point of $\mathfrak{a}$ is the point of $\mathfrak{a}$ of abscissa $\boldsymbol{i}-1$. The partial composition $\mathfrak{a} \circ_{i} \mathfrak{b}$ of two Motzkin paths $\mathfrak{a}$ and $\mathfrak{b}$ consists in replacing the $\boldsymbol{i}$-th point of $\mathfrak{a}$ by $\mathfrak{b}$. For instance, in Motz, one has


For any set $\mathscr{C}$ of colors, the bud operad $\operatorname{Bud}_{\mathscr{C}}(\operatorname{Motz})$ is the $\mathscr{C}$-colored graded collection of all Motzkin paths $\mathfrak{a}$ where all points of $\mathfrak{a}$ (inputs) and the whole path $\mathfrak{a}$ (output) are labeled on $\mathscr{C}$.
4.2. Series on colored operads. Here, some examples of series on colored operads are constructed, as well as examples of series of colors, series of color types, and pruned series.
4.2.1. Series of trees. Let $\mathscr{G}$ be the free $\mathscr{C}$-colored operad over $C$ where $\mathscr{C}:=\{1,2\}$ and $C$ is the $\mathscr{C}$-graded collection defined by $C:=C(2) \sqcup C(3)$ with $C(2):=\{\mathrm{a}\}, C(3):=\{\mathrm{b}\}$, out $(\mathrm{a}):=1$, $\operatorname{out}(\mathrm{b}):=2, \operatorname{in}(\mathrm{a}):=21$, and $\operatorname{in}(\mathrm{b}):=121$. Let $\mathbf{f}_{\mathrm{a}}\left(\right.$ resp. $\left.\mathbf{f}_{\mathrm{b}}\right)$ be the series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ where for any syntax tree $\mathfrak{t}$ of $\mathfrak{G},\left\langle\mathfrak{t}, \mathbf{f}_{\mathrm{a}}\right\rangle$ (resp. $\left.\left\langle\mathfrak{t}, \mathbf{f}_{\mathrm{b}}\right\rangle\right)$ is the number of internal nodes of $\mathfrak{t}$ labeled by a (resp. b). The series $f_{a}$ and $f_{b}$ are of the form


The sum $f_{a}+f_{b}$ is the series wherein the coefficient of any syntax tree $\mathfrak{t}$ of $\mathscr{G}$ is its degree. Let also $\mathbf{f}_{\left.\right|_{1}}$ (resp. $\mathbf{f}_{l_{2}}$ ) be the series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ where for any syntax tree $\mathfrak{t}$ of $\mathfrak{G},\left\langle\mathfrak{t}, \mathbf{f}_{\left.\right|_{1}}\right\rangle$ (resp. $\left.\left\langle t, f_{l_{2}}\right\rangle\right)$ is the number of inputs colors 1 (resp. 2) of $\mathfrak{t}$. The sum $\mathbf{f}_{l_{1}}+\mathbf{f}_{l_{2}}$ is the series wherein the coefficient of any syntax tree $\mathfrak{t}$ of $\mathcal{G}$ is its arity. Moreover, the series $\mathbf{f}_{a}+\mathbf{f}_{\mathrm{b}}+\mathbf{f}_{11}+\mathbf{f}_{l_{2}}$ is the series wherein the coefficient of any syntax tree $\mathfrak{t}$ of $\mathcal{G}$ is its total number of nodes.

The series of colors of $f_{a}$ is of the form

$$
\begin{equation*}
\mathfrak{c o l}\left(\mathbf{f}_{\mathrm{a}}\right)=(1,21)+2(1,221)+3(1,2221)+(2,2121)+(2,1221)+(1,1211)+\cdots \tag{4.2.2}
\end{equation*}
$$

and the series of color types of $\mathbf{f}$ is of the form

$$
\begin{equation*}
\mathfrak{c o l t}\left(\mathbf{f}_{\mathrm{a}}\right)=\mathrm{x}_{1} y_{1} y_{2}+2 \mathrm{x}_{1} y_{1} y_{2}^{2}+\mathrm{x}_{1} y_{1}^{3} y_{2}+3 x_{1} y_{1} y_{2}^{3}+2 x_{2} y_{1}^{2} y_{2}^{2}+\cdots \tag{4.2.3}
\end{equation*}
$$

4.2.2. Series of Motzkin paths. Let $\mathscr{G}$ be the $\mathscr{C}$-bud operad Bud $\mathscr{C}$ (Motz), where $\mathscr{C}:=\{-1,1\}$. Let $\mathbf{f}$ be the series of $\mathbb{K}\langle\langle\mathscr{G}\rangle\rangle$ defined for any Motzkin path $\mathfrak{b}$, input color $a \in \mathscr{C}$, and word of input colors $u \in \mathscr{C}^{|\mathfrak{b}|}$ by

$$
\begin{equation*}
\langle(a, \mathfrak{b}, u), \mathbf{f}\rangle:=\frac{1}{2^{|\mathfrak{b}|+1}}\left(\prod_{i \in[|\mathfrak{b}|]} q_{\mathrm{ht}_{\mathfrak{b}}(i)}^{u_{i}}\right)^{a} \tag{4.2.4}
\end{equation*}
$$

where $h t_{\mathfrak{b}}(i)$ is the ordinate of the $i$-th point of $\mathfrak{b}$. One has for instance, where the notation $\overline{1}$ stands for -1 ,

$$
\begin{array}{r}
\langle(1,0000,1 \overline{1} 111 \overline{1} 1), \mathbf{f}\rangle=\frac{1}{2^{8}}\left(q_{0} q_{0}^{\overline{1}} q_{1} q_{1} q_{2} q_{1}^{\overline{1}} q_{0}\right)^{1}=\frac{q_{0} q_{1} q_{2}}{2^{8}}, \\
\left\langle\left(\overline{1}, 0_{0}^{00}, \overline{1} 1 \overline{1} 11 \overline{1} 1 \overline{1} 1\right), \mathbf{f}\right\rangle=\frac{1}{2^{10}}\left(q_{0}^{\overline{1}} q_{1} q_{0}^{\overline{1}} q_{1} q_{2} q_{2}^{\overline{1}} q_{1} q_{1}^{\overline{1}} q_{0}\right)^{\overline{1}}=\frac{q_{0}}{2^{10} q_{1}^{2}} . \tag{4.2.5b}
\end{array}
$$

Moreover, the coefficients of the pruned series of $\mathbf{f}$ satisfy, by definition of $\mathfrak{p r u}$ and $\mathbf{f}$,

$$
\begin{equation*}
\langle\mathfrak{b}, \mathfrak{p r u}(\mathbf{f})\rangle=\sum_{\substack{a \in\{\overline{1}, 1\} \\ u \in\{\overline{1}, 1\}^{|\mathfrak{b}|}}}\langle(a, \mathfrak{b}, u), \mathbf{f}\rangle=\frac{1}{2^{|\mathfrak{b}|+1}} \sum_{u \in\{\overline{1}, 1\}^{|\mathfrak{b}|}}\left(\left(\prod_{i \in[|\mathfrak{b}|]} q_{\mathrm{ht}_{\mathfrak{b}}(i)}^{u_{i}}\right)+\left(\prod_{i \in[|\mathfrak{b}|]} q_{\mathrm{ht}_{\mathfrak{b}}(i)}^{-u_{i}}\right)\right) . \tag{4.2.6}
\end{equation*}
$$

These coefficients seem to factorize nicely. For instance,

$$
\begin{align*}
& \langle\mathrm{O}, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{1+q_{0}^{2}}{2 q_{0}}, \quad \text { (4.2.7a) } \\
& \langle 00, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{2}}{4 q_{0}^{2}}, \\
& \langle 000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}}{8 q_{0}^{3}}, \\
& \langle O 0, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{2}\left(1+q_{1}^{2}\right)}{8 q_{0}^{2} q_{1}} \text {, } \\
& \langle 0000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{4}}{16 q_{0}^{4}}, \\
& \langle 000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}\left(1+q_{1}^{2}\right)}{16 q_{0}^{3} q_{1}} \text {, } \\
& \left\langle 0^{0} 00, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}\left(1+q_{1}^{2}\right)}{16 q_{0}^{3} q_{1}} \text {, } \\
& \left\langle 0^{0} 0, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{2}\left(1+q_{1}^{2}\right)^{2}}{16 q_{0}^{2} q_{1}^{2}} \text {, } \\
& \langle 00000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{5}}{32 q_{0}^{5}},  \tag{4.2.7q}\\
& \langle 0000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{4}\left(1+q_{1}^{2}\right)}{32 q_{0}^{4} q_{1}}, \\
& \left\langle 00^{0} 00, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{4}\left(1+q_{1}^{2}\right)}{32 q_{0}^{4} q_{1}}, \quad \text { (4.2.7k) } \\
& \left\langle 00^{00} O_{0}, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}\left(1+q_{1}^{2}\right)^{2}}{32 q_{0}^{3} q_{1}^{2}}, \quad \text { (4.2.71) } \\
& \langle 0000, \mathfrak{p r u}(\mathbf{f})\rangle=\frac{\left(1+q_{0}^{2}\right)^{4}\left(1+q_{1}^{2}\right)}{32 q_{0}^{4} q_{1}}, \quad \text { (4.2.7m) } \\
& \left\langle 0^{0} 0, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}\left(1+q_{1}^{2}\right)^{2}}{32 q_{0}^{3} q_{1}^{2}}, ~(4.2 .7 n \text { ) } \\
& \left\langle 0^{00} 0_{0}, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{3}\left(1+q_{1}^{2}\right)^{2}}{32 q_{0}^{3} q_{1}^{2}}, \quad \text { (4.2.7o) } \\
& \left\langle 0^{000}, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{2}\left(1+q_{1}^{2}\right)^{3}}{32 q_{0}^{2} q_{1}^{3}}, \quad \text { (4.2.7p) } \\
& \left\langle\sigma_{0}^{0}, \mathfrak{p r u}(\mathbf{f})\right\rangle=\frac{\left(1+q_{0}^{2}\right)^{2}\left(1+q_{1}^{2}\right)^{2}\left(1+q_{2}^{2}\right)}{32 q_{0}^{2} q_{1}^{2} q_{2}} \text {. }
\end{align*}
$$

Observe that the specializations at $q_{0}:=1, q_{1}:=1$, and $q_{2}:=1$ of all these coefficients are equal to 1.
4.3. Bud generating systems. We rely on the monochrome operads defined in Section 4.1 to construct several bud generating systems. We review some properties of these, leaving the proofs to the reader.
4.3.1. Monochrome bud generating systems from Dias $\gamma_{\gamma}$. Let $\gamma$ be a nonnegative integer and consider the monochrome bud generating system $\mathscr{\Re}_{\mathrm{w}, \gamma}:=\left(\operatorname{Dias}_{\gamma}, \mathfrak{R}_{\gamma}\right)$ where

$$
\begin{equation*}
\mathfrak{R}_{\gamma}:=\{0 a, a 0: a \in[\gamma]\} . \tag{4.3.1}
\end{equation*}
$$

The derivation graph of $\mathscr{B}_{\mathrm{w}, 1}$ is depicted by Figure 2 and the one of $\mathscr{G}_{\mathrm{w}, 2}$, by Figure 3 .


Figure 2. The derivation graph of $\mathscr{B}_{\mathrm{w}, 1}$.


Figure 3. The derivation graph of $\mathscr{B}_{\mathrm{w}, 2}$.

Proposition 4.3.1. For any $\gamma \geqslant 0$, the monochrome bud generating system $\mathscr{B}_{w, \gamma}$ satisfies the following properties.
(i) It is faithful.
(ii) The set $\mathrm{L}\left(\mathscr{\Im}_{\mathrm{w}, \gamma}\right)$ is equal to the underlying monochrome graded collection of Dias ${ }_{\gamma}$.
(iii) The set of rules $\mathfrak{R}_{\gamma}(1)$ factorizes finitely Dias ${ }_{\gamma}$.

Property (i) of Proposition 4.3.1 is a consequence of the fact that $\mathscr{B}_{\mathrm{w}, \gamma}$ is monochrome and Property (ii) is implied by the fact that $\mathfrak{R}_{\gamma}$ is a generating set of Dias ${ }_{\gamma}$ [Gir15]. Moreover, observe that since the word $\gamma 0 \gamma$ of $\operatorname{Dias}_{\gamma}(3)$ admits exactly the two $\mathfrak{R}_{\gamma}$-treelike expressions $0 \gamma \circ_{1} \gamma 0$ and $\gamma 0 o_{2} 0 \gamma$, by Theorem 3.3.9, $\left\langle\gamma 0 \gamma, \operatorname{synt}\left(\mathscr{B}_{\mathrm{w}, \gamma}\right)\right\rangle=2$. Hence, $\mathscr{B}_{\mathrm{w}, \gamma}$ is not unambiguous.
4.3.2. A bud generating system for Motzkin paths. Consider the bud generating system $\mathscr{G}_{\mathrm{p}}:=$ (Motz, $\{1,2\}, \mathfrak{R},\{1\},\{1,2\}$ ) where

$$
\begin{equation*}
\mathfrak{R}:=\left\{(1, \infty, 22),\left(1,0_{0}, 111\right)\right\} . \tag{4.3.2}
\end{equation*}
$$

Figure 4 shows a sequence of derivations in $\mathscr{B}_{\text {p }}$ and Figure 5 shows the derivation graph of $\mathscr{B}_{p}$.

Figure 4. A sequence of derivations in $\mathscr{B}_{\mathrm{p}}$. The input colors of the elements of $\operatorname{Bud}\{1,2\}(\mathbf{M o t z})$ are depicted below the paths. The output color of all these elements is 1 .


Figure 5. The derivation graph of $\mathscr{S}_{\mathrm{p}}$. The input colors of the elements of Bud $\{1,2\}($ Motz $)$ are depicted below the paths. The output color of all these elements is 1 .

Let $L_{\mathscr{G}_{\mathrm{p}}}$ be the set of Motzkin paths with no consecutive horizontal steps.
Proposition 4.3.2. The bud generating system $\mathscr{G}_{\mathrm{p}}$ satisfies the following properties.
(i) It is faithful.
(ii) The restriction of the pruning map $\mathfrak{p r u}$ on the domain $L\left(\mathscr{G}_{\mathrm{p}}\right)$ is a bijection between $\mathrm{L}\left(\mathscr{B}_{\mathrm{p}}\right)$ and $L_{\mathscr{G}_{\mathrm{p}}}$.
(iii) The set of rules $\mathfrak{R}(1)$ finitely factorizes Bud $_{\{1,2\}}($ Motz $)$.

Properties (i) and (ii) of Proposition 4.3.2 together say that the sequence enumerating the elements of $L\left(\mathscr{G}_{p}\right)$ with respect to their arity is the one enumerating the Motzkin paths with no consecutive horizontal steps. This sequence is Sequence A104545 of [Slo], starting by

$$
\begin{equation*}
1,1,1,3,5,11,25,55,129,303,721,1743,4241,10415,25761,64095 . \tag{4.3.3}
\end{equation*}
$$

Moreover, since the Motzkin path $\mathrm{O}_{0}$ of $\operatorname{Motz}(5)$ admits exactly the two $\mathfrak{R}$-treelike expres-
 Hence $\mathscr{\mathscr { B }}_{\mathrm{p}}$ is not unambiguous.
4.3.3. A bud generating system for unary-binary trees. Let $C$ be the monochrome graded collection defined by $C:=C(1) \sqcup C(2)$ where $C(1):=\{\mathrm{a}, \mathrm{b}\}$ and $C(2):=\{\mathrm{c}\}$. Let $\mathscr{B}_{\mathrm{bu}}:=$ (Free $(C),\{1,2\}, \mathfrak{R},\{1\},\{2\})$ be the bud generating system where

$$
\begin{equation*}
\mathfrak{R}:=\{(1, \quad \mathrm{a}, 2),(1, \quad \mathrm{~b}, 2),(2, \mathrm{c}, 11)\} . \tag{4.3.4}
\end{equation*}
$$

Figure 6 shows a sequence of derivations in $\mathscr{\mathscr { B }}_{\text {bu }}$.


Figure 6. A sequence of derivations in $\mathscr{B}_{\mathrm{bu}}$. The input colors of the elements of $\operatorname{Bud}_{\{1,2\}}(\operatorname{Free}(C))$ are depicted below the leaves. The output color of all these elements is 1 . Since all input colors of the last tree are 2, this tree is in $\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)$.

A unary-binary tree is a planar rooted tree $t$ such that all internal nodes of $t$ are of arities 1 or 2 , all nodes of $t$ of arity 1 have a child which is an internal node of arity 2 or is a leaf, and all nodes of $\mathfrak{t}$ of arity 2 have two children which are internal nodes of arity 1 or are leaves.

Let $L_{\mathscr{B}_{\mathrm{bu}}}$ be the set of unary-binary trees with a root of arity 1 , all parents of the leaves are of arity 1 , and unary nodes are labeled by a or b.

Proposition 4.3.3. The bud generating system $\mathscr{B}_{\text {bu }}$ satisfies the following properties.
(i) It is faithful.
(ii) It is unambiguous.
(iii) The restriction of the pruning map pru on the domain $\mathrm{L}\left(\mathscr{\Im}_{\mathrm{bu}}\right)$ is a bijection between $\mathrm{L}\left(\mathscr{\Im}_{\mathrm{bu}}\right)$ and $L_{\mathscr{G}_{b u}}$.
(iv) The set of rules $\mathfrak{R}(1)$ finitely factorizes Bud $_{\{1,2\}}($ Free $(C))$.
4.3.4. A bud generating system for B-perfect trees. Let $B$ be a finite set of positive integers and $C_{B}$ be the monochrome graded collection defined by $C_{B}:=\bigsqcup_{n \in B} C_{B}(n):=\bigsqcup_{n \in B}\left\{\mathrm{a}_{n}\right\}$. We consider the monochrome bud generating system $\mathscr{B}_{\mathrm{bt}, B}:=\left(\right.$ Free $\left.\left(C_{B}\right), \mathfrak{R}_{B}\right)$ where $\mathfrak{R}_{B}$ is the set of all corollas of Free $\left(C_{B}\right)(1)$. Figure 7 shows the synchronous derivation graph of $\mathscr{B}_{\mathrm{bt},\{2,3\}}$.

A B-perfect tree is a planar rooted tree $\mathfrak{t}$ such that all internal nodes of $\mathfrak{t}$ have an arity in $B$ and all paths connecting the root of $\mathfrak{t}$ to its leaves have the same length. These trees and their generating series have been studied for the particular case $B:=\{2,3\}$ [MPRS79, CLRS09] and appear as data structures in computer science (see [Odl82, Knu98, FS09]).

Proposition 4.3.4. For any finite set $B$ of positive integers, the bud generating system $\mathscr{B}_{\mathrm{bt}, B}$ satisfies the following properties.


Figure 7. The synchronous derivation graph of $\mathscr{B}_{\mathrm{bt},\{2,3\}}$.
(i) It is synchronously faithful.
(ii) It is synchronously unambiguous.
(iii) The synchronous language $\mathrm{L}_{\mathrm{S}}\left(\mathscr{B}_{\mathrm{bt}, \mathrm{B}}\right)$ of $\mathscr{B}_{\mathrm{bt}, B}$ is the set of all B-perfect trees.
(iv) The set of rules $\mathfrak{R}_{B}(1)$ finitely factorizes Free $\left(C_{B}\right)$.
(v) When $1 \notin B$, the generating series $\mathbf{s}_{\mathrm{L}_{\mathrm{s}}\left(\mathscr{B}_{\mathrm{b}, B}\right)}$ of the synchronous language of $\mathscr{B}_{\mathrm{bt}, B}$ is well-defined.

Property $(v)$ of Proposition 4.3.4 is a consequence of the fact that when $1 \notin B$, Free $\left(C_{B}\right)$ is locally finite and therefore there are only finitely many elements in $L_{S}\left(\mathscr{B}_{\mathrm{bt}, \mathrm{B}}\right)$ of a given arity. By Property (iii) of Proposition 4.3.4, the sequences enumerating the elements of $\mathrm{L}_{\mathrm{S}}\left(\mathscr{H}_{\mathrm{bt}, B}\right)$ with respect to their arity are, for instance, Sequence A014535 of [Slo] for $B=\{2,3\}$ which starts by

$$
\begin{equation*}
1,1,1,1,2,2,3,4,5,8,14,23,32,43,63,97,149,224,332,489 \tag{4.3.5}
\end{equation*}
$$

and Sequence $A 037026$ of [Slo] for $B=\{2,3,4\}$ which starts by

$$
\begin{equation*}
1,1,1,2,2,4,5,9,15,28,45,73,116,199,345601,1021,1738,2987,5244 . \tag{4.3.6}
\end{equation*}
$$

4.3.5. A bud generating system for balanced binary trees. Consider the bud generating system $\mathscr{G}_{\text {bbt }}:=(\mathbf{M a g},\{1,2\}, \mathfrak{R},\{1\},\{1\})$ where

$$
\begin{equation*}
\mathfrak{R}:=\left\{\left(1, \text { 。 }_{\text {, }}, 11\right),\left(1, \text { o }_{\text {, }}, 12\right),(1, \text { o, } 21),(2, \text { 立, } 1)\right\} . \tag{4.3.7}
\end{equation*}
$$

Figure 8 shows a sequence of synchronous derivations in $\mathscr{G}_{\text {bbt }}$.


Figure 8. A sequence of synchronous derivations in $\mathscr{\Re}_{\mathrm{bbt}}$. The input colors of the elements of $\operatorname{Bud}_{\{1,2\}}(\mathbf{M a g})$ are depicted below the leaves. The output color of all these elements is 1. Since all input colors of the last tree are 1 , this tree is in $\mathrm{L}_{\mathrm{S}}\left(\mathscr{\Re}_{\mathrm{bbt}}\right)$.

The height of a binary tree $\mathfrak{t}$ is the height of $\mathfrak{t}$ seen as a monochrome syntax tree. A balanced binary tree [AVL62] is a binary tree $\mathfrak{t}$ wherein, for any internal node $x$ of $\mathfrak{t}$, the difference between the height of the left subtree and the height of the right subtree of $x$ is $-1,0$, or 1 .

Proposition 4.3.5. The bud generating system $\mathscr{B}_{\text {bbt }}$ satisfies the following properties.
(i) It is synchronously faithful.
(ii) It is synchronously unambiguous.
(iii) The restriction of the pruning map pru on the domain $\mathrm{L}_{\mathrm{S}}\left(\mathscr{\Im}_{\mathrm{bbt}}\right)$ is a bijection between $\mathrm{L}_{\mathrm{S}}\left(\mathscr{\Im}_{\mathrm{bbt}}\right)$ and the set of balanced binary trees.
(iv) The set of rules $\mathfrak{R}(1)$ finitely factorizes Bud $_{\{1,2\}}(\mathbf{M a g})$.

Properties (ii) and (iii) of Proposition 4.3 .5 are based upon combinatorial properties of a synchronous grammar $\mathcal{G}$ of balanced binary trees defined in [Gir12] and satisfying $\mathrm{SG}(\mathcal{G})=\mathscr{G}_{\mathrm{bbt}}$ (see Section 2.3.3 and Proposition 2.3.3). Besides, Properties (i) and (iii) of Proposition 4.3.5 together imply that the sequence enumerating the elements of $L_{S}\left(\mathscr{\Re}_{\mathrm{bbt}}\right)$ with respect to their arity is the one enumerating the balanced binary trees. This sequence in Sequence A006265 of [Slo], starting by

$$
\begin{equation*}
1,1,2,1,4,6,4,17,32,44,60,70,184,476,872,1553,2720,4288,6312,9004 \tag{4.3.8}
\end{equation*}
$$

4.4. Series of bud generating systems. We now consider the bud generating systems constructed in Section 4.3 to give some examples of hook generating series. We also put into practice what we have exposed in Sections 3.3.4 and 3.4.2 to compute the generating series of languages or synchronous languages of bud generating systems by using syntactic generating series and synchronous generating series.
4.4.1. Hook coefficients for binary trees. Let us consider the hook bud generating system $\mathrm{HS}_{\text {Mag, } G}$ where $G:=\left\{\begin{array}{c}\delta_{G}\end{array}\right\}$. This bud generating system leads to the definition of a statistic on binary trees, provided by the coefficients of the hook generating series hook $\left(\mathrm{HS}_{\mathrm{Mag}, \mathrm{G}}\right)$ which begins by

Theorem 3.3.5 implies that for any binary tree $\mathfrak{t}$, the coefficient $\left\langle\mathfrak{t}\right.$, hook $\left.\left(\operatorname{HS}_{\text {Mag, } G}\right)\right\rangle$ can be obtained by the usual hook-length formula of binary trees. This explains the name of hook generating series for hook( $\mathscr{B})$, when $\mathscr{B}$ is a bud generating system. Alternatively, the coefficient $\left\langle\mathfrak{t}\right.$, hook $\left.\left(\mathrm{HS}_{\text {Mag,G }}\right)\right\rangle$ is the cardinal of the sylvester class [HNT05] of permutations encoded by $t$.
4.4.2. Hook coefficients for words of Dias $\gamma$. Let us consider the monochrome bud generating system $\mathscr{G}_{w, \gamma}$ and its set of rules $\mathfrak{R}_{\gamma}$ introduced in Section 4.3.1. Since, by Proposition 4.3.1, $\Re_{\gamma}$ generates Dias $_{\gamma}, \mathscr{B}_{w, \gamma}$ is a hook bud generating system $\mathrm{HS}_{\text {Dias }_{\gamma}, \Re_{\gamma}}$ (see Section 3.3.2). This leads to the definition of a statistic on the words of Dias $\gamma_{\gamma}$, provided by the coefficients of the hook generating series hook $\left(\mathrm{HS}_{\operatorname{Dias}_{\gamma}, \Re_{\gamma}}\right)$ of $\mathrm{HS}_{\text {Dias }_{\gamma}, \Re_{\gamma}}$ which begins, when $\gamma=1$, by

$$
\begin{align*}
& \text { hook }\left(\mathrm{HS}_{\text {Dias }_{1}, \Re_{1}}\right)=(0)+(01)+(10)+3(011)+2(101)+3(110)+15(0111)+9(1011) \\
& +9(1101)+15(1110)+105(01111)+60(10111)+54(11011)+60(11101)+105(11110) \\
+ & 945(011111)+525(101111)+45(110111)+450(111011)+525(111101)+945(111110)+\cdots . \tag{4.4.2}
\end{align*}
$$

Let us set, for all $0 \leqslant a \leqslant n-1, h_{n, a}:=\left\langle 1^{a} 01^{n-a-1}\right.$, hook $\left.\left(\operatorname{HS}_{\text {Dias }_{1}, \Re_{1}}\right)\right\rangle$. By Lemmas 3.2.3 and 3.3.3, the $h_{n, a}$ satisfy the recurrence

$$
h_{n, a}= \begin{cases}1 & \text { if } n=1  \tag{4.4.3}\\ (2 a-1) h_{n-1, a-1} & \text { if } a=n-1 \\ (2 n-2 a-3) h_{n-1, a} & \text { if } a=0 \\ (2 a-1) h_{n-1, a-1}+(2 n-2 a-3) h_{n-1, a} & \text { otherwise }\end{cases}
$$

The numbers $h_{n, a}$ form a triangle beginning by

| 1 |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 |  |  |  |  |  |  |
| 3 | 2 | 3 |  |  |  |  |  |
| 15 | 9 | 9 | 15 |  |  |  |  |
| 105 | 60 | 54 | 60 | 105 |  |  |  |
| 945 | 525 | 450 | 450 | 525 | 945 |  |  |
| 10395 | 5670 | 4725 | 4500 | 4725 | 5670 | 10395 |  |
| 135135 | 72765 | 59535 | 55125 | 55125 | 59535 | 72765 | 135135 |.

These numbers form Sequence A059366 of [Slo].
4.4.3. Hook coefficients for Motkzin paths. It is proven in [Gir15] that $G:=\{00,00\}$ is a generating set of Motz. Hence, $\mathrm{HS}_{\text {Motz,G }}$ is a hook generating system. This leads to the definition of a statistic on Motzkin paths, provided by the coefficients of the hook generating series hook $\left(\mathrm{HS}_{\text {Motz,G }}\right)$ of $\mathrm{HS}_{\text {Motz,G }}$ which begins by

$$
\begin{align*}
& \operatorname{hook}\left(\mathrm{HS}_{\text {Motz,G }}\right)=0+00+2000+00+60000+2000+2000 \\
& +000+2400000+60000+60000+30000+60000 \\
& +20_{0}^{000}+30000+20000+0000000 \tag{4.4.5}
\end{align*}
$$

4.4.4. Generating series of some unary-binary trees. Let us consider the bud generating system $\mathscr{G}_{\text {bu }}$ introduced in Section 4.3.3. We have, for all $a \in\{1,2\}$ and $\alpha \in \mathscr{T}_{\{1,2\}}$,

$$
\chi_{a, \alpha}= \begin{cases}2 & \text { if }(a, \alpha)=(1,01)  \tag{4.4.6}\\ 1 & \text { if }(a, \alpha)=(2,20) \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\begin{equation*}
\mathbf{g}_{1}\left(y_{1}, y_{2}\right)=2 y_{2}, \quad(4.4 .7 a) \quad \mathbf{g}_{2}\left(y_{1}, y_{2}\right)=y_{1}^{2} . \tag{4.4.7b}
\end{equation*}
$$

Since, by Proposition 4.3.3, $\mathscr{B}_{\text {bu }}$ satisfies the conditions of Proposition 3.3.12, by this last proposition and (3.3.27), the generating series $\mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}$ of $\mathrm{L}\left(\mathscr{\Im}_{\text {bu }}\right)$ satisfies $\mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}=\mathbf{f}_{1}(0, t)$ where

$$
\begin{equation*}
\mathbf{f}_{1}\left(y_{1}, y_{2}\right)=y_{1}+2 \mathbf{f}_{2}\left(y_{1}, y_{2}\right), \quad(4.4 .8 a) \quad \mathbf{f}_{2}\left(y_{1}, y_{2}\right)=y_{2}+\mathbf{f}_{1}\left(y_{1}, y_{2}\right)^{2} \tag{4.4.8b}
\end{equation*}
$$

We obtain that $\mathbf{f}_{1}\left(y_{1}, y_{2}\right)$ satisfies the functional equation

$$
\begin{equation*}
y_{1}+2 y_{2}-\mathbf{f}_{1}\left(y_{1}, y_{2}\right)+2 \mathbf{f}_{1}\left(y_{1}, y_{2}\right)^{2}=0 \tag{4.4.9}
\end{equation*}
$$

Hence, $\mathbf{s}_{\mathrm{L}\left(\mathscr{T}_{\text {bu }}\right)}$ satisfies

$$
\begin{equation*}
2 t-\mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}+2 \mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}^{2}=0, \tag{4.4.10}
\end{equation*}
$$

showing that the elements of $\mathrm{L}\left(\mathscr{G}_{\mathrm{bu}}\right)$ are enumerated, arity by arity, by Sequence A052707 of [Slo] starting by

$$
\begin{equation*}
2,8,64,640,7168,86016,1081344,14057472,187432960,2549088256 . \tag{4.4.11}
\end{equation*}
$$

Besides, since by Proposition 4.3.3, $\mathscr{B}_{\text {bu }}$ satisfies the conditions of Theorem 3.3.14, by this last theorem and (3.3.33), $\mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}$ satisfies, for any $n \geqslant 1$,

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{L}\left(\mathscr{B}_{\mathrm{bu}}\right)}\right\rangle=\left\langle\mathrm{x}_{1} \mathrm{y}_{2}^{n}, \mathbf{f}\right\rangle, \tag{4.4.12}
\end{equation*}
$$

where $\mathbf{f}$ is the series satisfying, for any $a \in \mathscr{C}$ and any type $\alpha \in \mathscr{T}_{\{1,2\}}$, the recursive formula

$$
\begin{align*}
& \left\langle\mathrm{x}_{a} y^{\alpha_{1}} \mathrm{y}^{\alpha_{2}}, \mathbf{f}\right\rangle=\delta_{\alpha, \operatorname{type}(a)}+\delta_{a, 1} 2\left\langle\mathrm{x}_{2} y_{1}^{\alpha_{1}} \mathrm{y}_{2}^{\alpha_{2}}, \mathbf{f}\right\rangle \\
& +\delta_{a, 2} \sum_{\substack{d_{1}, d_{2}, d_{3}, d_{4} \in \mathbb{N} \\
\alpha_{1}=d_{1}+d_{2} \\
\alpha_{2}=d_{3}+d_{4} \\
\left(d_{1}, d_{3}\right) \neq(0,0) \neq\left(d_{2}, d_{4}\right)}}\left\langle\mathrm{x}_{1} \mathrm{y}_{1}^{d_{1}} \mathrm{y}_{2}^{d_{3}}, \mathbf{f}\right\rangle\left\langle\mathrm{x}_{2} \mathrm{y}_{1}^{d_{2}} y_{2}^{d_{4}}, \mathbf{f}\right\rangle . \tag{4.4.13}
\end{align*}
$$

4.4.5. Generating series of B-perfect trees. Let us consider the monochrome bud generating system $\mathscr{G}_{\mathrm{bt}, B}$ and its set of rules $\mathfrak{R}_{B}$ introduced in Section 4.3.4. By Proposition 4.3.4, the generating series $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{G}_{\mathrm{b}, B}\right)}$ is well-defined when $1 \notin B$. For this reason, in all this section we restrict ourselves to the case where all elements of $B$ are greater than or equal to 2. To maintain here homogeneous notations with the rest of the text, we consider that the set of colors $\mathscr{C}$ of $\mathscr{G}_{\mathrm{bt}, B}$ is the singleton $\{1\}$. We have, for all $\alpha \in \mathscr{T}_{\{1\}}$,

$$
\chi_{1, \alpha}= \begin{cases}1 & \text { if }(a, \alpha)=(1, b) \text { with } b \in B  \tag{4.4.14}\\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\begin{equation*}
\mathbf{g}_{1}\left(y_{1}\right)=\sum_{b \in B} y_{1}^{b} . \tag{4.4.15}
\end{equation*}
$$

Since by Proposition 4.3.4, $\mathscr{B}_{\mathrm{bt}, B}$ satisfies the conditions of Proposition 3.4.7, by this last proposition and (3.4.12), the generating series $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{B}_{\mathrm{b}, B}\right)}$ of $\mathrm{L}_{\mathrm{S}}\left(\mathscr{B}_{\mathrm{bt}, B}\right)$ satisfies $\mathbf{s}_{\mathrm{Ls}}\left(\mathscr{B}_{\mathrm{bt}, B}\right)=\mathbf{f}_{1}(t)$ where

$$
\begin{equation*}
\mathbf{f}_{1}\left(y_{1}\right)=y_{1}+\mathbf{f}_{1}\left(\sum_{b \in B} y_{1}^{b}\right) \tag{4.4.16}
\end{equation*}
$$

This functional equation for the generating series of $B$-perfect trees, in the case where $B=$ $\{2,3\}$, is the one obtained in [Odl82,FS09, Gir12] by different methods.

Besides, since by Proposition 4.3.4, $\mathscr{B}_{\mathrm{bt}, B}$ satisfies the conditions of Theorem 3.4.8, by this last theorem and (3.4.14), $\mathbf{s}_{\mathrm{Ls}\left(\mathscr{G}_{\mathrm{b}, B}\right)}$ satisfies, for any $n \geqslant 1$, the recursive formula

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{LS}}\left(\mathscr{B}_{\mathrm{b}, B}\right)\right\rangle=\delta_{n, 1}+\sum_{\substack{d_{b} \in \mathbb{N}, b \in B \\ n=\sum_{b \in B} b d_{\mathrm{b}}}}\left\langle d_{b}: b \in B \int!\left\langle t^{\sum_{b \in B} d_{\mathrm{b}}}, \mathbf{s}_{\mathrm{L}_{\mathrm{s}}\left(\mathscr{B}_{\mathrm{b}, B}\right)}\right\rangle\right. \tag{4.4.17}
\end{equation*}
$$

For instance, for $B:=\{2,3\}$, one has

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{Ls}\left(\mathscr{B}_{\mathrm{b},\{2,3)}\right)}\right\rangle=\delta_{n, 1}+\sum_{\substack{d_{2}, d_{3} \geqslant 0 \\ n=2 d_{2}+3 d_{3}}}\binom{d_{2}+d_{3}}{d_{2}}\left\langle t^{d_{2}+d_{3}}, \mathbf{s}_{\mathrm{Ls}\left(\mathscr{B}_{\mathrm{bt},(2,3)\}}\right)}\right\rangle \tag{4.4.18}
\end{equation*}
$$

which is a recursive formula to enumerate the $\{2,3\}$-perfect trees known from [MPRS79], and for $B:=\{2,3,4\}$,

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{B}_{\mathrm{b},\{2,3,4\}}\right)}\right\rangle=\delta_{n, 1}+\sum_{\substack{d_{2}, d_{3}, d_{4} \geqslant 0 \\ n=2 d_{2}+3 d_{3}+4 d_{4}}}\left\{d_{2}, d_{3}, d_{4} \int!\left\langle t^{d_{2}+d_{3}+d_{4}}, \mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{G}_{\mathrm{b},\{(2,3,4\}}\right)}\right\rangle .\right. \tag{4.4.19}
\end{equation*}
$$

Moreover, it is possible to refine the enumeration of $B$-perfect trees to take into account of the number of internal nodes with a given arity in the trees. For this, we consider the series $\mathbf{s}_{q}$ satisfying the recurrence

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{q}\right\rangle=\delta_{n, 1}+\sum_{\substack{d_{b} \in \mathbb{N}, b \geqslant 2 \\ n=\sum_{b \geqslant 2} b d_{b}}}\left\langle d_{b}: b \geqslant 2 \int!\left(\prod_{b \geqslant 2} q_{b}^{d_{b}}\right)\left\langle t^{\sum_{b \geqslant 2} d_{b}}, \mathbf{s}_{q}\right\rangle\right. \tag{4.4.20}
\end{equation*}
$$

The coefficient of $\left(\prod_{b \geqslant 2} q_{b}^{d_{b}}\right) t^{n}$ in $\mathbf{s}_{q}$ is the number of $\mathbb{N} \backslash\{0,1\}$-perfect trees with $n$ leaves and with $d_{b}$ internal nodes of arity $b$ for all $b \geqslant 2$. The specialization of $\mathbf{s}_{q}$ at $q_{b}:=0$ for all $b \notin B$ and $q_{b}:=t$ for all $b \in B$ is equal to the series $\mathbf{s}_{\mathrm{Ls}\left(\mathscr{B}_{\mathrm{b}, B}\right)}$.

First coefficients of $\mathbf{s}_{q}$ are

$$
\begin{array}{rlr}
\left\langle t, \mathbf{s}_{q}\right\rangle=1, & (4.4 .21 \mathrm{a}) & \left\langle t^{4}, \mathbf{s}_{q}\right\rangle=q_{2}^{3}+q_{4}, \\
\left\langle t^{2}, \mathbf{s}_{q}\right\rangle=q_{2}, & (4.4 .21 \mathrm{~b}) & \left\langle t^{5}, \mathbf{s}_{q}\right\rangle=2 q_{2}^{2} q_{3}+q_{5} \\
\left\langle t^{3}, \mathbf{s}_{q}\right\rangle=q_{3}, & (4.4 .21 \mathrm{c}) & \left\langle t^{6}, \mathbf{s}_{q}\right\rangle=q_{2}^{3} q_{3}+q_{2} q_{3}^{2}+2 q_{2}^{2} q_{4}+q_{6} \\
\left\langle t^{7}, \mathbf{s}_{q}\right\rangle=3 q_{2}^{2} q_{3}^{2}+2 q_{2} q_{3} q_{4}+2 q_{2}^{2} q_{5}+q_{7}, \\
\left\langle t^{9}, \mathbf{s}_{q}\right\rangle=4 q_{2}^{6} q_{3}+4 q_{2}^{3} q_{3} q_{4}+q_{3}^{4}+6 q_{2} q_{3}^{2} q_{4}+3 q_{2}^{2} q_{3} q_{5}+2 q_{2} q_{4} q_{5}+2 q_{2} q_{3} q_{6}+2 q_{2}^{2} q_{7}+q_{9}
\end{array}
$$

4.4.6. Generating series of balanced binary trees. Let us consider the bud generating system $\mathscr{G}_{\text {bbt }}$ introduced in Section 4.3.5. We have

$$
\chi_{a, \alpha}= \begin{cases}1 & \text { if }(a, \alpha)=(1,20)  \tag{4.4.22}\\ 2 & \text { if }(a, \alpha)=(1,11) \\ 1 & \text { if }(a, \alpha)=(2,10) \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\begin{equation*}
\mathbf{g}_{1}\left(y_{1}, y_{2}\right)=y_{1}^{2}+2 y_{1} y_{2}, \quad(4.4 .23 a) \quad \mathbf{g}_{2}\left(y_{1}, y_{2}\right)=y_{1} \tag{4.4.23b}
\end{equation*}
$$

Since by Proposition 4.3.5, $\mathscr{G}_{\mathrm{bbt}}$ satisfies the conditions of Proposition 3.4.7, by this last proposition and (3.4.12), the generating series $\mathbf{s}_{\mathrm{L}_{\mathrm{s}}\left(\mathscr{G}_{\mathrm{bbt}}\right)}$ of $\mathrm{L}_{\mathrm{S}}\left(\mathscr{\mathscr { G }}_{\mathrm{bbt}}\right)$ satisfies $\mathbf{s}_{\mathrm{Ls}}\left(\mathscr{B}_{\mathrm{bbt}}\right)=\mathbf{f}_{1}(t, 0)$ where

$$
\begin{equation*}
\mathbf{f}_{1}\left(y_{1}, y_{2}\right)=y_{1}+\mathbf{f}_{1}\left(y_{1}^{2}+2 y_{1} y_{2}, y_{1}\right) . \tag{4.4.24}
\end{equation*}
$$

This functional equation for the generating series of balanced binary trees is the one obtained in [BLL88, BLL97, Knu98, Gir12] by different methods. As announced in Section 3.4.2, the coefficients of $\mathbf{f}_{1}$ (and hence, those of $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{G}_{\mathrm{bbt}}\right)}$ ) can be computed by iteration. This consists in defining, for any $\ell \geqslant 0$, the polynomials $\mathbf{f}_{1}^{(\ell)}\left(y_{1}, y_{2}\right)$ as

$$
\mathbf{f}_{1}^{(\ell)}\left(y_{1}, y_{2}\right):= \begin{cases}y_{1} & \text { if } \ell=0  \tag{4.4.25}\\ y_{1}+f_{1}^{(\ell-1)}\left(y_{1}^{2}+2 y_{1} y_{2}, y_{1}\right) & \text { otherwise } .\end{cases}
$$

Since

$$
\begin{equation*}
\mathbf{f}_{1}\left(y_{1}, y_{2}\right)=\lim _{\ell \rightarrow \infty} \mathbf{f}_{1}^{(\ell)}\left(y_{1}, y_{2}\right) \tag{4.4.26}
\end{equation*}
$$

Equation (4.4.25) provides a way to compute the coefficients of $\mathbf{f}_{1}\left(y_{1}, y_{2}\right)$. First polynomials $\mathbf{f}_{1}^{(\ell)}\left(y_{1}, y_{2}\right)$ are

$$
\begin{array}{r}
\mathbf{f}_{1}^{(0)}\left(y_{1}, y_{2}\right)=y_{1}, \quad(4.4 .27 a) \quad \mathbf{f}_{1}^{(1)}\left(y_{1}, y_{2}\right)=y_{1}+y_{1}^{2}+2 y_{1} y_{2}, \\
\mathbf{f}_{1}^{(2)}\left(y_{1}, y_{2}\right)=y_{1}+y_{1}^{2}+2 y_{1} y_{2}+2 y_{1}^{3}+4 y_{1}^{2} y_{2}+y_{1}^{4}+4 y_{1}^{3} y_{2}+4 y_{1}^{2} y_{2}^{2}, \\
\mathbf{f}_{1}^{(3)}\left(y_{1}, y_{2}\right)=y_{1}+y_{1}^{2}+2 y_{1} y_{2}+2 y_{1}^{3}+4 y_{1}^{2} y_{2}+y_{1}^{4}+4 y_{1}^{3} y_{2}+4 y_{1}^{2} y_{2}^{2}+4 y_{1}^{5} \\
+16 y_{1}^{4} y_{2}+16 y_{1}^{3} y_{2}^{2}+6 y_{1}^{6}+28 y_{1}^{5} y_{2}+40 y_{1}^{4} y_{2}^{2}+16 y_{1}^{3} y_{2}^{3}+4 y_{1}^{7}+24 y_{1}^{6} y_{2} \\
+48 y_{1}^{5} y_{2}^{2}+32 y_{1}^{4} y_{2}^{3}+y_{1}^{8}+8 y_{1}^{7} y_{2}+24 y_{1}^{6} y_{2}^{2}+32 y_{1}^{5} y_{2}^{3}+16 y_{1}^{4} y_{2}^{4} . \tag{4.4.27d}
\end{array}
$$

Besides, since by Proposition 4.3.5, $\mathscr{G}_{\mathrm{bbt}}$ satisfies the conditions of Theorem 3.4.8, by this last theorem and (3.4.14), $\mathbf{s}_{\mathrm{Ls}\left(\mathscr{B}_{\mathrm{bbt}}\right)}$ satisfies, for any $n \geqslant 1$,

$$
\begin{equation*}
\left\langle t^{n}, \mathbf{s}_{\mathrm{L}_{\mathrm{S}}\left(\mathscr{B}_{\mathrm{bbt}}\right)}\right\rangle=\left\langle y_{1}^{n} y_{2}^{0}, \mathbf{f}\right\rangle, \tag{4.4.28}
\end{equation*}
$$

where $\mathbf{f}$ is the series satisfying, for any type $\alpha \in \mathscr{T}_{\{1,2\}}$, the recursive formula

$$
\begin{equation*}
\left\langle y_{1}^{\alpha_{1}} y_{2}^{\alpha_{2}}, \mathbf{f}\right\rangle=\delta_{\alpha,(1,0)}+\sum_{\substack{d_{1}, d_{2}, d_{3} \in \mathbb{N} \\ \alpha_{1}=2 d_{1}+d_{2}+d_{3} \\ \alpha_{2}=d_{2}}}\binom{d_{1}+\alpha_{2}}{d_{1}} 2^{d_{2}}\left\langle y_{1}^{d_{1}+d_{2}} y_{2}^{d_{3}}, \mathbf{f}\right\rangle . \tag{4.4.29}
\end{equation*}
$$

## Conclusion and perspectives

In this paper, we have presented a framework for the generation of combinatorial objects by using colored operads. The described devices for combinatorial generation, called bud generating systems, are generalizations of context-free grammars [Har78, HMU06] generating words, of regular tree grammars [GS84, $\left.\mathrm{CDG}^{+} 07\right]$ generating planar rooted trees, and of synchronous grammars [Gir12] generating some treelike structures. We have provided tools to enumerate the objects of the languages of bud generating systems or to define new statistics on these by using formal power series on colored operads and several products on these. There are many ways to extend this work. Here follow some few further research directions.

First, the notion of rationality and recognizability in usual formal power series [Sch61,Sch63, Eil74, BR88], in series on monoids [Sak09], and in series of trees [BR82] are fundamental. For instance, a series $\mathbf{s} \in \mathbb{K}\langle\langle\mathcal{J} J\rangle\rangle$ on a monoid $\mathfrak{J}$, is rational if it belongs to the closure of the set $\mathbb{K}\langle\mathfrak{J L}\rangle$ of polynomials on $\mathfrak{N}$ with respect to the addition, the multiplication, and the Kleene star operations. Equivalently, s is rational if there exists a $\mathbb{K}$-weighted automaton accepting it. The equivalence between these two properties for the rationality property is remarkable. We ask here for the definition of an analogous and consistent notion of rationality for series on a colored operad $\mathfrak{G}$. By consistent, we mean a property of rationality for $\mathfrak{G}$-series which can be defined both by a closure property of the set $\mathbb{K}\langle\mathscr{G}\rangle$ of the polynomials on $\mathfrak{G}$ with respect to some operations, and, at the same time, by an acceptance property involving a notion of a $\mathbb{K}$-weighted automaton on $\mathfrak{G}$. The analogous question about the definition of a notion of recognizable series on colored operads also seems worth investigating.

A second research direction fits mostly in the contexts of computer science and compression theory. A straight-line grammar (see for instance [ZL78, SS82, Ryt04]) is a context-free grammar with a singleton as language. There exists also the analogous natural counterpart for regular tree grammars [LM06]. One of the main interests of straight-line grammars is that they offer a way to compress a word (resp. a tree) by encoding it by a context-free grammar (resp. a regular tree grammar). A word $u$ can potentially be represented by a context-free grammar (as the unique element of its language) with less memory than the direct representation of $u$, provided that $u$ is made of several repeating factors. The analogous definition for bud generating systems could potentially be used to compress a large variety of combinatorial objects. Indeed, given a suitable monochrome operad $\Theta$ defined on the objects we want to compress, we can encode an object $x$ of $\mathcal{O}$ by a bud generating system $\mathscr{B}$ with $\mathcal{O}$ as ground operad and such that the language (or the synchronous language) of $\mathscr{G}$ is a singleton $\{y\}$ and $\mathfrak{p r u}(y)=x$. Hence, we can hope to obtain a new and efficient method to compress arbitrary combinatorial objects.

Let us finally describe a third extension of this work. Pros are algebraic structures which naturally generalize operads. Indeed, a pro is a set of operators with several inputs and several outputs, unlike in operads where operators have only one output (see for instance [Mar08]). Surprisingly, pros appeared earlier than operads in the literature [ML65]. It seems fruitful to translate the main definitions and constructions of this work (as e.g., bud operads, bud generating systems, series on colored operads, pre-Lie and composition products of series, star
operations, etc.) with pros instead of operads. We can expect to obtain an even more general class of grammars and obtain a more general framework for combinatorial generation.

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[^0]:    ${ }^{\mathrm{a}}$ See examples of monochrome operads and their bud operads in Section 4.1.

[^1]:    ${ }^{\mathrm{b}}$ See examples of bud generating systems and derivation graphs in Sections 4.3.1, 4.3.2, and 4.3.3.
    ${ }^{\mathrm{c}}$ See examples of bud generating systems and synchronous derivation graphs in Sections 4.3.4 and 4.3.5.

[^2]:    ${ }^{\mathrm{d}}$ See examples of series on the bud operad of the operad Motz of Motzkin paths in Section 4.2.2.
    ${ }^{\mathrm{e}}$ See an example of pruned series in Section 4.2.2.

[^3]:    ${ }^{\mathrm{f}}$ See examples of series of colors in Section 4.2.1.
    ${ }^{9}$ See an example of a series of color types in Section 4.2.1.

[^4]:    ${ }^{\mathrm{h}}$ See examples of these definitions in Sections 4.4.4, 4.4.5, and 4.4.6.

[^5]:    ${ }^{\text {i }}$ See examples of definitions of a hook-length statistics for binary trees, words of Dias ${ }_{\gamma}$, and for Motzkin paths in Sections 4.4.1, 4.4.2, and 4.4.3.

[^6]:    ${ }^{\mathrm{j}}$ See an example of computation of a series $\mathbf{s}_{\mathrm{L}(\mathscr{B})}$ in Section 4.4.4.

[^7]:    ${ }^{\mathrm{k}}$ See example of a computation of a series $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}(\mathscr{B})}$ by iteration in Section 4.4.6.

[^8]:    ${ }^{\mathrm{l}}$ See examples of computations of series $\mathbf{s}_{\mathrm{L}_{\mathrm{S}}(\mathscr{B})}$ in Sections 4.4.5 and 4.4.6.

