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# Stochastic Multi-Object Guidance Laws for Interception and Rendezvous Problems

Daniel E. Clark

**Abstract**—This paper considers the problem of guiding an unknown number of controllable interceptors to rendezvous with the same target at the same time. It is assumed that the all of the interceptors and the target are described by linear dynamics with Gaussian noise, though the theory presented does not preclude more general models. This extends the work of Athans to consider a scenario where the number of interceptors is unknown and time-varying. In particular, the focus is on the development of a stochastic multi-object guidance law for simultaneous rendezvous and interception. The work is presented as an homage to Athans’ original work which was published nearly 50 years ago.

**Index Terms**—linear-quadratic control, guidance, multi-object estimation, point processes, interception.

## I. INTRODUCTION

There has been recent interest in the problem of controlling a number of interceptors for a co-ordinated and co-operative simultaneous attack on the same target by a number of missiles at the same time [1]–[4]. These systems are either given target information independently and they devise their own guidance separately, or develop co-operative strategies sharing information between missiles [5]–[8].

Athans proposed the rendezvous and interception problem of controlling the motion of  $N$  dynamical interceptors to the state of  $N$  dynamical targets, giving the example of the barrage of  $N$  targets [9]. Taking into account the uncertainty of the motions of both the interceptors and the targets, the initial allocation of interceptors and targets may become suboptimal over time and thus recalculation of the cost function may be required. Athans’ cost function involved evaluation over all permutations of the interceptors to targets, where the number of targets and number of interceptors are the same and known.

This paper develops a stochastic multi-object guidance approach where the number of interceptors is potentially unknown and time-varying. Hence, within the window, new interceptors may appear and can be guided to the target, and some targets may expire during the period. This paper explores an extension of Athans’ approach to introduce uncertainty in the number and states of controllable interceptors and potentially the number of targets. The approach is based on point process theory [10] which accounts for uncertainty in the number of objects and their state vectors.

Algorithms for detecting and tracking multiple targets have been developed since the 1970s, principally motivated by applications in aerospace for surveillance applications. Methods

based on Multiple Hypothesis Tracking [11], [12] and Joint Probabilistic Data Association [13] underpin such systems, as well as methods founded on point process theory [14]–[17]. Despite the widespread use of the Kalman filter coupled with the linear-quadratic regulator, this approach has not been applied for a general multi-target tracking problem. Some results have been demonstrated for a simple scenario where the number of targets and interceptors is known [9], and in other target-tracking related applications [18]–[23], though a systematic approach to a controlled multi-target problems has not yet been demonstrated for the linear-quadratic regulator (LQR). One application recently applied the LQR to a multi-target scenario [24], though did not exploit the classical analytic results. Recent work on multi-target control has typically focussed on developing information-theoretic approaches for point process models, including Rényi divergence [25], [26], Cauchy-Schwartz divergence [27], [28], and Kullback-Leibler divergence [29], as well as using the second-order statistics of point processes to enable decision-making [30], [31].

Most multi-target tracking algorithms, eg. [32]–[34] assume that the each target state or hypothesised target is described with a Gaussian. We shall exploit this constraint to determine analytic forms for stochastic multi-target control applied to different multi-target representations. In particular, we show that when all of the targets attempt to minimise the same cost function, then the entire system of targets can be controlled by the same linear-quadratic gain matrix.

The paper is structured as follows: The next section summarises Athans’ stochastic rendez-vous problem [9] of guiding an interceptor to a target in the original continuous-time context. The discrete-time version is also presented which enables the connection to be made with multi-object tracking algorithms. The following section describes the stochastic rendezvous problem for guiding a known number of interceptors to a target and connects this to multi-object tracking algorithm implementations. Section IV extends the known many-to-one scenario to an unknown number of interceptors which are described by a point process. Section V describes the problem of pure interceptor birth, that is the problem of guiding newly incoming interceptors not visible at the current time to a target. Section VI describes the problem of guiding an unknown number of interceptors, which may expire in the time window. Section VII combines the results of section V and section VI to deal with the usual model in multi-object tracking, where there are births and deaths of targets. Section VIII discusses possible extensions for dealing with an unknown number of target interceptors and an unknown number of targets.

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## II. THE STOCHASTIC RENDEZVOUS PROBLEM

This section describes Athans' stochastic rendezvous problem [9] that underpins the approach taken in the paper. It is assumed that these are detected separately, and the interceptor is controlled towards the target.

### A. Target dynamic and observation models

1) *Target dynamics*: Suppose that the target state vector  $\mathbf{z}(t) \in \mathbb{R}^n$  satisfies the linear stochastic differential equations

$$\dot{\mathbf{z}}(t) = \mathbf{G}(t)\mathbf{z}(t) + \xi(t), \quad \mathbf{z}(t_0) = \mathbf{z}_0,$$

where the initial target state  $\mathbf{z}_0$  is a Gaussian random variable with known mean and covariance matrix  $\Sigma_0$ , and  $\mathbf{G}(t)$  is the target transition matrix. The process noise  $\xi(t)$  is assumed to be a Gaussian noise process with zero mean and known covariance matrix  $\Xi(t)$  which is independent of the target state. This can also be described in a discrete-time formulation where  $\mathbf{z}_k \in \mathbb{R}^n$  is the state of the target at time-step  $k$ ,  $\mathbf{G}_k$  is the transition matrix at time-step  $k$ . The Markov transition density describing the stochastic evolution of the target is described with

$$f_{k|k-1}^{\mathbf{z}}(\mathbf{z}_k|\mathbf{z}_{k-1}) = \mathcal{N}(\mathbf{z}_k; \mathbf{G}_{k-1}\hat{\mathbf{z}}_{k-1}, \Xi_{k-1}),$$

where  $\mathcal{N}(\cdot; m, P)$  denotes a Gaussian density with mean  $m$  and covariance  $P$ , where  $\Xi_k$  is the process noise at time-step  $k$ . The posterior density at time-step  $k-1$  is given by

$$p_{k-1}^{\mathbf{z}}(\mathbf{z}_{k-1}) = \mathcal{N}(\mathbf{z}_{k-1}; \hat{\mathbf{z}}_{k-1}, \Sigma_{k-1}),$$

where  $\hat{\mathbf{z}}_{k-1}$  is the mean of the posterior at time-step  $k-1$ . The predicted density from time-step  $k-1$  to time-step  $k$  is described by the Chapman-Kolmogorov equation, i.e.

$$p_{k|k-1}^{\mathbf{z}}(\mathbf{z}_k) = \int f_{k|k-1}^{\mathbf{z}}(\mathbf{z}_k|\mathbf{z}_{k-1}) p_{k-1}^{\mathbf{z}}(\mathbf{z}_{k-1}) d\mathbf{z}_{k-1}.$$

Since the dynamic and observation models are linear and Gaussian, this is described by the discrete-time Kalman filter prediction. We present this formulation since it will be useful to relate to the multi-target case in the following sections.

2) *Target observations*: The target state  $\mathbf{z}(t)$  is observed indirectly via a noisy sensor observation described with relation

$$\mathbf{w}(t) = \mathbf{H}(t)\mathbf{z}(t) + \gamma(t),$$

where the target measurement noise  $\gamma(t)$  is assumed to be a Gaussian noise process with zero mean and known covariance  $\Gamma(t)$  which is independent of the target state. In the discrete-time case, we describe this relation through a Gaussian likelihood function

$$\ell_k^{\mathbf{z}}(\mathbf{w}_k|\mathbf{z}_k) = \mathcal{N}(\mathbf{w}_k; \mathbf{H}_k\mathbf{z}_k, \Gamma_k),$$

where  $\mathbf{w}_k$  is the observation vector at time-step  $k$ ,  $\mathbf{H}_k$  is the observation matrix that projects the target state onto the observation space, and  $\Gamma_k$  is the observation noise covariance matrix. In the Bayesian formulation, the likelihood is used to update the predicted distribution, due to the properties of Gaussians, the posterior density is also Gaussian.

### B. Interceptor dynamic and observation models

1) *Interceptor dynamics*: Assume that the interceptor state vector  $\mathbf{x}(t) \in \mathbb{R}^n$ , and the interceptor control vector  $\mathbf{u}(t) \in \mathbb{R}^m$ , are related by means of the linear stochastic differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \phi(t); \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

The initial interceptor state  $\mathbf{x}_0$  is assumed to be a Gaussian random variable with known mean  $\hat{\mathbf{x}}_0$  and covariance  $\Phi_0$ . The interceptor driving noise  $\phi(t)$  is a vector-valued Gaussian noise process with zero mean and known covariance matrix  $\Phi(t)$ , and  $\phi(t)$  is independent of  $\mathbf{z}_0, \mathbf{x}_0, \mathbf{z}(t)$ , and  $\gamma(t)$ . The additive noise term  $\phi(t)$  models uncertainties in the interceptor dynamical model.

In the discrete-time formulation, we consider the state of the target  $\mathbf{x}_k \in \mathbb{R}^n$  at time-step  $k$ , which evolves according to state transition matrix  $\mathbf{A}_k$ . The posterior density at time-step  $k-1$  is assumed to be Gaussian, i.e.

$$p_{k-1}^{\mathbf{x}}(\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1}, \Pi_{k-1})$$

is the Gaussian posterior density, with mean  $\hat{\mathbf{x}}_{k-1}$ , and covariance  $\Phi_{k-1}$ . The state evolves according to Markov transition  $f_{k|k-1}^{\mathbf{x}}(\mathbf{x}_k|\mathbf{u}_{k-1}, \mathbf{x}_{k-1})$ , which is equal to

$$\mathcal{N}(\mathbf{x}_k; \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1}, \Phi_{k-1}),$$

which includes the control vector  $\mathbf{u}_{k-1}$ . The predicted density then becomes

$$p_{k|k-1}^{\mathbf{x}}(\mathbf{x}_k) = \int f_{k|k-1}^{\mathbf{x}}(\mathbf{x}_k|\mathbf{u}_{k-1}\mathbf{x}_{k-1}) p_{k-1}^{\mathbf{x}}(\mathbf{x}_{k-1}) d\mathbf{x}_{k-1}.$$

2) *Interceptor observations*: The interceptor state  $\mathbf{x}(t)$  is observed via the interceptor data vector  $\mathbf{y}(t) \in \mathbb{R}^q$ , defined with

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \theta(t),$$

where  $\theta(t)$  is the interceptor Gaussian measurement noise.

In the discrete-time case, we describe this relation through a Gaussian likelihood function

$$\ell_k^{\mathbf{x}}(\mathbf{y}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{y}_k; \mathbf{H}_k\mathbf{x}_k, \Theta_k),$$

where  $\mathbf{y}_k$  is the observation vector at time-step  $k$ ,  $\mathbf{H}_k$  is the observation matrix that projects the target state onto the observation space, and  $\Theta_k$  is the observation noise covariance matrix.

### C. Dynamics of the Error Vector

Since both  $\mathbf{z}(t)$  and  $\mathbf{x}(t)$  have the same dimension, the error vector  $\mathbf{e}(t)$  is can be described with the difference between  $\mathbf{z}(t)$  and  $\mathbf{x}(t)$ , i.e.

$$\mathbf{e}(t) \triangleq \mathbf{z}(t) - \mathbf{x}(t); \quad \mathbf{e}_k \triangleq \mathbf{z}_k - \mathbf{x}_k.$$

The error vector  $\mathbf{e}(t)$  satisfies the differential equation

$$\dot{\mathbf{e}}(t) = \mathbf{A}(t)\mathbf{e}(t) - \mathbf{B}(t)\mathbf{u}(t) + [\mathbf{G}(t) - \mathbf{A}(t)]\mathbf{z}(t),$$

with the initial condition

$$\mathbf{e}_0 = \mathbf{z}_0 - \mathbf{x}_0.$$

#### D. The Target Kalman Filter

The minimum variance estimate  $\hat{\mathbf{z}}(t)$  of the target state  $\mathbf{z}(t)$  is generated by the target Kalman filter,

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{z}}(t) &= [\mathbf{G}(t) - \Sigma(t)\mathbf{H}^\top(t)\Gamma^{-1}(t)]\hat{\mathbf{z}}(t) \\ &\quad + \Sigma(t)\mathbf{H}(t)\Gamma^{-1}(t)\mathbf{w}(t); \hat{\mathbf{z}}(t_0) = \hat{\mathbf{z}}_0, \end{aligned}$$

where the superscript  $\top$  denotes the transpose,  $\Sigma(t)$  is the covariance matrix of the target estimation error  $\hat{\mathbf{z}}(t) - \mathbf{z}(t)$  and is the solution of the matrix Riccati differential equation

$$\begin{aligned} \dot{\Sigma}(t) &= \mathbf{G}(t)\Sigma(t) + \Sigma(t)\mathbf{G}^\top(t) + \Xi(t) \\ &\quad - \Sigma(t)\mathbf{H}^\top(t)\Gamma^{-1}(t)\mathbf{H}(t)\Sigma(t); \Sigma(t_0) = \Sigma_0. \end{aligned}$$

The prediction step from time-step  $k-1$  to time-step  $k$  for the target is described with the following equations.

$$\begin{aligned} \hat{\mathbf{z}}_{k|k-1} &= \mathbf{G}_{k-1}\hat{\mathbf{z}}_{k-1}, \\ \Sigma_{k|k-1} &= \mathbf{G}_{k-1}\Sigma_{k-1}\mathbf{G}_{k-1}^\top + \Xi_{k-1}, \end{aligned}$$

where  $\hat{\mathbf{z}}_{k|k-1}$  and  $\Sigma_{k|k-1}$  are the discrete-time predicted mean and covariance respectively,  $\mathbf{G}_{k-1}$  is the state transition matrix,  $\Sigma_{k-1}$  is the covariance at time-step  $k-1$ , and  $\Xi_{k-1}$  is the process noise covariance. When measurement  $\mathbf{w}_k$  is received at time-step  $k$ , the update equations are

$$\begin{aligned} \hat{\mathbf{z}}_k &= \hat{\mathbf{z}}_{k|k-1} + \mathbf{U}_k (\mathbf{w}_k - \mathbf{H}_k\hat{\mathbf{z}}_{k|k-1}), \\ \Sigma_k &= (\mathbf{I} - \mathbf{U}_k\mathbf{H}_k)\Sigma_{k|k-1}, \\ \mathbf{U}_k &= \Xi_{k|k-1}\mathbf{H}_k^\top (\mathbf{H}_k\Xi_{k|k-1}\mathbf{H}_k^\top + \Gamma_k)^{-1}, \end{aligned}$$

where  $\mathbf{H}_k$  is the observation matrix,  $\Gamma_k$  is the observation noise covariance, and  $\mathbf{U}_k$  is the Kalman gain matrix.

#### E. The Interceptor Kalman Filter

Similarly, the minimum variance estimate  $\hat{\mathbf{x}}(t)$  of the interceptor state  $\mathbf{x}(t)$  is generated by the interceptor Kalman filter specified below where  $\Pi(t)$  is the covariance matrix of the interceptor estimation error  $\mathbf{x}(t) - \hat{\mathbf{x}}(t)$  and is the solution of the matrix Riccati differential equation

$$\begin{aligned} \frac{d}{dt}\Pi(t) &= \mathbf{A}(t)\Pi(t) + \Pi(t)\mathbf{A}^\top(t) + \Phi(t), \\ &\quad - \Pi(t)\mathbf{C}^\top(t)\Theta^{-1}(t)\mathbf{C}(t)\Pi(t), \\ \Pi(t_0) &= \Pi_0. \end{aligned}$$

The discrete-time interceptor Kalman filter is computed with

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1} &= \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1}, \\ \Pi_{k|k-1} &= \mathbf{A}_{k-1}\Pi_{k-1}\mathbf{A}_{k-1}^\top + \Phi_{k-1}, \end{aligned}$$

where  $\hat{\mathbf{x}}_{k|k-1}$  and  $\Pi_{k|k-1}$  are the predicted mean and covariance respectively,  $\mathbf{G}_{k-1}$  is the state transition matrix,  $\mathbf{A}_{k-1}$  is the covariance at time-step  $k-1$ , and  $\Phi_{k-1}$  is the process noise covariance. When measurement  $\mathbf{y}_k$  is received at time-step  $k$ , the update equations are

$$\begin{aligned} \hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{V}_k (\mathbf{y}_k - \mathbf{C}_k\hat{\mathbf{x}}_{k|k-1}), \\ \Pi_k &= (\mathbf{I} - \mathbf{V}_k\mathbf{C}_k)\Pi_{k|k-1}, \\ \mathbf{V}_k &= \Pi_{k|k-1}\mathbf{C}_k^\top (\mathbf{C}_k\Pi_{k|k-1}\mathbf{C}_k^\top + \Theta_k)^{-1}, \end{aligned}$$

where  $\mathbf{C}_k$  is the observation matrix,  $\Theta_k$  is the observation noise covariance, and  $\mathbf{V}_k$  is the Kalman gain matrix.

#### F. The Continuous-time Stochastic Control

Suppose that at time  $t \in [t_0, \top]$ , we have available the target state estimate  $\hat{\mathbf{z}}(t)$  and the interceptor state estimate  $\hat{\mathbf{x}}(t)$ . Then the minimum value of the *cost to go*  $\tilde{J}(t)$  is given by

$$\begin{aligned} \tilde{J}(t) &\triangleq \mathbb{E} \left[ \frac{1}{2}\mathbf{e}^\top(T)\mathbf{M}\mathbf{e}(T) + \frac{1}{2}\int_t^T \mathbf{u}^\top(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)d\tau \right] \\ &= \frac{1}{2}\hat{\mathbf{x}}^\top(t)\mathbf{K}(t)\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}^\top(t)\mathbf{S}(t)\hat{\mathbf{z}}(t) \\ &\quad + \frac{1}{2}\hat{\mathbf{z}}^\top(t)\mathbf{P}(t)\hat{\mathbf{z}}(t) + \beta(t), \end{aligned}$$

where the scalar  $\beta(t)$  is independent of  $\hat{\mathbf{x}}(t)$  and  $\hat{\mathbf{z}}(t)$ , and depends on the error covariance matrices  $\Sigma(t)$  and  $\Pi(t)$ .

The optimal interceptor control exists, is unique, and is given in feedback form by

$$\mathbf{u}^*(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)[\mathbf{S}(t)\mathbf{z}(t) - \mathbf{K}(t)\mathbf{x}(t)].$$

The matrices  $\mathbf{K}(t)$ ,  $\mathbf{S}(t)$ , and  $\mathbf{P}(t)$  are determined with the solutions to the following matrix Riccati nonlinear differential equations

$$\begin{aligned} \dot{\mathbf{K}}(t) &= -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^\top(t)\mathbf{K}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^\top(t)\mathbf{K}(t), \\ \mathbf{K}(T) &= \mathbf{M}, \\ \dot{\mathbf{S}}(t) &= -\mathbf{S}(t)\mathbf{G}(t) - \mathbf{A}^\top(t)\mathbf{S}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^\top(t)\mathbf{S}(t), \\ \mathbf{S}(T) &= \mathbf{M}, \\ \dot{\mathbf{P}}(t) &= -\mathbf{S}(t)\mathbf{G}(t) - \mathbf{G}^\top(t)\mathbf{P}(t) + \mathbf{S}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^\top(t)\mathbf{S}(t), \\ \mathbf{P}(T) &= \mathbf{M}. \end{aligned}$$

#### G. The Discrete-time Stochastic Control

The discrete-time rendezvous interceptor control is computed with [20],

$$\tilde{J}(t) = \mathbb{E} \left[ \mathbf{e}_K^\top \mathbf{M} \mathbf{e}_K + \sum_{k=0}^K \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k \right].$$

The optimal interceptor control for the discrete-time filter is given by

$$\mathbf{u}_k = \mathbf{R}_k^{-1}\mathbf{B}_k^\top[\mathbf{S}_{k+1}\hat{\mathbf{z}}_k - \mathbf{K}_{k+1}\hat{\mathbf{x}}_k],$$

where the matrices  $\mathbf{K}_k$  and  $\mathbf{S}_k$  are given by

$$\begin{aligned} \mathbf{K}_k &= -\mathbf{K}_{k+1}\mathbf{A}_k - \mathbf{A}_k^\top\mathbf{K}_{k+1} + \mathbf{K}_{k+1}\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^\top\mathbf{K}_{k+1}, \\ \mathbf{K}_K &= \mathbf{M}, \\ \mathbf{S}_k &= -\mathbf{S}_{k+1}\mathbf{G}_k - \mathbf{A}_k^\top\mathbf{S}_{k+1} + \mathbf{K}_k\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^\top\mathbf{S}_{k+1}, \\ \mathbf{S}_K &= \mathbf{M}, \\ \mathbf{P}_k &= -\mathbf{S}_{k+1}\mathbf{G}_k - \mathbf{G}_k^\top\mathbf{P}_{k+1} + \mathbf{S}_k\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^\top\mathbf{S}_{k+1}, \\ \mathbf{P}_K &= \mathbf{M}. \end{aligned}$$

These are the matrix Riccati difference equations that run back in time obtained by the discrete-time linear quadratic regulator. In the following sections, these will be used to determine the controls of a number of interceptors.

### III. THE KNOWN MANY-TO-ONE STOCHASTIC RENDEZVOUS PROBLEM

In this section we assume that  $N$  interceptors are to be guided toward a dynamic target. Let  $i$  index the interceptor,  $i = 1, 2, \dots, N$ , where  $N$  is known. This is the same approach as the many-to-many approach of Athans [9], though each interceptor is controlled to the same target. This is presented in anticipation of the unknown many-to-one approach to be taken in the following sections, where these concepts will be generalised to accommodate an unknown number of interceptors that are driven to a rendezvous with the same target.

#### A. Target models

We assume that there is a single target and that it is characterised as in the previous section. Hence, we assume that the target is characterized by the following equations:

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{G}(t)\mathbf{z}(t) + \xi(t), & \mathbf{z}(t_0) &= \mathbf{z}_0, \\ \mathbf{w}(t) &= \mathbf{H}(t)\mathbf{z}(t) + \gamma(t).\end{aligned}$$

#### B. Interceptor models

We assume that there are  $N$  interceptors, and assume that each interceptor, indexed by  $i$ , is characterized by the following equations:

$$\begin{aligned}\dot{\mathbf{x}}^{(i)}(t) &= \mathbf{A}(t)\mathbf{x}^{(i)}(t) + \mathbf{B}(t)\mathbf{u}^{(i)}(t) + \phi^{(i)}(t); & \mathbf{x}^{(i)}(t_0) &= \mathbf{x}_0^{(i)}, \\ \mathbf{y}^{(i)}(t) &= \mathbf{C}(t)\mathbf{x}^{(i)}(t) + \theta^{(i)}(t).\end{aligned}$$

We also assume that  $\mathbf{x}_0^{(i)}$  is Gaussian with

$$\begin{aligned}\mathbb{E}[\mathbf{x}_0^{(i)}] &= \hat{\mathbf{x}}_0^{(i)}, \\ \text{cov}[\mathbf{x}_0^{(i)} \mathbf{x}_0^{(i)\top}] &= \Pi_0,\end{aligned}$$

for all  $i = 1, 2, \dots, N$  that  $\phi^{(i)}(t)$  and  $\theta^{(i)}(t)$  are mutually independent Gaussian noise processes, both of zero mean. All of the interceptors are assumed to be identical and observed by the same sensor (or a sensor with identical characteristics). It is assumed that any false alarms on the sensor do not affect the observations of the targets.

#### C. The target state estimates

As in the previous section, the target state estimates are generated by the following differential equation:

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{z}}(t) &= [\mathbf{G}(t) - \Sigma(t)\mathbf{H}^\top(t)\Gamma^{-1}(t)]\hat{\mathbf{z}}(t) \\ &+ \Sigma(t)\mathbf{H}^\top(t)\Gamma^{-1}(t)\mathbf{w}(t).\end{aligned}$$

#### D. Interceptor state estimates

There are  $N$  interceptor Kalman filters, and the interceptor state estimates are generated by

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{x}}^{(i)}(t) &= [\mathbf{A}(t) - \Pi(t)\mathbf{C}^\top(t)\Theta^{-1}(t)\mathbf{C}(t)]\hat{\mathbf{x}}^{(i)}(t) \\ &+ \mathbf{B}(t)\mathbf{u}^{(i)}(t) + \Pi(t)\mathbf{C}^\top(t)\mathbf{S}^{-1}(t)\mathbf{C}(t)\mathbf{y}^{(i)}(t).\end{aligned}$$

Again, our assumptions on the interceptors imply that the gain matrices in the interceptor Kalman filters are identical in particular, the interceptor error covariance matrix  $\Pi(t)$  is the same for each interceptor.

#### E. Discrete-time Kalman filter for the interceptors

The discrete-time interceptor Kalman filter of the  $i^{\text{th}}$  interceptor is computed with the predicted mean and covariance,

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1}^{(i)} &= \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1}^{(i)} + \mathbf{B}_{k-1}\mathbf{u}_{k-1}^{(i)}; \\ \Pi_{k|k-1}^{(i)} &= \mathbf{A}_{k-1}\Pi_{k-1}^{(i)}\mathbf{A}_{k-1}^\top + \Phi_{k-1};\end{aligned}$$

and updated mean, covariance, and Kalman gain.

$$\begin{aligned}\hat{\mathbf{x}}_k^{(i)} &= \hat{\mathbf{x}}_{k|k-1}^{(i)} + \mathbf{V}_k \left( \mathbf{y}_k^{(i)} - \mathbf{C}_k\hat{\mathbf{x}}_{k|k-1}^{(i)} \right); \\ \Pi_k^{(i)} &= \left( \mathbf{I} - \mathbf{V}_k^\top \mathbf{C}_k \right) \Pi_{k|k-1}^{(i)}; \\ \mathbf{V}_k^{(i)} &= \Pi_{k|k-1}^{(i)} \mathbf{C}_k^\top \left( \mathbf{C}_k \Pi_{k|k-1}^{(i)} \mathbf{C}_k^\top + \Theta_k \right)^{-1}.\end{aligned}$$

#### F. The optimal guidance of each interceptor

We are interested in minimizing the expected cost to go for each interceptor:

$$\mathbb{E} \left[ \frac{1}{2} \mathbf{e}^{(i)\top}(T) \mathbf{M} \mathbf{e}^{(i)}(T) + \frac{1}{2} \int_t^T \mathbf{u}^{(i)\top}(\tau) \mathbf{R}(\tau) \mathbf{u}^{(i)}(\tau) d\tau \right].$$

The optimal control for the  $i^{\text{th}}$  interceptor is given by

$$\mathbf{u}^{(i)}(t) = \mathbf{R}^{-1}(t) \mathbf{B}^\top(t) \left[ \mathbf{S}(t) \hat{\mathbf{z}}(t) - \mathbf{K}(t) \hat{\mathbf{x}}^{(i)}(t) \right],$$

where  $\mathbf{K}(t)$ ,  $\mathbf{S}(t)$ , and  $\mathbf{P}(t)$  are independent of the interceptor.

#### G. Discrete-time control

The optimal interceptor control for the  $i^{\text{th}}$  discrete-time interceptor filter is given by

$$\mathbf{u}_k^{(i)} = \mathbf{R}_k^{-1} \mathbf{B}_k^\top [\mathbf{S}_{k+1} \hat{\mathbf{z}}_k - \mathbf{K}_{k+1} \hat{\mathbf{x}}_k^{(i)}].$$

where the matrices  $\mathbf{K}_k$  and  $\mathbf{S}_k$  for each filter. Note here that the calculation of the control  $\mathbf{u}_k^{(i)}$  of the  $i^{\text{th}}$  interceptor requires knowledge of  $\mathbf{R}_k$ ,  $\mathbf{B}_k$ ,  $\mathbf{S}_{k+1}$ ,  $\mathbf{K}_{k+1}$ , which are the same for each interceptor,  $\hat{\mathbf{z}}_k$ , which is the estimated state of the target, and  $\hat{\mathbf{x}}_k^{(i)}$ , the estimated state of the interceptor. Thus, if  $\mathbf{R}_k$ ,  $\mathbf{B}_k$ ,  $\mathbf{S}_{k+1}$ ,  $\mathbf{K}_{k+1}$ , and  $\hat{\mathbf{z}}_k$  were broadcast to each interceptor, then each interceptor can independently calculate their own required control vector.

#### H. The Global Cost Function

The overall cost functional  $\tilde{J}(t)$  is given by

$$\mathbb{E} \left[ \sum_{i=1}^N \frac{1}{2} \mathbf{e}^{(i)\top}(T) \mathbf{M} \mathbf{e}^{(i)}(T) + \frac{1}{2} \int_t^T \mathbf{u}^{(i)\top}(\tau) \mathbf{R}(\tau) \mathbf{u}^{(i)}(\tau) d\tau \right],$$

which, due to independence of the trajectories, is equal to

$$\sum_{i=1}^N \mathbb{E} \left[ \frac{1}{2} \mathbf{e}^{(i)\top}(T) \mathbf{M} \mathbf{e}^{(i)}(T) + \frac{1}{2} \int_t^T \mathbf{u}^{(i)\top}(\tau) \mathbf{R}(\tau) \mathbf{u}^{(i)}(\tau) d\tau \right].$$

In the following section, this approach is extended to a scenario where  $N$  is not necessarily known. Hence, the number of interceptors may not be known to the controller. This is possible by modelling the population of objects as a point process.

#### IV. THE UNKNOWN MANY-TO-ONE STOCHASTIC RENDEZVOUS PROBLEM WITH A FIXED NUMBER OF INTERCEPTORS

In this section we assume that there are a finite but unknown number of interceptors described by a set of target states  $\varphi = \{x_1, \dots, x_N\}$  that are to be guided toward a dynamic target. This extends the scenario described in the previous section to account for an unknown number of targets. In this section, we assume that the number of interceptors does not change in the time period  $[t, T]$ . This scenario has not previously been considered.

##### A. The cost function and optimal control

**Theorem IV.1.** *The Global Cost Function: Assume that we have a finite set of interceptors described with random finite set  $\varphi = \{x_1, \dots, x_N\}$ , where  $\varphi$  is described with a point process. The overall cost functional  $\tilde{J}_\varphi(t)$  is given by*

$$\tilde{J}_\varphi(t) = \mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi} J_{\mathbf{x}}(t) \right] = \int_{\mathcal{X}} \tilde{J}_{\mathbf{x}}(t) \Lambda(d\mathbf{x}),$$

where  $\Lambda(d\mathbf{x})$  is the first-order moment measure of the point process describing the set of interceptors evolving from time  $t$  to time  $T$ ,

$$J_{\mathbf{x}}(t) = \frac{1}{2} \mathbf{e}^{\mathbf{x}^\top(T)} \mathbf{M} \mathbf{e}^{\mathbf{x}}(T) + \frac{1}{2} \int_t^T \mathbf{u}^{\mathbf{x}}(\tau)^\top \mathbf{R}(\tau) \mathbf{u}^{\mathbf{x}}(\tau) d\tau,$$

is the deterministic cost function for interceptor with state  $\mathbf{x}$ , and  $\tilde{J}_{\mathbf{x}}(t) = \mathbb{E}[J_{\mathbf{x}}(t)]$ .

*Proof.* The theorem is an application of Campbell's theorem, eg. [35, p103].  $\square$

**Corollary IV.2** (Mixture of Gaussian intensity function). *Suppose that  $\Lambda(d\mathbf{x})$  can be described with a mixture of Gaussian processes of the form*

$$\Lambda(d\mathbf{x}) = \sum_{j=1}^N w^{(j)} \mathcal{N}(d\mathbf{x}^{(j)}, \hat{\mathbf{x}}^{(j)}, \Pi^{(j)}),$$

evolving from time  $t$  to time  $T$ . Then the cost function is determined with

$$\sum_{j=1}^N w^{(j)} \mathbb{E} \left[ J^{(j)}(t) \right],$$

where  $J^{(j)}(t)$  is the cost function equal to

$$\frac{1}{2} \mathbf{e}^{(i)^\top(T)} \mathbf{M} \mathbf{e}^{(j)}(T) + \frac{1}{2} \int_t^T \mathbf{u}^{(j)}(\tau)^\top \mathbf{R}(\tau) \mathbf{u}^{(j)}(\tau) d\tau.$$

Hence, the control for  $j^{\text{th}}$  term in the summation is computed with

$$\mathbf{u}^{(j)}(t) = \mathbf{R}^{-1}(t) \mathbf{B}^\top(t) \left[ \mathbf{S}(t) \hat{\mathbf{z}}(t) - \mathbf{K}(t) \hat{\mathbf{x}}^{(j)}(t) \right].$$

*Proof.* This follows directly by considering each term in the summation and its control trajectory.  $\square$

Note that mathematically there is very little difference between the known many-to-one stochastic rendezvous problem

and this scenario: we use the same control calculations. However, conceptually, there is a significant difference here. This Theorem and its Corollary show that we do not need to know how many interceptors there are in order to control them. Just as in the calculation of the control  $\mathbf{u}_k^{(i)}$  of the  $i^{\text{th}}$  interceptor in the preceding section, the computation of the control for each Gaussian requires knowledge of  $\mathbf{R}_k, \mathbf{B}_k, \mathbf{S}_{k+1}, \mathbf{K}_{k+1}$ , which are the same for each Gaussian,  $\hat{\mathbf{z}}_k$ , which is the estimated state of the target, and  $\hat{\mathbf{x}}_k^{(i)}$ , the estimated state of the  $i^{\text{th}}$  Gaussian. Thus, if  $\mathbf{R}_k, \mathbf{B}_k, \mathbf{S}_{k+1}, \mathbf{K}_{k+1}$ , and  $\hat{\mathbf{z}}_k$  were broadcast then each interceptor can independently calculate their own required control vector irrespective of whether the operator knows whether the interceptor exists or not.

##### B. Application to multi-target tracking models

To demonstrate the approach for multi-target tracking algorithms, we consider the different parametrisations used to represent them in a point process parametric.

1) *Multiple hypothesis process:* Consider  $M$  possible hypotheses, where for hypothesis  $j$  there are  $N^{(j)}$  known targets. Suppose that hypothesis  $j$  has hypothesis weight  $w^{(j)}$ , such that  $\sum_{j=1}^M w^{(j)} = 1$  and conditioned on a particular hypothesis  $j$ , there are  $N^{(j)}$  independent Gaussians. This is the case with Reid's multiple hypothesis tracking algorithm [34]. The intensity function for this example averages the intensities according to the hypothesis weights, i.e.

$$\Lambda(\mathbf{x}) = \sum_{j=1}^M w^{(j)} \sum_{i=1}^{N^{(j)}} \mathcal{N}(\mathbf{x}; m^{(i,j)}, P^{(i,j)}).$$

2) *Bernoulli process:* Now consider a process which has at most one object, where the existence probability is  $p$ . Then the intensity becomes

$$\Lambda(\mathbf{x}) = p \mathcal{N}(\mathbf{x}; m, P).$$

This was used in Musicki's IPDA filter [36].

3) *Poisson-binomial process:* The Poisson-binomial process [37]–[40] is the process describing the superposition of a given number of Bernoulli distributions. This is of particular interest in multi-target tracking applications since it is able to describe a set of target tracks and their existence probabilities. As seen in the  $N$ -object process, the intensity function is the sum over the independent intensities, so that

$$\Lambda(\mathbf{x}) = \sum_{i=1}^N p^{(i)} \mathcal{N}(\mathbf{x}; m^{(i)}, P^{(i)}).$$

4) *Mixture of Poisson-binomial processes:* Following the approach for multiple hypotheses, we can describe a similar process with  $M$  Poisson-binomial process hypotheses. This is similar to the process used in the Generalized Labelled Multi-Bernoulli filter by Vo and Vo [32], and the Poisson multi-Bernoulli mixture filter by Williams [16]. The intensity function of the Gaussian density case becomes

$$\Lambda(\mathbf{x}) = \sum_{j=1}^M w^{(j)} \sum_{i=1}^{N^{(j)}} p^{(i,j)} \mathcal{N}(\mathbf{x}; m^{(i,j)}, P^{(i,j)}).$$

## V. THE UNKNOWN MANY-TO-ONE STOCHASTIC RENDEZVOUS PROBLEM WITH PURE INTERCEPTOR BIRTH

Here we determine the control for interceptors that appear in the time interval  $[t, T]$  according to a spatio-temporal point process. It is assumed that there are no current interceptors under surveillance, and hence it is called a *pure birth process*.

### A. The cost function and optimal control

**Theorem V.1** (Cost function for the birth process). *Let us suppose that interceptors appear in the time interval  $[t, T]$  according to a spatio-temporal point process with intensity measure  $\Lambda_b(d\mathbf{x}, dr)$ , where  $d\mathbf{x} \subset \mathcal{X}$  describes the spatial distribution of and  $dr \subset [t, T]$  describes the time window of interceptor appearance. Then the cost functional  $\tilde{J}(t)$  of ensuring that these interceptors rendezvous with the target is given by*

$$\mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi_b} J_{\mathbf{x}}(t) \right] = \int_{\mathcal{X} \times [t, T]} \tilde{J}_{\mathbf{x}}(r) \Lambda_b(d\mathbf{x}, dr),$$

where

$$J_{\mathbf{x}}(t) = \frac{1}{2} \mathbf{e}^{\mathbf{x}\top} \mathbf{M} \mathbf{e}^{\mathbf{x}} + \frac{1}{2} \int_t^T \mathbf{u}^{\mathbf{x}}(\tau) \top \mathbf{R}(\tau) \mathbf{u}^{\mathbf{x}}(\tau) d\tau$$

is the cost function of interceptors, and  $\tilde{J}_{\mathbf{x}}(t) = \mathbb{E} [J_{\mathbf{x}}(t)]$ .

*Proof.* This is a direct application of Campbell's theorem for a spatio-temporal point process.  $\square$

*Example 1* (Spatio-temporal Poisson birth process). *Suppose that the birth process is a spatio-temporal Poisson point process, homogeneous in time with rate  $\lambda_b$ , spatially distributed with a mixture of Gaussians, so that*

$$\Lambda_b(d\mathbf{x}, dt) = \lambda_b dt \sum_{j=1}^M w_b^{(j)} \mathcal{N}(d\mathbf{x}; \hat{\mathbf{x}}_b^{(j)}, \Sigma_b^{(j)}).$$

Hence the cost function becomes

$$\mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi_b} J_{\mathbf{x}}(t) \right] = \lambda_b (T - t) \sum_{j=1}^{N_b} w_b^{(j)} \int_t^T \tilde{J}_b^{(j)}(r) dr,$$

where, in a similar manner to the previous example,  $\tilde{J}_b^{(j)}(r)$  is the cost function associated with the  $j^{\text{th}}$  component, and  $\sum_{j=1}^{N_b} w_b^{(j)} = 1$ .

**Corollary V.2** (Discrete-time model). *Let the process be homogeneous in time with rate  $\lambda_{b,k} = \lambda_b(t_k - t_{k-1})$  between time-steps  $k-1$  and time-step  $k$  at time  $t_k$ , and that the spatial distribution is a mixture of Gaussians, so that*

$$\Lambda_{b,k}(d\mathbf{x}) = \lambda_{b,k} \sum_{j=1}^{N_b} w_b^{(j)} \mathcal{N}(d\mathbf{x}; \hat{\mathbf{x}}^{(j)}, \Sigma^{(j)}).$$

Then the cost function is computed with

$$\mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi_b} J_{\mathbf{x}}(t) \right] = \sum_{j=1}^{N_b} w_b^{(j)} \sum_{k=1}^K \lambda_{b,k} \int_{t_k}^T \tilde{J}_b^{(j)}(t) dt,$$

where  $\tilde{J}_b^{(j)}(t)$  is of the sum form in section II.G.

## VI. THE UNKNOWN MANY-TO-ONE STOCHASTIC RENDEZVOUS PROBLEM WITH INTERCEPTOR DEATH

In this section, we deal with the problem of interceptors that expire before the end of the time window. This is a common modelling scenario in multi-target tracking. More explicitly, these are controllable interceptors that existed at some point in the time window  $[t, T]$  but it is not guaranteed that they will be available to the controller at time  $T$ . The cost function is modified to account for this attrition.

### A. The cost function and optimal control

Consider the deterministic cost function

$$\frac{1}{2} \mathbf{e}^{\mathbf{x}\top} \mathbf{M} \mathbf{e}^{\mathbf{x}} \mathbf{1}_{\tau \leq r} + \frac{1}{2} \int_t^{\min(r, T)} \mathbf{u}^{\mathbf{x}}(\tau) \top \mathbf{R}(\tau) \mathbf{u}^{\mathbf{x}}(\tau) d\tau,$$

which indicates that the interceptor is controlled from time  $t$  to the expiry time  $r$  if  $r < T$ , or up to time  $T$  if it does not expire in the window  $[t, T]$ , i.e.  $r \geq T$ . For simplicity, let us assume that the probability of interceptor survival is independent of its state in space and is distributed according to an exponential distribution. Then, we can compute a cost function marginalising the cost with respect to the probability of object survival. For instance, suppose that the expected time to target expiration is exponential distribution with expected time  $1/\alpha$ , then

$$J_{\mathbf{x},c}(t) = \frac{1}{2} \mathbf{e}^{\mathbf{x}\top} \mathbf{M} \mathbf{e}^{\mathbf{x}} (1 - e^{-\alpha(T-t)}) + \frac{1}{2} \int_t^T \alpha e^{-\alpha\tau} \mathbf{u}^{\mathbf{x}}(\tau) \top \mathbf{R}(\tau) \mathbf{u}^{\mathbf{x}}(\tau) d\tau.$$

**Theorem VI.1.** *Assume that we have a finite set of interceptors described with random finite set  $\varphi = \{x_1, \dots, x_N\}$ , where  $\varphi$  is described with a point process. The overall cost functional  $\tilde{J}_{\varphi}(t)$  is given by*

$$\mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi} J_{\mathbf{x},c}(t) \right] = \int \tilde{J}_{\mathbf{x},c}(t) \Lambda(d\mathbf{x}),$$

where  $\Lambda(d\mathbf{x})$  is the first-order moment measure of the point process describing the set of interceptors.

*Proof.* The proof follows from Theorem IV.1.  $\square$

**Corollary VI.2** (Mixture of Gaussian intensity function). *Suppose that  $\Lambda(d\mathbf{x})$  has a density function  $\Lambda(\mathbf{x})$  that can be described with a mixture of Gaussian processes of the form*

$$\Lambda(\mathbf{x}) = \sum_{j=1}^N w^{(j)} \mathcal{N}(\mathbf{x}^{(j)}, \hat{\mathbf{x}}^{(j)}, \Pi^{(j)}),$$

evolving from time  $t$  to time  $T$ . Then the cost function is determined with

$$\lambda(T-t) \sum_{j=1}^N w^{(j)} \mathbb{E} \left[ J_c^{(j)}(t) \right].$$

*Proof.* This follows directly by considering each term in the summation and its control trajectory.  $\square$

## VII. THE UNKNOWN MANY-TO-ONE STOCHASTIC RENDEZVOUS PROBLEM WITH INTERCEPTOR BIRTH AND DEATH

This section combines the results of the previous two sections to develop a guidance approach with Poisson interceptor arrivals and exponential interceptor death. This is analogous to the  $M/M/\infty$  queueing model [41], [42], which models Poisson arrivals into an infinite number of servers (interceptors), each of which has an exponential service time (lifetime).

### A. The cost function

**Theorem VII.1** (Cost function for the process). *Let us suppose that interceptors appear in the time interval  $[t, T]$  according to a spatio-temporal point process with intensity measure  $\Lambda_b(d\mathbf{x}, dr)$ , where  $d\mathbf{x} \subset \mathcal{X}$  describes the spatial distribution of and  $dr \subset [t, T]$  describes the time window of interceptor appearance.*

$$\tilde{J}_\varphi(t) = \int_{\mathcal{X} \times [t, T]} \tilde{J}_{\mathbf{x}, c}(r) \Lambda_b(d\mathbf{x}, dr) + \int_{\mathcal{X}} \tilde{J}_{\mathbf{x}, c}(t) \Lambda(d\mathbf{x}).$$

*Proof.* The proof is an application of Theorem V.1 and Theorem VI.1. Since the birth process is independent of the surviving process, the result is just the summation of these two processes.  $\square$

### B. Application to multi-target tracking algorithms

The method can be applied directly to the multi-target tracking models to control a population of interceptors to a target. For example, we can use the Gaussian mixture model of Vo and Ma [43] as follows.

**Corollary VII.2** (Discrete-time Gaussian mixture model). *Consider the combination of models in sections V and VI, with Gaussian mixtures*

$$\Lambda_b(d\mathbf{x}, dt) = \lambda_b dt \sum_{j=1}^{N_b} w_b^{(j)} \mathcal{N}(d\mathbf{x}; \hat{\mathbf{x}}_b^{(j)}, \Sigma_b^{(j)}),$$

and

$$\Lambda(\mathbf{x}) = \lambda \sum_{j=1}^N w^{(j)} \mathcal{N}(\mathbf{x}^{(j)}, \hat{\mathbf{x}}^{(j)}, \Pi^{(j)}).$$

Then the cost function is computed with

$$\begin{aligned} \mathbb{E} \left[ \sum_{\mathbf{x} \in \varphi_b} J_{\mathbf{x}}(t) \right] &= \lambda(T-t) \sum_{j=1}^N w^{(j)} \int_t^T \tilde{J}_c^{(j)}(t) dt \\ &+ \sum_{j=1}^{N_b} w_b^{(j)} \sum_{k=1}^K \lambda_{b,k} \int_{t_k}^T \tilde{J}_{b,c}^{(j)}(t) dt. \end{aligned}$$

*Proof.* This is a direct application of Corollary V.2 and Corollary VI.2.  $\square$

The same approach can be applied to any multi-target tracking algorithm where the target population is described with the intensity function, eg. in section IV.B.

## VIII. THE UNKNOWN MANY-TO-UNKNOWN MANY STOCHASTIC RENDEZVOUS PROBLEM

Athans posed three challenges for the many-to-many interception problem where the number of targets and interceptors are known and the same: 1) *How to decide which interceptor to intercept which target?*; 2) *How to find the optimal allocation of interceptors to targets?*; 3) *How to guide the interceptors to minimize the cost functional?*. To solve this problem, he proposed to minimise the following cost function

$$J_{\varphi_{\mathbf{x}}, \varphi_{\mathbf{z}}}(t) = \min_{\pi \in S_N} \sum_{i=1}^N J_{\mathbf{x}_i, \mathbf{z}_{\pi(i)}}(t),$$

where  $S_N$  is the set of permutations of order  $N$ , and  $\varphi_{\mathbf{x}} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ , and  $\varphi_{\mathbf{z}} = \{\mathbf{z}_1, \dots, \mathbf{z}_N\}$  are the sets of interceptors and targets. In the unknown many-to-unknown many problem,  $\varphi_{\mathbf{x}}$  and  $\varphi_{\mathbf{z}}$  become random sets and the cardinalities of  $\varphi_{\mathbf{x}}$  and  $\varphi_{\mathbf{z}}$  may be different.

Strategies for resolving this problem could include:

1) **Make a hard decision about the number of targets and interceptors** and assign interceptors to targets. In the case where the number of interceptors is greater than the number of targets, the choice could be to determine the minimum cost for the available interceptors and proceed as in the known case. In the case where the number of interceptors is less than the number of targets, then the cost function could involve a penalty for not intercepting targets. The OSPA metric is a multi-target miss distance metric that was designed to account for penalties in mismatched cardinalities and could be adapted for this purpose.

Let two sets of points be denoted  $\varphi_{\mathbf{x}} = \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$  and  $\varphi_{\mathbf{z}} = \{\mathbf{z}_1, \dots, \mathbf{z}_n\}$  with  $m, n \in \mathbb{N}$ ,  $S_k$  denotes the set of permutations on  $\{1, \dots, k\}$ ,  $k \in \mathbb{N}$ . The order parameter is constrained as  $0 < p < \infty$ , and the cut-off parameter is constrained as  $c > 0$ . For  $m \leq n$ , the OSPA distance is given by considering a combination of localisation and cardinality error [44]. Note that  $J_{\mathbf{x}_i, \mathbf{y}_{\pi(i)}}$  is a quadratic distance function, so that we can determine an OSPA distance with

$$J_{\varphi_{\mathbf{x}}, \varphi_{\mathbf{z}}}(t) = \left[ \frac{1}{n} \left( \min_{\pi \in S_n} \sum_{i=1}^m J_{\mathbf{x}_i, \mathbf{z}_{\pi(i)}}(t) + c^2(n-m) \right) \right]^{\frac{1}{2}},$$

where  $J_{\mathbf{x}, \mathbf{z}}(t)$  denotes the distance between the points  $\mathbf{x}$  and  $\mathbf{z}$ , subject to the maximum cut-off  $c$

2) **Design a fully stochastic solution.** To determine a fully stochastic solution to the problem, the obvious cost function would be the evaluation of the expected cost function, i.e.

$$\hat{J}_{\varphi_{\mathbf{x}}, \varphi_{\mathbf{z}}}(t) = \mathbb{E} [J_{\varphi_{\mathbf{x}}, \varphi_{\mathbf{z}}}(t)].$$

However, this is likely to be costly to evaluate in practice. We leave the determination of a stochastic solution to this problem to future study.

3) **Design an information-theoretic solution.** The final suggestion would be to develop a solution based on information divergences, eg. [25], [26], Cauchy-Schwartz divergence [27], [28], and Kullback-Leibler divergence [29]. One recent approach was developed by Doerr and Lineres [24] for guiding a swarm of interceptors to a fixed number of targets.



## IX. CONCLUSION

A stochastic multi-object guidance law for simultaneous rendezvous and interception of a dynamic target by an unknown and time-varying number of targets is proposed based on Athans' linear-quadratic Gaussian guidance laws for interception and rendezvous. The optimal control is determined by applying Campbell's theorem from point process theory.

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