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Path large deviations for stochastic evolutions driven by the square of a Gaussian process.

Freddy Bouchet, Roger Tribe and Oleg Zaboronski

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Abstract

Many dynamics are random processes with increments given by a quadratic form of a fast Gaussian process. We find that the rate function which describes path large deviations can be computed from the large interval asymptotic of a certain Fredholm determinant. The latter can be evaluated explicitly using Widom's theorem which generalizes the celebrated Szego-Kac formula to the multi-dimensional case. This provides a large class of dynamics with explicit path large deviation functionals. Inspired by problems in hydrodynamics and atmosphere dynamics, we present the simplest example of the emergence of metastability for such a process.

1 Introduction

Large deviation theory recently became a key theoretical tool for the statistical mechanics of non equilibrium systems. Describing path large deviations for the dynamics of effective degrees of freedom leads to a precise understanding of typical and rare trajectories of physical, biological or economic processes. A paradigm example for the effective descriptions of complex systems using large deviation theory is the macroscopic fluctuation theory of systems of interacting particles [1]. However, for genuine non-equilibrium processes, without local detailed balance, the class of systems for which the Hamiltonian for path large deviations can be described explicitly is extremely limited.

In this paper, we consider a class of systems for which the effective dynamics has increments which are given by a quadratic form of a fast Gaussian process. This strong hypothesis is relevant for many applications. Quadratic interactions are common in many physical examples: in hydrodynamics, plasma described by the Vlasov equation, magneto hydrodynamics, self gravitating systems, the KPZ equation, the physics described by quadratic networks (for instance heat transfer among quadratic networks [2]), to cite just a few. In all these systems with quadratic nonlinearities, in some regime a separation of time scale exists and the effective degrees of freedom are coupled to fast evolving Gaussian processes. This is the case, for example, for the kinetic theories of plasma [3, 4], self gravitating systems [5], geostrophic turbulence [6], wave turbulence [7] for some specific dispersion relations, among many other examples. From a theoretical and mathematical perspective, the hypothesis that effective degrees of freedom are

driven by a quadratic form of a fast Gaussian process is a decisive simplification. With this assumption, we will be able to write explicit formulas for the path large deviation Hamiltonian, and proceed to its analysis in many interesting examples.

The study of a slow process coupled to a fast one is a classical paradigm of physics and mathematics. For such fast/slow dynamics, one can study the averaging of the effect of the fast variable on the slow one (law of large numbers), or the typical fluctuations (stochastic averaging [8]), or the rare fluctuations described by large deviation theory [9]. The large deviation theory has been developed for slow/fast Markov processes [10, 11] or deterministic systems [12, 13]. We do not know however many examples of large deviation studies for which the fast process is not a Markov process. We stress that the fast Gaussian processes we use in this paper are not necessarily Markov.

The current paper can be considered as a continuation of the study carried out in [14], of the large deviation principles for systems with two significantly different time scales, when the drift for a slow process is given by a second degree polynomial of the fast process. However, in this new paper we consider a larger class of systems, for which the fast dynamics is not necessarily a Markov process. The main new contribution of the current work is to apply the asymptotic theory of Fredholm determinants to the calculation of the large deviation rate function. The main result is an explicit formula for the Hamiltonian which characterizes path large deviations for the slow process, which is valid for both Markov and non-Markov fast processes. Using those explicit formulas we can study a simple example of bistability motivated by hydrodynamic applications [15, 16].

We start with the definition of the model in Section 2 and give a heuristic derivation of the corresponding large deviation principle in Section 3. The highlight of this section is the application of Widom's theorem for the asymptotics of Fredholm determinants to the calculation of the rate function. In Section 4 we show the emergence of metastability for a particular representative of our class of models and study the corresponding 'instanton' trajectories. Brief conclusions are presented in Section 5. The Appendices A and B contain some technical derivations for Section 3. Appendix C contains a review of Widom's theorem.

2 Slow dynamics quadratically driven by a fast Gaussian process

Consider the following stochastic model:

$$\begin{cases} \dot{X}(t) &= Y^T\left(\frac{t}{\epsilon}, X(t)\right) M Y\left(\frac{t}{\epsilon}, X(t)\right) - \nu X(t), \\ X(0) &= x_0, \end{cases} \quad (1)$$

where $\{X(t)\}_{t \geq 0}$ is an \mathbb{R}^n -valued random process, ϵ is a parameter, which determines time scale separation between the processes X and Y , $0 < \epsilon \ll 1$; for a fixed $x \in \mathbb{R}^n$, $Y(t, x)$ is an N -dimensional stationary centred Gaussian process with covariance $C(\tau, x)$,

$$\mathbb{E}[Y_i(t, x) Y_j(t + \tau, y)] = C_{ij}(\tau, x, y), \quad \tau \geq 0, 1 \leq i, j \leq N, \quad (2)$$

which is assumed to be continuous in all the arguments τ, x, y for some of the heuristic arguments below to be rigourisable. As we will see, only $C(\tau, x, x)$ enters the final expression for the large deviation rate function, which justifies our shorthand notation $C(\tau, x) := C(\tau, x, x)$. Finally, M is a $n \times N \times N$ matrix, symmetric with respect to the permutation of the last two indices, and $\nu > 0$ is a parameter. Notice that the (X, Y) process need not be Markov.

We assume that $C(\tau, x)$ decays sufficiently fast (e. g. exponentially) with τ , perhaps uniformly with respect to x . Then, in the limit of $\epsilon \rightarrow 0$, the slow random process $X(t)$ stays near the solution to the deterministic equation

$$\begin{cases} dx(t) &= \text{Tr}[MC(0, x(t))] - \nu x(t), \\ x(0) &= x_0, \end{cases} \quad (3)$$

where (Tr) is the trace over N 'fast' indices. The *typical* fluctuations of $X(t)$ around $x(t)$ are Gaussian, with covariance of order ϵ (more precisely, the distribution of $\lim_{\epsilon \rightarrow 0} \frac{X(t) - x(t)}{\sqrt{\epsilon}}$ is centred Gaussian). Here we are interested in the statistics of *large deviations* of $X(t)$ when $X(t) - x(t) = O(1)$, which are no longer Gaussian.

3 Large deviation principle for paths of the slow process

The following computation is not a proof, just a heuristic argument devised to give an intuitive feel for the final form of the large deviation principle. We feel that its inclusion is necessary, due to the difference between our model and Markov models with time scale separation, for which the large deviation principle is known, see [14] for references: all we require of the fast process Y is to be mixing.

Let us fix time $t > 0$, choose a large integer $P \in \mathbb{N}$ and define

$$\Delta t = \frac{t}{P}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_P$ be a sequence of n -dimensional vectors with non-negative components. Let \mathbb{P}_{x_0} be the probability measure for the process X started from x_0 . Let \mathbb{E}_{x_0} be the expectation over the joint distribution of processes X, Y , where X is started from x_0 . Let \mathbb{E} be the expectation with respect to the Gaussian measure corresponding to Y . Then

$$\epsilon \log \mathbb{P}_{x_0} [X(k\Delta t) \in dx_k, k = 1, \dots, P] \leq - \sum_{k=1}^P \lambda_k^T (x_k - x_{k-1}) + \epsilon \log \mathbb{E} \left[\prod_{k=1}^P e^{\frac{\lambda_k^T}{\epsilon} F_k(Y, x_{k-1})} \right] \quad (4)$$

where

$$F_k(Y, x) = \int_{(k-1)\Delta t}^{k\Delta t} d\tau Y^T(\tau/\epsilon, x) M Y(\tau/\epsilon, x) - \nu x \Delta t + O(\Delta t^2), \quad (5)$$

see Appendix A for the derivation. Proceeding as informally as above, let us evaluate the expectation in the right hand side of the above bound. Choose $\Delta t = \sqrt{\epsilon}$. Then

$P = t/\sqrt{\epsilon} = O(\epsilon^{-1/2})$. Approximating the random process Y by a sequence of bounded random variables with a finite dependency range $\delta > 0$, we find¹

$$\begin{aligned} & \log \mathbb{E} \exp \left[\sum_{k=1}^P \frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t}^{k\Delta t} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) + O(\Delta T^2) \right) \right] \\ &= \sum_{k=1}^P \log \mathbb{E} \exp \left[\lambda_k^T \left(\int_{(k-1)\Delta t/\epsilon}^{k\Delta t/\epsilon} d\tau Y^T(\tau, x_{k-1}) M Y(\tau, x_{k-1}) \right) \right] + O(\epsilon^{-1/2}), \end{aligned} \quad (6)$$

see Appendix B for derivation. The remaining expectation is easy to compute using the fact that the process Y_t is stationary Gaussian. Let us define $m := \lambda^T M$, an $N \times N$ symmetric matrix. Let us present it in the form $m = S^T S$, where S is possibly complex Cholesky factor of m . Let us also rewrite

$$\exp \left[\lambda^T \left(\int_0^T d\tau Y^T(\tau, x) M Y(\tau, x) \right) \right] = \int \prod_{\tau=0}^T \mathfrak{D}q(\tau) e^{-\frac{1}{4} \int_0^T d\tau q^T(\tau) q(\tau) + \int_0^T d\tau q^T(\tau) S Y(\tau, x)}$$

(Hubbard-Stratonovich transformation.) Then, for sufficiently small components of λ ,

$$\begin{aligned} & \mathbb{E} \exp \left[\lambda^T \left(\int_0^T d\tau Y^T(\tau, x) M Y(\tau, x) \right) \right] = \int \prod_{\tau=0}^T \mathfrak{D}q(\tau) e^{-\frac{1}{4} \int_0^T d\tau q^T(\tau) q(\tau)} \mathbb{E} \left(e^{\int_0^T d\tau q^T(\tau) S Y(\tau, x)} \right) \\ &= \int \prod_{\tau=0}^T \mathfrak{D}q(\tau) e^{-\frac{1}{4} \int_0^T d\tau q^T(\tau) q(\tau) + \frac{1}{2} \int_0^T d\tau_1 \int_0^T d\tau_2 q^T(\tau_1) S C(t_1 - t_2, x) S^T q(\tau_2)} \\ &= \text{Det}^{-\frac{1}{2}} \left(I - 2S \hat{C}_T(x) S^T \right) = \text{Det}^{-\frac{1}{2}} \left(I - 2m \hat{C}_T(x) \right). \end{aligned} \quad (7)$$

Here $m \hat{C}_T(x)$ is an integral operator acting on (square integrable) \mathbb{R}^N -valued functions as follows:

$$f_\alpha(t) \mapsto m \hat{C}_T(x)(f)_\alpha(t) = \sum_{\beta, \delta=1}^N \int_0^T d\tau m_{\alpha\beta} C_{\beta\delta}(t - \tau, x) f_\delta(\tau), \quad \alpha = 1, \dots, N; t \in \mathbb{R}. \quad (8)$$

Remark. In what follows we will use capital Det and Tr to denote operator determinant and trace, and lowercase det and tr for the determinant and the trace of finite-dimensional matrices.

Thus it turns out, the limit of $\epsilon \rightarrow 0$ in our computation of the large deviation rate function corresponds to the large T asymptotics of the Fredholm determinant entering (7), see the derivation leading to (11) below. Luckily, such an asymptotic can be computed using Widom's theorem, which generalises the celebrated Szego-Kac formula for

¹It is the absence of error estimates associated with this approximation which makes our present discussion non-rigorous.

Fredholm determinants, see [17]: for a sufficiently small m (e. g. with respect to matrix norm),

$$\log \text{Det} \left(I - 2m\hat{C}_T(x) \right) = T \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2m\tilde{C}(k, x) \right) + O(T^0), \quad (9)$$

where

$$\tilde{C}(k, x) = \int_{\mathbb{R}} d\tau e^{ik\tau} C(\tau, x) \quad (10)$$

is the Fourier transform of the autocorrelation function $C(\tau, x)$. This remarkable statement is reviewed in Appendix C. Substituting (7,9) into (4,6) we find

$$\begin{aligned} & \epsilon \log \mathbb{P}_{x_0} [X(p\Delta t) \in dx_p, p = 1, \dots, P] \\ & \stackrel{(4,6)}{\leq} - \sum_{p=1}^P \Delta t \lambda_p^T \left(\frac{x_p - x_{p-1}}{\Delta t} + \nu x_{p-1} \right) + \sum_{p=1}^P \epsilon \log \mathbb{E} \exp \left[\int_0^{\Delta t/\epsilon} d\tau Y^T(\tau, x_{p-1}) m Y(\tau, x_{p-1}) \right] + O(\sqrt{\epsilon}) \\ & \stackrel{(7)}{=} - \sum_{p=1}^P \Delta t \lambda_p^T \left(\frac{x_p - x_{p-1}}{\Delta t} + \nu x_{p-1} \right) - \frac{1}{2} \sum_{p=1}^P \epsilon \log \text{Det} \left(I - 2m\hat{C}_{\frac{\Delta t}{\epsilon}}(x_{p-1}) \right) + O(\sqrt{\epsilon}) \\ & \stackrel{(9)}{=} - \sum_{p=1}^P \Delta t \lambda_p^T \left(\frac{x_p - x_{p-1}}{\Delta t} + \nu x_{p-1} \right) - \frac{1}{2} \sum_{p=1}^P \epsilon \left[\frac{\Delta t}{\epsilon} \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2m\tilde{C}(k, x_{p-1}) \right) + O(\epsilon^0) \right] + O(\sqrt{\epsilon}) \\ & = - \sum_{p=1}^P \Delta t \lambda_p^T \left(\frac{x_p - x_{p-1}}{\Delta t} + \nu x_{p-1} \right) - \frac{1}{2} \sum_{p=1}^P \Delta t \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2\lambda_p^T M \tilde{C}(k, x_{p-1}) \right) + O(\epsilon^{1/2}), \quad (11) \end{aligned}$$

where $P = t/\sqrt{\epsilon}$, $\Delta t = \sqrt{\epsilon}$. Finally, taking the limit $\epsilon \rightarrow 0$, we find

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon \mathbb{P}_{x_0} [X(\tau) \in dx(\tau), 0 \leq \tau \leq t] & \leq - \int_0^t d\tau \left(\lambda^T(\tau) \dot{x}(\tau) + \nu \lambda^T(\tau) x(\tau) \right) \\ & \quad - \frac{1}{2} \int_0^t d\tau \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2\lambda^T(\tau) M \tilde{C}(k, x(\tau)) \right). \end{aligned}$$

Therefore, our informal calculations based on Widom's theorem lead to the following result: the process $X(t)$ satisfies the large deviation principle with rate ϵ and the rate function

$$I[x] = \sup_{\lambda(\tau), 0 \leq \tau \leq t} \left[\int_0^t d\tau \lambda^T(\tau) (\dot{x}(\tau) + \nu x(\tau)) + \frac{1}{2} \int_0^t d\tau \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2\lambda^T(\tau) M \tilde{C}(k, x(\tau)) \right) \right]. \quad (12)$$

We would like to stress that from the mathematical point of view (12) is still a conjecture. Even less formally one can write

$$\mathbb{P}_{x_0} [X(\tau) \in dx(\tau), 0 \leq \tau \leq t] \sim e^{-\frac{1}{\epsilon} I[x]} \quad (13)$$

A typical application of the rate functional guessed above is to estimate the probability of transitioning between fixed points of the typical evolution (3). If x_0, x_1 are two such points, then

$$\log \mathbb{P}_{x_0} [X(t) \in dx_1] \sim -\frac{1}{\epsilon} \left[\inf_{x(\tau), 0 \leq \tau \leq t: x(0)=0, x(t)=x_1} \left(\sup_{\lambda(\tau), 0 \leq \tau \leq t} S_{eff}(\lambda, x) \right) \right], \quad (14)$$

where

$$\begin{aligned} S_{eff}(\lambda, x) &= \int_0^t d\tau \lambda^T(\tau) (\dot{x}(\tau) + \nu x(\tau)) \\ &+ \frac{1}{2} \int_0^t d\tau \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - 2\lambda^T(\tau) M \tilde{C}(k, x(\tau)) \right). \end{aligned} \quad (15)$$

As a self-consistency check, let us verify that the average evolution equation (3) appears as an equation for a typical trajectory for the large deviation principle (14), (15). A typical trajectory $(\lambda_c, x_c)_{0 \leq \tau \leq t}$ is a solution to Euler-Lagrange equations associated with S_{eff} such that

$$S_{eff}(\lambda_c, x_c) = 0.$$

Examining the derivation of the large deviation principle, it is reasonable to expect that $\lambda_c = 0$. Expanding (15) around $\lambda = 0$ we find

$$S_{eff} = \int_0^t d\tau \lambda^T(\tau) (\dot{x}(\tau) + \nu x(\tau) - \text{Tr} [MC(0, x(t))]) + O(\lambda^2), \quad (16)$$

where we used that $\int_{\mathbb{R}} \frac{dk}{2\pi} \tilde{C}(k, x) = C(0, x)$. Therefore, $\lambda = 0$ solves the Euler-Lagrange equations if

$$\dot{x}(\tau) + \nu x(\tau) - \text{Tr} [MC(0, x(t))] = 0, \quad x(0) = x_0,$$

which coincides with (3).

In particular, the fixed points of the slow dynamics are solutions to

$$\nu x = \text{Tr} [MC(0, x)] \quad (17)$$

Remarks.

1. If $N = 1$, and Y solves an Ornstein-Uhlenbeck SDE with X -dependent drift, the corresponding large deviation principle was derived in [15] and is consistent with conjecture (12) for all values of λ . However, in general one has to check that the optimal λ belongs to the domain of applicability of Widom's theorem, which is one of the challenges for the rigorous justification of the conjecture. A natural guess is that the minimizer must be small enough to ensure positive definiteness of the quadratic form in the functional integral (7).

2. If Y appears as a solution to an Ornstein-Uhlenbeck system of stochastic differential equations, then (12) can be viewed as a closed form answer for the trace of the asymptotic solution to the time-dependent matrix Riccati equation derived in [14].
3. In the context of modeling of two-dimensional turbulent flows, equation (1) can be interpreted as follows: Y is a Gaussian model of fast small-scale velocity field whose evolution depends on the static background created by X ; X is a large scale velocity field slowly evolving under the influence of Y . Thus the model can be thought of as a non-linear generalisation of the passive vector advection model. The shape of C reflects the nature of the small scale turbulent flow (compressibility, isotropy, etc.)

4 An example inspired by multistability in hydrodynamic and geostrophic turbulence

The aim of this section is to present an example of the use of the large deviation principle (15). We are specifically interested in multistability phenomena observed in two dimensional [15] and geostrophic [16] turbulent flows. In previous works, we have studied multistability for geostrophic dynamics [18], in cases when the turbulent flows is forced by white noises, and the stochastic process is an equilibrium one with detailed balance or generalized detailed balance. The large deviation principle (15) opens the possibility for studying multistability for turbulents flows modelled as a non-equilibrium process. The aim of this section is not to work out exactly multistabilty for turbulent flows, but rather to devise the simplest possible example, amenable to explicit analysis, with the same properties as the dynamics of turbulent flows.

To formulate the example, it will be easier to use complex notations. The fast variable $Y(\cdot, x) \in \mathbb{C}^N$ will be the analogous of a set of Fourier components that describe the turbulent fluctuations. It is characterised as the stationary solution of the complex Ornstein-Uhlenbeck process,

$$\begin{cases} dY(t, x) = -\Gamma(x)Y(t, x)dt + \sigma dW(t), \\ dX(t) = Y(t/\epsilon, X(t))^* M Y(t/\epsilon, X(t))dt - \nu X(t)dt, \end{cases} \quad (18)$$

where M is a self-adjoint $N \times N$ matrix; dW is the \mathbb{C}^N -valued Brownian motion, with the non-trivial covariance

$$d\bar{W}_i dW_j = \delta_{ij} dt, \quad (19)$$

$\Gamma(x)$ is a complex matrix, whose eigenvalues have positive real parts,

$$\Gamma(x) = \Gamma^{(0)} + ix^T \Gamma^{(1)}, \quad (20)$$

where $\Gamma^{(0)}$ is a real positive definite $N \times N$ matrix, $\Gamma^{(1)}$ is a real $n \times N \times N$ matrix. The former describes dissipation, whereas the latter corresponds to the ‘rotational’ advection of Y by the slow field X . All the coefficients are polynomials of degree at most one

in x . This structure of the system of SDE's (18) resembles that of the quasi-linear approximation to the Navier-Stokes equation, see [6] for details: the non-linearity in the right hand side is quadratic, the evolution of the slow variable is driven by the term quadratic in the fast variable, the drift of the fast variable resembles advection by the slow field X . Let us stress that model does not have any artificial “built-in” non-linearity, but respect strictly the algebraic structure of the Navier-Stokes equation or quasigeostrophic equations, see [6], although the value of the coefficient will be chosen arbitrarily.

Some standard computations lead to formulae for the correlation and auto-correlation functions, $C(0, x) := \mathbb{E}(Y(0, x) \otimes Y^*(0, x))$, $C(\tau, x) := \mathbb{E}(Y(\tau, x) \otimes Y^*(0, x))$. $C(0, x)$ solves the Lyapunov equation,

$$\Gamma(x)C(0, x) + C(0, x)\Gamma^*(x) = \sigma\sigma^*, \quad (21)$$

whereas

$$C(\tau, x) = e^{-\Gamma(x)\tau}C(0, x), \quad \tau \geq 0. \quad (22)$$

If $\tau < 0$, then $C(\tau, x) = C(0, x)e^{\Gamma^*(x)\tau}$. The effective Hamiltonian re-written in complex terms is

$$H(\lambda, x) = -\lambda^T \nu x - \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det \left(I - \lambda^T M \tilde{C}(k, x) \right), \quad (23)$$

where M is an $N \times N$ self-adjoint matrix and

$$\tilde{C}(k, x) := \int_{\mathbb{R}} d\tau e^{ik\tau} C(\tau) = (\Gamma(x) - ik)^{-1}C(0, x) + C(0, x)(\Gamma^*(x) + ik)^{-1}, \quad (24)$$

is the Fourier transform of the auto-correlation function.

Keeping the matters as simple as possible, let us choose $\Gamma^{(0)}$ and $\Gamma^{(1)}$ to be the diagonal matrices with real entries $\{\gamma_p^{(0)}, \gamma_p^{(1)}\}_{1 \leq p \leq N}$, where $\gamma^{(0)}$'s are all positive. The fixed point equation (17) takes the form

$$\sum_{j,k=1}^N \frac{(\sigma\sigma^*)_{jk} (M_\alpha)_{kj}}{\left(\gamma_j^{(0)} + \gamma_k^{(0)} + i \sum_{\beta=1}^n \left[(\gamma_\beta^{(1)})_j - (\gamma_\beta^{(1)})_k \right] x_\beta \right)} = \nu x_\alpha, \quad 1 \leq \alpha \leq n. \quad (25)$$

Notice that if either the noise covariance matrix $\sigma\sigma^*$, or the interaction matrix M_α is diagonal, there is a unique solution for the α -th component of the fixed point. Indeed, if $(M_\alpha)_{kj} = 0$ for all $k \neq j$, then the left hand side of equation (25) becomes x -independent and the equation becomes linear w.r.t x_α . The same remark applies if $\sigma\sigma^*$ is diagonal. Similarly, the fixed point is unique if $(\gamma_\beta^{(1)})_j - (\gamma_\beta^{(1)})_k = 0$ for all j, k, β . However, for general correlated noise, interaction and an inhomogeneous rotation matrix $\gamma^{(1)}$, there are multiple solutions to (26). Moreover, it is easy to choose the coefficients in such a way that there are multiple real solutions, see Fig. 1 for an $n = 1, N = 3$ example

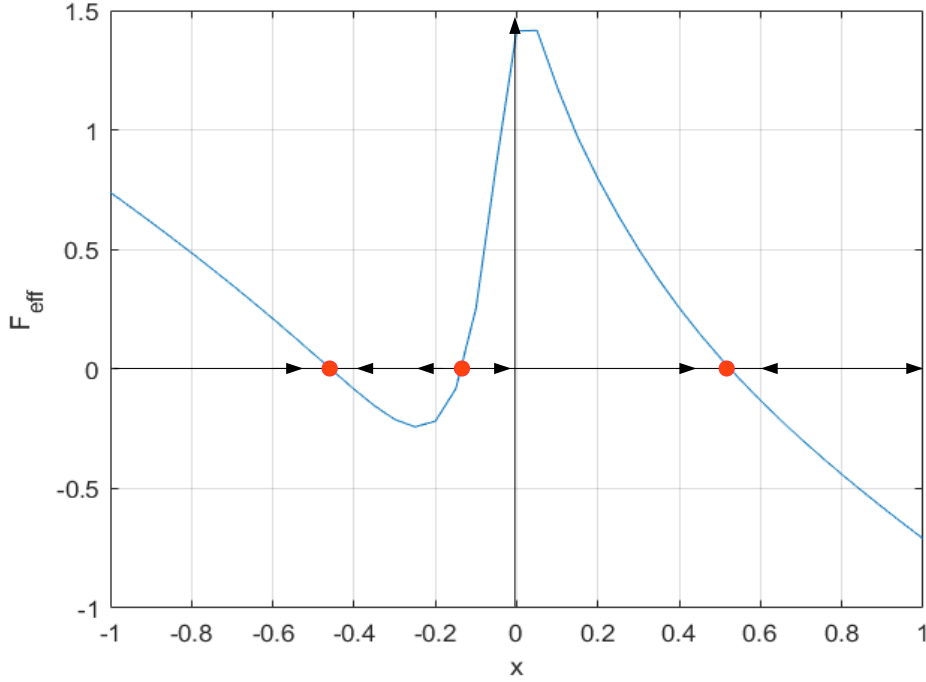


Figure 1: The effective force for the model (26). Notice a pair of stable fixed points of the averaged dynamics separated by an unstable fixed point.

with two stable and one unstable fixed point. The chosen model parameters are: $n = 1$, $N = 3$, $\nu = I_3$

$$\sigma = \frac{1}{2^{1/4}} \begin{pmatrix} -1-i & 0 & 0 \\ 1-i & -1-i & 0 \\ -1-i & 1-i & -1-i \end{pmatrix}, M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \gamma^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \gamma^{(1)} = \pi^2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (26)$$

(The appearance of powers of 2 and π in the above parameterisation has no special meaning. The choice $M_{ii} = 0$ and $M_{ij} = \text{const}$ for $i \neq j$ reflects some properties of the interaction matrix for 2-dimensional Navier-Stokes equation, but it is also not essential for the appearance of multiple equilibria.)

The fact that multiple equilibria appear naturally in the model (18) together with its link to quasi-linear hydrodynamics explained above makes us hope that the large deviation principle (12) might prove useful in studying realistic hydrodynamic phenomena of metastability, such as the zonal-dipole transition discovered in [15]. Euler-Lagrange equations associated with the effective action functional (15) are Hamiltonian with the Hamiltonian (23). Therefore, each solution lies on a constant energy surface $H(\lambda, x) = E$. If there is a single slow variable, the trajectories coincide with constant

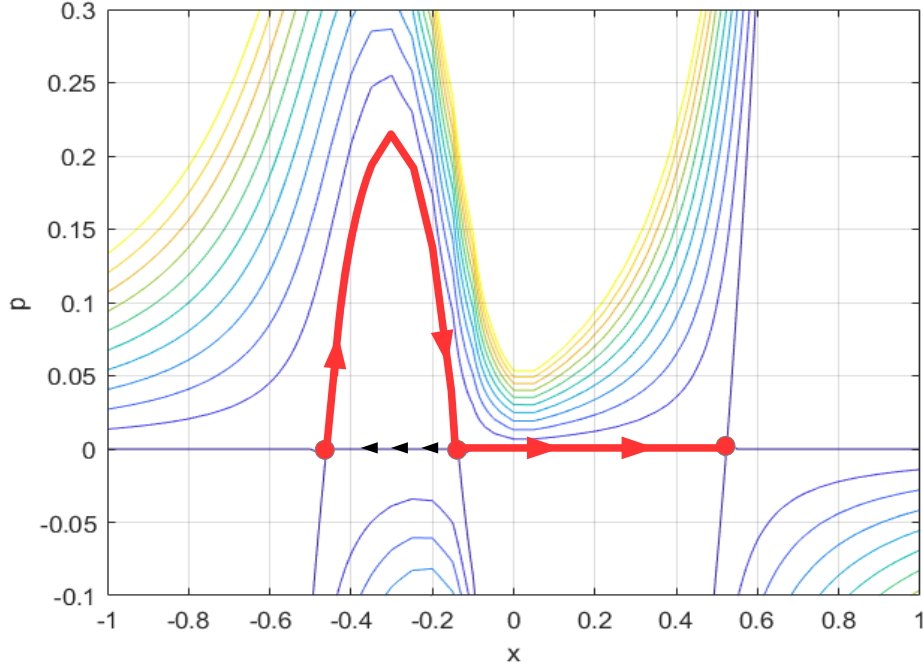


Figure 2: Contour lines of H_{eff} for the model (26). Contour lines in the upper half plane serve as optimal trajectories for transitions between the stable fixed points in a finite time. The red curve is the infinite-time optimal transition curve. Black arrows mark the typical trajectory connecting the unstable and stable fixed points.

energy surfaces. This allows one to determine a family of the most likely transition paths between the fixed points (the instanton trajectories) by building the contour plot of H numerically, see Fig. 2.

5 Conclusions. Outlook

Motivated by hydrodynamic applications, we have considered a model with two-time scales, where the slow variable is driven by a quadratic function of a fast Gaussian process with rapidly decaying auto-correlations. A natural question of computing the probabilities of rare events in this model reduces to the computation of large-interval asymptotics for a certain Fredholm determinant. To the leading order, such a computation can be easily carried out using Widom's theorem. To apply the resulting large deviation principle, we considered a special case of the fast field being a complex Ornstein-Ohlenbeck process with the rotational component of the drift given by a linear function of the slow process. As it turns out, the average slow dynamics for such

a model exhibits multiple equilibria, the transitions between which can be studied using large deviation theory.

There are many natural further questions to ask. Firstly, it should be a straightforward task to furnish a rigorous proof or provide a counter-example to the statement of the conjecture (12). Secondly, for the cases, when the fast process conditional on the value of the slow process is an Ornstein-Uhlenbeck process, it might be interesting to consider finite- ϵ corrections to the leading order answer. Albeit known, the sub-leading terms in the Widom asymptotic are only characterised as solutions to a certain matrix Wiener-Hopf integral equation. There is however a chance of finding these corrections rather more explicitly as solutions to time-dependent Riccati equations derived in [14].

Finally, the model considered has the general structure of many equations of hydrodynamics, plasma dynamics, self-gravitating systems, wave turbulence or other physical system with quadratic couplings or interactions. It would therefore be extremely interesting to analyse metastability for such physical systems, in the presence of time scale separation, using the findings of the present paper.

A The derivation of (4)

$$\begin{aligned}
\mathbb{P}_{x_0} [X(k\Delta t) \in dx_k, k = 1, 2, \dots, P] &= \mathbb{E}_{x_0} \left[\prod_{k=1}^P \mathbb{1}(X(k\Delta t) \in dx_k) \right] \\
&\leq \mathbb{E}_{x_0} \left[e^{\frac{\lambda_P^T}{\epsilon}(X(P\Delta t) - x_P)} \prod_{k=1}^{P-1} \mathbb{1}(X(k\Delta t) \in dx_k) \right] \\
&= e^{-\lambda_P^T \frac{x_P - x_{P-1}}{\epsilon}} \mathbb{E}_{x_0} \left[e^{\frac{\lambda_P^T}{\epsilon}(X(P\Delta t) - x_{P-1})} \prod_{k=1}^{P-1} \mathbb{1}(X(k\Delta t) \in dx_k) \right] \\
&= e^{-\lambda_P^T \frac{x_P - x_{P-1}}{\epsilon}} \mathbb{E}_{x_0} \left[e^{\frac{\lambda_P^T}{\epsilon} F_P(Y, x_{P-1})} \prod_{k=1}^{P-1} \mathbb{1}(X(k\Delta t) \in dx_k) \right].
\end{aligned}$$

The first inequality is due to Chebyshev, the last equality follows from solving (1) over a short time interval. Repeating the above steps $(P-1)$ times, we find that

$$\mathbb{P}_{x_0} [X(k\Delta t) \in dx_k, k = 1, \dots, P] \leq \exp \left[-\frac{1}{\epsilon} \sum_{k=1}^P \lambda_k^T (x_k - x_{k-1}) \right] \mathbb{E} \left[\prod_{k=1}^P e^{\frac{\lambda_k^T}{\epsilon} F_k(Y, x_{k-1})} \right],$$

which is equivalent to (4).

B The derivation of (6)

Recall that for this derivation $Y(\tau, x_k)$'s are approximated by a sequence of bounded random variables with a finite dependency length δ .

$$\begin{aligned}
& \log \mathbb{E} \exp \left[\sum_{k=1}^P \frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t}^{k\Delta t} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) + O(\Delta T^2) \right) \right] \\
&= \log \mathbb{E} \exp \left[\sum_{k=1}^P \frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t + \delta\epsilon}^{k\Delta t - \delta\epsilon} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) + O(\epsilon) \right) \right] \\
&= \log \mathbb{E} \exp \left[\sum_{k=1}^P \frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t + \delta\epsilon}^{k\Delta t - \delta\epsilon} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) \right) \right] \exp(O(\epsilon^{-1/2})) \\
&= \sum_{k=1}^P \log \mathbb{E} \exp \left[\frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t + \delta\epsilon}^{k\Delta t - \delta\epsilon} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) \right) \right] + O(\epsilon^{-1/2}) \\
&= \sum_{k=1}^P \log \mathbb{E} \exp \left[\frac{\lambda_k^T}{\epsilon} \left(\int_{(k-1)\Delta t}^{k\Delta t} d\tau Y^T(\tau/\epsilon, x_{k-1}) M Y(\tau/\epsilon, x_{k-1}) \right) \right] + O(\epsilon^{-1/2}) \\
&= \sum_{k=1}^P \log \mathbb{E} \exp \left[\lambda_k^T \left(\int_{(k-1)\Delta t/\epsilon}^{k\Delta t/\epsilon} d\tau Y^T(\tau, x_{k-1}) M Y(\tau, x_{k-1}) \right) \right] + O(\epsilon^{-1/2}),
\end{aligned}$$

which is (6).

C Widom's theorem

We will follow the original paper by Kac [19] and state the simplest set of conditions leading to the formula (9).

Let $K : \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$ be an $N \times N$ matrix-valued function of one variable. Assume that K is even ($K(t) = K(-t)$, for any $t \in \mathbb{R}$) and non-negative ($K_{ij}(t) \geq 0$ for any $t \in \mathbb{R}$ and $1 \leq i, j \leq N$). Assume in addition that

$$\int_{\mathbb{R}} |t| K(t) dt < \infty, \quad (27)$$

$$\int_{\mathbb{R}} \sum_{k=1}^N K_{ki} \leq 1, \quad 1 \leq i \leq N. \quad (28)$$

The function K can be regarded as a kernel of an integral operator \hat{K} acting on square-integrable functions from \mathbb{R} to \mathbb{R}^N ,

$$f \mapsto \hat{K}f(t) = \int_{\mathbb{R}} d\tau K(t - \tau) f(\tau), \quad t \in \mathbb{R}. \quad (29)$$

Then there is $\lambda_{max} > 0$ such that for any $\lambda : |\lambda| < \lambda_{max}$ the Fredholm determinant $\text{Det}(I - \lambda \hat{K}_T)$ exists and

$$\log \text{Det}(I - \lambda \hat{K}_T) = T \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det(1 - \lambda \tilde{K}(k)) + O(T^0), \quad (30)$$

where \hat{K}_T is the restriction of \hat{K} to functions on $[0, T]$ and

$$\tilde{K}(k) = \int_{\mathbb{R}} dx e^{-ikx} K(x), \quad k \in \mathbb{R}. \quad (31)$$

Remarks.

1. In [17] Widom presents a stronger version of the above statement which characterises the $O(T^0)$ term fully. For the current paper we only need the leading term.
2. The actual statement of Widom's theorem does not require the positivity of the kernel. In fact, all steps of the proof presented below go through for signed kernels as well, but the probabilistic intuition guiding these steps is lost. See also [19] for similar remarks about the original proof of Szego's theorem by Marc Kac.

Let us sketch the proof of the theorem using, as we already mentioned, the probabilistic method used in [19] to prove a continuous version of Szegő's formula for the asymptotics of Toeplitz determinants. For a sufficiently small $|\lambda|$ we can calculate the Fredholm determinant using the trace-log formula,

$$\log \text{Det}(I - \lambda \hat{K}_T) = - \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n \text{Tr} \hat{K}_T^n, \quad (32)$$

where

$$\text{Tr} \hat{K}_T^n = \int_{[0, T]^n} dx_1 dx_2 \dots dx_n \text{tr} K(x_1 - x_2) K(x_2 - x_3) \dots K(x_n - x_1).$$

Using the cyclic property of trace and the fact that the function K is even, we find

$$\frac{d}{dT} \text{Tr} \hat{K}_T^n = n \int_{[0, T]^n} dx_2 dx_3 \dots dx_n \text{tr} K(0 - x_2) K(x_2 - x_3) \dots K(x_{n-1} - x_n) K(x_n - 0). \quad (33)$$

Consider the following discrete time Markov chain $\{X_n, S_n\}_{n \geq 0}$ on the state space $\mathbb{R} \times \{1, 2, \dots, N\}$:

1. $(X_0, S_0) \sim (\delta_0, U_N)$, where U_N is the uniform distribution on $\{1, 2, \dots, N\}$.
2. At each time step, the transition $(x, i) \rightarrow (y, k)$ happens with probability $K_{ki}(y - x) dy$.

Notice that this is a Markov chain with killing, the survival probability when transitioning from state (x, i) is $g_i(x) := \sum_{k=1}^N \int_{\mathbb{R}} K_{ki}(y-x) dy \leq 1$. Examining the expression (33) for the derivative of the trace of the n -th power of \hat{K} , we see that it can be interpreted as the following expectation with respect to the law of the chain $\{X_n, S_n\}_{n \geq 0}$:

$$\frac{d}{dT} \text{Tr} \hat{K}_T^n = Nn \mathbb{E} (\mathbb{1}(X_n \in d0) \mathbb{1}(S_n = S_0) \mathbb{1}(\tau = n)), \quad (34)$$

where τ is the first exit time of the chain from the interval $(0, T) \times \{1, 2, \dots, N\}$. To derive the above expression we exploited the identity $\mathbb{1}(X_n \in d0) \mathbb{1}(\tau \geq n) = \mathbb{1}(X_n \in d0) \mathbb{1}(\tau = n)$. Substituting (34) into (33) and then (32), we find that

$$\begin{aligned} \frac{d}{dT} \log \text{Det}(I - \lambda \hat{K}_T) &= -N \mathbb{E} (\lambda^\tau \mathbb{1}(X_\tau \in d0) \mathbb{1}(S_\tau = S_0)) \\ &= -N \mathbb{E} (\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) \mathbb{1}(M_{\tau_0} < T) \mathbb{1}(S_{\tau_0} = S_0)), \end{aligned}$$

where τ_0 is the first exit time from $(0, \infty) \times \{1, 2, \dots, N\}$, $M_{\tau_0} = \max_{1 \leq n < \tau_0} (X_n)$. As $\log \det(I - \lambda \hat{K}_0) = 0$, we can integrate the last expression to find

$$\log \text{Det}(I - \lambda \hat{K}_T) = -N \mathbb{E} (\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) (T - M_{\tau_0})_+ \mathbb{1}(S_{\tau_0} = S_0)),$$

where $(x)_+ := \max(x, 0)$. Noticing that $T - (T - M)_+ = \min(T, M)$, we can re-arrange the above expression as follows:

$$\begin{aligned} \log \text{Det}(I - \hat{K}_T) &= -NT \mathbb{E} (\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) \mathbb{1}(S_{\tau_0} = S_0)) \\ &\quad + N \mathbb{E} (\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) \min(T, M_{\tau_0}) \mathbb{1}(S_{\tau_0} = S_0)), \end{aligned}$$

This is an exact expression for the Fredholm determinant as an expectation with respect to the law of the Markov chain we defined. In many cases it allows for an efficient computation of the large- T expansion of the Fredholm determinant using purely probabilistic methods. For us it is sufficient to check that $\lim_{T \rightarrow \infty} \min(T, M_{\tau_0}) = M_{\tau_0}$, which implies that

$$\log \text{Det}(I - \lambda \hat{K}_T) = -NT \mathbb{E} (\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) \mathbb{1}(S_{\tau_0} = S_0)) + O(T^0). \quad (35)$$

To calculate the expectation entering the leading term we use the following combinatorial lemma (see e. g. [20], volume 2): Let $(0, R_1, R_1 + R_2, \dots, R_1 + R_2 + \dots + R_{n-1}, 0)$ be the first n \mathbb{R} -projections of the states of the chain with $\tau_0 = n$. Then

$$\sum_{p=0}^{n-1} \prod_{k=1}^{n-1} \mathbb{1}(R_{1+p} + R_{2+p} + \dots + R_{k+p} > 0) = 1 \text{ a. s., } 0 \leq p \leq n-1. \quad (36)$$

The addition of subscripts in the above formula should be understood modulo n . The above statement is very general and relies only on the absence of atoms in the transition probabilities $K(y-x)dy$.

In this case, for any sequence $(0, R_1, R_1 + R_2, \dots, R_1 + R_2 + \dots + R_{n-1}, 0)$, its graph will almost surely have a unique global minimum, so there will be a unique cyclic permutation

$(0, R_{1+p}, R_{1+p} + R_{2+p}, \dots, R_{1+p} + R_{2+p} + \dots + R_{n-1+p}, 0)$, whose graph will stay positive between times 1 and $n - 1$. Then

$$\begin{aligned}
N\mathbb{E}(\lambda^{\tau_0} \mathbb{1}(X_{\tau_0} \in d0) \mathbb{1}(S_{\tau_0} = S_0)) &= N \sum_{n=1}^{\infty} \lambda^n \mathbb{E}(\mathbb{1}(X_{\tau_0} \in d0) \mathbb{1}(S_{\tau_0} = S_0) \mathbb{1}(\tau_0 = n)) \\
&= \sum_{n=1}^{\infty} \lambda^n \int_{\mathbb{R}^n} dr_1 \dots dr_n \text{tr}(K(r_1) \dots K(r_n)) \delta(r_1 + \dots + r_n) \prod_{k=1}^{n-1} \mathbb{1}(r_1 + \dots + r_k > 0) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \int_{\mathbb{R}^n} dr_1 \dots dr_n \text{tr}(K(r_1) \dots K(r_n)) \delta(r_1 + \dots + r_n) \sum_{p=0}^{n-1} \prod_{k=1}^{n-1} \mathbb{1}(r_{1+p} + \dots + r_{k+p} > 0) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \int_{\mathbb{R}^n} dr_1 \dots dr_n \text{tr}(K(r_1) \dots K(r_n)) \delta(r_1 + \dots + r_n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \int_{\mathbb{R}} \frac{dk}{2\pi} \int_{\mathbb{R}^n} dr_1 \dots dr_n e^{-ik(r_1 + \dots + r_n)} \text{tr}(K(r_1) \dots K(r_n)) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \int_{\mathbb{R}} \frac{dk}{2\pi} \text{tr}(\tilde{K}(k_1) \dots \tilde{K}(k_n)) = - \int_{\mathbb{R}} \frac{dk}{2\pi} \log \det(I - \lambda \tilde{K}(k)). \tag{37}
\end{aligned}$$

The third inequality is the symmetrisation of the integrand with respect to all cycling permutations, the fourth inequality is due to the combinatorial lemma (36). Substituting (37) into (35), we arrive at the statement (30) of Widom's theorem.

References

- [1] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2015 *Reviews of Modern Physics* **87** 593
- [2] Saito K and Dhar A 2011 *Physical Review E* **83** 041121
- [3] Lifshitz E M and Pitaevskii L P 1981 *Physical kinetics* (Course of theoretical physics, Oxford: Pergamon Press, 1981)
- [4] Nicholson D 1983 *Introduction to plasma theory* (Wiley, New-York)
- [5] Binney J and Tremaine S 1987 *Galactic dynamics* (Princeton, NJ, Princeton University Press, 1987, 747 p.)
- [6] Bouchet F, Nardini C and Tangarife T 2013 *J. Stat. Phys.* **153** 572–625
- [7] Nazarenko S 2011 *Wave turbulence* vol 825 (Springer Science & Business Media)
- [8] Pavliotis G and Stuart A 2008 *Multiscale methods: averaging and homogenization* (Springer Science & Business Media)

- [9] Freidlin M I, Sziucs J and Wentzell A D 2012 *Random perturbations of dynamical systems* vol 260 (Springer Science & Business Media)
- [10] Freidlin M I 1978 *Russian Mathematical Surveys* **33** 117–176
- [11] Veretennikov A Y 2000 *Stochastic Processes and their Applications* **89** 69–79
- [12] Kifer Y 1992 *Inventiones Mathematicae* **110** 337–370
- [13] Kifer Y 2004 *Ergodic Theory and Dynamical Systems* **24** 847–871
- [14] Bouchet F, Grafke T, Tangarife T and Vanden-Eijnden E 2016 *Journal of Statistical Physics* **162** 793–812
- [15] Bouchet F and Simonnet E 2009 *Physical Review Letters* **102** 094504
- [16] Bouchet F, Rolland J and Simonnet E 2019 *Physical Review Letters* **122** 074502
- [17] Widom H 1980 *Journal of Functional Analysis* **39** 182–198
- [18] Bouchet F, Laurie J and Zaboronski O 2014 *JOURNAL OF STATISTICAL PHYSICS* **156** 1066–1092 ISSN 0022-4715
- [19] Kac M *et al.* 1954 *Duke Mathematical Journal* **21** 501–509
- [20] Feller W 2008 *An introduction to probability theory and its applications, vol 2* (John Wiley & Sons)